AN EXACT PENALTY FUNCTION ALGORITHM FOR CONTROL PROBLEMS WITH STATE AND CONTROL CONSTRAINTS

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AN EXACT PENALTY FUNCTION ALGORITHM FOR CONTROL PROBLEMS WITH STATE AND CONTROL CONSTRAINTS[1]

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1. INTRODUCTION

This paper deals with optimal control problems with state constraints and control constraints. To place the contributions of this paper in context it is useful to consider a few earlier results.

For optimal control problems with (hard) control constraints (i.e. constraints of the form $u(t) \in \Omega$ for all t) only it is possible, as Bertsekas [1] has shown, to employ the Goldstein Levitin-Polyak Gradient Projection Method. Accumulation points, if they exist (in L_∞), satisfy a weak condition of optimality. An alternative approach, based on differentiable dynamic programming [2], is described in [3]. In this approach the Hamiltonian is minimized at each time, and the new control u_λ is set equal to the minimizing control on a subset I_λ of the total time interval T and the original control elsewhere. The step length λ (the total "length" of I_λ) is chosen to minimize (approximately) the cost $g^0(u_\lambda)$. It is shown in [3] that L_∞ accumulation points of sequences generated by the algorithm (if they exist) satisfy a strong condition of optimality (the maximum principle). A further study of this algorithm in [4] showed that if the controls generated by the algorithm were regarded as degenerate relaxed controls, then accumulation points, in the sense of control measures, always exist and satisfy a strong condition of optimality for the relaxed problem (i.e. the relaxed maximum principle). Warga describes a steepest descent algorithm which he employs for solving optimal control problems with control constraints using relaxed controls and establishes a similar result [5]. This kind of approach is attractive in that accumulation points always exist and satisfy a strong condition of optimality; however the determination of a (relaxed) control which minimizes the Hamiltonian (at each t) is computationally expensive.

Control problems with terminal constraints (e.g. constraints on control energy) can be formulated as

$$\min\{g^0(u) \mid g^j(u) \leq 0, \quad j \in M\}$$

where g^0, g^j are Frechet differentiable and can thus be handled by the natural analogs of finite dimensional algorithms.

Problems with terminal constraints and (hard) control constraints are more complex because the control constraints are not differentiable in the space of control functions (e.g. L_∞). Hence the following approach was adopted in [6,7]. A convex optimal control problem is defined whose (approximate) solution yields a search direction which, to first order, satisfies the control constraints and which either reduces the cost and nearby active terminal constraints for problems with terminal inequality constraints or reduces an exact penalty function for problems involving terminal equality constraints. The components of this sub problem are a linear dynamic system, a cost function which is the maximum of several functions affine in the search direction s and a hard constraint on s. To facilitate implementation weak variations were employed so that accumulation points (which always exist in the sense of control measures) satisfy a weak condition of optimality for the relaxed problem. Although convergence (in the above sense) was proven in [6,7], later numerical studies showed that convergence could be slow; the difficulty appeared to lie in the fact that the search direction does not converge to zero as a solution is approached. An improved version of this algorithm [8] utilizes automatic scaling of the search direction.

A substantial advance has been recently made by Warga [9] for control problems with state constraints. Relaxed controls are employed and, perhaps more importantly, it is shown how a satisfactory search direction for this problem can be obtained by solving a convex optimal control problem as in [6,7]; however in this case the associated multipliers (defining a supporting hyperplane to the reachable set of the "linearized" system) are functions rather than scalars. The resultant algorithm provides a potentially very useful method for state constrainted optimal control problems; accumulation points of sequences generated by the algorithm satisfy a strong condition of optimality. However, the algorithm requires (exact) minimization of the Hamiltonian at each t which is computationally very demanding in most applications and does not handle equality constraints.

The algorithm presented in this paper lies in the same family as those in [6-9], i.e. they determine a search direction by (approximately) solving a convex optimal control problem. In order to avoid excessive computation, the linearized model of the system equations are linear in state <u>and</u> control, avoiding the necessity for global minimization of the Hamiltonian; an inevitable consequence is that accumulation points of infinite sequence generated by the algorithm satisfy a weaker condition of optimality. The algorithms cope with terminal equality and inequality constraints in addition to the control and state constraints; an exact penalty function is employed to handle constraints except the control constraint. The algorithm includes automatic scaling of the search direction which, as shown in [8], improves numerical performance.

Large motions of robot arms and large space structures are governed by nonlinear dynamics and are best carried out open loop via optimal control. In the case of robots, the presence of obstacles leads to state space constraints on these motions. While the need to control vibrations generates state space constraints in large space structures the requirement to position or aim these devices exactly at the end of a large manoeuver results in terminal equality constraints. Clearly, control constraints are always present in these positioning problems. As a result there is renewed interest in developing efficient optimal control algorithms which can cope with complex constraints.

2. THE PROBLEM

We consider the following optimal control problem

$$P: \inf\{g_0(u) \mid g_1(u) = 0,\ g_2(u) \leq 0,\ g_3(u) \leq 0,\ u \in G\} \tag{1}$$

where G, the (convex) set of admissible controls, is defined by

$$G \triangleq \{u \in L_\infty \mid u(t) \in \Omega \text{ for all } t \in T \triangleq [0, 1]\} \tag{2}$$

and g_0: $G \to R$, g_1: $G \to R^{m_E}$, g_2: $G \to R^{m_I}$, g_3: $G: \to C^{m_s}$ are defined by:

$$g_0(u) \triangleq h_0(x^u(1)), \tag{3}$$

$$g_i(u) \triangleq h_i(x^u(1)),\ i = 1, 2, \tag{4}$$

$$g_3(u)(t) \triangleq h_3(x^u(t)). \tag{5}$$

In these definitions C^{m_s} denotes the Banach space of continuous functions from T to R^{m_s} with the sup norm $\|\cdot\|_\infty$ and x^u: $T \to R^n$ is the solution of

$$\dot{x}(t) = f(x(t), u(t), t) \tag{6}$$

$$x(0) = x_0. \tag{7}$$

Hence the constraints $g_1(u) = 0$, $g_2(u) \leq 0$ are conventional terminal constraints whereas $g_3(u) \leq 0$ is a state constraint. The functions g_i, $i = 0,\ldots,4$, are Frechet differentiable. To deal with the terminal and state constraints we introduce the penalty function γ_c: $G \to R$ defined, for all $c > 0$, by:

$$\gamma_c(u) \underline{\Delta}\ g_0(u)/c + \gamma(u) \tag{8}$$

where γ: $G \to R$ is defined by:

$$\gamma(u) \underline{\Delta} \max\{0; |g_1{}^j(u)|,\ j \in \underline{m}_E;\ g_2{}^j(u),\ j \in \underline{m}_I;$$

$$g_3{}^j(u)(t),\ j \in \underline{m}_S,\ t \in T\} \tag{9}$$

where

$$\underline{m}_E \underline{\Delta} \{1,2,\ldots,m_E\} \tag{10}$$

with similar definitions for $\underline{m}_I$, $\underline{m}_S$. The term 0 is included in (10) in case there are no equality constraints. To solve problem P we solve P_c:

$$P_c\text{: } \inf\{\gamma_c(u) | u \in G\} \tag{11}$$

increasing c adaptively finitely often when required to ensure convergence to a solution of P (more precisely, to a point satisfying necessary conditions of optimality for P).

