THE ROLE OF NONLINEAR THEORIES IN DYNAMIC ANALYSIS OF ROTATING STRUCTURES

by

J. C. Simo and L. Vu Quoc

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The Role of Nonlinear Theories in Dynamic Analysis of Rotating Structures

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Abstract

It is explicitly shown that modeling of the influence of centrifugal force on the bending stiffness in fast rotating structures necessitates a geometrically nonlinear theory. Explicit partial differential equations (PDE) of motion are derived to demonstrate how linear structural theories result in an unphysical loss of bending stiffness. This loss is quadratic with the angular velocity of revolution. Such a spurious destabilizing phenomenon, however, should not be attributed to a fundamental flaw in linear theories, but merely expresses the inadequacy of geometrically linear formulations to account for nonlinear geometric effects. The correct set of linear PDE's is obtained as a consistent first order linearization of the fully nonlinear theory.

1. Introduction

Recently, it has been pointed out by Kane, Ryan & Banerjee [1986] (KRB) that existing approaches to the dynamics of flexible bodies necessitate fundamental modification in order to capture the centrifugal stiffening effect in fast rotating beams. Our purpose is to show that

(i) Accounting for the stiffening effect in fast rotating structures requires a higher order (geometrically nonlinear) theories, hence necessarily *nonlinear* in the strain measures. A hierarchy of beam theories, from the linear to a fully nonlinear formulation, can be systematically developed by successive approximations in terms of a small perturbation parameter (e.g., Truesdell & Noll [1965], page 219).

(ii) Current approaches based on linearized strain measures are not, by design, conceived to capture such a stiffening effect, nor to account for any other nonlinear phenomena involving change in stiffness due to axial loading. In fact, use of a geometrically linear beam theory in the modeling of a fast rotating beam leads to a spurious loss of bending stiffness, which is quadratic with the angular velocity. This effect was numerically documented in Kane, Ryan & Banerjee [1985]. Herein, this phenomenon is quantified analytically by providing the relevant partial differential equation of motion for the transverse vibration.

(iii) The KRB approach may be viewed as a reparametrization of a higher order beam theory of the von Karman type, along with a subsequent truncation of nonlinear terms. Specifically, in the case of a beam, the axial displacement field is replaced by the elongation along the line of centroids, with the net result of rendering the stiffness matrix identical to that of a linear Timoshenko beam. This approach, however, ignores the effect of axial forces other than those coming from inertia effects.

(iv) A set of linear partial differential equations of motion is derived as a consistent first order linearization of the nonlinear theory. These linear PDE's capture correctly the action of the centrifugal force on the bending stiffness, and in fact, for the von Karman type model, are the exact counterpart of the KRB discrete approach. However, by contrast with the KRB approach, the Galerkin spatial discretization of these PDE's is straightforward. In addition, explicit expressions of the linear semi-discrete equations of motion in the present context are given.

(v) In cases where modeling of the above nonlinear geometric effects is desired, the use of a fully nonlinear beam or plate theory does not involve more computational effort than the use of a higher order nonlinear theory. In fact, by referring the dynamics of the beam directly to the inertial frame, the inertia operator becomes linear, hence simplifying considerably the task of integrating of the equations of motion.

For simplicity in the exposition, but without loss of generality, we shall consider the model problem a rotating beam whose motion is restricted to a plane. The result obtained in this paper can be generalized without difficulty to the three-dimensional motion. After considering in detail the developments pertaining to a fast rotating beam, we show that the conclusion obtained in this one-dimensional case essentially carry over without modification to the more general case of a plate.

2. Consistent higher order theories

Model problem and notation. Consider the rotating beam shown in Figure 1. Let $\{O; \mathbf{e}_1, \mathbf{e}_2\}$ be the inertial frame with base point $O \in \mathbf{R}^2$ and orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$. Let (X_1, X_2) denote the coordinates along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. The domain of the undeformed beam with length L and depth d is $B := [0, L] \times [-\frac{d}{2}, \frac{d}{2}] \subset \mathbf{R}^2$. Here we consider the case where one end of the beam is attached to the origin O, and the other end free. The motion of the beam is assumed to be restricted to the plane $(\mathbf{e}_1, \mathbf{e}_2)$, with a prescribed angle of revolution ψ. Points in the undeformed (reference) configuration, are denoted by $\mathbf{X} = X_1\, \mathbf{e}_1 + X_2\, \mathbf{e}_2$. In addition, we introduce a floating frame $\{O; \mathbf{a}_1, \mathbf{a}_2\}$ that follows the rigid body motion of the beam, often referred to as the shadow beam, or the *locally attached frame* (Canavin & Likins [1977]). The position vector of the deformed line of centroids is given by

$$\mathbf{x}_o := [X_1 + u_1(X_1,t)]\, \mathbf{a}_1(t) + u_2(X_1, t)\, \mathbf{a}_2(t)\,, \tag{1a}$$

where u_1 and u_2 designate the displacement components along the axes $\mathbf{a}_1$ and $\mathbf{a}_2$ of a material point initially located at position $(X_1\, \mathbf{a}_1)$ on the line of centroids. Since the motion is assumed to be planar, we have $\mathbf{e}_3 \equiv \mathbf{t}_3 \equiv \mathbf{a}_3$. The orientation of a cross section is defined by a moving frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ such that

$$\begin{Bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{Bmatrix} = \mathbf{A}^T \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix}, \quad \text{where} \quad \mathbf{A} := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1b}$$

where α denotes the rotation of the beam cross section with respect to the floating frame $\{\mathbf{a}_1, \mathbf{a}_2\}$. The deformation map is thus given by

$$\mathbf{x} = \mathbf{x}_o + X_2\, \mathbf{t}_2 \tag{1c}$$

Higher order beam theories, including the geometrically linear theory, can be systematically obtained by successive approximations of the strain measures for the fully linear theory.

