DECOUPLING OF LINEAR MULTIVARIABLE PLANTS BY
DYNAMICAL OUTPUT FEEDBACK: AN ALGEBRAIC THEORY

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Decoupling of Linear Multivariable Plants By Dynamical Output Feedback: An Algebraic Theory

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Abstract

This paper presents an algebraic theory for the design of a decoupling compensator for linear time-invariant multivariable systems. The design method uses a two-input one-output compensator, which gives a convenient parametrization of all diagonal I/O maps and all disturbance-to-output (D/O) maps achievable by a stabilizing compensator for a given plant. It is shown that this method has two degrees of freedom: any achievable diagonal I/O map and any achievable D/O map can be realized simultaneously by a choice of an appropriate compensator. The difference between all achievable diagonal and nondiagonal I/O maps and the "cost" of decoupling is discussed for some particular algebraic settings.

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I. Introduction

In the design theory of linear time-invariant multi-input multi-output systems, the characterization of all designs which can be achieved by a stabilizing controller for a given plant is a subject of great interest because it shows the limitations on achievable performance imposed by the plant model and the constraints of linearity and stability. The first results were obtained by Youla, et al. [You. 1] for the lumped continuous and discrete-time cases. Later, an algebraic formulation was given by Desoer, et al. [Des. 1] to include the lumped and distributed, the continuous-time and discrete-time cases. Using algebraic tools, Zames [Zam. 1] considered stable plants, characterized all stabilizing compensators and established bounds or closed-loop system performance. His methods were used for design in [Des. 2]. Further results in parametrized form were given in [Per. 1], [Che. 1], [Sae. 1] and [Vid. 1], until finally a general algebraic design procedure, which enables design with non-square plants and controllers and extends the parametrizations of [You. 1] and [Per. 1], was obtained in [Des. 3].

This paper presents a general algebraic design method for all <u>diagonal</u> input-output (I/O) maps which can be achieved by a stabilizing two-input one-output controller for a given plant. Such controllers were used for example in [Åst. 1], [Per. 1] and [Des. 3]. It is of great engineering interest to have an input-output map which is decoupled and to be able to design the disturbance-to-output (D/O) map independently of the I/O map. The system $\Sigma(P,K)$ shown in Fig. 1 represents a very general case in that y_2, the output-of-interest, is not necessarily the same as z, the measured output, i.e., the input to the compensator; furthermore the disturbance d is applied directly to the pseudo-state of the plant rather than being an additive input as, for example, in [Des. 3].

Decoupling of linear time-invariant multivariable systems over unique factorization domains is considered in [Dat. 1]; necessary and sufficient conditions are established for the existence of a decoupling dynamic or static state feedback in the case that the system is internally stable and reachable. Furthermore, the stability preserving stable compensator is required to be invertible over the unique factorization domain. In the present paper, the plant is not assumed to be stable, dynamic output-feedback is used, the compensator is not required to be stable, and if stable, it is not required to be invertible over the principal ring. Our plant has an output-of-interest y_2 and a measured output z.

The paper is organized as follows:

Section II defines the problem and states the stabilizibility conditions. Section III builds the necessary structures for decoupling the I/O map. Section IV presents the main results: the achievable diagonal I/O maps and the achievable D/O maps. Section V considers some examples and contains the conclusions.

The following is a list of the commonly used symbols:

a := means a denotes b. θ_n is the n-vector of zeros. W.l.o.g. means without loss of generality. If $\mathcal{G}$ is a ring, then $\mathcal{E}(\mathcal{G})$ denotes the set of matrices having all entries in $\mathcal{G}$. $\mathcal{R}_{\mathcal{U}}$ denotes the proper rational functions analytic in the region $\mathcal{U} \subset \mathbb{C}$, a symmetric subset of $\mathbb{C}$ which contains $\mathbb{C}_+$ and $\bar{\mathcal{U}} = \mathbb{C}_+ \cup \{\infty\}$. $\mathbb{R}(s)$ denotes the scalar rational functions in s with real coefficients, and $\mathbb{R}[s]$ denotes the scalar polynomials in s with real coefficients.

Throughout the paper, the properties of groups and of commutative rings are used; these and other standard algebraic terms can be found for example in [Bou. 1], [Coh. 1], [Jac. 1], [Lang 1], [Mac. 1], [Sig. 1]. The algebraic structure used here is similar to that of [Des. 3].

Algebraic Structure: [Bou. 1, p. 55], [Coh. 1, p. 395], [Jac. 1, p. 393], [Lang 1, p. 69].

$\mathcal{H}$: A principal ring (principal ideal domain), i.e., an entire commutative ring in which every ideal is principal (e.g., $\mathcal{R}_{\mathcal{U}}$).

$\tilde{\mathcal{G}}$: The field of fractions over $\mathcal{H}$ (e.g. $\mathbb{R}(s)$).

I : A multiplicative subset of $\mathcal{H}$, equivalently, $I \subset \mathcal{H}$, $0 \notin I$, $1 \in I$ and $x,y \in I$ implies that $xy \in I$ (e.g., $f \in I$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(\infty) = 1$).

$\mathcal{G} := \{n/d : n \in \mathcal{H}, d \in I\}$, a subring of $\tilde{\mathcal{G}}$ (e.g. $\mathbb{R}_p(s)$, the ring of proper scalar rational functions).

$\mathcal{U}(\mathcal{H}) := \{m \in \mathcal{H} : m^{-1} \in \mathcal{H}\}$, the group of units in $\mathcal{H}$ (e.g., $f \in \mathcal{U}(\mathcal{H})$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(s) \neq 0$ for all $s \in \bar{\mathcal{U}}$).

II. Problem Description

We consider the multi-input multi-output linear, time-invariant system $\Sigma(P,K)$ (${}^1\Sigma(P,K)$) shown in Fig. 1 (Fig. 2). Given a plant P, we wish to design a controller K with two inputs and one output such that the resulting feedback system is stable, K is proper, and the I/O map $v \mapsto y_2$ is nonsingular and decoupled, i.e., diagonal. We make the following assumptions on $\Sigma(P,K)$:

Assumptions on the System $\Sigma(P,K)$:

(P) $P \in \mathcal{G}^{2n \times n}$ has a right-coprime factorization (r.c.f.)

$$\begin{bmatrix} N^o_{pr} \\ \hdashline N^m_{pr} \end{bmatrix} D^{-1}_{pr} = \begin{bmatrix} P^o \\ \hdashline P^m \end{bmatrix}$$ with $D_{pr}, N^o_{pr}, N^m_{pr} \in \mathcal{H}^{nxn}$; $\det N^o_{pr} \not\equiv 0$ and $\det D_{pr} \in \mathcal{I}$.

(K) $K \in \mathcal{G}^{nx2n}$ has a left-coprime factorization (l.c.f.) $D^{-1}_{c\ell}[N_{\pi\ell} \vdots N_{f\ell}]$ with $D_{c\ell}, N_{\pi\ell}, N_{f\ell} \in \mathcal{H}^{nxn}$, $\det D_{c\ell} \in \mathcal{I}$, and $\det(D_{c\ell}D_{pr} + N_{f\ell}N^m_{pr}) \in \mathcal{I}$.

