STABILIZING EFFECTS OF FINITE-AMPLITUDE
RF WAVES ON THE INTERCHANGE INSTABILITY

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Stabilizing Effects of Finite-Amplitude RF Waves on the Interchange Instability

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The effects of the presence of a finite-amplitude RF wave on the interchange instability are studied theoretically in a three-dimensional quasineutral Darwin two-fluid system. An exact energy conservation law is developed for the system equations. The perturbed dynamical quantities are expanded to second order in the fluid displacements and it is demonstrated that a Lagrangian exists for the system. The interchange perturbation energy is calculated and terms associated with the perturbed magnetic potential energy are analyzed. For the terms studied, it is found that no stabilization of the incompressible interchange mode occurs unless the RF wave has a parallel-propagating component, and that coupling to perpendicular-propagating sidebands does not influence the stability of perpendicularly-propagating interchange modes. The stabilizing influence of these terms also does not depend directly on the presence of RF field gradients. The analysis suggests a physical mechanism by which RF stabilization can occur—the RF wave stabilizes the interchange mode by forcing the mode to bend magnetic field lines.

I. INTRODUCTION

The possibility of stabilizing the interchange mode with RF waves in the ion-cyclotron frequency range in axisymmetric mirror geometry was recently demonstrated in a series of experiments at the Phaedrus tandem mirror at Wisconsin [1]. The results of these experiments have led to considerable theoretical activity aimed at understanding the nature of the stabilizing mechanism. Ponderomotive effects of the RF wave on interchange stability have been studied by D'Ippolito and Myra [2] with a quasilinear kinetic model. Both ponderomotive and sideband coupling effects have been examined by Cohen and Rognlien [3] in the fluid approximation. Non-resonant [4] and resonant [5] sideband coupling effects have been considered by McBride and Stefan. Recently, Myra and D'Ippolito [6] also examined sideband effects. Finally, Similon and Kaufman [7] employed a two-timescale variational approach based on an appropriate ponderomotive potential to study the RF stabilization problem.

All theoretical models which examine the ponderomotive effects find the influence of an outwardly-directed RF wave electric field gradient to be stabilizing above the ion-cyclotron frequency ω_{ci} and destabilizing below it. The change in the stabilizing effect is generally attributed to the cyclotron-resonance; Similon and Kaufman however conclude that the transition occurs because resonances of the slow wave occur only below the cyclotron frequency. There is little agreement as yet on the effects of sideband coupling; the effect is found under various approximations to be stabilizing below ω_{ci} [4], stabilizing above ω_{ci} [5], and stabilizing on both sides of ω_{ci} [3]. Similon and Kaufman make no specific predictions concerning sideband coupling, but find the effect of the reaction of the interchange wave back on the RF wave, a related phenomenon, to be comparable with the ponderomotive effect [7].

In this paper, we also examine ponderomotive and sideband effects within the fluid approximation, but take an approach somewhat different from those advanced so far. First, a set of quasineutral fluid equations are developed which include the physics of both the interchange mode and waves in the ion-cyclotron frequency range in the Darwin approximation. The equations are a 3-d extension of the equations modeled by a computer simulation reported previously [8]. Next, an exact energy conservation law is derived for these equations. In considering the effect of an interchange-like perturbation on the resulting energy integral, neither the electrostatic approximation nor any approximation with regard to separation of timescales is used. The interacting effects of the RF wave and the interchange wave may thus be treated simultaneously. In avoiding these approximations we allow for, for example, the possibility of electromagnetic sidebands to be associated with the low-frequency, predominantly electrostatic component of the interchange mode. Also, since low frequency and sideband effects are treated together, ambiguities associated with the definition of these effects, which depend on the set of independent perturbation variables used, are avoided.

II. THE EQUATIONS

By assuming inertialess, zero-temperature electrons, we find from the electron momentum equation that the electric field may be expressed as

$$\mathbf{E} = -\frac{1}{c}\mathbf{u}_e \times \mathbf{B}, \tag{1}$$

where $\mathbf{u}_e$ is the electron fluid velocity. Using Eq. (1) in Ampère's Law, we see that

$$\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B} = \frac{1}{4\pi en}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2}$$

where, for brevity, we have defined $\mathbf{u} \equiv \mathbf{u}_i$ to be the *ion* fluid velocity. Substituting Eq. (1) in Faraday's Law and Eq. (2) into the pressureless fluid ion momentum equation, we obtain the system equations,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot (\mathbf{B}\mathbf{u}_e - \mathbf{u}_e\mathbf{B}), \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi mn}(\mathbf{B} \cdot \nabla \mathbf{B} - \tfrac{1}{2}\nabla B^2) + \mathbf{g}, \tag{4}$$

$$\mathbf{u}_e = \mathbf{u} - \frac{c}{4\pi en}\nabla \times \mathbf{B}, \tag{5}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0. \tag{6}$$

Here $\mathbf{g}(\mathbf{x}) \equiv -\nabla\phi_g$ is a fixed gravitational field, m is the *ion* mass, and $n = n_e = n_i$ is the density of both electrons and ions as required by the quasineutrality assumption. From Eq. (5), it is clear that the electron continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}_e) = 0, \tag{7}$$

is also valid, and may be substituted for Eq. (6).

An energy conservation law exists for this system of equations. It is shown in Appendix A that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnu^2 + \frac{B^2}{8\pi} + mn\phi_g\right) + \nabla \cdot \left(\frac{1}{2}mnu^2\mathbf{u} + \frac{B^2\mathbf{u}_e}{4\pi} - \frac{\mathbf{B}(\mathbf{B}\cdot\mathbf{u}_e)}{4\pi} + mn\phi_g\mathbf{u}\right) = 0. \tag{8}$$

which, of course, also implies conservation of the energy integral

$$\mathcal{E} \equiv \int d\mathbf{x} \left(\frac{1}{2} mnu^2 + \frac{B^2}{8\pi} + mn\phi_g \right), \tag{9}$$

up to a flux across the boundary

$$S \equiv \oint dS\, \hat{\mathbf{n}} \cdot \left(\frac{1}{2} mnu^2 \mathbf{u} + \frac{B^2 \mathbf{u}_e}{4\pi} - \frac{\mathbf{B}(\mathbf{B} \cdot \mathbf{u}_e)}{4\pi} + mn\phi_g \mathbf{u} \right), \tag{10}$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector from the boundary surface.

In this study, we concentrate on stabilizing processes which occur totally within the plasma region and do not rely on transfer of energy across the boundary. Processes involving the energy flux will in general be important but can be deferred since, within the formalism developed here, they may be treated separately by examining the flux integral Eq. (10). In the meantime, we ignore the energy flux contribution, or equivalently, assume it to be zero. The analysis will then be applicable to periodic systems, which have no boundaries, or to undriven, plasma-filled systems bounded by perfectly-conducting walls, for which $\hat{\mathbf{n}} \cdot \mathbf{u}$ and $\hat{\mathbf{n}} \times \mathbf{E}$ and therefore $\hat{\mathbf{n}} \cdot \mathbf{u}_e$ and the energy flux vanish at the boundaries.

III. THE PERTURBATION EXPANSION

We now expand the system quantities n, $\mathbf{u}$, $\mathbf{u}_e$, and $\mathbf{B}$ in orders of the interchange perturbation, e.g.,

$$\mathbf{B} = \mathbf{B}_0(\mathbf{x}, t) + \delta\mathbf{B}^{(1)}(\mathbf{x}, t) + \delta\mathbf{B}^{(2)}(\mathbf{x}, t) + \ldots. \tag{11}$$

Note that the zero-order quantities contain the time-independent equilibrium *and the finite-amplitude RF wave* and therefore depend on both time and space.

We find that all of the perturbed quantities may be expressed in terms of the electron and ion fluid displacements $\boldsymbol{\xi}_e(\mathbf{x}, t)$ and $\boldsymbol{\xi}_i(\mathbf{x}, t)$. By considering the change in volume occupied by an infinitesimal fluid element of species s located at some unperturbed location $\mathbf{x}_0$ under the spatial transformation defined by the displacement field $\boldsymbol{\xi}_s(\mathbf{x})$, we can calculate the perturbed density to be

$$n(\mathbf{x} + \boldsymbol{\xi}_s(\mathbf{x})) = \frac{n_0(\mathbf{x})}{\left[\hat{\mathbf{x}} + \frac{\partial \boldsymbol{\xi}_s}{\partial x}, \hat{\mathbf{y}} + \frac{\partial \boldsymbol{\xi}_s}{\partial y}, \hat{\mathbf{z}} + \frac{\partial \boldsymbol{\xi}_s}{\partial z} \right]}, \tag{12}$$

where $[,,]$, the triple scalar product, is the Jacobian of the transformation

$$\mathbf{x}_{new}(\mathbf{x}) = \mathbf{x} + \boldsymbol{\xi}_s(\mathbf{x}). \tag{13}$$

Expanding Eq. (12) as a Taylor series in $\boldsymbol{\xi}_s$, we obtain the desired expressions

$$\delta n^{(1)} = -\nabla \cdot (n_0 \boldsymbol{\xi}_s), \tag{14}$$

and

$$\delta n^{(2)} = \tfrac{1}{2}\nabla\nabla : (n_0 \boldsymbol{\xi}_s \boldsymbol{\xi}_s). \tag{15}$$

Since the continuity equation holds for both species, the formulas derived here for n, $\delta n^{(1)}$, and $\delta n^{(2)}$ hold equally well for the displacement field of either species.

The expressions for the perturbations to the species fluid velocities $\delta \mathbf{u}_s$ as functionals of the displacements are obtained by considering the new velocity field experienced by an infinitesimal fluid element resulting from displacement of the fluid element and perturbations to the velocity field. That is, as illustrated in Fig. 1, a fluid element located at $\mathbf{x}$ with velocity $\mathbf{u}_{0s}(\mathbf{x},t)$ at time t in the unperturbed system will instead find itself at $\mathbf{x}+\boldsymbol{\xi}_s(\mathbf{x},t)$ with velocity $(\mathbf{u}_{0s}+\delta\mathbf{u}_{0s})(\mathbf{x}+\boldsymbol{\xi}_s,t)$ at time t in the perturbed system. A similar statement holds at time $t+\Delta t$, where Δt is infinitesimally small. By formulating the relation among the vectors shown in Fig. 1 as a first order Taylor series in Δt and then allowing $\Delta t \to 0$, we can obtain for $\delta \mathbf{u}_s$ the expression

$$\begin{aligned}\delta\mathbf{u}_s(\mathbf{x},t) = &\frac{\partial \boldsymbol{\xi}_s}{\partial t} + \mathbf{u}_{s0}\cdot\nabla\boldsymbol{\xi}_s \\ &- [\mathbf{u}_{s0}(\mathbf{x}+\boldsymbol{\xi}_s,t) - \mathbf{u}_{s0}(\mathbf{x},t)] - [\delta\mathbf{u}_s(\mathbf{x}+\boldsymbol{\xi}_s,t) - \delta\mathbf{u}_s(\mathbf{x},t)].\end{aligned} \tag{16}$$

First- and second-order expressions for the perturbed fluid velocities may be obtained by first evaluating Eq. (16) to first order,

$$\delta\mathbf{u}_s^{(1)} = \frac{\partial \boldsymbol{\xi}_s}{\partial t} + \mathbf{u}_{s0}\cdot\nabla\boldsymbol{\xi}_s - \boldsymbol{\xi}_s\cdot\nabla\mathbf{u}_{s0}, \tag{17}$$

and then substituting this expression into Eq. (16) expanded to second order:

$$\delta\mathbf{u}_s^{(2)} = -\boldsymbol{\xi}_s\cdot\nabla\left(\frac{\partial \boldsymbol{\xi}_s}{\partial t} + \mathbf{u}_{s0}\cdot\nabla\boldsymbol{\xi}_s - \boldsymbol{\xi}_s\cdot\nabla\mathbf{u}_{s0}\right) - \tfrac{1}{2}\boldsymbol{\xi}_s\boldsymbol{\xi}_s : \nabla\nabla\mathbf{u}_{s0}. \tag{18}$$

When physical quantities are expressible in terms of fluid displacements, it means that those quantities depend only on the instantaneous configuration of the system without regard to how the system has evolved. That this property should hold for densities and fluid velocities is in fact quite obvious, and indeed, the formulas derived above hold quite generally for any system of fluids for which the image of the displacement-generated coordinate transformation Eq. (13) does not "fold" on itself. Mathematically this requires that the transformation be one-to-one with its image, or equivalently, that $\nabla\cdot\boldsymbol{\xi}_s > -1$ everywhere.

