STABILITY REGIONS OF NONLINEAR AUTONOMOUS

DYNAMICAL SYSTEMS

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Hsiao-Dong Chiang[1], Morris W. Hirsch[2], Felix F. Wu[1]

ABSTRACT

A topological and dynamical characterization of the stability boundaries for a fairly large class of nonlinear autonomous dynamic systems is presented. The stability boundary of a stable equilibrium point is shown to consist of the stable manifolds of all the equilibrium points on the stability boundary. Several necessary and sufficient conditions are derived to determine whether a given equilibrium point is on the stability boundary. A method to find the stability region based on these results is proposed. The method, when feasible, will find the exact stability region, rather than a proper subset of it as in the Lyapunov theory approach. Several examples are given to illustrate the theoretical prediction.

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I. INTRODUCTION

The problem of determining the stability region (region of attraction) of a stable equilibrium point for a nonlinear autonomous dynamical system is an important one in many applications, such as electric power systems [1,2], economics [3], ecology [4] etc. The numerous methods proposed in the literature for estimating the stability region can be roughly divided into two classes[6] : those using Lyapunov functions, and all others. Most of the methods belong to the Lyapunov function approach, which is based mainly on La Salle's extension of Lyapunov theory [7-10]. The estimated stability region based on these methods usually is only a subset of the true stability region. Recently methods using computer generated Lyapunov functions [11,12] have been proposed. Another method, belonging to the Lyapunov function approach, is the Zubov's method[8]. Theoretically, this method provides the true stability region via the solution of a partial differential equation. Recent advance includes the maximal Lyapunov function [30]. One of the early non-Lyapunov methods proposed for planar systems [31] requires the construction of a non-trivial integral function. The method of sinks [13], also for planar systems, utilizes the analogy between the vector field and the velocity field of an incompressible fluid. An iterative procedure using Volterra series for estimating the stability region was proposed [14]. Another method, called the trajectory-reversing method, was recently proposed [5,6], in which the estimation of the stability region is synthesized from a number of system trajectories obtained by integrating the system equations.

In this paper a comprehensive analysis of the stability region is conducted. Several necessary and sufficient conditions for an equilibrium point (or closed orbit) to lie on the stability boundary are derived. A complete characterization of the stability boundary is presented for a fairly general class of nonlinear autonomous dynamical systems. It is shown that the stability boundary of this class of systems consists of the union of the stable manifolds of all equilibrium points on the stability boundary. A method to find the stability region based on these results is proposed; this method belongs to the non-Lyapunov function approach. The method is applied to several examples studied in the literature.

The organization of the paper is as follows. Some fundamental concepts in the theory of mathematical dynamical systems that are essential in the subsequent development in this paper are introduced in Sec. 2. In Sec. 3, topological properties of the equilibrium point on the stability boundary are presented. In Sec.4, a complete characterization of the stability boundary of a class of systems is given. The class of systems is examined in Sec. 5 and is shown to be fairly large. In Sec. 6 a new method for determining stability region is proposed. In Sec. 7, the method is applied to several examples.

2. SOME CONCEPTS IN DYNAMICAL SYSTEMS

In this section we introduce some concepts that play a central role in the theory of dynamical systems. For general background on the theory of mathematical dynamical systems the reader is advised to consult the survey paper by Smale [15], or the books by Guckenheimer and Holmes[28], or Palis and De Melo[20].

Abstractly, a dynamical system (M,f) is characterized by :

(1) A state space M of the possible states for the system under consideration.

(2) A vector field f, defined on M, which generates the time evolution of the states x in M.

The state space M is assumed to be Haudsdorff; usually M is a manifold or an open subset of some topological vector space. In this section the state space M is a C^2 manifold without boundary. The time evolution is a map from $M \times I \rightarrow M$, defined by $(x,t) \rightarrow \Phi_t(x)$, where I is an interval of R and $\Phi_t(\cdot)$ is called the flow (induced by the vector field f). A vector field is said to be *complete* if $\Phi_t(x)$ is defined on $M \times R$. If M is compact, all its vector fields are complete. We may write $\Phi_t(x) = x(t)$, the map $t \rightarrow x(t)$ is the *trajectory* of $x \in M$, the image of this map is called the *orbit*. The set of all trajectories is the phase portrait of f.

When the vector field f does not depend on time the dynamical system is said to be autonomous. A nonlinear autonomous dynamical system can be described by a set of differential equation

$$\dot{x} = f(x) \qquad x \in M \tag{2-1}$$

We shall assume that the vector field f is C^1; this is a sufficient condition for existence and uniqueness of solution. In this case the solution passing through x_0 at time t=0 is denoted by $\Phi_t(x_0)$.

A zero of a vector field is referred to as an equilibrium point(e.p.) or simply an equilibrium point. It is a solution of the equation

$$f(x) = 0 \tag{2-2}$$

We shall denote the set of equilibrium points of (2-1) by $E := \{ x : f(x) = 0 \}$.

An equilibrium point x of f is said to be *hyperbolic* if, in local coordinates, none of the eigenvalues of the Jacobian matrix $J_x f$ at x have zero real part. For a hyperbolic equilibrium point x, we can decompose the state space uniquely as a direct sum of two subspaces $E^s + E^u$ such that each subspace is invariant under the linear operator $J_x f$, the eigenvalues of $J_x f$ restricted to E^s have negative real part and the eigenvalue of $J_x f$ restricted to E^u have positive real part. Letting the dimension of E^s be n_s and the dimension of E^u be n_u, we can express each subspace as following :

the stable subspace $E^s = \text{span} \{ v^1, v^2, \ldots, v^{n_s} \}$

the unstable subspace $E^u = \text{span} \{ w^1, w^2, \ldots, w^{n_u} \}$

where $v^1, v^2, \ldots, v^{n_s}$ are the n_s (generalized) eigenvectors whose eigenvalues have negative real parts, $w^1, w^2, \ldots, w^{n_u}$ are the n_u (generalized) eigenvectors whose eigenvalues have positive real parts, obviously, $n_s + n_u = n$

We call the value n_u the *type* of x. An equilibrium point of type 0 is called a *sink* : one of type n is called a *source;* all others are called *saddle. Type-one* equilibrium point ($n_u = 1$) will be of some importance. It is well known that sinks are stable equilibrium points, while sources and saddles are unstable equilibrium points.

By a *closed orbit* of a dynamical system we mean the image of a nonconstant periodic solution of (2-1), i.e. a trajectory γ is a closed orbit if γ is not an equilibrium point point and $\Phi_t(x) = x$ for some $x \in \gamma$, $t \neq 0$. A closed orbit is said to be *hyperbolic* if for any $p \in \gamma$, n-1 of the eigenvalues of the Jacobian of $\Phi(\tau(\gamma))$ at p have modulus not equal to 1 (one eigenvalue must always be 1). A *critical element* of f is an orbit which is either a closed orbit or an equilibrium point.

Let $\hat{x}$ be an equilibrium point. Its stable and unstable manifolds $W^s(\hat{x})$, $W^u(\hat{x})$ are defined as follows

$$W^s(\hat{x}) = \{ x \in M : \Phi_t(x) \rightarrow \hat{x} \quad \text{as } t \rightarrow \infty \} \tag{2-3a}$$

$$W^u(\hat{x}) = \{ x \in M : \Phi_t(x) \rightarrow \hat{x} \quad \text{as } t \rightarrow -\infty \} \tag{2-3b}$$

Similarly the stable and unstable manifolds of a hyperbolic closed orbit can be defined. Since the stable manifold of $\hat{x}$ for the flow $\Phi_t(\cdot)$ coincides with the unstable manifold of $\hat{x}$ for the flow $\Phi_{-t}(\cdot)$, this dual property enable us to translate each property of stable manifold into that of unstable manifold. Obviously, these two sets $W^s(\cdot)$, $W^u(\cdot)$ are invariant sets and it is known that $W^s(\cdot)$, $W^u(\cdot)$ are the image of injective C^1 immersions of R^{n_s}, R^{n_u}[15].

The long-term behavior of the trajectory can be studied in terms of its *ω-limit set* $\omega(x)$. We say y is in the ω-limit set of x if there is a sequence $\{ t_i \}$ in R , $t_i \rightarrow \infty$, such that

$$y = \lim_{i \rightarrow \infty} \Phi_{t_i}(x)$$

The α-limit set $\alpha(x)$ is defined similarly by letting $t_i \rightarrow -\infty$. It can be shown that these limit sets are closed invariant subsets of M [27,p.198]. For example, an equilibrium point is its own ω-limit set; it is also the ω-limit set of trajectories in its stable manifold and the α-limit set of trajectories in its unstable manifold. A closed orbit γ is the ω-limit set and the α-limit set of every point on γ.

The idea of transversality is basic in the study of dynamical systems. If A,B are injectively immersed manifold in M, we say they satisfy the *transversality condition* if either (i) at every point of intersection $x \in A \cap B$, the tangent spaces of A and B span the tangent space of M at x.

i.e. $T_x(A) + T_x(B) = T_x(M) \qquad \text{for} \quad x \in A \cap B$

or (ii) they do not intersect at all.

One of the most important features of a hyperbolic equilibrium point $\hat{x}$ is that its stable and unstable manifolds intersect transversely at $\hat{x}$. This transverse intersection is important because it persists under perturbation of the vector field.

3. EQUILIBRIA ON THE STABILITY BOUNDARY

We will show in section 4 that under fairly general conditions, the stability boundary of a stable equilibrium is the union of stable manifolds of the equilibria on the stability boundary. Therefore in this section we derive conditions to characterize the equilibria on the stability boundary. The necessary and sufficient conditions for an equilibrium to be on the stability boundary are derived in terms of both the stable manifold and the unstable manifold of the equilibrium. We also study the number of equilibria on the stability boundary.

Consider a nonlinear autonomous dynamical system described by the differential equation

$$\dot{x} = f(x) \tag{3-1}$$

where x is a n-dimensional vector and the vector field f is C^1.