2.1. Fully nonlinear theory. Within the scope of planar motion considered in this paper, a fully nonlinear dynamic beam theory accounting for finite strain — shearing, extension and bending — is given by the system of partial differential equations of motion (Simo & Vu-Quoc [1985])

$$\begin{Bmatrix} A_\rho\,[\ddot{u}_1 - \ddot{\psi}\, u_2 - 2\,\dot{\psi}\,\dot{u}_2 - \dot{\psi}^2\,(X_1 + u_1)] \\ A_\rho\,[\ddot{u}_2 + \ddot{\psi}\,(X_1 + u_1) + 2\,\dot{\psi}\,\dot{u}_1 - \dot{\psi}^2\, u_2)] \\ I_\rho\,(\ddot{\alpha} + \ddot{\psi}\,) \end{Bmatrix} - \mathbf{A} \begin{Bmatrix} N_{,x} - \alpha_{,x}\, V \\ V_{,x} + \alpha_{,x}\, N \\ M_{,x} + \Gamma_1\, V - \Gamma_2\, N \end{Bmatrix} = \mathbf{O} \tag{2}$$

where A_ρ denotes the mass per unit reference length, and I_ρ the mass moment of inertia of the cross section. The superposed "dot" indicates derivative with respect to time t, while the notation $(\,\cdot\,)_{,x}$ corresponds to the first derivative with respect to X_1. In (2), $(N,\ V,\ M)$ represent the axial force, shear force, and bending moment relative to the local frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, respectively, with $(\Gamma_1, \Gamma_2, \kappa)$ being their respective conjugate strain measures such that

$$\begin{aligned} N &= EA\,\Gamma_1 \\ V &= GA\,\Gamma_2\,, \\ M &= EI\,\kappa \end{aligned} \quad \text{and} \quad \begin{Bmatrix} \Gamma_1 \\ \Gamma_2 \\ \kappa \end{Bmatrix} := \mathbf{A}^T \begin{Bmatrix} 1 + u_{1,x} \\ u_{2,x} \\ \alpha_{,x} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{3}$$

where $\mathbf{A}$ is the orthogonal matrix defined earlier in (1b). Recall that, since the function ψ is prescribed, $\dot{\psi}$ and $\ddot{\psi}$ are known functions. We note that the above definition of the strain measures $(\Gamma_1, \Gamma_2, \kappa)$ is unique in the sense that the resulting reduced expended power of the beam is identical to the exact stress power of the three-dimensional continuum theory. The static version of equations (2) was developed in Reissner [1972] for the planar problem, and in Reissner [1973,1981], Antman [1974] from the three dimensional case. The dynamic case along with the development of a computationally suitable paramatrization is treated in Simo [1985]. Successive approximations to the nonlinear theory can be constructed via standard power series expansion in terms of a "small" parameter $\epsilon > 0$. The series expansion of the strain measures defined in (3) are given up to second order by

$$\begin{Bmatrix} \Gamma_{1\epsilon} \\ \Gamma_{2\epsilon} \\ \kappa_\epsilon \end{Bmatrix} := \mathbf{A}(\epsilon\,\alpha)^T \begin{Bmatrix} 1 + \epsilon\, u_{1,x} \\ \epsilon\, u_{2,x} \\ \epsilon\alpha_{,x} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \epsilon \begin{Bmatrix} u_{1,x} \\ u_{2,x} - \alpha \\ \alpha \end{Bmatrix} + \frac{\epsilon^2}{2} \begin{Bmatrix} (u_{2,x})^2 - (u_{2,x} - \alpha)^2 \\ -2\alpha\, u_{1,x} \\ 0 \end{Bmatrix} + \mathbf{O}(\epsilon^3) \tag{4}$$

In what follows, we denote the first order (ϵ) approximation to the nonlinear strain measures by ($\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\kappa}$), and the second order (ϵ^2) approximation by ($\bar{\bar{\Gamma}}_1$,$\bar{\bar{\Gamma}}_2$,$\bar{\bar{\kappa}}$); for example, $\bar{\Gamma}_1 = u_{1,x}$ and $\bar{\bar{\Gamma}}_1 = u_{1,x} + [(u_{2,x})^2 - (u_{2,x} - \alpha)^2]/2$. Clearly, ($\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\kappa}$) are the usual strain measures employed in the linear Timoshenko beam theory.

2.2. Linear (first order) beam theory. The equations of motion corresponding to a geometrically linear beam theory are obtained simply by retaining the first order approximation in (4):

$$\begin{Bmatrix} A_\rho\, [\ddot{u}_1 - \ddot{\psi}\, u_2 - 2\, \dot{\psi}\, \dot{u}_2 - \dot{\psi}^2\, (X_1 + u_1)] \\ A_\rho\, [\ddot{u}_2 + \ddot{\psi}\, (X_1 + u_1) + 2\, \dot{\psi}\, \dot{u}_1 - \dot{\psi}^2\, u_2)] \\ I_\rho\, (\ddot{\alpha} + \ddot{\psi}\,) \end{Bmatrix} - \begin{Bmatrix} \bar{N}_{,x} \\ \bar{V}_{,x} \\ \bar{M}_{,x} + \bar{V} \end{Bmatrix} = \mathbf{O} \tag{5a}$$

where the first column corresponds to the inertia operator, and the second column to the stiffness operator. In (5a), the internal forces (N, V, M) are given by the usual linear constitutive law

$$\begin{aligned} \bar{N} &= EA\, \bar{\Gamma}_1 = EA\, u_{1,x} \\ \bar{V} &= GA_s\, \bar{\Gamma}_2 = GA_s\, (u_{2,x} - \alpha) \\ \bar{M} &= EI\, \bar{\kappa} = EI\, \alpha_{,x} \end{aligned} \tag{5b}$$

To see the effect of centrifugal force on the bending deformation of the beam, we differentiate $(5a)_3$ and make use of $(5a)_2$ to obtain the equation of motion for the transverse displacement,

$$A_\rho\, \ddot{u}_2 + EI\, u_{2,xxxx} - A_\rho\, \dot{\psi}^2 u_2 \;=\; -2A_\rho\, \dot{\psi}\, \dot{u}_1\,, \tag{6}$$

in which, for simplicity, we have made the assumption of steady state revolution ($\ddot{\psi} = 0$) and negligible shear deformation, $u_{2,x} = \alpha + O(\epsilon)$. The destabilizing effect due to the use of linear beam theory mentioned earlier can now be clearly identified: The term ($-A_\rho\, \dot{\psi}^2\, u_2$) induces a loss of stiffness, which is quadratic in the angular velocity of revolution $\dot{\psi}$. This observation is indeed corroborated by the numerical experiments in Kane *et al* [1985, Figure 8]. Note that only the transverse component along $\mathbf{a}_2$ of the centrifugal force in the shear equation $(5)_2$, represented by the term $(-\dot{\psi}\, u_2)$, is transferred to the bending equation $(5a)_3$. The contribution of the axial component along $\mathbf{a}_1$ of the centrifugal force, represented by the term $[\dot{\psi}^2\,(X_1 + u_1)]$, on the other hand, exerts no influence on the bending. It should be noted here that this term is in fact of order 1, while the term $(-\psi\, u_2)$ is of order ϵ. Thus, from a physical standpoint, the loss of stiffness results precisely from this partial transfer of the action of centrifugal force to the bending equation. Moreover, there is a value of the angular velocity of revolution that renders the stiffness matrix singular.