Under assumptions (P) and (K) the system $\Sigma(P,K)$ in Fig. 1 is completely described by

$$\left[\begin{array}{c:c} I_n & -D_{pr} \\ \hdashline D_{c\ell} & N_{f\ell}N^m_{pr} \end{array}\right] \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \left[\begin{array}{c:c:c:c} 0 & 0 & -I_n & 0 \\ \hdashline N_{\pi\ell} & N_{f\ell} & 0 & -N_{f\ell}N^m_{pr} \end{array}\right] \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \tag{2.1}$$

$$\left[\begin{array}{c:c} I_n & 0 \\ \hdashline 0 & N^o_{pr} \\ \hdashline 0 & N^m_{pr} \end{array}\right] \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ z \end{bmatrix} + \left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -N^o_{pr} \\ \hdashline 0 & 0 & 0 & -N^m_{pr} \end{array}\right] \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \tag{2.2}$$

Let $u := (v^T, u_1^T, u_2^T, d^T)^T$, $\xi := (y_1^T, \xi_p^T)^T$, $y := (y_1^T, y_2^T, z^T)^T$. Then equations (2.1) and (2.2) are of the form

$$D\xi = N_\ell u \tag{2.3}$$

$$N_r\xi = y + Eu \tag{2.4}$$

where the matrices D, N_ℓ, N_r, E, defined in an obvious manner from (2.1) and (2.2), have their elements in $\mathcal{H}$.

For any $D_{c\ell} \in \mathcal{H}^{nxn}$ and any $N_{f\ell} \in \mathcal{H}^{nxn}$, define

$$D_h := D_{c\ell}D_{pr} + N_{f\ell}N_{pr}^m \in \mathcal{H}^{nxn}. \tag{2.5}$$

Note that det D = det D_h = $\det(D_{c\ell}D_{pr}+N_{f\ell}N_{pr}^m) \in \mathcal{I}$ by assumption (K).

Definition 2.1 ($\mathcal{H}$-stability): The system $\Sigma(P,K)$ is called $\mathcal{H}$-stable if and only if $H_{yu} : u = (v^T,u_1^T,u_2^T,d^T)^T \mapsto y = (y_1^T,y_2^T,z^T)^T$ satisfies $H_{yu} \in \mathcal{H}^{3nx4n}$.

Let assumptions (P) and (K) hold. Note that assumption (K) requires that det $D \in \mathcal{I}$, hence from equations (2.3) and (2.4) we obtain

$$H_{yu} = N_r D^{-1} N_\ell + E \in \mathcal{E}(\mathcal{G}). \tag{2.6}$$

Thus det $D \in \mathcal{I}$ is a sufficient condition for the well-posedness of $\Sigma(P,K)$.

From (2.1) and (2.5) it is easily seen that $\Sigma(P,K)$ is $\mathcal{H}$-stable if and only if det $D_h \in \mathcal{U}(\mathcal{H})$ [Des. 1, Corollary 3.1]. Hence w.l.o.g. $\Sigma(P,K)$ is $\mathcal{H}$-stable if and only if we can take $D_h = I$ [Vid. 1].

Proposition 2.2 (Stabilizibility of P): Let P satisfy assumption (P), and let $P^m \in \mathcal{G}_s^{nxn}$, where $\mathcal{G}_s$:= Jacobson Radical of $\mathcal{G}$ [Jac. 1]. Then P is stabilizable (equivalently, there is a compensator K which satisfies assumption (K) such that $\Sigma(P,K)$ is $\mathcal{H}$-stable) if and only if $N_{pr}^m D_{pr}^{-1}$ is a r.c.f. of P^m.

Proof

($\Leftarrow$) The pair (N_{pr}^m, D_{pr}) is right-coprime (r.c.), which implies that there exist U_{pr}^m, $V_{pr}^m \in \mathcal{H}^{nxn}$ such that

$$U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I_n. \tag{2.7}$$

Choose a compensator $K = D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ with $D_{c\ell} := V_{pr}^{m}$, $N_{f\ell} := U_{pr}^{m}$. Clearly $D_{c\ell}, N_{f\ell} \in \mathcal{H}^{nxn}$. Then (2.5) and (2.7) imply that $D_h = I$. Hence, from (2.1), (2.2), (2.5) and (2.6) it follows that, for <u>any</u> $N_{\pi\ell} \in \mathcal{H}^{nxn}$, the system $\Sigma(P,K)$ is $\mathcal{H}$-stable.

($\Longrightarrow$) For a proof by contradiction, suppose that the pair (N_{pr}^{m}, D_{pr}) is not r.c. Then N_{pr}^{m} and D_{pr} have a greatest common right factor R such that $N_{pr}^{m} = \hat{N}_{pr}^{m}R$, $D_{pr} = \hat{D}_{pr}R$, $(\hat{N}_{pr}^{m}, \hat{D}_{pr})$ is a r.c. pair and $\det R \notin U(\mathcal{H})$; equivalently, $(\det R)^{-1} \notin \mathcal{H}$ or $R^{-1} \notin \mathcal{H}^{nxn}$. Now, defining $\hat{D}_h$ in an obvious manner,

$$\det D_h = \det(D_{c\ell}D_{pr} + N_{f\ell}N_{pr}^{m}) = \det[(D_{c\ell}\hat{D}_{pr} + N_{f\ell}\hat{N}_{pr}^{m})R] = \det \hat{D}_h \det R. \tag{2.8}$$

Since $\det R \notin U(\mathcal{H})$ and $\det \hat{D}_h \in \mathcal{H}$, we have that $\det D_h \notin U(\mathcal{H})$.[†] Thus (2.8) shows that for all $D_{c\ell}$, and $N_{f\ell}$, the system $\Sigma(P,K)$ is not $\mathcal{H}$-stable, i.e., P is not stabilizable. □

<u>III. The Construction of Δ_L and Δ_R:</u>

We now construct a diagonal matrix Δ_L and a diagonal matrix Δ_R using N_{pr}^{o}.

Let $N_{pr}^{o} =: \begin{bmatrix} \longleftarrow n_{p1} \longrightarrow \\ \vdots \\ \longleftarrow n_{pn} \longrightarrow \end{bmatrix}$ where, for $k = 1, \ldots, n$,

$n_{pk} \in \mathcal{H}^{1xn}$ denotes the k-th row of N_{pr}^{o}. Since $\mathcal{H}$ is a principal ring, we can define, for $k = 1, \ldots, n$, Δ_{Lk} to be a greatest common divisor

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[†]To see this, suppose that $\det D_h \in U(\mathcal{H})$. Then from (2.8), $\det \hat{D}_h = \det D_h (\det R)^{-1}$ implies that $(\det R)^{-1} = (\det D_h)^{-1} \det \hat{D}_h \in \mathcal{H}$ since $(\det D_h)^{-1} \in \mathcal{H}$; this contradicts $\det R \notin U(\mathcal{H})$.