Suppose x_s is a stable equilibrium of the vector field f. The *stability region* (or *region of attraction*) of x_s is defined to be $W^s(x_s)$, that is, the set of all points x such that

$$\lim_{t\to\infty} \Phi_t(x) \to x_s \tag{3-2}$$

We will also denote the stability region of x_s by $A(x_s)$, its boundary and its closure by $\partial A(x_s)$ and $\bar{A}(x_s)$, respectively. When it is clear from the context, we write A for $A(x_s)$, etc.. Alternatively, the stability region can be expressed as

$$A(x_s) = \{x \in R^n : \omega(x) = x_s\} \tag{3-3}$$

Based on the properties of the stable manifold of x_s, we have the following proposition[15]:

Proposition 3-1 : $A(x_s)$ is an open, invariant set diffeomorphic to R^n.

Since the boundary of an invariant set is also invariant and the boundary of an open set is closed, therefore we have :

Proposition 3-2 : $\partial A(x_s)$ is a closed invariant set of dimension $< n$. If $A(x_s)$ is not dense in R^n, then $\partial A(x_s)$ is of dimension n-1.

proof: The second part of this proposition is from a general result[29,p.40] which states that, if U is an open set in R^n, then ∂U is of dimension $< n$; moreover if U is not dense in R^n, then ∂U is of dimension n-1.

Remark :

If there are at least two stable equilibrium points, then the dimension of stability boundary of each of them is n-1; in particular stability boundaries are nonempty in this case.

Next, we give conditions for an equilibrium point to be on the stability boundary, which is a key step in the characterization of the stability region $A(x_s)$. We do this in two steps. First we impose only one assumption on the dynamical system (3-1), namely, that equilibrium points are hyperbolic, and derive conditions for an equilibrium point to be on the stability boundary in terms of both its stable and unstable manifolds (Theorem 3-3). Additional conditions are then imposed on the dynamical system and the results are further sharpened. (Theorem 3-5). It should be noted that these characterizations are also applicable to closed orbits; the obvious generalization is omitted here. We use the notation A-B to denote those elements which belong to A but not to B.

Theorem 3-3: (Characterization of equilibrium point on the stability boundary)

Let $A(x_s)$ be the stability region of a stable equilibrium point x_s. Let $\hat{x} \neq x_s$ be a hyperbolic equilibrium point. Then

(i) $\hat{x} \in \partial A(x_s)$ if and only if $\{W^u(\hat{x})-\hat{x}\} \bigcap \bar{A}(x_s) \neq \phi$

(ii) Suppose $\hat{x}$ is not a source (i.e. $\{W^s(\hat{x})-\hat{x}\} \neq \phi$). Then $\hat{x} \in \partial A(x_s)$ if and only if $\{W^s(\hat{x})-\hat{x}\} \bigcap \partial A(x_s) \neq \phi$

Proof:(i) If $y \in W^u(\hat{x}) \bigcap \bar{A}$, then

$$\lim_{t \to \infty} \Phi_{-t}(y) = \hat{x}$$

But since $\bar{A}$ is invariant, we have

$$\Phi_{-t}(y) \in \bar{A}.$$

It follows that

$$\hat{x} \in \bar{A}$$

Since $\hat{x}$ can not be in the stability region, $\hat{x}$ is on the stability boundary.

Suppose conversely that $\hat{x} \in \partial A$. Let $G \subset \{W^u(\hat{x}) - \hat{x}\}$ be a fundamental domain for $W^u(\hat{x})$ with respect to the time-one map Φ_1; this means that G is a compact set such that

$$\bigcup_{t \in R} \Phi_t(G) = \{W^u(\hat{x}) - \hat{x}\} \tag{3-4}$$

Let G_ϵ be the ϵ-neighborhood of G in R^n. Then $\bigcup_{t<0} \Phi_t(G_\epsilon)$ contains a set of the form $\{U - W^s(\hat{x})\}$, where U is a neighborhood of $\hat{x}$ [20, corollary 2, p.86]. Since $\hat{x} \in \partial A$, it follows that $U \bigcap A = \phi$. But, by assumption, $\hat{x} \in \partial A$ (i.e. $W^s(\hat{x}) \bigcap A \neq \phi$). Therefore we have

$$\{ U - W^s(\hat{x})\} \bigcap A \neq \phi \tag{3-5}$$

or

$$\bigcup_{t<0} \Phi_t(G_\epsilon) \bigcap A \neq \phi \tag{3-6}$$

This implies that $G_\epsilon \bigcap \Phi_t(A) \neq \phi$ for some t. Since $A(x_s)$ is invariant under the flow it follows that

$$G_\epsilon \bigcap A \neq \phi$$

Since $\epsilon > 0$ is arbitrary and G is a compact set, we conclude that G contains at least a point of $\bar{A}$.

The proof of (ii) is similar to the proof of (i). ##

As a corollary to Theorem 3-4, if $\{W^u(\hat{x}) - \hat{x}\} \bigcap A(x_s) \neq \phi$, then $\hat{x}$ must be on the stability boundary. Since any trajectory in $A(x_s)$ approaches x_s, we see that a sufficient condition for $\hat{x}$ to be on the stability boundary is the existence of a trajectory in $W^u(\hat{x})$ which approaches x_s. The nice thing about this condition is that it can be checked numerically. From practical point of view, therefore, we would like to see when this condition is also necessary. We are going to show this condition becomes necessary under two additional assumptions which are reasonable.

So far we have assumed only that the equilibrium points are hyperbolic. This is a generic property for dynamical systems. Roughly speaking we say a property is generic for a class of systems if that property is true for "almost all" systems in this class. A formal definition is given in [15]. It has been shown [16] that among $C^r(r \geqslant 1)$ vector field, the following properties are generic: (i) all equilibrium points and closed orbits are hyperbolic and (ii) the intersections of the stable and unstable manifolds of critical elements satisfy the transversality condition. Theorem 3-3 can be sharpened under two conditions, one of which is generic for the dynamical system (3-1). That is the transversality condition. The other condition requires that every trajectory on the stability boundary approach one of the critical elements.

The following Lemma, which is a consequence of λ-lemma[20,p.86] is interesting in itself and useful in the proof of next theorem. Recall that the type of an equilibrium point is the dimension of its unstable manifold. An m-disk is a disk of dimension m.

Lemma 3-4 :

Let $\hat{z}$ be an equilibrium point of type m on the stability boundary ∂A such that $W^u(\hat{z}) \bigcap A \neq \phi$. Let $y \in \{W^s(\hat{z}) - \hat{z}\}$ and let D be an m-disk centered at y transverse to $W^s(\hat{z})$ at y. Then $D \bigcap A \neq \phi$.

Now, we present the key theorem of this section which characterizes an equilibrium point being on the stability boundary, in terms of both its stable and unstable manifolds. From the practical point of view, this result is more useful than Theorem 3-3.

Theorem 3-5 : (Further characterization of equilibrium point on the stability boundary)

Let $A(x_s)$ be the stability region of a stable equilibrium point x_s. Let $\hat{x} \neq x_s$ be an equilibrium point. Assume

(i) All the equilibrium points on $\partial A(x_s)$ are hyperbolic.

(ii) The stable and unstable manifolds of equilibrium points on $\partial A(x_s)$ satisfy the transversality condition.

(iii) Every trajectory on $\partial A(x_s)$ approaches one of the equilibrium points as $t \to \infty$.

Then

(1) $\hat{x} \in \partial A(x_s)$ if and only if $W^u(\hat{x}) \bigcap A(x_s) \neq \phi$.

(2) $\hat{x} \in \partial A(x_s)$ if and only if $W^s(\hat{x}) \subseteq \partial A(x_s)$.

Proof : (1) Because of Theorem 3-3, we only need to prove that, under these assumptions, $\hat{x} \in \partial A(x_s)$ implies $W^u(\hat{x}) \bigcap A(x_s) \neq \phi$. We use the notation $n_u(x)$ to denote the type of an equilibrium point x. It follows from assumption (i) that $n_u(x) \geqslant 1$ for all equilibrium points $x \in \partial A(x_s)$. Let $\hat{x} \in \partial A(x_s)$ and $n_u(\hat{x}) = h$. By Theorem 3-3 there exists a point $y \in \{W^u(\hat{x}) - \hat{x}\} \bigcap \bar{A}(x_s)$. If $y \in A(x_s)$, the proof is complete. Suppose $y \in \partial A$, by assumption (iii) there exists an equilibrium point $\hat{z} \in \partial A$ and $y \in \{W^s(\hat{z}) - \hat{z}\}$. Let $n_u(\hat{z}) = m$, by assumption (ii) $W^u(\hat{x})$ and $W^s(\hat{z})$ meet transversely at y, thus $h > m$ [15]. Now, consider two cases : (i) $h = 1$ case; then m must be zero (i.e. $\hat{z}$ must be a stable equilibrium point., which is a contradiction. Consequently, $W^u(\hat{x}) \bigcap A(x_s) \neq \phi$. (ii) $h > 1$ case, without loss of generality, we assume inductively that $W^u(\hat{z}) \bigcap A(x_s) \neq \phi$. Therefore $W^u(\hat{x})$ contains an m-disk D centered at y, transverse to $W^s(\hat{z})$. Applying lemma 3-4, we have $D \bigcap A \neq \phi$, which implies $W^u(\hat{x}) \bigcap A(x_s) \neq \phi$. This completes the proof.

(2) As pointed out in part (1) we only prove that $\hat{x} \in \partial A(x_s)$ implies $W^s(\hat{x}) \subseteq \partial A(x_s)$. By part (a) we have $W^u(\hat{x}) \bigcap A(x_s) \neq \phi$. Now applying the lemma 3-4 (with $\hat{z} = \hat{x}$) we conclude that $\{W^s(\hat{x}) - \hat{x}\} \subseteq \bar{A}(x_s)$. Since $W^s(\hat{x}) \bigcap A(x_s) = \phi$ we completes this proof.

Remarks:

(1) Fig. 1 shows an example for which the assumption that every trajectory on the stability boundary approaches one of the equilibrium points does not hold. For this system, the unstable manifold of x_1 does not intersect with the stability region (see Theorem 3-5) and a part of the stable manifold of x_1 is not on the stability boundary (see Theorem 3-5).