Our algorithm therefore has two components: one solves P_c for fixed c and the other adjusts c. The first component generates at u a search direction $v - u$ which is admissible ($v \in G$) and which is a descent direction for γ_c at u, i.e. v satisfies

$$\hat{\gamma}_c(v, u) < \gamma_c(u)$$

where, for all u,v in G, $\hat{\gamma}_c(v, u)$ is a first order estimate of $\gamma_c(v)$ defined by:

$$\hat{\gamma}_c(v,u) \triangleq [g^0(u) + Dg^0(u; v-u)]/c + \hat{\gamma}(v,u) \tag{12}$$

where

$$\hat{\gamma}(v,u) \triangleq \max\{0; |g_1^j(u) + Dg_1^j(u,v-u)|, j \in \underline{m}_E; [g_2^j(u) + Dg_2^j(u,v-u)],$$

$$j \in \underline{m}_I; [g_3^j(u)(t) + Dg_3^j(u,v-u)(t)], j \in \underline{m}_S, t \in T\}. \tag{13}$$

A suitable step along this direction is then obtained by a modified version of Armijo's rule. The second component of the algorithm chooses the penalty coefficient c to satisfy a test of the form $t_c(u) \leq 0$.

3. ASSUMPTIONS AND PRELIMINARY RESULTS

The following assumptions are made

A1 The functions $f: R^n \times \Omega \times T \to R^n$

$h_0: R^n \to R$, $h_1: R^n \to R^{m_E}$, $h_2: R^n \to R^{m_I}$ and

$h_3: R^n \to R^{m_S}$ are continuously differentiable.

A2 There exists a $M \in (0, \infty)$ such that

$$\|f(x,u,t)\| \leq M(1 + \|x\|)$$

for all $(x,u,t) \in R^n \times \Omega \times T$.

A3 The set Ω is compact and convex.

It follows from our assumptions that, for each u in G, there exists a unique absolutely continuous solution x^u; an elementary application of Gronwall's lemma yields

Proposition 1

There exists a compact subset X of R^n such that $x^u(t)$ lies in X for all u in G and all t in T. ∇

For all u,v in G let $z^{v,u}: T \to R^n$ denote the (unique, absolutely continuous) solution of:

$$\dot{z}(t) = A^u(t)z(t) + B^u(t)[v(t) - u(t)] \quad (14)$$

$$z(0) = 0 \quad (15)$$

where $A^u: T \to R^{nxn}$ and $B^u: T \to R^{nxn}$ are defined by:

$$A^u(t) \triangleq f_x(x^u(t), u(t), t) \quad (16)$$

$$B^u(t) \triangleq f_u(x^u(t), u(t), t). \quad (17)$$

Clearly $z^{v,u}$ is an estimate (in the sense to be made precise later) of $x^v - x^u$.

Proposition 2

There exists a $d \in (0, \infty)$ such that

(a) $\|x^v - x^u\|_\infty \leq d\|v - u\|_\infty$ for all $u, v \in G$,

(b) $\|z^{v,u}\|_\infty \leq d\|v - u\|_\infty$ for all $u, v \in G$.

(c) For all $\eta > 0$, there exists a $\delta > 0$ such that

$$\|(x^w - x^v) - z^{w,v}\|_\infty \leq \eta\|w - v\|_\infty \tag{18}$$

for all w,v satisfying

$$\|w - v\|_\infty \leq \delta. \tag{19}$$

Proof

(a), (b) These results follow from the Gronwall lemma and the fact that f_x and f_u are bounded in the compact set $X \times \Omega \times T$ (see proof of (c)).

(c) Since

$$\begin{aligned}(x^w - x^v)(t) &= \int_0^t [f(x^w(s), w(s),s) - f(x^v(s),v(s),s)]ds \\ &= \int_0^t [\int_0^1 f_x(x^v(s) + \alpha(x^w(s) - x^v(s)), w(s) + \alpha(w(s) - \\ &\qquad v(s)),s)d\alpha] [x^w(s) - x^v(s)] \\ &+ [\int_0^1 f_u(x^v(s) + \alpha(x^w(s) - x^v(s), w(s) + \alpha(w(s) \\ &\qquad - v(s)), s)d\alpha] [w(s) - v(s)]]ds\end{aligned}$$

and

$$z^{w,v}(t) = \int_0^t [f_x(x^v(s), v(s), s)z^{w,v}(s) + f_u(x^v(s), v(s), s)(w(s) - v(s))]ds$$

it follows from (b) and the uniform continuity of f_x and f_u in $X \times \Omega \times T$ that for all $\eta' > 0$ there exists a $\delta > 0$ such that $e \triangleq (x^w - x^v) - z^{w,v}$ satisfies:

$$\|e(t)\| \leq \int_0^t [\|A^v(s)\| \; \|e(s)\| + \eta'[\|x^w(s) - x^v(s)\| + \|w(s) - v(s)\|]]ds$$
$$\leq e^{at} \eta' \; d\|w - v\|_\infty$$

for all w,v in $B_\delta(u)$ where a is an upper bound for $\|f_x\|$ on $X \times \Omega \times T$. The desired result follows easily. ∇

It follows that $u \to x^u$, $G \to L_\infty$ is Frechet differentiable, the differential at u in the direction $v-u$ being $z^{v,u}$. Similarly the functions $u \to g_j(u)$, $j = 0\text{-}3$ are Frechet differentiable; for $j = 0,1,2$ the differential at u in the direction $v-u$ is $Dg_j(u; v-u) = \langle\nabla h_j(x^u(1)), z^{v,u}(1)\rangle$ whereas the differential of g_3 is given by

$$Dg_3(u; v-u)(t) = \langle\nabla h_3(x^u(t)), z^{v,u}(t)\rangle. \tag{20}$$

These differentials will later be expressed in terms of adjoints. Note that $t \to Dg_3(u, v-u)(t)$ is continuous.

For any u in G let $R(u)$ denote the set of admissible trajectories of the linearized system, that is

$$R(u) \triangleq \{z^{v,u} | v \in G\}. \tag{21}$$

Proposition 3

(a) For all u in G the set $R(u)$ is convex.

(b) For all u,v,w in G

$$\|z^{v,w} - z^{u,w}\|_\infty \to 0,$$

uniformly in w as $v \to u$,

Proof

(a) This follows from the linearity (in z and v) of the linearized system (14) and the convexity of G.

(b) The proof of this result is essentially the same as the proof of Proposition 2.3 in [7]. ∇

We shall use the latter result, which ensures, in a certain sense, the continuity of R(u), to establish the convergence of an algorithm to solve P_c.

4. AN ALGORITHM FOR SOLVING P_c

The master algorithm has two major components, a subalgorithm for solving P_c (for fixed c) and a rule for updating the penalty parameter c. We consider first a convergent algorithm for problem P_c defined in (11).

The procedure for solving P_c is, in principle, very simple. Given a control u in G it determines a search direction $s_c(u) \triangleq v-u$ where v is an control in G which satisfies

$$\hat{\gamma}_c(v,u) - \gamma_c(u) \leq \phi_c(u)/2 \tag{22}$$

where

$$\phi_c(u) \underline{\Delta} \inf\{(1/2c)\|v-u\|_2^2 + \hat{\gamma}_c(v,u) | v \in \bar{G}\} - \gamma_c(u) \leq 0. \tag{23}$$

Let $w^0(u)$ denote the minimizer in (23), i.e.

$$w^0(u) \underline{\Delta} \arg\min \{(1/2c)\|v-u\|_2^2 + \hat{\gamma}_c(v,u) | v \in \bar{G}\}. \tag{24}$$

