ON THE DESIGN OF STABILIZING COMPENSATORS

VIA SEMI-INFINITE OPTIMIZATION

by

E. Polak and T. L. Wuu

Memorandum No. UCB/ERL M86/102

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ON THE DESIGN OF STABILIZING COMPENSATORS VIA SEMI-INFINITE OPTIMIZATION

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Abstract

The design of stabilizing compensators for linear, time invariant feedback systems, by means of semi-infinite optimization algorithms, requires a stability test in the form of a finite or infinite set of differentiable inequalities. The classical Nyquist stability criterion leads to an integer valued function and hence cannot be used in the design of stabilizing compensators via semi-infinite optimization. This paper evolves the classical Nyquist stability criterion into a semi-infinite inequality which is a necessary and sufficient condition of stability, and which is compatible with the use of semi-infinite optimization. Computational aspects of the new stability are discussed and design examples are given.

1. INTRODUCTION

The most basic requirement in control system design is exponential stability of the closed loop system. The manner in which this requirement is fulfilled depends on the synthesis techniques adopted by the designer. For very good reasons, the Nyquist stability criterion [Nyq.1], has served for many years as a principal "manual" tool for ensuring stability in linear time-invariant systems. Unfortunately, as was pointed out in [Pol.1], the Nyquist stability criterion cannot be used in conjunction with computer-aided design techniques which make use of semi-infinite optimization. The reason for this is easy to see. Consider a linear, time invariant feedback system $\Sigma(p)$, where p is a design parameter vector which determines the coefficients of the compensator blocks. Let $\chi(p,s) = n(p,s)+d(p,s)$ be the characteristic polynomial of $\Sigma(p)$, with the polynomials $n(p,s)$, $d(p,s)$ such that $n(p,s)/d(p,s)$ is a proper rational function in s. The coefficients of $n(p,s)$, $d(p,s)$ can be assumed to be differentiable functions of the design parameter $p \in \mathbb{R}^m$. Let $q(p)$ be the number of $\mathbb{C}_+$ zeros of $d(p,s)$ and let

$$N(p) \triangleq \{ \lim_{\omega \to \infty} \arg[\chi(p,j\omega)/d(p,j\omega)] - \arg[\chi(p,0)/d(p,0)] \}/2\pi - q(p). \tag{1.1}$$

Then the Nyquist stability criterion [Nyq.1] states that $\chi(p,s)$ has no zeros in $\mathbb{C}_+$ if and only if

$$N(p) = 0. \tag{1.2}$$

Since the function $N(p)$ is integer valued, equation (1.2) cannot be used as a constraint in an optimization problem solvable either by nonlinear programming or semi-infinite optimization algorithms.

A first attempt to modify the Nyquist stability criterion into a form compatible with optimization-based control system design was proposed by Polak in [Pol.1]. In particular, it was shown in [Pol.1] that if $\tilde{d}(s)$ is a polynomial of the same degree as $\chi(p,s)$, whose zeros are all in $\overset{o}{\mathbb{C}}_-$, then $\chi(p,s)$ has no $\mathbb{C}_+$ zeros if and only if the Nyquist locus of $T(p,j\omega) \triangleq \chi(p,j\omega)/\tilde{d}(j\omega)$, traced out for $\omega\in(-\infty,\infty)$ does not encircle the origin. It was shown in [Pol.1] that a *sufficient condition* for this to hold can be expressed in the form (by no means unique)

$$\mathrm{Im}[T(p,j\omega)] - c_1 \mathrm{Re}[T(p,j\omega)]^2 + c_2 \le 0\,,\ \ \forall\ \omega\in(-\infty,\infty)\,, \tag{1.3}$$

where $c_1, c_2 > 0$.

The modified Nyquist stability criterion in [Pol.1] satisfies the requirements of semi-infinite optimization. However, it does suffer from four drawbacks: (i) it is only a sufficient condition, which at times can be very conservative, (ii) it involves an arbitrary polynomial $\tilde{d}(s)$ whose definition requires judgement, (iii) the selection of a form such (1.3) involves some skill, e.g., in some cases it may be advantageous to use the inequality

$$-c_1 \mathrm{Im}[T(p,j\omega)]^2 - \mathrm{Re}[T(p,j\omega)] + c_2 \le 0\ \forall\ \omega\in[0,\infty)\,, \tag{1.4}$$

where $c_1, c_2 > 0$, instead of (1.3), and (iv) it proved to be sensitive to the frequency discretization step size.

In this paper, we begin by establishing the critical role that a computational stability test plays in the design of linear feedback systems via semi-infinite optimization. In particular, we show that the function of the computational stability test is not only to ensure stability, but also to confine the computation to a subset of the design parameter space where design specification inequalities are differentiable. Next we present a new computational stability test which can be viewed as a *modified* Nyquist stability test and which is both a *necessary and sufficient* condition of stability. The new stability test is compatible with semi-infinite optimization requirements and does not suffer from any of the drawbacks of the computational stability test presented in [Pol.1]. Finally, we pressent a method for efficient numerical implementation of the new stability test and present two computational examples to illustrate its performance as a design tool for obtaining a stabilizing compensator for a control system.

2. A MODIFIED NYQUIST STABILITY CRITERION

Consider a parametrized, linear, time-invariant, interconnected, finite dimensional dynamical system, $\Sigma(p)$, described by a set of state equations:

$$\begin{aligned} \dot{x}_i(t) &= A_i(p)x_i(t) + B_i(p)u_i(t), \ i = 1,2,3, \cdots, k, \\ y_i(t) &= C_i(p)x_i(t) + D_i(p)u_i(t), \ i = 1,2,3, \cdots, k, \end{aligned} \tag{2.1a}$$

together with a set of interconnection equations:

$$u_i(t) = \sum_{j=1}^{k} E_{ij} y_j(t) + F_i r_i(t), \ i = 1,2,3, \cdots, k, \tag{2.1b}$$

where the $r_i(t)$ are external, vector valued inputs (many of which are usually identically zero), and the matrices $A_i(p)$, $B_i(p)$, $C_i(p)$ and $D_i(p)$ are continuously differentiable with respect to the design parameter $p \in \mathbf{R}^{n_\Sigma}$. In fact, except for the the indices i corresponding to compensator subsystems, the matrices $A_i(p)$, $B_i(p)$, $C_i(p)$ and $D_i(p)$ are constant.

Let $E \triangleq [E_{ij}]_{i,j \in \underline{k}}$, where $\underline{k} \triangleq \{1,2, \cdots, k\}$, and let $D_o \triangleq \mathrm{diag}(D_1(p), \cdots, D_k(p))$. When the the matrix $(I - ED_o(p))$ is nonsingular, the interconnection equations (2.1b) can be eliminated, to obtain a "closed loop" state space representation for the system $\Sigma(p)$ of the form

$$\begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)r(t), \\ y(t) &= C(p)x(t) + D(p)r(t), \end{aligned} \tag{2.1c}$$

where $x = (x_1, , ... x_k)$, $y = (y_1, , ... y_k)$, and $r = (r_1, , ... r_k)$. The matrices $A(p)$, $B(p)$, $C(p)$ and $D(p)$ are continuously differentiable in p.