(g.c.d.) over $\mathcal{H}$ of the elements of n_{pk} [Lang 1, p. 71]. Then for a suitable (row-vector) $\tilde{n}_{pk} \in \mathcal{H}^{1xn}$, $n_{pk} = \Delta_{Lk}\, \tilde{n}_{pk}$ and

$$N_{pr}^{o} = \begin{bmatrix} \Delta_{L1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \Delta_{Ln} \end{bmatrix} \begin{bmatrix} \leftarrow \tilde{n}_{p1} \rightarrow \\ \vdots \\ \leftarrow \tilde{n}_{pn} \rightarrow \end{bmatrix} =: \Delta_L \tilde{N}_{pr}^{o} \tag{3.1}$$

where Δ_L, $\tilde{N}_{pr}^{o} \in \mathcal{H}^{nxn}$ are not unique, since each Δ_{Lk} is only defined within a factor in $U(\mathcal{H})$. (In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, Δ_{Lk} "bookkeeps" the plant zeros in $\overline{\mathcal{U}}$ that are common to all elements of row k of N_{pr}^{o}). Datta and Hautus [Dat. 1] used a similar factorization.

The matrix $\tilde{N}_{pr}^{o} \in \mathcal{H}^{nxn}$ is not necessarily invertible over $\mathcal{H}^{nxn}$; but by assumption (P), and since det $\tilde{N}_{pr}^{o} \in \mathcal{H}$, $(\tilde{N}_{pr}^{o})^{-1}$ has elements in the field of fractions $[\mathcal{H}][\mathcal{H} \setminus 0]^{-1}$ of the entire ring $\mathcal{H}$ [Lang 1, p. 69]. From (3.1), det N_{pr}^{o} = det Δ_L det $\tilde{N}_{pr}^{o}$ where Δ_L is nonsingular by construction. Let a_{ij} denote the ij-th element of $(\tilde{N}_{pr}^{o})^{-1}$ and for i,j = 1, ..., n, define

$$(\tilde{N}_{pr}^{o})^{-1} = [a_{ij}] = [\frac{m_{ij}}{d_{ij}}], \qquad m_{ij}, d_{ij} \in \mathcal{H}. \tag{3.2}$$

For j = 1, ..., n, let Δ_{Rj} be a least common multiple (l.c.m.) of $(d_{ij})_{i=1}^{n}$; i.e., Δ_{Rj} is a l.c.m. of $d_{1j}, d_{2j}, \ldots, d_{nj}$ of the elements of the j-th column of $(\tilde{N}_{pr}^{o})^{-1}$ [Lang 1, p. 72]. Note that each Δ_{Rj} is only defined within a factor in $U(\mathcal{H})$. Define

$$\Delta_R := \text{diag}\,(\Delta_{R1}, \Delta_{R2}, \ldots, \Delta_{Rn}) \in \mathcal{H}^{nxn} \tag{3.3}$$

An extraction of a diagonal factor as Δ_R analogous to the present one is found in [Des. 2].

Fact 3.1: Let $\tilde{N}^o_{pr}$ and Δ_R be defined by (3.1) and (3.3). Then $(\tilde{N}^o_{pr})^{-1}\cdot\Delta_R \in \mathcal{H}^{n\times n}$.

Proof

Since Δ_{Rj} is a l.c.m. of the $(d_{ij})^n_{i=1}$, for i = 1, ..., n we have $\bar{d}_{ij} \in \mathcal{H}$ such that

$$\Delta_{Rj} = d_{ij}\bar{d}_{ij}, \qquad d_{ij},\bar{d}_{ij} \in \mathcal{H}. \tag{3.4}$$

Then for i,j = 1, ..., n, the ij-th element of $(\tilde{N}^o_{pr})^{-1}\Delta_R$ is

$$\frac{m_{ij}}{d_{ij}}\Delta_{Rj} = m_{ij}\bar{d}_{ij} \in \mathcal{H} \tag{3.5}$$

by (3.2) and (3.4).

IV. Achievable Performance of $\Sigma(P,K)$.

The I/O Map H_{y_2v} and the D/O Map H_{y_2d}

For any $\Sigma(P,K)$ satisfying assumptions (P) and (K) (hence for which $\det D_h \in \mathcal{I}$), the I/O map $H_{y_2v} : v \mapsto y_2$ and the D/O map $H_{y_2d} : d \mapsto y_2$ are given by

$$H_{y_2v} = N^o_{pr}D_h^{-1}N_{\pi\ell} \tag{4.1}$$

$$H_{y_2d} = N^o_{pr}[I-D_h^{-1}N_{f\ell}N^m_{pr}] = N^o_{pr}D_h^{-1}D_{c\ell}D_{pr} \tag{4.2}$$

Now, $\Sigma(P,K)$ is $\mathcal{H}$-stable if and only if $\det D_h \in \mathcal{U}(\mathcal{H})$; consequently, if $\Sigma(P,K)$ is $\mathcal{H}$-stable we may take $D_h = I$ [Vid. 1]. Using this, (2.5) and (3.1), we obtain

$$H_{y_2v} = N^{o}_{pr}N_{\pi\ell} = \Delta_L\tilde{N}^{o}_{pr}N_{\pi\ell} \tag{4.3}$$

$$H_{y_2d} = N^{o}_{pr}[I-N_{f\ell}N^{m}_{pr}] = N^{o}_{pr}D_{c\ell}D_{pr} \tag{4.4}$$

<u>Definition 4.1 (Achievable Maps)</u>: Let P be given and satisfy assumption (P); let K satisfy assumption (K) and let K be such that the system $\Sigma(P,K)$ is $\mathcal{H}$-stable.

Roughly speaking, let $\mathcal{H}_{y_2v}(P)$ denote the set of all <u>achievable diagonal</u> I/O maps of $\Sigma(P,K)$; more precisely,

$\mathcal{H}_{y_2v}(P) := \{H_{y_2v}$: for the given P, there exists a compensator K satisfying assumption (K) such that $\Sigma(P,K)$ is $\mathcal{H}$-stable with H_{y_2v} <u>diagonal</u> and <u>nonsingular</u>$\}$.

Let $\mathcal{H}_{y_2d}(P)$ denote the set of all <u>achievable</u> D/O maps of $\Sigma(P,K)$; more precisely,

$\mathcal{H}_{y_2d}(P) := \{H_{y_2d}$: for the given P, there exists a compensator K satisfying assumption (K) such that $\Sigma(P,K)$ is $\mathcal{H}$- stable with H_{y_2v} <u>diagonal</u> and <u>nonsingular</u>$\}$.

<u>Achievable I/O Map and D/O Map</u>:

The following theorem characterizes all the achievable diagonal I/O maps and the achievable D/O maps for $\Sigma(P,K)$.

