ROBUST STABILITY UNDER ADDITIVE

PERTURBATIONS: THE NONLINEAR CASE

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Robust Stability under Additive Perturbations: The Nonlinear Case

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Abstract

We consider a MIMO nonlinear feedback system.[1] $S(\underset{\sim}{P},\underset{\sim}{C})$ which is assumed to be S-stable. The plant $\underset{\sim}{P}$ is subjected to an *arbitrary* additive (resp. multiplicative) perturbation $\Delta\underset{\sim}{P}$ (resp. $\underset{\sim}{M}$). We prove necessary and sufficient conditions for the S-stability of the perturbed system. As a special case, we obtain a generalization of our earlier result (on linear time-invariant systems) to linear time-varying systems.

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I. Introduction

One of the main purposes of feedback is to reduce the sensitivity of the closed-loop system to changes in the plant, and it is very important to determine whether a feedback system remains stable after being subjected to changes in the plant. There is an abundant literature on this subject with various restrictions imposed on the nature of i) the plant: linear lumped [Des. 1], [Åst. 1], [Fra. 1] [Doy. 1]; linear distributed [Chen 1], [Chen 2]; nonlinear and time-varying [Zam. 1], [San. 1]), ii) the perturbation: a) stable perturbation: ([Åst. 1], [Fra. 1], [Cru. 1] [Pos. 1], [Zam. 2]) *-all giving only sufficient conditions;* b) a class of possibly *unstable linear* perturbations [Doy. 1], [Chen 1] with *necessary and sufficient conditions* (n.a.s.c.); c) *linear fractional* possibly unstable perturbations [Chen 2] with *n.a.s.c.*

In an earlier paper [Bha. 1], we considered MIMO *linear time-invariant* systems (lumped or distributed, discrete- or continuous-time) and a simple algebraic proof of a n.a.s.c. for stability of ${}^1S(P,C)$, (Fig. 1, solid lines only), when P is subjected to an additive perturbation ΔP which is *proper* but *not necessary stable*. In this paper, we extend this result to MIMO *nonlinear* systems (Theorem 1, Sec. 3) and when we specialize this result to the linear case, (Sec. 4, Cor. 1), we obtain a generalization of the result of [Bha. 1] which applies to *time-varying* systems as well. We state and give a simple algebraic proof of a *necessary and sufficient condition* for S-stability (see Sec. II for Defn.) of the feedback system ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ under *arbitrary* additive (resp. multiplicative) perturbations $\Delta\underset{\sim}{P}$, (resp. $\underset{\sim}{M}$), (i.e., $\Delta\underset{\sim}{P}$, (resp. $\underset{\sim}{M}$), is *not* required to be S-stable).

In Section III we formalize the following intuitive argument: (the nominal system ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is assumed S-stable) a) the addition of $\Delta\underset{\sim}{P}$ to ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ (as shown by the dotted lines in Fig. 1) creates a new loop; b) the "gain seen by $\Delta\underset{\sim}{P}$," through ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is equal to $-\underset{\sim u_2}{Q}$, c) since the nominal system ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is S-stable, $\underset{\sim u_2}{Q}$ is S-stable, $\forall\ u_2 \in L_e^{n_i}$, d) view the new loop as ${}^1S(\Delta\underset{\sim}{P},\underset{\sim u_2}{Q})$ (Fig. 2): "clearly" $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ is S-stable $\iff$ $\forall\ u_2 \in L_e^{n_i}$, ${}^1S(\Delta\underset{\sim}{P},\underset{\sim u_2}{Q})$ is S-stable.

II. Definitions and Notations

U.t.c. means under these conditions. Let $(L,\|\cdot\|)$ be a normed space of "time functions": $T \to V$ where T is the time set (typically $\mathbb{R}_+$ or $\mathbb{N}$), V is a normed space typically $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\cdots$) and $\|\cdot\|$ is the chosen norm in L. Let L_e be the corresponding extended space (see e.g., [Des. 2]).