2.3. Second order beam theory. A second order theory can be consistently derived by retaining second order terms in the approximation to the strain measures, according to

$$\begin{Bmatrix} \bar{\bar{\Gamma}}_1 \\ \bar{\bar{\Gamma}}_2 \\ \bar{\bar{\kappa}} \end{Bmatrix} = \begin{Bmatrix} u_{1,x} \\ u_{2,x} - \alpha \\ \alpha \end{Bmatrix} + \begin{Bmatrix} \alpha\, u_{2,x} - \frac{1}{2}\alpha^2 \\ -\alpha\, u_{1,x} \\ 0 \end{Bmatrix} . \tag{7}$$

In addition, the second order approximation to the equations of motion (2) now takes the form

$$\begin{Bmatrix} A_\rho\,[\ddot{u}_1 - \ddot{\psi}\, u_2 - 2\,\dot{\psi}\,\dot{u}_2 - \dot{\psi}^2\,(X_1 + u_1)] \\ A_\rho\,[\ddot{u}_2 + \ddot{\psi}\,(X_1 + u_1) + 2\,\dot{\psi}\,\dot{u}_1 - \dot{\psi}^2\, u_2)] \\ I_\rho\,(\ddot{\alpha} + \ddot{\psi}\,) \end{Bmatrix} - \begin{Bmatrix} \bar{\bar{N}}_{,x} \\ \bar{\bar{V}}_{,x} \\ \bar{\bar{M}}_{,x} + \bar{\bar{V}} \end{Bmatrix} + \begin{Bmatrix} (\alpha\, \bar{V})_{,x} \\ -(\alpha\, \bar{N})_{,x} \\ -\bar{\Gamma}_1\, \bar{V} + \bar{\Gamma}_2\, \bar{N} \end{Bmatrix} = \mathbf{O}\,, \tag{8}$$

where $\bar{\bar{N}} = EA\,\bar{\bar{\Gamma}}_1$, $\bar{\bar{V}} = GA_s\,\bar{\bar{\Gamma}}_2$ and $\bar{\bar{M}} = EI\,\alpha_{,x}$. To obtain the equation governing the transverse vibration u_2, we proceed as follows: (i) Make use of $(8)_1$ to express $\bar{N}_{,x}$ in terms of the $\mathbf{a}_1$ component of the inertia force, (ii) Substitute the result into $(8)_2$ and solve for $\bar{\bar{V}}_{,x}$, and (iii) Differentiate $(8)_3$ and make use the expression for $\bar{\bar{V}}_{,x}$ obtained previously in (ii). Observe that the procedure is analogous to that employed in the first order approximation. The only crucial difference here is that the axial component of the inertia force along the axis $\mathbf{a}_1$ is now transferred to the bending equation due to the presence of the term $(\alpha\,\bar{N})_{,x}$ in $(8)_2$. This term accounts for the contribution to the transverse momentum of the axial (along $\mathbf{a}_1$) forces in the beam. Again, as in previous section, neglecting higher order terms $O(\epsilon^2)$ in the final equation, considering constant angular velocity of revolution, and assuming for simplicity negligible shear deformation, one obtains

$$A_\rho \ddot{u}_2 + EI\, u_{2,xxxx} + A_\rho\, \dot{\psi}^2 (X_1\, u_{2,x} - u_2) = -2\, A_\rho\, \dot{\psi}\, \dot{u}_1 \tag{9}$$

Note that equation (9), resulting from the foregoing second order approximation, is substantially different than its counterpart equation (6), which results from the first order approximation to the nonlinear theory. Now both components of the centrifugal force are completely transferred to the bending equation: The term $(A_\rho \dot{\psi}^2 X_1\, u_{2,x})$ in (9) dominates the term $(A_\rho\, \dot{\psi}^2\, u_2)$ — the latter is the only term present in (6) — and appropriately accounts for the stiffening effect due to centrifugal force. Conceptually, the transferring of the action of axial load to the bending equation is analogous to the effect of axial force in the linearized buckling analysis (e.g., beam-column equation); the only difference being the dynamic origin of the axial loading.

2.4. Consistent linear partial differential equations. We shall obtain the first order partial differential equations governing the motion of the beam by consistent linearization of (8). Before truncating the terms of order ϵ^2, it is crucial to note that the term $(\alpha\, N_{,x})$ in $(8)_2$ is actually of first order (ϵ), and not of second order (ϵ^2). It follows from the equation for axial vibration $(8)_1$ that

$$\alpha\, N_{,x} = -\alpha\, (A_\rho\, \dot{\psi}\, X_1) + O(\epsilon) \tag{10}$$

and therefore must be retained in the first order approximation to the nonlinear equations of motion (8). After regrouping terms according to their nature, we obtain the following linear PDE's

$$\begin{Bmatrix} A_\rho\, \ddot{u}_1 \\ A_\rho\, \ddot{u}_2 \\ I_\rho\, \ddot{\alpha} \end{Bmatrix} + \begin{Bmatrix} -2A_\rho\, \dot{\psi}\, \dot{u}_2 \\ 2A_\rho\, \dot{\psi}\, \dot{u}_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -EA\, u_{1,xx} - A_\rho\, (\dot{\psi}^2\, u_1 + \ddot{\psi}\, u_2) \\ -GA_s\, (u_{2,x} - \alpha)_{,x} + A_\rho\, (\ddot{\psi}\, u_1 - \dot{\psi}^2\, u_2 + \dot{\psi}^2\, X_1\, \alpha) \\ -EI\, \alpha_{,xx} - GA_s\, (u_{2,x} - \alpha) \end{Bmatrix} = \begin{Bmatrix} A_\rho\, \dot{\psi}^2\, X_1 \\ -A_\rho\, \ddot{\psi}\, X_1 \\ -I_\rho\, \ddot{\psi} \end{Bmatrix} \tag{11}$$

where the 4 columns correspond respectively to the inertia, gyroscopic, material stiffness and geometric stiffness, and inertia force due to revolution effects.

