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MULTIVARIABLE FEEDBACK SYSTEMS

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ALGEBRAIC DESIGN OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS

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1. INTRODUCTION

This paper considers exclusively linear time-invariant systems with the configuration $\Sigma(P,K)$ of Fig. 1, where the plant P has an output y_0 and a measured output y_m and the controller K has two inputs: the exogeneous input v and the feedback signal e_1. This configuration is a slight extension of the standard one considered in most textbooks and papers [Blo. 1, Cal. 1, Kai. 1, Per. 1, Ros. 1, Vid. 1, You. 1]. It is simpler than that considered in [Net. 3]. Algebraic techniques are systematically used in this paper [Des. 1, Des. 3, Des. 4, Net. 2, Vid. 1, Vid. 2]. The contribution of this work lies in its more general configuration and its standardized proofs. For previous work on decoupling, see [Ham. 1] and the references therein.

Six theorems address the crucial issues in the design of control systems: stability; achievable I/O and D/O maps; achievable decoupled I/O maps; robustness of stability; asymptotic tracking: necessary conditions; and sufficient conditions for (robust) asymptotic tracking.

The following is a list of the commonly used symbols:

a := means a denotes b. θ_n is the n-vector of zeros. W.l.o.g. means without loss of generality. U.t.c. means under these conditions. If $\mathcal{G}$ is a ring, then $\mathcal{M}(\mathcal{G})$ denotes the set of matrices having all entries in $\mathcal{G}$. $\mathcal{R}_{\mathcal{U}}$ denotes the proper rational functions analytic in the region $\mathcal{U} \subset \mathbb{C}$, a symmetric subset of $\mathbb{C}$ which contains $\mathbb{C}_+$ and $\overline{\mathcal{U}} = \mathbb{C}_+ \cup \{\infty\}$. $\mathbb{R}(s)$ denotes the scalar rational functions in s with real coefficients, and $\mathbb{R}[s]$ denotes the scalar polynomials in s with real coefficients.

Algebraic Structure: [Bou. 1, p. 55], [Jac. 1, p. 393], [Lang 1, p. 69].

$\mathcal{H}$: A principal ring (principal ideal domain), i.e., an entire commutative ring in which every ideal is principal (e.g., $\mathcal{R}_{\mathcal{U}}$).

$\tilde{\mathcal{G}}$: The field of fractions over $\mathcal{H}$ (e.g. $\mathbb{R}(s)$).

$\mathcal{I}$: A multiplicative subset of $\mathcal{H}$, equivalently, $\mathcal{I} \subset \mathcal{H}$, $0 \notin \mathcal{I}$, $1 \in \mathcal{I}$ and $x,y \in \mathcal{I}$ implies that $xy \in \mathcal{I}$ (e.g., $f \in \mathcal{I}$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(\infty) = 1$).

$\mathcal{G} := \{n/d : n \in \mathcal{H}, d \in \mathcal{I}\}$, a subring of $\tilde{\mathcal{G}}$ (e.g. $\mathbb{R}_p(s)$, the ring of proper scalar rational functions).

$\mathcal{U}(\mathcal{H}) := \{m \in \mathcal{H} : m^{-1} \in \mathcal{H}\}$, the group of units in $\mathcal{H}$ (e.g., $f \in \mathcal{U}(\mathcal{H})$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(s) \neq 0$ for all $s \in \overline{\mathcal{U}}$).

$\mathcal{G}_s := \{ x \in \mathcal{G} : (1+xy)^{-1} \in \mathcal{G}, \forall y \in \mathcal{G} \}$ (Jacobson radical of $\mathcal{G}$).

Four examples of this algebraic structure are given in [Des. 3, Table I].

II. DESIGN THEORY

2.1. Problem Description

We consider the multi-input-multi-output (MIMO) linear, time-invariant system $\Sigma(P,K)$ (${}^1\Sigma(P,K)$) shown in Fig. 1 (Fig. 2). Given a plant P we wish to design a controller K with two inputs and one output such that the resulting feedback system is *stable* and K has elements in $\mathcal{G}$. We make the following assumptions on $\Sigma(P,K)$:

Assumptions on the System Σ(P,K)

(P) $P = \begin{bmatrix} P^o \\ P^m \end{bmatrix} \in \mathcal{G}^{2n_o \times n_i}$ has a right-coprime factorization (r.c.f.) $\begin{bmatrix} N_{pr}^o \\ \cdots \\ N_{pr}^m \end{bmatrix} D_{pr}^{-1}$ with $D_{pr} \in \mathcal{H}^{n_i \times n_i}$, N_{pr}^o, $N_{pr}^m \in \mathcal{H}^{n_o \times n_i}$ and $\det D_{pr} \in \mathcal{J}$.

(K) $K \in \mathcal{G}^{n_i \times (n_v + n_o)}$ has a left-coprime factorization (l.c.f.) $D_{cl}^{-1}[N_{\pi l} \ \vdots \ N_{fl}]$ with $D_{cl} \in \mathcal{H}^{n_i \times n_i}$, $N_{\pi l} \in \mathcal{H}^{n_i \times n_v}$, $N_{fl} \in \mathcal{H}^{n_i \times n_o}$ and $\det D_{cl} \in \mathcal{J}$, $\det(D_{cl} D_{pr} + N_{fl} N_{pr}^m) \in \mathcal{J}$.

It is understood that the subsystems P and K, specified by their transfer functions, do not have any unstable hidden modes [Cal. 1 sec. 4.2].

Under assumptions (P) and (K) the system $\Sigma(P,K)$ in Fig. 1 is completely described by

$$\begin{bmatrix} I_{n_i} & \vdots & -D_{pr} \\ \cdots & \vdots & \cdots \\ D_{cl} & \vdots & N_{fl} N_{pr}^m \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & \vdots & 0 & \vdots & -I_{n_i} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ N_{\pi l} & \vdots & N_{fl} & \vdots & 0 & \vdots & -N_{fl} N_{pr}^m \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} I_{n_i} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N_{pr}^o \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N_{pr}^m \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_o \\ y_m \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 & \vdots & 0 & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & -N_{pr}^o \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & -N_{pr}^m \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \tag{2.2}$$

Let $u := (v^T, u_1^T, u_2^T, d^T)^T$, $\xi := (y_1^T, \xi_p^T)^T$, $y := (y_1^T, y_o^T, y_m^T)^T$. Then equations (2.1) and (2.2) are of the form

$$D\xi = N_l u \tag{2.3}$$

$$N_r \xi = y + Eu \tag{2.4}$$

where the matrices D, N_l, N_r, E, defined in an obvious manner from (2.1) and (2.2), have their elements in .

For any $D_{cl} \in \mathcal{H}^{n_i \times n_o}$ and any $N_{fl} \in \mathcal{H}^{n_i \times n_o}$, define

$$D_h := D_{cl} D_{pr} + N_{fl} N_{pr}^m \ . \tag{2.5}$$

Note that $\det D = \det D_h$ and, by assumption (K), $\det D \in \mathcal{J}$.

Definition 2.1. ($\mathcal{H}$-stability): The system $\Sigma(P,K)$ is said to be $\mathcal{H}$-stable if and only if $H_{yu} : u \mapsto y$ satisfies $H_{yu} \in \mathcal{E}(\mathcal{H})$;

Let assumptions (P) and (K) hold; then from equations (2.3) and (2.4) we obtain

$$H_{yu} = N_r D^{-1} N_l + E \in \mathcal{E}(\mathcal{G}) \ . \tag{2.6}$$

Definition 2.2 (Stabilizing Controller): The controller K *is said to stabilize* P if K satisfies assumption (K) and the resulting system $\Sigma(P,K)$ is $\mathcal{H}$-stable.

Theorem 2.3 ($\mathcal{H}$-stability)

Consider the system $\Sigma(P,K)$ where P satisfies (P), and K will be specified later.