A function $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to *class* K iff φ is φ *continuous* and *increasing*. φ is said to belong to *class* K_o iff $\varphi \in K$ and $\varphi(0) = 0$. If φ_1 and $\varphi_2 \in K_o$, then $\varphi_1 + \varphi_2$ and $\alpha \mapsto \varphi_1(\varphi_2(\alpha)) \in K_0$. A nonlinear causal map $\underset{\sim}{H}: L_e^{n_i} \to L_e^{n_o}$ is said to be S-*stable* iff $\exists\ \varphi \in K$ s.t. $\forall\ x \in L_e^{n_i}$, $\forall\ T \in T$,

$$\|\underset{\sim}{H}x\|_T \leq \varphi(\|x\|_T)\ .$$

$\underset{\sim}{H}$ is said to b *incrementally* S-*stable (incr.* S-*stable)* iff (i) $\underset{\sim}{H}$ is S-stable,

(ii) $\exists\ \tilde{\varphi} \in K_o$ s.t. $\forall\ x,\ x' \in L_e^{n_i}$, $\forall\ T \in T$,

$$\|\underset{\sim}{H}x - \underset{\sim}{H}x'\|_T \leq \tilde{\varphi}(\|x - x'\|_T)$$

Let β, γ, and $\tilde{\gamma}$ be constants: if $\varphi(x) = \beta + \gamma x$, then S-stability reduces to *finite-gain stability* [Saf. 1], [Des. 2]; if $\tilde{\varphi}(x) = \tilde{\gamma}x$, then incr. S-stability reduces to finite-gain incr. stability. It can be shown that if the nonlinear causal maps $\underset{\sim 1}{H}$ and $\underset{\sim 2}{H}$ are S-stable, (inc. S-stable), then $\underset{\sim 1}{H} + \underset{\sim 2}{H}$ and $\underset{\sim 1}{H} \circ \underset{\sim 2}{H}$ are S-stable, (incr. S-stable, resp.). (For simplicity, in what follows we drop the symbol "$\circ$"

denoting the composition of the maps.)

A feedback system is said to be *well-posed* iff the relation from the exogenous inputs into each subsystem† variable (i.e., subsystem input and subsystem output) is a well-defined nonlinear causal map between the corresponding extended spaces. More precisely, the system ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ of Fig. 1, where $\underset{\sim}{P}: L_e^{n_i} \to L_e^{n_o}$, $\underset{\sim}{C}: L_e^{n_o} \to L_e^{n_i}$ are causal maps, is said to be *well-posed* iff $\underset{\sim}{H}: (u_1,u_2) \mapsto (e_1,e_2,y_1,y_2)$ is well-defined and causal. Note that ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is well-posed implies that† $(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$ and $(\underset{\sim}{I}+\underset{\sim}{C}\underset{\sim}{P})^{-1}$ are well-defined and causal. We say that a well-posed nonlinear feedback system is S-*stable* (incr. S-stable) iff the map from the exogenous inputs to any subsystem variable is S-stable (incr. S-stable, resp.).

III. Statement and Proof of the Theorems

We will need the following assumptions:

A1. $\underset{\sim}{P}: L_e^{n_i} \to L_e^{n_o}$ and $\underset{\sim}{C}: L_e^{n_o} \to L_e^{n_i}$ are nonlinear causal maps between the appropriate extended spaces and the nominal system

$$ {}^1S(\underset{\sim}{P},\underset{\sim}{C}) \text{ is well posed.} \tag{1}$$

A2.

$$ {}^1S(\underset{\sim}{P},\underset{\sim}{C}) \text{ is S-stable.} \tag{2}$$

A2'. ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is S-stable and $\underset{\sim 1}{\Psi}: (u_1,u_2) \mapsto y_1$, defined by ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$, is *incr. S-stable*.

---

†By subsystem we mean any block of the block diagram of the feedback system.

†The meaning of $(\underset{\sim}{I}+\underset{\sim}{P}\underset{\sim}{C})^{-1}$ deserves clarification: the map $\underset{\sim}{C}$ is composed with $\underset{\sim}{P}$ then the identity is added, and the resulting map is inverted. Although this formula has the same form as the linear case, it has a completely different interpretation.