The Galerkin spatial discretization of the linear PDE's (11) is standard. For completeness, we shall simply give, without derivation, the expressions of the matrices resulting from applying a Galerkin finite element method. Such procedure has been applied to the spatial discretization of a fully nonlinear beam model (Simo & Vu-Quoc [1985,86]). Upon defining the following quantities,

$$\mathbf{d} := (u_1,\, u_2,\, \alpha)\ , \qquad \mathbf{I} := Diag[A_\rho,\, A_\rho,\, I_\rho]\ , \tag{12a}$$

$$\mathbf{g} := 2\, A_\rho\, \dot{\psi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Psi} := \begin{bmatrix} \dfrac{d}{dX_1} & 0 & 0 \\ 0 & \dfrac{d}{dX_1} & -1 \\ 0 & 0 & \dfrac{d}{dX_1} \end{bmatrix} \tag{12b}$$

$$\mathbf{C} := Diag[EA,\ GA_s,\ EI]\ , \qquad \mathbf{B}_1 := A_\rho \begin{bmatrix} -\dot{\psi}^2 & -\ddot{\psi} & 0 \\ \ddot{\psi} & -\dot{\psi}^2 & (\dot{\psi}^2\, X_1) \\ 0 & 0 & 0 \end{bmatrix}, \tag{12c}$$

and introducing the discretization,

$$\mathbf{d}(X_1,t) \cong \sum_{I=1}^{\nu} [N_I(X_1)\ \mathbf{1}]\ \mathbf{q}_I(t)\ , \quad \text{with } \mathbf{1} := Diag[1,\ 1,\ 1] \tag{13}$$

the resulting linear semi-discrete equations of motion can be written as

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{D}\,\dot{\mathbf{q}} + [\mathbf{S} + \mathbf{G}]\,\mathbf{q} = \mathbf{P} \tag{14a}$$

where **M** designates the mass matrix, **D** the gyroscopic matrix, **S** the material stiffness matrix, **G** the dynamic geometric stiffness matrix, and **P** the applied force; **q** denotes the vector of all generalized coordinates. It is easy to verify that the following expressions for the (3×3) submatrices coupling the generalized coordinates $\mathbf{q}_I$ to $\mathbf{q}_J$ hold

$$\mathbf{M}_{IJ} = \int_{[0,L]} [N_I(X_1)\ \mathbf{1}] \bullet \mathbf{I}\ [N_J(X_1)\ \mathbf{1}]\ dX_1\ , \tag{14b}$$

$$\mathbf{D}_{IJ} = \int_{[0,L]} [N_I(X_1)\ \mathbf{1}] \bullet \mathbf{g}\ [N_J(X_1)\ \mathbf{1}]\ dX_1\ , \tag{14c}$$

$$\mathbf{S}_{IJ} = \int_{[0,L]} \boldsymbol{\Psi}\ [N_I(X_1)\ \mathbf{1}] \bullet \mathbf{C}\ \boldsymbol{\Psi}\ [N_J(X_1)\ \mathbf{1}]\ dX_1\ , \tag{14d}$$

$$\mathbf{G}_{IJ} = \int_{[0,L]} [N_I(X_1)\ \mathbf{1}] \bullet \mathbf{B}_1\ [N_J(X_1)\ \mathbf{1}]\ dX_1\ . \tag{14e}$$

Observe the non-symmetry of the matrices in (14), except for the mass matrix **M** and the material stiffness matrix **S**. In addition, we note that the geometric stiffness **G** comes from purely dynamic origin.

In what follows, we shall interpret the KRB approach in the context of a similar setting.

3. The Kane-Ryan-Banerjee approach

We shall re-examine the discrete approach proposed in Kane, Ryan & Banerjee [1985] and, by deriving the appropriate PDE, show that this approach essentially amounts to a re-parametrization of a nonlinear structural model of the von Karman type. These authors consider a potential energy function given by

$$\Pi := \int_{[0,L]} [EA(s_{,x})^2 + GA_s(u_{2,x} - \alpha)^2 + EI(\alpha_{,x})^2]\ dX_1 \tag{15a}$$

where $s_{,x}$ denotes the partial derivative of the elongation of the center line with respect to X_1, and is given by

$$s(X_1, t) = \int_0^{X_1} \sqrt{J(X_1, t)}\, dX_1 - X_1 \,, \quad J(X_1, t) := [(1 + u_{1,x})^2 + (u_{2,x})^2] \,,$$

$$\text{and} \quad s_{,x}(X_1,t) = \sqrt{(1 + u_{1,x})^2 + (u_{2,x})^2} - 1 \tag{15b}$$

The essential feature that distinguishes expression (15a) from its counterpart in the linearized theory is the use of $s_{,x}$ instead of $\bar{\Gamma}_1 = u_{1,x}$ in the contribution of the axial strain to the potential energy Π of the system. By the change of variable, $\sigma = \bar{\sigma}(X_1) := X_1 + u_1(X_1, t)$, the elongation $s(X_1, t)$ given by (15b) can be recast into the form

$$s(X_1, t) = \int_0^{X_1 + u_1(X_1, t)} \left[1 + \left(\frac{\partial u_2(\bar{\sigma}^{-1}(\sigma), t)}{\partial \sigma} \right)^2 \right]^{\frac{1}{2}} d\sigma - X_1 \,, \tag{16}$$

where we have assumed the boundary condition $u_1(0, t) \equiv 0$. Relation (16) is the one essentially used in Kane *et al* [1985] with an additional assumption that $u_2(\bar{\sigma}^{-1}(\sigma),t) \equiv u_2(\sigma,t)$. On the other hand, the kinetic energy of the system is given by

$$K = \frac{1}{2} \int_{[0,\, L]} A_\rho \{ [\dot{u}_1 - \dot{\psi}\, \dot{u}_2]^2 + [\dot{u}_2 + \dot{\psi}\,(X_1 + \dot{u}_1)]^2 \}\, dX_1$$

$$+ \frac{1}{2} \int_{[0,\, L]} I_\rho\, [\dot{\alpha} + \dot{\psi}]^2\, dX_1 \tag{17}$$

It should be noted that the same expression for K holds in the nonlinear theory. The Lagrangian of the system is given by $L := K - \Pi$. Note that $\{u_1, u_2, \alpha\}$ are the independent variables in L. However, in place of u_1, Kane *et al* choose to select the *elongation* $s(X_1, t)$ of the line of centroids as independent variable. The basic variables in the KRB approach are thus (s, u_2, α), u_1 being implicitly defined in terms of (s, u_2) using the nonlinear relation $s_{,x} = \sqrt{J} - 1$, where J is in turn defined in terms of (u_1, u_2) by (15b). An explicit expression of the resulting system of PDE's is difficult to obtain because of the complexity of the inertia operator. These authors proceed numerically and derive linear semi-discrete system of equations, $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}$ by first introducing the discretization

$$\begin{aligned} s(X_1, t) &\cong \sum_{I=1}^{\nu} N_{1I}(X_1) q_{1I}(t) \\ u_2(X_1, t) &\cong \sum_{I=1}^{\nu} N_{2I}(X_1) q_{2I}(t) \,. \\ \alpha(X_1, t) &\cong \sum_{I=1}^{\nu} N_{3I}(X_1) q_{3I}(t) \end{aligned} \tag{18}$$

where $N_{1l}(X_1)$ are prescribed independent basis functions, such as the eigenfunctions of a cantilever beam, and then linearizing the resulting nonlinear inertia operator. Recall that in discretizing (16), the additional assumption that $\dfrac{\partial u_2(\bar{\sigma}^{-1}(\sigma), t)}{\partial \sigma} = \sum_{l=1}^{\nu} N_{2l,x}(\sigma) q_{2l}(t)$ is made.

