# TOPOLOGICAL PROOF OF THE NIELSEN-WILLSON THEOREM

by

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Abstract

This paper proves that the recent result in [1] when specialized to transistor circuits is equivalent to the Nielsen-Willson theorem. The proof is graph-theoretic in nature.

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## I. Introduction

Nielsen and Willson gave the necessary and sufficient condition for a transistor circuit to have a unique solution for all values of circuit parameters [2]. Although their result is stated in topological terms, their proof is analytical rather than topological.

Recently, Nishi and Chua [1] gave the necessary ansd sufficient conditions for a more general class of nonlinear resistive circuits containing current-controlled current sources (or voltage-controlled voltage sources) to have a unique solution for all values of circuit parameters. This result is stated in terms of the fundamental cutset or loop matrix of the associated graph.

The result in [1] differs from that in [2] in the following two respects: i) The proof of [1] uses only matrix manipulations familiar in graph theory and can be easily extended to more general classes of circuits; ii) The circuits treated in [1] contain as a special case the transistor circuits treated in [2] (where each transistor is represented by the Ebers-Moll model). However the relationship between the results in [1] and [2] is not clear because the conditions are stated in quite different forms.

Our main objective of this paper is to prove that the results in [1] and [2] are in fact equivalent to each other for transistor circuits modelled by the Ebers-Moll equation.

## II. Theorem

In this section we first briefly review both the Nielsen-Willson theorem [2] and a recent theorem in [1]. We will then present a new theorem which relates these two theorems.

Assumption 1. All circuits are connected.

Theorem 1. (Nielsen and Willson [2]).

A transistor circuit $N_T$ has a unique solution for all values of circuit parameters if and only if we cannot obtain the feedback structure shown in Fig. 1 by applying the following three operations (I), (II) and (III) to $N_T$:

(I) Short-circuit all voltage sources and open-circuit all current sources.

(II) Short-circuit or open-circuit each linear or nonlinear resistor.

(III) Replace all transistors except two by one of the five circuits shown in Figs. 2(b)-(f).

Consider next a CCCS circuit $N_c$ made of linear positive resistors, strictly monotone-increasing nonlinear resistors, dc sources, and linear CCCS with current gains $\alpha$ satisfying $0 < \alpha < 1$. The associated graph G of the circuit $N_c$ is obtained from $N_c$ as follows:

(i) Short-circuit all voltage sources and open-circuit all current sources.

(ii) Replace each linear or nonlinear resistor by a nonoriented branch (which is called a resistor branch).

(iii) Replace each CCCS $\mu$ ($\mu$=1,2,... ) by a pair of directed branches $a_\mu$ and $b_\mu$ as shown in Fig. 3. The branches $a_\mu$ and $b_\mu$ are called an a- and a b- branch, respectively.

Note that an a- and a b- branch correspond to an input and an output port branch of a CCCS. Note also that the directions of the branches in Fig. 3 are identical to those of the currents in the CCCS. This is different from those in the previous paper [4], which requires an unconventional graph in order to include all 4 types of controlled sources.

From Assumption 1, the graph G is connected.

The notations $\mathcal{O}(\cdot)$, $\mathcal{S}(\cdot)$, $\mathcal{O}/\mathcal{S}(\cdot)$ and $Z(\cdot)$ are introduced as in [4]. Note in particular that zero operation $Z(\cdot)$ is applied only to a pair of CCCS branches and that $Z(\mu)$ means "$\mathcal{S}(a_\mu)$ <u>and</u> $\mathcal{O}(b_\mu)$."

We apply the following operations to G.

(I)' Apply $\mathcal{O}/\mathcal{S}(\cdot)$ to each resistor branch.

(II)' Apply $Z(\cdot)$ to some (possibly none) pairs of CCCS branches.

After the application of operations (I)' and (II)' we obtain a graph composed only of some pairs of CCCS branches. Consider such a graph $G_0$ composed of n pairs of CCCS branches.

<u>Assumption 2</u>. The a-branches form a tree of $G_0$ and $G_0$ is connected.

Let the fundamental cutset matrix $C_f$ of $G_0$ be

$$C_f = \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cc} a_1 \dots a_n & b_1 \dots b_n \end{array} \\ \left[ \begin{array}{c:c} \quad I \quad & \quad Q \quad \end{array} \right] \end{array} \tag{1}$$

We define $\Delta(G_0)$ by

$$\Delta(G_0) = |I+Q| \tag{2}$$

Remember that $\Delta(G_0)$ is defined only for a graph satisfying Assumption 2.

Theorem 2 [1]. A CCCS circuit $N_c$ has a unique solution for all values of circuit parameters if and only if G satisfies the following three conditions:

(i) There exists no loop composed exclusively of a-branches.

(ii) There exists no cutset composed exclusively of b-branches.

(iii) By applying operations (I)' and (II)' to G we cannot obtain any graph $G_0$ satisfying $\Delta(G_0) < 0$.

In Theorem 1 the transistors are assumed to be represented by the Ebers-Moll model shown in Fig. 4(a). Let N denote the circuit obtained from a transistor circuit $N_T$ by replacing each transistor with the Ebers-Moll model. Since N is a CCCS circuit, we can apply Theorem 2 to N. Note that G, the associated graph of N, always satisfies conditions (i) and (ii) of Theorem 2. To prove Theorem 1 by using Theorem 2, it is sufficient for us to prove the following theorem.

Theorem 3. We can obtain a graph $G_0$ satisfying $\Delta(G_0) < 0$ by applying operations (I)' and (II)' to G if and only if a feedback structure can be obtained from the original transistor circuit $N_T$ by applying operations (I), (II), and (III) of Theorem 1.

## III. Proof of Theorem 3

### 3.1. Necessity

For the subsequent discussion let $G(N_T)$ denote the associated graph of the circuit obtained from a transistor circuit $N_T$ by replacing each transistor with the Ebers-Moll model. Suppose the following assumption holds:

Assumption 3. Applying operations (I)' and (II)' to the associated graph $G(N_T)$, we can obtain a graph such that $\Delta(G_0) < 0$.