<u>Theorem 4.2 (Achievable Diagonal I/O Maps and Achievable D/O Maps)</u>:

Consider the system $\Sigma(P,K)$ of Fig. 1: let P and K satisfy assumptions (P) and (K). Let $D^{-1}_{p\ell}N^{m}_{p\ell} = P^{m}$, where $D_{p\ell}, N^{m}_{p\ell} \in \mathcal{H}^{nxn}$, det $D_{p\ell} \in \mathcal{I}$, be a l.c.f. of P^m, and let $N^{m}_{pr}D^{-1}_{pr}$ be a r.c.f. of P^m. Let Δ_L and Δ_R be defined by (3.1) and (3.3) above. Then,

i) the map $H_v \in \mathcal{H}^{nxn}$ is an achievable, diagonal, nonsingular I/O map of the $\mathcal{H}$-stable system $\Sigma(P,K)$ if and only if $H_v \in \mathcal{H}_{y_2v}(P)$, where

$$H_{y_2v}(P) = \{\Delta_L\Delta_R Q_d : Q_d \in \mathcal{H}^{nxn},\ Q_d \text{ is } \underline{\text{diagonal}} \text{ and } \underline{\text{nonsingular}}\}. \tag{4.5}$$

ii) the map $H_d \in \mathcal{H}^{nxn}$ is an achievable D/O map of the $\mathcal{H}$-stable system $\Sigma(P,K)$ if and only if $H_d \in H_{y_2d}(P)$, where

$$H_{y_2d}(P) = \{N^o_{pr}[I-(U^m_{pr}+RD_{p\ell})N^m_{pr}] = N^o_{pr}(V^m_{pr}-RN^m_{p\ell})D_{pr} : R \in \mathcal{H}^{nxn} \text{ s.t. } \det(V^m_{pr}-RN^m_{p\ell}) \in I\} \tag{4.6}$$

and V^m_{pr}, U^m_{pr}, N^m_{pr}, D_{pr} are as in (2.7).

Comments:

1) If decoupling were not required, the set of achievable I/O maps of $\Sigma(P,K)$ would be given by

$$H_{y_2v}(P) = \{N^o_{pr}Q = \Delta_L\tilde{N}^o_{pr}Q : Q \in \mathcal{H}^{nxn}\} \tag{4.5a}$$

and the set of achievable D/O maps would still be given by (4.6) [Des. 3]. Requiring the I/O map to be decoupled adds a number of constraints: i) $Q_d \in \mathcal{H}^{nxn}$ must be diagonal; ii) we must have $\Delta_L\Delta_R$ as a left factor of H_{y_2v} instead of just $N^o_{pr} = \Delta_L\tilde{N}^o_{pr}$. In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, we can interpret the cost of decoupling as follows: The $\overline{\mathcal{U}}$-zeros of $P^o : e_2 \mapsto y_2$ will be the zeros of H_{y_2v}, the closed-loop system I/O map, whether it is decoupled or not. However, with decoupling, the multiplicity of these $\overline{\mathcal{U}}$-zeros may be greater than that of P^o. This is due to Δ_R: indeed, from (4.5) and (4.5a) we see that Δ_L is a left-factor of any achievable I/O map with or without decoupling. On the other hand Δ_R is required to guarantee that $N_{\pi\ell} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d \in \mathcal{E}(\mathcal{H})$, and therefore may have a greater multiplicity of the same $\overline{\mathcal{U}}$-zeros than $\tilde{N}^o_{pr}$ has.

2) If $\det \tilde{N}^o_{pr} \in U(\mathcal{H})$, equivalently if $(\tilde{N}^o_{pr})^{-1} \in \mathcal{H}^{nxn}$, then $\Delta_R = I$ and the diagonal I/O maps are of the form $\Delta_L Q_d$.

3) The diagonalization of the I/O map H_{y_2v} is achieved by choosing $N_{\pi\ell}$; this choice is independent of that of $D_{c\ell}$ and $N_{f\ell}$. Similarly, $N_{\pi\ell}$ does not affect the D/O map. Thus the I/O map and the D/O map of the $\mathcal{H}$-stable $\Sigma(P,K)$ can be specified independently: it is a two-degrees-of-freedom design [Hor. 1].

4) It is important to note the constraints imposed on H_{y_2d} by the $\overline{\mathcal{U}}$-zeros and the $\mathcal{U}$-poles of the plant when $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$. If $\Sigma(P,K)$ is $\mathcal{H}$-stable and if $PF := PD_{c\ell}^{-1}N_{f\ell}$ is full normal rank in $\mathcal{G}$, then

a) if z_o is a $\overline{\mathcal{U}}$-zero of N_{pr}^{o} (equivalently, $\exists\ \alpha \neq \theta_n$ such that $\alpha^* N_{pr}^{o}(z_o) = \theta_n$, then

$$\alpha^* N_{pr}^{o}(I-N_{f\ell}N_{pr}^{m})(z_o) = \alpha^* H_{y_2d}(z_o) = \theta_n^*. \tag{4.7}$$

b) if N_{pr}^{m} has full normal rank and if z_m is a $\overline{\mathcal{U}}$-zero of N_{pr}^{m} (equivalently, $\exists\ \beta \neq \theta_n$ such that $N_{pr}^{m}(z_m)\beta = \theta_n$) then

$$N_{pr}^{o}(I-N_{f\ell}N_{pr}^{m})(z_m)\beta = N_{pr}^{o}(z_m)\beta = H_{y_2d}(z_m)\beta. \tag{4.8}$$

c) if p_o is a $\mathcal{U}$-pole of P (equivalently, $\exists\ \gamma \neq \theta_n$ such that $D_{pr}(p_o)\gamma = \theta_n$) then

$$N_{pr}^{o}D_{c\ell}D_{pr}(p_o)\gamma = H_{y_2d}(p_o)\gamma = \theta_n \tag{4.9}$$

Thus, whenever either N_{pr}^{o} or N_{pr}^{m} has a $\overline{\mathcal{U}}$-zero or when P has a $\mathcal{U}$-pole, the D/O map is constrained by a vector-equality such as (4.7), (4.8) or (4.9) respectively.

Proof of Theorem 4.2

($\Longrightarrow$) We are given P and K such that the $\mathcal{H}$-stable system $\Sigma(P,K)$ achieves the diagonal I/O map $H_v \in \mathcal{H}^{n\times n}$ and the D/O map $H_d \in \mathcal{H}^{n\times n}$.

We have to show that H_v is the form $\Delta_L \Delta_R Q_d$ and H_d is of the form $N^o_{pr}[I-(U^m_{pr}+RD_{p\ell})N^m_{pr}]$ for some $R \in \mathcal{H}^{nxn}$ and some diagonal $Q_d \in \mathcal{H}^{nxn}$.

Since $\Sigma(P,K)$ is $\mathcal{H}$-stable, w.l.o.g. we take $D_h = I$. From (4.3), the diagonal matrix Δ_L is obviously a left-factor of H_v; it remains to show that H_v has $\Delta_L \Delta_R$ as a left-factor. For a proof by contradiction, suppose that H_v is of the form

$$H_v = \Delta_L \tilde{\Delta}_R Q_d \tag{4.10}$$

where $\tilde{\Delta}_R$ is a <u>proper</u> factor of Δ_R and $Q_d \in \mathcal{H}^{nxn}$, nonsingular, diagonal; for example suppose that

$$\tilde{\Delta}_R = \mathrm{diag}(\Delta_{R1},\ldots,\Delta_{Rj-1},\tilde{\Delta}_{Rj},\Delta_{Rj+1},\ldots,\Delta_{Rn}) \tag{4.11}$$

where, for a <u>non-unit prime</u> element δ_j in $\mathcal{H}$, [Lang 1, p. 72],

$$\Delta_{Rj} = \delta_j \tilde{\Delta}_{Rj} \tag{4.12}$$

Since H_v is the I/O map of $\Sigma(P,K)$, from (4.3) and (4.10)

$$H_v = \Delta_L \tilde{N}^o_{pr} N_{\pi\ell} = \Delta_L \tilde{\Delta}_R Q_d \tag{4.13}$$

