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Abstract

We examine the diffusion in action of a generic Hamiltonian mapping over a range of actions for which the local standard mapping approximation exhibits stable accelerator modes. For the generic mapping the corresponding orbits are not completely stable, but the orbits may appear to be stable over many mapping periods. We examine the effect of these "quasi-accelerator mode" orbits on the diffusion. We find the most important physical effect in enhancing the diffusion to be the growth and decay of the locally stable mode (island) area, which leads to trapping and detrapping of phase space within the mode. An analytic theory is developed to incorporate this effect into a Fokker-Planck description of the diffusion. The solution for the distribution function, generated by a constant flux across the mode, is calculated and compared to numerical results. Provided a reasonable approximation to the locally stable island area is used in the theory, good agreement between theoretical and numerical results is obtained in a parameter range for which the effect of the mode is significant.

I. INTRODUCTION

The study of nonlinear dynamical systems has revealed many examples of chaotic behavior. The simplest systems in which such behavior is observed are Hamiltonian systems having two degrees of freedom. Two-dimensional area-preserving mappings which have their own Hamiltonian structure may be used to model such systems.

In many problems, such as ion or electron cyclotron resonance heating, the evolution of only one of the two phase-space variables, the action (or the energy), is of interest. We assume that the other variable, angle or phase, is randomized much more rapidly than the action. Based on these assumptions we can describe the dynamics using a Fokker-Planck equation in action alone.[1] The Fokker-Planck equation describes the evolution of the distribution of actions, as represented by the distribution function $f(u, n)$ where u is the action and n is the "time."

For a Hamiltonian system, the Fokker-Planck equation is specified by giving the diffusion coefficient $D(u)$. The quasilinear diffusion coefficient D_{QL} has been used by many authors[1,2] to describe the evolution of the action in an area-preserving map. However there are stringent limits on the validity of the quasilinear approximation, which assumes phase randomization on each mapping iteration. The global diffusion coefficient of the action for the standard map

$$\begin{aligned} I_{n+1} &= I_n + K \sin\theta_n \pmod{2\pi}, \\ \theta_{n+1} &= \theta_n + I_{n+1} \pmod{2\pi}, \end{aligned} \tag{1}$$

has been calculated by Rechester *et al.*,[3,4] including phase correlations over many mapping periods, by the Fourier path method, which relies on the doubly periodic nature of the mapping. We have shown that the results of this calculation can be applied to a more generic map, such as the Fermi map,

$$\begin{aligned} u_{n+1} &= u_n + \sin\psi_n, \\ \psi_{n+1} &= \psi_n + \frac{2\pi M}{u_{n+1}} \pmod{2\pi}, \end{aligned} \tag{2}$$

by using a local approximation for u in the neighborhood of a fixed point, $u_{n+1} = u_n$, $\psi_{n+1} = \psi_n \pmod{2\pi}$.[5]

For certain ranges of K, for which there are stable "fixed points" of the standard map that jump by $2\pi m$ in action on each iteration, the Fourier path method fails if the action is not taken modulo 2π. Phase points within the stable islands about these fixed points stream, rather than diffuse, in action, leading to a singularity in the usually defined diffusion. The singularity can be removed by adding noise to the standard map. This has been done by Karney *et al.*[6] to study the enhanced diffusion over ranges of K values for which stable fixed points or "accelerator modes" exist.

For more generic mappings, such as (2), the locally approximated standard mapping has a K value which is a function of the original action u. This can be seen by linearizing around a given fixed point $u_l = M/l$, with l an integer, and $\psi_1 = \pi$, to obtain

$$\begin{aligned} \Delta u_{n+1} &= \Delta u_n - \sin\theta_n, \\ \theta_{n+1} &= \theta_n - \frac{2\pi M}{u_l^2}\,\Delta u_{n+1} (\mathrm{mod} 2\pi). \end{aligned} \tag{3}$$

Letting

$$K = 2\pi M/u_l^2 \tag{4}$$

and

$$I_n = -K\Delta u_n \tag{5}$$

puts the map in the standard form (1). From (4) we see that K depends on u_l and therefore the effective K of the mapping of a given trajectory changes as it diffuses or streams. Since K changes, the accelerator modes of (1) are not truly stable modes of (2), and may therefore be designated as quasi-accelerator modes. The larger the mapping parameter M, the closer the mapping remains near a given value of K over a range of l-values.

For quasi-accelerator modes there are two mechanisms by which diffusion is enhanced. (1) Particles on locally unstable orbits become trapped on locally stable orbits as they are transported in the direction of increasing stable island size (phase space area). They then stream through values of u corresponding to increasing and then decreasing island size until they are detrapped at values of u having the same island size at which they were originally trapped. (2) Trajectories near stable islands are only weakly unstable, and therefore shadow the stable trajectories over many mapping periods. This leads to long-time correlations in the phase which again result in streaming. This latter effect has been considered by Karney[7] to qualitatively explain the peaks in the diffusion for the standard map for the case in which no orbits have initial phase space coordinates lying within stable accelerator-mode islands.

For generic maps, such as (2), we shall present scaling arguments to show that the first effect usually dominates over the second. We then develop Fokker-Planck theory, including sources and sinks, to describe the phase space transport. The analytical results are then compared with numerical calculation.

II. SCALING OF THE DIFFUSION

For the accelerator modes of the standard map the fixed points of a single mapping iteration are located at

$$I_0 = 2\pi m, \quad K \sin\theta_0 = 2\pi l \tag{6}$$

and are stable for

$$|2 + K\cos\theta_0| < 2 \tag{7a}$$

Applying (6) to (7a) we find a window of stability (in K) for

$$2\pi l < K < \sqrt{(2\pi l)^2 + 16}, \tag{7b}$$

corresponding to $\pi/2 < \theta_0 < 2.1376$. The mode is born with an inverse tangent bifurcation at $K = 2\pi l$ and period doubles at $K = \sqrt{(2\pi l)^2 + 16}$. The first mode ($l$=1) has the largest stable range of K-values and the largest maximum island size, and consequently is most important for modifying the local diffusion.