(i) Let K satisfy (K). Then $\Sigma(P,K)$ is $\mathcal{H}$-stable if and only if $\det D_h \in \mathcal{U}(\mathcal{H})$.

(ii) Let, in addition, $P^m \in \mathcal{G}_s^{n_o \times n_i}$, where $\mathcal{G}_s :=$ Jacobson radical of $\mathcal{G}$. Then there is a compensator K which stabilizes P if and only if (N_{pr}^m, D_{pr}) is a right-coprime (r.c.) pair.

Proof: **(i) (=>)** To prove that $\det D_h \in \mathcal{U}(\mathcal{H})$, we first show that (D,N_l) is left-coprime (l.c.) and (N_r,D) is r.c., where N_r, D, N_l are defined by (2.1)-(2.4).

Let $L,R \in \mathcal{E}(\mathcal{H})$ be products of elementary row and column matrices, respectively. Then using Bezout identities it can be shown that (D,N_l) is l.c. <=> $(\hat{D},\hat{N}_l)$ is l.c. where $[\hat{D} \ \vdots \ \hat{N}_l] = [D \ \vdots \ N_l][R]$; similarly (N_r,D) is r.c. <=> $(\bar{N}_r,\bar{D})$ is r.c. where $\begin{bmatrix} \bar{N}_r \\ \cdots \\ \bar{D} \end{bmatrix} = [L]\begin{bmatrix} N_r \\ \cdots \\ D \end{bmatrix}$. By elementary column operations on $[D \ \vdots \ N_l]$ of equation (2.3), we obtain

$$\hat{D} = \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ D_{cl} & \vdots & 0 \end{bmatrix}, \quad \hat{N}_l = \begin{bmatrix} 0 & \vdots & 0 & \vdots & I_{n_i} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ N_{\pi l} & \vdots & N_{fl} & \vdots & 0 & \vdots & -N_{fl} \end{bmatrix} \tag{2.10}$$

By assumption (K), $(\hat{D},\hat{N}_l)$ in (2.10) is l.c. By elementary row operations on $\begin{bmatrix} N_r \\ \cdots \\ D \end{bmatrix}$ of equations (2.3)and (2.4), we obtain

$$\bar{N}_r = \begin{bmatrix} I_{n_i} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N_{pr}^o \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N_{pr}^m \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & \vdots & D_{pr} \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \tag{2.11}$$

In view of assumption (P), equation (2.11) shows that, $(\bar{N}_r,\bar{D})$ is r.c.

Now for a proof by contradiction, suppose that $\det D_h \notin \mathcal{U}(\mathcal{H})$; then $\det D_h = \det D \notin \mathcal{U}(\mathcal{H})$. Hence, $D^{-1} \notin \mathcal{E}(\mathcal{H})$ since $\mathcal{H}$ is a commutative ring [Jac. 1, p. 94]. Using Bezout identities it is easy to show that, since (N_r,D) are r.c., $N_r D^{-1} \in \mathcal{E}(\mathcal{H})$ <=> $D^{-1} \in \mathcal{E}(\mathcal{H})$, and that, since (D,N_l) are l.c., $N_r D^{-1} N_l \in \mathcal{E}(\mathcal{H})$ <=> $N_r D^{-1} \in \mathcal{E}(\mathcal{H})$. Therefore, $H_{yu} = (N_r D^{-1} N_l + E) \notin \mathcal{E}(\mathcal{H})$, which implies that $\Sigma(P,K)$ is *not* $\mathcal{H}$-stable. Since this is a contradiction, $\det D_h \in \mathcal{U}(\mathcal{H})$. **(<=)** Since $\det D_h \in \mathcal{U}(\mathcal{H})$, and $\det D = \det D_h$, we have $(\det D)^{-1} \in \mathcal{H}$, and $D^{-1} \in \mathcal{E}(\mathcal{H})$. Consequently, $H_{yu} = (N_r D^{-1} N_l + E) \in \mathcal{E}(\mathcal{H})$ and $\Sigma(P,K)$ is $\mathcal{H}$-stable.

(ii) (=>) For a proof by contradiction, suppose that the pair (N_{pr}^m, D_{pr}) is not r.c. Then (N_{pr}^m, D_{pr}) have a greatest-common-right-divisor (gcrd) R such that $\det R \notin \mathcal{U}(\mathcal{H})$, $N_{pr}^m = \tilde{N}_{pr}^m R$, $D_{pr} = \tilde{D}_{pr} R$ and $(\tilde{N}_{pr}^m, \tilde{D}_{pr})$ are r.c. Defining $\tilde{D}_h$ is an obvious manner, we write

$$\det D_h = \det[(D_{cl}\hat{D}_{pr} + N_{fl}\hat{N}_{pr}^m)R] = \det \hat{D}_h \det R \tag{2.16}$$

where $\det \hat{D}_h \in \mathcal{H}$ and $(\det R)^{-1} \notin \mathcal{H}$. Then $\det D_h \notin \mathcal{U}(\mathcal{H})$, because if $\det D_h \in \mathcal{U}(\mathcal{H})$, then from (2.16), $(\det R)^{-1} = (\det \hat{D}_h)(\det D_h)^{-1} \in \mathcal{H}$, which is a contradiction. Therefore, *for all* D_{cl}, N_{fl}, $\det D_h \notin \mathcal{U}(\mathcal{H})$, including those D_{cl} and N_{fl} for which K satisfies (K). Then by part (i), the system $\Sigma(P,K)$ is not $\mathcal{H}$-stable for all K which satisfy (K). In other words, there is no such K that stabilizes P.
(<=) By assumption, the pair (N_{pr}^m, D_{pr}) is r.c.; hence, there exist U_{pr}^m, $V_{pr}^m \in \mathcal{E}(\mathcal{H})$ such that

$$U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I_{n_i} \tag{2.17}$$

As a compensator choose $K := (V_{pr}^m)^{-1}[N_{\pi l} \vdots U_{pr}^m]$, where $N_{\pi l} \in \mathcal{E}(\mathcal{H})$ is arbitrary. From (2.5) and (2.17), $D_h = I$ and $\det D_h = 1 \in \mathcal{U}(\mathcal{H})$.

It remains to show that $\det V_{pr}^m \in \mathcal{J}$: For the chosen compensator, (2.5) implies

$$V_{pr}^m D_{pr} = I - U_{pr}^m N_{pr}^m \tag{2.18}$$

and taking determinants of both sides of (2.18) we obtain

$$\det V_{pr}^m = \det(I - U_{pr}^m N_{pr}^m)(\det D_{pr})^{-1} \tag{2.19}$$

By assumption, $P^m \in \mathcal{E}(\mathcal{G}_s)$; by the properties of the Jacobson radical $\mathcal{G}_s$, we have $P^m D_{pr} = N_{pr}^m \in \mathcal{E}(\mathcal{G}_s)$ and $U_{pr}^m N_{pr}^m \in \mathcal{G}_s^{n_i \times n_i}$ since D_{pr} and $U_{pr}^m \in \mathcal{E}(\mathcal{H})$. Using standard determinant expansion formulas, we see that $\det(I - N_{fl} N_{pr}^m) \in \mathcal{J}$ and hence, $[\det(I - N_{fl} N_{pr}^m)]^{-1} \in \mathcal{G}$. Since $\det D_{pr} \in \mathcal{J}$, equation (2.19) shows that $(\det V_{pr}^m)^{-1} \in \mathcal{G}$ and hence, $\det V_{pr}^m \in \mathcal{J}$. Thus, the compensator K chosen above has all its elements in $\mathcal{G}$ and for this K, $\det D_h \in \mathcal{U}(\mathcal{H})$. Therefore $\Sigma(P,K)$ is $\mathcal{H}$-stable. ∎

III. ACHIEVABLE PERFORMANCE OF Σ(P,K)

We now use the relationships between the stabilizing controller K and $\det D_h$ to give global parametrizations of a) the family of *all* I/O maps possible for a given plant with some *stabilizing controller* b) the family of *all* disturbance-to-output (D/O) maps possible for a given plant with some *stabilizing controller*.