In (23), (24) $\bar{G}$ denotes the set of relaxed controls and $\hat{\gamma}_c(v,u)$, where v is a relaxed control, is interpreted in the usual way by replacing any function $\phi(x(t),v(t),t)$ (for example, $f(x(t),v(t),t)$, $f_x(x(t),v(t),t)$ and $f_u(x(t),v(t),t)$) occurring in the evaluation of (23)) by $\int_\Omega \phi(x(t),\omega,t)dv(t)(\omega)$; similarly $\|v-u\|_2^2 = \int_0^1 \|v(t) - u(t)\|^2 dt$ is replaced by $\int_0^1 \int_\Omega \|\omega-u(t)\|^2 v(t)(\omega)d\omega.dt$. It is easily established that the minimum in (24) is achieved so that $w^0(u)$ exists. Since $w^0(u)$ can be approximated (by means of the proximity algorithm desribed later) arbitrarily closely it is clear that a control problem v in G satisfying (22) can be computed in a finite number of iterations. The step length $\lambda_c(u)$ is then determined by a step size rule of the Armijo type, i.e. $\lambda_c(u)$ is the largest λ in $S \underline{\Delta} \{1,\beta,\beta^2,...\}$, $\beta \in (0, 1)$, satisfying

$$\gamma_c(u + \lambda s_c(u)) - \gamma_c(u) \leq \lambda\phi_c(u)/4 \tag{25}$$

Obviously the constants (1/2) and (1/4) in (22) and (24) may be replaced by any other constants $c_1 > c_2$ in (0, 1). Let A_c: G → G denote the map

$$A_c(u) = u + \lambda_c(u)s_c(u) \tag{26}$$

To analyse the algorithm it is convenient to introduce a few terms. Let the functions y_0, y_1, y_2, y_3 be defined by:

$$y_0(v;u) \triangleq [(1/2)\|v-u\|_2^2 + g_0(u) + Dg_0(u, v-u)] \tag{27}$$

$$y_j(v;u) \triangleq g_j(u) + Dg_j(u, v-u), \; j = 1\text{-}3. \tag{28}$$

Clearly, y_0 and y_j, $j = 1,2$ are mappings from G x G into, respectively, R, R^{m_E} and R^{m_I} while y_3 is a mapping from G x G into $C^{m_S} * L_2^{m_S}$ (with the norm $\|\cdot\|_2$). Let Y(u) denote the "reachable set" achieved by these functions, i.e.

$$Y(u) \triangleq y(G;u) \triangleq \{y(v;u) | v \in G\} \tag{29}$$

where

$$y(v;u) \triangleq (y_0(v;u), y_1(v;u), y_2(v;u), y_3(v;u)). \tag{30}$$

Let $\overline{Y}(u)$ denote

$$\overline{Y}(u) \triangleq y(\overline{G};u) \triangleq \{y(v;u) | v \in \overline{G}\} \tag{31}$$

where y(v;u), when v is a relaxed control, is interpreted as described earlier It is known that $\overline{Y}(u)$ is the closure of Y(u).

Proposition 4

For all u in G the set $\overline{Y}(u)$ is a closed, bounded and convex subset of $F \triangleq R \times R^{m_E} \times R^{m_I} \times C^{m_S}$ and, hence, of $H \triangleq R \times R^{m_E} \times R^{m_I} \times L_2^{R^{m_S}}$.

Proof

See Warga [8]. Convexity follows from the fact that the variational equation (14) is already linear in z and v and the fact that

$\phi(v) \triangleq \int_0^1 \int_\Omega \|\omega-u(t)\|^2 v(t)(\omega)d\omega$ (v is a relaxed control) has the property that $\phi(v) = \alpha\phi(v_1) + (1-\alpha)\phi(v_2)$ if $v = \alpha v_1 + (1-\alpha)v_2$. ∇

We next note from (23) and (24) that

$$\theta_c(u) \leq \phi_c(u) \tag{32}$$

where $\theta_c: G \to R$ is defined by

$$\theta_c(u) \triangleq \hat{\gamma}_c(w^0(u),u) - \gamma_c(u) \tag{33}$$

We can express ϕ_c in terms of $\overline{Y}$ as follows:

$$\phi_c(u) = \min\{\eta_c(y) | y \in \overline{Y}(u)\} - \gamma_c(u) \tag{34}$$

where $y = (y_0, y_1, y_2, y_3) \in D$ and $\eta_c: F \to R$ is defined by

$$\eta_c(y) \triangleq y_0/c + \max\{0; |y_1^j|,\ j \in \underline{m}_E;\ y_2^j,\ j \in \underline{m}_I;\ y_3^j(t),\ j \in \underline{m}_S,\ t \in T\}.$$

.....(35)

Proposition 5

For all $c \geq 0$, $\phi_c: G \to R$ is continuous.

Proof

Let $\psi: \overline{G} \times G \to R$ be defined by

$$\psi(v,u) \triangleq \eta_c(y(v,u))$$

so that

$$\phi_c(u) = \min\{\psi(v,u) | v \in \overline{G}\} - \gamma_c(u).$$

By slight extension to Proposition 3 (see [7]), $z^{w,v} \to z^{w,u}$ in L_∞, uniformly in w in $\overline{G}$, as $v \to u$ in G. It follows from the definition of y and the continuity of ∇h_j, j = 0-3, that $y^{w,v} \to y^{w,u}$ and $\psi(w,v) \to \psi(w,u)$, uniformly in w in $\overline{G}$, as $v \to u$ in G. Since

$$\phi_c(v) - \phi_c(u) \leq \psi(w^0(u), v) - \psi(w^0(u), u)$$

and

$$\phi_c(v) - \phi_c(u) \geq \psi(w^0(v), v) - \psi(w^0(v), u)$$

it follows that $\phi_c(v) \to \phi_c(u)$ as $v \to u$ in G. ∇

We can now commence establishing the convergence properties of the algorithm map A_c.

Proposition 6

(a) For any u in G, $\theta_c(u) < 0$ if and only if $\phi_c(u) < 0$.

(b) Suppose that u* in G is optimal for P_c. Then $\theta_c(u^*) = \phi_c(u^*) = 0$.

Proof

(a) Since $\theta_c(u) \leq \phi_c(u)$ it is clear that $\theta_c(u) < 0$ of $\phi_c(u) < 0$. Suppose than that $\theta_c(u) < 0$. From (33) there exists a v in G such that

$$\hat{\gamma}(v,u) - \gamma_c(u) \leq \theta_c(u)/2 < 0.$$

For all α in [0, 1] let v_α in G be defined by

$$v_\alpha = u + \alpha(v - u).$$

Then, by (23), for all α in [0, 1],

$$\phi_c(u) \leq (1/2c)\|v_\alpha - u\|_2^2 + \hat{\gamma}_c(v_\alpha, u) - \gamma_c(u)$$

$$\leq (\alpha^2/2c)\|v - u\|_2^2 + \alpha\theta_c(u)$$

so that $\phi_c(u) < 0$ for α sufficiently small.

(b) If $\theta_c(u^*) \leq \phi_c(u^*) < 0$ there exists a v in G such that $\hat{\gamma}_c(v, u^*) - \gamma_c(u^*) \leq \phi_c(u^*)/2$. The desired result can be shown to follow from Proposition 2(c). ∇

Proposition 7

For all u in G such that $\phi_c(u) < 0$ (equivalently $\theta_c(u) < 0$) there exists a $\delta > 0$ and a $\lambda_1 > 0$, λ_1 in S, such that

$$\lambda_c(v) \geq \lambda_1 \qquad (36)$$

for all $v \in B_\delta(u) \triangleq \{w \in G \,|\, \|w-v\|_\infty \leq \delta\}$.

Proof

For all λ in [0, 1]:

$$\gamma_c(v+\lambda s_c(v)) - \gamma_c(v) \leq \hat{\gamma}_c(v+\lambda s_c(v),v) - \gamma_c(v)$$

$$+ \left|\gamma_c(v+\lambda s_c(v)) - \hat{\gamma}_c(v+\lambda s_c(v),v)\right|$$

$$\leq \lambda\phi_c(v)/2 + e(v, \lambda)$$

where $e(v, \lambda) \triangleq \left|\gamma_c(v + \lambda s_c(v)) - \hat{\gamma}_c(v + \lambda s_c(v), v)\right|$.

