LIMITS OF LINEAR RESPONSE OF A VLASOV DISTRIBUTION

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Abstract

The linear response of a periodic Vlasov plasma distribution function is computed to second order in the electric field (this procedure is also given justification). The results for a specific electric field are then compared with the results of computer simulation for different amplitudes of the electric field. The onset of the deviations from linear theory as the amplitude increases are correctly predicted by trapping theory, indicating that trapping is responsible for limiting the validity of linear theory.

This research was supported by Department of Energy Contract DE-AT03-76ET53064.

LIMITS OF LINEAR RESPONSE OF A VLASOV DISTRIBUTION

Introduction

The amplitude at which a wave in a collisionless (Vlasov) plasma ceases to obey linear theory is a question of general interest. Although this question is well understood theoretically for the case to be considered here, the techniques learned in simulating it will be needed in attacking the more difficult magnetized case.

An electrostatic wave in an unmagnetized plasma is considered here. The wave is imposed, i.e. the distribution function has no effect on the electric field. Thus the wave may represent *any* electrostatic wave, but self-consistent effects will not be present. The boundary conditions are periodic, and the wave amplitude varies in time.

In the report which follows, first the linear theory is worked out in detail so that it may be compared with simulation to ensure the correctness of the simulation (this work is certainly not new, as it is implicit in quasi-linear theory, but I have found no references which treat it). Next, trapping (which is expected to be responsible for the onset of non-linearity in the particle response) is considered, and a simple rule for when trapping should be important is derived. (This rule is also well known.) Finally, a simulation is performed to examine the transition from linearity to non-linearity. The distribution function and its first two moments are used as diagnostics.

Analytic Theory and Calculation of Second-Order Perturbed Distribution Function

The interaction of a wave with a plasma is through the distribution function, and all information regarding that interaction is contained in it. For a linear wave, the distribution function is only perturbed slightly, and can be expanded as a power series in some parameter related to the electric field strength. The first three orders of this expansion have important meanings. The zero order distribution function is just the unperturbed distribution function, and for the cases of interest here, is constant in space and time. The first order perturbed distribution function supports the wave, and is proportional to the first order electric field. Like the first order electric field, the first order perturbed distribution function is sinusoidal in time and space. The second order perturbed distribution function has a non-zero average, and therefore represents the first term in the expansion which can give rise to long-term deviations from the initial distribution. The second order perturbed distribution function can be used to compute several diagnostics of interest; for instance, the average velocity of the distribution (assuming the average velocity of the zero order distribution was chosen as zero), and the perturbed kinetic energy density of the distribution to highest order will be

$$\langle \vec{v} \rangle = \iiint \vec{v} f_2(\vec{v})\, d^3\vec{v} \tag{1}$$

and

$$\mathcal{E} = \frac{1}{2} nm \iiint v^2 f_2(\vec{v})\, d^3\vec{v} \tag{2}$$

(It would be a surprise if the wave kinetic energy of a linear wave were anything but of second order in the wave electric field.) The second order perturbed distribution function is therefore the order of most interest in the context of the interaction of a linear wave with a plasma, and specifically for current drive applications.

Linear theory usually deals only with the first order perturbed distribution function. To calculate the second order perturbed distribution function, it is necessary to extend the usual linear theory to quantities of second order. It is to be emphasized that these quantities are *not* of a non-linear nature; they are simply second-order consequences of linear theory (i.e., they depend only on the first order electric field). This concept will be clear to those who are familiar with quasi-linear theory (although this method is not strictly speaking quasi-linear).

The method of computing the second order distribution function will be illustrated for the 1-d electrostatic case, but first, some justification must be given for the claim that the second order perturbed distribution function is truly derivable from linear quantities.

Second Order Correction to the Electric Field

The procedures for calculating the kinetic energy from perturbed velocities and distribution functions require some justification for ignoring the second-order correction to the electric field, which is a non-linear quantity. While in this model the electric field is imposed, in general, the self-consistency requirement will dictate the existence of a correction which is of second order in the first-order field. (Since the electric field is now being expanded, it is fair to ask what the expansion parameter really is; it is sufficient to second order to consider the expansion parameter to be the part of the electric field which varies sinusoidally at the desired frequency and wave number.)