Since $\mathcal{H}$ is a principal ring, we may cancel the nonsingular left-factor Δ_L and invert $\tilde{N}^o_{pr}$ in (4.13) to obtain

$$N_{\pi\ell} = (\tilde{N}^o_{pr})^{-1}\tilde{\Delta}_R Q_d. \tag{4.14}$$

From (3.2)

$$N_{\pi\ell} = \left[\frac{m_{ij}}{d_{ij}}\right]\mathrm{diag}(\Delta_{R1},\ldots,\Delta_{Rj-1},\tilde{\Delta}_{Rj},\Delta_{Rj+1},\ldots,\Delta_{Rn})\cdot Q_d \tag{4.15}$$

Consider the j-th column of $N_{\pi\ell}$ and recall that Δ_{Rj} is by definition a l.c.m. of $(d_{ij})_{i=1}^{n}$. Then since $\Delta_{Rj} = \delta_j\tilde{\Delta}_{Rj}$, w.l.o.g.

$$d_{ij} = \delta_j\tilde{d}_{ij}, \quad \tilde{d}_{ij} \in \mathcal{H} \tag{4.16}$$

where $\tilde{d}_{ij}$ is a factor of $\tilde{\Delta}_{Rj}$, i.e., there exists a $\tilde{c}_{ij} \in \mathcal{H}$, possibly a unit, such that

$$\tilde{\Delta}_{Rj} = \tilde{d}_{ij}\tilde{c}_{ij}. \tag{4.17}$$

Hence, with $q_j \in \mathcal{H}$ denoting the j-th diagonal entry of some general diagonal $Q_d \in \mathcal{H}^{nxn}$, from (4.15), (4.16) and (4.17) we obtain the ij-th entry of $N_{\pi\ell}$ as

$$\frac{m_{ij}}{d_{ij}} \cdot \tilde{\Delta}_{Rj}q_j = \frac{m_{ij}}{\delta_j}\tilde{c}_{ij}q_j. \tag{4.18}$$

Since $\delta_j \notin U(\mathcal{H})$ and in general δ_j is not a factor of q_j, the right-hand side of (4.18) is <u>not</u> in $\mathcal{H}$. Therefore, except when the prime δ_j is a factor of q_j, $N_{\pi\ell} \notin \mathcal{H}^{nxn}$; thus with $N_{\pi\ell}$ as in (4.14), there is a diagonal $Q_d \in \mathcal{H}^{nxn}$ such that $\Sigma(P,K)$ is not $\mathcal{H}$-stable. Therefore H_v must be of the form $\Delta_L\Delta_R Q_d$ and $H_v \in H_{y_2v}(P)$ must be given by (4.5).

Now consider H_d. Recalling proposition 2.2, since $\Sigma(P,K)$ is $\mathcal{H}$-stable, the pair (N_{pr}^{m}, D_{pr}) is r.c. and hence satisfies (2.7). We can take $D_h = I$; equivalently, from (2.5)

$$N_{f\ell}N_{pr}^{m} + D_{c\ell}D_{pr} = I \tag{4.19}$$

Viewing (4.19) as a <u>linear</u> matrix equation in $\mathcal{H}^{nxn}$, we solve for $(N_{f\ell}, D_{c\ell})$ subject to det $D_{c\ell} \in I$ so that $D_{c\ell}^{-1}N_{f\ell} \in \mathcal{G}^{nxn}$: from (2.7) we have

$$U_{pr}^{m}N_{pr}^{m} + V_{pr}^{m}D_{pr} = I \tag{4.20}$$

and since $N_{pr}^{m}D_{pr}^{-1} = D_{p\ell}^{-1}N_{p\ell}^{m} = P^{m}$ we have

$$D_{p\ell}N_{pr}^{m} - N_{p\ell}^{m}D_{pr} = 0. \tag{4.21}$$

The pair (U_{pr}^{m},V_{pr}^{m}) in (4.20) is a particular solution to $(N_{f\ell},D_{c\ell})$ in (4.19) and the pair $(D_{p\ell},-N_{p\ell}^{m})$ is a particular solution to the homogeneous equation (4.21). Hence, any general solution of (4.19) is given by

$$N_{f\ell} = U_{pr}^{m} + RD_{p\ell} \tag{4.22a}$$

$$D_{c\ell} = V_{pr}^{m} - RN_{p\ell}^{m} \tag{4.22b}$$

where $R \in \mathcal{H}^{nxn}$. Now, from assumption (K), we see that $K \in \mathcal{G}^{nx2n}$ if and only if $\det D_{c\ell} \in \mathcal{I}$. So we must require that the arbitrary $R \in \mathcal{H}^{nxn}$ satisfies

$$\det(V_{pr}^{m}-RN_{p\ell}^{m}) \in \mathcal{I}. \tag{4.23}$$

(Note that if $P \in \mathcal{G}_{s}^{2nxn}$ where $\mathcal{G}_{s}$:= Jacobson radical of $\mathcal{G}$, then N_{pr}^{m} and $N_{p\ell}^{m} \in \mathcal{G}_{s}^{nxn}$ and (4.23) is automatically satisfied $\forall R \in \mathcal{H}^{nxn}$ [Des. 3, proof of Theorem 3.1]). So, by (4.4)

$$H_{d} = N_{pr}^{o}[I-(U_{pr}^{m}+RD_{p\ell})N_{pr}^{m}] \tag{4.24}$$

and by (4.19) and (4.20)

$$H_{d} = N_{pr}^{o}[V_{pr}^{m}D_{pr}-RN_{p\ell}^{m}D_{pr}] = N_{pr}^{o}(V_{pr}^{m}-RN_{p\ell}^{m})D_{pr} \tag{4.25}$$

Therefore H_{d} given by (4.24) and (4.25) is an element of $\mathcal{H}_{y_{2}d}(P)$ given by (4.6)

($\Leftarrow$) By assumption, for some diagonal nonsingular $Q_d \in \mathcal{H}^{nxn}$, we are given $H_v = \Delta_L \Delta_R Q_d$ and for some $R \in \mathcal{H}^{nxn}$ we are given $H_d = N^o_{pr}[I-(U^m_{pr}+RD_{p\ell})N^m_{pr}] = N^o_{pr}(V^m_{pr}-RN^m_{p\ell})D_{pr}$. We have to show that there exists some compensator K such that the I/O map H_v and the D/O map H_d are achieved by an $\mathcal{H}$-stable $\Sigma(P,K)$.

Choose the controller K as $K := D^{-1}_{c\ell}[N_{\pi\ell}\vdots N_{f\ell}]$ with $N_{f\ell}$ and $D_{c\ell}$ as in (4.22a-b) and choose $N_{\pi\ell}$ as

$$N_{\pi\ell} := (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d \tag{4.26}$$

where, by Fact 3.1, $N_{\pi\ell} \in \mathcal{H}^{nxn}$.

$$D_h := D_{c\ell}D_{pr} + N_{f\ell}N^m_{pr} = (V^m_{pr}-RN^m_{p\ell})D_{pr} + (U^m_{pr}+RD_{p\ell})N^m_{pr}$$

and by (4.19) and (4.20), $D_h = I$. Consequently det $D_h \in \mathcal{U}(\mathcal{H})$ and $\Sigma(P,K)$, specified by $N_{\pi\ell}$, $N_{f\ell}$, $D_{c\ell}$ in equations (4.26) and (4.22a-b), is $\mathcal{H}$-stable.