To show the necessity we have to show that $N_T$ can be reduced to a feedback structure by applying operations (I)-(III) under Assumption 3. For the moment we adopt the following operations:

(A) Apply the operation in Fig. 2(f) to some (possibly none) transistor of $N_T$. Let $\tilde{N}_T$ denote the resulting transistor circuit.

(B) Apply operations (I)' and (II)' to the associated graph $G(\tilde{N}_T)$.

As a result of operations (A) and (B) we can obtain a graph $G_0$ such that

$$\Delta(G_0) < 0.^{\dagger} \tag{3}$$

In general there exist many graphs satisfying (3). So we impose the following:

Assumption 4. The graph $G_0$ has a minimum number of vertices among all such graphs satisfying (3).

Let n denote the number of a-branches in $G_0$ and let the fundamental cut-set matrix of $G_0$ be given by (1). Then (3) implies

$$|I+Q| < 0. \tag{4}$$

The proof of necessity will be accomplished by proving the following two propositions:

Proposition 1. Operations (A) and (B) are equivalent to operations (II) and (III) of Theorem 1.

Proposition 2. The graph $G_0$ corresponds to a feedback structure.

The following lemmas prove these propositions.

Lemma 1. Any principal minor of $I+Q$ (excluding $|I+Q|$ itself) is nonnegative.

Proof. See Appendix 1.

Lemma 2. For our purpose, applying operations (I)' and (II)' to the Ebers-Moll model is equivalent to replacing the graph in Fig. 4(b) by the graphs in Figs. 5(a)-(d).

Proof. See Appendix 2.

Figures 5(c) and (d) are the same as Figs. 2(e) and (b), respectively. If we apply operation $Z(\cdot)$ to Figs. 5(a) and (b), we would obtain Figs. 2(c) and (d), respectively. Thus operation (II)' contains the operations in Figs. 2(b)-(e) but not the operation of Fig. 2(f). From this we conclude that Proposition 1 holds.

Remark 1: A pair of CCCS branches $(a_\mu, b_\mu)$ of $G_0$ corresponds to a transistor.

Let $G_{0t}$ denote the graph obtained from $G_0$ by deleting each b-branch $b_\mu$ ($\mu=1,\ldots,n$) and by adding an e-branch, $e_\mu$ ($\mu=1,2,\ldots,n$), as shown in Fig. 6.

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† Remember that $\Delta(\cdot)$ is defined only for graphs satisfying Assumption 2. The existence of a graph $G_0$ satisfying (3) is guaranteed by Assumption 3, since the case where $N_T = \tilde{N}_T$ is included in operation (A).

We call $G_{0t}$ a transition graph of $G_0$.[†] Since there is a one-to-one correspondence between $G_0$ and $G_{0t}$, we write this relation as $G_0 \sim G_{0t}$. Note that the a-branches form a tree even in $G_{0t}$.[††] Let the fundamental cutset matrix $C_{tf}$ of the graph $G_{0t}$ be

$$C_{tf} = \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cc} a_1 \ldots a_n & e_1 \ldots e_n \end{array} \\ \left[ \begin{array}{c|c} \; I \; & \; P \; \end{array} \right] \end{array} \quad . \tag{5}$$

Lemma 3.

$$P = I+Q \tag{6}$$

Proof. See Appendix 3.

Lemma 4.

$$|P| = |I+Q| = -1 \tag{7}$$

Proof. Since P is a totally unimodular matrix, each minor of P is 0 or ± 1. From this and from (4), (7) follows.

Lemma 5. The graph $G_{0t}$ contains no loop composed only of e-branches. Thus the e-branches form a tree of $G_{0t}$.

Proof. See Appendix 4.

Lemma 6. No principal minor of P(=I+Q) is positive.

Proof. See Appendix 5.

Lemma 7. No principal minor of Q is positive.

Proof. See Appendix 6.

Lemma 8. Each diagonal element of Q equals -1.

This lemma follows from Lemma 6.

The following identity holds:

$$|I+Q| = 1 + \text{the sum of all principal minors of } Q \tag{8}$$

From (7), (8) and Lemma 7 we conclude that

$$n = 2$$

and

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[†] The subscript t of $G_{0t}$ means a transition graph.

[††] Recall the a-branches form a tree of $G_0$.

$$Q = \begin{bmatrix} -1 & \pm 1 \\ \pm 1 & -1 \end{bmatrix}. \tag{9}$$

It is easily seen that the ± signs in (9) should be taken as + because of the directions of the branches in Fig. 6(a). Thus we have

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{10}$$

The matrix (10) corresponds to the graph in Fig. 7. Since each pair of CCCS branches corresponds to one transistor (see Fig. 6(a) and Remark 1), we have:

Lemma 9. The graph $G_0$ in Fig. 7 represents a feedback structure.
Lemma 9 proves Proposition 2. Thus we conclude that under Assumption 3 a feedback structure can be derived by applying operations (I)-(III) to $N_T$.

### 3.2 Sufficiency

Suppose that a feedback structure F can be obtained from a transistor circuit $N_T$ by applying operations (I)-(III). We will prove that we can obtain a graph $G_0$ satisfying $\Delta(G_0) < 0$ by applying operations (I)' and (II)' to G. Since operations (I) and (II) are the same as operation (i) in Section II and operation (I)', we consider mainly operations (III) and (II)'. As is seen from the preceding necessity proof, the essential difference between operations (III) and (II)' is the fact that the former include the operation in Fig. 2(f) but the latter does not.

Case 1: F can be obtained without using the operation in Fig. 2(f).

In this case we can also obtain F by applying operation (II)'. Figure 8 shows F with each transistor replaced by the Ebers-Moll model. Here the transistor $T_\mu$ ($\mu$=1,2) is represented by two pairs of CCCS branches, $(a_\mu, b_\mu)$ and $(\tilde{a}_\mu, \tilde{b}_\mu)$ and two resistor branches $R_\mu$ and $\tilde{R}_\mu$.† We use the same notation below. By applying operations (I)' and (II)' to Fig. 8, we can obtain a graph $G_0$ in Fig. 7, for which $\Delta(G_0) < 0$.

Case 2: F can be obtained only by applying the operation in Fig. 2(f).