The argument begins by expanding the Vlasov equation in orders of the electric field:

$$\frac{\partial f_0}{\partial t} + \vec{v} \cdot \frac{\partial f_0}{\partial \vec{x}} + \frac{q}{m}(\vec{v} \times \vec{B}) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0 \tag{3}$$

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} + \frac{q}{m}(\vec{v} \times \vec{B}) \cdot \frac{\partial f_1}{\partial \vec{v}} = -\frac{q}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}} \tag{4}$$

$$\frac{\partial f_2}{\partial t} + \vec{v} \cdot \frac{\partial f_2}{\partial \vec{x}} + \frac{q}{m}(\vec{v} \times \vec{B}) \cdot \frac{\partial f_2}{\partial \vec{v}} = -\frac{q}{m}\vec{E}_1 \cdot \frac{\partial f_1}{\partial \vec{v}} - \frac{q}{m}\vec{E}_2 \cdot \frac{\partial f_0}{\partial \vec{v}} \tag{5}$$

Here $\vec{E}_1$ is the expansion parameter. It is assumed that there is no equilibrium electric field ($\vec{E}_0 = 0$).

$\vec{E}_2$ is the part of the electric field which in a self-consistent model would be generated by second-order perturbations in the charge density arising from $\vec{E}_1$ (and self-consistently from $\vec{E}_2$ itself). Poisson's equation dictates its existence, and puts some constraints on its temporal behavior, specifically, it, like all other second order quantities may have a part which is slowly varying in space and time, and a part with twice the wavelength and twice the frequency of the linear field. Its average (in space and time over the short time-scale, i.e., its slowly varying part) is chosen to be zero. This *is* an assumption, and deserves some discussion.

The assumption that $\vec{E}_2$ has an average of zero can be understood intuitively. Any electric field which has a non-zero average represents a *global* effect, as opposed to the local effect of any sinusoidally varying field. $\vec{E}_2$ can only arise due to the flow and build-up of charge on a large scale. In a periodic model, for instance, the average of $\vec{E}_2$ is constrained to be zero by the periodicity requirement on the electrostatic potential. In the general case, the average value of $\vec{E}_2$ will depend on the boundary conditions of the model.

The right-hand side of equation (5) can now be split into a slowly varying and a rapidly oscillating part, with the slowly varying part depending on $\vec{E}_1$ but not $\vec{E}_2$. f_2 can likewise be split into a slowly varying part $\bar{f}_2$ and a rapidly oscillating part $\tilde{f}_2$ splitting equation (5) in two. The average value of $\tilde{f}_2$ will be small, and if the wave is of constant amplitude in either space or time, the appropriate average of $\tilde{f}_2$ will be zero. It is the average value of f_2 which is of interest. This is now plainly $\bar{f}_2$, which does not depend on $\vec{E}_2$ at all. Thus, provided only $\bar{f}_2$ is considered, $\vec{E}_2$ is irrelevent, and the second order perturbed distribution function depends only on the linear electric field amplitude.

Theory Applied to Electrostatic Waves in an Unmagnetized Plasma

To illustrate how the second order perturbed distribution function is calculated, and to test the correctness of the method, the second order perturbed distribution function will be calculated for a

Langmuir wave which is spatially uniform. The electric field is chosen to be $E(x,t) = E_1 \exp[\gamma t + i(kx - \omega t)]$. The electric field is imposed rather than calculated self-consistently for several reasons. First, imposing self-consistency is nothing more than choosing the correct "imposed" field, i.e., self-consistency (in linear problems) is in effect a condition on γ. Second, a self-consistent wave must (given a Maxwellian plasma) have a negative growth rate (i.e. a positive damping rate). This would require that the distribution function which we are trying to find, be used as the initial condition. The obvious initial condition to use is $f_2 = 0$ at $t = -\infty$. Naturally, one cannot have a damping rate as well as no excitation at $t = -\infty$, but physically speaking one must have some way of exciting the wave which is outside the infinite homogeneous model such as an antenna or ponderomotive force; thus, it is reasonable to assume a driving field at some time during the evolution of the wave and a positive growth rate γ.