Any design specifications will require that the closed loop system (2.1c) be exponentially stable. In addition, we can expect to have requirements of robustness, disturbance rejection (in the frequency domain) and plant saturation avoidance (in the time domain). The disturbance rejection requirement (in the output y_l with respect to the disturbance input r_m) is commonly expressed in the form

$$\overline{\sigma}[\hat{H}_{y_l r_m}(j\omega,p)] - b_{lm}(\omega) \le 0\,,\ \ \forall \omega \in [\omega',\omega'']\,, \tag{2.2a}$$

where $\hat{H}_{y_l r_m}(j\omega,p)$ is the appropriate *transfer function* matrix and $\overline{\sigma}[H]$ denotes the maximum singular value of the matrix H. The plant saturation avoidance requirement can be expressed in the form

$$\overline{\sigma}[H_{y_s r_q}(t,p)] - b_{sr}(t) \le 0\,,\ \ \forall t \in [0,\infty)\,, \tag{2.2b}$$

where $H_{y_s r_q}(t,p)$ is the appropriate *impulse response* matrix.

We note that both $\overline{\sigma}[\hat{H}_{y_l r_m}(j\omega,p)]$ and $\overline{\sigma}[H_{y_s r_q}(t,p)]$ can become unbounded when p is such that $\Sigma(p)$ is not exponentially stable. Because of this, semi-infinite optimization algorithms for control system design are invariably multiphase, with the first one or two phases devoted to computing a stabilizing compensator p_0 and the remaining (usually two) phases devoted to satisfying other design requirements and to optimization. During the last two phases, the algorithms produce a sequence of design vectors $\{p_i\}_{i=0}^{\infty}$ which are all stabilizing and yield progressively better performance. Consequently, the manner in which one ensures stability has a critical effect on the overall numerical behavior of a semi-infinite optimization algorithm in the design of a closed loop system. We now proceed to develop a new stability test for linear, time invariant mutivariable systems, which is efficient from the point of view of semi-infinite optimization.

We shall denote the characteristic polynomial of $\Sigma(p)$ by $\chi(s,p)$. Clearly, the coefficients of $\chi(s,p)$ are continuously differentiable in p. We shall discuss efficient methods of evaluating $\chi(s,p)$ and its derivatives with respect to p later.

When, it is desired to ensure not only exponential stability of a closed loop system, but also to exercise some control over the location of its poles, it is convenient to make use of the following definition of S-stability.

Definition 2.1 *(S-stability):* Consider a linear, time-invariant, finite dimensional dynamical system Σ of the form (2.1c). Let S be an open unbounded subset of $\mathbb{C}$ (e.g., as shown in Fig. 2.1) which is symmetrical with respect to the real axis, and such that $S^c \supset \mathbb{C}_+$, where S^c is the complement of S and $\mathbb{C}_+$ is the closed right half of the complex plane.

We say that the system Σ is S–stable if all the zeros of its characteristic polynomial are in S. ∎

In practice, S-stability can only be ensured with respect to a set S having a reasonably simple characterization. We shall assume that the set S has a right boundary ∂S which is given by an expression of the form

$$\partial S = \{ s \in \mathbb{C} \mid s = \sigma + j\omega, \sigma = f(\omega), -\infty < \omega < \infty \}, \tag{2.3a}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a negative valued, piecewise continuously differentiable function. In this case, the set S can be described by a simple inequality:

$$S = \{ s \in \mathbb{C} \mid s = \sigma + j\omega, \sigma - f(\omega) < 0, -\infty < \omega < \infty \}. \tag{2.3b}$$

Next, we shall denote by $\mathbb{C}[s]$ the ring of polynomials in $s \in \mathbb{C}$ with coefficients in $\mathbb{C}$, by $\mathbf{R}[s]$ the ring of polynomials in $s \in \mathbb{C}$ with coefficients in $\mathbf{R}$, and by P_N the set of monic polynomials of degree N contained in $\mathbf{R}[s]$. Furthermore, for any polynomial $\chi(s)$ in $\mathbb{C}[s]$ or in $\mathbf{R}[s]$, we shall denote by $Z[\chi(s)]$ the set of zeros of $\chi(s)$.

We now state our new modified Nyquist stability test.

Theorem 2.1 *(Modified Nyquist Stability Criterion):* Let $S \subset \mathbb{C}$ be as specified in Definition 2.1 and let $\mathbf{B} \subset \mathbb{C}$ be any simply connected set satisfying $(0,0) \notin \mathbf{B}$. Suppose that $D(s,q) \in \mathbb{C}[s]$ is a parametrized polynomial of degree N, whose coefficients depend on the parameter vector $q \in \mathbf{R}^{n_D}$ in such a way that for every $\chi(s) \in P_N$ satisfying $Z[\chi(s)] \subset S$, there exists a $q_\chi \in \mathbf{R}^{n_D}$ such that

$$(i) \quad Z[D(s,q_\chi)] \subset S, \tag{2.4a}$$

$$(ii) \quad \chi(s)/D(s,q_\chi) \in \mathbf{B}, \ \forall \ s \in \partial S. \tag{2.4b}$$

Then, given a polynomial $\chi(s) \in P_N$, $Z[\chi(s)] \subset S$ if and only if there exists a $q_\chi \in \mathbf{R}^{n_D}$ such that (2.4a,b) hold.

Proof : (=>) Suppose that $Z[\chi(s)] \subset S$. Then, by assumption, there exists a $q_\chi \in \mathbf{R}^{n_D}$ such that (2.4a), (2.4b) hold.

(<=) Next, suppose that (2.4a), (2.4b) hold. Then, because $\mathbf{B}$ is a simply connected set which does not contain the origin, the locus traced out in the complex plane by $\chi(s)/D(s,q_\chi)$, for $s \in \partial S$, does not encircle the origin. It now follows from (2.4a) and the Argument Principle [Mar.1] that $Z[\chi(s)] \subset S$. ∎

It is clear from Theorem 2.1 that an acceptable parametrization of the polynomial $D(s,q)$ depends on the shape of the set S and the choice of the set **B**. A further requirement is imposed by semi-infinite optimization: the parametrization must be such that it is easy to ensure that the zeros of $D(s,q)$ are in S. We shall see in the next section that the selection of a parametrization of $D(s,q)$ and of the set **B** can be fairly easy.