The island phase space area is found in the usual way by constructing a perturbation Hamiltonian $\hat{H}$ in the neighborhood of the stable fixed point (I_0, θ_0). We write

$$\hat{H} = \frac{\hat{I}^2}{2} + K(\cos\theta + \theta\sin\theta_0) = \hat{H}_{sx}, \tag{8}$$

where $\hat{H}_{sx}$ is the value of $\hat{H}$ on the separatrix

$$\hat{H}_{sx} = K(\cos\theta_x + \theta_x \sin\theta_0), \tag{9}$$

$\hat{I}$ is the deviation from the phase stable action

$$\hat{I} = I - I_0 \tag{10}$$

and θ_x is the value of θ at the unstable fixed point, given by

$$\theta_x = \pi - \theta_0. \tag{11}$$

Solving for $\hat{I}_{max}$ we obtain

$$\hat{I}_{max} = (2K)^{1/2}[-2\cos\theta_0 + (\pi - 2\theta_0)\sin\theta_0]^{1/2} \tag{12}$$

We note that θ_0 remains in the neighborhood of $\pi/2$, and θ remains within the island near θ_0, allowing the following expansions. Setting $\theta = \theta_0 + \hat{\theta}$, and $\theta_0 = \pi/2 + \Delta_0$, we have $\Delta_0 = 0$ at the island birth and $\Delta_0 \approx 0.567$ at its bifurcation. Expanding (8) and (9) to cubic terms in Δ_0 and $\hat{\theta}$, we obtain

$$\frac{3}{2\pi l}\hat{I}^2 = (\Delta_0 - \hat{\theta})(2\Delta_0 + \hat{\theta})^2 \tag{13}$$

where, from (6), to second order in Δ_0,

$$\Delta_0 = 2^{1/2}\left[1 - \frac{2\pi l}{K}\right]^{1/2}. \tag{14}$$

Setting $\hat{\theta} = 0$ in (13), we obtain

$$\hat{I}_{max} = (8\pi l\ \Delta_0^3/3)^{1/2} \tag{15}$$

which yields the scaling of $\hat{I}_{max}$ with Δ_0 and the approximate value $\hat{I}_{max} \approx 1.24$ (for $l = 1$) at the bifurcation (the exact result from (8) and (9) is $\hat{I}_{max} \approx 1.32$). The island area A_I is found by integration of $\hat{I}(\hat{\theta})$. Using (13), we obtain

$$A_I = \frac{24}{5}(2\pi l\,\Delta_0^5)^{1/2}. \tag{16}$$

We can now estimate the effective diffusion arising from the accelerator mode due to trapping, in the original space, from the relation

$$D_T \approx \frac{(\Delta u)^2}{2\tau}\, g(K) \tag{17}$$

where τ is the average streaming time, Δu is the average distance over which streaming takes place, and

$$g(K) = A_I(I)/(2\pi)^2 \tag{18}$$

is the fraction of participating phase space. Since for a single iteration I steps by $2\pi l$, $\Delta I = 2\pi l \tau$, we then have, using (5)

$$\Delta u \approx \tau, \tag{19}$$

since $K \approx 2\pi l$. From the equilibrium condition (4) we obtain, by differentiation,

$$\Delta u = \frac{(2\pi M)^{1/2}}{2K^{3/2}} \Delta K. \tag{20}$$

From (7b) we calculate the value of $\Delta K = K - 2\pi l$ at the middle of the mode, for large K,

$$\Delta K_{ave} = 4/K. \tag{21}$$

Evaluating the average island area (16) in the large K limit, where, from (14), $\Delta_0 = 8^{1/2}/K$, we obtain

$$g(K)_{ave} = \frac{A_I(\Delta K_{ave})}{(2\pi)^2} = \frac{1}{(2\pi)^2} \frac{24}{5} \frac{8^{5/4}}{K^2}. \tag{22}$$

Substituting (22) into (17), together with (19), (20) and (21) gives

$$D_T \approx \frac{(2\pi M)^{1/2}}{K^{5/2}} \frac{1}{(2\pi)^2} \frac{24}{5} \frac{8^{5/4}}{K^2} \tag{23}$$

We can compare the value and the scaling of D_T in (23), with the value obtained from the long-time correlations, but ignoring the trapping and detrapping. To do this we use the result of Karney[7] for the correlation time of a trajectory in the neighborhood of a small island to remain in that neighborhood. He finds numerically a probability distribution

$$p(\tau) = a\tau^{-1.4}$$

a result also obtained numerically by Chirikov and Shepelyansky[8] for a somewhat different configuration. Using this proportionality, the part of the diffusion coefficient due to long time correlations near but outside of the accelerator mode can be written

$$D_c = \int_1^{\tau_{max}} \frac{(\Delta u)^2}{2\tau} p(\tau)\, g(K) d\tau \tag{24}$$

where τ_{max} is the time to stream across the mode. Using our previous results we observe that $\Delta u = \tau$ such that (24) is easily integrated to obtain

$$D_c = \frac{a}{1.2} (\tau_{max}^{0.6} - 1) \, g(K).$$

Dropping the 1 ($\tau_{max} \gg 1$), using the usual normalization for $p(\tau)$ to obtain $a = 0.4$, and obtaining τ_{max} from our previous expressions we have

$$D_c = \frac{8}{3} \left[\frac{(2\pi M)^{1/2}}{K^{5/2}} \right]^{0.6} \frac{24}{5} \frac{8^{5/4}}{(2\pi)^2 K^2} \tag{25}$$

Comparing (25) with (23) we note that, except for a numerical factor of order unity, the two expressions differ only in that the factor $(2\pi M)^{1/2}/K^{5/2}$ is taken to the 0.6 power in (25), while it is linear in (23). Since this factor is assumed large for the effect of the accelerator mode to be important, we conclude that the trapping and detrapping dominates the effect of correlations. However, the requirement that $(2\pi M)^{1/2}/K^{5/2} \gg 1$ puts a severe requirement on the local uniformity (size of M) for the effect of the accelerator mode to be significant. For example, for $K = 2\pi$, the first mode, we require that $M \gg 1.5.10^3$. The higher modes (in l) become increasingly dependent on having a large value of M. To have large effects requires a more stringent condition on M, as we shall see in the next section.