Choose $\delta > 0$ so that $\phi_c(v) \leq \phi_c(u)/2 < 0$ (and $|\phi_c(u)| \leq 2|\phi_c(v)|$) for all $v \in B_\delta(u)$. Let r denote $\sup\{\|v - u\|_\infty | u, v \in G\}$ so that r is an upper bound for $s_c(u)$. Since $\|\lambda s_c(u)\|_\infty \leq \lambda r$ it follows from Proposition 2 and the continuity of γ_c and $\hat{\gamma}_c$ that there exists a λ_1 in S such that $e(v, \lambda) \leq \lambda[(1/8)|\phi_c(u)|] \leq \lambda|\phi_c(v)/4|$ for all $\lambda \in [0, \lambda_1]$ and all v. Hence

$$\gamma_c(v + \lambda s_c(v)) - \gamma_c(v) \leq \lambda_1\phi_c(v)/4$$

for all v in $B_\delta(u)$. ∇

We can now establish our main result.

Theorem 1

Let $\{u_i\}$ be an infinite sequence in G satisfying $u_{i+1} = A_c(u_i)$. Then any L_∞ accumulation point u^* satisfies $\phi_c(u^*) = 0$ ($\theta_c(u^*) = 0$). ∇

Proof

Suppose $u_i \to u^*$ in G along a subsequence I and that, contrary to what is to be proven, $\phi_c(u^*) < 0$. It follows from Proposition 7 that there exist an i_1 in I and a $\lambda_1 > 0$ such that

$$\gamma_c(u_{i+1}) - \gamma_c(u_i) \leq \lambda_1 \phi_c(u_i)/4$$

$$\leq \lambda_1 \phi_c(u^*)/8 < 0$$

for all i in I, $i \geq i_1$. But this contradicts the convergence of $\gamma_c(u_i)$ to $\gamma_c(u^*)$ as $i \to \infty$, $i \in I$. Hence $\theta_c(u^*) = \phi_c(u^*) = 0$. ∇

5. THE TEST FUNCTION t_c

The second ingredient of the algorithm is a test function t_c. At iteration i, the penalty parameter c_i is left unchanged if $t_{c_i}(u_i) \leq 0$; otherwise it is increased. In order to ensure that accumulation points satisfy necessary conditions of optimality for P (rather than P_c) the test function must have certain properties. These are [10]:

(i) for all $c > 0$, $t_c: G \to R$ is continuous.

(ii) for all $c > 0$, $\phi_c(u^*) = 0$ and $t_c(u^*) \leq 0$ implies that u^* satisfies necessary conditions of optimality for P.

(iii) for all u^* in G there exists a $c^* > 0$ and a $\delta > 0$ such that $t_c(u) \leq 0$ for all u in $B_\delta(u^*)$ and all $c \geq c^*$.

The test function that we propose is modelled on that employed in [8]; it is:

$$t_c(u) = \phi_c(u) + \gamma(u)/c. \tag{37}$$

Before proceeding it is necessary to introduce a constraint qualification. In conventional mathematical programming this usually takes the form of (positive) linear independence of the (most) active constraint gradients. Because the control constraint is not differentiable, a more sophisticated test is necessary. Let A denote the set of all possible vectors in R^{m_E} whose elements are ± 1. Our constraint qualification can now be specified.

A4 Constraint Qualification

(a) For all u such that $\gamma(u) > 0$

$$\chi(u) \triangleq \inf\{\hat{\gamma}(v,u) - \gamma(u) | v \in G\} < 0.$$

(b) For all u such that $\gamma(u) = 0$ there exists an $\varepsilon > 0$ and, for each α in A, a v_α in G such that

$$\alpha^j Dg_1^j(u;\ v_\alpha-u) \leq -\varepsilon,\ j = 1,\dots,m_E,$$
$$g_2^j(u) + Dg_2^j(u;\ v_\alpha-u) \leq -\varepsilon,\ j = 1,\dots,m_I, \tag{38}$$
$$g_3^j(u)(t) + Dg_3^j(u;\ v_\alpha-u)(t) \leq -\varepsilon,\ j = 1,\dots,m_S,\ t \in T$$

The first condition ensures that γ can be reduced if $\gamma(u) > 0$. Since $\hat{\gamma}(v, u) = 0$ if $\gamma(u) = 0$, an alternative qualification is needed for this case (if there are no equality constraints, condition (b) reduces to condition (a)). If there are no inequality or state constraints then condition (b) is equivalent to the linear independence of the equality constraints.

Necessary conditions of optimality for P are well known (see [7,9]).

Proposition 8

Let u* be optimal for P. Then $\gamma(u^*) = 0$ and there exists a $\lambda_0 \geq 0$, multipliers λ_1^j, $j = 1 - m_E$, multipliers $\lambda_2^j \geq 0$, $j = 1 - m_I$ and a positive Radon measure $\lambda_3 = (\lambda_3^1,\dots,\lambda_3^{m_S})$ on T with values in R^{m_S} such that

(a) $\lambda = (\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3) \neq 0$

(b) $\lambda_2^j = 0$ if $g_2^j(u^*) < 0$, $j = 1 - m_I$

(c) $\lambda_3^j(\{t \in T | g_3^j(u^*)(t) < 0\}) = 0$, $j = 1 - m_S$

(d) $$\lambda_0 Dg_0(u,\ v-u) + \langle\lambda_1,\ Dg_1(u,\ v-u)\rangle + \langle\lambda_2,\ Dg_2(u,\ v-u)\rangle + \int_0^1 \langle Dg_3(u,\ v-u)(t),\ \lambda_3(dt)\rangle \geq 0$$

for all v in G. ∇

Corollary

If u* in G is optimal for P and the constraint qualification holds, then $\bar{\lambda}_0 > 0$ and may be normalized to unity.

Proof

Suppose u* ε G but $\lambda_0 = 0$. Then there exist $\lambda_1 = (\lambda_1, \ldots, \lambda_1^{m_E})$, $\lambda_2 = (\lambda_2^1, \ldots, \lambda_2^{m_I}) \geq 0$ and a positive Radon measure $\lambda_3 = (\lambda_3^1, \ldots, \lambda_3^{m_S})$ satisfying (a), (b) and (c) in Proposition 8 and:

$$\langle\lambda_1, Dg_1(u^*, v-u^*)\rangle + \langle\lambda_2, Dg_2(u^*, v-u^*)\rangle$$

$$+ \int_0^1 \langle Dg_3(u^*, v-u^*)(t), \lambda_3(dt)\rangle \geq 0 \tag{39}$$

for all v in G. But, by the constraint qualification there exists a v in G and a $\varepsilon > 0$ such that

$$(\text{sgn } \lambda_1^j) Dg_1^j(u^*, v-u^*) \leq -\varepsilon, \; j \in \underline{m}_E$$

$$Dg_2^j(u^*, v-u^*) \leq -\varepsilon, \; j \in \{j \in \underline{m}_I \,|\, g_2^j(u^*) = 0\}$$

$$Dg_3^j(u^*, v-u^*)(t) \leq -\varepsilon, \; t \in \{t \,|\, g_3^j(u^*)(t) = 0\}, \; j \in \underline{m}_S.$$

which contradicts (39). Hence $\lambda_0 > 0$. ∇

We now examine the properties of our proposed test function t_c. We assume in the sequel that assumptions A1-A4 hold.

Proposition 9

For all $c > 0$ the function t_c: $G \to R$ is continuous.