3. PARAMETRIZATION OF THE NORMALIZING POLYNOMIAL D(s, q)

Because it leads to the simplest semi-infinite inequalities, we shall set $\mathbf{B} = \overset{o}{\mathbb{C}}_+$ (the open right half plane), and we shall give a few examples of parametrizations of the normalizing polynomial $D(s,q)$. The simplest parametrization of $D(s,q)$, which can be used with *any* set $S \subset \overset{o}{\mathbb{C}}_-$, is

$$D(s,q) = a_0 + a_1 s^1 + \cdots + a_{N-1} s^{N-1} + s^N , \tag{3.1}$$

where N is the degree of $D(s,q)$, and $q \triangleq [a_0, a_1, \cdots, a_{N-1}]^T \in \mathbf{R}^N$. Since for any characteristic polynomial $\chi(s) \in P_N$, we can choose a coefficients vector q_χ for $D(s,q)$ in (3.1), such that $D(s,q_\chi) \equiv \chi(s)$ and since $(0,1) \in \overset{o}{\mathbb{C}}_+$, it is clear that (2.4a), (2.4b) are satisfied by this parametrization for any set $S \subset \overset{o}{\mathbb{C}}_-$ and $\mathbf{B} = \overset{o}{\mathbb{C}}_+$.

Unfortunately, from an optimization point of view, the parametrization (3.1) is not at all satisfactory for two reasons. The first is that the zeros of $D(s,q)$ may turn out to be unacceptably sensitive to variations in the parameter q. The second one is that, for a given set S, there appears to be no simple way of ensuring that $Z[D(s,q)] \subset S$. In fact, with the parametrization in (3.1), finding a $q \in \mathbf{R}^{n_D}$, such that $Z[D(s,q)] \subset S$ is almost as difficult as placing the zeros of a parametrized characteristic polynomial $\chi(s,p)$ in S.

A much better, general purpose parametrization of $D(s,q)$, for $\mathbf{B} = \overset{o}{\mathbb{C}}_+$ and any $S \subset \overset{o}{\mathbb{C}}_-$, is the following one:

$$D(s,q) = \prod_{i=1}^{N} (s - \alpha_i - j\beta_i), \tag{3.2}$$

where N is the degree of $D(s,q)$ and $q \triangleq [\alpha_1, \alpha_2, \cdots, \alpha_N, \beta_1, \beta_2, \cdots, \beta_N]^T \in \mathbf{R}^{2N}$.

Suppose that S is defined as in (2.3b), that $\mathbf{B} = \overset{o}{\mathbb{C}}_+$ and that $D(s,q)$ is parametrized as in (3.2). Referring to the requirements in Theorem 2.1, we see that given any $\chi(s) \in P_N$ such that $Z[\chi(s)] \subset \mathbf{S}$, the parametrization (3.2) allows us to choose a $q_\chi \in \mathbf{R}^{2N}$ such that $D(s, q_\chi) \equiv \chi(s)$. Hence (2.4a) is satisfied. Next, since $\chi(s)/D(s, q_\chi) \equiv 1$ and $(0,1) \in \overset{o}{\mathbb{C}}_+$, we see that (2.4b) is satisfied. It therefore follows from Theorem 2.1 that a stabilizing parameter $p_S \in \mathbf{R}^{n_\Sigma}$ for the parametrized system $\Sigma(p)$ can be obtained by solving the following set of ordinary and semi-infinite inequalities for a $p_S \in \mathbf{R}^{n_\Sigma}$ and a $q_S \in \mathbf{R}^{n_D}$:

$$\alpha_i - f(\beta_i) + \varepsilon \le 0, \quad \text{for } i = 1, 2, \cdots, N, \tag{3.3a}$$

$$\chi(f(\omega) + j\omega)/D(f(\omega) + j\omega, q) - \varepsilon \ge 0, \quad \forall\, \omega \in (-\infty, \infty), \tag{3.3b}$$

for some $\varepsilon > 0$. Current semi-infinite optimization algorithms have no difficulty solving this system of inequalities.

Note that as defined in (3.2), $D(s,q)$ is a polynomial with complex coefficients whose zeros need not occur in complex conjugate pairs, and that $n_D = 2N$. There are a number of cases where one can use a parametrization of $D(s,q)$ with fewer than $2N$ parameters. We shall give two examples.

Example 3.1 : Consider the case shown in Fig. 3.1a, where S is defined as a sector with damping angle $\Theta \in (0, \pi/2)$, i.e.,

$$\mathbf{S} \triangleq \{ s \in \mathbb{C} \mid \sigma + |\omega/\tan\Theta| < 0 \}. \tag{3.4}$$

Before giving a parametrization for $D(s,q)$ of arbitrary order, let us examine linear and quadratic polynomials. We see that when $a, b \in \mathbf{R}$, $Z[(s+a)] \subset \mathbf{S}$ if and only if

$$a > 0, \tag{3.5a}$$

and that $Z[(s^2 + as + b)] \subset \mathbf{S}$ if and only if

$$a > 0, \tag{3.5b}$$

$$b > 0, \tag{3.5c}$$

$$a^2 - 4b\cos^2\Theta > 0. \tag{3.5d}$$

To take advantage of this observation, when the degree of $D(s,q)$ is even, we set

$$D(s,q) \triangleq \prod_{i=1}^{n}(s^2+a_i s+b_i), \tag{3.6a}$$

where $q \triangleq [a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n]^T \in \mathbb{R}^{2n}$, and $N = 2n$. When the degree of $D(s,q)$ is odd, we set

$$D(s,q) \triangleq (s+a_0)\prod_{i=1}^{n}(s^2+a_i s+b_i), \tag{3.6b}$$

where $q \triangleq [a_0,a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n]^T \in \mathbb{R}^{2n+1}$, and $N = 2n+1$. Hence, when the polynomial $D(s,q)$ is expressed as a product of linear and quadratic polynomials, as in (3.5a) or (3.5b), it is very easy to ensure that its zeros are in S (defined in (3.4)).

Next we return to the requirements of Theorem 2.1. Since characteristic polynomials have real coefficients, their zeros occur in complex conjugate pairs. Therefore, given any $\chi(s) \in P_N$, such that $Z[\chi(s)] \subset \mathbf{S}$, there exists a parameter vector q_χ such that $D(s,q_\chi) = \chi(s)$ (with $D(s,q)$ defined as in (3.6a) or (3.6b), as appropriate). Therefore, if we set $\mathbf{B} = \overset{o}{\mathbb{C}}_+$, we see that (2.4a), (2.4b) are satisfied. It therefore follows from Theorem 2.1 that a stabilizing parameter p_S for the system $\Sigma(p)$ can be obtained by solving the following set of inequalities for a $p_S \in \mathbb{R}^{n_\Sigma}$ and a $q_S \in \mathbb{R}^N$:

$$q^i - \varepsilon \geq 0, \qquad \text{for } i = 1,2,\cdots,N, \tag{3.7a}$$

$$a_i^2 - 4b_i\cos^2\Theta > 0, \qquad \text{for } i = 1,2,\cdots,n, \tag{3.7b}$$

$$\mathrm{Re}[\chi(re^{j(\pi-\Theta)},p)/D(re^{j(\pi-\Theta)},q)] - \varepsilon \geq 0, \qquad \forall\, r \in [0,\infty), \tag{3.7c}$$

for some $\varepsilon > 0$. Note that because of symmetry, if (3.7c) is satisfied, then it is also satisfied when Θ is replaced by $-\Theta$.