(2) To show that the transversality condition is needed in Theorem 3-5, let us consider the example taken from [17]. In Fig. 2 the transversality condition is not satisfied because the intersection of the unstable manifold of x_1 and the stable manifold of x_2 is a portion of the manifold whose tangent space has dimension 1. Note that the unstable manifold of x_1 intersects with the stability boundary (see Theorem 3-3), but not the stability region (see Theorem 3-5). A part of the stable manifold of x_1 (upper part in Fig. 2) is not in the stability boundary (see Theorem 3-5).

The next result concerns the number of equilibrium points on the stability boundary.

Theorem 3-6 : (Number of equilibrium points on the stability boundary)

If the stability boundary ∂A is a smooth compact manifold and all the equilibrium points of vector field f on ∂A are hyperbolic, then the number of equilibrium points on ∂A is even.

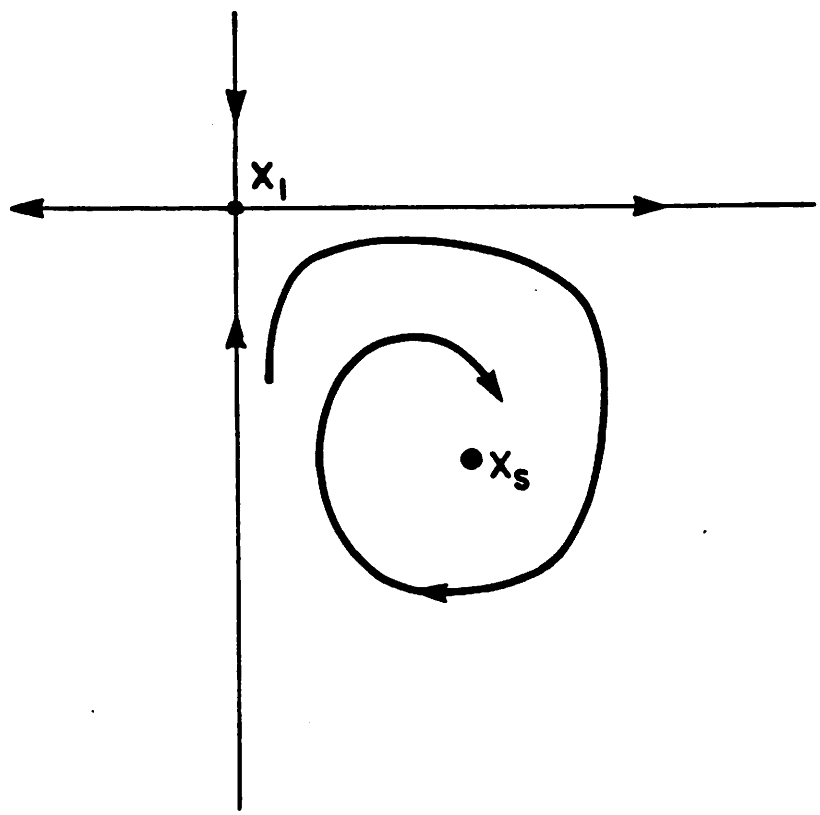

Fig. 1. An example of dynamical system whose trajectories on the stability boundary does not all approach its critical elements.

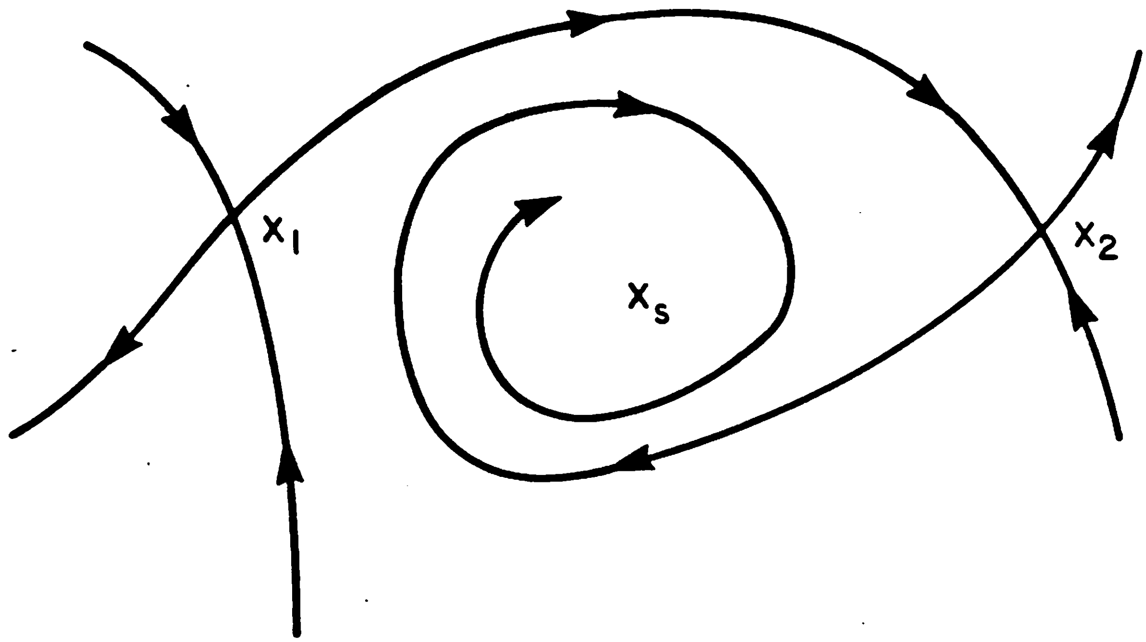

Fig. 2. The intersection between the unstable manifold of x_1 and the stable manifold of x_2 does not satisfy the transversality condition.

Proof : The proof is based on the following fact [24, Exercise 7, p.139]: the Euler characteristic of the boundary of a compact manifold is even. From the Poincare-Hopf Index Theorem [25,pp.134], it follows that the sum of the indices of equilibrium points of f on the stability boundary ∂A is even, but the index of f at a hyperbolic equilibrium point is either +1 or -1 [26,p.37]. Consequently, Theorem 3-7 follows.##

Remarks:

(1) Genesio and Vicino [22] have shown that theorem 3-6 is true for a special case, namely; an odd order system ($n \neq 5$) without "degenerate" equilibrium point.

(2) Theorem 3-6 is false if it is not assumed that ∂A is smooth. (see Fig. 3)

(3) Theorem 3-6 is also true under the weaker assumption that every equilibrium point is *nondegenerate* in the sense that $J_p f$ is invertible. The proof is the same.

4. STABILITY BOUNDARY

In this section we characterize the stability boundary for a fairly large class of nonlinear autonomous dynamical systems (3-1) whose stability boundary is nonempty. We make the following assumptions concerning the vector field:

(A1): All the equilibrium points on the stability boundary are hyperbolic.

(A2): The stable and unstable manifolds of equilibrium points on the stability boundary satisfy the transversality condition.

(A3): Every trajectory on the stability boundary approaches one of the equilibrium points as $t \to \infty$.

Theorem 4-1 asserts that if assumptions (A1) to (A3) are satisfied, then the stability boundary *is* the union of the stable manifolds of the equilibrium points on the stability boundary.

Theorem 4-1 : (Characterization of stability boundary)

For the nonlinear autonomous dynamical system (3-1) that satisfies assumptions (A1) to (A3), let x_i ,i=1,2... be the equilibrium points on the stability boundary $\partial A(x_s)$ of the stable equilibrium point x_s. Then

$$\partial A(x_s) = \bigcup_i W^s(x_i) \tag{4-1}$$

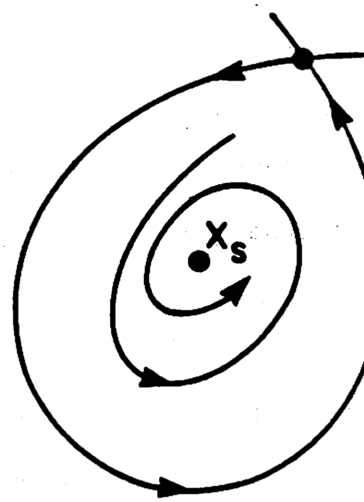

Fig. 3. The stability boundary of x_s is not smooth.

Proof: Let x_i, i=1,2,... be the equilibrium points on the stability boundary. Theorem 3-5 implies

$$\bigcup_i W^s(x_i) \subseteq \partial A(x_s) \tag{4-2}$$

The assumption (A3) implies

$$\partial A(x_s) \subseteq \bigcup_i W^s(x_i) \tag{4-3}$$

Combining (4-2) and (4-3) we have the required result. ##

Theorem 4-1 can be generalized to allow closed orbits to exist on the stability boundary. The following theorem, which we shall not use, is stated below without proof.

Theorem 4-2 : (Characterization of stability boundary)

Consider the dynamical systems (3.1) whose vector field satisfies the following assumptions.

(B1): All the critical elements on the stability boundary are hyperbolic.

(B2): The stable and unstable manifolds of critical elements on the stability boundary satisfy the transversality condition.

(B3): Every trajectory on the stability boundary approaches one of the critical elements as $t \to \infty$.

Let x_i ,i=1,2... be the equilibrium points and γ_j, j=1,2,... be the closed orbits on the stability boundary $\partial A(x_s)$ of the stable equilibrium point x_s. Then

$$\partial A(x_s) = \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j)$$

Returning now to assumptions (A1) to (A3), let

$$B = \bigcup_i W^s(x_i) \tag{4-4}$$

where i ranges over the equilibrium points on ∂A whose type is greater than one. Now the dimension of the stable manifold of a type-one equilibrium point is n-1 and the dimensions of the stable manifolds of other equilibrium points (with $n_u > 1$) are lower than n-1. It follows from the Baire theorem and Proposition 3-2 that the set B is nowhere dense in ∂A. Thus, we have the next corollary:

Corollary 4-3 : For the nonlinear autonomous dynamical system (3-1), if assumptions (A1) to (A3) are satisfied, then

$$\partial A(x_s) = \bigcup_j \overline{W^s(x_j)} \tag{4-5}$$

where j ranges over the type-one equilibrium points on $\partial A(x_s)$.