Proof

This follows from the continuity of ϕ_c and γ.

Proposition 10

If u^* in G satisfies $\phi_c(u^*) = 0$ ($\theta_c(u^*) = 0$) and $t_c(u^*) \leq 0$ then u^* is desirable for P(i.e. u^* satisfies the conditions in Proposition 8).

Proof

Suppose u^* in G and $c > 0$ satisfy $\phi_c(u^*) = 0$ and $t_c(u^*) \leq 0$. It follows from the definition of t_c that $\gamma(u^*) = 0$. Suppose, contrary to what is to be proven, that u^* is not desirable for P. Since $\gamma(u^*) = 0$ it follows (from the usual arguments in deriving necessary conditions of optimality) that there exists a v^* in G such that

(a) $Dg_0(u^*;\ v^*-u^*) < 0$

(b) $Dg_1(u^*;\ v^*-u^*) = 0$

(c) $g_2(u^*) + Dg_2(u^*;\ v^*-u^*) \leq 0$

(d) $g_3(u^*)(t) + Dg_3(u^*;\ v^*-u^*)(t) \leq 0, \quad t \in T.$

It follows that $\hat{\gamma}_c(v^*, u^*) < \gamma_c(u^*)$ (see (12), (13)) so that $\theta_c(u^*) < 0$ ($\phi_c(u^*) < 0$), contradicting our hypothesis. Hence u* is desirable. ∇

The final property required of t_c is the most difficult to establish. The proof is given in the Appendix.

Proposition 11

For all u* in G there exists a $c^* > 0$ and a $\delta > 0$ such that $t_c(u^*) \leq 0$ for all u in $B_\delta(u^*)$, $c \geq c^*$.

6. AN ALGORITHM FOR SOLVING P

The algorithm for solving the original problem P can now be stated. A proximity algorithm is employed to solve P_c approximately (full details are given later). The procedure for choosing the penalty parameter c employs a monotonically increasing sequence $\{c_j\}$ ($c_j \nearrow \infty$ as $j \to \infty$).

Algorithm 1

Parameters: $S = (1, \beta, \beta^2, \ldots)$, $\beta \in (0, 1)$, $C = \{c_i\}_0^\infty$ where $c_i \nearrow \infty$, as $i \to \infty$.

Data: $u_1 \in G$.

Step 0: Set $i = 1$.

Step 1: If $t_{c_{i-1}}(u_i) \leq 0$ set $c_i = c_{i-1}$;

if $t_{c_{i-1}}(u_i) > 0$ set c_i equal to the smallest c in C

such that $t_{c_i}(u_i) \leq 0$.

Step 2: Compute a control v_i in G satisfying
$\hat{\gamma}_{c_i}(v_i, u_i) - \gamma_{c_i}(u_i) \leq \phi_c(u_i)/2$.

Step 3: Compute λ_i, the smallest λ in S such that

$\gamma_{c_i}(u_i + \lambda(v_i - u_i)) - \gamma_{c_i}(u_i) \leq \lambda\phi_c(u_i)/4$.

Step 4: Set $u_{i+1} = u_i + \lambda_i(v_i - u_i)$.
Set $i = i + 1$

Go to Step 1. ∇

Theorem 2

Let $\{u_i\}$ be an infinite sequence of controls generated by Algorithm 2.

(i) If c_{i-1} is increased finitely often and then remains constant, then any L_∞ accumulation point $\hat{u}$ of $\{u_i\}$ is desirable for P.

(ii) If c_{i-1} is increased infinitely often (i.e. $J \triangleq \{i \mid t_{c_{i-1}}(u_i) > 0\}$ has infinite cardinality) then the subsequence $\{u_i \mid i \in J\}$ has no L_∞ accumulation points.

Proof

The result follows directly from Theorem 1, Proposition 9, 10 and 11 and Theorem 4 in [10]. ∇

The only remaining task is the specification of an algorithm for determining, given any u in G and any c > 0, a control v satisfying

$$\hat{\gamma}_c(v,u) - \gamma_c(u) \leq \phi_c(u)/2. \tag{40}$$

This is required for Step 2 of Algorithm 1. We obtain v by using a proximity algorithm described below. The algorithm generates, at each iteration, a $y_i \in Y(u)$ and $z_i \notin Y(u)$ such that;

$$\gamma_c(u) + \phi_c(u) \in [\eta_c(z_i), \eta_c(y_i)] \tag{41}$$

The algorithm stops when

$$[\eta_c(y_i)-\gamma_c(u)] \leq [\eta_c(z_i)-\gamma_c(u)]/2 \leq \phi_c(u)/2. \tag{42}$$

Since $y_i \in Y(u)$, the control v generating y_i ($y_i = y(u, v)$) satisfies (40) and hence may be used in the main algorithm.

To describe the algorithm it is useful to introduce, for all $\alpha > 0$, the set $D(\alpha)$ in H defined by:

$$D(\alpha) \triangleq \{y \in H | \eta_c(y) \leq \alpha;\ y_0/c \geq \alpha_0;\ y_2^j \geq a_2,$$

$$j \in \underline{m}_I;\ y_3^j(t) \geq a_3,\ j \in \underline{m}_S,\ t \in T\} \tag{43}$$

where α_0, a_2 and a_3 (which depend on u) are defined in the Appendix. Recalling the definition of η_c in (35) it is easily seen that this set is convex and bounded.

The algorithm employed is a fairly standard proximity algorithm, except for two modifications. At iteration i the sub-algorithm determines α_{i+1}, y_i and $\bar{y}_i$, both generated by ordinary controls (i.e.lying in Y(u)) and z_i and $\bar{z}_i$ lying in $D(\alpha_{i+1})$. Then $\tilde{y}_i \in [y_i, \bar{y}_i] \triangleq \{y | y = y_i + a(\bar{y}_i - y_i), a \in T\}$ and $\tilde{z}_i \in [z_i, \bar{z}_i]$ are chosen to minimise $\|y-z\|_2$ subject to the constraint $y \in [y_i, \bar{y}_i]$, $z \in [z_i, \bar{z}_i]$. A conventional proximity algorithm would set $y_{i+1} = \tilde{y}_i$, $z_{i+1} = \tilde{z}_i$. However, although Y(u) is convex, it is difficult, owing to the presence of the quadratic term in the first component of y(v,u) (see (27)-(30)) to determine a v in G such that $y(v,u) = \tilde{y}_i$. The sub-algorithm proceeds as follows. Suppose $\tilde{y}_i = y_i + a_i(\bar{y}_i - y_i)$ where $y_i = y(v_i, u)$ and $\bar{y}_i = y(\bar{v}_i, u)$, v_i, $\bar{v}_i$ lie in G, and $a_i \in T$. Set $v_{i+1} = v_i + a_i(\bar{v}_i - v_i)$. Because of the convexity of G, v_{i+1} lies in G. Set $y_{i+1} = y(v_{i+1}, u)$. Because of the quadratic term in the first component of y_{i+1}, $(y_{i+1})_0 < (\tilde{y}_i)_0$, $(y_i = ((y_i)_0, (y_i)_1, (y_i)_2, (y_i)_3) \in F = R \times R^{E^m} \times R^{I^m} \times C^{m_s})$ etc); however $(y_{i+1})_j = (\tilde{y}_i)_j$, j = 1-3. because these components of y(v;u) are linear in (v-u). Hence set $z_{i+1} = ((z_{i+1})_0, (z_{i+1})_1, (z_{i+1})_2, (z_{i+1})_3)$

where:

$$(z_{i+1})_0 = (\tilde{z}_i)_0 + (y_{i+1})_0 - (\tilde{y}_i)_0 \qquad (44)$$

$$(z_{i+1})_j = (\tilde{z})_j \ , \ j = 1,2 \qquad (45)$$

and

$$(z_{i+1})_3 = \arg\min\{\|(y_{i+1})_3 - w\|_2 \,|\, ((z_{i+1})_0, (z_{i+1})_1, (z_{i+1})_2, w) \in D(\alpha_{i+1})\} \quad (46)$$

Since $(y_{i+1})_0 = (\tilde{y}_i)_0 + (y_{i+1})_0 - (\tilde{y}_i)_0$ and

$(y_{i+1})_j = (\tilde{y}_i)_j$, $j = 1\text{-}3$ it follows that

$$\|z_{i+1} - y_{i+1}\|_2 \leq \|\tilde{z}_i - \tilde{y}_i\|_2$$

which is the essential property required by the algorithm. The computation of the function $(z_{i+1})_3$ is simple (see Appendix).