To show that the KRB approach outlined above amounts to employing a geometrically nonlinear theory, we obtain below the system of governing PDE's in the variables (u_1, u_2, α). Making use of Hamilton's principle along with the expression (15b) for $s_{,x}$ in terms of (u_1, u_2) we obtain, after standard manipulation, the system

$$\left\{ \begin{array}{c} A_\rho \, [\ddot{u}_1 - \ddot{\psi} \, u_2 - 2 \, \dot{\psi} \, \dot{u}_2 - \dot{\psi}^2 \, (X_1 + u_1)] \\ A_\rho \, [\ddot{u}_2 + \ddot{\psi} \, (X_1 + u_1) + 2 \, \dot{\psi} \, \dot{u}_1 - \dot{\psi}^2 \, u_2)] \\ I_\rho \, (\ddot{\alpha} + \ddot{\psi}) \end{array} \right\} - \left\{ \begin{array}{c} EA \, [(1 - J^{-\frac{1}{2}}) \, (1 + u_{1,x})]_{,x} \\ GA_s \, (u_{2,x} - \alpha)_{,x} - EA \, [(1 - J^{-\frac{1}{2}}) \, u_{2,x}]_{,x} \\ EI \, \alpha_{,xx} - GA_s \, (u_{2,x} - \alpha) \end{array} \right\} = 0 \quad (19)$$

It should be noted that equations (19) are *nonlinear* in the stiffness operator, and closely related to the von Karman second order model. Conceptually, by using relation (15b), one could recast this system of equations in terms (s, u_2, α). To see this, we introduce the perturbation parameter $\epsilon > 0$. Assuming that $\{u_1, u_2, \alpha\}$ are of order ϵ, by expanding $s(X_1, t)$ in powers of ϵ, we find

$$s_{\epsilon,x} := [(1 + \epsilon \, u_{1,x})^2 + (\epsilon \, u_{2,x})^2]^{\frac{1}{2}} - 1 = \epsilon \, u_{1,x} + \frac{\epsilon^2}{2} (u_{2,x})^2 + O(\epsilon^3) \qquad (20)$$

Thus $s_{,x}$ agrees with the consistent second order strain $\bar{\bar{\Gamma}}_1$ only if shear deformation is of second order, i.e., $u_{2,x} = \alpha + O(\epsilon)$. In addition, we have the following expansions

$$(1 - J_\epsilon^{-\frac{1}{2}}) \, (1 + \epsilon \, u_{1,x}) = \epsilon \, u_{1,x} + \frac{\epsilon^2}{2} (u_{2,x})^2 + O(\epsilon^3) \qquad (21a)$$

$$EA \, [(1 - J_\epsilon^{-\frac{1}{2}}) \, \epsilon \, u_{2,x}]_{,x} = \epsilon^2 \, EA \, u_{1,xx} \, u_{2,x} + O(\epsilon^3) \qquad (21b)$$

The term (21b) is precisely the one responsible for the transfering of the axial force acting on the beam to the transverse equilibrium.

Since the direct contribution of the axial component along $\mathbf{a}_1$ of the centrifugal force to the transverse equilibrium given by $(EA \, u_{1,xx} \, \alpha)$ is absent from the KRB approach — here (s, u_2, α) are chosen as independent variables — the question arises as to how centrifugal stiffening is accounted for in this formulation. This is accomplished through the inertia operator by expressing u_1 and $\dot{u}_1$ in terms of (s, u_2) and their derivatives with the aid of the nonlinear relation (16). Upon introducing the discretization (18), the resulting discrete nonlinear inertia operator is then linearized to obtain the linear semi-discrete equations of motion.

Remarkably enough, we obtain from equations (19) exactly the same PDE's for the transverse vibration (9) after some manipulation similar to what described in Section 2.3. This result therefore shows that the use of the nonlinear von Karman type model can also appropriately account for the action of the centrifugal force to the transverse vibration as manifest in the expression for the first order dynamic geometric stiffness. From the expansion (21b) of the term $(EA\ [(1 - J^{-\frac{1}{2}})\ u_{2,x}]_{,x})$ in $(19)_2$, and from the equation for axial vibration $(19)_1$, we note that this term is in fact of first order (ϵ), similar to (10), i.e.,

$$EA\ [(1 - J_\epsilon^{-\frac{1}{2}})\ \epsilon\ u_{2,x}]_{,x} = -\epsilon\ A_\rho\ \dot{\psi}\ X_1\ u_{2,x} + O(\epsilon^2)\ . \tag{22}$$

Note that instead of α in the consistent approximation from fully nonlinear theory, we obtain $u_{2,x}$ in this von Karman type model. This is valid only when shear deformation is of second order, i.e., $u_{2,x} = \alpha + O(\epsilon)$. We then arrive at the following linear PDE's,

$$\begin{Bmatrix} A_\rho\, \ddot{u}_1 \\ A_\rho\, \ddot{u}_2 \\ I_\rho\, \ddot{\alpha} \end{Bmatrix} + \begin{Bmatrix} -2A_\rho\, \dot{\psi}\, \dot{u}_2 \\ 2A_\rho\, \dot{\psi}\, \dot{u}_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -EA\ u_{1,xx} - A_\rho\,(\dot{\psi}^2\, u_1 + \ddot{\psi}\, u_2) \\ -GA_s\,(u_{2,x} - \alpha)_{,x} + A_\rho\,(\ddot{\psi}\, u_1 - \dot{\psi}^2\, u_2 + \dot{\psi}^2\, X_1\, u_{2,x}) \\ -EI\ \alpha_{,xx} - GA_s\,(u_{2,x} - \alpha) \end{Bmatrix} = \begin{Bmatrix} A_\rho\, \dot{\psi}^2\, X_1 \\ -A_\rho\, \ddot{\psi}\, X_1 \\ -I_\rho\, \ddot{\psi} \end{Bmatrix} \tag{23}$$

as a first order consistent linearization of (19). This system is entirely equivalent to the KRB discrete approach. The only difference, as noted above, is that the dynamic geometric stiffness operator $\mathbf{B}_1$ in (12c) must now be redefined as