Remark :

Same conclusion, (4-5), in Corollary 4-3 has been derived by Tsolas, Arapostathis and Varaiya [17] under different assumptions. Similar results of Theorem 3-5 and Theorem 4-2 under a stronger condition than (A3) have been derived previously [32].

The following theorem gives an interesting result on the structure of the equilibrium points on the stability boundary. Moreover, it presents a necessary condition for the types of equilibrium points on a bounded stability boundary.

Theorem 4-4: (Structure of equilibrium points on the stability boundary)

For the nonlinear autonomous dynamical system (3-1), if assumptions (A1) to (A3) are satisfied, then the stability boundary must contain at least one type-one equilibrium point. If, furthermore,

(a) the stability region is bounded,

and

(b) $\Phi_t(x)$ approaches an equilibrium point as $t \to -\infty$, for all $x \in \partial A$

then ∂A must contain at least one type-one equilibrium point and one source.

Proof : Since (a) implies $A(x_s)$ is not dense in R^n, it follows that the dimension of $\partial A(x_s)$ is (n-1) (see proof of Proposition 3-2). Since $\partial A(x_s) = \bigcup W^s(x_j)$, where $x_j \in \partial A(x_s)$, at least one of the x_j must be a type-one equilibrium point, say x_1, so that the dimension of $\bigcup W^s(x_j)$ is (n-1). Repeating the same argument, if $\partial W^s(x_1)$ is nonempty, then the dimension of $\partial W^s(x_1)$ is $\leqslant$ (n-2), say (n-k). The application of Theorem 4-1 yields $\partial W^s(x_1) = \bigcup W^s(x_j)$, $x_j \in \partial W^s(x_1)$. In order for $\bigcup W^s(x_j)$ to have dimension (n-k), at least one of the x_j must be a type-k equilibrium point. If the stability region is bounded, the same argument can be repeated until we reach an type-n equilibrium point(a source).

Remark:

The hypothesis (b) of this theorem follows from the condition (1) in Theorem 5-1 or conditions (1) and (2) in Theorem 5-3.

The contrapositive of Theorem 4-4 leads to the following corollary, which is useful in predicting unboundedness of the stability region.

Corollary 4-5 : (Sufficient condition for stability region to be unbounded)

For the nonlinear autonomous dynamical systems (3-1), if assumptions (A1) to (A3) and condition (b) in Theorem 4-4 are satisfied and if ∂A contains no source, then the stability region is unbounded.

5. SUFFICIENT CONDITION FOR ASSUMPTION (A3)

The characterization of stability boundary in the previous section is valid for dynamical systems satisfying assumptions (A1) to (A3). Since assumptions (A1) and (A2) are generic properties, assumption (A3) is the crucial one in the application of Theorem 4-1. In this section, we will show that many dynamical systems arising from physical system models satisfy assumption (A3). We first present two theorems that give sufficient conditions for this assumption.

It should be stressed that the main results in this paper are independent of the existence of Lyapunov functions. For a convenient sufficient condition for guaranteeing assumption (A3), however, we will introduce a function in the following theorems which bears some resemblance to a Lyapunov function. Recall that E denotes the set of equilibrium points of (3-1). If V is a function on R^n, then $\dot{V}(x) := \frac{d}{dt}\big|_{t=0} V(\Phi_t(x)) = \nabla V(x) \cdot f(x)$.

Theorem 5-1 : Suppose there exists a C^1 function $V : R^n \to R$ for the system (3-1) such that

$$(1)\ \dot{V}(x) < 0 \qquad \text{if } x \notin E.$$

Suppose also there exists $\delta > 0$ such that for any $\hat{x} \in E$, the open ball $B_\delta(\hat{x}) := \{ x : |x - \hat{x}| < \delta \}$ contains no other point in E and the distance between any two such balls is at least δ. Furthermore, suppose that there exist a positive continuous function $\alpha : R^n \to R^+$ and two constants, $c_1 > 0$ and $c_2 > 0$, such that

$$(2)\ \alpha(x)\,|f(x)| < c_1 \qquad \text{for all } x \in R^n;$$

and

$$(3)\ \alpha(x)\,\dot{V}(x) < c_2 \qquad \text{unless } x \in B_\delta(\hat{x}) \text{ for some } \hat{x} \in E.$$

Then the assumption (A3) is true.

Proof : Let $x(t) := \Phi_t(x)$ be a trajectory on the stability boundary. If $x(t)$ does not approach one of the equilibrium points, then it must approach infinity owing to condition (1). We show this leads to a contradiction. The trajectory $x(t)$ may pass through a finite or infinite number of balls $B_\delta(\hat{x})$. We consider these two cases separately.

Case 1. $x(t)$ passes through a finite number of balls $B_\delta(\hat{x})$.

In this case, we know by condition (1) that there exists a T such that $\Phi_t(x)$ is not in any $B_\delta(\hat{x})$, for all $t > T$. Therefore, by condition (3) we have

$$\dot{V}(x(t)) < -\frac{c_2}{\alpha(x(t))} \qquad \text{for all } t > T$$

We estimate for $t > T$

$$\begin{aligned} V(x(t)) - V(x(T)) &= \int_t^T \dot{V}(x(\tau))d\tau \qquad (5\text{-}1) \\ &< -c_2 \int_T^t \frac{1}{\alpha(x(\tau))} d\tau \\ &< -\frac{c_2}{c_1} \int_T^t |f(x(\tau))|\, d\tau \\ &= -\frac{c_2}{c_1} \int_T^t |\dot{x}(\tau)|\, d\tau \\ &\leqslant -\frac{c_2}{c_1} \left| \int_T^t \dot{x}(\tau) d\tau \right| \\ &= -\frac{c_2}{c_1} |x(t) - x(T)| \end{aligned}$$

This shows that $\lim_{t \to \infty} V(x(t)) = -\infty$. But this contradicts the fact that $V(\cdot)$ is bounded below (by $V(x_s)$) along any trajectory on the stability boundary, which follows from condition (1) and the continuity property of the function $V(\cdot)$.

Case 2. $x(t)$ passes through infinitely many of balls $B_\delta(\hat{x})$.

Let $\{\hat{p}_1, \hat{p}_2, \ldots\}$ be a sequence of distinct equilibrium points through whose δ-ball $B_\delta(\hat{p}_i)$ the trajectory $x(t)$ passes. Let us define two increasing sequences $\{t_i\}$ and $\{s_i\}$; where t_i is the first time $x(t)$ enters the δ-ball $B_\delta(\hat{p}_i)$ and s_i is the first time $> t_i$ that $x(t)$ leaves the 2δ-ball $B_{2\delta}(\hat{p}_i)$.

Fix an integer $m > 0$; then for $t \geqslant t_{m+1}$ we have :

$$V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(\tau))d\tau$$

$$< \sum_{i=1}^{m} \int_{t_i}^{s_i} \dot{V}(x(\tau))d\tau$$

$$< -\frac{c_2}{c_1} \sum_{i=1}^{m} \left| \int_{t_i}^{s_i} \dot{x}(\tau)d\tau \right|$$

$$< -\frac{c_2}{c_1}\, m\, \delta$$

Letting $m \to \infty$, we contradict the boundedness of $V(\cdot)$ along the trajectory on the stability boundary. Therefore every trajectory on the stability boundary must approach one of the equilibrium points. ##

Remark :

If the number of equilibrium points of the vector field f is finite, then condition (3) in Theorem 5-1 is satisfied with $\alpha(x) = 1$ because of the continuity property of $\dot{V}(\cdot)$ along the trajectory. Therefore an important special case of Theorem 5-1 leads to the following corollary.

Corollary 5-2 : Suppose that the system (3-1) has a finite number of equilibrium points on its stability boundary and there exists a C^1 function $V : R^n \to R$ for the system (3-1) such that

$$\dot{V}(x) < 0 \qquad \text{if } x \notin E; \tag{5-2}$$

and

$$|f(x)| \text{ is bounded for } x \in R^n. \tag{5-3}$$

Then assumption (A3) is true.

Theorem 5-3 : Suppose there exists a C^1 function $V : R^n \to R$ for the system (3-1) such that

(1) $\dot{V}(x) \leqslant 0$ at every point $x \notin E$;

(2) if $x \notin E$, then the set $\{t \in R : \dot{V}(\Phi_t(x)) = 0\}$ has measure 0 in R;

and either

(3) the map $V: R^n \to R$ is proper;

or

(3') : For each $x \in R^n$, if $\{V(\Phi_t(x))\}_{t \geqslant 0}$ is bounded, then $\{\Phi_t(x)\}_{t \geqslant 0}$ is bounded.

Then the assumption (A3) is true.

Proof: From the well-known Lyapunov-type argument, the conditions (1) and (2) imply that all the limit sets of trajectories consist of equilibrium points [27,p.203]. Since the stability boundary is a closed invariant set, by the continuity property of the function $V(\cdot)$ and the conditions (1) and (2) we have the value of V along every trajectory on the stability boundary is bounded below by $V(x_s)$. Hence, condition (3) or condition (3') implies $\{x(t)\}_{t \geqslant 0}$ is bounded. Since the limit set of any compact trajectory is non-empty, thus (A3) follows.##

Remarks:

(1) It can be shown, by applying Corollary 5-2, that the following dynamical systems satisfy the assumption (A3),

$$\dot{x} = D\,f(x)$$

where $f: R^n \to R^n$ is a bounded gradient vector field with only finitely many equilibrium points on its stability boundary and the matrix D is a positive diagonal matrix.

(2) It has been shown [19] that many second-order dynamical systems frequently encountered in physical system models satisfy the conditions in Theorem 5-3,

$$M\ddot{x} + D\dot{x} + f(x) = 0$$

whose state space representation is

$$\dot{x} = y$$

$$M\dot{y} = -Dy - f(x)$$

where M, D are positive diagonal matrices. $f:R^n \rightarrow R^n$ is a bounded gradient vector field with bounded Jacobian, and the number of equilibrium points on any stability boundary is finite.