Without loss of generality we assume the following:

Assumption 5. F is obtained from $N_T$ by the operation in Fig. 2(f) only.

Assumption 6. There is no other way to get F than applying the operation in Fig. 2(f).

Referring to the Assumption 5, we can assume $N_T$ consists of n (> 2)

†Two pairs of branches $(a_\mu, b_\mu)$ and $(\tilde{a}_\mu, \tilde{b}_\mu)$ are called "complementary" to each other.

transistors only. Let two transistors in F be $T_1$ and $T_2$, and let the others in $N_T$ be $T_3, T_4, \ldots, T_n$. It follows from Case 1 that by applying operations (I)' and (II)' to F we can obtain the graph in Fig. 7 which satisfies $\Delta(G_0) < 0$. Let $G_{0t}$ be the transition graph of $G_0$.

Suppose that we replace the transistors $T_1$ and $T_2$ in $N_T$, as shown in Fig. 6(a), by two pairs of CCCS branches $(a_1, b_1)$ and $(a_2, b_2)$, respectively. Remember that $(a_1, b_1)$ and $(a_2, b_2)$ were in $G_0$. Replace furthermore each remaining transistor $T_\mu (\mu=3,4,\ldots,n)$ by either one of two complementary pairs of CCCS branches $(a_\mu, b_\mu)$ and $(\tilde{a}_\mu, \tilde{b}_\mu)$, as shown in Fig. 6(a). Let the graph obtained above be $\hat{G}_0$. Of course $\hat{G}_0$ consists of n pairs of CCCS branches. Let $\hat{G}_{0t}$ be the transition graph of $\hat{G}_0$. In general a-branches or e-branches do not necessarily form a tree of $\hat{G}_{0t}$. However we have:

Lemma 10. By appropriately replacing $T_\mu$ $(\mu=3,4,\ldots,n)$, by a-branches and e-branches, we can get a transition graph $\hat{G}_{0t}$ such that

i) a-branches form a tree of $\hat{G}_{0t}$

ii) e-branches form a tree of $\hat{G}_{0t}$

Proof. See Appendix 7.

Let the fundamental cutset matrices of $\hat{G}_0$ and $\hat{G}_{0t}$ be

$$\hat{C}_f = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} \begin{array}{c|c|c|c} a_1 a_2 & a_3 \cdots a_n & b_1 b_2 & b_3 \cdots b_n \\ \hline \begin{matrix} 1 & \\ & 1 \end{matrix} & 0 & Q_{11} & Q_{12} \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & Q_{21} & Q_{22} \end{array}$$

$$\hat{C}_{tf} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} \begin{array}{c|c|c|c} a_1 a_2 & a_3 \cdots a_n & e_1 e_2 & e_3 \cdots e_n \\ \hline \begin{matrix} 1 & \\ & 1 \end{matrix} & 0 & P_{11} & P_{12} \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & P_{21} & P_{22} \end{array}$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = I + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} .$$

From Lemma 10 it follows that

$$|P_{22}| \neq 0 \tag{11}$$

Since by Assumptions 5 and 6 $G_{0t}$ is obtained from $\hat{G}_{0t}$ by applying operations $O(a_\mu)$ and $S(e_\mu)$ ($\mu=3,4,\ldots,n$), the fundamental cutset matrix of $G_{0t}$ is given by

$$C_{tf} = \begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 a_2 & e_1 & e_2 \end{array} \\ \left[ \begin{array}{c:c} I & P_{11}-P_{12}P_{22}^{-1}P_{21} \end{array} \right] \end{array} \tag{12}$$

Since $\Delta(G_0) < 0$, we have

$$|P_{11}-P_{12}P_{22}^{-1}P_{21}| < 0. \tag{13}$$

From Lemma 10 it follows that

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \neq 0$$

We have to consider two cases:

(i) $\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} < 0$

In this case we have

$\Delta(\hat{G}_0) < 0$

(ii) $\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0$

In this case we conclude from (13) that

$|P_{22}| < 0.$

Let $\tilde{G}_0$ denote the graph obtained from $\hat{G}_0$ by applying $Z(\cdot)$ to two pairs of CCCS branches $(a_1,b_1)$ and $(a_2,b_2)$. Then we have

$\Delta(\tilde{G}_0) = |P_{22}| < 0$

In any case we can get a graph satisfying $\Delta(\cdot) < 0$.

This completes the proof of Theorem 3.

## Appendix 1. Proof of Lemma 1

Let

$$Q = \begin{array}{cc} \overbrace{\phantom{Q_{11}}}^{k} & \overbrace{\phantom{Q_{12}}}^{n-k} \\ \end{array}$$

$$Q = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array}\right] \begin{array}{l} \}\ k \\ \}\ n-k \end{array} \tag{A1.1}$$

where $Q_{11}$ and $Q_{22}$ are square matrices. Assume that $|I+Q_{11}| < 0$. Let $\hat{G}_0$ denote the graph obtained by applying $Z(\mu)$ $(\mu=k+1,k+2,\ldots,n)$ to $G_0$. Then $\hat{G}_0$ is connected and its fundamental cutset matrix is given by $[I \vdots Q_{11}]$. Therefore we have $\Delta(\hat{G}_0) < 0$. Now $\hat{G}_0$ has fewer vertices than $G_0$, which contradicts Assumption 4.

## Appendix 2. Proof of Lemma 2

By applying operations (I)' and (II)' to a graph of the Ebers-Moll model (see Fig. 4(b)), we can obtain 16 distinct graphs, some of which are illustrated in Figs. 5(a)-(g). It is sufficient for us to consider only the cases where

1) no b-branch is a self-loop as shown in Fig. 5(g).
2) no a-branch is a bridge (which means a branch which itself forms a cutset) as shown in Fig. 5(f).
3) 2 pairs of CCCS branches as shown in Fig. 5(e) do not remain simultaneously in $G_0$.