The functional dependence of the density and fluid velocities on the displacements outlined here is just a kinematic relationship. In contrast, the functional dependence of the magnetic field on the electron displacement is a field-dynamic physical property of our system, and does not hold in general. For example, if our equations included resistivity, the magnetic field would be able to relax to its lowest energy state, even if the fluids were held fixed. Thus, an infinite number of magnetic field configurations correspond to a single fluid state during the course of the evolution of a single system. In contrast, in our system, the magnetic field configuration is completely determined by the electron configuration.

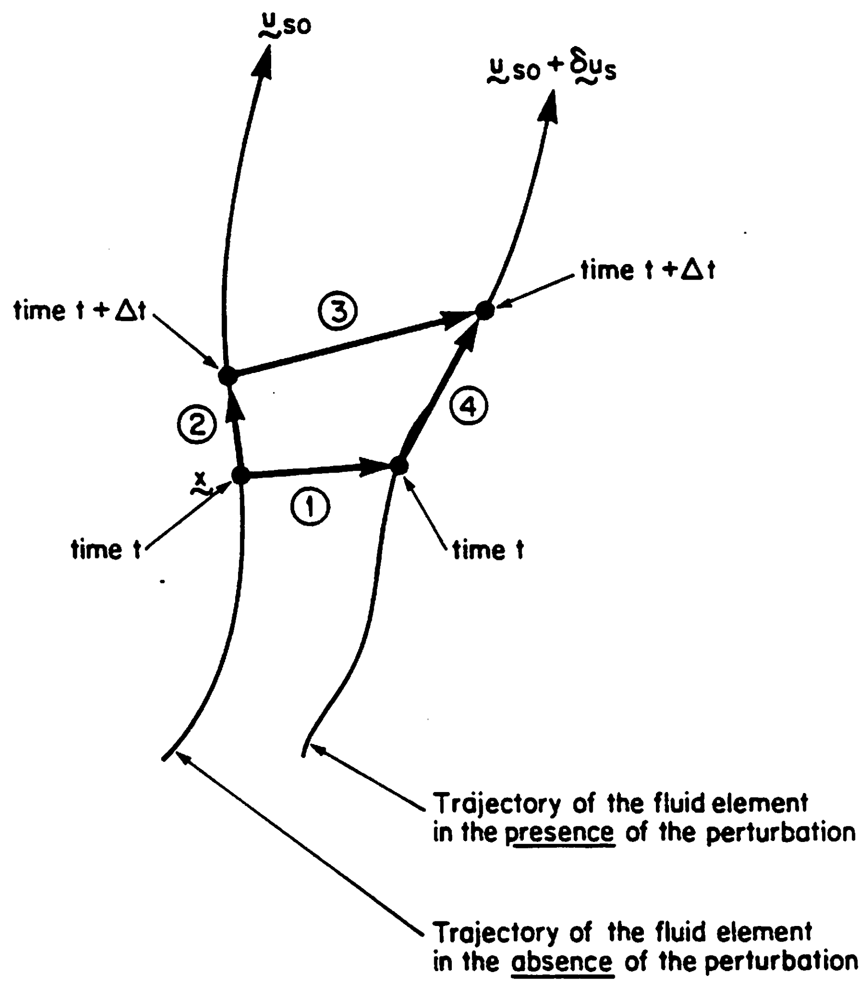

FIG. 1. Schematic illustration of the relationship among the unperturbed and perturbed fluid velocities $\mathbf{u}_{s0}$ and $\delta\mathbf{u}_s$ of a single fluid element at time t, and the associated fluid displacement $\boldsymbol{\xi}_s$ at times t and $t + \Delta t$, where s indexes the fluid species and Δt is considered infinitesimally small. Vectors in the diagram are defined by ① $= \boldsymbol{\xi}_s(\mathbf{x}, t)$, ② $= \mathbf{u}_{s0}(\mathbf{x}, t)\Delta t$, ③ $= \boldsymbol{\xi}_s(\mathbf{x} + \mathbf{u}_{s0}\Delta t, t + \Delta t)$, and ④ $= (\mathbf{u}_{s0} + \delta\mathbf{u}_s)(\mathbf{x} + \boldsymbol{\xi}_s(\mathbf{x}, t), t)\,\Delta t$.

We can motivate this by a simple argument. Consider two identical systems, both initiated with identical magnetic fields $\mathbf{B}(\mathbf{x},0)$. Let the electron fluid of one system, designated the primed system, evolve a factor of $1/\alpha$ more slowly than the unprimed system. The electron fluid velocities of the two systems are then related by

$$\mathbf{u}_e'(\mathbf{x},t) = \alpha \mathbf{u}_e(\mathbf{x},\alpha t). \tag{19}$$

At time t, we find the magnetic field of the primed system evolving according to

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}'(\mathbf{x},t) &= \nabla \cdot (\mathbf{B}'\mathbf{u}_e' - \mathbf{u}_e'\mathbf{B}') \\ &= \alpha \nabla \cdot (\mathbf{B}'\mathbf{u}_e(\mathbf{x},\alpha t) - \mathbf{u}_e(\mathbf{x},\alpha t)\mathbf{B}'),\end{aligned} \tag{20}$$

i.e., the evolution of $\mathbf{B}'$ at time t is $1/\alpha$ times slower than that of $\mathbf{B}$ at time αt but is otherwise identical. With both the electron fluid and the magnetic field evolving $1/\alpha$ times more slowly in the primed system, it is clear for this subset of system evolutions that the magnetic field configuration is determined by the electron fluid configuration.

It is therefore with some confidence that we assume $\mathbf{B}(\mathbf{x},t)$ is a functional of $\boldsymbol{\xi}_e(\mathbf{x},t)$. Armed with this assumption, we can derive formulas for the magnetic field perturbation quantities to any order. At first order, the perturbation magnetic field is

$$\frac{\partial}{\partial t}\delta\mathbf{B}^{(1)} = \nabla \cdot (\mathbf{B}_0 \delta\mathbf{u}_e^{(1)} + \delta\mathbf{B}^{(1)}\mathbf{u}_{e0} - \delta\mathbf{u}_e^{(1)}\mathbf{B}_0 - \mathbf{u}_{e0}\delta\mathbf{B}^{(1)}). \tag{21}$$

Since, by assumption, $\delta\mathbf{B}^{(1)}$ cannot depend functionally on $\mathbf{u}_{e0}$ but only on $\boldsymbol{\xi}_e$, we can set $\mathbf{u}_{e0} = 0$ in solving for $\delta\mathbf{B}^{(1)}$ and then expect the resulting expression, which will depend only on $\boldsymbol{\xi}_e$, to hold even when $\mathbf{u}_{e0} \neq 0$. By substituting the electron version of Eq. (17) in Eq. (21), setting $\mathbf{u}_{e0} = 0$, noting that the latter implies $\partial\mathbf{B}_0/\partial t = 0$, and integrating once with respect to t, we obtain

$$\delta\mathbf{B}^{(1)} = \nabla \cdot (\mathbf{B}_0\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\mathbf{B}_0), \tag{22}$$

which we then easily check does satisfy the full equation Eq. (21) with $\mathbf{u}_{e0} \neq 0$. Using the same method for $\delta\mathbf{B}^{(2)}$ in which expressions for $\delta\mathbf{B}^{(1)}$, $\delta\mathbf{u}_e^{(1)}$, and $\delta\mathbf{u}_e^{(2)}$ are used, we find

$$\delta\mathbf{B}^{(2)} = \tfrac{1}{2}\nabla\nabla\!:(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\mathbf{B}_0) - \nabla \cdot [\boldsymbol{\xi}_e(\mathbf{B}_0 \cdot \nabla)\boldsymbol{\xi}_e], \tag{23}$$

which again may be checked in analogous fashion, this time by means of a fairly lengthy calculation outlined in Appendix B.

The form of the first-order expressions for δn, $\delta\mathbf{u}_s$, and $\delta\mathbf{B}$ allows the linear evolution of the interchange mode in the presence of the RF wave to be described by a single vector equation linear in the electron displacement $\boldsymbol{\xi}_e$. The equation is time-periodic with the frequency of the RF wave and is thus suitable for study by means of an analytic method described in a related report [8]. It is expressible as the ion momentum equation,

$$\frac{\partial \delta \mathbf{u}}{\partial t} + \delta \mathbf{u} \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \delta \mathbf{u} = \frac{1}{4\pi m n_0} \left[\mathbf{B}_0 \cdot \nabla \delta \mathbf{B} + \delta \mathbf{B} \cdot \nabla \mathbf{B}_0 - \nabla \left(\mathbf{B}_0 \cdot \delta \mathbf{B} \right) \right]$$
$$- \frac{1}{4\pi m n_0} \frac{\delta n}{n_0} \left(\mathbf{B}_0 \cdot \nabla \mathbf{B}_0 - \frac{1}{2} \nabla B_0^2 \right), \tag{24}$$

with the substitutions

$$\delta \mathbf{u} \equiv \frac{\partial \boldsymbol{\xi}_e}{\partial t} + \mathbf{u}_{e0} \cdot \nabla \boldsymbol{\xi}_e - \boldsymbol{\xi}_e \cdot \nabla \mathbf{u}_{e0} + \frac{c}{4\pi e n_0} \nabla \times \delta \mathbf{B} - \frac{\delta n}{n_0} \frac{c \nabla \times \mathbf{B}_0}{4\pi e n_0}, \tag{25}$$

which follows from Ampère's Law,

$$\delta \mathbf{B} \equiv \nabla \cdot \left(\mathbf{B}_0 \boldsymbol{\xi}_e - \boldsymbol{\xi}_e \mathbf{B}_0 \right), \tag{26}$$

and

$$\delta n \equiv -\nabla \cdot \left(n_0 \boldsymbol{\xi}_e \right). \tag{27}$$

Expressions for the perturbed densities and fluid velocities also allow a Lagrangian formulation of our problem. By trial and error, a Lagrangian of the form $L = \int d\mathbf{x}\, \mathcal{L}$ has been found, where

$$\mathcal{L}\left(n_i, \mathbf{u}_i, n_e, \mathbf{u}_e, \mathbf{A}, \partial \mathbf{A}/\partial \mathbf{x}\right) = n_i(\mathbf{x},t) \left[\frac{1}{2} m_i \mathbf{u}_i^2(\mathbf{x},t) + \frac{e}{c} \mathbf{u}_i(\mathbf{x},t) \cdot \mathbf{A}(\mathbf{x},t) - m_i \phi_g(\mathbf{x}) \right]$$
$$- n_e(\mathbf{x},t) \frac{e}{c} \mathbf{u}_e(\mathbf{x},t) \cdot \mathbf{A}(\mathbf{x},t) - \frac{\left(\nabla \times \mathbf{A}(\mathbf{x},t)\right)}{8\pi}, \tag{28}$$

is the Lagrangian density and the ion index i is temporarily indicated explicitly. The system equations (3)–(6) are recovered by setting the variation of the path integral $\int dt\, L$ equal to zero and restricting solutions to those which obey quasineutrality as initial conditions:

$$n_i(\mathbf{x}, t = 0) = n_e(\mathbf{x}, t = 0). \tag{29}$$

Allowed variations are defined in terms of the assumed-independent functions $\boldsymbol{\xi}_i(\mathbf{x},t)$, $\boldsymbol{\xi}_e(\mathbf{x},t)$, and $\mathbf{A}(\mathbf{x},t)$ which in turn determine the allowed variations of the dynamical variables appearing in $\mathcal{L}$ via the relations

$$\delta n_s = -\nabla \cdot \left(n_s \boldsymbol{\xi}_s \right), \qquad s = i, e, \tag{30}$$

and

$$\delta \mathbf{u}_s = \frac{\partial \boldsymbol{\xi}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \boldsymbol{\xi}_s - \boldsymbol{\xi}_s \cdot \nabla \mathbf{u}_s, \qquad s = i, e. \tag{31}$$