6. AN ALGORITHM TO DETERMINE THE STABILITY REGION

Theorem 3-5 and Theorem 4-1 lead to the following conceptual algorithm to determine the stability boundary of a stable equilibrium point, *assuming* that assumptions (A1) to (A3) of section 4 hold.

Algorithm

Step 1: Find all the equilibrium points.

Step 2: Identify those equilibrium points whose unstable manifolds contain trajectories approaching the stable equilibrium point x_s.

Step 3: The stability boundary of x_s is the union of the stable manifolds of the equilibrium points identified in step 2.

Remark :

In view of Corollary 4-3, the foregoing conceptual algorithm may be modified in such a way that only *type-one* equilibrium points are considered in steps 1 and 2, and the union of the *closures* of the stable manifolds are used in step 3.

Step 1 in the algorithm foregoing involves finding all the solutions of $f(x)=0$. Step 2 can be accomplished numerically. The following procedure is suggested:

(i) Find the Jacobian at the equilibrium point (say, $\hat{x}$).

(ii) Find many of the generalized unstable eigenvectors of the Jacobian having unit length.

(iii) Find the intersection of each of these normalized, generalized unstable eigenvectors (say, y_i) with the boundary of an ϵ -ball of the equilibrium point. (the intersection points are $\hat{x} + \epsilon y_i$ and $\hat{x} - \epsilon y_i$)

(iv) Integrate the vector field backward (reverse time) from each of these intersection points up to some specified time. If the trajectory remains inside this this ϵ -ball, then go to next step. Otherwise, we replace the value ϵ by $\alpha \epsilon$ and also the intersection points $\hat{x} \pm \epsilon y_i$ by $\hat{x} \pm \alpha \epsilon y_i$, where $0 < \alpha < 1$. Repeat this step.

(v) Numerically integrate the vector field starting from these intersection points.

(vi) Repeat the steps (iii) through (v). If any of these trajectories approaches x_s, then the equilibrium point is on the stability boundary.

For a planar system, the type of the equilibrium point on the stability boundary is either one(saddle) or two(source). The stable manifold of a type-one equilibrium point in this case has dimension one, which can easily be determined numerically as follows :

(a) Find a normalized stable eigenvector y of the Jacobian at the equilibrium point $\hat{x}$.

(b) Find the intersection of this stable eigenvector with the boundary of an ϵ-ball of the equilibrium point $\hat{x}$. (the intersection points are $\hat{x} + \epsilon y$ and $\hat{x} - \epsilon y$)

(c) Integrate the vector field from each of these intersection points after some specified time. If the trajectory remains inside this this ϵ-ball, then go to next step. Otherwise, we replace the value ϵ by $\alpha \epsilon$ and also the intersection points $\hat{x} \pm \epsilon y_i$ by $\hat{x} \pm \alpha \epsilon y_i$, where $0 < \alpha < 1$. Repeat this step.

(d) Numerically integrate the vector field backward (reverse time) starting from these intersection points.

(e) The resulting trajectories are the stable manifold of the equilibrium point.

For higher dimensional systems, the numerical procedure similar to the one above can only provide a set of trajectories on the stable manifold. To find the stable manifold and unstable manifold of an equilibrium point is a nontrivial problem. A power series expansion of the stable manifold of an equilibrium point is derived in [18].

7. EXAMPLES

The method for the determination of stability region proposed in Section 6 has been applied to some examples we have found in the literature, almost all of them are planar systems. In this section we present these examples to illustrate the results of this paper. In each example we give two figures; one compares the estimated stability region by previous methods and the present one, the other gives the phase portrait of the system to verify the results of this paper. Throughout these examples we *assume* the transversality condition is satisfied. The assumptions (A1) and (A3) have been checked for these examples; the details are omitted.

Example 1 : This is an example studied in [22,10]

$$\dot{x}_1 = -2x_1 + x_1x_2 \tag{7-1}$$

$$\dot{x}_2 = -x_2 + x_1x_2$$

There are two equilibrium points : (0.0,0.0) is a stable equilibrium point and (2,1) is a type-one equilibrium point. The trajectory on the unstable manifold of (2,1) converges to the stable equilibrium point (0.0,0.0), hence (2,1) is on the stability boundary (Theorem 3-5). Therefore the stability boundary is the stable manifold of (2,1) (Theorem 4-1), which is the curve C in Fig. 4(a). Because there is no source, the stability region is unbounded (Corollary 4-5). Curves A and B in Fig. 4(a) are obtained by the methods in [10] and [22], respectively. The approximately true stability boundary mentioned in [22] seems to agree with curve C. Fig. 4(b) is the phase portrait of this system.

Example 2 : The following system is considered in [10]

$$\dot{x}_1 = x_2 \tag{7-2}$$

$$\dot{x}_2 = 0.301 - \sin(x_1+0.4136) + 0.138\sin 2(x_1+0.4136) - 0.279x_2$$

The equilibrium points of (7-2) are periodic on the subspace $\{(x_1,x_2) \mid x_2 = 0\}$ and there exists a V-function

$$V(x_1,x_2) = 0.5x_2^2 + 0.301x_1 - \cos(x_1+0.4136) + 0.069\cos 2(x_1+0.4136) \tag{7-3}$$

We have

$$\dot{V}(x_1,x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \tag{7-4}$$

$$= -0.279x_2^2 \leqslant 0$$

the Jacobian matrix of (7-2) at (x_1,x_2) is

$$J(x) = \begin{bmatrix} 0 & 1 \\ a & -0.279 \end{bmatrix} \tag{7-5}$$

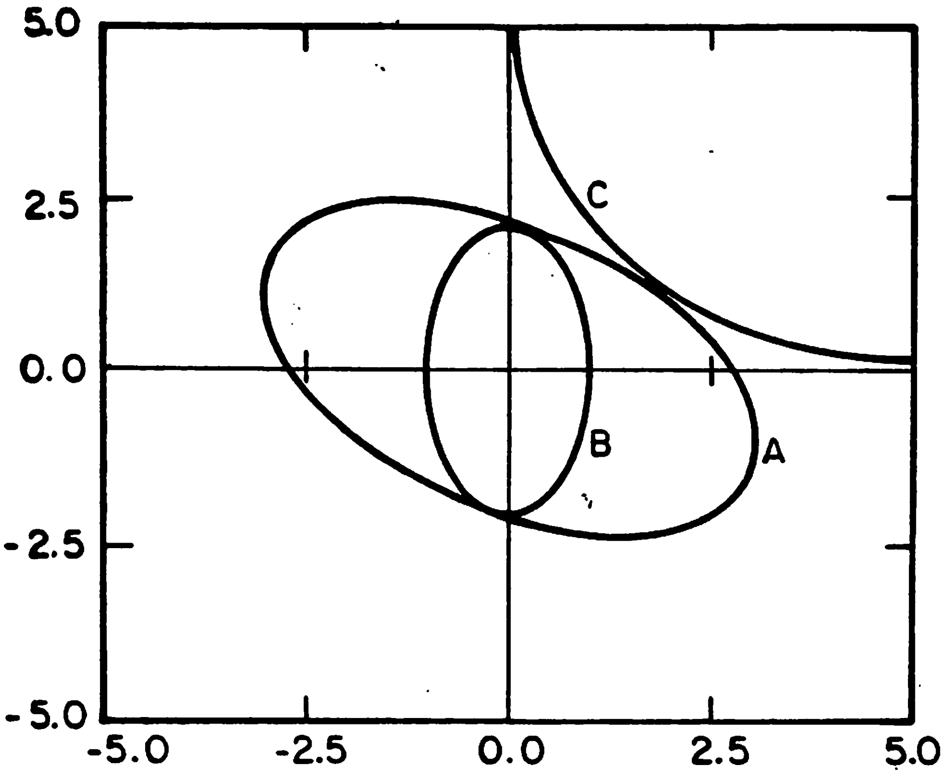

Fig. 4(a). Predictions of the stability region of Example 1 by different methods. Curves A and B are obtained by the methods in [10] and [22]. Curve C is obtained by the present method.

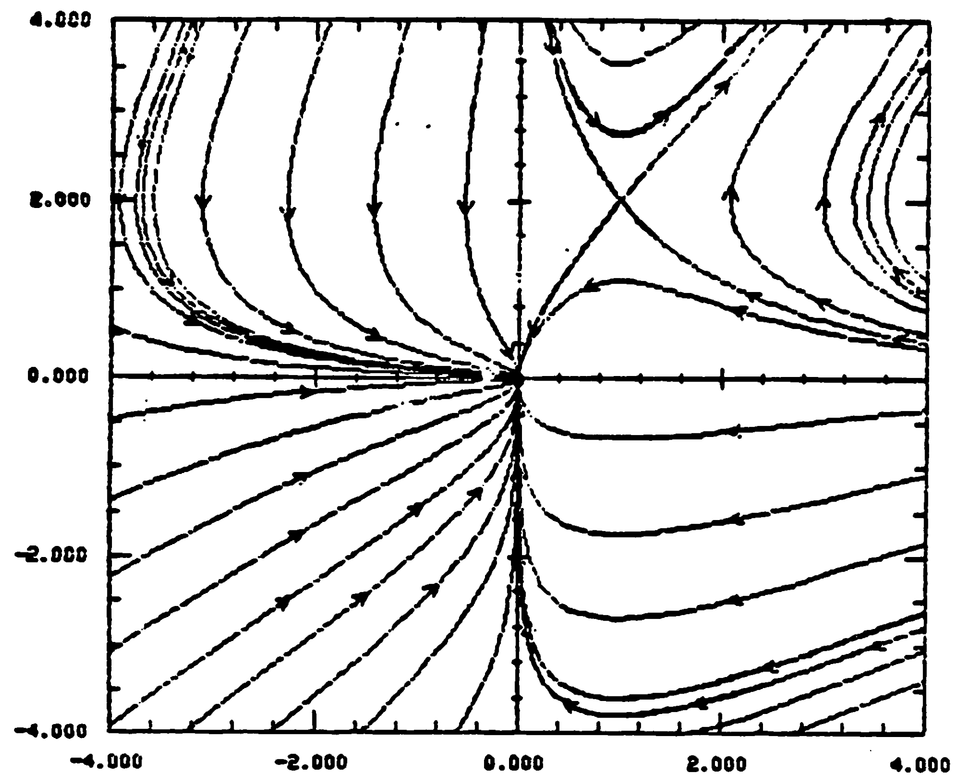

Fig. 4(b). The phase portrait of this system. Note that all the points inside the curve C converge to the stable equilibrium point which verifies that curve C is the true stability region.

where $a = -\cos(x_1+0.4136) + 0.276\cos 2(x_1+0.4136)$

Let λ_1, λ_2 be the eigenvalue of $J(x)$:

$$\lambda_1 + \lambda_2 = -0.279 \tag{7-6a}$$

$$\lambda_1 \times \lambda_2 = -a \tag{7-6b}$$

The following observations are immediate

(1) At least one of the eigenvalue must be negative which implies there is no source in the system (7-2). By Corollary 4-5 we conclude that the stability region (with respect to any stable equilibrium point) is unbounded.