As a trivial corollary of the above, we find that when $\mathbf{S} = \overset{o}{\mathbb{C}}_-$, we can compute a $p_S \in \mathbb{R}^{n_\Sigma}$ such that $\Sigma(p_S)$ is exponentially stable, by solving the following set of inequalities for a $p_S \in \mathbb{R}^{n_\Sigma}$ and a $q_S \in \mathbb{R}^N$:

$$q^i - \varepsilon \geq 0\,, \quad \text{for } i = 1,2,\cdots,N\,, \tag{3.8a}$$

$$\mathrm{Re}[\chi(j\omega,p)/D(j\omega,q)] - \varepsilon \geq 0\,, \quad \forall\, \omega \in [0,\infty)\,. \quad \blacksquare \tag{3.8b}$$

Example 3.2 : We saw in Example 3.1 that the parametrization (3.6a), (3.6b) has the property that for any $\chi(s) \in P_N$ there exists a $q_\chi \in \mathbf{R}^{n_D}$ such that $D(s,q_\chi) \equiv \chi(s)$. Hence it satisfies (2.4a), (2.4b) for any "stability" set $\mathbf{S} \subset \overset{0}{\mathbb{C}}_-$ (defined as in Definition 2.1) and any simply connected set $\mathbf{B} \subset \mathbb{C}$ such that $(0,0) \notin \mathbf{B}$ and $(0,1) \in \mathbf{B}$. However, as a practical matter, it can only be used for ensuring $\mathbf{S}$ -stability in the case where one can construct simple inequalities for ensuring that $Z[D((s,q) \subset \mathbf{S}$. In Example 3.1, we gave such a case. We shall now show that the parametrization (3.6a), (3.6b) can be used with a whole family of sets $\mathbf{S}$.

Consider the case where

$$\mathbf{S} = \{ s = (\sigma + j\omega) \in \mathbb{C} \mid \sigma < f(\omega),\ \omega \in (-\infty,\infty) \} \tag{3.9a}$$

with

$$f(\omega) = g(\omega^2) = \min_{j \in \underline{m}} g^j(\omega^2)\,. \tag{3.9b}$$

where $\underline{m} \triangleq \{1,2,\cdots,m\}$. We make three assumptions: for all $j \in \underline{m}$, (i) the functions $g^j : \mathbf{R} \to \mathbf{R}$ are continuous and negative valued, (ii) $g^j(\omega^2) \leq g(0)$ for all $\omega \in \mathbf{R}$, and (iii) $g^j(-\omega^2) \geq g(0)$ for all $\omega \in \mathbf{R}$.

The following three examples illustrate the kind of sets $\mathbf{S}$ can one define within the above framework:

$$g^i(z) = -\alpha\,, \ \forall\, z \in \mathbf{R}; \tag{3.10a}$$

$$g^j(z) = -z - \alpha\,, \ \forall\, z \in \mathbf{R}; \tag{3.10b}$$

$$g^k(z) = \begin{cases} -\beta\sqrt{z} + \gamma_1\,, & \forall\ z > \delta \\ -\alpha z + \gamma_2\,, & \forall\ |z| <= \delta \\ \beta\sqrt{-z} + \gamma_1\,, & \forall\ z < -\delta \end{cases} \tag{3.10c}$$

To set up the inequalities for ensuring that $Z[D(s,q)] \subset \mathbf{S}$, we again consider linear and quadratic factors. Now $-a$ is the only zero of $(s+a)$. Hence, $Z[(s+a)] \subset \mathbf{S}$ if and only if

$$-a < g(0). \tag{3.11a}$$

Next, the zeros of (s^2+as+b) are $s_1 = -a/2+\sqrt{a^2/4-b}$, and $s_2 = -a/2-\sqrt{a^2/4-b}$. Hence $Z[(s^2+as+b)] \subset \mathbf{S}$ if and only if

$$a^2/4-b \le 0 \ \text{ and } \ -a/2 < g(b-a^2/4) \tag{3.11b}$$

or

$$a^2/4-b > 0 \ \text{ and } \ -a/2 \pm \sqrt{a^2/4-b} < g(0). \tag{3.11c}$$

The relations (3.11b), (3.11c) cannot be used in conjunction with a semi-infinitie optimization algorithm. Therefore we propose to replace them with the following set of three inequalities which we shall show to be equivalent to (3.11b), (3.11c):

$$-a/2-g(0) < 0 \tag{3.12a}$$

$$-a/2-g(b-a^2/4) < 0 \tag{3.12b}$$

$$(a^2/4-b)-[g(0)+a/2]^2 < 0. \tag{3.12c}$$

Proposition 3.1 : The systems of inequalities (3.11b), (3.11c) and (3.12a), (3.12b), (3.12c) are equivalent.

Proof : (=>) Suppose that $s_1 = -a/2+\sqrt{a^2/4-b}$ satisfies either (3.11b) or (3.11c)(This ensures that s_2 satisfies either (3.11b) or (3.11c) as well). Suppose that (3.11b) holds. Then (3.12b) and (3.12c) hold automatically, while (3.12a) holds because $g(0) \ge g(b-a^2/4)$. Next suppose that (3.11c) holds. Then (3.12a) and (3.12c) holds automatically. By assumption, because $a^2/4-b > 0$, we have that $g(b-a^2/4) \ge g(0)$. Hence $g(b-a^2/4)+a/2 \ge g(0)+a/2 \ge 0$, which shows that (3.12a) holds.

(<=) Suppose that $s_1 = -a/2+\sqrt{a^2/4-b}$ satisfies (3.12a) - (3.12c). Then, if $a^2/4-b \le 0$, then (3.12b) implies that (3.11b) is satisfied. If $a^2/4-b > 0$, then (3.12a), (3.12c) imply that (3.11c) is satisfied. Hence the two systems are equivalent. ■

We conclude that if we set $\mathbf{B} = \overset{o}{\mathbb{C}}_+$, then a stabilizing parameter p_S for the system $\Sigma(p)$ can be obtained by solving the following set of inequalities for a $p_S \in \mathbf{R}^{n_\Sigma}$ and a $q_S \in \mathbf{R}^N$ (we state the inequalities for the case where N is odd) :

$$-a_0-g(0)+\varepsilon \le 0, \tag{3.13a}$$

$$-a_i/2-g(0)+\varepsilon \le 0, \text{ for } i = 1,2,\dots,n, \tag{3.13b}$$

$$-a_i/2-g(b_i-a_i^2/4)+\varepsilon \le 0, \text{ for } i = 1,2,\dots,n, \tag{3.13c}$$

$$(a_i^2/4-b_i)-[g(0)+a_i/2]^2+\varepsilon \le 0, \text{ for } i = 1,2,\dots,n, \tag{3.13d}$$

$$\mathrm{Re}[\chi(f(\omega)+j\omega),p)/D(f(\omega)+j\omega),q)]-\varepsilon \ge 0, \quad \forall\ \omega\in[0,\infty), \tag{3.13e}$$

for some $\varepsilon > 0$.