Application of this prescription leads to the Euler-Lagrange equations for this system:

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial u_{sj}} + \frac{\partial}{\partial x_k} \left(u_{sk} \frac{\partial \mathcal{L}}{\partial u_{sj}} \right) + \frac{\partial u_{sk}}{\partial x_j} \frac{\partial \mathcal{L}}{\partial u_{sk}} = n_s \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial n_s}, \qquad s = i, e, \tag{32a}$$

and

$$\frac{\partial \mathcal{L}}{\partial A_j} = \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial (\partial A_j / \partial x_k)}. \tag{32b}$$

where the indices j and k run through x, y, and z, and sums over k are implied. Evaluation of these equations for the Lagrangian density $\mathcal{L}$ given by Eq. (28) yields equations which preserve quasineutrality if so initialized. Setting $n_i = n_e \equiv n$, defining $\mathbf{E} = -(1/c)(\partial \mathbf{A}/\partial t)$ and $\mathbf{B} = \nabla \times \mathbf{A}$, and invoking the continuity equation for each species as implied by Eq. (30) then yields Eqs. (3)–(6).

IV. THE EXPANSION ORDERING

The previous section has described derivations of first- and second-order expressions for the dynamical quantities δn_s, $\delta \mathbf{u}_s$, and $\delta \mathbf{B}$. The derivations of the expressions for $\delta n^{(1)}$, $\delta n^{(2)}$, $\delta \mathbf{u}_s^{(1)}$, and $\delta \mathbf{u}_s^{(2)}$ are proof of their consistency with our perturbation model, represented schematically by Fig. 1 and Eqs. (12) and (16), while the derivations of the perturbed magnetic field quantities $\delta \mathbf{B}^{(1)}$ and $\delta \mathbf{B}^{(2)}$ may be considered proof of the consistency of the corresponding expressions with Faraday's Law. We also have found that the expressions for δn and $\delta \mathbf{u}_s$ are consistent with the continuity equation through at least second order. However, while there are no apparent problems with the fluid kinematics and Faraday's Law, inconsistencies do appear, starting at second order, when the ion fluid dynamics are considered.

Demonstration of the presence of inconsistencies for our system of equations requires a simple but lengthy calculation. This calculation has not been performed for our system, but observation of the inconsistencies in analogous, simplified models leaves little doubt that similar difficulties will appear. A few illustrative examples of the problem are given in the two models described in Appendix C. The difficulty underlying this problem may be traced to the system of ordering employed. As explained in the Appendix, formal ordering in powers of $\boldsymbol{\xi}_s$ apparently leads to inconsistencies when the zero-order system is time-dependent. It becomes necessary in this case to order everything *including* $\boldsymbol{\xi}_s$ itself in powers of a parameter, say ϵ, i.e., $\boldsymbol{\xi}_s(\mathbf{x}) = \epsilon \boldsymbol{\xi}_s^{(1)}(\mathbf{x}) + \epsilon^2 \boldsymbol{\xi}_s^{(2)}(\mathbf{x}) + \ldots$, and then, as usual, set $\epsilon = 1$. The resulting ordering scheme removes all objections raised in the Appendix and does not change our first-order expressions if we identify $\boldsymbol{\xi}_s = \boldsymbol{\xi}_s^{(1)}$. The second order expressions, however, are modified. Specifically, the second order expression for each quantity is corrected by adding the quantity's own first-order expression with $\boldsymbol{\xi}_s^{(2)}$ substituted for $\boldsymbol{\xi}_s$. By this prescription, we obtain,

$$\delta n^{(2)} = \tfrac{1}{2} \nabla\nabla : (\boldsymbol{\xi}_s^{(1)} \boldsymbol{\xi}_s^{(1)} n_0) - \nabla \cdot (\boldsymbol{\xi}_s^{(2)} n_0), \tag{33}$$

$$\begin{aligned} \delta \mathbf{u}_s^{(2)} = & - \boldsymbol{\xi}_s^{(1)} \cdot \nabla \left(\frac{\partial \boldsymbol{\xi}_s^{(1)}}{\partial t} + \mathbf{u}_{s0} \cdot \nabla \boldsymbol{\xi}_s^{(1)} - \boldsymbol{\xi}_s^{(1)} \cdot \nabla \mathbf{u}_{s0} \right) - \tfrac{1}{2} \boldsymbol{\xi}_s^{(1)} \boldsymbol{\xi}_s^{(1)} : \nabla\nabla \mathbf{u}_{s0} \\ & + \frac{\partial \boldsymbol{\xi}_s^{(2)}}{\partial t} + \mathbf{u}_{s0} \cdot \nabla \boldsymbol{\xi}_s^{(2)} - \boldsymbol{\xi}_s^{(2)} \cdot \nabla \mathbf{u}_{s0}, \end{aligned} \tag{34}$$

and

$$\delta \mathbf{B}^{(2)} = \tfrac{1}{2} \nabla\nabla \cdot (\boldsymbol{\xi}_e^{(1)} \boldsymbol{\xi}_e^{(1)} \mathbf{B}_0) - \nabla \cdot [\boldsymbol{\xi}_e^{(1)} \cdot (\mathbf{B}_0 \cdot \nabla) \boldsymbol{\xi}_e^{(1)}] + \nabla \cdot (\mathbf{B}_0 \boldsymbol{\xi}_e^{(2)} - \boldsymbol{\xi}_e^{(2)} \mathbf{B}_0), \tag{35}$$

as our corrected second order expressions.

V. THE PERTURBED ENERGY

The first-order and corrected second-order expressions for the perturbed quantities may now be used to compute expressions for the energy integral to various orders of $\xi_s^{(1)}$, the main interest of this report. To zero order,

$$\mathcal{E}^{(0)} = \int d\mathbf{x} \left(\tfrac{1}{2} n_0 m u_0^2 + \frac{B_0^2}{8\pi} + n_0 m \phi_g \right). \tag{36}$$

Straightforward substitution of perturbation quantities into

$$\mathcal{E}^{(1)} = \int d\mathbf{x} \left(\tfrac{1}{2} \delta n^{(1)} m u_0^2 + n_0 m \mathbf{u}_0 \cdot \delta \mathbf{u}^{(1)} + \frac{1}{4\pi} \mathbf{B}_0 \cdot \delta \mathbf{B}^{(1)} + \delta n^{(1)} m \phi_g \right), \tag{37}$$

yields, with some integration by parts,

$$\mathcal{E}^{(1)} = \int d\mathbf{x} \left[n_0 m \mathbf{u}_0 \cdot \left(\frac{\partial \xi_i}{\partial t} + \mathbf{u}_0 \cdot \nabla \xi_i \right) - n_0 m \xi_e \cdot \left(\frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 \right) \right], \tag{38}$$

while substitution into

$$\mathcal{E}^{(2)} = \int d\mathbf{x} \left[\tfrac{1}{2} \delta n^{(2)} m u_0^2 + n_0 m \mathbf{u}_0 \cdot \delta \mathbf{u}^{(2)} + \frac{1}{4\pi} \mathbf{B}_0 \cdot \delta \mathbf{B}^{(2)} + \delta n^{(2)} m \phi_g \right.$$
$$\left. + \tfrac{1}{2} n_0 m (\delta \mathbf{u}^{(1)})^2 + \delta n^{(1)} m \mathbf{u}_0 \cdot \delta \mathbf{u}^{(1)} + \frac{1}{8\pi} (\delta \mathbf{B}^{(1)})^2 \right], \tag{39}$$

leads to

$$\mathcal{E}^{(2)} = \mathcal{E}_{1,1}^{(2)} + \mathcal{E}_2^{(2)}, \tag{40}$$

where

$$\mathcal{E}_{1,1}^{(2)} = \int d\mathbf{x} \left\{ \frac{1}{2} n_0 m \left(\frac{\partial \xi_i}{\partial t} + \mathbf{u}_0 \cdot \nabla \xi_i \right)^2 \right.$$
$$+ \frac{1}{8\pi} (\mathbf{B}_0 \cdot \nabla \xi_e)^2 - \frac{1}{4\pi} (\nabla \cdot \xi_e) \mathbf{B}_0 \cdot (\mathbf{B}_0 \cdot \nabla \xi_e) + \frac{B_0^2}{16\pi} \left[(\nabla \cdot \xi_e)^2 + \sum_{j,k} \frac{\partial \xi_{ej}}{\partial x_k} \frac{\partial \xi_{ek}}{\partial x_j} \right]$$
$$\left. + \frac{1}{2} n_0 m \xi_i \xi_i \colon \nabla \nabla \phi_g \right\}, \tag{41}$$

where the sums over j and k are meant to indicate sums over the x-, y-, and z-components, and

$$\mathcal{E}_2^{(2)} = \int d\mathbf{x} \left[n_0 m \mathbf{u}_0 \cdot \left(\frac{\partial \xi_i^{(2)}}{\partial t} + \mathbf{u}_0 \cdot \nabla \xi_i^{(2)} \right) \right.$$
$$\left. + \frac{1}{4\pi} \mathbf{B}_0 \cdot (\mathbf{B}_0 \cdot \nabla \xi_e^{(2)}) - \frac{1}{8\pi} B_0^2 (\nabla \cdot \xi_e^{(2)}) + n_0 m \xi_i^{(2)} \cdot \nabla \phi_g \right]. \tag{42}$$

In all these expressions, $\xi_s^{(1)}$ has been shortened to ξ_s. In Eqs. (40) through (42), $\mathcal{E}_{1,1}^{(2)}$ is meant to represent those terms which are formally square in the $\xi_s^{(1)}$'s, while $\mathcal{E}_2^{(2)}$ stands for that portion of the second-order energy which is linear in the $\xi_s^{(2)}$'s. In deriving these expressions, surface terms are assumed to vanish, consistent with the boundary conditions described earlier. Details of calculations leading to Eqs. (40)–(42) are presented in Appendix D.