By using (4.3) and (4.26) we calculate the I/O map of this $\Sigma(P,K)$ as

$$H_{y_2v} = \Delta_L \tilde{N}^o_{pr} N_{\pi\ell} = \Delta_L \tilde{N}^o_{pr}(\tilde{N}^o_{pr})^{-1}\Delta_R Q_d = \Delta_L \Delta_R Q_d = H_v,$$

and by using (4.4) and (4.23) we calculate the D/O map of this $\Sigma(P,K)$ as

$$H_{y_2d} = N^o_{pr}(I-N_{f\ell}N^m_{pr}) = N^o_{pr}[I-(U^m_{pr}+RD_{p\ell})N^m_{pr}] = H_d$$

or

$$H_{y_2d} = N^o_{pr}D_{c\ell}D_{pr} = N^o_{pr}(V^m_{pr}-RN^m_{p\ell})D_{pr} = H_d. \qquad \square$$

V. Examples and Conclusions

In the following examples we focus our attention on the diagonal

I/O map of $\Sigma(P,K)$. Since the design has two degrees of freedom, only $N_{\pi\ell}$ is calculated: indeed, the compensator parameters used to design the D/O map are not needed for the I/O map.

Example 1: In this example, $\mathcal{H} := \mathcal{R}(s,e^{-\tau s})$ is the principal ring where $\mathcal{R}(s,e^{-\tau s})$ denotes the rational functions which are proper in s, analytic in $\mathbb{C}_+$ and have coefficients in $\mathbb{R}[e^{-\tau s}]$. ($\mathbb{R}[e^{-\tau s}]$ is the ring of polynomials in $e^{-\tau s}$ with real coefficients.) Consider the P^o given by (5.1) below: it is strictly proper but not $\mathcal{H}$-stable and it has a simple zero[†] at $s = 3$.

$$P^o(s,e^{-\tau s}) = \begin{bmatrix} \frac{e^{-s}}{s-1} & \frac{1}{s-2} \\ \frac{e^{-2s}}{s+1} & \frac{e^{-s}}{s-1} \end{bmatrix} \notin \mathcal{H}^{2\times 2} \tag{5.1}$$

A r.c.f. of P^o is given by $P^o = N^o_{pr} D^{-1}_{pr} = \begin{bmatrix} \frac{e^{-s}}{s+2} & \frac{s-1}{(s+1)^2} \\ \frac{(s-1)e^{-2s}}{(s+1)(s+2)} & \frac{(s-2)e^{-s}}{(s+1)^2} \end{bmatrix}$

$\cdot \operatorname{diag}\left[\frac{s-1}{s+2}, \frac{(s-1)(s-2)}{(s+1)^2}\right]^{-1}$. Then $N^o_{pr} = \Delta_L \tilde{N}^o_{pr} = \operatorname{diag}\left[\frac{1}{s+2}, \frac{e^{-s}}{s+1}\right]$

$\cdot \begin{bmatrix} e^{-s} & \frac{(s-1)(s+2)}{(s+1)^2} \\ \frac{(s-1)e^{-s}}{s+2} & \frac{s-1}{s+1} \end{bmatrix}$. Here Δ_L and $\tilde{N}^o_{pr}$ are not unique; Δ_L extracts a zero at infinity from the rational part of each row of N^o_{pr}.

[†]By "the zeros of P^o" we mean the zeros of the rational function det N^o_{pr}.

Now $(\tilde{N}^o_{pr})^{-1} = \left[\begin{array}{c:c} \dfrac{(s-2)(s+1)}{(s-3)e^{-s}} & \dfrac{-(s-1)(s+2)}{(s-3)e^{-s}} \\ \hdashline \dfrac{-(s-1)(s+1)^2}{(s-3)(s+2)} & \dfrac{(s+1)^2}{(s-3)} \end{array}\right] \notin \mathcal{H}^{2x2}$, from which we

obtain $\Delta_R = \text{diag}\left[\dfrac{(s-3)e^{-s}}{(s+1)^2}, \dfrac{(s-3)e^{-s}}{(s+1)^2}\right]$ and $N_{\pi\ell} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d$

$= \left[\begin{array}{c:c} \dfrac{s-2}{s+1} & \dfrac{-(s-1)(s+2)}{(s+1)^2} \\ \hdashline \dfrac{-(s-1)e^{-s}}{s+2} & e^{-s} \end{array}\right] Q_d$. Note that each diagonal entry of Δ_R

is equal to det $\tilde{N}^o_{pr}$. In fact, it can be shown that in the <u>2x2</u> case, each diagonal entry of Δ_R is <u>always</u> equal to det $\tilde{N}^o_{pr}$ modulo a unit factor in $\mathcal{U}(\mathcal{H})$. Consequently, det $\Delta_R = (\det \tilde{N}^o_{pr})^2$ modulo a unit in $\mathcal{U}(\mathcal{H})$, and the number of $\mathbb{C}_+$-zeros of the diagonal closed-loop I/O map is increased. In the example, $H_{y_2v} = \Delta_L\Delta_R Q_d = \text{diag}\left[\dfrac{(s-3)e^{-s}}{(s+2)(s+1)^2}, \dfrac{(s-3)e^{-2s}}{(s+1)^3}\right]$ $\cdot Q_d$, where $Q_d \in \mathcal{H}^{2x2}$ is diagonal and nonsingular. Here, H_{y_2v} has a zero of multiplicity <u>two</u> at $s = 3$ and it may have other $\mathbb{C}_+$-zeros due to Q_d. Comparing this to the $\mathbb{C}_+$-zeros of det N^o_{pr} we see that the cost of decoupling is the increased number of $\mathbb{C}_+$-zeros (due to Δ_R) and the restriction that Q_d be diagonal.

<u>Example 2</u>: In this example, let $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$ where $\mathcal{U} = \mathbb{C}_+$. P^o is given by (5.2): it is proper but not $\mathcal{H}$-stable; P^o has a zero of multiplicity two at $s = 1$, a zero at $s = 2$ and two zeros at infinity.

$$P^{o}(s) = \left[\begin{array}{c:c:c} \frac{s-1}{(s-3)(s+2)} & \frac{1}{s+2} & \frac{(s-1)(s-2)}{(s+1)(s+2)} \\ \hdashline \frac{s+1}{s-3} & 1 & \frac{s-2}{s+2} \\ \hdashline 0 & \frac{1}{(s-1)(s+1)} & \frac{s-2}{(s+1)(s+2)} \end{array}\right] \not\in \mathcal{H}^{3\times 3}. \qquad (5.2)$$