For a given system $\Sigma(P,K)$ satisfying (P) and (K), and $\det D_h \neq 0$, equations (2.1) and (2.2) show that the I/O map $H_{y_o v} : v \mapsto y$ and the D/O map $H_{y_o d} : d \mapsto y$ are given by:

$$H_{y_o v} = N_{pr}^o D_h^{-1} N_{\pi l} \tag{3.1}$$

$$H_{y_o d} = N_{pr}^o [I - D_h^{-1} N_{fl} N_{pr}^m] = N_{pr}^o D_h^{-1} D_{cl} D_{pr} \tag{3.2}$$

Definition 3.1 (Achievable Maps)

Let P be a given plant that satisfies assumption (P); hence the specification of the controller K determines the system $\Sigma(P,K)$. Roughly speaking, let $\mathcal{H}_{y_o v}(P)$ denote the set of all *achievable I/O maps* of $\Sigma(P,K)$, and let $\mathcal{H}_{y_o d}(P)$ denote the set of all *achievable D/O* maps of $\Sigma(P,K)$; more precisely,

$$\mathcal{H}_{y_o v}(P) := \{H_{y_o v} : \text{K stabilizes the given plant P}\} \tag{3.3}$$

$$\mathcal{H}_{y_o d}(P) := \{H_{y_o v} : \text{K stabilizes the given plant P}\} \tag{3.4}$$

The following theorem characterizes all the achievable I/O maps and the achievable D/O maps for $\Sigma(P,K)$.

Normalization Assumption: Since by Theorem 2.3, K stabilizes P if and only if $\det D_h \in \mathcal{U}(\mathcal{H})$, we take w.l.o.g.

$$D_h = I_{n_i} \tag{3.5}$$

whenever K stabilizes P [Vid. 2].

Theorem 3.2 (Achievable I/O Maps and Achievable D/O Maps)

Consider the system $\Sigma(P,K)$ of Fig. 1. Let P satisfy assumption (P) and let (N_{pr}^m, D_{pr}) be a r.c. pair. Let $D_{pl}^{-1}N_{pl}^m$ be a l.c.f. of P^m. Then

$$\text{(i)}\ \mathcal{H}_{y_o v} = \{N_{pr}^o Q : Q \in \mathcal{H}^{n_i \times n_v}\} \tag{3.6}$$

equivalently, any map $H_v \in \mathcal{H}^{n_o \times n_v}$ is an achievable I/O map of the $\mathcal{H}$-stable system $\Sigma(P,K)$ if and only if $H_v = N_{pr}^o Q$ for some $Q \in \mathcal{H}^{n_i \times n_v}$.

$$\begin{aligned}\text{(ii)}\ \mathcal{H}_{y_o d} = \{N_{pr}^o[I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^o(V_{pr}^m - RN_{pl}^m)D_{pr} :\\ R \in \mathcal{H}^{n_i \times n_o} \text{ s.t. } \det(V_{pr}^m - RN_{pl}^m) \in \mathcal{J}\}\end{aligned} \tag{3.7}$$

where V_{pr}^m, U_{pr}^m, N_{pr}^m, D_{pr} are as in (2.16); equivalently, any map $H_d \in \mathcal{H}^{n_o \times n_i}$ is an achievable D/O map of the $\mathcal{H}$-stable system $\Sigma(P,K)$ if and only if $H_d = N_{pr}^o[I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^o(V_{pr}^m - RN_{pl}^m)D_{pr}$ for some $R \in \mathcal{H}^{n_i \times n_o}$ which satisfies $\det(V_{pr}^m - RN_{pl}^m) \in \mathcal{J}$.

Comments: 1) In the case that $y_o = y_m$ (i.e., $N_{pr}^o = N_{pr}^m =: N_{pr}$) the set of achievable I/O maps and the set of achievable D/O maps reduce to those in [Des. 3]:

$$\mathcal{H}_{y_o v}(P) = \{N_{pr} Q : Q \in \mathcal{H}^{n_i \times n_v}\}$$

$$\mathcal{H}_{y_o d_o}(P) = \{I - N_{pr}(U_{pr}^m + RD_{pl}) : R \in \mathcal{H}^{n_i \times n_o}, \text{ and } R \text{ is s.t. } \det D_{cl} \in \mathcal{J}\}$$

where $d_o := N_{pr} d$. 2) $H_{y_2 d}$ by the $\bar{\mathcal{U}}$-zeros and the $\mathcal{U}$-poles of the plant when $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$. If $\Sigma(P,K)$ is $\mathcal{H}$-stable and if PF $:= PD_{cl}^{-1}N_{fl}$ is full normal rank in $\mathcal{G}$, then

a) if z_o is a $\bar{\mathcal{U}}$-zero of N_{pr}^o (equivalently, $\exists\ \alpha \neq \vartheta_{n_o}$ such that $\alpha^* N_{pr}^o(z_o) = \vartheta_{n_i}$) then

$$\alpha^* N_{pr}^o(I - N_{fl}N_{pr}^m)(z_o) = \alpha^* H_{y_o d}(z_o) = \vartheta_{n_i}\ . \tag{3.8}$$

b) if N_{pr}^m has full normal rank and if z_m is a $\bar{\mathcal{U}}$-zero of N_{pr}^m (equivalently, $\exists\,\beta \neq \vartheta_n$ such that $N_{pr}^m(z_m)\beta = \vartheta_{n_i}$) then

$$N_{pr}^o(I - N_{fl}N_{pr}^m)(z_m)\beta = N_{pr}^o(z_m)\beta = H_{y_o d}(z_m)\beta\ . \tag{3.9}$$

c) if p_o is a $\mathcal{U}$-pole of P (equivalently, $\exists\ \gamma \neq \vartheta_n$ such that $D_{pr}(p_o)\gamma = \vartheta_{n_i}$) then

$$N_{pr}^{o} D_{cl} D_{pr}(p_o)\gamma = H_{y_o d}(p_o)\gamma = \vartheta_{n_o} \tag{3.10}$$

Thus, whenever either N_{pr}^{o} or N_{pr}^{m} has a $\bar{\mathcal{U}}$-zero or when P has a $\mathcal{U}$-pole, the D/O map is constrained by a vector-equality such as (3.8), (3.9) or (3.10) respectively.

Proof of Theorem 3.2: **(=>)** We are given P satisfying (P) and a controller K which stabilizes P. Let H_v be the I/O map and H_d be the D/O map of this $\Sigma(P,K)$. We must show that H_v is of the form $N_{pr}^{o} Q$ for some $Q \in \mathcal{H}^{n_i \times n_v}$ and H_d is of the form $N_{pr}^{o}[I - (U_{pr}^{m} + RD_{pl})N_{pr}^{m}] = N_{pr}^{o}(V_{pr}^{m} - RN_{pl}^{m})D_{pr}$ for some $R \in \mathcal{H}^{n_i \times n_o}$ satisfying $\det(V_{pr}^{m} - RN_{pl}^{m}) \in \mathcal{J}$.

Since K satisfies (K), $N_{\pi l} \in \mathcal{H}^{n_i \times n_v}$ and by Theorem 2.3, $\det D_h \in \mathcal{U}(\mathcal{H})$. Let $Q := D_h^{-1} N_{\pi l} = N_{\pi l}$; then $Q \in \mathcal{H}^{n_i \times n_v}$ and by (3.1), $H_v = N_{pr}^{o} D_h^{-1} N_{\pi l} = N_{pr}^{o} Q$.