The reasons are as follows:

Suppose that a b-branch, say $b_1$, is a self-loop in $G_0$ satisfying (3). Then the fundamental cutset matrix of $G_0$ is given by

$$C_f = \begin{array}{c|cccc|c|c} & a_1 & \cdots & & a_n & b_1 & b_2 \cdots b_n \\ \hline a_1 & 1 & & & & 0 & \\ \hline a_2 & & 1 & & & & \\ \vdots & & & \ddots & & 0 & Q_1 \\ a_n & & & & 1 & & \end{array} .$$

Since $|I+Q| = |I+Q_1|$, we have $|I+Q_1| < 0$, which contradicts Lemma 1. We can disucss in a dual way the case where an a-branch is a bridge.

Suppose next that Fig. 5(e) remains in $G_0$. Then the fundamental cutset matrix of $G_0$ is given by

$$C_f = \begin{array}{c} \begin{array}{ccccc|cc|c} a_1 & a_2 & a_3 & \cdots & a_n & b_1 & b_2 & b_3 \cdots b_n \end{array} \\ \left[\begin{array}{ccccc|cc|c} 1 & & & & & 0 & -1 & \\ & 1 & & & & -1 & 0 & \\ \hline & & 1 & & & & & \\ & & & \ddots & & & 0 & \\ & & & & 1 & & & \end{array}\right] \end{array}$$

from which we have

$$|I+Q| = 0$$

This contradicts $\Delta(G_0) < 0$.

Among the 16 graphs cited above, only the graphs in Figs. 5(a)-(d) satisfy 1)-3) above.

## Appendix 3. Proof of Lemma 3

We can identify the branches $a_\mu$ and $b_\mu$ with the $\mu$-th and the $(n+\mu)$-th column of $C_f$ in (1), respectively. Similarly we identify a branch $e_\mu$ with the $\mu$-th column of P in (5). Since from Fig. 6 it follows that

$$e_\mu = a_\mu + b_\mu \tag{A3.1}$$

we have Lemma 3.

Appendix 4. Proof of Lemma 5

Suppose that branches $e_\mu$ ($\mu=1,2,\ldots,k$) form a loop. Then we have

$$e_1 \pm \ldots \pm e_k = 0 \tag{A4.1}$$

Here the $\pm$ sign should be taken appropriately. (Note that $e_\mu$ is identified with the $\mu$-th column of $P$). Equation (A4.1) implies that columns of $P$ are not independent. Thus we see $|P| = |I+Q| = 0$, which contradicts (4).

## Appendix 5. Proof of Lemma 6

Before we prove Lemma 6, we will define the following two operations:

Operation (A'). Let $(a_{\mu 1}, e_{\mu 1}) \ldots (a_{\mu h}, e_{\mu h})$ (h may possibly be zero) be some pairs of branches of the transition graph $G_{0t}$. Then apply $\mathcal{O}(a_{\mu_i})$ and $\mathcal{S}(e_{\mu_i})$ $(i=1,2,\ldots,h)$.

Operations (A"). Let $(a_{\mu_1}, b_{\mu_1}) \ldots (a_{\mu_h}, b_{\mu_h})$ (h may possibly be zero) be some pairs of CCCS branches of $G_0$. Then delete $a_{\mu_i}$ and $b_{\mu_i}$ $(i=1,\ldots,h)$ and coalesce the initial vertex of $a_{\mu_i}$ and the end vertex of $b_{\mu_i}$ (that is, merge vertices C and E in Fig. 6(a)).

Then it is apparent that

Lemma A.1. Operations (A), (A') and (A") are in essence equivalent to each other in the sense that we can get the same graphs from the original transistor circuit $N_T$ by applying operation (B) and any of operations (A), (A') and (A").

By introducing operations (A') and (A") it becomes easy to handle operation (A) algebraically. Let $\hat{G}_{0t}$ denote a graph obtained from $G_{0t}$ by applying operation (A'). Note that in $\hat{G}_{0t}$ a-branches do not necessarily form a tree of $\hat{G}_{0t}$.

Lemma A.2. In $\hat{G}_{0t}$ a-branches form a tree if and only if the principal submatrix of P corresponding to branches $a_{\mu_i}$ and $b_{\mu_i}$ $(i=1,\ldots,h)$ is nonsingular.

Let us prove Lemma 6. Let

$$P = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{P_{11}}}^{k} & \overbrace{\phantom{P_{12}}}^{n-k} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{array} \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

and suppose that

$$|P_{11}| = 1 \tag{A5.1}$$

Let

$$T = \begin{bmatrix} P_{11}^{-1} & 0 \\ -P_{21}P_{11}^{-1} & I \end{bmatrix}. \tag{A5.2}$$

Then we have

$$|T| = |P_{11}^{-1}| = 1 \tag{A5.3}$$

and

$$
(\tilde{C}_{tf} =)T\, C_{tf} = \begin{array}{cccc} a_1 \dots a_k & a_{k+1} \dots a_n & e_1 \dots e_k & e_{k+1} \dots e_n \\ \end{array}
$$
$$
\left[\begin{array}{c|c|c|c} P_{11}^{-1} & 0 & I & P_{11}^{-1}P_{12} \\ \hline -P_{21}P_{11}^{-1} & I & 0 & \hat{P} \end{array}\right] \tag{A5.4}
$$

where

$$
\hat{P} = P_{22} - P_{21}P_{11}^{-1}P_{12} \tag{A5.5}
$$

Since from (A5.4) and (A5.5)

$$
0 > \Delta(G_0) = |I+Q| = |P| = |TP|
$$
$$
= \begin{vmatrix} I & P_{11}^{-1}P_{12} \\ 0 & \hat{P} \end{vmatrix} = |\hat{P}|, \tag{A5.6}
$$

we have

$$
|\hat{P}| < 0.
$$

The matrix $\tilde{C}_{tf}$ is the fundamental cutset matrix of the graph $G_{0t}$ with respect to the tree $\{a_{k+1},\dots a_n,\ e_1,\dots,e_k\}$. Let $\hat{G}_{0t}$ denote the graph obtained from $G_{0t}$ by applying operation (A') to the branches $(a_1,e_1)$, $(a_2,e_2),\dots,(a_k,e_k)$.† Then a-branches $\{a_{k+1},\dots,a_n\}$ form a tree $\hat{T}$ in $\hat{G}_{0t}$ and the fundamental cutset matrix of $\hat{G}_{0t}$ with respect to $\hat{T}$ is given by $[I\vdots\hat{P}]$. Let $\hat{G}_0 \sim \hat{G}_{0t}$. Then $\Delta(\hat{G}_0) = |\hat{P}| < 0$ and $\hat{G}_0$ has fewer vertices than $G_0$, which contradicts Assumption 4.