4. EVALUATION OF $\chi(s,p)$ AND ITS DERIVATIVES

The use of the stability test (2.4a), (2.4b), in the form (3.3a), (3.3b), or in the form (3.13a) -(3.13e), in conjunction with a semi-infinite optimization algorithm, requires the evaluation of the characteristic polynomial $\chi(s,p)$ and its partial derivatives. We shall now describe the method we chose to perform these evaluations.

The characteristic polynomial of the interconnected system Σ in (2.1c) is given by $\chi(s,p) = \det[sI-A(p)]$. For $s\in \mathbb{C}$, the evaluation of $\det[sI-A(p)]$ can be very time consuming, partly because it involves complex arithmetic and partly because the dimension of the matrix A may be large. Hence, to ensure the efficient evaluation of $\det[sI-A(p)]$ over a range of values of s, matrix decomposition methods must be used.

Matrix decomposition methods are based on similarity transformations yielding a matrix

$$\tilde{A}(p) = V(p)^{-1}A(p)V(p). \tag{4.1}$$

For the purpose of facilitating the evaluation of $\det[sI-A(p)]$, it is necessary to find a transformation matrix $V(p)$ which results in a matrix $\tilde{A}(p)$ such that $\det[sI-\tilde{A}]$ is easy to compute. The simplest situation occurs when the matrix $A(p)$ is diagonalizable, i.e., there exists a matrix of eigenvectors $V(p)$ such that $\tilde{A}(p) = \Lambda(p)$, with $\Lambda(p)$ a diagonal matrix whose diagonal elements are the eigenvalues $\lambda_j(p)$ of the matrix $A(p)$. In that case, assuming that $A(p)$ is an $N\times N$ matrix,

$$\chi(s,p) = \det[sI-A(p)] = \det[sI-\Lambda(p)] = \prod_{j=1}^{N}[s-\lambda_j(p)]. \tag{4.2}$$

Unfortunately, the problem involving the diagonalization of a nonsymmetric matrix $A(p)$ can be arbitrary ill-conditioned. Consequently, diagonalization cannot be used reliably to compute $\det[sI - A(p)]$ when $A(p)$ has fewer than N eigenvectors or when the transformation matrix $V(p)$ is close to singular. Thus formula (4.2) should not be used when the condition number $cond(V(p)) \triangleq ||V(p)|| \, ||V(p)^{-1}||$ is large.

When diagonalization cannot be used, one can simplify the computation of $\det[sI - A(p)]$ by first reducing $A(p)$ to upper Hessenberg form $H(p)$ by means of an orthogonal similarity transformation:

$$H(p) = V(p)^T A(p) V(p), \tag{4.3a}$$

where $V(p)$ is a Hermitian matrix, so that $cond(V(p)) = 1$. This leads to the formula

$$\chi(s,p) = \det[sI - A(p)] = \det[sI - H(p)]. \tag{4.3b}$$

The Hessenberg form $H(p)$ is cheaper to compute than the diagonal form $\Lambda(p)$. Furthermore, the computation of $H(p)$ is very stable and the computation of $\det[sI - H(p)]$ only requires some simple pivoting. Consequently, the computation of $\det[sI - A(p)]$ by computing $\det[sI - H(p)]$ is numerically very stable and, *for a single value of* s, it is definitely less costly than the computation of $\det[sI - A(p)]$ by evaluation of $\det[sI - \Lambda(p)]$. However, when, as in our case, one needs to evaluate $\det[sI - A(p)]$ for *many* values of s, the cost of pivoting in the computation of $\det[sI - H(p)]$ dominates and it is preferable to use formula (4.2), provided that $A(p)$ is not near defective.

Next we turn to the computation of the partial derivatives of $\chi(s,p)$. When the eigenvalues $\lambda_j(p)$ of $A(p)$ are distinct, they are differentiable (see [Kat.1]) and their partial derivatives are given by

$$\frac{\partial \lambda_j(p)}{\partial p^i} = \frac{\langle u_j, \frac{\partial A(p)}{\partial p^i} v_j \rangle}{\langle u_j, v_j \rangle}, \tag{4.4}$$

where v_j and u_j are the right and left eigenvectors, respectively, of $A(p)$, corresponding to the eigenvalue $\lambda_j(p)$. Therefore when the eigenvalues of $A(p)$ are distinct, the partial derivatives of $\chi(s,p)$ can be computed making use of the following formula:

$$\frac{\partial \chi(s,p)}{\partial p^i} = \sum_{j=1}^{N} \left\{ \frac{\partial \lambda_j(p)}{\partial p^i} \prod_{\substack{k=1 \\ k \neq j}}^{N} [s-\lambda_k(p)] \right\}$$

$$= \det[sI-A(p)] \sum_{j=1}^{N} \frac{\partial \lambda_j(p)}{\partial p^i} \frac{1}{s-\lambda_j(p)} , \tag{4.5}$$

When the eigenvalues of $A(p)$ are not distinct, the computation of $\partial \det[sI-A(p)]/\partial p^i$ becomes much more difficult and requires the use of a general formula which we shall now develop.

Proposition 4.1 : Let $M(s,p) \triangleq sI-A(p)$, let $m_{j,\bullet}$ and $m_{\bullet,j}$ denote the j-th row and the j-th column of $M(s,p)$, let $M^i_{j,\bullet}(s,p)$ denote the matrix obtained from $M(s,p)$ by replacing its $j-th$ row by $\frac{\partial m_{j,\bullet}}{\partial p^i}$, and let $M^i_{\bullet,j}(s,p)$ denote the matrix obtained from $M(s,p)$ by replacing its $j-th$ column by $\frac{\partial m_{\bullet,j}}{\partial p^i}$.

Then,

$$\frac{\partial \det[sI-A(p)]}{\partial p^i} = \sum_{j=1}^{N} \det[M^i_{j,\bullet}(s,p)] = \sum_{j=1}^{N} \det[M^i_{\bullet,j}(s,p)] . \tag{4.6}$$

Proof : We shall give a proof by induction. Equation (4.6) is clearly true for 2×2 matrices. Expanding a $(k+1)\times(k+1)$ determinant in terms of $k\times k$ cofactors, we see that if formula (4.6) holds for $k\times k$ matrices, it must also hold for $(k+1)\times(k+1)$ matrices.

∎

The computation of $\partial \det[sI-A(p)]/\partial p^i$ according to (4.6) is very expensive. Fortunately, closed loop control systems seldom, if ever, have repeated eigenvalues. In addition, the matrix $\partial A(p)/\partial p^i$ is sometimes very sparse, with the likely consequence that $\partial m_{j,\bullet}/\partial p^i = 0$ for some j and hence that $\det[M^i_{j,\bullet}(s,p)] = 0$.

To conclude this section, we shall summarize our suggestion for evaluating $\chi(s,p)$ and its partial derivatives in the form of an algorithm. We make use of the fact that one of the most robust methods (used in EISPACK [Smi.1]) for diagonalizing a matrix $A(p)$ is first to reduce it to upper Hessenberg form $H(p)$, using orthogonal similarity transformations, and then to reduce the Hessenberg form $H(p)$ to Schur form $S(p)$ by iterative unitary similarity transformations of the QR method. Finally, back substitutions are applied to accumulate eigenvectors.