Now from (2.5) and (3.5)

$$N_{fl} N_{pr}^{m} + D_{cl} D_{pr} = I \tag{3.11}$$

Viewing (3.11) as a *linear* matrix equation in $\mathcal{E}(\mathcal{H})$, we solve for (D_{cl}, N_{fl}) subject to $\det D_{cl} \in \mathcal{J}$ so that $D_{cl}^{-1} N_{fl} \in \mathcal{G}^{n_i \times n_o}$: since (N_{pr}^{m}, D_{pr}) is a r.c. pair, from (2.17) we have

$$U_{pr}^{m} N_{pr}^{m} + V_{pr}^{m} D_{pr} = I \tag{3.12}$$

and since $N_{pr}^{m} D_{pr}^{-1} = D_{pl}^{-1} N_{pl}^{m} = P^{m}$, we have

$$D_{pl} N_{pr}^{m} - N_{pl}^{m} D_{pr} = 0 \tag{3.13}$$

The pair (U_{pr}^{m}, V_{pr}^{m}) is (3.12) is a particular solution to (N_{fl}, D_{cl}) in (3.11) and the pair $(D_{pl}, -N_{pl}^{m})$ is a particular solution to the *homogeneous* equation (3.13). Hence, any general solution of (3.11) is given by

$$N_{fl} = U_{pr}^{m} + RD_{pl} \tag{3.14a}$$

$$D_{cl} = V_{pr}^{m} - RN_{pl}^{m} \tag{3.14b}$$

We now show that $R \in \mathcal{E}(\mathcal{H})$. Since K satisfies (K), $\det D_{cl} \in \mathcal{J}$; therefore $\det(V_{pr}^{m} - RN_{pl}^{m}) \in \mathcal{J}$. Since (D_{pl}, N_{pl}^{m}) are l.c., there exist $V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H})$ such that

$$D_{pl} V_{pl} + N_{pl}^{m} U_{pl} = I \tag{3.15}$$

Thus, by (3.14a-b) and (3.15), $R = R(D_{pl} V_{pl} + N_{pl}^{m} U_{pl}) = (N_{fl} - U_{pr}^{m}) V_{pl} + (V_{pr}^{m} - D_{cl}) U_{pl} \in \mathcal{E}(\mathcal{H})$ since $N_{fl}, D_{cl}, U_{pr}^{m}, V_{pr}^{m}, V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H})$.

From (3.2) and (3.14a-b), $H_d = N_{pr}^{o}[I - (U_{pr}^{m} + RD_{pl})N_{pr}^{m}] = N_{pr}^{o}(V_{pr}^{m} - RN_{pl}^{m})D_{pr}$. Therefore the given H_v and H_d are elements of the sets (3.6) and (3.7) respectively.

(<=) For some $Q \in \mathcal{H}^{n_i \times n_v}$, we are given $H_v = N_{pr}^{o}$, and for some $R \in \mathcal{H}^{n_i \times n_o}$, we are given $H_d = N_{pr}^{o}[I - (U_{pr}^{m} + RD_{pl})N_{pr}^{m}] = N_{pr}^{o}(V_{pr}^{m} - RN_{pl}^{m})D_{pr}$, where $\det(V_{pr}^{m} - RN_{pl}^{m}) \in \mathcal{J}$. We must show that there exists a compensator K which stabilizes P and the $\mathcal{H}$-stable $\Sigma(P,K)$ achieves the given H_v and H_d.

Choose the controller $K := D_{cl}^{-1}[N_{\pi l} \vdots N_{fl}]$ with N_{fl} and D_{cl} as in (3.14a-b) and $N_{\pi l} = Q$. Clearly, $D_{cl}, N_{\pi l}, N_{fl} \in \mathcal{E}(\mathcal{H})$. Note that $\det D_{cl} \in \mathcal{J}$ is guaranteed by the R that was chosen. Now, by (2.5)

$$D_h = (V_{pr}^m - RN_{pl}^m)D_{pr} + (U_{pr}^m + RD_{pl})N_{pr}^m$$

By (3.12) and (3.13), $D_h = I$. Rewriting (3.16) as

$$(V_{pr}^m - RN_{pl}^m)D_{pr} + [Q \vdots (U_{pr}^m + RD_{pl})]\begin{bmatrix} \vartheta_{n_v \times n_i} \\ \cdots\cdots \\ N_{pr}^m \end{bmatrix} = I \ ,$$

we see that $(D_{cl}, [N_{\pi l} \vdots N_{fl}])$ are l.c., and this K satisfies (K). since $\det D_h \in \mathcal{U}(\mathcal{H})$, $\Sigma(P,K)$ is $\mathcal{H}$-stable by Theorem 2.3(i).

By (3.1) and with $D_h = I$, we calculate the I/O map: $H_{y_o v} = N_{pr}^o N_{\pi l} = N_{pr}^o Q = H_v$. By (3.2), the D/O map is $H_{y_o d} = N_{pr}^o[I - N_{fl}N_{pr}^m] = N_{pr}^o[I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^o D_{cl} D_{pr} = N_{pr}^o(V_{pr}^m - RN_{pl}^m)D_{pr} = H_d$. ∎

Summary: Given the set up of Theorem 3.2 and in particular the Q and the R of (3.6) and (3.7), the compensator that achieves the specified H_v and H_d and that stabilizes P is given by the coprime factorization $D_{cl} = V_{pr}^m - RN_{pl}^m$, $[N_{\pi l} \vdots N_{fl}] = [Q \vdots U_{pr}^m + RD_{pl}]$.

IV. DECOUPLING

In this section we characterize all *diagonal* I/O maps which can be achieved by ${}^1\Sigma(P,K)$ for the given plant P.

Let $P \in \mathcal{G}^{2n \times n}$; i.e., $n_o = n_i = n$, $K \in \mathcal{G}^{n \times 2n}$ and $n_v = n$. Let assumption (P) and (K) hold with these new dimensions.

Let $n_{pk} \in \mathcal{H}^{1 \times n}$ denote the k-th row of $N_{pr}^o \in \mathcal{H}^{n \times n}$. For k = 1, ..., n, define Δ_{Lk} as a greatest common divisor (g.c.d.) over $\mathcal{H}$ of the elements of n_{pk} [Lang 1, p. 71]. Δ_{Lk} exists since $\mathcal{H}$ is a principal ring. Then the row-vector $\tilde{n}_{pk}$ is uniquely defined by $n_{pk} = \Delta_{Lk}\tilde{n}_{pk}$ and $\tilde{n}_{pk} \in \mathcal{H}^{1 \times n}$. Let $\tilde{N}_{pr}^o \in \mathcal{H}^{n \times n}$ be defined as the matrix which has $\tilde{n}_{pk}$ as its k-th row, k = 1, ..., n. Then

$$N_{pr}^o = diag(\Delta_{L1}, \ldots, \Delta_{Lk}, \ldots, \Delta_{Ln})\tilde{N}_{pr}^o =: \Delta_L \tilde{N}_{pr}^o \ . \tag{4.1}$$

Note that Δ_L and $\tilde{N}_{pr}^o$ are unique within unimodular factors. (In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, Δ_{Lk} "book-keeps" the plant zeros in $\mathcal{U}$ that are common to all elements of the k-th row of N_{pr}^o). A similar factorization is used in [Dat. 1].

The matrix $\tilde{N}_{pr}^o$ is not necessarily invertible over $\mathcal{H}^{n \times n}$. But by assumption (P) and since $\det N_{pr}^o \in \mathcal{H}$, $(\tilde{N}_{pr}^o)^{-1}$ has elements in the field of fractions $[\mathcal{H}][\mathcal{H} \setminus 0]^{-1}$ of the entire ring $\mathcal{H}$ [Lang 1, p. 69]. Let $\frac{m_{ij}}{d_{ij}}$ denote the ij-th element of $(\tilde{N}_{pr}^o)^{-1}$, i,j = 1, ..., n, where $m_{ij}, d_{ij} \in \mathcal{H}$ and m_{ij}, d_{ij} are coprime; thus

$$(\tilde{N}_{pr}^o)^{-1} =: \left[\frac{m_{ij}}{d_{ij}}\right] \ , \ i,j = 1,\ldots,n \ . \tag{4.2}$$

For j = 1, ..., n, let Δ_{Rj} be a least common multiple (l.c.m.) of $d_{1j}, d_{2j}, \ldots, d_{nj}$ of the j-th column of $(\tilde{N}_{pr}^o)^{-1}$ [Lang 1, p. 72]. Define

$$\Delta_R := diag(\Delta_{R1},\ldots,\Delta_{Rj},\ldots,\Delta_{Rn}) \in {}^{n\times n} \tag{4.3}$$

Δ_R is unique within a unimodular factor.