Algorithm 2 (to determine $\phi_c(u)$ approximately)

Parameters:

Data: $u \in G$, $c > 0$, $v_0 \in G$ (e.g. $v_0 = u$)).

Step 0 : (Initialization)

Set $i = 0$.

Compute $\alpha_0 = \arg\min\{y_0/c \,|\, y = (y_0, y_1, y_2, y_3) \in Y(u)\}$.

Set $y_0 = y(v_0, u)$.

Set $z_0 = (\alpha_0, 0, 0, 0)$.

Step 1: Compute $s_i = y_i - z_i$.

Compute $\eta_i = \min\{\langle s_i, y\rangle_2 \,|\, y \in Y(u)\}$.

Set $\bar{y}_i = \arg\min\{\langle s_i, y\rangle_2 \,|, y \in Y(u)\}$

and set $\bar{v}_i$ equal to corresponding control (i.e. $\bar{y}_i = y(v_i;u)$).

Compute $\eta_i' = \max\{\langle s_i, y_i\rangle_2 \,|\, y \in D(\alpha_i)\}$

Set $\bar{z}_i = \arg\max\{\langle s_i, y\rangle_2 \,|\, y \in D(\alpha_i)\}$

If $\eta_i' \geq \eta_i$, set $\alpha_{i+1} = \alpha_i$

If $\eta_i' < \eta_i$ choose α_{i+1} to satisfy

$\max\{\langle s,y\rangle \,|\, y \in D(\alpha_{i+1})\} = \eta_i$ and set $\bar{z}_i = \arg\max\{\langle s,y\rangle \mid y \in D(\alpha_{i+1})\}$.

Step 2: Compute $(\tilde{y}_i, \tilde{z}_i) = \arg\min [\|y-z\|_2 \mid y \in [y_i, \bar{y}_i], z \in [z_i, \bar{z}_i]]$

and $a_i \in T$ satisfying

$$\tilde{y}_i = y_i + a_i[y_i - \bar{y}_i]$$

Set $v_{i+1} = v_i + a_i[v_i - \bar{v}_i]$

Set $y_{i+1} = y(v_{i+1}, u_i)$

Compute z_{i+1} (see (44) - (46)).

Step 3: If $[\eta_c(y_{i+1}) - \gamma_c(u)] \leq [\eta_c(z_{i+1}) - \gamma_c(u)]/2$ stop.

Else set $i = i+1$ and go to Step 1. ∇

Proposition 12

For all u such that $\phi_c(u) < 0$, algorithm 2 terminates in a finite number of iterations yielding a v in G satisfying:

$$\hat{\gamma}_c(v,u) - \gamma_c(u) \leq \phi_c(u)/2.$$

Proof

Let α^0 solve $\inf\{\alpha \mid D(\alpha)\ \bar{Y}(u) \neq \emptyset\}$. Consider the cost function $c(y,z)$; $\bar{Y}(u) \times D(\alpha^0) \to R^+$ defined by;

$$c(y, z) = (1/2)\|y - z\|_2^2 .$$

At any point (y, z) in $\bar{Y}(u) \times D(\alpha^0)$ the gradient of $c(y, z)$ is $(y-z, z-y)$. The algorithm first computes a $(\bar{y}, \bar{z})$ satisfying

$$\langle y-z,\ \bar{y} - \bar{z}\rangle_2 \leq 0$$

Hence

$$DC((y,z);(\bar{y},\bar{z}) - (y,z)) = \langle y-z,\ \bar{y} - y\rangle_2 + \langle z-y,\ \bar{z}-z\rangle_2$$

$$= \langle y-z,\ (\bar{y} - y) - (\bar{z} - z)\rangle_2$$

$$= - \langle y-z,\ y-z\rangle_2 + \langle y-z,\ \bar{y} - z\rangle_2$$

$$\leq -2\ c(y,z).$$

The successor point (y',z') minimizes $c(y'',z'')$ subject to the constraints $y'' \in [y,\bar{y}]$, $z'' \in [z,\ \bar{z}]$ and, hence, satisfies:

$$c(y',z') - c(y,z) \leq \min_{\lambda \in T}\{\ - 2\lambda c(y,z) + (1/2)d\lambda^2\}$$

where the upper bound $(1/2)d\lambda^2$ on the second order term in the expansion of $c(y',z' - c(y,z)$ arises from the boundedness of $\bar{Y}(u)$ and $D(\alpha^0)$. It easily follows that

$$c(y',z') - c(y,z) \leq -\phi(y,z)$$

where

$$\phi(y,z)\ \underline{\Delta}\ c(y,z)^2/d \text{ if } c(y,z) \leq d$$

$$\underline{\Delta}\ \ d \text{ if } c(y,z) > d.$$

Clearly $\phi(y,z) = 0$ if and only if $c(y,z) = 0$.

Consider now a sequence $\{\omega_i\}$, $\omega_i \triangleq (y_i,z_i)$, generated by the algorithm. From the above $\{c(\omega_i)\}$ is non-increasing and bounded below by zero, so that the sequences $\{c(\omega_i)\}$ and $\{\phi(\omega_i)\}$ converge. If $c(\omega_i) \not\to 0$ so that $\phi(\omega_i) \not\to 0$ as $i \to \infty$ there exist a $\delta > 0$ and an integer i, such that $\phi(\omega_i) \geq \delta$ for all i. But this implies that $c(\omega_i) \to -\infty$, a contradiction since c is bounded below by 0. Hence $c(\omega_i) \to 0$ as $i \to \infty$, i.e. $\|y_i - z_i\|_2 \to 0$ as $i \to \infty$.

The sequences $\{(y_3)_i\}$ and $\{(z_3)_i\}$ are uniformly Lipschitz continuous and hence equicontinuous so (by Ascoli's Theorem) there exist a y^* and z^* and a subsequence I such that

$y_i \overset{I}{\to} y^*$ and $z_i \overset{I}{\to} z^*$ in F and in H. Since $\|y_i - z_i\|_2 \to 0$ as $i \to 0$ it follows that $y^* = z^*$ and that $|\eta_c(y_i) - \eta_c(z_i)| \to 0$ as $i \to \infty$.