VI. STABILITY ANALYSIS

Having now obtained a second-order expression for the perturbed energy associated with the interchange plasma displacements, we would normally proceed by analyzing the stability of the various terms. The presence, however, of terms involving $\xi_s^{(2)}$ (i.e., the terms in the integral in Eq. (42)), makes this task difficult. In fact, such a task might well be either analytically impossible or meaningless for general evolutions of the zero-order system; thus, further analysis of the entire expression Eq. (40) would likely require use of some of the various properties of the zero-order system of our problem. The time-dependence of our zero-order system is weak for example, due to the presence of the weak finite-amplitude RF wave, and time-periodic, allowing the use of time-Fourier components. Additionally, the system is assumed spatially-periodic which would eliminate all spatial harmonic dependence from all terms, greatly simplifying the analysis. A number of possibilities then exist, but will be left for future pursuit.

For the present, we will be content in examining a simpler problem—that of the stability properties implied by the perturbed magnetic potential energy terms formally square in the linear interchange displacements in Eq. (41). Since the terms we are considering represent a perturbation to a potential energy quantity, $B^2/8\pi$, we can be safe in assuming that any analysis demonstrating its increase in the presence of an interchange perturbation has demonstrated a stabilizing effect on that perturbation. In so doing, we are conceding the possibility that other terms, in particular, those involving $\xi_s^{(2)}$, may contribute other stabilizing or destabilizing effects in addition to those described here. It is likely in fact that such additional terms exist. At present, there exists both a theory and a computer simulation study [8] suggesting that perpendicularly-propagating RF waves have some stabilizing influence on interchange modes, whereas the present analysis would predict no such effect. Thus, we attempt in this analysis only to demonstrate the nature of some of the available stabilizing mechanisms, not to predict the sum effect of all such mechanisms.

Consider then the perturbation magnetic potential energy terms from Eq. (41):

$$\left(\frac{B^2}{8\pi}\right)^{(2)} = \frac{1}{8\pi}(\mathbf{B}_0\cdot\nabla\xi_e)^2 - \frac{1}{4\pi}(\nabla\cdot\xi_e)\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\xi_e) + \frac{B_0^2}{16\pi}\left[(\nabla\cdot\xi_e)^2 + \sum_{j,k}\frac{\partial\xi_{ej}}{\partial x_k}\frac{\partial\xi_{ek}}{\partial x_j}\right], \quad (43)$$

As discussed in Appendix D, the first of these terms is the square-first-order line-bending energy. The second contains both second-order line-bending perturbations and cross terms from first-order compressional and first-order line-bending perturbations, while the third magnetic energy term is the second-order magnetic compressional energy term.

We restrict our analysis to incompressible modes since these are generally the most unstable. The second and third magnetic energy terms are then zero, and we can conclude that RF stabilization comes only from the line-bending term $(\mathbf{B}_0 \cdot \nabla \xi_e)^2/8\pi$ in Eq. (41). This has a few important implications. First, for this term to be stabilizing, the RF wave must have a wave propagation component along the background magnetic field. The only perpendicularly-propagating wave existing in our system is the compressional Alfvén wave, for which the wave magnetic field is aligned with the background magnetic field. $\mathbf{B}_0$ always points in a constant direction in this case, and a growing interchange displacement is therefore free to propagate perpendicular to $\mathbf{B}_0$ since the line-bending term then vanishes. In contrast, in the case of non-perpendicular RF wave propagation, the RF wave magnetic field has a component perpendicular to the background field, the direction of $\mathbf{B}_0$ thus oscillates (i.e., line-bending is occurring), and the line-bending term is then positive-definite and therefore stabilizing.

A physical picture of the stabilizing mechanism is illustrated in Fig. 2. For simplicity, assume that the background magnetic field points in the z-direction, the density gradient and gravity are oriented respectively antiparallel and parallel to the x-axis, and the RF wave is a shear Alfvén wave propagating along $\hat{\mathbf{z}}$ with wave magnetic field pointing in the y-direction. The interchange mode thus propagates predominately in the y-direction. In this geometry, a single magnetic field line, bent by the shear Alfvén wave, instantaneously sees different phases of the interchange mode. Plasma motion of the interchange mode, typically in the x-direction, then tends to further bend the field line, thus drawing kinetic energy out the mode. Put another way, the RF wave stabilizes the interchange mode by forcing it to use energy bending field lines.

A second property of the stabilizing mechanism associated with the line-bending term is that perpendicularly-propagating interchange modes can not be stabilized by coupling to perturbation-induced perpendicularly-propagating RF sidebands. Such a stabilizing mechanism is only available when one or the other is obliquely-propagating. This may be seen by expanding both the zero-order magnetic field and the electron displacement in orders of the RF wave amplitude:

$$\mathbf{B}_0(\mathbf{x}, t) = \mathbf{B}_{0(0)}(\mathbf{x}) + \mathbf{B}_{0(1)}(\mathbf{x}, t) + \ldots, \tag{44}$$

$$\xi_e(\mathbf{x}, t) = \xi_{e(0)}(\mathbf{x}, t) + \xi_{e(1)}(\mathbf{x}, t) + \ldots, \tag{45}$$

where the subscript in parentheses denotes the order. The time dependence of these expansion quantities takes the form

$$\mathbf{B}_{0(1)} \sim \exp(\mp i\omega_{RF} t), \tag{46a}$$

$$\xi_{e(0)} \sim \exp(\mp i\omega t), \tag{46b}$$

$$\xi_{e(1)} \sim \exp[\mp i(\omega \pm \omega_{RF})t], \tag{46c}$$

where ω_{RF} is the RF wave frequency and ω is the interchange frequency. To second order in the RF wave, the sideband contributes to the d.c. line-bending energy only through terms

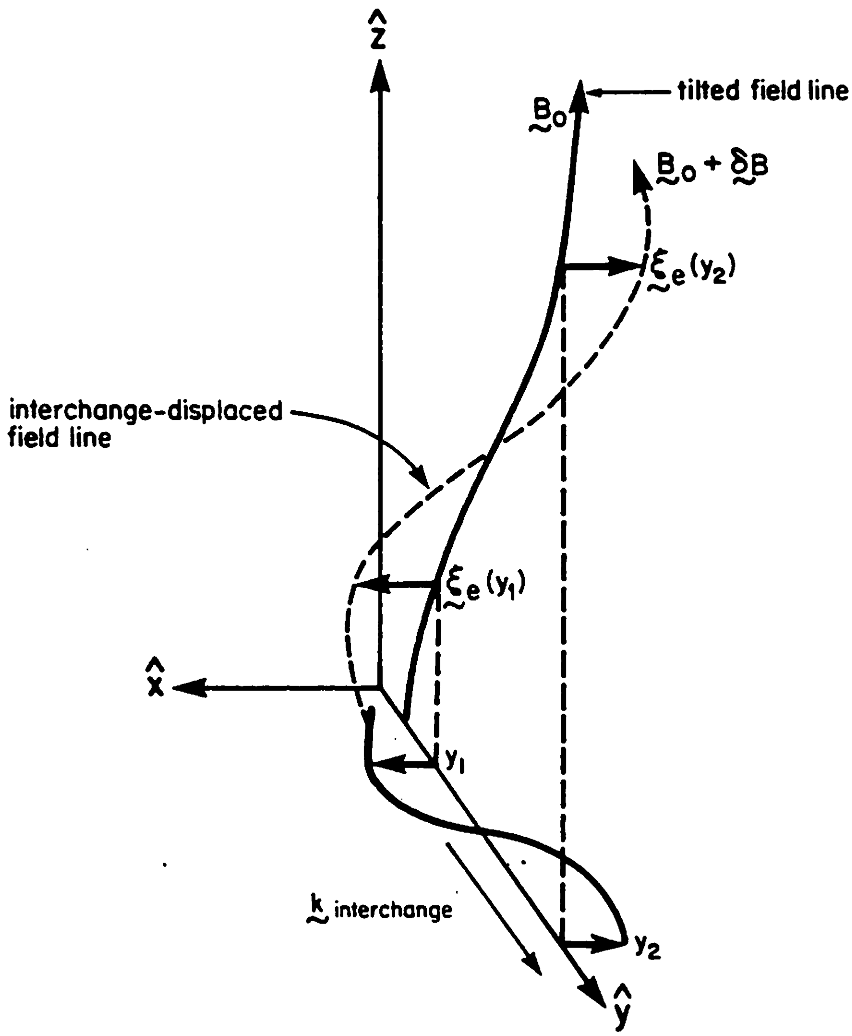

FIG. 2. Schematic of the line-bending stabilizing mechanism. In the presence of a parallel-propagating finite-amplitude shear Alfvén wave propagating along $\hat{z}$ with wave magnetic field in the $\hat{y}$-direction, a typical field line will be "wiggling" in the $\hat{y}$-$\hat{z}$ plane, as illustrated, with the wave frequency. At a given instant, the field line will take on a range of values for its y-coordinate at various points along its length. Two such coordinates, y_1 and y_2, are labeled. These two y-positions lie in *different* parts of the interchange wave, which we assume is propagating in the $\hat{y}$-direction. Electron plasma displacements associated with the interchange wave are then directed primarily in the $\pm\hat{x}$-direction, and thus will tug on the field line in that direction. Since y_1 and y_2 lie in different parts of the interchange wave, different portions of the field line will tend to experience different displacements $\xi_e(y_1)$ and $\xi_e(y_2)$, thus stretching the field line in the x-direction and drawing energy out of the interchange mode. This picture is confused somewhat by the fact that the field line actually oscillates with the Alfvén wave frequency, but the argument still stands; the field line sees different parts of the interchange wave and thus forces the wave to expend energy bending it.

of the form

$$\frac{1}{4\pi}(\mathbf{B}_{0(0)} \cdot \nabla \xi_{e(0)})(\mathbf{B}_{0(1)} \cdot \nabla \xi_{e(1)}), \tag{47a}$$

$$\frac{1}{4\pi}(\mathbf{B}_{0(0)} \cdot \nabla \xi_{e(1)})(\mathbf{B}_{0(1)} \cdot \nabla \xi_{e(0)}), \tag{47b}$$

and

$$\frac{1}{8\pi}(\mathbf{B}_{0(0)} \cdot \nabla \xi_{e(1)})^2, \tag{47c}$$

none of which contribute if both the interchange wave and its RF sidebands propagate perpendicular to the background field $\mathbf{B}_{0(0)}$. We note however, that the term

$$\frac{1}{8\pi}(\mathbf{B}_{0(1)} \cdot \nabla \xi_{e(0)})^2, \tag{47d}$$

which contains no sideband terms, can contribute.

Finally, as is clear from the form of the line-bending term, the stabilizing effect does not depend directly on spatial gradients of the wave magnetic field. This does not rule out the dependence of stability on the gradient of the wave electric field, nor does it discount the possibility that the mechanics of the propagation of the RF wave indirectly connects the magnitude of this term with wave field gradients.

V. SUMMARY

The stability of the interchange mode in the presence of a finite-amplitude RF wave has been examined in a three-dimensional quasineutral two-fluid system. An exact energy conservation law has been shown to hold for the equations governing the system. Additionally, all the dynamical quantities of the system associated with the interchange mode—the perturbed density, fluid velocities, and magnetic field—are demonstrated to be functionals of the electron and ion fluid displacements, and expansions for these dynamical quantities in the displacements are calculated to second order. Furthermore, at first order, it is shown that the ion displacements may be eliminated in favor of the electron displacements, allowing linear perturbations in a finite RF wave in our system to be described by a single linear vector equation with time-periodic coefficients. Linear expressions for the perturbed density and velocities are also used to demonstrate the existence of a Lagrangian for the system.