III. PHASE SPACE TRANSPORT

We wish to describe the evolution of a distribution function $f(u, n)$ in the action alone. We assume that the phase evolves randomly and the evolution in action is a Markov process. In addition we assume that the change in action is small on the time scale over which the phases become random. These assumptions lead to a Fokker-Planck equation for the action

$$\frac{\partial f(u, n)}{\partial n} = -\frac{\partial}{\partial u} [B(u) f(u, n)] + \frac{1}{2} \frac{\partial^2}{\partial u^2} [D(u) f(u, n)] , \tag{26}$$

where $D(u)$ is the local diffusion coefficient

$$D(u) = \frac{1}{\Delta n} \int du'(u' - u)^2 W_t(u, 0; u', \Delta n) , \tag{27a}$$

and $B(u)$ is the local friction coefficient

$$B(u) = \frac{1}{\Delta n} \int du'(u' - u) W_t(u, 0; u', \Delta n). \tag{27b}$$

The transition probability $W_t(u, 0; u', \Delta n)$ is the probability density that a particle has action u' at time Δn given that it had action u at time 0. The time Δn is assumed to be small compared to the evolution time of the action distribution function, but must be longer than the phase-relaxation. We assume that only the first and second moments of W_t are proportional to Δn. For Hamiltonian systems with action-angle variables (i.e., with periodic dependence on the angles) and assuming random phases, it may be shown that[5]

$$B(u) = \frac{1}{2} \frac{dD(u)}{du} . \tag{28}$$

In principle we could treat the entire problem within the context of (26) through (28) by taking the integration time long enough that the entire period of streaming over the mode is included. However, this would require averaging over a range of action space large compared to the mode. To obtain the distribution more locally, we treat the streaming explicitly as sources and sinks in the Fokker-Planck equation. This is a natural description when only the trapping-detrapping mechanism is considered. The diffusion coefficient is then that of the non-streaming particles, as previously obtained.[9] The Fokker-Planck equation, including sources and sinks, can be written

$$\frac{\partial f(u)}{\partial t} = \frac{\partial}{\partial u} \left[\frac{1}{2} D(u) \frac{\partial f(u)}{\partial u} \right] + S(u) - L(u), \tag{29}$$

where

$$D(u) = \frac{1}{K^2(u)} D_\infty[K(u)] \tag{30}$$

and D_∞ is obtained from the standard-map approximation[3-4,9]

$$D_\infty(K) = K^2[\tfrac{1}{2} - J_2(K) + J_2^2(K) + O(1/K^2)], \tag{31}$$

with K dependent on u as given in (4). D_∞ is obtained in the long-time limit, implying averaging over many 2π intervals in the action I of the standard map. Since we are interested in intervals of u corresponding to many 2π intervals in I, we can obtain the variation of f over the mode. The sources and sinks are straightforwardly determined, in principle, from the amount of phase space captured or lost from the islands. We assume that the island area $A(u)$ has a single maximum within the mode. Since there are two symmetric islands at a given u in the mode near $\theta = \pm\frac{\pi}{2}$, transporting phase space in opposite directions, the loss and source terms are closely related, as

$$L = a'(u) f(u, n) \tag{32}$$

$$S = a'(u) f(\overline{u}, \overline{n}) \tag{33}$$

where $a'(u) = |A'(u)/2\pi|$, $a(u)$ is the fraction of phase space area occupied by the island, and where $A'(u)$ is the derivative with respect to u of the island area. The barred quantities are defined by

$$A(\overline{u}) = A(u) \tag{34}$$

and

$$\overline{n} = n - \tau(|\overline{u} - u|), \tag{35}$$

where $\tau(|\overline{u} - u|)$ is the time to stream the distance $|\overline{u} - u|$. We will only consider the steady-state, such that f is independent of n. If $f(\overline{u}) = f(u)$, then there is no transport due to the accelerator mode. The picture, then is of two sets of islands, carrying phase space in opposite directions, in which the islands grow to a maximum locally-stable value and then decay back to zero, over the range of the mode. The islands transport flux, with the maximum flux being transported when the island size is greatest. In a steady-state problem we have constant total flux and thus we expect the gradient of the distribution to be a minimum (a minimum of diffusively transported phase space) when the island has maximum area. We shall see that this is, in fact, the case.

To obtain a solution of the Fokker-Planck equation we need to have explicit expressions for $D(u)$ and $A(u)$. In general, these are complicated functions allowing only numerical solutions. To obtain an analytical solution we shall assume that D is constant and that A' is antisymmetric about some maximum value A_m, at u_m. Then, making the change in variable $u = u_m - x$, $\overline{u} = u_m + x$, we expand $f(u)$ and $f(\overline{u})$ in Taylor series about $u = u_m$, to obtain

$$S - L = a'(x) f'(0) 2x \tag{36}$$

Substituting (36) into the Fokker-Planck equation, we have, in the variable x

$$\frac{1}{2}\frac{d}{dx}D\frac{df}{dx}+a'(x)f'(0)2x=0 \tag{37}$$

We integrate (37) once, to obtain

$$f'(x)=-\frac{4}{D}f'(0)\int_0^x a'(x)x\,dx \tag{38}$$

where Δu is the half-width of the mode. To proceed further, we must specify $a'(x)$. We take a symmetric form for the normalized island area

$$a(x)=a_m\left[1-\left|\frac{x}{\Delta u}\right|^n\right] \tag{39}$$

with a_m and n to be specified, which can be matched either to the analytical value from (16) or to the numerically determined value as we do in the next section. We substitute (39) in (38), and perform the integration, to obtain

$$f'(x)=\left[1+\frac{4\,a_m\,\Delta u}{D}\,\frac{n}{n+1}\left[\frac{x}{\Delta u}\right]^{n+1}\right]f'(0). \tag{40}$$

We solve for $f'(0)$ by requiring that

$$f'(\Delta u)=\Gamma/D \tag{41}$$

where Γ is the flux and D, given by (30) evaluated at $u=u_m+\Delta u$, is the diffusion coefficient just at the edge of the mode. Outside of the mode, we have the constant flux solution

$$D\ f'(x)=\Gamma=\text{const.} \tag{42}$$

The result is

$$f'(0)=\frac{\Gamma/D}{1+\dfrac{4a_m\Delta u}{D}\,\dfrac{n}{n+1}}\left[1+\frac{4a_m\Delta u}{D}\,\frac{n}{n+1}\left|\frac{x}{\Delta u}\right|^{n+1}\right]. \tag{43}$$

We estimate the value of M required for the accelerator mode to have a significant effect on the diffusion by setting

$$\frac{4a_m\Delta u}{D}\,\frac{n}{n+1}=1. \tag{44}$$

We choose Δu from (21), using (20), to obtain

$$\Delta u = \frac{2(2\pi M)^{1/2}}{K^{5/2}}. \tag{45}$$

We then calculate $a_m = a(\Delta u)$, from (16), where $\Delta_0 = (2\Delta K/K)^{1/2}$ from (14), to obtain

$$a_m = \frac{24}{5} \frac{2^{5/2} K^{9/8}}{(2\pi M)^{5/8}} (\Delta u)^{5/4}. \tag{46}$$