Lemma 4.1: Let $\tilde{N}^o_{pr}$ and Δ_R be defined by (4.1) and (4.3). Then $(\tilde{N}^o_{pr})^{-1}\Delta_R \in \mathcal{H}^{n\times n}$.

Proof: Since Δ_{Rj} is a l.c.m. of $(d_{ij})_{i=1}^n$, we have $\bar{d}_{ij} \in \mathcal{H}$ such that $\Delta_{Rj} = d_{ij}\bar{d}_{ij}$ for i = 1, ..., n. Then, for i,j = 1, ..., n, the ij-th element of $(\tilde{N}^o_{pr})^{-1}\Delta_R = \dfrac{m_{ij}}{d_{ij}}\Delta_{Rj} = m_{ij}\bar{d}_{ij} \in \mathcal{H}$ by (4.2)

Definition 4.2 (Achievable diagonal I/O map): Let P be a given plant that satisfies assumption (P). Roughly speaking, let $\mathcal{H}^d_{y_ov}(P)$ denote the set of all *achievable diagonal* I/O maps of $\Sigma(P,K)$; more precisely,

$$\mathcal{H}^d_{y_ov}(P) := \{H^d_{y_ov} : \text{K stabilizes P and the resulting I/O map } H^d_{y_ov} \text{ is diagonal and nonsingular.}\}$$

Theorem 4.3 (Achievable Diagonal I/O Maps)

Consider the system $\Sigma(P,K)$ of Fig. 1. Let P satisfy assumption (P) and let (N^m_{pr}, D_{pr}) be r.c. Let $D_{pl}^{-1}N^m_{pl}$ be a l.c.f. of P^m. Then

$$\mathcal{H}^d_{y_ov}(P) = \{\Delta_L\Delta_R Q_d : Q_d \in \mathcal{H}^{n\times n}, \text{with } Q_d \text{ diagonal and nonsingular}\} \tag{4.4}$$

equivalently, the map $H^d_v \in \mathcal{H}^{n\times n}$ is an achivevable I/O map of the $\mathcal{H}$-stable system $\Sigma(P,K)$ if and only if $H^d_v = \Delta_L\Delta_R Q_d$ for some nonsingular, diagonal $Q_d \in \mathcal{H}^{n\times n}$.

Proof: (=>) We are given P satisfying (P) and K which stabilizes P. Let $H^d_v \in \mathcal{H}^{n\times n}$ be the diagonal I/O map of this $\Sigma(P,K)$. We must show that H^d_v is of the form $\Delta_L\Delta_R Q_d$ for some diagonal, nonsingular $Q_d \in \mathcal{H}^{n\times n}$.

Since $\Sigma(P,K)$ is $\mathcal{H}$-stable, we use (3.5). By (3.1) and (4.1), the diagonal matrix Δ_L is obviously a left-factor of H^d_v. It remains to show that H^d_v has $\Delta_L\Delta_R$ as a left-factor. For a contradiction, suppose that H^d_v is of the form

$$H^d_v = \Delta_L\tilde{\Delta}_R Q_d \tag{4.5}$$

where $\tilde{\Delta}_R$ is a *proper* factor of Δ_R, and $Q_d \in \mathcal{H}^{n\times n}$ is nonsingular and diagonal. W.l.o.g. suppose, for example, that

$$\tilde{\Delta}_R = diag(\Delta_{R1},\ldots,\Delta_{Rj-1},\tilde{\Delta}_{Rj},\Delta_{Rj+1},\ldots,\Delta_{Rn}) \tag{4.6}$$

where, for a *non-unit prime* element $\delta_j \in \mathcal{H}$ [Lang. 1, p. 72],

$$\Delta_{Rj} = \delta_j\tilde{\Delta}_{Rj} \ . \tag{4.7}$$

Then by (3.1) and (4.5)

$$\Delta_L\tilde{N}^o_{pr}N_{\pi l} = \Delta_L\tilde{\Delta}_R Q_d \tag{4.8}$$

Since $\mathcal{H}$ is a principal ring, we may cancel the nonsingular left-factor Δ_L and invert $\tilde{N}^o_{pr}$ in (4.8) to obtain

$$N_{\pi l} = (\tilde{N}^o_{pr})^{-1}\tilde{\Delta}_R Q_d \ . \tag{4.9}$$

By (4.2) and (4.6)

$$N_{\pi l} = \left[\frac{m_{ij}}{d_{ij}}\right] \cdot diag\,(\Delta_{R1},\ldots,\widetilde{\Delta}_{Rj},\ldots,\Delta_{Rn}) \cdot Q_d \quad . \tag{4.10}$$

Recalling that Δ_{Rj} is by definition a l.c.m. of $(d_{ij})_{i=1}^{n}$ and by (4.7), for some i, we have

$$d_{ij} = \delta_j \widetilde{d}_{ij} \tag{4.11}$$

where $\widetilde{d}_{ij} \in \mathcal{H}$ is a factor of $\widetilde{\Delta}_{Rj}$; i.e., there is a $\widetilde{c}_{ij} \in \mathcal{H}$, possibly a unit, such that

$$\widetilde{\Delta}_{Rj} = \widetilde{d}_{ij}\widetilde{c}_{ij} \quad . \tag{4.12}$$

Hence, with $q_j \in \mathcal{H}$ denoting the j-th (non-zero) diagonal entry of some general non-singular diagonal $Q_d \in \mathcal{H}^{n\times n}$, we obtain the ij-th element of $N_{\pi l}$ from (4.10), (4.11) and (4.12) as

$$\frac{m_{ij}}{\delta_j}\widetilde{c}_{ij}q_j \quad . \tag{4.13}$$

Since $\delta_j \notin \mathcal{U}(\mathcal{H})$ and in general δ_j is not a factor of q_j, (4.13) is *not* in $\mathcal{H}$. Therefore, except when the prime non-unit δ_j is a factor of q_j, $N_{\pi l} \notin \mathcal{H}^{n\times n}$, thus with $N_{\pi l}$ as in (4.10), there is a diagonal, nonsingular $Q_d \in \mathcal{H}^{n\times n}$ such that K does not satisfy assumption (K). This contradicts the assumption that K stabilizes P. Therefore, H_v^d must be an element of the set (4.4). **(<=)** For some diagonal, nonsingular $Q_d \in \mathcal{H}^{n\times n}$, we are given $H_v^d = \Delta_L \Delta_R Q_d$. We must show that there exists a compensator K which stabilizes P, and the $\mathcal{H}$-stable $\Sigma(P,K)$ achieves the given H_v^d.

Choose the controller $K := D_{cl}^{-1}[N_{\pi l} \vdots N_{fl}]$ with

$$N_{\pi l} := (\widetilde{N}_{pr}^{o})^{-1}\Delta_R Q_d \tag{4.14}$$

where, by Lemma 4.1, $N_{\pi l} \in \mathcal{H}^{n\times n}$, and choose N_{fl}, D_{cl} as in (3.14a-b) with $n_i = n_o$. To prove that this K satisfies (K) and that $\Sigma(P,K)$ is $\mathcal{H}$-stable, one uses the same reasoning as in Theorem 3.2 (ii). Hence, we omit the proof.

By (3.1), (3.5) and (4.14) we calculate the diagonal I/O map as

$$H_{y_2v}^d = N_{pr}^o D_h^{-1} N_{\pi l} = \Delta_L \widetilde{N}_{pr}^o(\widetilde{N}_{pr}^o)^{-1}\Delta_R Q_d = \Delta_L \Delta_R Q_d = H_v^d \quad .$$

■

V. ROBUST STABILITY

The following robust stability theorem considers multiple perturbations (both plant *and* compensator) for the system $\Sigma(P,K)$.