The desired result follows easily. ∇

We have now to show how the various steps on Algorithm 2 can be performed. In Step 1 we require the solution of $\min\{\langle s,y\rangle_2 | y \in Y(u)\}$ making use of the characterization of Y(u) we can see that the problem is equivalent to

$$\min \{\int_0^1 s_0[(1/2)\|v(t) - u(t)\|^2 dt + s_0 g_0(u) + s_0 h_{0x}(x^u(1)) z^{v,u}(1))$$

$$+ \int_0^1 \langle s_3(t), h_{3x}(x^u(t))\, z^{v,u}(t) dt + \langle s_1, h_{1x}(x^u(1))\, z^{v,u}(1)\rangle$$

$$+ \langle s_2, h_{2x}(x^u(1))\, z^{v,u}(1)\rangle | v \in G\}$$

subject to the system equations

$$\dot{z}^{v,u}(t) = A^u(t)z^{v,u}(t) + B^u(t)[v(t) - u(t)]$$

$$z^{v,u}(t) = 0$$

and the control constraint

$$v(t) \in \Omega, \quad t \in T.$$

This is a standard optimal control problem. The minimizer v satisfies

$$v^0(t) = \arg\min[(1/2)\|v(t) - u(t)\|^2 +$$

$$\langle\lambda(t), A^u(t)z^{v,u}(t) + B^u(t)[v(t)-u(t)]\,|\, v(t) \in \Omega]. \tag{47}$$

where $\lambda(t)$ is the solution of

$$-\dot{\lambda}(t) = [A^u(t)]^T\lambda(t) + h_{3x}(x^u(t))^T s_3(t)$$

$$\lambda(1) \quad = h_{0x}(x^u(1))^T s_0 \; h_{1x}(x^u(1))^T s_1 + h_{2x}(x^u(1))^T s_2$$

It is known that there exists a control in G satisfying (47).

Step 1 also requires the solution of $\max\{\langle s,y\rangle_2 | y \in D(\alpha)\}$. The computation of this is discussed in the appendix.

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APPENDIX

1. Computation of $\max\{\langle s,y\rangle_2 \mid y \in D(\alpha)\}$.

Using the characterisation of $D(\alpha)$ this reduces to

$$\max\{s_O y_O + \langle s_1, y_1\rangle + \langle s_2, y_2\rangle + \langle s_3, y_3\rangle_2 \,|$$
$$y_O/c + |y_1^j| \leq \alpha,\ j \in \underline{m}_E;$$
$$y_2^j \geq a_2, y_O/c + y_2^j \leq \alpha,\ j \in \underline{m}_I;$$
$$y_3^j \geq a_3,\ y_O/c + y_3^j \leq \alpha,\ j \in \underline{m}_S;$$
$$y_O/c \in [\alpha_O, \alpha]\}$$

where $a_2 \ll 0$ and $a_3 \ll 0$. Since the term y_O/c is common, we consider the equivalent problem P_β in which the constraints are replaced by $|y_1^j| \leq \beta$, $y_2^j \in [a_2, \beta]$, $y_3^j \in [a_3, \beta]$, where $\beta \,\underline{\Delta}\, \alpha - y_O/c \in [0, \alpha - \alpha_O]$. P_β can be decomposed into the following independent problems:

$$P_\beta^1\colon \max\{\langle s_1, y_1\rangle \,\big|\, |y_1^j| \leq \beta,\ j \in m_E\} = f_1(\beta)$$
$$P_\beta^2\colon \max\{\langle s_2, y_2\rangle \,\big|\, y_2^j \in [a_2, \beta],\ j \in m_I\} = f_2(\beta)$$
$$P_3^3\colon \max\{\langle s_3, y_3\rangle_2 \,\big|\, y_3^j \in [a_3, \beta],\ j \in \underline{m}_S\} = f_3(\beta).$$

Let d_1, d_2, d_3, e_2, e_3 be defined by

$$d_1 \,\underline{\Delta}\, \sum_{j=1}^{m_E} |s_1^j|,$$

$$d_2 \,\underline{\Delta}\, \sum_{j=1}^{m_I} (s_2^j)_+, \quad e_2 \,\underline{\Delta}\, \sum_{j=1}^{m_j} (s_2^j)_-$$

$$d_3 \,\underline{\Delta}\, \sum_{j=1}^{m_S} \int_0^1 (s_3^j(t))_+\, dt,\ e_3 \,\underline{\Delta}\, \sum_{j=1}^{m_S} \int_0^t (s_3^j(t))_-\, dt$$

where $(\alpha)_+ \,\underline{\Delta}\, \max\{0, \alpha\}$, and $(\alpha)_- \,\underline{\Delta}\, \max\{0, -\alpha\}$. Then

$$s_O y_O = s_O c\alpha - s_O c\beta$$
$$f_1(\beta) = d_1\beta$$
$$f_2(\beta) = d_2\beta - e_2 a_2$$
$$f_3(\beta) = d_3\beta - e_3 a_3$$

β is chosen to solve

$$\max\{s_0y_0+f_1(\beta) + f_2(\beta) + f_3(\beta) | \beta \in [0, \alpha - \alpha_0]\}$$

$$= \max\{\rho + \beta[d_1 + d_2 + d_3 - s_0c] | \beta \in [0, \alpha - a_0]\}$$

where $\rho \triangleq s_0c\alpha - e_2a_2 - e_3a_3$. Clearly β^0, the maximising β, satisfies

$$\beta^0 = \alpha - \alpha_0 \text{ if } d_1 + d_2 + d_3 > s_0c$$

$$= 0 \quad \text{ if } d_1 + d_2 + d_3 \leq s_0c$$

and the maximizing y_0, y_1, y_2, y_3 satisfy

$$y_0 = c(\alpha - \beta^0)$$

$$y_1^j = \beta^0 \operatorname{sgn}(s_1^j),\ j \in \underline{m}_E$$

$$y_2^j = \beta^0 \text{ if } s_2^j > 0$$

$$= a_2 \text{ if } s_2^j \leq 0,\quad j \in \underline{m}_I$$

$$y_3^j(t) = \beta^0 \text{ if } s_3^j(t) > 0$$

$$= a_3 \text{ if } s_3^j(t) \leq 0,\ j \in \underline{m}_S,\ t \in T.$$

$$\max\{\langle s,y\rangle | y \in D(\alpha)\} = s_0c(\alpha-\beta^0)+f_1(\beta^0)+f_2(\beta^0)+f_3(\beta^0)$$

2. Computation of $(z_{i+1})_3$

The constraint in (50) is equivalent to:

$$w^j(t) \geq a_3$$

$$w^j(t) \leq m(t)$$

where

$$m(t) \triangleq \alpha - (z_{i+1})_0/c$$

Hence $(z_{i+1})_3$ defined by

$$(z_{i+1})_3^j(t) \triangleq m \quad \text{if } (y_{i+1})_3^j(t) \geq m$$

$$\triangleq (y_{i+1})_3^j(t) \text{ if } (y_{i+1})_3^j(t) \in [a_3, m]$$

$$\triangleq a_3 \text{ if } (y_{i+1})_3^j(t) \leq a_3$$

for all $j \in \underline{m}_S$ and all $t \in T$ minimises both $\|(y_{i+1})_3 - w\|_2$ and $\|(y_{i+1})_3 - w\|_\infty$ subject to the constraint in (50). Since $\{(y_i)_3\}$ is uniformly Lipschitz continuous it follows that $\{(z_i)_3\}$ is also uniformly Lipschitz continuous.

3. Proof of Proposition 11

(a) Suppose $\gamma(u^*) > 0$. From constraint qualification (a), there exists a $\delta > 0$ and a v^* in G such that $\hat{\gamma}(v^*, u^*) - \gamma(u^*) \leq -3\delta$. It is easily established that $u \to \hat{\gamma}(v^*, u)$, $G \to R$, is continuous. Hence there exists an $\varepsilon > 0$ such that $\hat{\gamma}(v^*, u) - \gamma(u) \leq -2\delta$ for all u in $B_\varepsilon(u^*)$. Since $\|v^* - u\|_2^2$ and $Dg^0(u, v^*-u)$ are uniformly bounded in $B_\varepsilon(u^*)$, there exists a $c_1 \geq 0$ such that:

$$\phi_c(u) \leq [(1/2c)\|v^*-u\|_2^2 + [g^0(u)/c + Dg^0(u;\ v^*-u)/c$$
$$+ \hat{\gamma}(v^*, u)] - [g^0(u)/c + \gamma(u)]] < -\delta$$

for all $u \in B_\varepsilon(u^*)$, $c \geq c_1$. Choosing $c^* \geq c_1$ so that $\gamma(u)/c^* \leq \delta/2$ for all u in $B_\varepsilon(u^*)$ yields $t_c(u) \leq -\delta/2$ for all u in $B_\varepsilon(u^*)$, all $c \geq c^*$.