From (16), the analytical value of $a(x)$ is

$$a(x) = a_m \left[1 - \frac{x}{\Delta u}\right]^{5/4}. \tag{47}$$

Noting that

$$\int_0^{\Delta u} a'(x)\, x\, dx \equiv -\int_0^{\Delta u} a(x)\, dx$$

we determine n by equating the integral of (47) from $x = 0$ to $x = \Delta u$ to the integral over the same range of (39), to obtain $n = 4/5$. For this value, together with (45) and (46), we obtain from (44),

$$M = \frac{K^9}{234}, \tag{48}$$

which yields, for $K = 2\pi$, $M = 6.5 \cdot 10^4$. This is the value of M for which the gradient of the distribution function is depressed by a factor ½ at the center of the mode. We should also note that the Taylor series expansion used in (36) limits the value of M to this same order. In the next section we compare these results with numerical calculations, but replace the analytical estimates of island area with numerically determined values. Due to stochasticity near the separatrix, the locally stable area within the separatrix is about a factor of 10 smaller, at u_m, than that calculated from (16). Consequently, the estimate of M from (48) is a lower bound and is approximately a factor of 100 smaller than that required to obtain a factor ½ local depression of the slope, as we shall see in the following section.

VI. COMPARISON OF ANALYTIC AND NUMERICAL RESULTS

We first numerically calculate the area of the accelerator mode, as a function of K, for the standard map. The procedure is to distinguish regular island orbits from the stochastic sea for a large set of initial conditions and determine the ratio of areas from the ratio of initial conditions in each category. Of course, only a small portion of the phase space, in the neighborhood of the island need be explored.

The result of the numerical calculation for the total stable area of the mode versus K is shown in Fig. 1, for the lowest mode $l = 1$. We observe that the structure, as expected, is much more complicated than that assumed in the analytic calculation. The area is also smaller and the mode width broader than that assumed, if taken from the inverse tangent bifurcation to the period doubling bifurcation. The structure of the mode is qualitatively understood in terms of the various resonances, which are indicated on the figure. The initial mode growth versus K is similar to that expected, but decreased due to the growth of the stochastic layer near the mode separatrix. The first major dip in the regular area near $K \approx 6.6$ involves the 4:1 resonance between the mapping and local island frequencies. The growth of this island chain and its interaction with other second order islands rapidly increases the stochasticity within the separatrix of the main island. At the bottom of the dip in area most of the stable area lies in the 4-island chain, rather than in the main island. The reestablishment of the stable area near $K \approx 6.7$ is achieved by restabilization of the central island, as the 4-island chain moves outward. The first extinction of the stable island occurs due to a very strong 3:1 resonance interaction near $K \approx 7.0$. Again, the 3-island chain moves outward, reestablishing the central stability region. The 2:1 resonance, which is the period doubling bifurcation of the central fixed point appears at $K \approx 7.4$, and is, in fact, rather benign in that stable island trajectories (KAM curves) remain encircling the bifurcated system. However small additional increases in K destroy these trajectories and take the phase space through the usual period doubling sequences to chaos.

For comparison with the numerical diffusion the island area is approximated by the dashed curve, computed from (39), with $n = 2$. We choose $a_m = .004$ to approximately match the integrated area under the island area curve. Because the solution of the Fokker-Planck equation depends only on $\int_0^{\Delta u} a dx$, the exact form of the variation of the island area is of little importance, which is borne out by the numerical results. We also show the theoretical area (dash-dot line), from (16), used in the estimation of the minimum significant M-value obtained in (48).

For the numerical calculation of $f(u)$ we employ the Fermi acceleration map (2) with large M. We place a source at one side of a quasi-accelerator mode and a sink at the other side. We then calculate the distribution between the source and the sink. A second sink is employed at the other side of the source to keep the distribution bounded. We consider the lowest mode, $K = 2\pi$, and take $M = 10^7$ and 10^6, with the results shown in Figs. 2 and 3, respectively. The M-values satisfy condition (48) for obtaining a large effect. There are two effects to be noted. In the absence of the accelerator mode (no sources or sinks) from (29) we see that

$$\frac{df}{du} \propto \frac{1}{D(u)} \tag{48}$$

where D is given by (30) and (31). This accounts for a lowest order variation of slope with u which is independent of M. Comparing Fig. 2 ($M = 10^7$) with Fig. 3 ($M = 10^6$) we see a decidedly greater flattening of the distribution in the former case. This is expected from the solution (43) since the product $a_M \Delta u$ which determines the flattening is seen from (45) and (46) to be proportional to $M^{1/2}$.

Comparing the relative experimental slopes df/du (normalized to $D/D_{QL} = .61$ at $K = 2\pi$) with the ratio D_{QL}/D, as indicated by (48) and with $D(u)$ given by (30), we obtain the results in Table 1. First, looking at the results for $M = 10^7$, for which we expect better agreement with the theory, we see that df/du differs markedly from D_{QL}/D in the region of the mode ($2904 < u < 3162$). For that region, taking the approximation to the numerically determined area (dashed curve) and the local $D(u)$, our solution (43) gives a slope depression ratio of $R = 0.33$. This is to be compared with the numerically found depression of the slope from the value at $u_1 = 3162$, where the mode is born, to the value at $u_2 = 3010$, near the mode center. Over this range, $D(u) \approx 0.8$ from (31) is relatively constant. From the table, we find the ratio $R = 0.3$, in good agreement with the theoretical value. For $M = 10^6$, the theoretical value of slope depression ratio from, (43), is $R = 0.6$. This depression is somewhat less than the observed depression for that case, of $R = 0.52$, but the agreement with the theory is still quite good. However, the depression of the slope at the edge of the mode ($K \approx 7.5$) is still significant for both $M = 10^7$ and $M = 10^6$, indicating a larger range of u over which the mode can be felt. We present a qualitative explanation for this phenomenon, below.

We have already obtained an approximate equation for the area inside the accelerator separatrix in (16). A constant slope approximation to this area is shown as the dash-dot line in Fig. 1. Although the theoretical island area rises, initially, with K, at the same rate as the area found numerically the two curves rapidly diverge as stochasticity eats away the island area near the separatrix. The trajectories in this stochastic region within the separatrix are not trapped. Nevertheless, they may stream along with the mode for many iterations of the mapping, while the trajectory diffuses through the separatrix. Simple estimates of this diffusion indicate that, at $M = 10^6$, significant enhancement of the streaming is obtained. The effect is smaller for $M = 10^7$, since the time for particles to diffuse through the separatrix is constant (at a given island area) while the streaming time increases as $M^{1/2}$.