Expanding the unmagnetized Vlasov equation,

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = 0 \tag{6}$$

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = -\frac{q}{m}E\frac{\partial f_0}{\partial v} \tag{7}$$

$$\frac{\partial f_2}{\partial t} + v\frac{\partial f_2}{\partial x} = -\frac{q}{m}E\frac{\partial f_1}{\partial v} \tag{8}$$

These equations can be solved using the method of characteristics with characteristics defined by $x(t) = x(t_0) + v \cdot (t - t_0)$ and $v(t) = \text{constant}$ (see Appendix A for details of solution). The result is

$$\bar{f}_2(v) = \frac{\hat{E}^2}{4}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left[\frac{\partial f_0/\partial v}{(\omega - kv)^2 + \gamma^2}\right] \tag{9}$$

where the bar denotes averaging over position and phase angle.

Note that this formula is very similar to formulas from quasi-linear theory. This formula, however, results from a single coherent wave rather than a spectrum of incoherent waves.

Distribution Function After Passage of Wave

A generalization of this method is also useful. If a plasma is subjected to a wave pulse of limited duration, the form of the perturbed distribution function as $t \to \infty$ can be calculated for a pulse of arbitrary shape. Since the wave has disappeared at $t = \infty$, this represents *only* The change in the distribution function can be ascribed solely to Landau damping.

Let

$$E(t) = \hat{E}(t)e^{i(kx-\omega t)}$$

where $\hat{E}(t)$ is the pulse modulation function (no assumption about the rate of variation of this function is necessary). Using the same method of characteristics as before,

$$\begin{aligned} f_1(x,t) &= -\frac{q}{m}\frac{\partial f_0}{\partial v}\int_{-\infty}^{t} \hat{E}(t')e^{i[k(x+v(t'-t))-\omega t']}\,dt' \\ &= -\frac{q}{m}\frac{\partial f_0}{\partial v}e^{ik(x-vt)}\int_0^{\infty} \hat{E}(t-\tau)e^{i(\omega-kv)\tau}\,d\tau \end{aligned} \tag{10}$$

where $\tau = t - t'$. Taking the first formula for f_1 as $t \to \infty$,

$$\begin{aligned} f_1 &\to -\frac{q}{m}\frac{\partial f_0}{\partial v}e^{ik(x-vt)}\int_{-\infty}^{\infty} \hat{E}(t')e^{i(kv-\omega)t'}\,dt' \\ &= -\frac{q}{m}\frac{\partial f_0}{\partial v}e^{ik(x-vt)}\tilde{E}(kv-\omega) \end{aligned} \tag{11}$$

where $\tilde{E}$ is the Fourier transform of $\hat{E}$. Note the well-known rippling behavior of f_1 with the ripples becoming finer and finer as time passes [1].

To calculate the second-order perturbed distribution function, the second formula for $f_1(x,t)$ is substituted into the integral:

$$\begin{aligned} f_2(x,t) &= -\frac{q}{m}\int_{-\infty}^{t} E(x',t')\frac{\partial}{\partial v'}f_1(x',v',t')\,dt' \\ &= \frac{q^2}{m^2}\int_{-\infty}^{t} \mathrm{Re}\left[\hat{E}(t')e^{i(kx'-\omega t')}\right] \\ &\quad \cdot \mathrm{Re}\left[e^{i(kx'-\omega t')}\frac{\partial}{\partial v}\left(\frac{\partial f_0}{\partial v}\int_0^{\infty}\hat{E}(t'-\tau)e^{i(\omega-kv)\tau}\,d\tau\right)\right] \end{aligned} \tag{12}$$

$$\bar{f}_2 = \frac{1}{2}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left[\frac{\partial f_0}{\partial v}\int_{-\infty}^{t}\mathrm{Re}\left(\int_0^{\infty}\hat{E}(t')\hat{E}(t'-\tau)e^{i(\omega-kv)\tau}\,d\tau\right)dt'\right] \tag{13}$$