†Note that deleting the i-th row of a fundamental cutset matrix $C_f$ corresponds to short-circuiting the tree branch i and deleting the i-th column of the main part of $C_f$ corresponds to open-circuiting the link branch i.

## Appendix 6. Proof of Lemma 7

Suppose that in (1)

$$\beta_1(a_1+b_1) + \beta_2(a_2+b_2)+ \dots + \beta_{n-1}(a_{n-1}+b_{n-1}) + b_n = 0 \tag{A6.1}$$

or

$$\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_{n-1}e_{n-1} + b_n = 0 \tag{A6.2}$$

holds. Here $\beta_\mu$ ($\mu=1,2,\dots,n-1$) are real numbers and $a_\mu$ and $b_\mu$ are the $\mu$-th and the $(n+\mu)$-th column of $C_f$ in (1), respectively, and $e_\mu$ the $\mu$-th column of P in (5). By adding the product of the $\mu$-th column of P and $\beta_\mu$ to the n-th column of P, we have the matrix

$$\hat{P} = [e_1 \vdots e_2 \vdots \dots \vdots e_{n-1} \vdots a_n].$$

For, the n-th column of $\hat{P}$ is

$$\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_{n-1}e_{n-1} + e_n$$

$$= (\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_{n-1}e_{n-1} + b_n) + a_n = a_n.$$

We have, of course,

$$|I+Q| = |P| = |\hat{P}| \ (< 0)$$

Since $a_n = [0\ 0\ \dots\ 0\ 1]'$, let[†]

$$\hat{P} = \left[\begin{array}{c|c} P_{11} & \bigcirc \\ \hline /// & 1 \end{array}\right].$$

Therefore we have

$$|P_{11}| = |\hat{P}| < 0$$

On the other hand let us consider the graph $\hat{G}_0$ obtained from $G_0$ by applying $Z(n)$. Of course $\hat{G}_0$ has fewer vertices than $G_0$. The fundamental cutset matrix of $\hat{G}_0$ is given by $[I \vdots Q_{11}]$ and $P_{11} = I + Q_{11}$ holds. Thus we have $\Delta(\hat{G}_0) < 0$, which contradicts Assumption 4. Therefore we conclude that (A6.1) (or (A6.2)) does not hold.

Let us introduce some terminologies. Since the e-branches form a tree of $G_{0t}$ (see Lemma 5), we can partition the e-branches into two connected parts by

[†]The prime means the transpose of a matrix.

open-circuiting an arbitrary e-branch, say $e_\mu$. Note that the a-branches should be ignored in the above discussion, and an isolated vertex, if occurred, is regarded as one part. One connected part composed of the e-branches, to which the initial vertex of the branch $e_\mu$ belongs, is called the "back part" of $e_\mu$ and another connected part, to which the end vertex of $e_\mu$ belongs, is called the "front part" of $e_\mu$. If the end vertex of $a_\mu$ belongs to the front (resp., back) part of $e_\mu$, then $e_\mu$ is called a forward (resp., backward) branch in $G_{0t}$.

Lemma A.3. That (A6.1) holds is equivalent to the branch $e_n$ being a forward branch of $G_{0t}$.

Proof. We will explain Lemma A.3 by using Fig. A.1. We consider the branch $e_2$ in Fig. A.1. Then the branches $e_0$, $e_1$, $e_3$ belong to the front part of $e_2$ and the branches $e_4$ and $e_5$ belong to the back part of $e_2$. Since the branch $a_2$ is directed to the front part of $e_2$, $e_2$ is a forward branch. Now in Fig. A.1

$$-e_1 + e_3 + b_2 = 0$$

holds.

The general case follows along the reasoning as this example.

From Lemma A.3 we have Lemma A.4.

Lemma A.4. Every $e_\mu$ ($\mu$=1,2,...,n) in $G_{0t}$ are backward branches.

Lemma A.5. Let $(a_\mu, b_\mu)$ be a pair of CCCS branches in $G_0$ (see Fig. A.2(a)) and let $\tilde{G}_0$ be a graph obtained from $G_0$ by replacing branches $(a_\mu, b_\mu)$ with $(\tilde{a}_\mu, \tilde{b}_\mu)$. (Attention to the directions of branches.) Here $(a_\mu, b_\mu)$ and $(\tilde{a}_\mu, \tilde{b}_\mu)$ are two pairs of CCCS branches in an identical Ebers-Moll model and are called "complementary" to each other. Then the graph $\tilde{G}_0$ can be obtained from G by appropriately applying operations (A) and (B) in Section III.

Proof. It is apparent from the Ebers-Moll model. Note that the a-branches of the graph $\tilde{G}_0$ in Lemma A.5 do not necessarily form a tree.

Lemma A.6. Suppose that a principal minor, say the upper left-most k x k principal minor, of Q is positive (=1). Then the graph $\tilde{G}_0$ obtained from $G_0$ by replacing $(a_\mu, b_\mu)$ ($\mu$=1,2,...,k) with its complement $(\tilde{a}_\mu, \tilde{b}_\mu)$, as in Lemma A.5 satisfies

$$\Delta(G_0) = \Delta(\tilde{G}_0) \ (< 0) \qquad \text{(A6.3)}$$

Proof. Note that Lemma A.6 asserts that the a-branches (i.e., $\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k,$