To illustrate the complexity of the real problem we give a more detailed comparison of the numerical result with the simple theoretical model. To do this we use another presentation of the numerical results for $M = 10^7$, as shown in Fig. 4. Here the slope df/du is plotted for a fixed (arbitrary) flux. Considerable averaging of the data points is required because of noise, with the result, as seen in the figure, still somewhat noisy. However, the larger structures are real, as discussed below.

To compare these numerical results with theory, we note that the numerical slope is equal to $1/D(u)$ (the solid line from (31)) at the value of u for which the inverse tangent bifurcation occurs and the mode is first born ($K = 2\pi$). At this value of K there is not expected to be any accelerator mode streaming. We note that there is a slight offset in the break in the slope, occuring at slightly higher u (lower K) from that where the effect of the accelerator mode is first expected to manifest itself. We match the slope, at the inverse tangent bifurcation to the theoretical solution given in (40), with the mode area given by the approximate dashed curve in Fig. 1. The result of this matching gives the dashed curve in Fig. 4. We see that the fit is quite reasonable for that portion of the mode in which the stable island area is growing or nearly constant. The correlations within the unstable separatrix, although not dominating, play an important smoothing role.

A significant deviation from the smooth behavior occurs around the 3:1 second order island resonance catastrope ($K \approx 6.95$), in which the entire stable area is destroyed. However, the stable mode partially recovers, and the correlations within the separatrix are sufficient to enhance the streaming for u's near this value. There is still considerable depression in slope at the period doubling bifurcation, because stable area still exists. Even beyond the limit of the bifurcation sequence, where no stable area exists, there are residual correlations that keep the slope below the normalized value of $1/D$. The effect is qualitatively similar for the case of $M = 10^6$.

As a further confirmation of the general correctness of model, the locations (in u) of the source and sink were interchanged, for the case of $M = 10^7$, and the numerical problem run again. The result, corresponding to Fig. 4, is shown in Fig. 5. Except for minor variations, the depressions of the slope are quite similar in the two cases, consistent with the theoretical picture.

V. DISCUSSION AND CONCLUSIONS

We have seen that, for the Fermi acceleration map with large M, quasi-accelerator modes can have a significant effect on the local diffusion. This is true for a wide class of mappings whose structure varies with action, but which locally approximate the standard map. Two mechanisms for the enhanced diffusion were explored, that of trapping and detrapping in the quasi-mode, and that of long-time correlations for stochastic orbits in the neighborhood of locally regular orbits. The latter effect persists in the limit of $M = \infty$ (the standard map) and from numerically determined correlation-time probabilities, the diffusion rate approaches infinity as $M^{0.3}$. For large M the trapping and detrapping mechanism is shown to dominate, becoming large as $M^{1/2}$. However, the $M = \infty$ limit is singular (i.e. there is no trapping and detrapping for the standard map, since KAM curves are preserved).

An analytical method of computing the effect of quasi-accelerator modes on the diffusion is developed, using local sources and sinks to represent the effect of the mode. Although the calculation can be carried through completely analytically, with a suitably approximated variation of the locally stable island area of the quasi-mode, a better result is obtained if the numerically determined island area is used in constructing an analytic approximation. The exact shape of the variation of the island area with action is not important in determining the distribution function. This follows because the island shape enters into the coefficients of a first order differential equation whose solution is the gradient of the distribution function. Thus the distribution, itself, is sensitive to doubly integrated values of the island area, smoothing local anomalies.

The analytical solution for the distribution function agrees well with that determined numerically, provided M is sufficiently large, and a sufficiently accurate value of a'/D is used in the analytical theory. In our calculation a'/D was constructed from a numerical computation of the island area and a local, nearly constant value of $D = 1.6 D_{QL}$. The agreement was quite good for $M = 10^7$ and somewhat less so, as expected, for $M = 10^6$. The anomalies can be explained, qualitatively, in terms of the tendency for orbits in the stochastic portion of the mode to diffuse slowly away from the stable island.

ACKNOWLEDGEMENT

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REFERENCES

[1] M. A. Lieberman and A. J. Lichtenberg, *Phys. Rev. A* 5, 1852 (1972).

[2] B. V. Chirikov, *Phys. Rep.* 52, No. 5, 263 (1979).

[3] A. B. Rechester and R. B. White, *Phys. Rev. Lett.* 44, 1586 (1980).

[4] A. B. Rechester, M. N. Rosenbluth, and R. B. White, *Phys. Rev. A* 23, 2664 (1981).

[5] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).

[6] C. F. F. Karney, A. B. Rechester, and R. B. White, *Physica* 4D 425 (1982).

[7] C. F. F. Karney, *Physica* 8D 360 (1983).

[8] B. V. Chirikov and D. L. Shepelyansky, "Correlation Properties of Dynamical Chaos in Hamiltonian Systems," Institute of Nuclear Physics, Novosibirsk, Preprint 83-149 (1983).

[9] N. W. Murray, M. A. Lieberman, and A. J. Lichtenberg, *Phys. Rev. A* 32 2413 (1985).

Table 1

K	D_{QL}/D	$M = 10^7$		$M = 10^6$	
		u	$\left\|\frac{df}{du}\right\|$ (nor.)	u	$\left\|\frac{df}{du}\right\|$ (nor.)
2π	.61	3160	.61	1000	.61
6.9	.61	3010	.18	950	.32
7.5	.70	2860	.56	900	.58
8.5	1	2750	.71	860	1