Setting $t = \infty$ and interchanging the order of integration,

$$\begin{aligned} \bar{f}_2(t\to\infty) &= \frac{1}{2}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left[\frac{\partial f_0}{\partial v}\mathrm{Re}\left(\int_0^{\infty}\left\{\int_{-\infty}^{\infty}\hat{E}(t')\hat{E}(t'-\tau)\,dt'\right\}e^{i(\omega-kv)\tau}\,d\tau\right)\right] \\ &= \frac{1}{4}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left[\frac{\partial f_0}{\partial v}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\hat{E}(t')\hat{E}(t'+\tau)\,dt'\right\}e^{i(\omega-kv)\tau}\,d\tau\right] \end{aligned} \tag{14}$$

since $\int \hat{E}(t')\hat{E}(t'-\tau)\,dt'$ is even in τ. Now applying the convolution theorem gives

$$\bar{f}_2(t\to\infty) = \frac{1}{4}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left(|\tilde{E}(\omega-kv)|^2\frac{\partial f_0}{\partial v}\right) \tag{15}$$

(This formula can actually be used to compute $\bar{f}_2(t_0)$ by substituting $\hat{E}(t)\Theta(t_0 - t)$ for $\hat{E}(t)$ where $\Theta(t)$ is the Heavyside step function.) Again, note the similarity to the results of quasi-linear theory. Equation (15) has the form of a diffusion operator operating on f_0. This formula will also be used in the comparison with the simulation results.

One important conclusion which can be drawn from (15), is the relation between the damping rate and the width in velocity of the resonant region. When the envelope is rapidly varying, $\tilde{E}(\omega)$ is a wide function, and so a large part of velocity space will be perturbed significantly. When the envelope is slowly varying, only a narrow part of velocity space should be affected. This should mean that a strongly damped wave will spread out its energy over a wider region of velocity space, and therefore that the maximum magnitude of the perturbation will be smaller. This in turn should mean that strongly damped waves will follow the predictions of linear theory at higher amplitudes than weakly damped waves. This intuitive argument will become more concrete in the section devoted to trapping.

Analytic Checks of Perturbed Distribution Function

To check the accuracy of the linear calculation (and the correctness of the algebra) two separate calculations were made. Since neither of the methods generate the distribution function, the second order perturbed kinetic energy was chosen as the diagnostic for comparison between the calculations. The first method is to compute the first and second order perturbed velocities, and compute the energy from them. This calculation is closely related to the calculation by perturbed distribution functions, so only the correctness of the algebra is truly tested. The second method is completely different, and is valid only in the limit of small γ. It assumes only that the linear dielectric function is given, and that the field amplitude changes sufficiently slowly. This method should represent a truly independent verification of the correctness of the method of calculation by perturbed distribution functions.

Langmuir Energy by Method of Perturbed Velocities

This method can be derived as a canonical transformation of the method of perturbed distribution functions. Starting with the formal solution of the Vlasov equation,

$$f(x(t), v(t), t) = f_0(v(0), t = 0) \tag{16}$$

(note that f is independent of x at $t=0$ — it is only necessary that f be independent of x at *some* time, even $t=-\infty$) multiply both sides by $\frac{1}{2}nmv(t)^2$, integrate over all velocities, and over one period in $x(t)$:

$$\int \frac{1}{2}nmv^2 f(x,v,t)\, d^3v d^3x = \int \frac{1}{2}nmv(t)^2 f_0(v(0),t=0)\, d^3v(t) d^3x(t)$$

Now make the change of variables $x(t) \rightarrow x(0)$, $v(t) \rightarrow v(0)$. Since this is a canonical transformation, the phase space volume is unchanged, and

$$\int \frac{1}{2}nmv^2 f(x,v,t)\, d^3v d^3x = \int \frac{1}{2}nmv(t)^2 f_0(v(0),t=0)\, d^3v(0) d^3x(0)$$

Now expanding both sides in orders of the electric field and setting the second order terms equal (remembering that integrating over x is an averaging operation),

$$\int \frac{1}{2}nmv^2 \bar{f}_2(v,t)\, d^3v = \int \frac{1}{2}nm[\overline{v_1(t)^2} + 2\overline{v_0(t)v_2(t)}] f_0(v_0,t=0)\, d^3v_0 \tag{17}$$

The calculation by perturbed velocities begins by expanding the equation of motion

$$\frac{dv}{dt} = \frac{q}{m}E(x) \tag{18}$$

in orders of the electric field. The expansion is

$$\frac{dv_0}{dt} = 0 \tag{19}$$

$$\frac{dv_1}{dt} = \frac{q}{m}E(x_0) \tag{20}$$

$$\frac{dv_2}{dt} = \frac{q}{m}(x_1\frac{\partial}{\partial x_0})E(x_0) \tag{21}$$

Setting $E = E_1 \exp(ikx - i\omega t + \gamma t)$ (as in the calculation of the second order perturbed distribution function), turns the $x_1 \cdot \partial/\partial x$ into ikx_1. Some care must be taken in the interpretation of these formulae, as the velocities are taken to be complex (for now), but x_0 and x_1 must be purely real.