$a_{k+1},\ldots,a_n\}$) form a tree of $\tilde{G}_0$. Let the fundamental cutset matrix of $G_0$ be given by

$$C_f = \begin{array}{c} \\ a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cccc} a_1\ldots a_k & a_{k+1}\ldots a_n & b_1\ldots b_k & b_{k+1}\ldots b_n \end{array} \\ \left[\begin{array}{c|c|c|c} I & \bigcirc & Q_{11} & Q_{12} \\ \hline \bigcirc & I & Q_{21} & Q_{22} \end{array}\right] \end{array} \tag{A6.4}$$

where by assumption

$$|Q_{11}| = 1 \tag{A6.5}$$

By multiplying $C_f$ from the left by an appropriate nonsingular matrix we get

$$C_f^{(1)} = \begin{array}{c} \begin{array}{cccc} a_1\ldots a_k & a_{k+1}\ldots a_n & b_1\ldots b_k & b_{k+1}\ldots b_n \end{array} \\ \left[\begin{array}{c|c|c|c} Q_{11}^{-1} & 0 & I & Q_{11}^{-1}Q_{12} \\ \hline -Q_{21}Q_{11}^{-1} & I & 0 & Q_{22}-Q_{21}Q_{11}^{-1}Q_{12} \end{array}\right] \end{array} \tag{A6.6}$$

from which we see the branches $b_1$, $b_2$, ..., $b_k$ $a_{k+1}$ ,..., $a_n$ form a tree of $G_0$ and $C_f^{(1)}$ is the fundamental cutset matrix of $G_0$ with respect to this tree. It therefore follows that the branches $\tilde{a}_1$, $\tilde{a}_2,\ldots\tilde{a}_k$, $a_{k+1},\ldots,a_n$ form a tree T of $\tilde{G}_0$ and that the fundamental cutset matrix of $\tilde{G}_0$ with respect to T is given by

$$C_f = \begin{array}{c} \begin{array}{cccc} \tilde{a}_1\tilde{a}_2\ldots\tilde{a}_k & a_{k+1}\ldots a_n & \tilde{b}_1\ldots\tilde{b}_k & b_{k+1}\ldots b_n \end{array} \\ \left[\begin{array}{c|c|c|c} I & 0 & Q_{11}^{-1} & -Q_{11}^{-1}Q_{12} \\ 0 & I & Q_{21}Q_{11}^{-1} & Q_{22}-Q_{21}Q_{11}^{-1}Q_{12} \end{array}\right] \end{array} \tag{A6.7}$$

Therefore we have

$$\Delta(G_0) = \begin{vmatrix} I+Q^{-1}_{11} & -Q^{-1}_{11}Q_{12} \\ Q_{21}Q_{11}^{-1} & I+Q_{22}-Q_{21}Q_{11}^{-1}Q_{12} \end{vmatrix}$$

$$= \begin{vmatrix} Q_{11}^{-1} & 0 \\ 0 & I \end{vmatrix} \begin{vmatrix} I & 0 \\ Q_{21}Q_{11}^{-1} & -I \end{vmatrix} \begin{vmatrix} Q_{11}+I & Q_{12} \\ Q_{21} & Q_{22}+I \end{vmatrix} \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}$$

$$= |Q_{11}^{-1}| \, \Delta(G_0) = \Delta(G_0)$$

Lemma A.7. Suppose that the same assumption as in Lemma A.6 holds. Let $\tilde{G}_0$ be the graph obtained in Lemma A.6 and let $\tilde{G}_0 \sim \tilde{G}_{0t}$. Then the branches $\tilde{e}_\mu$ ($\mu=1,2,\ldots,k$) are forward branches of $\tilde{G}_{0t}$.

Proof Branches $e_\mu$ in $\tilde{G}_{0t}$, the transition graph of $G_0$, and $\tilde{e}_\mu$ in $\tilde{G}_{0t}$ have opposite directions. On the other hand the end vertices of $a_\mu$ and $e_\mu$ ($\mu=1,\ldots,k$) are the same vertices in $G_{0t}$ and $\tilde{G}_{0t}$. Since all $e_\mu$ ($\mu=1,\ldots,n$) are backward branches in $\tilde{G}_{0t}$ (see Lemma A.4), branches $\tilde{e}_\mu$ ($\mu=1,\ldots,k$) are forward branches in $\tilde{G}_{0t}$.

Suppose that Q has a positive principal minor. Then there exists another graph $\tilde{G}_0$ such that 1) $\tilde{G}_0$ is obtained from G by applying operations (A) and (B) (see Lemma A.5) and 2) $\Delta(G_0) = \Delta(\tilde{G}_0)$ (see Lemma A.6). Furthermore, since $\tilde{G}_{0t}$ has a forward branch (see Lemma A.7), we can derive another graph $\hat{G}_0$ which satisfies $\Delta(G_0) = \Delta(\hat{G}_0)$ (< 0) and which has fewer vertices than $G_0$. This contradicts Assumption 4. This completes the proof of Lemma 7.

## Appendix 7. Proof of Lemma 10

Suppose that $\hat{G}_{0t}$ contains a loop consisting only of e-branches. Then we can apply at least once the operation in Fig. 2(b) instead of that in Fig. 2(f) to obtain F. This contradicts Assumption 6.

Suppose next that there exists a cutset consisting only of a-branches in $\hat{G}_{0t}$. Then we cannot obtain a connected graph after we apply the operation in Fig. 2(f). This also leads to the contradiction.

Similar discussion applies for a cutset consisting of e-branches and a loop consisting of a-branches. Therefore Lemma 10 follows.

## References


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[2] R. O. Nielsen and A. N. Willson, "A fundamental result concerning the topology of transistor circuits with multiple equilibria," Proc. IEEE, Vol. 68, pp. 196-208, February 1980.

[3] T. Nishi and S. Uchiyama, "On the uniqueness of solution for the equations of nonlinear resistive network, Part II," presented at the Meeting of Technical Group on Circuits and Systems in Japan, April 1982 (internal report (in Japanese)).

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Figure Captions

Fig. 1. Feedback structure.

Fig. 2. Operation (III), which means replacing (a) by one of (b)-(f).

Fig. 3. Graph representation of a CCCS.

Fig. 4. Ebers-Moll model and its graph representation.

Fig. 5. Resulting graphs by applying operations (II)' to the Ebers-Moll model.

Fig. 6. Illustration of a transition graph.

Fig. 7. The graph $G_0$ corresponding to a feedback structure.

Fig. 8. Graph representation of a feedback structure.

Fig. A.1. Illustration for the proof of Lemma A.3.