Algorithm 4.1 : *(Evaluates $\chi(s,p)$ and its partial derivatives)*

Data : Matrix $A(p)$, $\zeta \gg 0$, an upper bound on acceptable condition number.

Step 1 : Reduce matrix $A(p)$ to Hessenberg form $H(p)$ by orthogonal transformation $Q(p)$ so that

$$H(p) = Q^T(p)A(p)Q(p). \tag{4.8}$$

Step 2 : Attempt to reduce Hessenberg form $H(p)$ to Schur form $S(p)$ by the QR method.

(i) If the QR method fails to converge, i.e. the eigenvalues of $A(p)$ cannot be computed, then compute $\det[sI - A(p)]$ using (4.3b) and $\partial \det[sI - A(p)]/\partial p^i$, $i = 1, \cdots, n_\Sigma$, using formula (4.6). Stop and exit.

(ii) If the reduction to Schur form was successful, then a complex orthogonal matrix $R(p)$ was constructed such that

$$S(p) = R(p)^H H(p) R(p). \tag{4.9}$$

The eigenvalues $\lambda_i(p)$ of $A(p)$ are given by

$$\lambda_i(p) = [S(p)]_{ii}, \ i = 1, 2, \cdots, N. \tag{4.10}$$

Step 3 : Compute a matrix of eigenvectors, $P(p)$, of the upper triangular Schur form $S(p)$ by back substitutions and compute its inverse $P(p)^{-1}$ (if it exists) and its condition number *cond* $(P(p))$.

Step 4 : If *cond* $(P(p)) \geq \zeta$ or $P(p)$ is singular, compute $\det[sI - A(p)]$ and its partial derivatives using (4.3b), (4.6).

Else construct the diagonal matrix $\Lambda(p) = diag(\lambda_1(p), \cdots, \lambda_N(p))$ which satisfies

$$\Lambda(p) = P(p)^{-1} S(p) P(p), \tag{4.11}$$

and compute the right and left eigenvector matrices

$$V(p) = Q(p)R(p)P(p), \tag{4.12a}$$

$$U(p) \triangleq V^{-1}(p) = P^{-1}(p)R^H(p)Q^T(p). \tag{4.12b}$$

Note : since Q and R are orthogonal matrices, $cond\,(V) = cond\,(P)$.

Step 5 : Set v_j to be the $j-th$ column of $V(p)$, set u_j^T to be the $j-th$ row of $U(p)$ and compute $\det[sI - A(p)]$ and its partial derivatives using (4.2), (4.5).

Stop and exit. ∎

5. DESIGN EXAMPLES

We shall now present two examples illustrating the use of our modified Nyquist stability criterion in control system design. We follow the design formulation methodology described in [Pol.3].

Example 5.1: Design of Control System for CH-47 Tandem Rotor Helicopter.

Consider the design of a control system, with configuration shown in Fig. 5.1, for a CH-47 tandem rotor helicopter. The control system controls two measured outputs, the vertical velocity and pitch attitude, by manipulating collective and differential collective rotor thrust commands. An abstracted, nominal plant model, denoted by P, for the dynamics relating these variables at 40 knot airspeed is given in [Doy.1]

$$\dot{x}_N = A_N x_N + B_N u_N \,, \tag{5.1a}$$

$$y_N = C_N x_N + D_N u_N \,, \tag{5.1b}$$

where

$$A_N = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32.0 \\ -0.14 & 0.44 & -1.3 & -30.0 \\ 0.0 & 0.018 & -1.6 & 1.2 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}, \qquad B_N = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0.0 & 0.0 \end{bmatrix}, \tag{5.1c}$$

$$C_N = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 57.3 \end{bmatrix}, \qquad D_N = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}. \tag{5.1d}$$

The eigenvalues of the plant are -2.22787, 0.0652232, and 0.491325 ± 0.415134i.

We assume that we are required to design a compensator which stabilizes the closed loop system, reduces sensitivity to output disturbances and avoids plant saturation by disturbances. We shall given a mathematical expression for these requirements shortly. First, we chose a compensator C, with a state space representation of the form

$$\dot{x}_C = A_C(p)x_C + B_C(p)u_C \tag{5.2a}$$

$$y_C = C_C(p)x_C + D_C(p)u_C \tag{5.2b}$$

with

$$A_C(p) = \begin{bmatrix} p^1 & p^2 & p^3 & p^4 \\ p^5 & p^6 & p^7 & p^8 \\ p^9 & p^{10} & p^{11} & p^{12} \\ p^{13} & p^{14} & p^{15} & p^{16} \end{bmatrix}, \qquad B_C(p) = \begin{bmatrix} p^{17} & p^{18} \\ p^{19} & p^{20} \\ p^{21} & p^{22} \\ p^{23} & p^{24} \end{bmatrix}, \tag{5.2c}$$

$$C_C(p) = \begin{bmatrix} p^{25} & p^{26} & p^{27} & p^{28} \\ p^{29} & p^{30} & p^{31} & p^{32} \end{bmatrix}, \qquad D_C(p) = \begin{bmatrix} p^{33} & p^{34} \\ p^{35} & p^{36} \end{bmatrix}. \tag{5.2d}$$

and $p = [\, p^1 , p^2 , \cdots , p^{36} \,]$, the design vector.

• *Closed-Loop System Stability Requirement:*

We begin with the most fundamental requirement: that of exponential stability of the nominal closed loop system. Let $\mathbf{B} = \overset{o}{\mathbb{C}}_+$, $\mathbf{S} = \overset{o}{\mathbb{C}}_-$, and let

$$D(s,q) \triangleq \prod_{i=1}^{4} (s^2 + a_i s + b_i) , \tag{5.3}$$

where $q \triangleq [\, a_1 , b_1 , a_2 , b_2 , a_3 , b_3 , a_4 , b_4 \,]$. By Theorem 3.2.4, exponential stability of the *nominal* closed-looped system will be ensured if

$$-q^i + \varepsilon < 0 , \qquad \text{for } i = 1 , 2, \cdots , 8 . \tag{5.4a}$$

$$-\operatorname{Re}[\chi(j\omega, p)/D(j\omega, q)] + \varepsilon < 0 , \qquad \forall\, \omega \in [0, \infty) , \tag{5.4b}$$

where $\varepsilon > 0$.