As demonstrated in Appendix C, the formal expansion in the displacements leads to inconsistencies in systems analogous to the one under study. The contradictions appear when the zero-order system is time-dependent and are apparently removed by defining a small parameter and ordering all dynamic quantities *including the displacements* in terms of this parameter. Expressions for the second-order energy are however changed. Terms involving the second-order displacements $\xi_s^{(2)}$ appear, and can probably be assumed to appear also in the problem system. These terms may, in general, be expected to contribute significantly to interchange stability, as evidenced by related theoretical and computer simulation studies [8].

Since the terms involving $\xi_s^{(2)}$ are difficult to treat, we focus our analysis on the perturbed magnetic energy terms expressible as bilinear forms in $\xi_e^{(1)}$, and leave open the possibility that other stabilizing or destabilizing effects exist. We find that, of these terms, only one contributes to the stability of incompressible interchange modes. It is found that the term only affects stability when the RF wave has a parallel-propagating component, and that the stabilizing mechanism does not directly depend on spatial gradients of the RF magnetic field. Furthermore, the term exhibits a stabilizing effect via sideband coupling only when either the interchange mode or the sideband itself possesses a parallel-propagating component.

Finally, a physical picture of the stabilizing mechanism associated with this term is discussed. The stabilizing effect is produced by forcing the interchange mode to give up energy bending magnetic field lines. This physical picture stands independently as a valid concept, irrespective of the validity of the underlying analysis, since it is motivated separately by a physically intuitive argument. That is, if the presence of the RF wave does not force the interchange mode to bend field lines, one can reasonably ask why not, and expect the answer to advance the understanding of the RF stabilization process. In fact, it cannot be denied that this mechanism is stabilizing; it appears after all as a positive definite term in the energy integral Eq. (41). It can only be argued, as already indicated, that other effects present may act either to cancel or dominate it.

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APPENDIX A. ENERGY CONSERVATION THEOREM

Conservation of energy for our system of equations is shown by demonstrating that the flow of magnetic field energy,

$$\frac{1}{4\pi}\nabla\cdot[B^2\mathbf{u}_e-\mathbf{B}(\mathbf{B}\cdot\mathbf{u}_e)]=-\frac{1}{4\pi}\nabla\cdot[(\mathbf{u}_e\times\mathbf{B})\times\mathbf{B}], \tag{A1}$$

which, from Eq. (1), is just the divergence of the Poynting flux $c\nabla\cdot(\mathbf{E}\times\mathbf{B})/4\pi$, is equal to the remaining terms in Eq. (8). Expanding the right-hand side of Eq. (A1) and using Ampère's Law in the second term, we find:

$$\begin{aligned}\frac{1}{4\pi}\nabla\cdot[B^2\mathbf{u}_e-\mathbf{B}(\mathbf{B}\cdot\mathbf{u}_e)]&=-\frac{1}{4\pi}\mathbf{B}\cdot\nabla\times(\mathbf{u}_e\times\mathbf{B})+\frac{1}{4\pi}\mathbf{u}_e\times\mathbf{B}\cdot(\nabla\times\mathbf{B})\\&=-\frac{1}{4\pi}\mathbf{B}\cdot[\nabla\cdot(\mathbf{B}\mathbf{u}_e-\mathbf{u}_e\mathbf{B})]-\frac{1}{4\pi}\left(\mathbf{u}-\frac{c}{4\pi en}\nabla\times\mathbf{B}\right)\cdot(\nabla\times\mathbf{B})\times\mathbf{B}.\end{aligned} \tag{A2}$$

Applying Faraday's Law to the first term on the right-hand side and the ion momentum equation (4) to the second, we obtain:

$$\frac{1}{4\pi}\nabla\cdot[B^2\mathbf{u}_e-\mathbf{B}(\mathbf{B}\cdot\mathbf{u}_e)]=-\frac{1}{8\pi}\frac{\partial}{\partial t}B^2-\frac{1}{2}mn\frac{\partial}{\partial t}u^2-mn\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2+\phi_g\right). \quad \text{(A3)}$$

Subtracting the quantity,

$$m\left(\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{u})\right)\left(\frac{1}{2}u^2+\phi_g\right), \quad \text{(A4)}$$

which is identically zero from the continuity equation (6), and using the fact the gravitational field in our system is static, we find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnu^2+\frac{B^2}{8\pi}+mn\phi_g\right)+\nabla\cdot\left(\frac{1}{2}mnu^2\mathbf{u}+\frac{B^2\mathbf{u}_e}{4\pi}-\frac{\mathbf{B}(\mathbf{B}\cdot\mathbf{u}_e)}{4\pi}+mn\phi_g\mathbf{u}\right)=0, \quad \text{(A5)}$$

the desired energy conservation law (Eq. (8)).

APPENDIX B. CHECK OF THE EXPRESSION FOR $\delta\mathbf{B}^{(2)}$

The expression for $\delta\mathbf{B}^{(2)}$ (Eq. (23)),

$$\delta\mathbf{B}^{(2)}=\tfrac{1}{2}\nabla\nabla:(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\mathbf{B}_0)-\nabla\cdot[\boldsymbol{\xi}_e(\mathbf{B}_0\cdot\nabla)\boldsymbol{\xi}_e], \quad \text{(B1)}$$

obtained by assuming $\mathbf{u}_{e0}=0$ in the expression

$$\frac{\partial}{\partial t}\delta\mathbf{B}^{(2)}=\nabla\cdot(\delta\mathbf{B}^{(2)}\mathbf{u}_{e0}+\delta\mathbf{B}^{(1)}\delta\mathbf{u}_e^{(1)}+\mathbf{B}_0\delta\mathbf{u}_e^{(2)}$$
$$-\mathbf{u}_{e0}\delta\mathbf{B}^{(2)}-\delta\mathbf{u}_e^{(1)}\delta\mathbf{B}^{(1)}-\delta\mathbf{u}_e^{(2)}\mathbf{B}_0), \quad \text{(B2)}$$

is verified by again considering this equation, but now allowing $\mathbf{u}_{e0}\neq 0$.

When expressions for $\delta\mathbf{u}_e^{(1)}$ (Eq. (17)), $\delta\mathbf{u}_e^{(2)}$ (Eq. (18)), $\delta\mathbf{B}^{(1)}$ (Eq. (22)), and $\delta\mathbf{B}^{(2)}$ (Eq. (B1)) are substituted into Eq. (B2), we find the terms of the vector equation obtained fall into four categories. These categories include (a) terms which contain the factor $\partial\boldsymbol{\xi}_e/\partial t$, and, to the exclusion of terms containing this factor, terms which acquire their vector character from the components of the factors (b) $\mathbf{u}_{e0}$, (c) $\boldsymbol{\xi}_e$, and (d) $\mathbf{B}_0$.

It can be shown that the terms belonging to each category separately satisfy the equation analogous to Eq. (B2). That is, we can show that

$$\nabla\nabla:\left(\boldsymbol{\xi}_e\frac{\partial\boldsymbol{\xi}_e}{\partial t}\mathbf{B}_0\right)-\nabla\cdot\left(\frac{\partial\boldsymbol{\xi}_e}{\partial t}\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e)\right)-\nabla\cdot\left[\boldsymbol{\xi}_e\nabla\cdot\left(\mathbf{B}_0\frac{\partial\boldsymbol{\xi}_e}{\partial t}\right)\right]=$$
$$\nabla\cdot\left[\left(\boldsymbol{\xi}_e\cdot\nabla\frac{\partial\boldsymbol{\xi}_e}{\partial t}\right)\mathbf{B}_0-\mathbf{B}_0\left(\boldsymbol{\xi}_e\cdot\nabla\right)\frac{\partial\boldsymbol{\xi}_e}{\partial t}\right], \quad \text{(B3a)}$$

for the terms in category (a),

$$\tfrac{1}{2}\nabla\nabla\colon[\boldsymbol{\xi}_e\boldsymbol{\xi}_e\nabla\cdot(\mathbf{B}_0\mathbf{u}_{e0})] =$$
$$\nabla\cdot\{\tfrac{1}{2}\nabla\nabla\colon(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\mathbf{B}_0)\,\mathbf{u}_{e0} - \nabla\cdot[\boldsymbol{\xi}_e\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e)]\,\mathbf{u}_{e0} - \nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\mathbf{B}_0)\,(\boldsymbol{\xi}_e\cdot\nabla\mathbf{u}_{e0})$$
$$- \tfrac{1}{2}\mathbf{B}_0\,(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\colon\nabla\nabla\mathbf{u}_{e0}) + \mathbf{B}_0\,(\boldsymbol{\xi}_e\cdot\nabla)\,(\boldsymbol{\xi}_e\cdot\nabla\mathbf{u}_{e0})\,\}, \tag{B3b}$$

for the terms in category (b),

$$-\nabla\cdot\{\boldsymbol{\xi}_e\nabla\cdot[\nabla\cdot(\mathbf{B}_0\mathbf{u}_{e0} - \mathbf{u}_{e0}\mathbf{B}_0)\,\boldsymbol{\xi}_e]\} =$$
$$\nabla\cdot\{\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\mathbf{B}_0)\,(\mathbf{u}_{e0}\cdot\nabla\boldsymbol{\xi}_e) - \mathbf{B}_0\,(\boldsymbol{\xi}_e\cdot\nabla)\,(\mathbf{u}_{e0}\cdot\nabla\boldsymbol{\xi}_e)$$
$$+ \mathbf{u}_{e0}\nabla\cdot[\boldsymbol{\xi}_e\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e)] - (\mathbf{u}_{e0}\cdot\nabla\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\cdot\nabla\mathbf{u}_{e0})\,\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e)\}, \tag{B3c}$$

for the terms in category (c), and

$$-\tfrac{1}{2}\nabla\nabla\colon[\boldsymbol{\xi}_e\boldsymbol{\xi}_e\nabla\cdot(\mathbf{u}_{e0}\mathbf{B}_0)] = \nabla\cdot\big[\tfrac{1}{2}\,(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\colon\nabla\nabla\mathbf{u}_{e0})\,\mathbf{B}_0 - \tfrac{1}{2}\mathbf{u}_{e0}\nabla\nabla\colon(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\mathbf{B}_0)$$
$$+ (\mathbf{u}_{e0}\cdot\nabla\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\cdot\nabla\mathbf{u}_{e0})\,\nabla\cdot(\boldsymbol{\xi}_e\mathbf{B}_0) + \boldsymbol{\xi}_e\cdot\nabla\,(\mathbf{u}_{e0}\cdot\nabla\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\cdot\nabla\mathbf{u}_{e0})\mathbf{B}_0\big], \tag{B3d}$$

for the terms in category (d), where the terms on the left-hand side of Eqs. (B3a)–(d) sum to the left-hand side of Eq. (B2) after the indicated substitutions are made, and similarly, the sum of terms on the right-hand side of Eqs. (B3a)–(d) equals to the right-hand side of Eq. (B2).