$$\mathbf{B}_2 := A_\rho \begin{bmatrix} -\dot{\psi}^2 & -\ddot{\psi} & 0 \\ \ddot{\psi} & [-\dot{\psi}^2 + \dot{\psi}^2\, X_1\, \dfrac{d}{dX_1}] & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

and hence a slight change in the dynamic geometric stiffness matrix $\mathbf{G}$ in (14e). The other matrices — mass, gyroscopic, material stiffness — remain the same as obtained in Section 2.4. One can easily verify the correspondence of the terms in the discrete equations of motion resulting from the linear PDE's (23) to those given in Kane *et al* [1985]. In addition, when there is no dynamic effects, the linear model governed by the PDE's (23) reduces exactly to the Timoshenko beam theory. It should be pointed out, however, that the choice of $s_{,x}$ and $(u_{2,x} - \alpha)$ as axial and shear strain measures does not agree with the consistent second order strain measures $\bar{\bar{\Gamma}}_1$ and $\bar{\bar{\Gamma}}_2$ unless $u_{2,x} = \alpha + O(\epsilon)$ (negligible shear deformation).

4. Extension to plate formulation

As a direct application of the foregoing discussion, we shall extend the results to the case of a plate undergoing three-dimensional rotating motion. Again, for clarity, we assume that the axis of revolution of the plate passes through an inertially fixed material point of the plate. The dynamics of this revolution is completely prescribed *a priori*; the orientation the axis of revolution, however, need not be fixed with respect to the inertial frame.

Model problem and notation. Consider the material frame $\{O;\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3\}$ with base point $O \in \mathbf{R}^3$ and an orthonormal basis $\{\mathbf{E}_\mathrm{I}\}$. Let the inertial frame be $\{O;\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ such that $\mathbf{e}_k \equiv \mathbf{E}_k$, for k=1,2,3. Coordinates with respect to $\{\mathbf{E}_\mathrm{I}\}$ are denoted by $(X_1,\ X_2,\ X_3)$; coordinates with respect to $\{\mathbf{e}_i\}$ are denoted by $(x_1,\ x_2,\ x_3)$. The domain of the undeformed plate is defined to be $B := \Omega\times[\frac{d}{2},\ \frac{d}{2}]$ with $O \in B$ and such that a point $X \in B$ has coordinates $(X_1,\ X_2) \in \Omega \subset \mathbf{R}^2$ and $X_3 \in [\frac{d}{2},\ \frac{d}{2}]$. Consider now a floating frame $\{O;\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\}$, attached to the deformed plate, and whose dynamics with respect to the inertial (material) frame $\{\mathbf{E}_\mathrm{I}\}$ is completely prescribed by an orthogonal matrix $\mathbf{Q}(t)$ such that $\mathbf{a}_\mathrm{I}(t) = \mathbf{Q}(t)\,\mathbf{E}_\mathrm{I}$. The map $t\rightarrow\mathbf{Q}(t)$ in fact describes the rigid body rotation of the (undeformed) plate about the origin O. The deformation of the plate relative to the floating frame $\{\mathbf{a}_\mathrm{I}\}$ is then given by

$$
\begin{aligned}
\mathbf{x} &= \mathbf{x}_o + X_3\,\mathbf{t}_3 \\
\mathbf{x}_o &:= \sum_{\gamma=1,2} [X_\gamma + u_\gamma(X_1,\ X_2,\ t)]\,\mathbf{a}_\gamma + u_3(X_1,\ X_2,\ t)\,\mathbf{a}_3\ ,\ddagger \qquad (25)\\
\mathbf{t}_3(X_1,\ X_2,\ t) &= \mathbf{\Lambda}(X_1,\ X_2,\ t)\,\mathbf{a}_3(t)
\end{aligned}
$$

where $(u_1,\ u_2\ ,\ u_3)$ are the displacement components of a point $X \in B$; $\mathbf{t}_3$ designates the normal to the deformed plate, and $\mathbf{\Lambda}$ an orthogonal transformation.

Consistent second order strain measures. It can be shown that up to second order, the two-dimensional counterpart of the one-dimensional strain measures in (7) are given by

$$\bar{\Gamma}_{\gamma\beta} = u_{[\gamma,\beta]} + \frac{1}{2}\,u_{3,\gamma}\,u_{3,\beta} - \frac{1}{2}\,(u_{3,\gamma} - \alpha_\gamma)\,(u_{3,\beta} - \alpha_\beta)\ , \qquad (26a)$$

$$\bar{\Gamma}_{3\beta} = (u_{3,\beta} - \alpha_\beta) - u_{\gamma,\beta}\,\alpha_\gamma\ , \qquad (26b)$$

$$\bar{\kappa}_{\gamma\beta} = \alpha_{[\gamma,\beta]}\ , \qquad (26c)$$

for $\gamma,\ \beta = 1,2$, where we have used the notation $u_{[\gamma,\beta]} := \frac{1}{2}(u_{\gamma\beta} + u_{\beta\gamma})$. Note that the strain measures in (26a) reduces exactly to the in-plane strain measures

‡ In what follows, subscripts in greek letters take values in {1,2}, while subscripts in roman letters take values in {1,2,3}.

of the von Karman plate model,

$$\bar{\bar{\Gamma}}_{\gamma\beta} = u_{[\gamma,\beta]} + \frac{1}{2}\, u_{3,\gamma}\, u_{3,\beta}\,, \tag{27}$$

with the assumption of negligible shear deformation, $u_{3,\beta} = \alpha_\beta + O(\epsilon)$. This is entirely analogous to the one-dimensional case of the beam considered in previous sections. Further, we recall that the first order strain measures are $\bar{\Gamma}_{\gamma\beta} = u_{[\gamma,\beta]}$, $\bar{\Gamma}_{3\beta} = u_{3,\beta} - \alpha_\beta$, and $\bar{\kappa}_{\gamma\beta} \equiv \bar{\bar{\kappa}}_{\gamma\beta}$.