A3. The additive perturbation $\Delta \underset{\sim}{P}: L_e^{n_i} \to L_e^{n_o}$ is a nonlinear causal map subject only to the restriction that the perturbed system

$$\tilde{S}(\underset{\sim}{P}, \Delta \underset{\sim}{P}, \underset{\sim}{C}) \text{ be well-posed.} \tag{3}$$

Remark 1: $\underset{\sim}{P}$, $\underset{\sim}{C}$, and $\Delta \underset{\sim}{P}$ are nonlinear, causal maps subject only to (1), (2), and (3). None, some or all of these three maps may be unstable.

Remark 2: The nominal S-stable system ${}^1S(\underset{\sim}{P}, \underset{\sim}{C})$ with u_1, u_2 as inputs and y_1, y_2 as outputs, defines the S-stable maps $\underset{\sim}{\Psi}_1 : L_e^{n_o} \times L_e^{n_i} \to L_e^{n_i}$ s.t. $y_1 = \underset{\sim}{\Psi}_1(u_1, u_2) := \underset{\sim}{Q}_{u_2}(u_1)$ and $\underset{\sim}{\Psi}_2 = L_e^{n_o} \times L_e^{n_i} \to L_e^{n_o}$ s.t. $y_2 = \underset{\sim}{\Psi}_2(u_1, u_2) := \underset{\sim}{H}_{u_2}(u_1)$.

Theorem 1: Let assumptions A1, A2 and A3 hold. U.t.c.,

(i) $\tilde{S}(\underset{\sim}{P}, \Delta \underset{\sim}{P}, \underset{\sim}{C})$ is S-stable $\iff \forall\, u_2 \in L_e^{n_i}$, ${}^1S(\Delta \underset{\sim}{P}, \underset{\sim}{Q}_{u_2})$ is S-stable.

(ii) If, instead of A2, the stronger assumption A2' holds and we let $\underset{\sim}{Q}_o$ denote the partial map $u_1 \to \underset{\sim}{\Psi}_1(u_1, 0)$, then,

$$\tilde{S}(\underset{\sim}{P}, \Delta \underset{\sim}{P}, \underset{\sim}{C}) \text{is S-stable} \iff {}^1S(\underset{\sim}{P}, \underset{\sim}{Q}_o) \text{is S-stable} .$$

Proof of (i): (=>) Consider the system $\tilde{S}(\underset{\sim}{P}, \Delta \underset{\sim}{P}, \underset{\sim}{C})$ (see Fig. 1), and write its equations in terms of the outputs y_1, y_2, and y_3:

$$y_1 = \underset{\sim}{C}(u_1 - y_3 - y_2) \tag{4}$$

$$y_2 = \underset{\sim}{P}(u_2 + y_1) \tag{5}$$

$$y_3 = \Delta \underset{\sim}{P}(u_3+y_1) \tag{6}$$

If, in Fig. 1, we delete the dotted part of the figure, we are left with ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ driven by u_1, u_2 with outputs y_1, y_2: ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is described by

$$y_1 = \underset{\sim}{C}(u_1-y_2) \tag{7}$$

$$y_2 = \underset{\sim}{P}(u_2+y_1) \tag{8}$$

By assumption, ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is S-stable, hence

$$y_i = \underset{\sim i}{\Psi}(u_1,u_2) \tag{9}$$

where for i = 1,2, $\underset{\sim i}{\Psi}$ is a causal S-stable map on $L_e^{n_o} \times L_e^{n_i}$. Using (9) in (4) and (5) we obtain a new equivalent representation for $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$:

$$y_1 = \underset{\sim 1}{\Psi}(u_1-y_3,u_2) := \underset{\sim u_2}{Q}(u_1-y_3) \tag{10}$$

$$y_2 = \underset{\sim 2}{\Psi}(u_1-y_3,u_2) \tag{11}$$

$$y_3 = \Delta\underset{\sim}{P}(u_3+y_1) \tag{12}$$

Note that (10) and (12) involve only the outputs y_1 and y_3 and that (10) and (12) describe the system ${}^1S(\Delta\underset{\sim}{P},\underset{\sim u_2}{Q})$ shown in Fig. 2. Now we obtain, successively:

$\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ is S-stable hence the map $(u_1,u_2,u_3) \mapsto (y_1,y_2,y_3)$ is S-stable which implies that the map $(u_1,u_2,u_3) \mapsto (y_1,y_3)$ defined by (10) and (12) is S-stable, which, in turn, is equivalent to $\forall\ u_2 \in L_e^{n_i}$, ${}^1S(\Delta\underset{\sim}{P},\underset{\sim u_2}{Q})$ is S-stable.

(<=) Assume that, $\forall\, u_2 \in L_e^{n_i}$, ${}^1S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_{u_2})$ is S-stable. Then, I) by the last equivalence, (10) and (12) specify, for the system $\tilde{S}(\underset{\sim}{P}, \underset{\sim}{\Delta P}, \underset{\sim}{C})$, an S-stable map from (u_1, u_2, u_3) to (y_1, y_3); II) since ${}^1S(\underset{\sim}{P}, \underset{\sim}{C})$ is S-stable by assumption, the function $\underset{\sim}{\Psi}_2$ in (11) is S-stable; III) consequently, the *three* equations (10)-(12) define an S-stable map from (u_1, u_2, u_3) to (y_1, y_2, y_3); equivalently $\tilde{S}(\underset{\sim}{P}, \underset{\sim}{\Delta P}, \underset{\sim}{C})$ is S-stable.

Proof of (ii): (=>) By the theorem, $\tilde{S}(\underset{\sim}{P}, \underset{\sim}{\Delta P}, \underset{\sim}{C})$ is S-stable, hence $\forall\, u_2 \in L_e^{n_i}$ ${}^1S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_{u_2})$ is S-stable. In particular, for $u_2 = 0$ we have that the *system* ${}^1S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_o)$ is S-stable.

(<=) By assumption ${}^1S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_o)$ is S-stable.

We show that because $(u_1, u_2) \rightarrow \underset{\sim}{\Psi}(u_1, u_2) =: \underset{\sim}{Q}_{u_2}(u_1)$ is incr. S-stable, then $\forall\, u_2 \in L_e^{n_i}$, $S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_{u_2})$ is S-stable. Then, by theorem 1, (i), $\tilde{S}(\underset{\sim}{P}, \underset{\sim}{\Delta P}, \underset{\sim}{C})$ is S-stable.

Consider ${}^1S(\underset{\sim}{\Delta P}, \underset{\sim}{Q}_{u_2})$

$$y_1 = \underset{\sim}{Q}_{u_2}(u_1 - y_3) \tag{13}$$

$$y_3 = \underset{\sim}{\Delta P}(u_3 + y_1) \tag{14}$$

We can rewrite $\underset{\sim}{Q}_{u_2}(u_1 - y_3)$ as follows

$$\underset{\sim}{Q}_{u_2}(u_1 - y_3) = \underset{\sim}{Q}_o(u_1 - y_3) + \tilde{y}_1 = y_1^o + \tilde{y}_1$$

where the last equalities define $\tilde{y}_1$ and y_1^o, resp. Now $\forall\, (u_1, u_2) \in L_e^{n_o} \times L_e^{n_i}$

and $\forall\ T$, we have:

$$\|\tilde{y}_1\|_T \le \|\underset{\sim}{Q}_{u_2}(u_1-y_3) - \underset{\sim}{Q}_o(u_1-y_3)\|_T = \|\underset{\sim}{\Psi}_1(u_1-y_3,u_2) - \underset{\sim}{\Psi}_1(u_1-y_3,0)\|_T$$
$$\le \tilde{\varphi}(\|u_2\|_T)$$