(2) The stable equilibrium points and the type-one equilibrium points are located alternately on the x_1-axis.

It can be shown that (6.284098,0.0) is a stable equilibrium point of (7-2). Let us consider its stability region. The application of Theorem 3-5 shows that the type-one equilibrium points (2.488345,0.0) and (8.772443,0.0) are on the stability boundary. The stability region is again unbounded owing to the absence of a source. The stability boundary obtained by the present method is the curve B shown in Fig. 5(a) which is the union of stable manifolds of the equilibrium points (2.488345,0.0) and (8.772443,0.0). Curve A is the stability boundary obtained in [10](after a shift in coordinates). It is clear from the phase portrait in Fig. 5(b) that all the points inside curve B converge to the stable equilibrium point which verifies that the curve B is the exact stability boundary.

Example 3 : The following system was also considered in [6]

$$\dot{x}_1 = -x_1 + x_2 \tag{7-7}$$

$$\dot{x}_2 = 0.1x_1 - 2.0x_2 - x_1^2 - 0.1x_1^3$$

There are three equilibrium points : (0.0,0.0) is stable, (-2.55 , -2.55) is type-one and (-7.45,-7.45) is also stable. We are interested in the stability region of (0.0,0.0). Again the stability region in this case is unbounded. Fig. 6(a) shows the stability region obtained by our method. Fig. 6(b) represents the phase portrait of this system.

Example 4 : A simple nonlinear speed-control system studied by Fallside etc. [21] and Jocic [6] shown in Fig. 7(a) can be described by the following equation

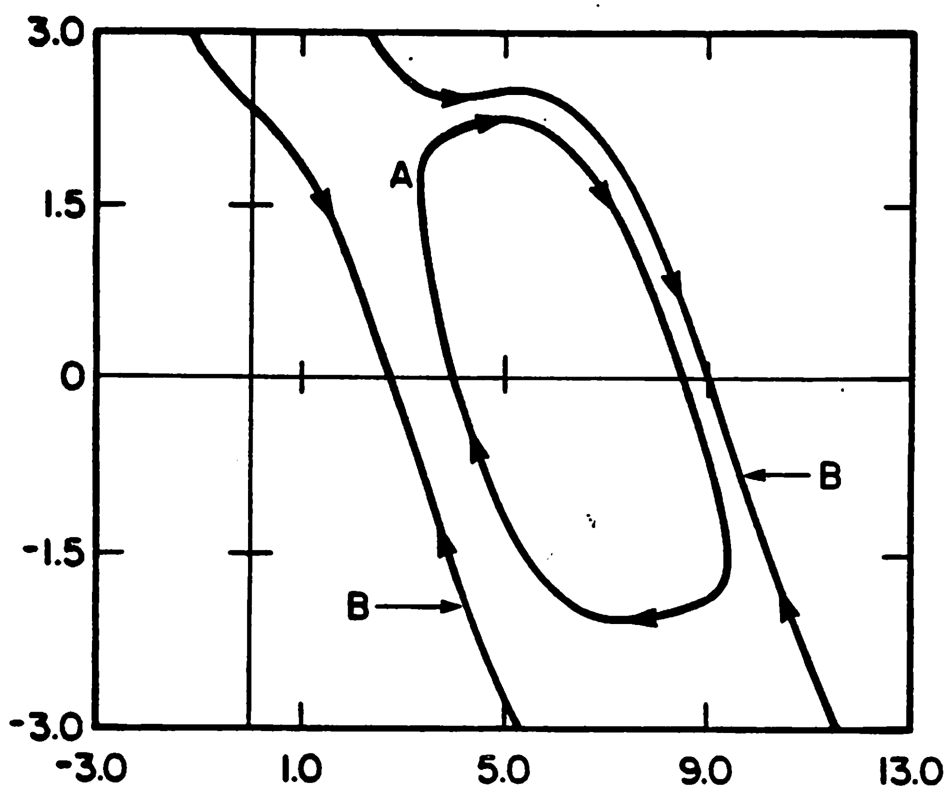

Fig. 5(a). Predictions of the stability region of Example 2 by different methods. Curves A is obtained by the methods in [10](after a shift in coordinates). Curve B is obtained by the present method.

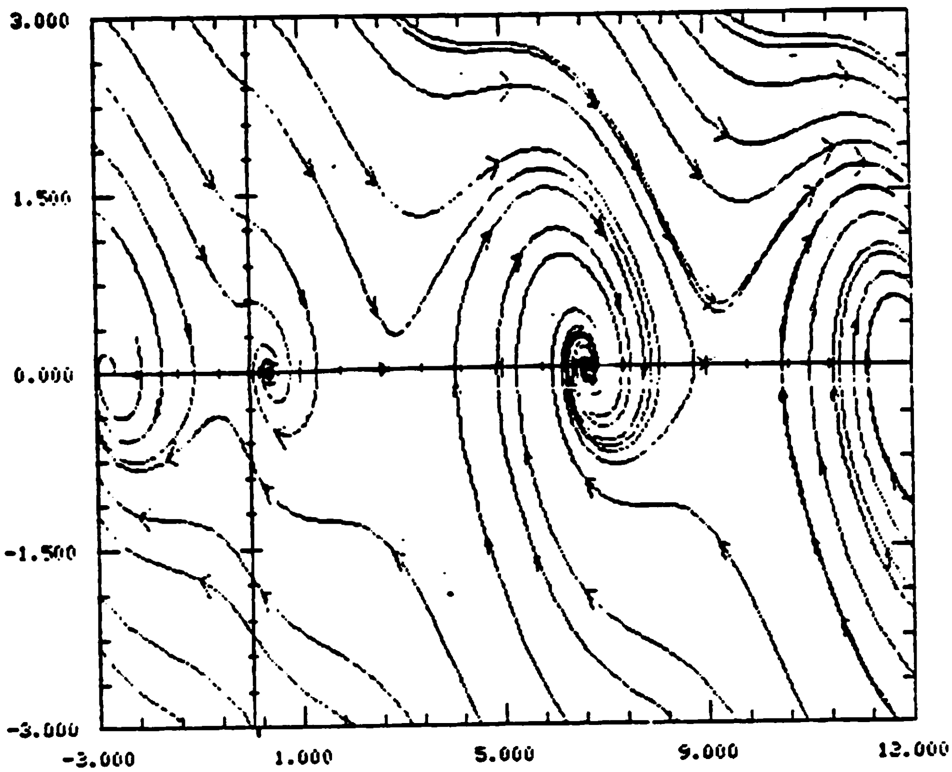

Fig. 5(b). The phase portrait of this system.

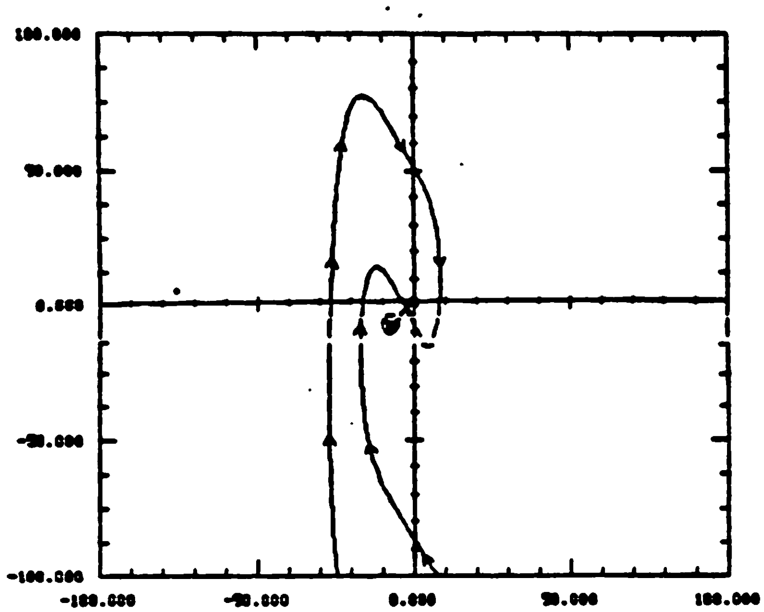

Fig. 6(a). Predictions of the stability region of Example 3 by the present method. The curves in this figure is the stable manifold of the type-one equilibrium point (-2.55,-2.55). The stability region of (0.0,0.0) is the region inside these curves which contains (-2.55,-2.55).