Some of the tricks used in verifying Eqs. (B3) were: (a) using $\nabla\cdot\mathbf{B}_0 = 0$ (necessary), (b) eliminating factors of 1/2 by, for example, noting that

$$\tfrac{1}{2}\partial_l\partial_k\,[\partial_j(\xi_l\xi_k)u_jB_i] = \partial_l\partial_k\,[\xi_l(\partial_j\xi_k)u_jB_i]\,, \tag{B4}$$

where $\partial_j \equiv (\partial/\partial x_j)$, etc. and sums over repeated indices are implied, and (c) consistently associating the same indices with the same vectors within an equation; e.g., indices are permuted so as to allow all terms in an equation to have the factors B_i, u_j, ξ_l, and ξ_k.

APPENDIX C. ANALYSIS OF THE ORDERING SCHEME

We can demonstrate the inconsistencies related to our original ordering scheme by examining two simple models.

Consider first a single particle moving in a static potential $U(\mathbf{x})$, i.e.,

$$\ddot{\mathbf{x}} = -\nabla U(\mathbf{x}). \tag{C1}$$

Allow the motion to be separated into a zero-order part $\mathbf{x}_0(t)$ and a perturbation $\boldsymbol{\xi}(t)$, so that $\mathbf{x}(t) = \mathbf{x}_0(t) + \boldsymbol{\xi}(t)$. We assume that there are zero-order flows $\dot{\mathbf{x}}_0(t) \neq 0$ and zero-order forces $\ddot{\mathbf{x}}_0(t) \neq 0$. Since $(d^2/dt^2)(\mathbf{x}_0 + \boldsymbol{\xi}) = -\nabla U(\mathbf{x}_0 + \boldsymbol{\xi})$, we have

$$\ddot{\mathbf{x}}_0(t) = -\nabla U(\mathbf{x}_0(t)), \tag{C2}$$

and

$$\ddot{\boldsymbol{\xi}}(t) = -\boldsymbol{\xi}\cdot\nabla\nabla U(\mathbf{x}_0(t)) - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon\nabla\nabla\nabla U(\mathbf{x}_0) + \ldots. \tag{C3}$$

There is of course an exact energy conservation law for this system—multiplying Eq. (C1) by $\dot{\mathbf{x}}$ gives

$$\frac{d}{dx}\left(\tfrac{1}{2}\dot{x}^2 + U(\mathbf{x})\right) = 0,$$

which implies

$$\tfrac{1}{2}\dot{x}^2 + U(\mathbf{x}) = \mathcal{E}. \tag{C4}$$

We now ask whether a conservation theorem exists for the energy associated with the perturbation. There are two ways to derive an equation for the energy. The first is to multiply Eq. (C3) by $\dot{\boldsymbol{\xi}}$. We then obtain

$$\dot{\boldsymbol{\xi}} \cdot \ddot{\boldsymbol{\xi}} + \dot{\boldsymbol{\xi}}\boldsymbol{\xi} \colon \nabla\nabla U(\mathbf{x}_0) = O(|\boldsymbol{\xi}|^3),$$

i.e.,

$$\frac{d}{dt}\left(\tfrac{1}{2}\dot{\xi}^2\right) + \dot{\boldsymbol{\xi}}\boldsymbol{\xi} \colon \nabla\nabla U(\mathbf{x}_0) = 0, \tag{C5}$$

to order $|\boldsymbol{\xi}|^2$.

The second method is to expand Eq. (C4) in orders of $|\boldsymbol{\xi}|$:

$$\tfrac{1}{2}(\dot{\mathbf{x}}_0 + \dot{\boldsymbol{\xi}})^2 + U(\mathbf{x}_0) + \boldsymbol{\xi} \cdot \nabla U(\mathbf{x}_0) + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi} \colon \nabla\nabla U(\mathbf{x}_0) + \ldots = \mathcal{E},$$

leading to, order by order in $|\boldsymbol{\xi}|$:

Zero-order:
$$\tfrac{1}{2}\dot{x}_0^2 + U(\mathbf{x}_0) = \mathcal{E}^{(0)}, \tag{C6}$$

First-order:
$$\dot{\mathbf{x}}_0 \cdot \dot{\boldsymbol{\xi}} + \boldsymbol{\xi} \cdot \nabla U(\mathbf{x}_0) = \mathcal{E}^{(1)}, \tag{C7}$$

Second-order:
$$\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi} \colon \nabla\nabla U(\mathbf{x}_0) = \mathcal{E}^{(2)}. \tag{C8}$$

It is, of course, this second-order energy which is of interest. But if we differentiate this expression with respect to time, we obtain

$$\frac{d}{dt}\left(\tfrac{1}{2}\dot{\xi}^2\right) + \dot{\boldsymbol{\xi}}\boldsymbol{\xi} \colon \nabla\nabla U(\mathbf{x}_0) + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\dot{\mathbf{x}}_0 \vdots \nabla\nabla\nabla U(\mathbf{x}_0) = 0, \tag{C9}$$

which does *not* agree with Eq. (C5) when zero-order forces $\mathbf{F}_0(\mathbf{x}_0) = -\nabla U(\mathbf{x}_0)$ are present, thus suggesting a problem with the expansion ordering. Note that Eq. (C3) relates the acceleration of $\boldsymbol{\xi}$ to a nonlinear force quadratic (and higher) in $\boldsymbol{\xi}$. The structure of the equations when a zero-order force is present requires us to make explicit the expansion of $\boldsymbol{\xi}$ itself in terms of its linear part:

$$\ddot{\boldsymbol{\xi}}^{(1)} = -\boldsymbol{\xi}^{(1)} \cdot \nabla\nabla U(\mathbf{x}_0), \tag{C10}$$

$$\ddot{\boldsymbol{\xi}}^{(2)} = -\tfrac{1}{2}\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)} \colon \nabla\nabla\nabla U(\mathbf{x}_0) - \boldsymbol{\xi}^{(2)} \cdot \nabla\nabla U(\mathbf{x}_0), \tag{C11}$$

where $\boldsymbol{\xi} \equiv \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + O(|\boldsymbol{\xi}^{(1)}|^3)$. This ordering does not change the form of linear equations; thus Eq. (C5), which was derived from a linear equation, remains intact. The ordering changes the expansion of $\mathcal{E}^{(2)}$, however. We now have

Zero-order: $$\tfrac{1}{2}\dot{x}_0^2 + U(\mathbf{x}_0) = \mathcal{E}^{(0)}, \tag{C12}$$

First-order: $$\dot{\mathbf{x}}_0 \cdot \dot{\boldsymbol{\xi}}^{(1)} + \boldsymbol{\xi}^{(1)} \cdot \nabla U(\mathbf{x}_0) = \mathcal{E}^{(1)}, \tag{C13}$$

Second-order: $$\dot{\mathbf{x}}_0 \cdot \dot{\boldsymbol{\xi}}^{(2)} + \boldsymbol{\xi}^{(2)} \cdot \nabla U(\mathbf{x}_0) + \tfrac{1}{2}\dot{\xi}^{(1)2} + \tfrac{1}{2}\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)}\colon \nabla\nabla U(\mathbf{x}_0) = \mathcal{E}^{(2)}. \tag{C14}$$

Now differentiating Eq. (C14) yields

$$\ddot{\mathbf{x}}_0 \cdot \dot{\boldsymbol{\xi}}^{(2)} + \dot{\mathbf{x}}_0 \cdot \ddot{\boldsymbol{\xi}}^{(2)} + \frac{d}{dt}\left(\tfrac{1}{2}\dot{\xi}^2\right) + \boldsymbol{\xi}\dot{\boldsymbol{\xi}}\colon \nabla\nabla U(\mathbf{x}_0) + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\dot{\mathbf{x}}_0 \vdots \nabla\nabla\nabla U(\mathbf{x}_0)$$
$$+\dot{\boldsymbol{\xi}}^{(2)} \cdot \nabla U(\mathbf{x}_0) + \boldsymbol{\xi}^{(2)}\dot{\mathbf{x}}_0\colon \nabla\nabla U(\mathbf{x}_0) = 0,$$

which is just Eq. (C5) (with $\boldsymbol{\xi} \leftrightarrow \boldsymbol{\xi}^{(1)}$) since

$$\ddot{\mathbf{x}}_0 \cdot \dot{\boldsymbol{\xi}}^{(2)} + \dot{\boldsymbol{\xi}}^{(2)} \cdot \nabla U(\mathbf{x}_0) = 0,$$

from Eq. (C2), and

$$\dot{\mathbf{x}}_0 \cdot \ddot{\boldsymbol{\xi}}^{(2)} + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\dot{\mathbf{x}}_0 \vdots \nabla\nabla\nabla U(\mathbf{x}_0) + \boldsymbol{\xi}^{(2)}\dot{\mathbf{x}}_0\colon \nabla\nabla U(\mathbf{x}_0) = 0,$$

from Eq. (C11). So Eq. (C14) is the correct form of the perturbation energy but is less useful due to the presence of the $\boldsymbol{\xi}^{(2)}$'s.

Now consider a simple fluid system represented by the equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{F}, \tag{C15}$$

and

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0. \tag{C16}$$

Again consider the evolution of the system to be composed of an unperturbed (zero-order) part and a perturbation. In the unperturbed system, consider the fluid element located at position $\mathbf{x}$ at time t. Define $\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ to be the location of the same fluid element at time t in the perturbed system. With this definition of $\boldsymbol{\xi}(\mathbf{x}, t)$, some careful thought about the acceleration of the head and tail of the vector $\boldsymbol{\xi}$, whose tail is being carried along by the motion of an unperturbed fluid element, produces

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla\right)\left(\frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u}_0 \cdot \nabla \boldsymbol{\xi}\right) = (\mathbf{F}_0 + \delta\mathbf{F})(\mathbf{x} + \boldsymbol{\xi}, t) - \mathbf{F}_0(\mathbf{x}, t), \tag{C17}$$

or

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla\right)\left(\frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u}_0 \cdot \nabla \boldsymbol{\xi}\right) =&\, \delta\mathbf{F}^{(1)} + \boldsymbol{\xi} \cdot \nabla \mathbf{F}_0 \\ &+ \delta\mathbf{F}^{(2)} + \boldsymbol{\xi} \cdot \nabla \delta\mathbf{F}^{(1)} + \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon \nabla\nabla\mathbf{F}_0 \\ &+ O(|\boldsymbol{\xi}|^3).\end{aligned} \tag{C18}$$

Order-by-order in $|\boldsymbol{\xi}|$ we obtain,

$$\left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)=\delta\mathbf{F}^{(1)}+\boldsymbol{\xi}\cdot\nabla\mathbf{F}_0, \tag{C19}$$

$$0=\delta\mathbf{F}^{(2)}+\boldsymbol{\xi}\cdot\nabla\delta\mathbf{F}^{(1)}+\tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon\nabla\nabla\mathbf{F}_0. \tag{C20}$$

In fact it can be shown that both of these equations are tautologies when

$$\mathbf{F}_0\equiv\frac{\partial\mathbf{u}_0}{\partial t}+\mathbf{u}_0\cdot\nabla\mathbf{u}_0, \tag{C21}$$

$$\delta\mathbf{F}^{(1)}\equiv\frac{\partial\delta\mathbf{u}^{(1)}}{\partial t}+\mathbf{u}_0\cdot\nabla\delta\mathbf{u}^{(1)}+\delta\mathbf{u}^{(1)}\cdot\nabla\mathbf{u}_0, \tag{C22}$$

$$\delta\mathbf{F}^{(2)}\equiv\frac{\partial\delta\mathbf{u}^{(2)}}{\partial t}+\mathbf{u}_0\cdot\nabla\delta\mathbf{u}^{(2)}+\delta\mathbf{u}^{(1)}\cdot\nabla\delta\mathbf{u}^{(1)}+\delta\mathbf{u}^{(2)}\cdot\nabla\mathbf{u}_0, \tag{C23}$$

where, as before,

$$\delta\mathbf{u}^{(1)}=\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla\mathbf{u}_0, \tag{C24}$$

$$\delta\mathbf{u}^{(2)}=-\boldsymbol{\xi}\cdot\nabla\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla\mathbf{u}_0\right)-\tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon\nabla\nabla\mathbf{u}_0. \tag{C25}$$

Nevertheless, as in the single particle case, we find inconsistencies with the perturbation energy. When Eq. (C19) is multiplied by the linear quantity $n_0(\partial/\partial t+\mathbf{u}_0\cdot\nabla)\boldsymbol{\xi}$, we obtain

$$\frac{\partial}{\partial t}\left[\frac{1}{2}n_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)^2\right]+\nabla\cdot\left[\frac{1}{2}n_0\mathbf{u}_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)^2\right]$$
$$=n_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)\cdot(\delta\mathbf{F}^{(1)}+\boldsymbol{\xi}\cdot\nabla\mathbf{F}_0). \tag{C26}$$

This equation should be right as it is analogous to Eq. (C5) in the single particle case; that is, Eq. (C19) is a linear equation.