Since $\tilde{\varphi} \in K_0 \subset K$, for the system ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_0)$, the map $(u_1,u_2,u_3) \mapsto \tilde{y}_1$ is S-stable. (13) and (14) can be rewritten as:

$$y_1^o = \underset{\sim}{Q}_o(u_1-y_3) \tag{15}$$

$$y_3 = \Delta\underset{\sim}{P}(u_3+\tilde{y}_1+y_1^o) \tag{16}$$

The systems (13)-(14) and (15)-(16) are equivalent. Since ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_o)$ is S-stable and since $(u_1,u_2,u_3) \mapsto \tilde{y}_1$ is S-stable we conclude that $\forall\ u_2 \in L_e^{n_i}$ ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_{u_2})$ is S-stable hence $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ is S-stable, by Theorem 1, (i).

■

We replace assumption A3 by:

A3'. The multiplicative perturation $\underset{\sim}{M} = L_e^{n_i} \to L_e^{n_o}$ is a nonlinear causal map subject only to the restriction that the perturbed system $\hat{S}(\underset{\sim}{P},\underset{\sim}{M},\underset{\sim}{C})$ (see Fig. 3) be well-posed. Recall that $y_2 = \underset{\sim}{\Psi}_2(u_1,u_2) := \underset{\sim}{H}_{u_2}(u_1)$ (see Remark 2). We now state:

Theorem 2: Let assumptions A1, A2 and A3' hold. U.t.c. $\hat{S}(\underset{\sim}{P},\underset{\sim}{M},\underset{\sim}{C})$ is S-stable $\Longleftrightarrow$ $\forall\ u_2 \in L_e^{n_i}$, ${}^1S(\underset{\sim}{M},\underset{\sim}{H}_{u_2})$ (see Fig. 2, caption) is S-stable.

Proof: From Fig. 3, we write just as in Theorem 1,

$$y_1 = \underset{\sim}{\Psi}_1(u_1-y_3,u_2) \tag{17}$$

$$y_2 = \underset{\sim}{\Psi}_2(u_1-y_3,u_2) =: \underset{\sim}{H}_{u_2}(u_1-y_3) \tag{18}$$

$$y_3 = \underset{\sim}{M}(u_3+y_2) \tag{19}$$

Then, observe that (18) and (19) define ${}^1S(\underset{\sim}{M},\underset{\sim}{H}_{u_2})$ and follow the proof of Theorem 1, replacing $\Delta\underset{\sim}{P}$ by $\underset{\sim}{M}$, $\underset{\sim}{Q}_{u_2}$ by $\underset{\sim}{H}_{u_2}$ and $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ by $\hat{S}(\underset{\sim}{P},\underset{\sim}{M},\underset{\sim}{C})$.

IV. The Linear Case

Suppose that, in addition to assumption A1, A2, and A3 above, $\underset{\sim}{P}$, $\underset{\sim}{C}$ and $\Delta\underset{\sim}{P}$ are *linear*. (Note that we say linear, *not* linear and time-invariant; thus $\underset{\sim}{P}: L_e^{n_i} \to L_e^{n_o}$ is a *linear map (not necessarily represented by a transfer function)*, etc. ..). Consequently, S-stability and incr. S-stability are equivalent to finite-gain stability. By linearity, $\underset{\sim}{\Psi}_1$ is linear and an easy calculation shows that

$$\underset{\sim}{\Psi}(u_1,u_2) = \underset{\sim}{Q}_o u_1 - \underset{\sim}{Q}_o \underset{\sim}{P} u_2 \tag{20}$$

where $\underset{\sim}{Q}_0$ and $\underset{\sim}{Q}_o \underset{\sim}{P}$ are finite-gain stable (by A2) linear maps. Thus the second part of Theorem 1 applies and we have: the linear system $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ is finite-gain stable <=> the linear system ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_o)$ is finite-gain stable. Now for the closed loop system ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_o)$ we have:

$$y_1 = \underset{\sim}{Q}_o(\underset{\sim}{I}-\underset{\sim}{M}\underset{\sim}{Q}_o)u_1 - \underset{\sim}{Q}_o\underset{\sim}{M}u_3 \tag{21}$$