Fig. A.2. Replacement of $(a_\mu, b_\mu)$ by its complement $(\tilde{a}_\mu, \tilde{b}_\mu)$

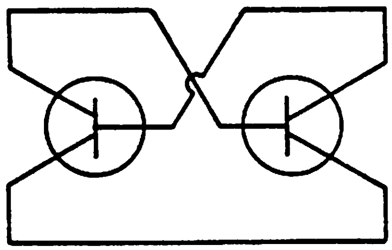

Fig. 1

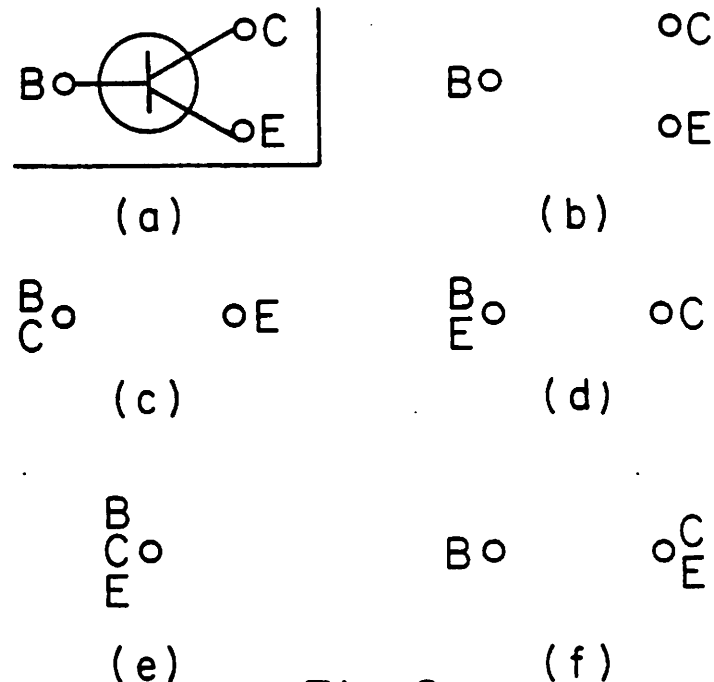


Fig. 2

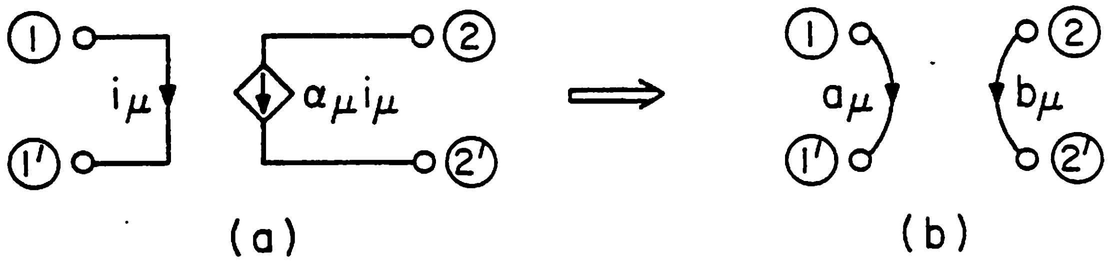


Fig. 3

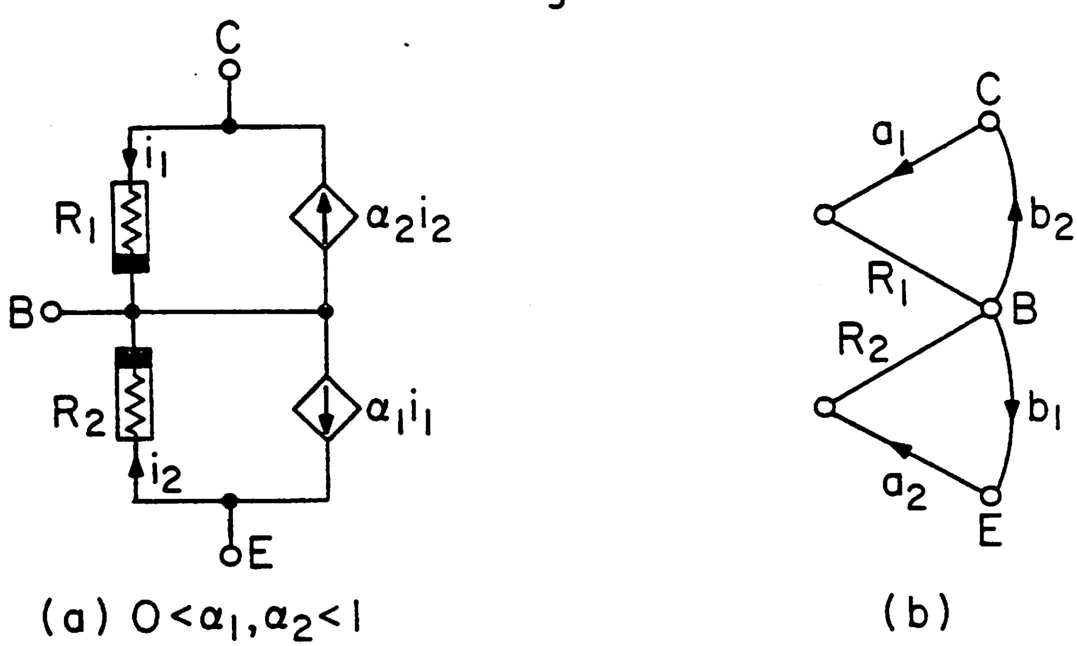


Fig. 4