• *Stability Robustness*

Major unstructured uncertainties associated with our hellicopter model are due to neglected rotor dynamics and unmodeled rate limit nonlinearities. These are discussed at length in [Ste.1]. For the purpose of our design example, it suffices to note that the modeling uncertainties are the same in both control channels and that if the actual plant model is expressed in the form $P(s)(I + \Delta(s))$, then we may assume that

$$\bar{\sigma}[\Delta(j\omega)] < b(\omega) \qquad \forall\, \omega > 0, \tag{5.10}$$

with $\bar{\sigma}$ denoting the maximum singular value of the matrix in question and $b(\omega) > 1$ for all $\omega > 10$ rad/sec. Hence (see [Doy.1]) to ensure that not only the *nominal* design, but also the worst case design is exponentially stable, we require that

$$\bar{\sigma}[\hat{H}_{yr}(j\omega,p)] \le 1/b(\omega) \qquad \forall\omega > 0\,. \tag{5.6}$$

- *Plant Saturation Avoidance*

To ensure avoidance of plant saturation by disturbances over the frequency range [0.01 , 100], we impose the condition:

$$\bar{\sigma}[\hat{H}_{u_2 r}(j\omega,p)] <= \; 6.0, \;\; \forall\omega\in[\,0.01\,,\,100\,]\,. \tag{5.7}$$

- *Desensitization to Output Disturbances in System Bandwidth*

We assume that the desired bandwidth of the closed loop system is [0 , 2.0]. We shall desensitize the system to output disturbances within this frequency interval by introducing an appropriate cost function. To ensure that the disturbances are not excessively amplified outside the closed loop system bandwidth, we require that

$$\bar{\sigma}[\hat{H}_{yd}(j\omega,p)] <= \; 2.40, \;\; \forall\omega\in[\,2.0\,,\,1000.0\,] \tag{5.8}$$

- *Objective : Desensitization to Output Disturbances*

Finally, we model the requirement of output disturbance rejection over the frequency interval [0.01 , 2.00], as a cost function $f^0 : \mathbf{R}^{36} \to \mathbf{R}$, i.e.:

$$f^0(p) \triangleq \max\{\, \bar{\sigma}[\hat{H}_{yd}(j\omega\,,p)] \mid \omega\in[\,0.01\,,\,2.00\,]\,\}\,. \tag{5.9}$$

The initial values for the compensator were chosen arbitrarily as follows:

$$A_C(p) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -20 & -8 \end{bmatrix}, \qquad B_C(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.10a}$$

$$C_C(p) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad D_C(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.10b}$$

The resulting closed-loop eigenvalues were -10.1505, -7.6577, -0.101678, -1.85382, −0.0943453 ± 1.16857i, and 0.3505 ± 4.52988i. The initial values of q for $D(s, q)$ were chosen so that the zeros of the polynomial $D(s, q)$ matched the stable closed-loop eigenvalues, above. Hence we set q = [17.8282 , 77.9079 , 1.9755 , 0.2081 , 0.2087 , 1.3764 , 2.7724 , 22.4414]. The modified nyquist diagram for the initial values is shown in Fig.5.2.

Design via semi-infinite optimization must be carried out in two stages. First a stabilizing compensator must be computed, then the remainder of the optimization is carried out while maintaining closed loop stability.

After 14 iterations of the phase I - phase II, feasible directions method [Gon.1], the eight inequalities in (5.4a) and one functional inequality (5.4b), ensuring stability, were satisfied. The stabilizing compensator had the following matrices:

$$A_C(p) = \begin{bmatrix} -0.0597886 & 1.02787 & -0.00941522 & -0.00805203 \\ -3.00538 & -9.93141 & 0.00238179 & -0.053201 \\ -0.118403 & -0.00252507 & -0.154035 & 1.48117 \\ 0.00701371 & -0.00863074 & -20.0169 & -7.95165 \end{bmatrix}, \tag{5.11a}$$

$$C_C(p) = \begin{bmatrix} 1.00561 & 0.00432885 & 1.00029 & -0.0159385 \\ -0.076972 & 0.9845 & -0.17239 & 0.850142 \end{bmatrix}, \tag{5.11b}$$

$$B_C(p) = \begin{bmatrix} 0.993688 & -0.100487 \\ -0.151619 & 0.996535 \\ 0.863102 & -0.206853 \\ -0.23829 & 0.9523 \end{bmatrix}, \quad D_C(p) = \begin{bmatrix} -0.111091 & 0.286772 \\ -1.48049 & -0.110133 \end{bmatrix}. \tag{5.11c}$$

The resulting closed-loop eigenvalues at this point were −13.4488 + 0i, −9.1549 + 0i, −0.0531313 + 0i, −0.686026 + 0i, −0.632693 ± 1.83429i, and −3.68045 ± 5.36757i.

The vector q at iteration 14, was q = [17.8781 , 77.9102 , 3.81775 , 3.52904 , 3.86632 , 4.98543 , 6.1543 , 29.4078]. The modified Nyquist diagram for the stabilized system is shown in Fig. 5.3a. The state of the other requirements is shown in Fig. 5.3b,c.

After 63 iterations, all the design constraints were satisfied and the cost $\max_{\omega \in [0.01, 2.0]} \{ \bar{\sigma}[\hat{H}_{yd}(j\omega, p)] \} = 0.3625$. The final compensator is described by the following matrices:

$$A_C(p) = \begin{bmatrix} 0.469015 & 0.858198 & 0.0985659 & -0.291412 \\ -2.95312 & -9.96053 & -0.071059 & -0.17516 \\ -0.110506 & 0.031315 & 0.0878297 & 1.20987 \\ 0.0129985 & -0.0674458 & -8.01688 & -3.74451 \end{bmatrix}, \tag{5.12a}$$

$$C_C(p) = \begin{bmatrix} 0.761554 & 0.229028 & -1.60218 & 0.213464 \\ 0.127167 & 0.654161 & -0.760495 & -0.278236 \end{bmatrix}, \tag{5.12b}$$

$$B_C(p) = \begin{bmatrix} -0.379027 & -0.319741 \\ -0.0424007 & 0.37653 \\ 0.260686 & 1.77696 \\ 0.238893 & -0.337195 \end{bmatrix}, \quad D_C(p) = \begin{bmatrix} -0.902073 & 2.25228 \\ -0.877849 & 0.233123 \end{bmatrix}. \tag{5.12c}$$

The results of the final design are shown in Fig. 5.4a,b,c in solid lines, which can be compared with the results of the initial design shown in dashed lines.

Example 5.2 : A Design with Time and Frequency Domain Constraints

Consider the unity-feedback configuration shown in Fig. 5.1. We assume that the plant P is given as in (5.1a), (5.1b), with

$$A_N = \begin{bmatrix} -3 & -4 & -2 \\ 1 & 0 & 0 \\ 0 & -2 & -4 \end{bmatrix}, \quad B_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_N = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 3 \end{bmatrix}, \quad D_N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.13}$$

The compensator C is assumed to be the form (5.2a), (5.2b), with

$$A_C(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_C(p) = \begin{bmatrix} p^1 & p^2 \\ p^3 & p^4 \end{bmatrix},$$
$$C_C(p) = \begin{bmatrix} p^5 & p^6 \\ p^7 & p^8 \end{bmatrix}, \quad D_C(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.14}$$

where $p = [p^1, p^2, \cdots, p^8]$ is the design vector.