$$y_3 = \underset{\sim}{M}\underset{\sim}{Q}_o u_1 + \underset{\sim}{M} u_3 \tag{22}$$

where $\underset{\sim}{M} = \Delta\underset{\sim}{P}(\underset{\sim}{I}+\underset{\sim}{Q}_o\,\Delta\underset{\sim}{P})^{-1}$. Since $\underset{\sim}{Q}_o$ is finite-gain stable, (21), (22) show that ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_o)$ is finite-gain stable if and only if $\underset{\sim}{M} := \Delta\underset{\sim}{P}(\underset{\sim}{I}+\underset{\sim}{Q}_o\,\Delta\underset{\sim}{P})^{-1}$ is finite-gain stable. We state this as:

Corollary 1: Let $\underset{\sim}{P}$, $\underset{\sim}{C}$, $\Delta\underset{\sim}{P}$ satisfy A1, A2, and A3 and let them be *linear* (but not necessarily time-invariant). U.t.c., $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ is finite-gain stable $\Longleftrightarrow$ the linear map $\Delta\underset{\sim}{P}(\underset{\sim}{I}+\underset{\sim}{Q}\Delta\underset{\sim}{P})^{-1}$ is finite-gain stable.

Note: Similar considerations apply to the case for multiplicative perturbations.

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Figure Captions

Fig. 1. The figure shows the system $\tilde{S}(\underset{\sim}{P},\Delta\underset{\sim}{P},\underset{\sim}{C})$ with inputs u_1, u_2 u_3 and, y_1, y_2, y_3. If the dotted part of the diagram is removed, we are left with ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ whose inputs are u_1, u_2 and outputs y_1, y_2.

Fig. 2. ${}^1S(\Delta\underset{\sim}{P},\underset{\sim}{Q}_{u_2})$: obtained from Fig. 1. The "gain seen by $\Delta\underset{\sim}{P}$" going through ${}^1S(\underset{\sim}{P},\underset{\sim}{C})$ is $-\underset{\sim}{Q}_{u_2}$. To obtain ${}^1S(\underset{\sim}{M},\underset{\sim}{H}_{u_2})$ replace $\Delta\underset{\sim}{P}$ by $\underset{\sim}{M}$ and $\underset{\sim}{Q}_{u_2}$ by $\underset{\sim}{H}_{u_2}$.

Fig. 3. The figure shows the multiplicatively perturbed system $\hat{S}(\underset{\sim}{P},\underset{\sim}{M},\underset{\sim}{C})$ with inputs u_1, u_2, u_3 and outputs y_1, y_2 and y_3.