Figure Captions

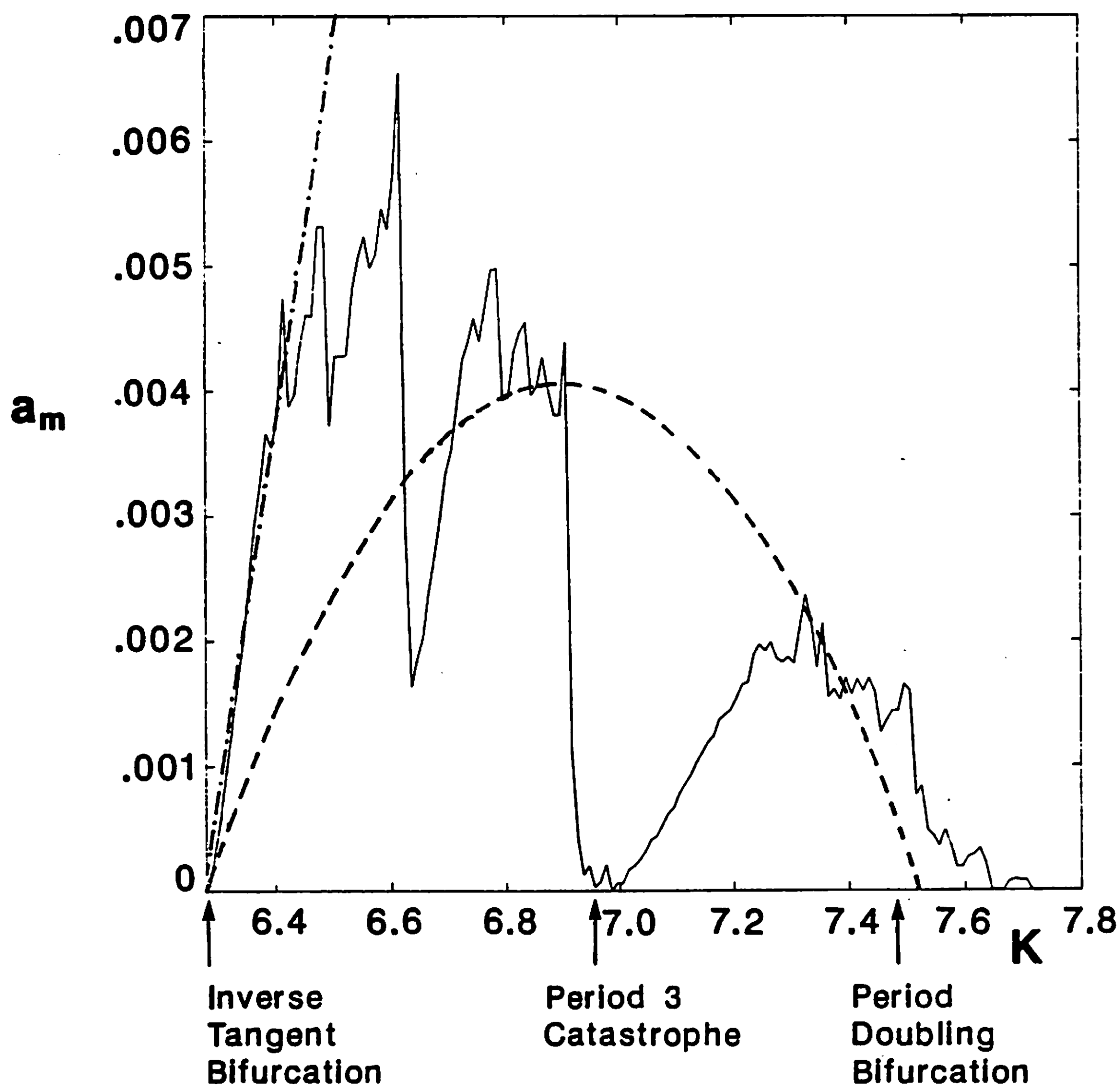
Fig. 1. Stable area of the first accelerator mode of the standard map as a function of K, normalized to the map area $(2\pi)^2$.

Fig. 2. Relative phase space density between a source and a sink, spanning the lowest accelerator mode for $M = 10^7$.

Fig. 3. The same as Fig. 2, for $M = 10^6$.

Fig. 4. Comparison of theoretical and numerical relative values of $|df/du|$. The theoretical value is shown as a dashed curve within the mode and a solid curve ($\propto 1/D$) outside of the mode ($M = 10^7$).

Fig. 5. Relative numerical values of $|df/du|$ with the same flux as Fig. 4, but the source and the sink reversed ($M = 10^7$).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.006
.005
.004
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.002
.001
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6.4
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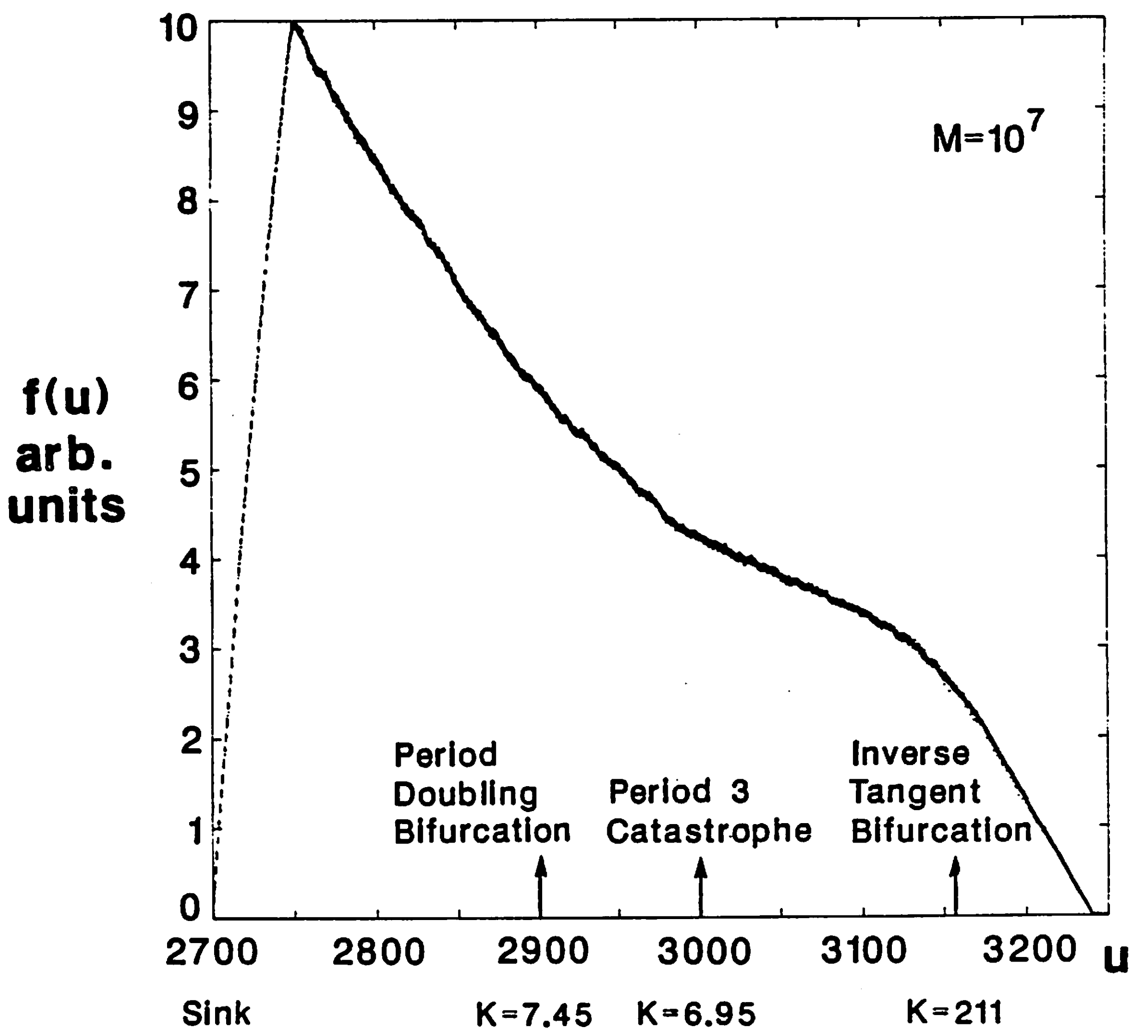
6.8
7.0
7.2
7.4
7.6
7.8
K
Inverse Tangent Bifurcation
Period 3 Catastrophe
Period Doubling Bifurcation