Now suppose the force in Eq. (C15) is conservative: $\mathbf{F}=-\nabla U(\mathbf{x})$. Then $\delta\mathbf{F}^{(1)}=\delta\mathbf{F}^{(2)}=0$. This produces an immediate contradiction in Eq. (C20) for a general potential $U(\mathbf{x})$. We would in fact expect $\ddot{\boldsymbol{\xi}}$ to be affected by $\frac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon\nabla\nabla\mathbf{F}_0$, so the ordering $\boldsymbol{\xi}=\boldsymbol{\xi}^{(1)}+\boldsymbol{\xi}^{(2)}+\ldots$ is probably appropriate again. Then Eq. (C20) would be modified as

$$\left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)\left(\frac{\partial\boldsymbol{\xi}^{(2)}}{\partial t}+\mathbf{u}_0\cdot\nabla\boldsymbol{\xi}^{(2)}\right)=\delta\mathbf{F}^{(1)}[\boldsymbol{\xi}^{(2)}]+\boldsymbol{\xi}^{(2)}\cdot\nabla\mathbf{F}_0$$
$$+\delta\mathbf{F}^{(2)}[\boldsymbol{\xi}^{(1)},\boldsymbol{\xi}^{(1)}]+\boldsymbol{\xi}^{(1)}\cdot\nabla\delta\mathbf{F}^{(1)}[\boldsymbol{\xi}^{(1)}]+\tfrac{1}{2}\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)}\colon\nabla\nabla\mathbf{F}_0. \tag{C27}$$

and now there is no obvious difficulty. Note that the contradiction of $\frac{1}{2}\boldsymbol{\xi}\boldsymbol{\xi}\colon\nabla\nabla\mathbf{F}_0=0$ is eliminated if there are no zero-order forces.

Now consider the energy for this system. Again, there is an exact conservation law:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}nu^2 + nU(\mathbf{x})\right) + \nabla\cdot\left(\tfrac{1}{2}nu^2\mathbf{u} + nU(\mathbf{x})\mathbf{u}\right) = 0. \tag{C28}$$

We have already shown in the original plasma system that expanding Eq. (C28) to second order in $|\boldsymbol{\xi}|$ leads to an expression of the form:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}n_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)^2 + \frac{1}{2}n_0\boldsymbol{\xi}\boldsymbol{\xi}:\nabla\nabla U(\mathbf{x})\right] + \nabla\cdot[\text{something}] = 0, \tag{C29}$$

but, again in contradiction, evaluation of Eq. (C26) with $\delta\mathbf{F}^{(1)} = 0$, $\mathbf{F}_0 = -\nabla U(\mathbf{x})$ yields

$$\frac{\partial}{\partial t}\left[\frac{1}{2}n_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)^2 + \frac{1}{2}n_0\boldsymbol{\xi}\boldsymbol{\xi}:\nabla\nabla U(\mathbf{x})\right]$$
$$+\nabla\cdot\left[\frac{1}{2}n_0\mathbf{u}_0\left(\frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}\right)^2 + \frac{1}{2}n_0\mathbf{u}_0\boldsymbol{\xi}\boldsymbol{\xi}:\nabla\nabla U(\mathbf{x})\right] = \frac{1}{2}n_0\mathbf{u}_0\boldsymbol{\xi}\boldsymbol{\xi}\vdots\nabla\nabla\nabla U(\mathbf{x}), \tag{C30}$$

the term on the right-hand side being the offending term. Here we have used

$$\delta n^{(1)} = -\nabla\cdot(\boldsymbol{\xi}n_0), \tag{C31}$$

and

$$\delta n^{(2)} = \tfrac{1}{2}\nabla\nabla:(\boldsymbol{\xi}\boldsymbol{\xi}n_0), \tag{C32}$$

derived earlier. If $\boldsymbol{\xi}$ must be ordered, as suggested above, Eqs. (C24), (C25), (C31), and (C32) would be modified as

$$\delta\mathbf{u}^{(1)} = \frac{\partial\boldsymbol{\xi}^{(1)}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}^{(1)} - \boldsymbol{\xi}^{(1)}\cdot\nabla\mathbf{u}_0, \tag{C33}$$

$$\delta\mathbf{u}^{(2)} = \frac{\partial\boldsymbol{\xi}^{(2)}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}^{(2)} - \boldsymbol{\xi}^{(2)}\cdot\nabla\mathbf{u}_0 - \boldsymbol{\xi}^{(1)}\cdot\nabla\delta\mathbf{u}^{(1)} - \tfrac{1}{2}\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)}:\nabla\nabla\mathbf{u}_0, \tag{C34}$$

$$\delta n^{(1)} = -\nabla\cdot(\boldsymbol{\xi}^{(1)}n_0), \tag{C35}$$

$$\delta n^{(2)} = -\nabla\cdot(\boldsymbol{\xi}^{(2)}n_0) + \tfrac{1}{2}\nabla\nabla:(\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)}n_0). \tag{C36}$$

We have not verified that this kind of ordering produces consistent equations in the fluid case, nor have we verified that a magnetic force of the form $\mathbf{F} = (\nabla\times\mathbf{B})\times\mathbf{B}/4\pi m n_0$ yields inconsistent equations when ordering in terms of $|\boldsymbol{\xi}|$. Both conjectures are likely, however, in view of this analysis.

In summary, in the single particle case, ordering of $\boldsymbol{\xi} \equiv \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \ldots$ is required to produce consistent expressions for the conserved perturbed energy. The resulting equation is not immediately applicable within our analysis, due to presence of the $\boldsymbol{\xi}^{(2)}$'s. In the fluid case, the ordering $\boldsymbol{\xi} \equiv \boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(2)} + \ldots$ is required to remove two contradictions in the example of a static potential-derived force. All contradictions examined are removed in both the particle and fluid case if there are no zero-order forces; in this case ordering formally by $|\boldsymbol{\xi}|$ seems adequate.

APPENDIX D. PERTURBATION ENERGY CALCULATIONS

First order. The first-order energy perturbation is obtained by substituting first-order expressions for the perturbed ion density and ion fluid velocity (Eqs. (14) and (17) applied to the ion species) and perturbed magnetic field (Eq. (22)) into Eq. (37). We find

$$\mathcal{E}^{(1)} = \int d\mathbf{x} \Big[-\frac{1}{2}\nabla\cdot(n_0\boldsymbol{\xi}_i)mu_0^2 + n_0 m\mathbf{u}_0\cdot\Big(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}_i - \boldsymbol{\xi}_i\cdot\nabla\mathbf{u}_0\Big) + \frac{\mathbf{B}_0}{4\pi}\cdot\nabla\cdot(\mathbf{B}_0\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\mathbf{B}_0) - \nabla\cdot(n_0\boldsymbol{\xi}_e)m\phi_g\Big]. \tag{D1}$$

Integrating by parts, assuming as before that surface terms vanish, and using the relation

$$\int d\mathbf{x}\,[-\tfrac{1}{2}\nabla\cdot(n_0\boldsymbol{\xi}_i)mu_0^2] = \int d\mathbf{x}\,(\tfrac{1}{2}n_0 m\boldsymbol{\xi}_i\cdot\nabla u_0^2) = \int d\mathbf{x}\, n_0 m\mathbf{u}_0\cdot(\boldsymbol{\xi}_i\cdot\nabla)\mathbf{u}_0, \tag{D2}$$

to cancel a term, we obtain

$$\mathcal{E}^{(1)} = \int d\mathbf{x}\Big[n_0 m\mathbf{u}_0\cdot\Big(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}_i\Big) - \boldsymbol{\xi}_e\cdot\Big(\frac{(\nabla\times\mathbf{B}_0)\times\mathbf{B}_0}{4\pi} - n_0 m\nabla\phi_g\Big)\Big], \tag{D3}$$

or, using the unperturbed ion momentum equation,

$$\mathcal{E}^{(1)} = \int d\mathbf{x}\Big[n_0 m\mathbf{u}_0\cdot\Big(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}_i\Big) - n_0 m\boldsymbol{\xi}_e\cdot\Big(\frac{\partial\mathbf{u}_0}{\partial t} + \mathbf{u}_0\cdot\nabla\mathbf{u}_0\Big)\Big]. \tag{D4}$$

Second order. Calculation of the second-order perturbation energy is more involved; thus we first split $\mathcal{E}^{(2)} = \mathcal{E}_{1,1}^{(2)} + \mathcal{E}_2^{(2)}$ as described in the main text, and then define $\mathcal{E}^{(2)} \equiv \mathcal{K}^{(2)} + \mathcal{U}_M^{(2)} + \mathcal{U}_g^{(2)}$, where $\mathcal{K}^{(2)}$, $\mathcal{U}_M^{(2)}$, and $\mathcal{U}_g^{(2)}$ are respectively the kinetic energy, and magnetic and gravitational potential energy contributions to $\mathcal{E}_{1,1}^{(2)}$. Calculation of $\mathcal{E}_2^{(2)}$ is similar to the first order calculations just outlined, since all terms are linear in $\boldsymbol{\xi}_s^{(2)}$. We therefore describe just the calculation of $\mathcal{E}_{1,1}^{(2)}$.