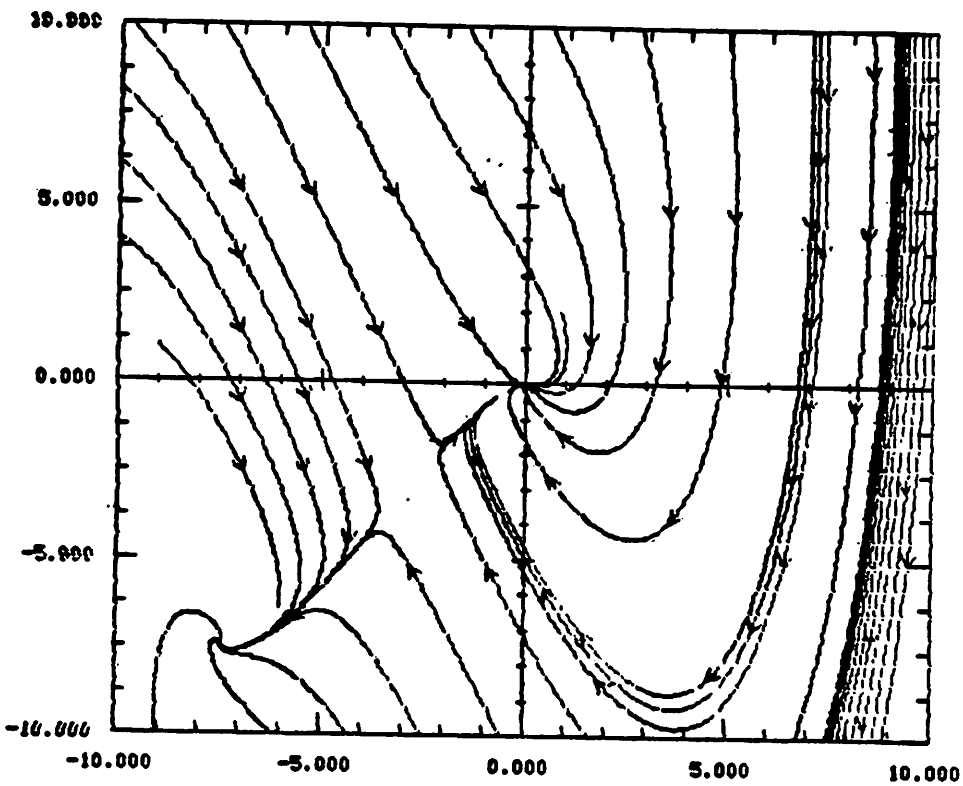

Fig. 6(b). The phase portrait of this system. Note that the coordinate system has been rescaled.

$$\dot{x}_1 = x_2 \tag{7-8}$$

$$\dot{x}_2 = -K_d x_2 - x_1 - g x_1^2 \left(\frac{x_2}{K_d} + x_1 + 1\right)$$

For $K_d = 1$ and $g=6$, there are three equilibrium points : (-0.78865, 0.0) is stable (the corresponding Jacobian has two real negative eigenvalues), (-0.21135,0.0) is type-one and (0.0,0.0) is also stable (the corresponding Jacobian has two complex eigenvalues with negative real parts). The type-one equilibrium point is on the stability boundary of (0.0,0.0) and also on the stability boundary of (-0.78865,0.0) because the two branches of its unstable manifold approaches them. Thus, by Theorem 4-1 we conclude that the stability region of (0.0,0.0) is the open set containing (0.0,0.0); its boundary is characterized by the stable manifold of (-0.211325,0.0); the stability region of (-0.78865,0.0) is the open set containing (-0.78865,0.0) with the same boundary as that of (0.0,0.0). The region in Fig. 7(b) is the stability region predicted by this method. The region denoted by A_J in Fig. 7(c) shows the stability region predicted by method of sinks [13], and the region A_w is predicted by [21]. The phase portrait of this control system is in Fig. 7(d).

Example 5 : Consider the following system which is similar to (7-8) except the term $-K_d x_2$ is replaced by $K_d x_2$.

$$\dot{x}_1 = x_2 \tag{7-9}$$

$$\dot{x}_2 = K_d x_2 - x_1 - g x_1^2 \left(\frac{x_2}{K_d} + x_1 + 1\right)$$

For $K_d = 1$ and $g=6$, there are three equilibrium points : (-0.78865, 0.0) is stable, (-0.21135,0.0) is type-one and (0.0,0.0) is a source. It can be shown that both the type-one equilibrium points and the source are on the stability boundary. Both parts of the unstable manifold of the type-one equilibrium point (-0.21135,0.0) approach the stable equilibrium point. We conclude that they both belong to the stability region ; consequently, the stability region is the whole state-space except for the stable manifold of (-0.21135, 0.0) and the source (0.0,0.0). Fig. 8(a) shows the stable manifold and unstable manifold of (-0.21135, 0.0). The phase portrait of this system is in Fig. 8(b) Compare system (7-9) to system (7-8) we found that the stability region of (0.0,0.0) for (7-8) is shrunk to a point for (7-9) while the stability region of the stable equilibrium point of (7-9) is expanded to fill almost all of the state space.

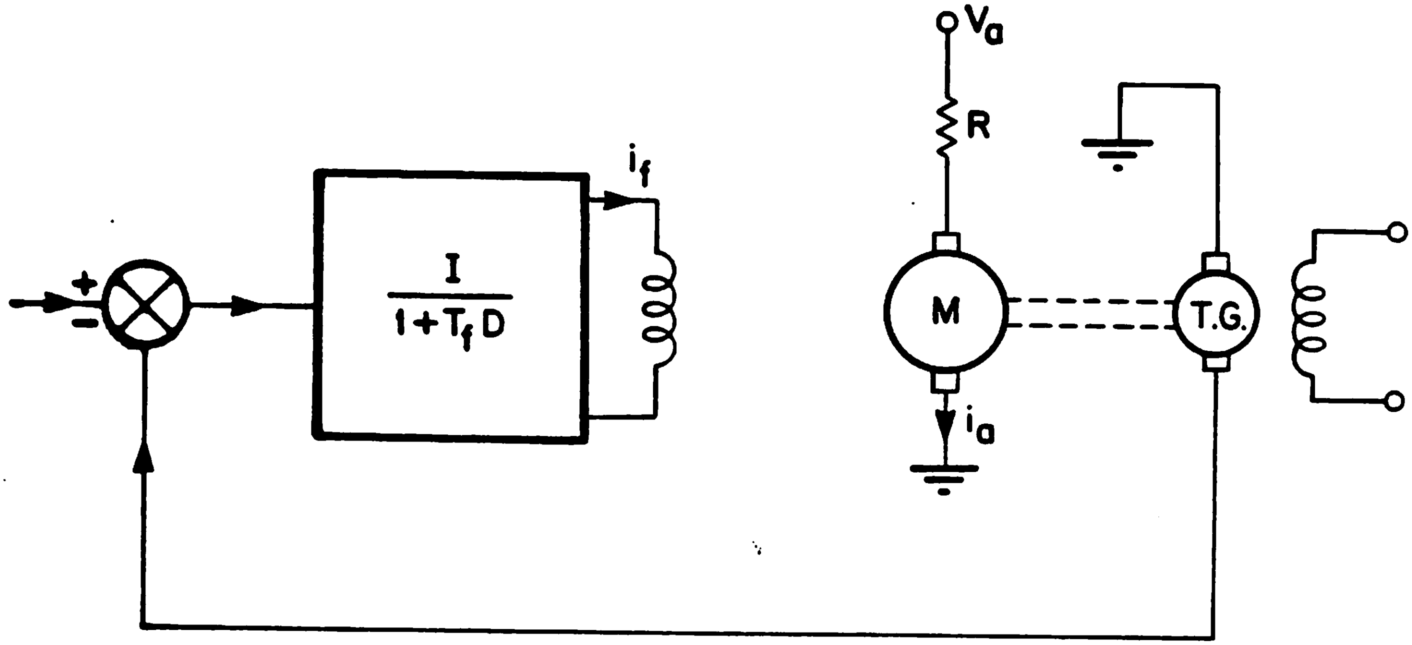

Fig. 7(a). A simple nonlinear speed-control system.

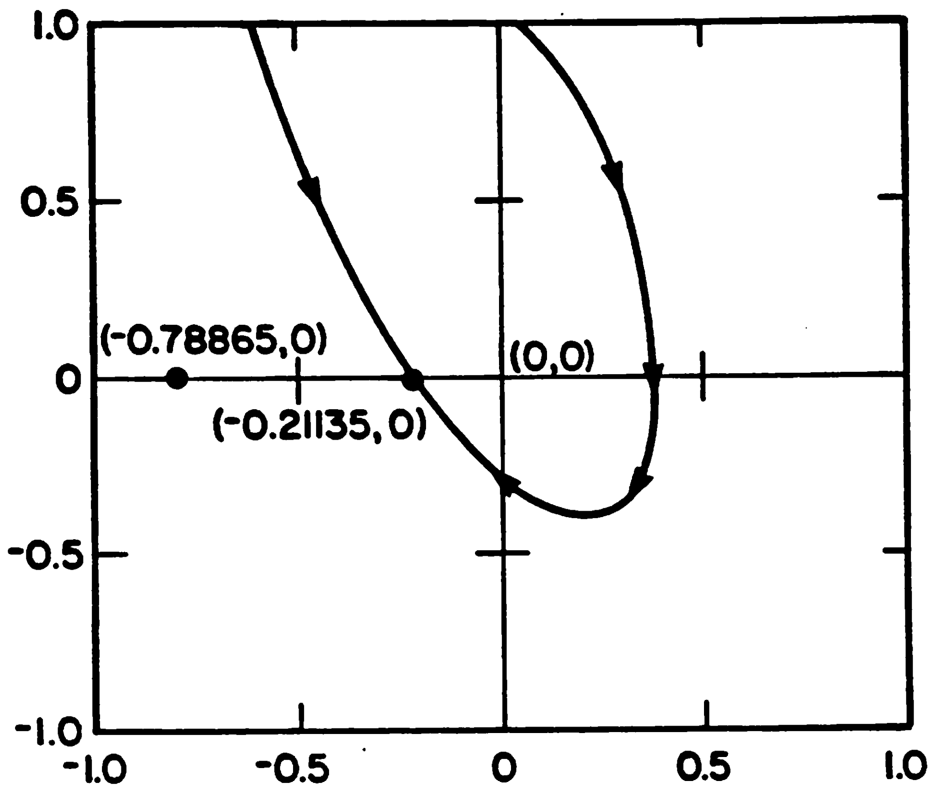

Fig. 7(b). The stability region of Example 4 predicted by the present method.