A r.c.f. of P^{o} is given by

$$P^{o} = N^{o}_{pr} D^{-1}_{pr} = \left[\begin{array}{c:c:c} \frac{s-1}{(s+1)(s+2)} & \frac{s-1}{(s+1)(s+2)} & \frac{(s-1)(s-2)}{(s+1)(s+2)} \\ \hdashline 1 & \frac{s-1}{s+1} & \frac{s-2}{s+2} \\ \hdashline 0 & \frac{1}{(s+1)^2} & \frac{s-2}{(s+1)(s+2)} \end{array}\right] \cdot \operatorname{diag}\left[\frac{s-3}{s+1}, \frac{s-1}{s+1}, 1\right]^{-1}.$$

$$\text{Then, } N^{o}_{pr} = \Delta_L \tilde{N}^{o}_{pr} = \operatorname{diag}\left[\frac{s-1}{s+2}, 1, \frac{1}{s+1}\right] \cdot \left[\begin{array}{c:c:c} \frac{1}{(s+1)} & \frac{1}{(s+1)} & \frac{(s-2)}{(s+1)} \\ \hdashline 1 & \frac{(s-1)}{(s+1)} & \frac{(s-2)}{(s+2)} \\ \hdashline 0 & \frac{1}{(s+1)} & \frac{(s-2)}{(s+2)} \end{array}\right].$$

Δ_L and N^{o}_{pr} are not unique and Δ_L extracts a zero at $s = 1$ from the first row of N^{o}_{pr} and a zero at infinity from the third row of N^{o}_{pr}. Now

$$(\tilde{N}^{o}_{pr})^{-1} = \left[\begin{array}{c:c:c} \frac{(s-2)(s+1)}{s-1} & \frac{1}{s-1} & \frac{-(s^2-3)}{s-1} \\ \hdashline \frac{-(s+1)^2}{s-1} & \frac{s+1}{s-1} & \frac{(s+1)^2}{s-1} \\ \hdashline \frac{(s+1)(s+2)}{(s-1)(s-2)} & \frac{-(s+2)}{(s-1)(s-2)} & \frac{-2(s+2)}{(s-1)(s-2)} \end{array}\right] \not\in \mathcal{H}^{3\times 3}$$

and $\Delta_R = \text{diag}\left[\dfrac{(s-1)(s-2)}{(s+1)^2(s+2)}, \dfrac{(s-1)(s-2)}{(s+1)(s+2)}, \dfrac{(s-1)(s-2)}{(s+1)^2(s+2)}\right]$.

(The first and the third diagonal entries of Δ_R are equal to det $\tilde{N}^o_{pr}$.)

$$\text{Then } N_{\pi\ell} = \left[\begin{array}{c|c|c} \dfrac{(s-2)^2}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} & \dfrac{-(s^2-3)(s-2)}{(s+1)^2(s+2)} \\ \hline \dfrac{-(s-2)}{(s+2)} & \dfrac{s-2}{s+2} & \dfrac{s-2}{s+2} \\ \hline \dfrac{1}{s+1} & \dfrac{-1}{s+1} & \dfrac{-2}{(s+1)^2} \end{array}\right] Q_d \text{ and}$$

$$H_{y_2v} = \Delta_L\Delta_R Q_d = \text{diag}\left[\frac{(s-1)^2(s-2)}{(s+1)^2(s+2)^2}, \frac{(s-1)(s-2)}{(s+1)(s+2)}, \frac{(s-1)(s-2)}{(s+1)^3(s+2)}\right] Q_d .$$

where $Q_d \in \mathcal{H}^{3x3}$ is diagonal and nonsingular. The closed-loop diagonal I/O map H_{y_2v} has a zero of multiplicity four at $s = 1$, a zero of multicipiicity three at $s = 2$ and three zeros at infinity. H_{y_2v} may have other $\mathbb{C}_+$-zeros due to Q_d. The cost of decoupling is the increased number of $\mathbb{C}_+$-zeros (due to Δ_R) and the restriction that Q_d be diagonal.

Example 3: In this example we design a decoupling compensator for the P^o given by (5.3), which is the model of a "boiler subsystem" in [Joh. 1]. Johansson and Koivo apply the Inverse Nyquist Array method of Rosenbrock in the design of a multivariable controller. Let $\mathcal{H}$: $\mathcal{R}_{\mathcal{U}}$ where $\mathcal{U} = \mathbb{C}_+$

$$P^o(s) = \begin{bmatrix} \dfrac{-e^{-2s}}{10s+1} & \dfrac{-1}{10s+1} \\ 0 & \dfrac{e^{-10s}}{60s+1} \end{bmatrix} \in \mathcal{H}^{2x2} \tag{5.3}$$

A r.c.f. of P^o is given by $D_{pr} = I$, $N^o_{pr} = P^o$. Then $\Delta_L = \operatorname{diag}\left[\frac{1}{7s+1}, \frac{1}{40s+1}\right]$ and $(\tilde{N}^o_{pr})^{-1} = \begin{bmatrix} \frac{-(10s+1)e^{2s}}{(7s+1)} & \frac{(60s+1)e^{12s}}{(40s+1)} \\ 0 & \frac{(60s+1)e^{10s}}{(40s+1)} \end{bmatrix}$. From this we obtain $\Delta_R = \operatorname{diag}[e^{-2s}, e^{-12s}]$ and $N_{\pi\ell} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d$

$$= \begin{bmatrix} \frac{-(10s+1)}{(7s+1)} & \frac{(60s+1)}{(40s+1)} \\ 0 & \frac{(60s+1)e^{-2s}}{(40s+1)} \end{bmatrix} Q_d,$$

where $Q_d \in \mathcal{H}^{2x2}$ is diagonal and nonsingular. Finally, $H_{\check{y}_2 v} = \Delta_L \Delta_R Q_d = \operatorname{diag}\left[\frac{e^{-2s}}{7s+1}, \frac{e^{-12s}}{40s+1}\right] \cdot Q_d$. Here, the closed-loop I/O map is diagonal and the time-constants are reduced from 10 sec. and 60 sec. to 7 sec. and 40 sec. respectively. We complete our design by giving a choice of $D_{c\ell}$ and $N_{f\ell}$ as

$$D_{c\ell} = V^m_{pr} - RN^m_{p\ell} = \begin{bmatrix} 1 & \frac{e^{-10s}}{60s+1} \\ \frac{e^{-12s}}{60s+1} & 1 \end{bmatrix} - RP^o$$

$$N_{f\ell} = U^m_{pr} + RD_{p\ell} = \begin{bmatrix} 0 & -1 \\ \frac{e^{-10s}(10s+1)}{60s+1} & 1 \end{bmatrix} + RI$$

where $R \in \mathcal{H}^{2x2}$ is such that $V^m_{pr} - RN^m_{p\ell} \in \mathcal{I}$.

Conclusions

Without decoupling, the set of all achievable I/O maps of $\Sigma(P,K)$ is given by (4.5a). The compensator parameter $N_{\pi\ell}$, which is used in designing the I/O map, is made $\mathcal{H}$-stable by an appropriate choice of a diagonal $\mathcal{H}$-stable matrix Δ_R defined by (3.3). Finally, the set of all achievable diagonal nonsingular I/O maps is given by (4.5), where Δ_L appears as a left-factor of both diagonal and non-diagonal achievable I/O maps.

The examples of this section clearly illustrate the cost involved in decoupling the I/O map while requiring that it be $\mathcal{H}$-stable; this cost is reflected by Δ_R and Q_d: Δ_R must be chosen so that $N_{\pi\ell}$ is $\mathcal{H}$-stable; $Q_d \in \mathcal{H}^{n\times n}$ must be diagonal. In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$ (or $\mathcal{H} = \mathcal{R}(s,e^{-\tau s})$ as in example 1) the presence of Δ_R in the diagonal I/O map results in increasing the number of $\mathcal{U}$-zeros. If $N^o_{pr} \in \mathcal{H}^{2\times 2}$, det Δ_R has exactly twice as many $\overline{\mathcal{U}}$-zeros as det $\tilde{N}^o_{pr}$ (for a proof see the Appendix.)