(b) Suppose $\gamma(u^*) = 0$ so that $g_1^j(u^*) = 0$, $j = 1, \ldots, m_E$. From constraint qualification (b) there exists a $\delta > 0$ and, for each α in A, a v_α in G such that

$$\alpha^j Dg_1^j(u^*;\ v_\alpha - u^*) \leq -2\delta, \quad j = 1,\ldots,m_E$$

$$g_2^j(u^*) + Dg_2^j(u^*;\ v_\alpha - u^*) \leq -2\delta, \quad j = 1,\ldots,m_I$$

$$g_3^j(u^*) + Dg_3^j(u^*;\ v_\alpha - u^*) \le -2\delta, \qquad j = 1,\dots,m_S$$

the last inequality being interpreted as holding for all t in T. From the continuity of g_i^j and $u \to Dg_i^j(u;\ v_\alpha - u)$ for all i, j, there exists an $\varepsilon > 0$ such that these inequalities hold with u* replaced by u and 2δ by δ for all u in $B_\varepsilon(u^*)$, all α in A. For each u choose $\alpha(u) = (\alpha^1(u),\dots,\alpha^{m_E}(u))$ in A so that $\alpha^j(u)\ g_1^j(u) = |g_1^j(u)|$, $j \in \underline{m}_E$. Since $|a| = \max\{a, -a\} = \max\{\alpha^j a, -\alpha^j a\}$ we have

$$\max\{|g_1^j(u) + Dg_1^j(u;\ s)|,\ j \in \underline{m}_E\}$$

$$= \max\{|g_1^j(u)| + \alpha^j(u)Dg_1^j(u;\ s), -|g_1^j(u)| - \alpha^j(u)Dg_1^j(u;\ s),\ j \in \underline{m}_E\}$$

Let d be an upper bound for $(1/2)\|v_\alpha - u\|_2^2$, and M an upper bound for $Dg_0(u,\ v_\alpha - u)$, $Dg_1^j(u;\ v_\alpha - u)$, $j \in \underline{m}_E$, as u ranges over G, α over A. Replacing (v−u) by $\lambda(v_{\alpha(u)} - u)$ in definition (23) of ϕ_c yields, for all u in $B_\varepsilon(u^*)$:

$$\begin{aligned}\phi_c(u) \le \min_{\lambda\varepsilon T} \max\ \{&d\lambda^2/c + M\lambda/c + (|g_1^j(u)| - \gamma(u)) - \lambda\delta,\\ &d\lambda^2/c + M\lambda/c - (|g_1^j(u)| + \gamma(u)) + M\lambda,\ j \in \underline{m}_E;\\ &d\lambda^2/c + M\lambda/c - \lambda\delta - \gamma(u);\ d\lambda^2/c + M\lambda/c - \gamma(u)\}\end{aligned} \qquad (A1)$$

In (40), $d\lambda^2/c$ is an upper bound for $(1/2c)\|\lambda(v_{\alpha(u)} - u))\|_2^2$ and $M\lambda/c$ an upper bound for $Dg_0(u;\ \lambda(v_{\alpha(u)} - u))/c$. The first two terms in (A1) arise from the terms involving g_1^j in the definition of ϕ_c, the third term from terms involving g_2^j and g_3^j and the last term from the constraint that $\gamma(v,u) \ge 0$. It follows that

$$\phi_c(u) \le \min_{\lambda\varepsilon T} \max\{d\lambda^2/c - \lambda[\delta - M/c];\ d\lambda^2/c + b\lambda - \gamma(u)\} \qquad (A2)$$

where $b \triangleq 2M$, for all u in $B_\varepsilon(u^*)$, all $c \geq 1$. Hence:

$$t_c(u) \leq \min_{\lambda \varepsilon T} \max\{d\lambda^2/c - \lambda[\delta - M/c] + \gamma(u)/c;$$

$$d\lambda^2/c + b\lambda - \gamma(u)[1 - 1/c]\} \qquad (A3)$$

for all u in $B_\varepsilon(u^*)$, all $c \geq 1$. The first term in (42) is negative if

$$\lambda[\delta - M/c - d\lambda/c] > \gamma(u)/c$$

i.e. if

$$\lambda > \gamma(u)/[c(\delta - M/c - d\lambda/c)]$$

and

$$[\delta - M/c - d\lambda/c] > 0$$

Since $\lambda \varepsilon [0,1]$, the latter two inequalities hold if:

$$\lambda > \lambda_1(u) \triangleq \gamma(u)/[c(\delta - M/c - d/c)] \text{ and } c > (M + d)/\delta \qquad (A4)$$

The second term in (42) is negative if

$$\lambda[b + d\lambda/c] \leq \gamma(u)[1 - 1/c]$$

and $c > 1$. Since $\lambda \varepsilon [0,1]$, these inequalities hold if

$$\lambda < \lambda_2(u) \triangleq \gamma(u)[1 - 1/c]/[b + d/c] \text{ and } c > 1. \qquad (A5)$$

Hence c^* can be chosen to satisfy $\lambda_1(u) < \lambda_2(u)$, $c > (M + d)/\delta$ and $c > 1$ thus ensuring $t_c(u) \leq 0$ for all u in $B_\varepsilon(u^*)$ and all $c \geq c^*$. ∇

Finally we show how (for any u in G) α_0, a_2 and a_3 may be computed. Recall that:

$$y_0(v;u) = (1/2)\|v-u\|_2^2 + g_0(u) + Dg_0(u, v-u)$$

$$y_j(v;u) = g_j(u) + Dg_j(u, v-u),\ j \varepsilon \underline{3}$$

where

$$Dg_j(u, v-u) = \langle \nabla h_j(x^u(1)), z^{v,u}(1)\rangle,\ j = 0,1,2$$

and

$$Dg_3(u,\ v-u) = \langle \nabla h_3(x^u(\cdot)),\ z^{v,u}(\cdot)\rangle.$$

For any given u $\|\nabla h_j(x^u(1))\|$, j = 0,1,2 and $\|\nabla h_3(x^u(\cdot))\|_2$ are easily calculated. Using Gronwall's inequality an upper bound $\pi(u)$ for $\|z^{v,u}(t)\|$ as v ranges over G and t ranges over T is easily computed. Finally a value r such that $u,v \in \Omega$ implies $(1/2)\|v - u\|_2^2 \leq r$ can be computed. Hence

$$y_0(v;\ u) \geq r + g_0(u) - \|\nabla h_0(x^u(1))\|\pi(u)$$

$$y_j(v;\ u) \geq g_j(u) - \|\nabla h_j(x^u(1))\|\pi(u),\ j = 1,2,$$

$$y_3(v;\ u) \geq g_3(u) - \|\nabla h_3(x^u(\cdot))\|_2\ \pi(u),$$

for all v in G. Hence we may employ

$$\alpha_0 = [r + g_0(u) - \|\nabla h_0(x^u(1))\|\ \pi(u)]/c$$

$$a_2 = [g_2(u) - \|\nabla h_2(x^u(1))\|\ \pi(u)]$$

$$a_3 = [g_3(u) - \|\nabla h_3(x^u(\cdot))\|_2\ \pi(u)]$$

in the definition of $D(\alpha)$.