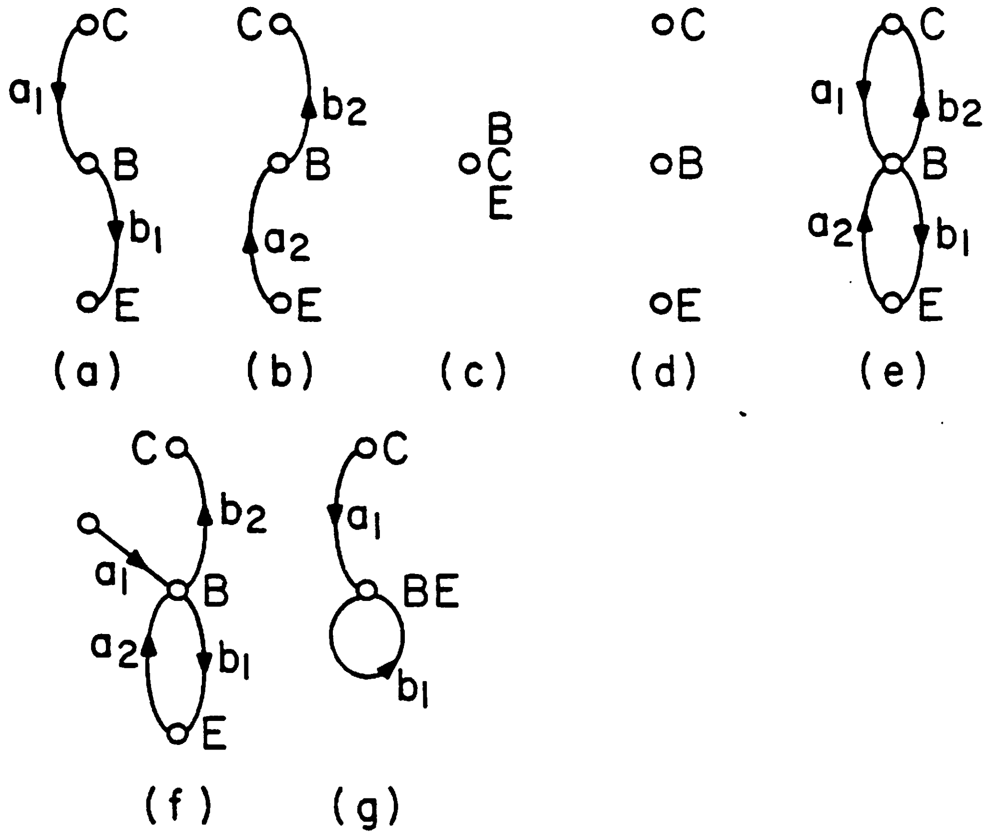


Fig. 5

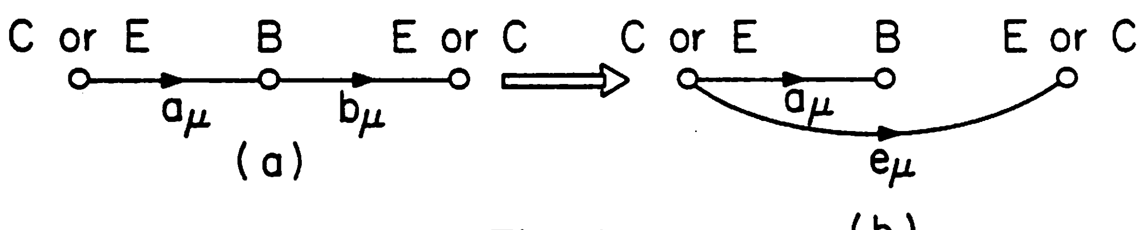


Fig. 6

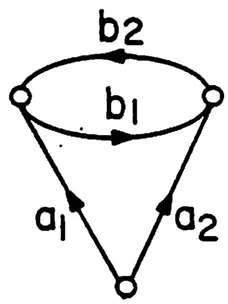


Fig. 7

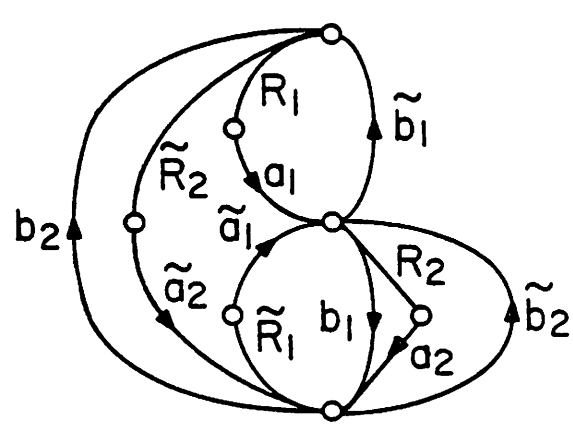


Fig. 8

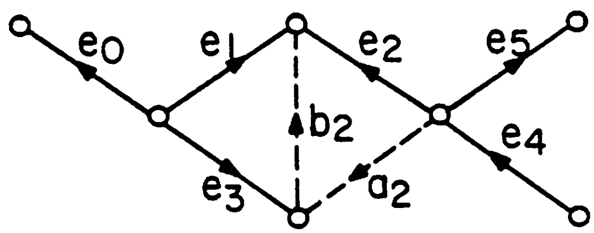


Fig. A.1

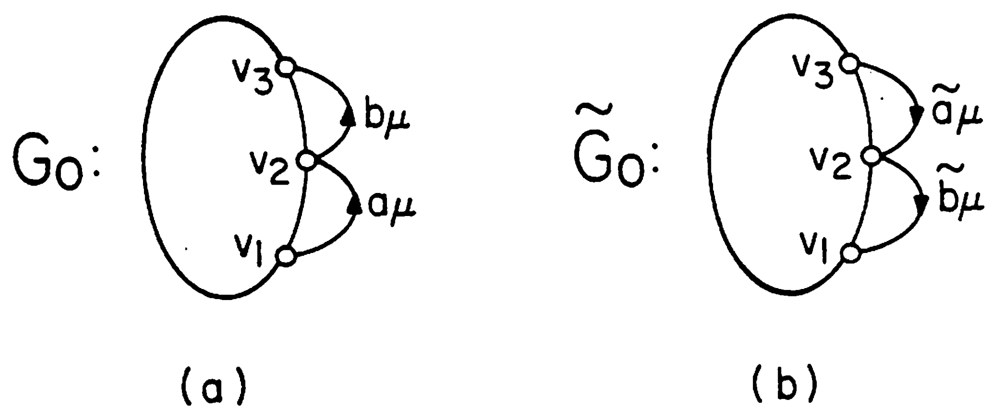


Fig. A.2