Constitutive laws. The elastic material internal forces $\bar{\bar{N}}_{\gamma\beta}$, $\bar{\bar{V}}_\gamma$, and moments $\bar{\bar{M}}_{\gamma\beta}$ are related to the strain measures (26) by a functional relation analogous to that of classical small deformation plate theory. That is, one assumes

$$\bar{\bar{N}}_{\gamma\beta} = \frac{Ed}{1-\nu^2}\,[\nu\, \bar{\bar{\Gamma}}_{\rho\rho}\, \delta_{\gamma\beta} + (1-\nu)\, \bar{\bar{\Gamma}}_{\gamma\beta}] \tag{28a}$$

$$\bar{\bar{V}}_\gamma = GA_\gamma\, \bar{\bar{\Gamma}}_{3\gamma} \tag{28b}$$

$$\bar{\bar{M}}_{\gamma\beta} = \frac{Ed^3}{12\,(1-\nu^2)}\,[\nu\, \bar{\bar{\Gamma}}_{\rho\rho}\, \delta_{\gamma\beta} + (1-\nu)\, \bar{\bar{\kappa}}_{\gamma\beta}] \tag{28c}$$

Here, E represents the Young's modulus, ν the Poisson's ratio, G the shear modulus, and A_γ may be taken to be $\frac{5}{6}d$. The same relationship holds for the first order internal forces $\bar{N}_{\gamma\beta}$, $\bar{V}_\gamma$, and internal moments $\bar{M}_{\gamma\beta}$ in terms of the first order strain measures $(\bar{\Gamma}_{\gamma\beta}, \bar{\Gamma}_{3,\gamma}, \kappa_{\gamma\beta})$.

Equations of motion. One can show that following system partial differential equations, analogous to (8), furnish the consistent second order approximation to the fully nonlinear equations of motion

$$\begin{aligned} F_\beta^{inertia} - \bar{\bar{N}}_{\beta\gamma,\gamma} + (\alpha_\gamma\, \bar{V}_\gamma)_{,\beta} &= 0\,, \\ F_3^{inertia} - \bar{\bar{V}}_{\beta,\beta} - (\alpha_\gamma\, \bar{N}_{\beta\gamma})_{,\beta} &= 0\,, \\ T_\beta^{inertia} - \bar{\bar{M}}_{\beta\gamma,\gamma} - \bar{\bar{V}}_\beta - \bar{\Gamma}_{\beta\gamma}\, \bar{V}_\gamma + \bar{\Gamma}_{3\gamma}\, \bar{N}_{\beta\gamma} &= 0 \end{aligned} \tag{29}$$

where $\mathbf{F}^{inertia} := F_i^{inertia}\, \mathbf{a}_i$ denotes the inertia operator for the translational part of the equations of motion, and $\mathbf{T}^{inertia} := T_i^{inertia}\, \mathbf{a}_i$ the inertia operator for the rotational part. To evaluate $\mathbf{F}^{inertia}$ and $\mathbf{T}^{inertia}$, one proceed as follows. Let $\mathbf{u} := u_i\, \mathbf{a}_i$, thus $\mathbf{x}_o := X_\gamma\, \mathbf{a}_\gamma + \mathbf{u}$, and define the angular velocity of the floating frame $\{\mathbf{a}_i\}$ relative to the inertial frame as

$$\mathbf{w} := w_i\, \mathbf{a}_i\,, \quad \text{such that} \quad \dot{\mathbf{Q}} = \mathbf{Q}\, \check{\mathbf{w}}\,, \tag{30a}$$

where $\check{\mathbf{w}}$ is a skew-symmetric tensor with components relative to $\{\mathbf{a}_i\}$ given by

$$\check{\mathbf{w}} = \check{w}_{ij}\, \mathbf{a}_i \otimes \mathbf{a}_j\,, \quad \text{and} \quad [\check{w}_{ij}] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{30b}$$

Let $\mathbf{L}$ be the linear momentum per unit of mid-surface area. Using the kinematic assumption (25) it follows that

$$\mathbf{L} := \int_{[-\frac{d}{2},\, \frac{d}{2}]} [\dot{\mathbf{x}}_o + X_3\, \dot{\mathbf{t}}_3]\, dX_3 = A_\rho\, [\overset{\nabla}{\mathbf{u}} + \mathbf{w} \times \mathbf{x}_o] \tag{31a}$$

where A_ρ now denotes the mass per unit of undeformed area, and a superposed "∇" represents time differentiation keeping fixed the floating frame $\{\mathbf{a}_i\}$. The inertia force $\mathbf{F}^{inertia}$ is then given by

$$\mathbf{F}^{inertia} = \dot{\mathbf{L}} \equiv A_\rho\, [\overset{\nabla\nabla}{\mathbf{u}} + \dot{\mathbf{w}} \times \mathbf{x}_o + 2\mathbf{w} \times \overset{\nabla}{\mathbf{u}} + \mathbf{w} \times (\mathbf{w} \times \mathbf{x}_o)]\ . \tag{31b}$$

Similarly, the couple $\mathbf{T}^{inertia}$ is obtained from the angular momentum per unit of mid-surface $\mathbf{H} := \int_{[-\frac{d}{2},\, \frac{d}{2}]} (\mathbf{x} - \mathbf{x}_o) \times \dot{\mathbf{x}}\; dX_3$ as $\mathbf{T}^{inertia} = \dot{\mathbf{H}}$. The expression for $\mathbf{T}^{inertia}$ is conveniently expressed in terms of the vector $\boldsymbol{\alpha} := \alpha_\gamma\, \mathbf{a}_\gamma$ that defines the infinitesimal rotation of the normal $\mathbf{t}_3$ of the plate. Note that $\alpha_3 \equiv 0$, i.e., there is no rotational degree of freedom along the axis $\mathbf{a}_3$ in the classical Mindlin-Reissner plate theory. In addition, let $\mathbf{J}_\rho$ denote the inertia dyadic of a transverse (undeformed) fiber of the plate: $\mathbf{r} := X_3\, \mathbf{a}_3$, where $X_3 \in [-\frac{d}{2}, \frac{d}{2}]$. By definition, we have

$$\mathbf{J}_\rho := \int_{[-\frac{d}{2},\, \frac{d}{2}]} [\|\mathbf{r}\|^2\, \mathbf{1} - \mathbf{r} \otimes \mathbf{r}]\; dX_3 = \frac{\rho\, d^3}{12}\, [\mathbf{1} - \mathbf{a}_3 \otimes \mathbf{a}_3]\ . \tag{32b}$$

It can be shown that the rate of angular momentum $\dot{\mathbf{H}}$ takes the form

$$\mathbf{T}^{inertia} = \dot{\mathbf{H}} \equiv [\mathbf{J}_\rho\, (\overset{\nabla\nabla}{\boldsymbol{\alpha}} + \mathbf{w} \times \overset{\nabla}{\boldsymbol{\alpha}} + \dot{\mathbf{w}}) + (\overset{\nabla}{\boldsymbol{\alpha}} + \mathbf{w}) \times \mathbf{J}_\rho\, (\overset{\nabla}{\boldsymbol{\alpha}} + \mathbf{w})]\ , \tag{32a}$$

The second order equations of motion for the plate are now completely defined.