Considering first $\mathcal{K}^{(2)}$, we find:

$$\begin{aligned}\mathcal{K}^{(2)} &= \int d\mathbf{x}\,[\tfrac{1}{2}n_0 m(\delta\mathbf{u}^{(1)})^2 + \tfrac{1}{2}\delta n_{1,1}^{(2)} mu_0^2 + n_0 m\mathbf{u}_0\cdot\delta\mathbf{u}_{1,1}^{(2)} + \delta n^{(1)} m\mathbf{u}_0\cdot\delta\mathbf{u}^{(1)}],\\ &= \int d\mathbf{x}\,[\tfrac{1}{2}n_0 m(\delta\mathbf{u}^{(1)})^2 + \tfrac{1}{2}\delta n_{1,1}^{(2)} m\mathbf{u}_0\\ &\quad - n_0 m\mathbf{u}_0\cdot(\tfrac{1}{2}\boldsymbol{\xi}_i\boldsymbol{\xi}_i\colon\nabla\nabla\mathbf{u}_0 + \boldsymbol{\xi}_i\cdot\nabla\delta\mathbf{u}^{(1)}) - \nabla\cdot(n_0\boldsymbol{\xi}_i)m\mathbf{u}_0\cdot\delta\mathbf{u}^{(1)}],\end{aligned} \tag{D5}$$

where here and in the following the subscript "1,1" refers to that part of the perturbed quantity formally bilinear in $\boldsymbol{\xi}_s$. When the last term is integrated by parts, the last two terms yield $n_0 m \delta\mathbf{u}^{(1)} \cdot (\boldsymbol{\xi}_i \cdot \nabla)\mathbf{u}_0$. When combined with the first term, we obtain,

$$\begin{aligned}\tfrac{1}{2}n_0 m(\delta\mathbf{u}^{(1)})^2 &+ n_0 m\delta\mathbf{u}^{(1)} \cdot (\boldsymbol{\xi}_i \cdot \nabla)\mathbf{u}_0 \\ &= \tfrac{1}{2}n_0 m\left(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0 \cdot \nabla\boldsymbol{\xi}_i - \boldsymbol{\xi}_i \cdot \nabla\mathbf{u}_0\right) \cdot \left(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0 \cdot \nabla\boldsymbol{\xi}_i + \boldsymbol{\xi}_i \cdot \nabla\mathbf{u}_0\right), \\ &= \tfrac{1}{2}n_0 m\left(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0 \cdot \nabla\boldsymbol{\xi}_i\right)^2 - \tfrac{1}{2}n_0 m(\boldsymbol{\xi}_i \cdot \nabla\mathbf{u}_0)^2. \end{aligned} \tag{D6}$$

Thus, when the remaining substitutions are made,

$$\begin{aligned}\mathcal{K}^{(2)} = \int d\mathbf{x}\,\Bigg[\frac{1}{2}n_0 m\left(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0 \cdot \nabla\boldsymbol{\xi}_i\right)^2 &- \frac{1}{2}n_0 m(\boldsymbol{\xi}_i \cdot \nabla\mathbf{u}_0)^2 \\ &+ \frac{1}{4}mu_0^2\nabla\nabla\colon(n_0\boldsymbol{\xi}_i\boldsymbol{\xi}_i) - \frac{1}{2}n_0 m\mathbf{u}_0 \cdot (\boldsymbol{\xi}_i\boldsymbol{\xi}_i\colon\nabla\nabla\mathbf{u}_0)\Bigg]. \end{aligned} \tag{D7}$$

The second and fourth terms of this expression simplify to ($\partial_j \equiv (\partial/\partial x_j)$, etc., sum over repeated indices implied) :

$$\begin{aligned}-\tfrac{1}{2}n_0 m(\boldsymbol{\xi}_i \cdot \nabla\mathbf{u}_0)^2 - \tfrac{1}{2}n_0 m\mathbf{u}_0 \cdot (\boldsymbol{\xi}_i\boldsymbol{\xi}_i\colon\nabla\nabla\mathbf{u}_0) &= -\tfrac{1}{2}n_0 m[\xi_j(\partial_j u_{0i})\xi_k(\partial_k u_{0i}) + u_{0i}\xi_j\xi_k\partial_j\partial_k u_{0i}] \\ &= -\tfrac{1}{2}n_0 m\xi_j\xi_k[\partial_j(u_{0i}\partial_k u_{0i})] = -\tfrac{1}{4}n_0 m\xi_j\xi_k\partial_j\partial_k u_{0i}^2 \\ &= -\tfrac{1}{4}n_0\boldsymbol{\xi}_i\boldsymbol{\xi}_i\colon\nabla\nabla(u_{0i}^2), \end{aligned} \tag{D8}$$

which, when integrated by parts two times, cancels the third term. Thus we obtain for the second-order kinetic energy:

$$\mathcal{K}^{(2)} = \int d\mathbf{x}\,\frac{1}{2}n_0 m\left(\frac{\partial\boldsymbol{\xi}_i}{\partial t} + \mathbf{u}_0 \cdot \nabla\boldsymbol{\xi}_i\right)^2. \tag{D9}$$

The second-order magnetic field energy is found by evaluating the expression

$$\mathcal{U}_M^{(2)} = \int d\mathbf{x}\left(\frac{\delta\mathbf{B}^{(1)}\delta\mathbf{B}^{(1)}}{8\pi} + \frac{\mathbf{B}_0 \cdot \delta\mathbf{B}_{1,1}^{(2)}}{4\pi}\right). \tag{D10}$$

Substituting the integral expressions for the perturbed magnetic field (Eqs. (22) and (23)), we obtain

$$\begin{aligned}\mathcal{U}_M^{(2)} = \int d\mathbf{x}\,\frac{1}{8\pi}\{[\nabla \cdot (\mathbf{B}_0\boldsymbol{\xi}_e)]^2 - 2[\nabla \cdot (\mathbf{B}_0\boldsymbol{\xi}_e)] \cdot [\nabla \cdot (\boldsymbol{\xi}_e\mathbf{B}_0)] + [\nabla \cdot (\boldsymbol{\xi}_e\mathbf{B}_0)]^2 \\ +\mathbf{B}_0 \cdot [\nabla\nabla\colon(\boldsymbol{\xi}_e\boldsymbol{\xi}_e\mathbf{B}_0)] - 2\mathbf{B}_0 \cdot \nabla \cdot [\boldsymbol{\xi}_e(\mathbf{B}_0 \cdot \nabla)\boldsymbol{\xi}_e]\}, \end{aligned} \tag{D11}$$

or, with the component indices displayed explicitly (sum over repeated indices implied),

$$\mathcal{U}_M^{(2)} = \int \frac{d\mathbf{x}}{8\pi} \{\partial_j(B_j\xi_i)\partial_k(B_k\xi_i) \\ -2[\partial_j(B_j\xi_i)\partial_k(\xi_k B_i) - (\partial_k B_i)\xi_k B_j(\partial_j\xi_i)] \\ +B_i\partial_j\partial_k(\xi_j\xi_k B_i) + \partial_j(\xi_j B_i)\partial_k(\xi_k B_i)]\}. \tag{D12}$$

For the first line in Eq. (D12), we obtain, using $\nabla \cdot \mathbf{B}_0 = 0$:

$$\int \frac{d\mathbf{x}}{8\pi}\, \partial_j(B_j\xi_i)\partial_k(B_k\xi_i) = \int \frac{d\mathbf{x}}{8\pi}\, (\mathbf{B}_0 \cdot \nabla\xi_e)^2, \tag{D13}$$

while for the second line, again using $\nabla \cdot \mathbf{B}_0 = 0$, we find

$$-\int \frac{d\mathbf{x}}{4\pi}\, \partial_j(B_j\xi_i)(\partial_k\xi_k)B_i = -\int \frac{d\mathbf{x}}{4\pi}\, \mathbf{B}_0(\nabla \cdot \xi_e)\cdot(\mathbf{B}_0 \cdot \nabla)\xi_e. \tag{D14}$$

The third line is analyzed by applying the identity:

$$\tfrac{1}{2}n_0(\nabla \cdot \xi)^2 + \tfrac{1}{2}n_0(\partial_j\xi_k)(\partial_k\xi_j) + \xi \cdot \nabla[\nabla \cdot (\xi n_0)] - \tfrac{1}{2}\xi\xi{:}\,\nabla\nabla n_0 = \tfrac{1}{2}\nabla\nabla{:}\,(\xi\xi n_0). \tag{D15}$$

Substituting for n_0 components of the zero-order magnetic field B_i and for ξ the electron displacement ξ_e, multiplying by B_i, summing over i, and integrating, we find

$$\int d\mathbf{x}\, \{-B_i\xi_e \cdot \nabla[\nabla \cdot (\xi_e B_i)] + \tfrac{1}{2}B_i\xi_e\xi_e{:}\,\nabla\nabla B_i + \tfrac{1}{2}B_i\nabla\nabla{:}\,(\xi_e\xi_e B_i)]\} = \\ \int d\mathbf{x}\, B_0^2[(\nabla \cdot \xi_e)^2 + (\partial_j\xi_k)(\partial_k\xi_j)]. \tag{D16}$$

Integrating the first and third terms on the left-hand side by parts, we find for the terms on the third line of Eq. (D12)

$$\frac{1}{8\pi}\int d\mathbf{x}\, \{[\nabla \cdot (\xi_e\mathbf{B}_0)]^2 + \mathbf{B}_0 \cdot (\xi_e\xi_e{:}\,\nabla\nabla\mathbf{B}_0)\} = \frac{1}{16\pi}\int d\mathbf{x}\, B_0^2\left[(\nabla \cdot \xi_e)^2 + \sum_{j,k}\frac{\partial\xi_{ej}}{\partial x_k}\frac{\partial\xi_{ek}}{\partial x_j}\right]. \tag{D17}$$

Now combining Eqs. (D13), (D14), and (D17), we obtain the second-order magnetic energy:

$$\mathcal{U}_M^{(2)} = \int d\mathbf{x}\left\{\frac{1}{8\pi}(\mathbf{B}_0 \cdot \nabla\xi_e)^2 - \frac{1}{4\pi}(\nabla \cdot \xi_e)\mathbf{B}_0 \cdot (\mathbf{B}_0 \cdot \nabla\xi_e) + \frac{B_0^2}{16\pi}\left[(\nabla \cdot \xi_e)^2 + \sum_{j,k}\frac{\partial\xi_{ej}}{\partial x_k}\frac{\partial\xi_{ek}}{\partial x_j}\right]\right\}. \tag{D18}$$

We can understand something of the nature of the terms appearing in the integral in Eq. (D18) by tracing each back to the terms in the integral in Eq. (D11). We find the $(\mathbf{B}_0 \cdot \nabla\xi_e)^2$ term represents the square-first-order line-bending energy, while the second term contains both second-order line-bending terms and cross terms between first-order line-bending and first-order compressional terms.

The last term in the integral represents second-order magnetic compressional energy. It is the only magnetic energy term remaining in the two-dimensional system oriented perpendicular to $\mathbf{B}_0$. In this system, Faraday's Law takes the same form as the continuity equation [8]. It is therefore not a coincidence that Eq. (D15), used in deriving the second-order modification to the density Eq. (15), also appears here. Furthermore, to the extent that the magnetic field behaves like a density through the form of Faraday's Law, it is not hard to understand this last term as the second-order magnetic compression term, since it comes from the second-order perturbation terms which resemble exactly terms obtained from the second order expansion of the displacement-induced perturbation density.

Finally, we find the gravitational potential energy

$$\mathcal{U}_g^{(2)} = \int dx\, \delta n_{1,1}^{(2)} m\phi_g. \tag{D19}$$

Substituting Eq. (15) with $s = i$ and integrating twice by parts, we easily obtain

$$\mathcal{U}_g^{(2)} = \int dx\, \tfrac{1}{2} n_0 m \xi_i \xi_i \colon \nabla\nabla\phi_g. \tag{D20}$$

By instead allowing $s = e$ in Eq. (15), we also find a similar expression holds for the electrons.

Combining the expressions for $\mathcal{K}^{(2)}$ (Eq. (D9)), $\mathcal{U}_M^{(2)}$ (Eq. (D18)), and $\mathcal{U}_g^{(2)}$ (Eq. (D20)), we obtain Eq. (41). The expression for the entire second-order perturbed energy $\mathcal{E}^{(2)}$ is then obtained by adding the expression for $\mathcal{E}_2^{(2)}$, completing the derivation.

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