In the following, let $\Sigma(\widetilde{P},\widetilde{K})$ denote the *perturbed system* where $\widetilde{P} = \begin{bmatrix} \widetilde{N}_{pr}^o \\ \cdots \\ \widetilde{N}_{pr}^m \end{bmatrix} \widetilde{D}_{pr}^{-1}$, $\widetilde{N}_{pr}^o = N_{pr}^o + \Delta N_{pr}^o$, $\widetilde{N}_{pr}^m = N_{pr}^m + \Delta N_{pr}^m$, $\widetilde{D}_{pr} = D_{pr} + \Delta D_{pr}$ and $\widetilde{K}$ is defined similarly. Assumptions (P) and (K) become $(\widetilde{P})$ and $(\widetilde{K})$ with all parameters replaced by their perturbed versions.

Theorem 5.1 (Robust Stability)

Consider the system $\Sigma(P,K)$ of Fig. 1, where P satisfies assumption (P) and K stabilizes P. Let D_{pr}, N_{pr}^{o}, N_{pr}^{m}, D_{cl}, N_{fl}, N_{rl} be additively perturbed by, respectively, ΔD_{pr}, ΔN_{pr}^{o}, ... etc., with $\det D_{pr} \in \mathcal{J}$, $\det D_{cl} \in \mathcal{J}$ and $\det(D_{cl}D_{pr}+N_{fl}N_{pr}^{m}) \in \mathcal{J}$.

(i) Let $\tilde{P}$ and $\tilde{K}$ satisfy assumptions $(\tilde{P})$ and $(\tilde{K})$. Then $\Sigma(\tilde{P},\tilde{K})$ is $\mathcal{H}$-stable if and only if $\det(\tilde{D}_{cl}\tilde{D}_{pr}+\tilde{N}_{fl}\tilde{N}_{pr}^{m}) \in \mathcal{U}(\mathcal{H})$.

(ii) Let $(\mathcal{H}, \|\cdot\|)$ be a Banach algebra and B(0;r) denote the open ball in $\mathcal{H}^{p\times q}$ of radius r centered at zero where p and q are specified by the context. Let $\rho_{dp} > 0$, $\rho_{np} > 0$, $\rho_{dc} > 0$, $\rho_{nf} > 0$ be such that

$$\|D_{cl}\|\rho_{dp} + \|N_{fl}\|\rho_{np} + \|D_{pr}\|\rho_{dc} + \|N_{pr}\|\rho_{nf} + \rho_{dp}\rho_{nc} + \rho_{np}\rho_{nf} < 1 \quad (5.1)$$

U.t.c. if

$$\Delta D_{pr} \in B(0;\rho_{dp}) \; ,\Delta D_{cl} \in B(0;\rho_{dc}) \; , \quad (5.2)$$

$$\Delta N_{pr}^{m} \in B(0;\rho_{np}) \; ,\text{and} \quad \Delta N_{fl} \in B(0;\rho_{nf})$$

then the perturbed system $\Sigma(\tilde{P},\tilde{K})$ is $\mathcal{H}$-stable.

Proof: (i) Same as the proof of Theorem 2.3(i), with all parameters replaced by the perturbed versions. (ii) The perturbed system $\Sigma(\tilde{P},\tilde{K})$ is $\mathcal{H}$-stable if and only if $\det \tilde{D}_h \in \mathcal{U}(\mathcal{H}) <=> \tilde{D}_h^{-1} \in \mathcal{E}(\mathcal{H})$ where $D_h := D_{cl}D_{pr} + N_{fl}N_{pr}^{m}$. By normalization of the unperturbed system, $D_h = I$. Then

$$\begin{aligned}\tilde{D}_h &= I + D_{cl}\Delta D_{pr} + N_{fl}\Delta N_{pr}^{m} + \Delta D_{cl}D_{pr} + \Delta N_{fl}N_{pr}^{m} + \Delta D_{cl}\Delta D_{pr} + \Delta N_{fl}\Delta N_{pr}^{m} \\ &=: I + R\end{aligned} \quad (5.3)$$

By (5.1) and (5.2), $\|R\| < 1$; hence, $(I+R)^{-1} \in \mathcal{E}(\mathcal{H})$ [Rud. 1, Theorem 18.3]. Therefore, $\tilde{D}_h^{-1} \in \mathcal{E}(\mathcal{H})$ and the conclusion follows.

■

Comments: 1) Similar results may be obtained for the case in which a *left-coprime* factorization (l.c.f.) of the plant P and a *right-coprime* factorization (r.c.f.) of the compensator K are used. 2) In the *lumped case*, the sufficiency result (ii) allows changes in the *number* and *location* of *poles* and *zeros* in both the stable and the unstable regions of the plane: this allows the consideration of systems of different orders having different number of unstable zeros and poles.

VI. ASYMPTOTIC TRACKING

For the tracking problem we consider the system $\Sigma(P,K)$ of Fig. 1 with $n_v = n_o$.

Definition 6.1 (Class of Inputs): The class A of inputs to be tracked consists of vectors $\alpha^{-1}u$ where $\alpha \in \mathcal{J}\setminus\mathcal{U}(\mathcal{H})$ and $u \in \mathcal{H}^{n_o}$, with the property that the vector u is *not* a multiple of α. Consequently, the vector $\alpha^{-1}u \notin \mathcal{E}(\mathcal{H})$: the inputs to be tracked are *not* stable time functions (typically steps, ramps, parabolas, sinusoids, etc.).

Definition 6.2 (Asymptotic Tracking): The closed-loop system $\Sigma(P,K)$ is said to *asymptotically track the class A* if and only if $v - y_o \in \mathcal{H}^{n}$, $\forall\, v \in A$.

Comments: 1) The function $v - y_o$ is the tracking error: if the class $\mathcal{H}$ is suitably chosen, $v - y_o \in \mathcal{H}^{n}$ implies that $v(t) - y_o(t) \to \vartheta_n$ as $t \to \infty$ (e.g., $\mathcal{H}=\mathcal{R}_{\mathcal{U}}$ with

$\mathcal{U} \supset \mathbb{C}_+$). **2)** Alternatively we could have used D_ℓ^{-1} driven by $u \in \mathcal{H}^{n_o}$ as the generator of tracked signals, where $(\det D_\ell)^{-1} \not\in \mathcal{H}$. With α defined as the largest invariant factor of D_ℓ, using the discussion in [Vid. 1] it can be shown that there is no loss of generality in adopting our definition as far as *robust* asymptotic tracking--to be precisely defined later--is concerned.

Theorem 6.3 (Necessary Conditions)

Let P satisfy (P). Let K stabilize P and have a l.c.f. $D_{cl}^{-1}[N_{\pi l} \vdots N_{fl}] \in \mathcal{G}^{n_i \times 2n_o}$; w.l.o.g. let $D_h = I_{n_i}$. U.t.c. if the system $\Sigma(P,K)$ asymptotically tracks the class A, then

(i) $n_i \geq n_o$ (6.1)

(ii) $(N_{pr}^{o} N_{\pi l}\ ,\ \alpha I)$ is r.c. (6.2)

Comments: **1)** By calculation, $H_{y_o v} = N_{pr}^{o} N_{\pi l} \in \mathcal{H}^{n_o \times n_o}$. Conclusion (ii) implies that $\det H_{y_o v} = \det(N_{pr}^{o} N_{\pi l})$ and α are coprime in $\mathcal{H}$. **2)** Let $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$. If $\Sigma(P,K)$ tracks the class A, the zeros off N_{pr}^{o}, the zeros of $N_{\pi l}$, and the zeros of $H_{y_2 v}$ are disjoint from those of α. In particular, if N_{pr}^{o} and α have some common zeros in $\mathcal{U}$, there exists no K such that $\Sigma(P,K)$ tracks A.