Fig. 1



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8
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M=10^7
Period
Doubling
Bifurcation
Period 3
Catastrophe
Inverse
Tangent
Bifurcation
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2800
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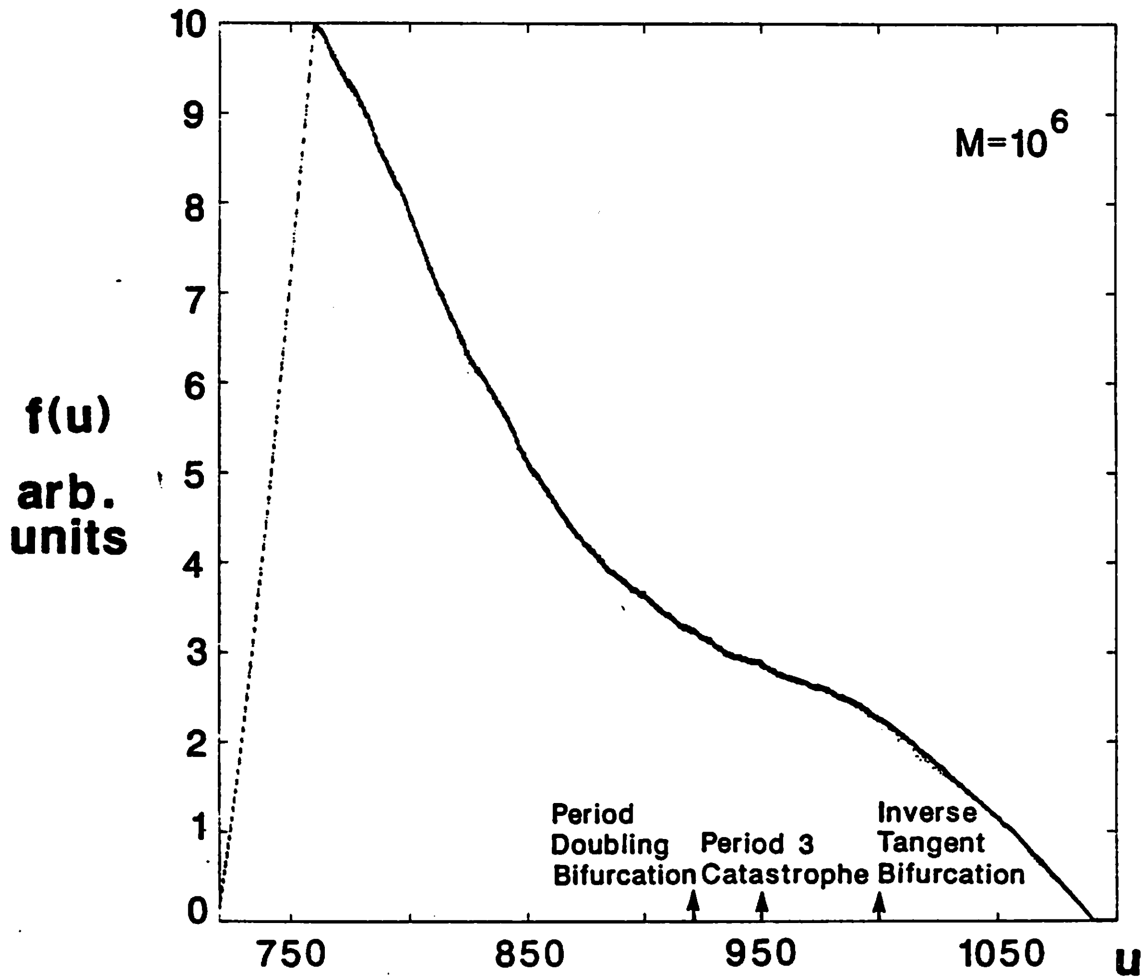
3000
3100
3200
u
Sink
K=7.45
K=6.95
K=211