Next, we derive the counterpart of equation (9) that governs, to first order approximation, the transverse vibration of the plate. This approximation is systematically obtained exactly as in Section **2.3**. We first note that the term $(\alpha_\gamma\, \overline{N}_{\beta\gamma,\beta})$ in the shear equations $(29)_2$ is of first order as a result of the centrifugal terms in the equations for in-plane forces $(29)_1$. We recall that this term allows an appropriate account for the action of the centrifugal force on the bending of the plate. For steady state revolution and negligible shear, the transverse vibration of the plate is governed, up to first order, by the linear PDE

$$\begin{aligned} A_\rho\, \ddot{u}_3 + D\, \Delta^2\, u_3 - [X_\gamma\, w_\gamma\, w_\beta - \|\mathbf{w}\|^2 X_\beta]\, u_{3,\beta} \\ - A_\rho\, (w_1^2 + w_2^2)\, u_3 = -w_\gamma\, \phi_\gamma\, w_3 - 2(w_1\, \dot{u}_2 - w_2\, \dot{u}_1) \end{aligned} \tag{33a}$$

$$D := \frac{E d^3}{12(1 - \nu^2)} \tag{33b}$$

with Δ denoting the Laplacian operator. A complete analogy with equation (9) should be observed: The term $[A_\rho\,(w_1^2 + w_2^2)\; u_3]$ gives rise to an unphysical loss of stiffness, quadratic with the angular velocity, when linear plate theory is used; a complete account for the action of the centrifugal force is realized up to the first order with the additional term $[X_\gamma\; w_\gamma\; w_\beta - \|\mathbf{w}\|^2 X_\beta]\; u_{3,\beta}$ when second order plate theory is employed. These two terms form the dynamic geometric stiffness operator for the fast rotating plate.

Conceptually the second order theory governed by (26),(28) and (29), with $\mathbf{F}^{inertia}$ and $\mathbf{T}^{inertia}$ given by (31) and (32) respectively, can be treated numerically by standard finite element procedures. From a computational standpoint the main issues concerns the development of the appropriate spatial discretization.

5. Conclusion

The present work demonstrates the limited range of application, and even inadequacy, of linear structural theories to model physically relevant situations. Our discussion shows that even for extremely stiff beams for which linear theories are expected to be valid, a high enough angular velocity of revolution will predict a physically inadmissible destabilization effect. Fully nonlinear models, on the other hand, are able to account for situations more general than that discussed herein. Efficient computational procedures based on the use of such theories have been recently developed, Simo & Vu-Quoc [1985]. In the context of a general three-dimensional finite-strain rod model, we refer to Simo & Vu-Quoc [1986a] for static analysis, and to Simo & Vu-Quoc [1986b] for the dynamic analysis of flexible rods performing large overall motions.

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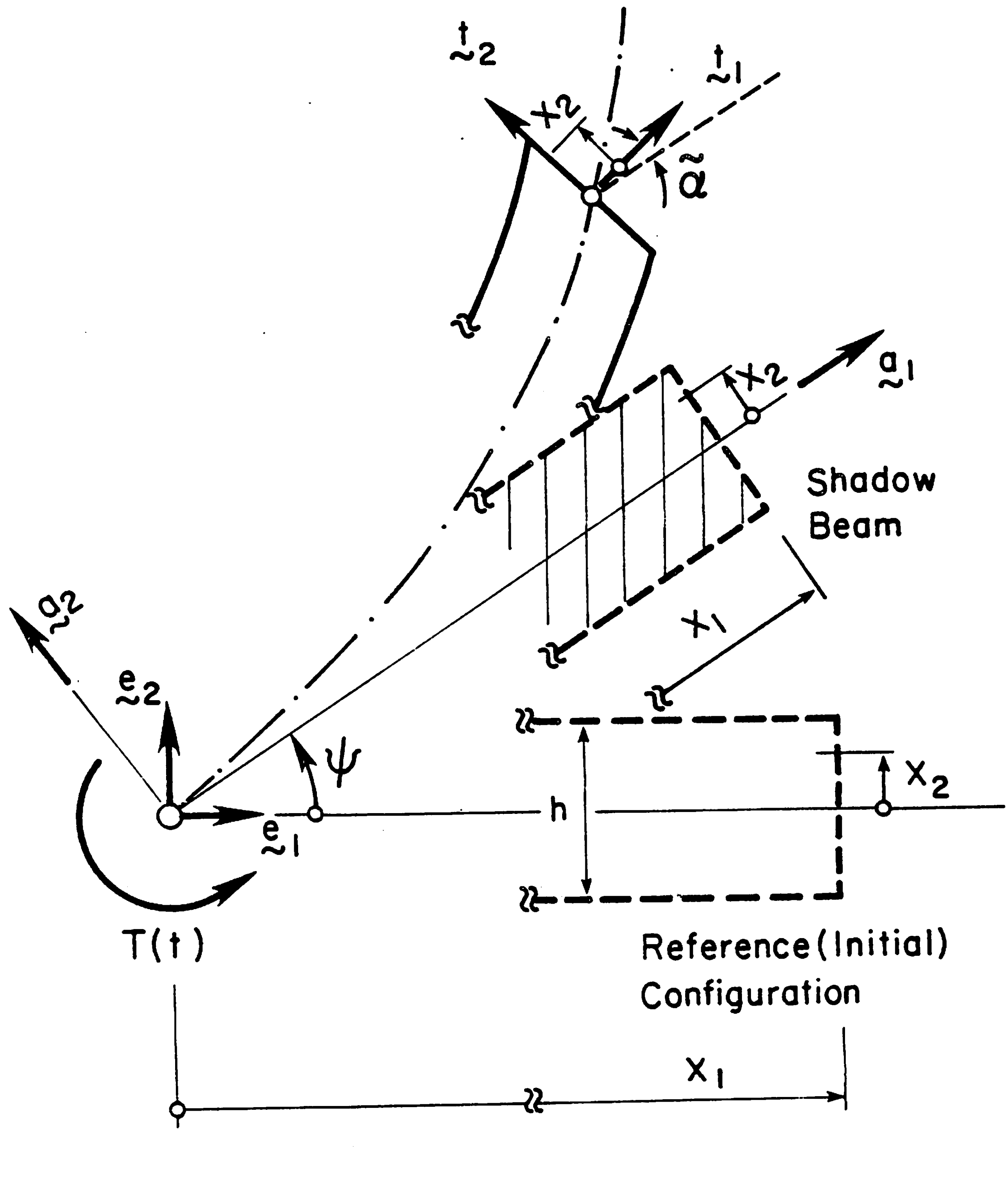

Figure 1. *Basic kinematics.* Floating and inertial frames.