This design method has two degrees of freedom: decoupling the I/O map has no effect on the D/O map. the D/O map is designed using the parameters $D_{c\ell}$ and $N_{f\ell}$ of the compensator. The only compensator parameter used in the I/O map is $N_{\pi\ell}$.

The results developed in this paper are valid for many classes of systems, some of which are listed in [Des. 3, Table I].

Appendix

Let $\mathcal{H} := \mathcal{R}\mathcal{U}$ and let $n = 2$. Let $\tilde{N}^o_{pr}$, Δ_L and Δ_R be defined by (3.1) and (3.3). Under these conditions, $\det \Delta_R = (\det \tilde{N}^o_{pr})^2 \cdot u$ where $u \in \mathcal{U}(\mathcal{H})$.

Proof. Let $\tilde{N}^o_{pr} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \in \mathcal{H}^{2x2}$ where, by the construction of Δ_L, (n_{11}, n_{12}) is a coprime pair and (n_{21}, n_{22}) is a coprime pair. With $\delta := \det \tilde{N}^o_{pr}$, the first and second columns of $(\tilde{N}^o_{pr})^{-1}$ are $(n_{22}/\delta, -n_{21}/\delta)$, $(-n_{12}/\delta, n_{11}/\delta)$, resp. Now, any irreducible common factor that cancels in n_{22}/δ, will not be a common factor in $-n_{21}/\delta$, since $(n_{22}, -n_{21})$ are coprime. Thus the least common denominator for the first column is δ. The same holds for the second column, hence $\Delta_R = \operatorname{diag}(\delta, \delta)$ and $\det \Delta_R = (\det \tilde{N}^o_{pr})^2$, modulo factors in $\mathcal{U}(\mathcal{H})$.

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Figure Captions

Fig. 1. The System $\Sigma(P,K)$.

Fig. 2. The System ${}^{1}\Sigma(P,K)$.

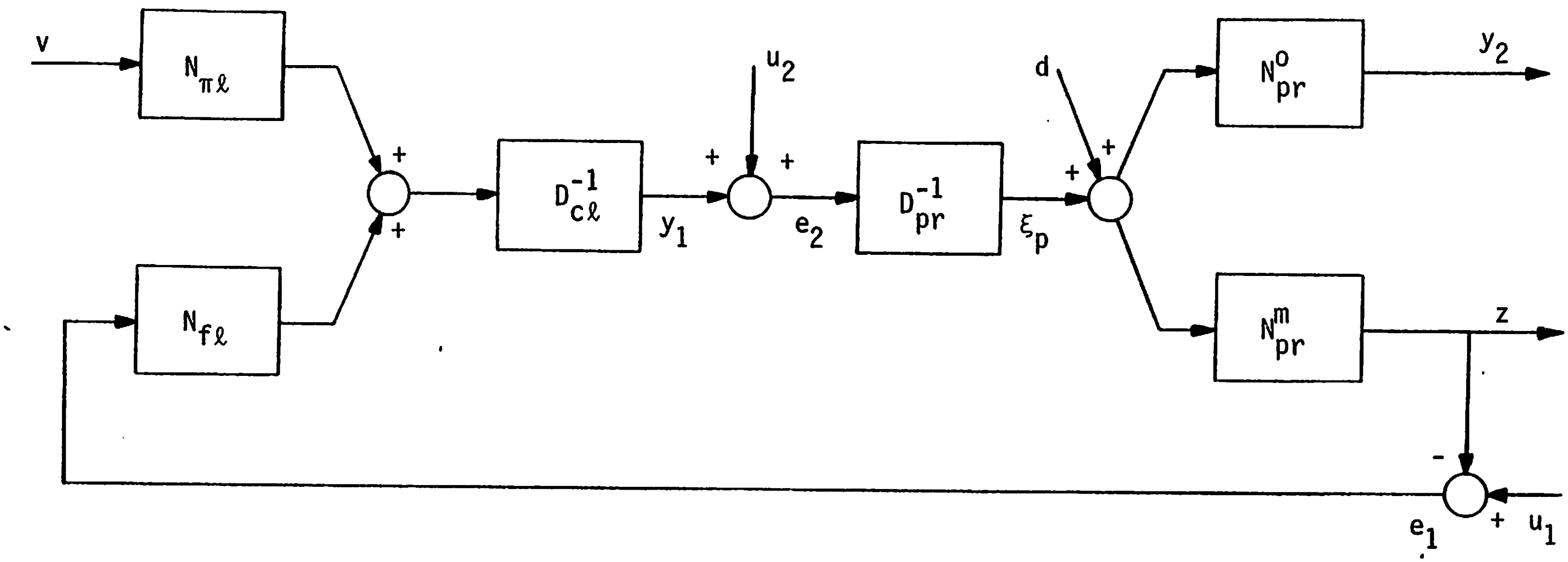
v
$N_{\pi\ell}$
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Fig. 1

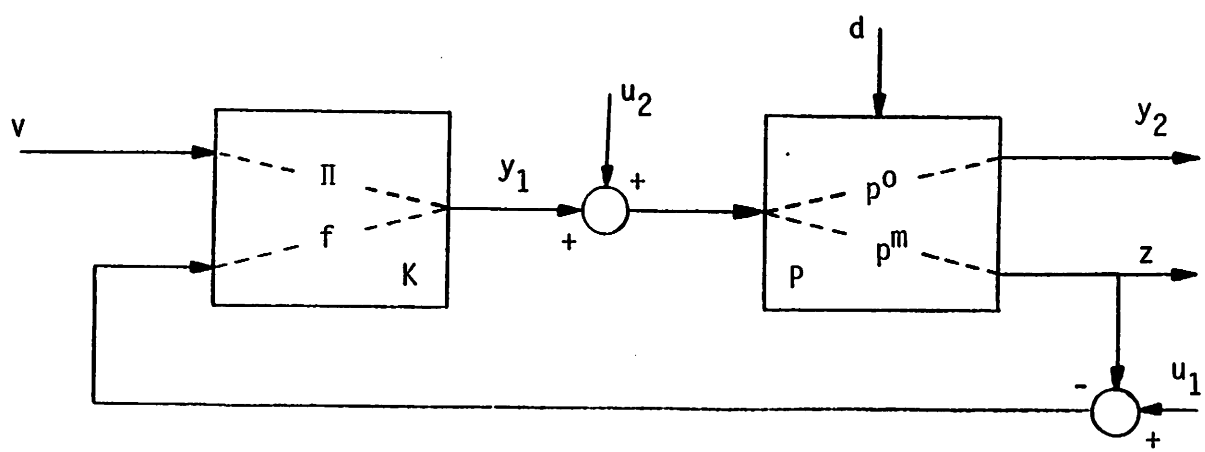
v
Π
f
K
y_1
u_2
+
+
d
p^o
p^m
P
y_2
z
u_1
-
+

Fig. 2