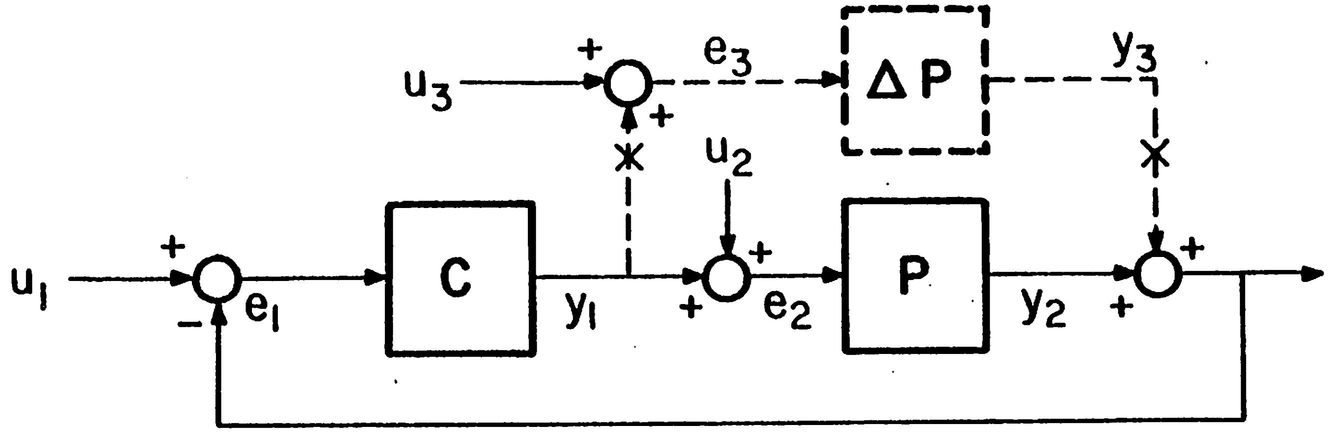

Fig. 1

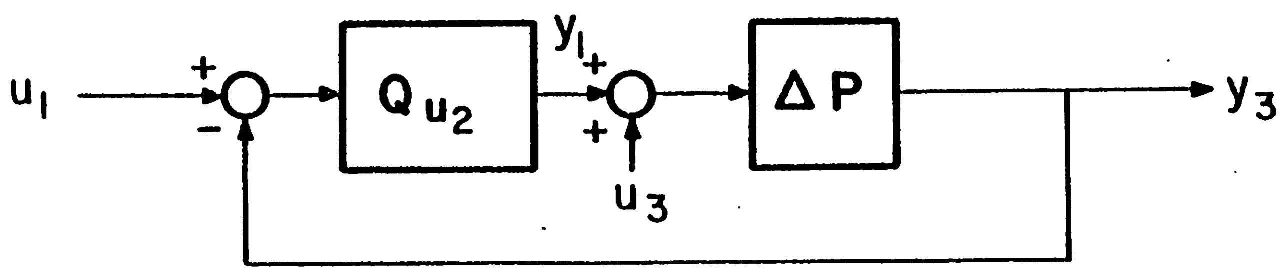

Fig. 2

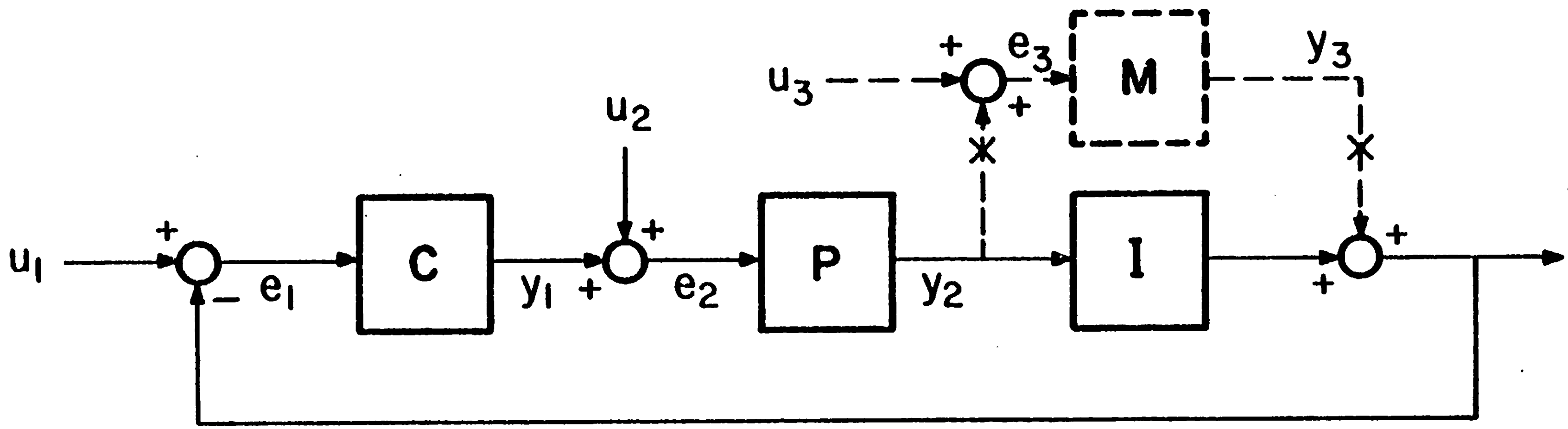

Fig. 3