Proof: Let u_t / α be an input to be tracked; thus, $u_t \in \mathcal{H}^{n_o}$. The transfer matrix $H_{e_t u_t} : u_t \mapsto (v - y_o) =: e_t$ is given by

$$H_{e_t u_t} = (I - N_{pr}^{o} N_{\pi l})\alpha^{-1} \tag{6.3}$$

By assumption, $H_{e_t u_t} \in \mathcal{H}^{n_o \times n_o}$ since asymptotic tracking is achieved.

(i) Suppose, for a proof by contradiction, that $n_i < n_o$. Then $rk(N_{pr}^{o} N_{\pi l}) \leq \min(rkN_{pr}^{o}, rkN_{\pi l}) \leq n_i < n_o$. Thus, there exists $\gamma \in \mathcal{H}^{n_o}$ such that [Bou. 1, Chap III, sec. 8, Prop. 14]

$$(a) N_{pr}^{o} N_{\pi l} \gamma = \vartheta_{n_o} \tag{6.4a}$$

$$(b) \gamma \text{ is not a multiple of } \alpha\ . \tag{6.4b}$$

If γ were a multiple of α, say $\gamma = \alpha^k \tilde{\gamma}$ where k is the multiplicity of α as a factor of γ, then $N_{pr}^{o} N_{\pi l} \tilde{\gamma} = \vartheta_{n_o}$, and $\tilde{\gamma} \in \mathcal{H}^{n_o}$ and $\alpha \in \mathcal{H}$ have no (non-trivial) common factors.

Apply the input $v = \alpha^{-1}\gamma$; $\alpha^{-1}\gamma \not\in \mathcal{E}(\mathcal{H})$ by (6.4b) above. Then $e_t = v - y_o = (I - N_{pr}^{o} N_{\pi l})\alpha^{-1}\gamma = \alpha^{-1}\gamma \not\in \mathcal{E}(\mathcal{H})$; which contradicts the assumption that $\Sigma(P,K)$ asymptotically tracks the class A.

(ii) Since $H_{e_t u_t} \in \mathcal{H}^{n_o \times n_o}$, let $(I - N_{pr}^{o} N_{\pi l})\alpha^{-1} =: M \in \mathcal{E}(\mathcal{H})$; equivalently,

$$N_{pr}^{o} N_{\pi l} + M\alpha = I \tag{6.5}$$

Hence, $(N_{pr}^{o} N_{\pi l}, \alpha I)$ is r.c. and since N_{pr}^{o}, $N_{\pi l} \in \mathcal{E}(\mathcal{H})$, $(N_{\pi l}, \alpha I)$ is also r.c. and $(\alpha I, N_{pr}^{o})$ is l.c.

∎

Theorem 6.4 (Robust asymptotic tracking: sufficient conditions)

Let P satisfy assumption (P) and let $N_{pr}^{m} D_{pr}^{-1} = D_{pl}^{-1} N_{pl}^{m}$ be a r.c.f. and a l.c.f., respectively, of P^m. Let K stabilize P and let $D_{cl}^{-1} N_{fl} = N_{fr} D_{cr}^{-1}$ be a l.c.f. and a r.c.f., respectively, of the feedback compensator.

I) (Tracking)

If **(i)** $N_{pr}^{m} N_{fl} - N_{pr}^{o} N_{\pi l} = \alpha N_c$ for some $N_c \in \mathcal{H}^{n_o \times n_o}$ **and**

(ii) $D_{cr} = \alpha D_c$ for some $D_c \in \mathcal{H}^{n_o \times n_o}$

then $\Sigma(P,K)$ tracks asymptotically the class A.

II) (Robust Tracking)

Let the plant P be perturbed to $\widetilde{P}$ and let the compensator K be perturbed to $\widetilde{K}$: let $\widetilde{P}$ and $\widetilde{K}$ be described by similar coprime factorizations (i.e., all N's become $\widetilde{N}$ and D's becomes $\widetilde{D}$'s) *but α is not perturbed. U.t.c.*

if **(i)** $\widetilde{K}$ stabilizes $\widetilde{P}$,

(ii) $\widetilde{N}_{pr}^{m} \widetilde{N}_{fl} - \widetilde{N}_{pr}^{o} \widetilde{N}_{\pi l} = \alpha \widetilde{N}_c$ for some $\widetilde{N}_c \in \mathcal{H}^{n_o \times n_o}$ **and**

(iii) $\widetilde{D}_{cr} = \alpha \widetilde{D}_c$ for some $\widetilde{D}_c \in \mathcal{H}^{n_o \times n_o}$

then $\Sigma(\widetilde{P},\widetilde{K})$ tracks asymptotically the class A.

■

Comments: **1)** Condition (ii) of part I requires that $1/\alpha$ appears in each input-channel of the compensator: the internal model must contain each unstable factor of α, the denominator of the signal generator. **2)** Condition (i) of part I means that the difference between the closed-loop gain $u_1 \mapsto y_m$ and the closed-loop gain $v \mapsto y_o$ must have α as a factor (if $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, $N_{pr}^{m} N_{fl} - N_{pr}^{o} N_{\pi l}$ must have a blocking zero at each $\mathcal{U}$-zero of α) for asymptotic tracking. **3)** For $^1S(P,C)$, [Des. 3], condition (ii) of part I becomes a tautology: ($N_{pr}^{o} = N_{pr}^{m}$ and $N_{fl} = N_{\pi l}$). **4)** As long as the α-factor conditions (ii) and (iii) of part II are obeyed, any perturbation of the plant and of the compensator *however large* they may be, robust asymptotic tracking will be maintained provided that stability is maintained. **5)** Condition (ii) of part I may not be minimal; i.e., some factor $\hat{\alpha}$ of α may already exist in the plant and thus $\hat{\alpha}^{-1} D_{pl} \in \mathcal{E}(\mathcal{H})$. However, if (ii) is satisfied, then the internal model is present in the compensator and thus allows the plant to be arbitrarily perturbed.

Proof of I: Since K stabilizes P, $D_h = I_{n_i}$ as in (3.5). Similarly, we can set

$$D_{pl} D_{cr} + N_{pl}^{m} N_{fr} = I_{n_o} \tag{6.6}$$

From the properties of r.c.f. and l.c.f. we have

$$\begin{bmatrix} D_{cl} & N_{fl} \\ -N_{pl}^{m} & D_{pl} \end{bmatrix} \begin{bmatrix} D_{pr} & -N_{fr} \\ N_{pr}^{m} & D_{cr} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} \tag{6.7}$$

$$\begin{bmatrix} D_{pr} & -N_{fr} \\ N_{pr}^{m} & D_{cr} \end{bmatrix} \begin{bmatrix} D_{cl} & N_{fl} \\ -N_{pl}^{m} & D_{pl} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} \tag{6.8}$$

Using (6.8), $H_{e_t u_t}$ of (6.3) can be written as

$$H_{e_t u_t} = (D_{cr} D_{pl} + N_{pr}^{m} N_{fl} - N_{pr}^{o} N_{\pi l})\alpha^{-1} \tag{6.9}$$

From (i) and (ii), we obtain $H_{e_t u_t} = D_c D_{pl} + N_c \in \mathcal{E}(\mathcal{H})$. Hence, $\Sigma(P,K)$ asymptotically tracks the class A.