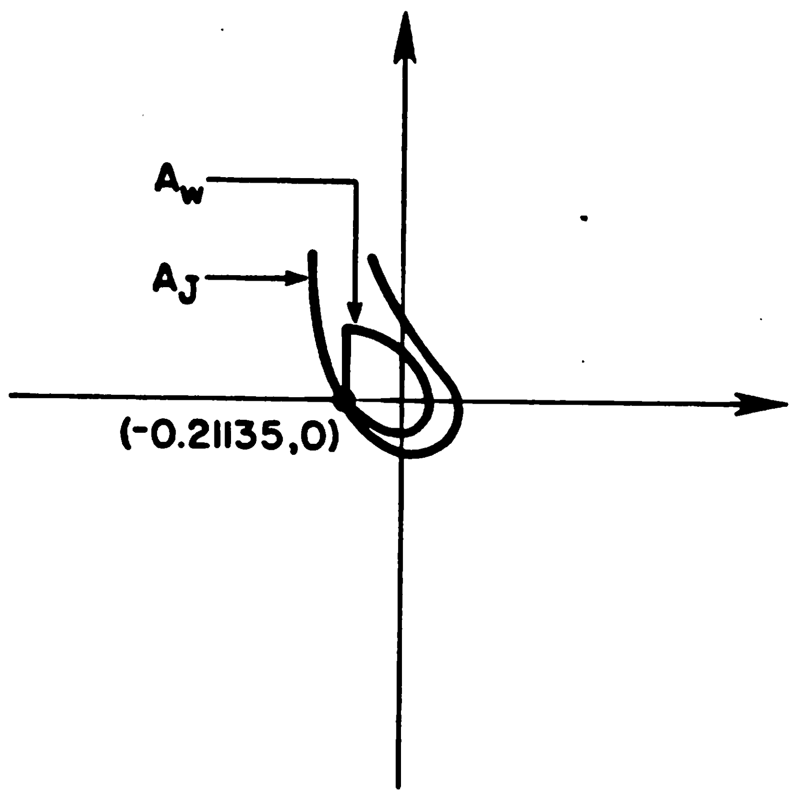

Fig. 7(c). Predictions of the stability region of Example 4 by different methods. The regions denoted by A_J and A_W are obtained by the methods in [13] and [21].

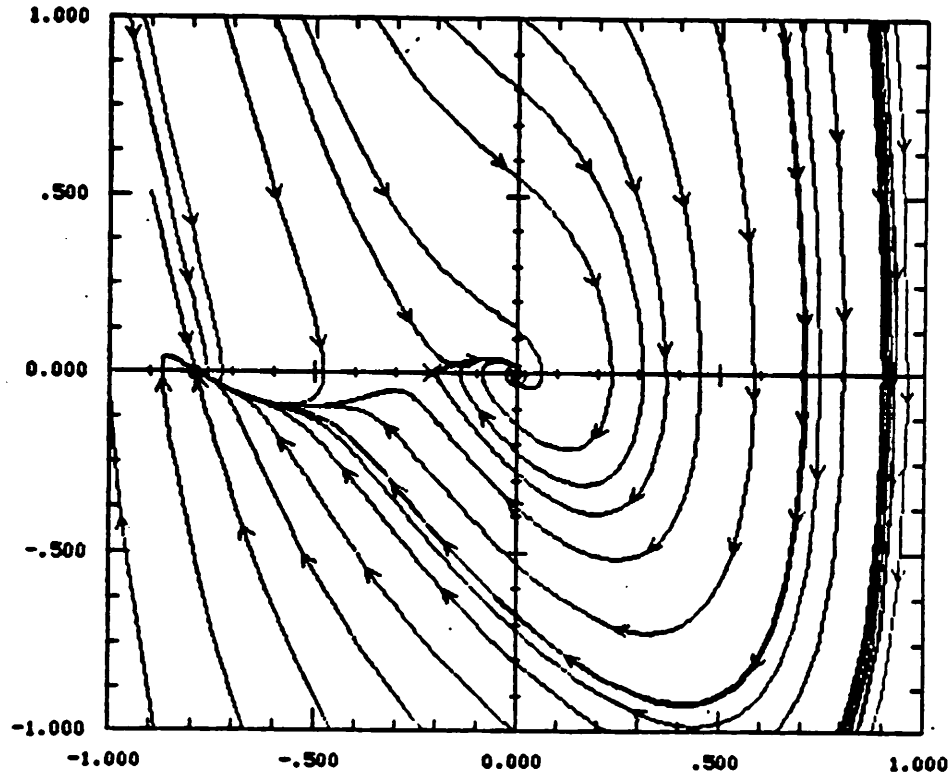

Fig. 7(d). The phase portrait of this system.

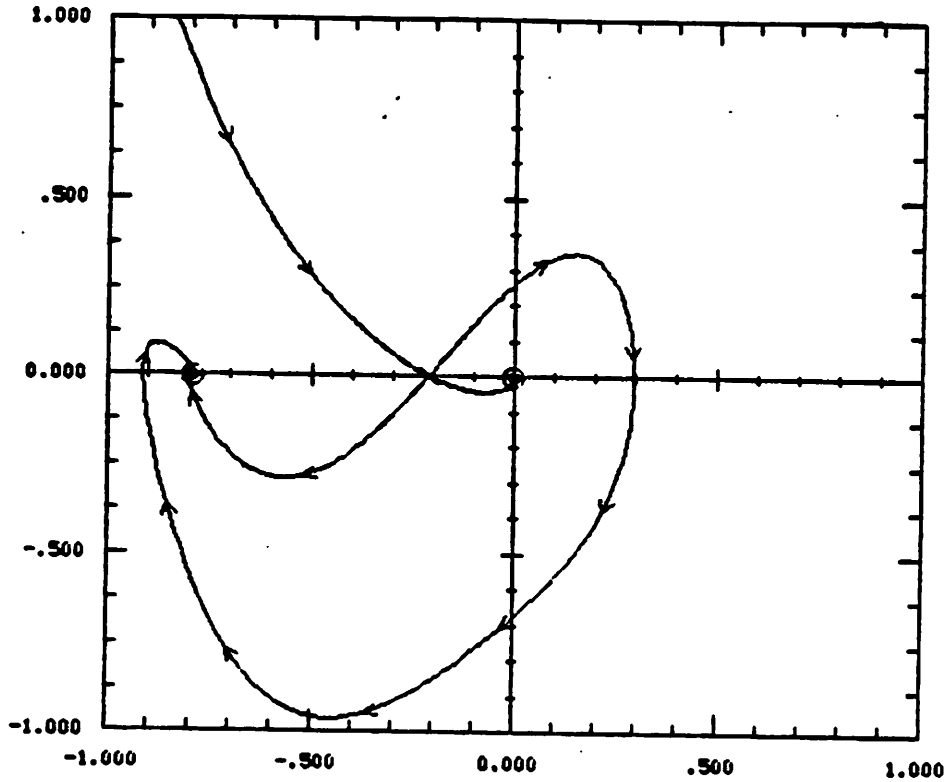

Fig. 8(a). The stability region of Example 5 predicted by the present method is the whole state space except for the stable manifold of (-0.21135, 0.0), denoted by the curve A and the source (0.0,0.0).

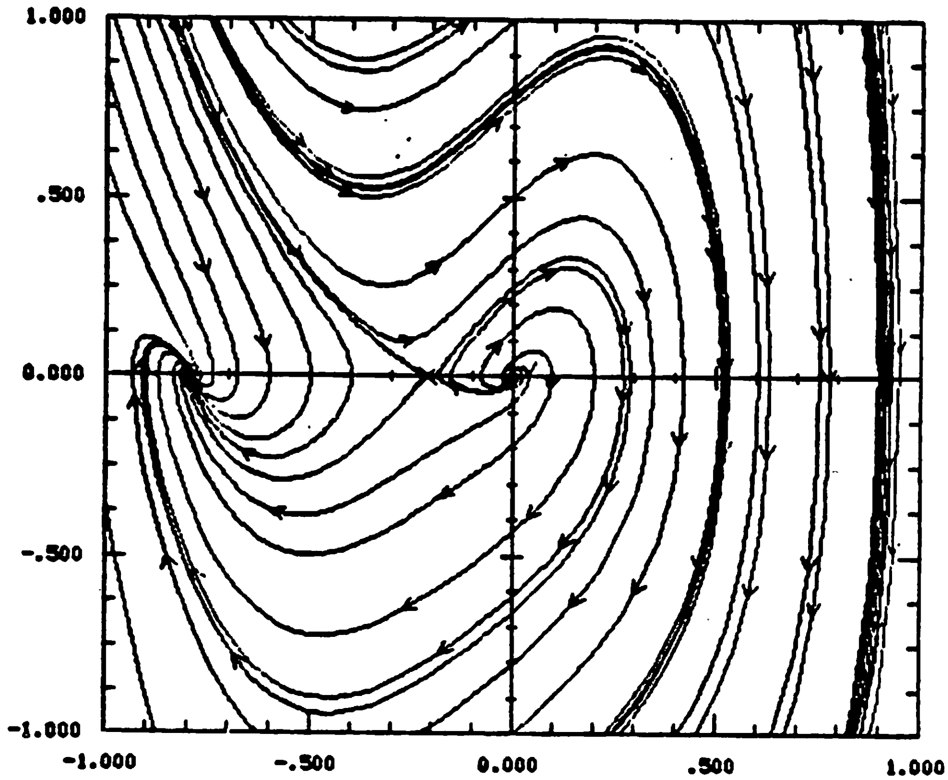

Fig. 8(b). The phase portrait of this system. Note that all the points except for the curve A converge to the stable equilibrium point.

8. CONCLUSION

A comprehensive theory of stability regions of stable equilibrium points for nonlinear autonomous dynamical systems is presented. A complete dynamical characterization of the stability boundary of a fairly large class of nonlinear autonomous dynamical systems is derived. A method for finding the stability region based on its topological properties is proposed.

The proposed method requires the determination of the stable manifold of an equilibrium point. For lower dimensional systems this may be done by numerical methods. For higher dimensional systems efficient computational methods to derive the stable manifolds are needed.

Acknowledgement

The research is sponsored by the Department of Energy, Division of Electric energy systems, under contract DE-AC01-84-CE76257. M.W. Hirsch was partially supported by the National Science Foundation under contract MCS 83-03283. The authors wish to thank P.P. Varaiya and L.O. Chua for their helpful discussions.

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