Fig. 2



f(u)
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Fig. 3

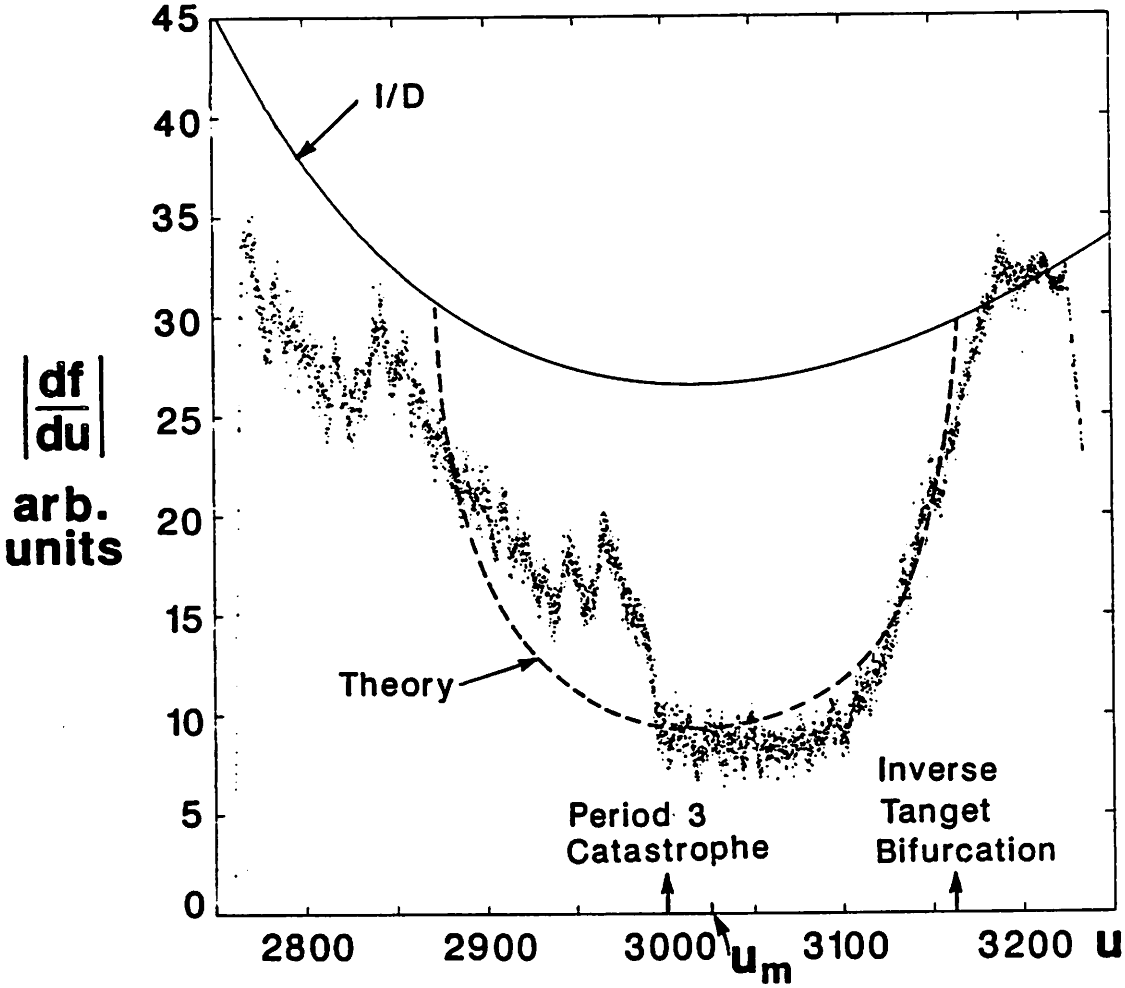
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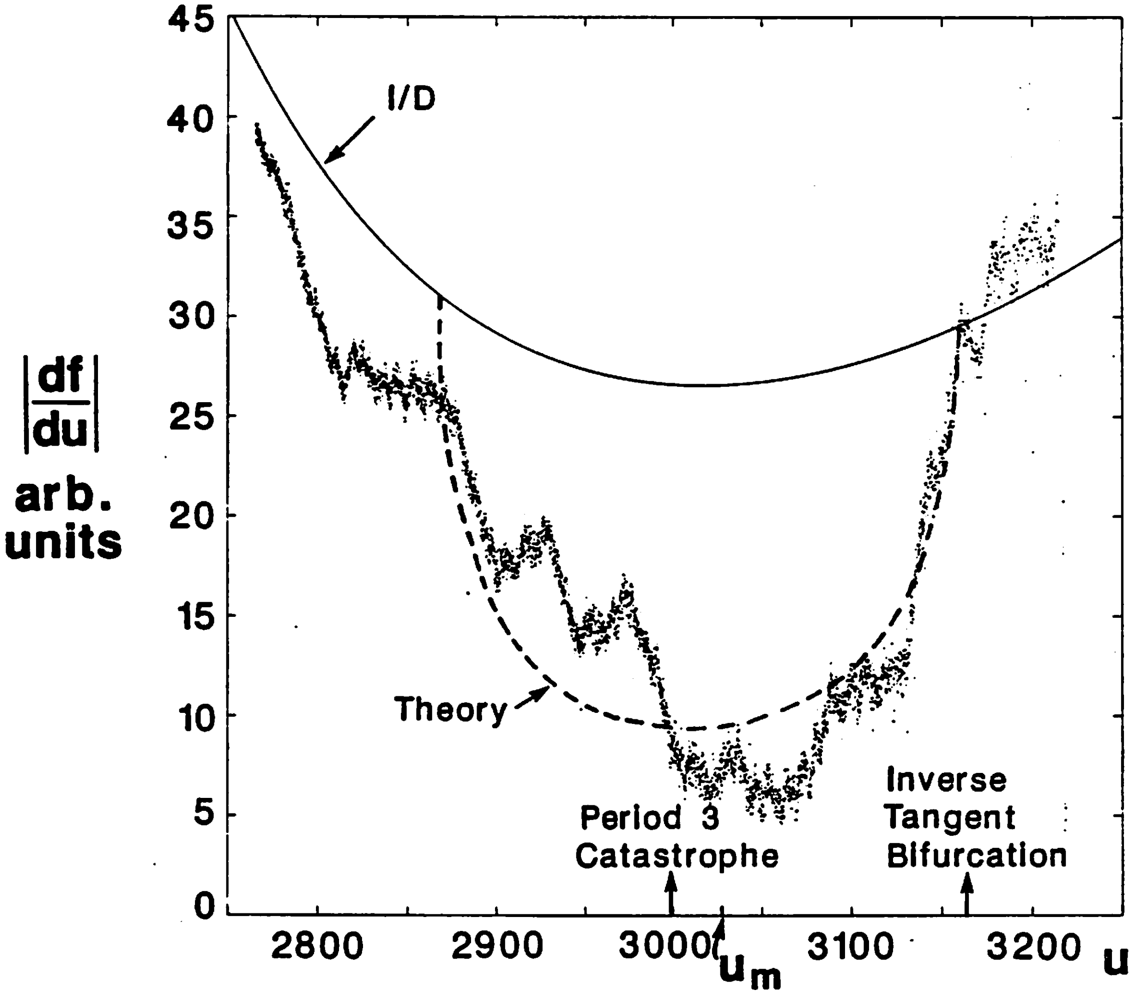
Fig. 4



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arb. units
Theory
Period 3
Catastrophe
Inverse
Tangent
Bifurcation
2800
2900
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3100
3200
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u_m

Fig. 5