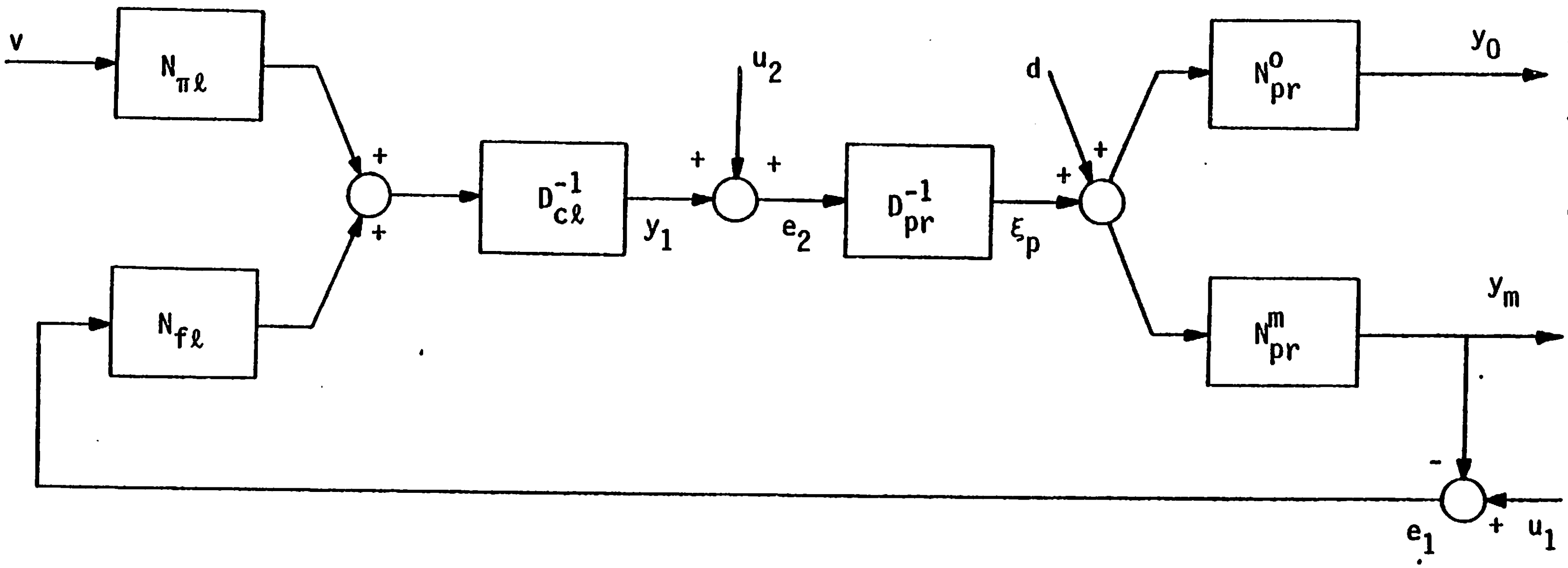

Fig. 1. The System $\Sigma(P,K)$.

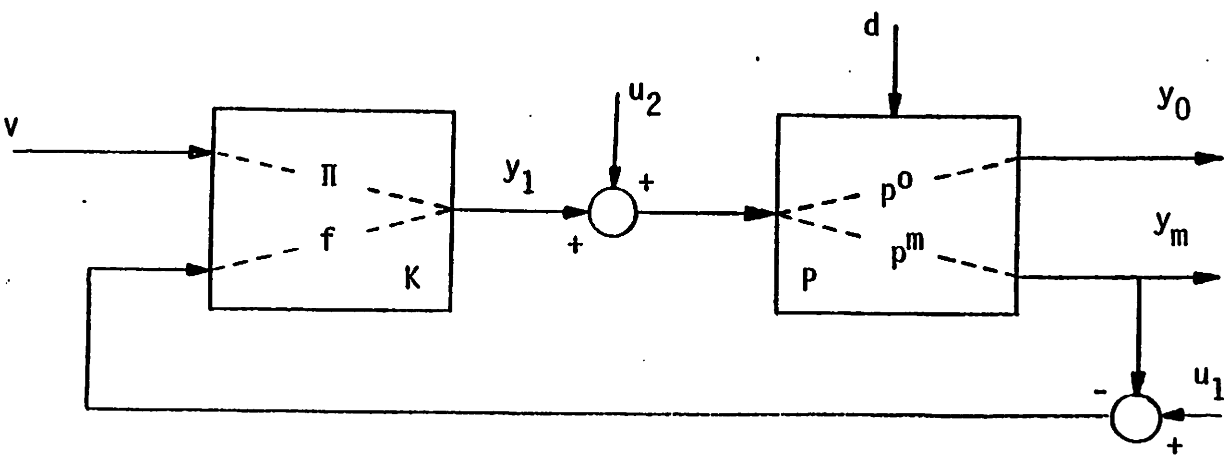

Fig. 2. The System ${}^{1}\Sigma(P,K)$.

Proof of II: The proof of I can be repeated word for word except that K's, P's, N's and D's are now $\tilde{K}$'s, $\tilde{P}$'s, $\tilde{N}$'s and $\tilde{D}$'s, respectively. ∎

Conclusions: This paper presents an algebraic design theory for linear feedback systems. The results obtained rely on linearity and time-invariance, and important factors such as saturation and noise are ignored.

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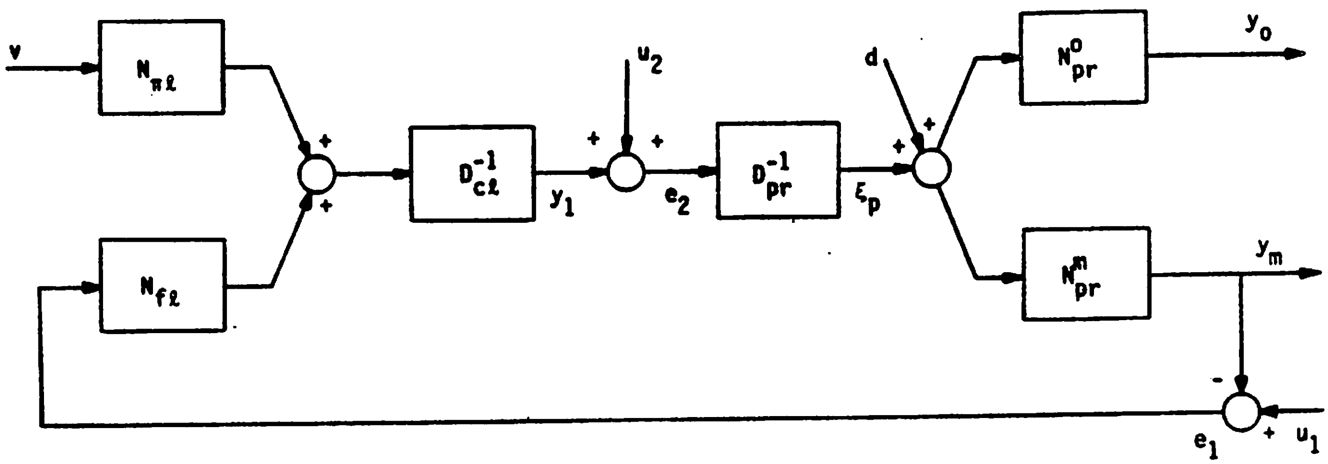

Fig. 1. The System Σ(P,K).

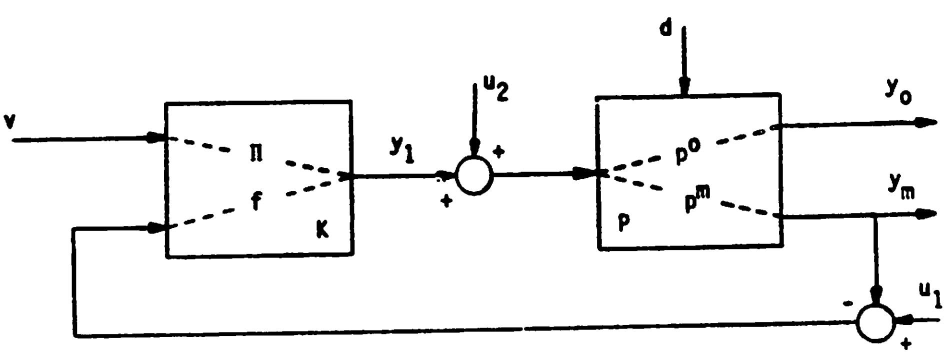

Fig. 2. The System $^1\Sigma(P,K)$.

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