Our design constraints and objective are as follows:

- *Time Domain Constraints:*

(1) The step response constraints on plant output $y(t) = (y^1(t), y^2(t))$, corresponding to an input $r(t) = (r^1(t), r^2(t))$, $r^i(t) = 1$, for $i = 1, 2$, were specified in terms of the following numbers:

rise time	1.000
settling time	3.000
final time	10.000
peak amplitude	1.090
rise amplitude	0.800
settling amplitude ration	0.040

These constraints lead to the following pair of semi-infinite inequalities:

$$\underline{b}(t) <= y^i(t,p) <= \overline{b}(t), \quad \forall t \in [0, 16.0], i = 1, 2. \tag{5.15a}$$

where

$$\underline{b}(t) = \begin{cases} 0, & \text{for all } t \in [0, 1.00] \\ 0.8, & \text{for all } t \in [1.00, 3.0] \\ 0.96, & \text{for all } t \in [3.0, 10.0] \end{cases} \tag{5.15b}$$

$$\overline{b}(t) = \begin{cases} 1.09, & \text{for all } t \in [0, 3.0] \\ 1.04, & \text{for all } t \in [3.0, 10.0] \end{cases} \tag{5.15c}$$

- *Frequency Domain Constraints:*

(1) Noninteraction constraint:

$$| \hat{H}_{y_i r_j}(j\omega, p) | <= 0.15, \quad \forall \omega \in [0.01, 200], (i,j) \in \{ (1,2), (2,1) \}. \tag{5.16}$$

(2) Plant saturation by disturbance avoidance constraint:

$$\overline{\sigma}[\hat{H}_{u_2 r}(j\omega, p)] <= 5.0, \quad \forall \omega \in [0.01, 200]. \tag{5.17}$$

(3) Output disturbance desensitization constraint:

$$\overline{\sigma}[\hat{H}_{yd}(j\omega, p)] <= 1.05, \quad \forall \omega \in [1.0, 1000.0]. \tag{5.18}$$

• *Exponential Stability Constraints:*

To apply the modified Nyquist criterion, we define $\mathbf{B} = \overset{o}{\mathbb{C}}_+$ and let

$$D(s,q) \triangleq (s + a_0)\prod_{i=1}^{2}(s^2 + a_i s + b_i)\,, \tag{5.19}$$

where $q \triangleq [a_0, a_1, a_2, b_1, b_2]$, and we require that

$$-q^i + \varepsilon \le 0\,, \qquad \forall\, i = 1, 2, \cdots, 5\,. \tag{5.20a}$$

$$-\operatorname{Re}[\chi(j\omega, p)/D(j\omega, q)] + \varepsilon \le 0\,,\ \ \forall \omega \in [0, \infty)\,, \tag{5.20b}$$

where $\varepsilon > 0$.

• *Cost function:*

We propose to achieve good output disturbances rejection by defining it as our cost. This leads to the cost function $f : \mathbf{R}^8 \to \mathbf{R}$ defined by:

$$f^{\,0}(p) \triangleq \max\{\ \bar{\sigma}[\hat{H}_{yd}(j\omega,p)]\ \mid\ \omega \in [\,0.001\,,\,1.00\,]\ \}\,. \tag{5.21}$$

First a stabilizing compensator was obtained by means of the semi-infinite optimization algorithm [Gon.1], which satisfied the stability constraints (5.20a,b) :

$$A_c(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_c(p) = \begin{bmatrix} 1.001 & -1.000 \\ -0.499 & 1.000 \end{bmatrix},$$

$$C_c(p) = \begin{bmatrix} 1.001 & -0.999 \\ 0.100 & 1.500 \end{bmatrix}, \qquad D_c(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.22}$$

The system responses corresponding to this compensator are shown in Fig.5.5a,b,c,d,e,f in dashed lines. We see that some of the design constraints are violated. After 12 iterations of the Polak-Wardi algorithm [Pol.4], all the constraints are satisfied. After another 59 iterations, during which the cost function $f^0(p)$ was minimized without constraint violation, we obtained a compensator defined by the matrices

$$B_C(p) = \begin{bmatrix} 9.4709 & -4.2233 \\ -2.9109 & 10.5181 \end{bmatrix}, \qquad C_C(p) = \begin{bmatrix} 2.60501 & -3.895 \\ 0.49817 & 1.38818 \end{bmatrix}, \tag{5.23}$$

the matrices A_C and D_C remained as in (5.22).

The final design can be evaluated by inspecting the system responses in Fig.5.5a,b,c,d,e,f in solid lines.

6. CONCLUSION

This paper has shown that the existing modified Nyquist stability test can be modified to allow verification of S-stability in a numerically well-conditioned manner. This was achieved by introducing a normalizing denominator $D(s,p_d)$ for the stability test. Under the parameterization of case 1,2 and 3, the constraints imposed on the parameter p_d in order to guarantee $Z[D(s,p_d)] \subset S$ are continuously differentiable. In the process, we have made it impossible to determine the usual gain and phase margins for the SISO system. However, this is not a great loss, since by means of semi-infinite optimization, stability robustness can be ensured in a much more sophisticated manner; see [Pol.2].

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7. REFERENCES

[Gon.1] C. Gonzaga, E. Polak, and R. Trahan, "An Improved Algorithm for Optimization Problems with Functional Inequality Constraints," *IEEE Trans. Automat. Contr.*, vol.AC-25, no.1, 1980.

[Doy.1] J. C. Doyle, G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", *IEEE Trans. on Control*, Vol. AC-26, No.1, pp.4-16, 1982.

[Kat.1] T. Kato, *Perturbation Theory of Matrices*, J. Assoc. Comp. Mach., 1958.

[Mar.1] J. E. Marsden, *Basic Complex Analysis*, W. H. Freeman & Company, 1973.

[Nyq.1] H. Nyquist, "Regeneration Theory", *Bell Syst. Tech. J.* vol. 2, pp. 126-147, Jan. 1932.

[Pol.1] E. Polak, "A Modified Nyquist Stability Test for Use in Computer Aided Design", *IEEE* AC-29, No.1, pp.91-93, Jan. 1984.

[Pol.2] E. Polak and R. Trahan, "An Algorithm for Computer Aided Design of Control Systems", *Proc. IEEE Conf. on Dec. and Control*, 1976.

[Pol.3] E. Polak and Y. Wardi, "A nondifferentiable optimization algorithm for the design of control systems subject to singular value inequalities over a frequency range", *Automatica*, Vol. 18, NO. 3, pp. 267-283, 1982.

[Pol.4] E. Polak, D. Q. Mayne and D. M. Stimler, "Control System Design via Semi-Infinite Optimization", *Proceedings of the IEEE*, pp 1777-1795, December 1984.

[Smi.1] B. T. Smith, J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema, C. B. Moler, *Matrix Eigensystem Routines - EISPACK Guide*, Springer-Verlag, 1976.

[Ste.1] G. Stein, J. C. Doyle, "Singular values and feedback: Design Examples," *Proc. Allerton Conf.*, Urbana, IL, 1978.