These equations can now be solved directly (see Appendix B for details of solution):

$$v_0 = \text{constant} \tag{22}$$

$$v_1 = i\frac{q}{m}E_1 \frac{e^{ikx_i - i(\omega - kv_0 + i\gamma)t}}{\omega - kv_0 + i\gamma} \tag{23}$$

$$\bar{v}_2 = \frac{E_1^2}{4}\frac{q^2}{m^2}e^{2\gamma t}\frac{d}{dv_0}\left[\frac{1}{(\omega - kv_0)^2 + \gamma^2}\right] \tag{24}$$

Here the bar denotes an average over space (it is assumed that the initial distribution is uniform in space).

Taking the real part of v_1,

$$\frac{1}{2}v_1^2 + v_0 v_2 = \frac{E_1^2}{4}\frac{q^2}{m^2}e^{2\gamma t}\frac{d}{dv_0}\left[\frac{v_0}{(\omega - kv_0)^2+\gamma^2}\right] \quad (25)$$

after some simple manipulation. Averaging over the distribution function and integrating by parts gives a total kinetic energy

$$W = -\frac{\hat{E}^2}{4}\frac{nq^2}{m}\int \frac{v_0 \partial f_0(v_0)/\partial v_0}{(\omega - kv_0)^2+\gamma^2}\,dv_0 \quad (26)$$

The kinetic energy according to the second order perturbed distribution function is

$$W = \frac{\hat{E}^2}{4}\frac{nq^2}{m}\int \frac{v^2}{2}\frac{\partial}{\partial v}\frac{\partial f_0/\partial v}{(\omega - kv)^2+\gamma^2}\,dv \quad (27)$$

which is the same as (26) when integrated by parts, so the algebra of the derivation of the second order perturbed distribution function can safely be assumed to be correct.

Derivation of Energy from Dielectric Function

The final test of the correctness of the method is the well-known formula

$$\mathcal{E}_{wave} = W_{wave} + \epsilon_0 \frac{\hat{E}^2}{4} = \frac{\hat{E}^2}{4}\frac{\partial}{\partial \omega}\left(\omega \epsilon_r(\omega)\right) \quad (28)$$

where ϵ_r is the real part of the dielectric function. As mentioned, this derivation makes stronger assumptions about the wave than the preceding ones. This will make the results harder to compare, but the comparison will represent a true test of the correctness of the concepts behind the use of the second order perturbed distribution function.

The derivation of (28) will now be outlined. It is assumed that $\vec{k}$ and ω_0 are real. The real and imaginary parts of $\epsilon(\vec{k},\omega)$ will be denoted ϵ_r and ϵ_i. (A scalar dielectric function is also assumed, though a more general result can be similarly derived for a tensor dielectric function.)

The only formulas needed are

$$\vec{D}(\vec{k},\omega) = \epsilon(\vec{k},\omega)\,\vec{E}(\vec{k},\omega) \quad (29)$$

and

$$\vec{\jmath}_{ext} + \frac{\partial \vec{D}}{\partial t} = 0 \quad (30)$$

Since $\vec{k}$ plays no role in the derivation, it will be dropped.

Since ϵ is a scalar, the vectors are unnecessary, and will be omitted from the rest of the derivation. Assume that E has the form

$$E(t) = \hat{E}(t)e^{-i\omega_0 t} \tag{31}$$

where $\hat{E}$ is real, and varies slowly, so that $\hat{E}(\omega)$ is sharply peaked around $\omega = 0$. Note that the existence of $E(\omega)$ here implies that $E \to 0$ as $t \to \pm\infty$.

The strategy of the derivation is to approximate $\partial D/\partial t$ as a function of time, then substitute this into the second formula and dot it with $\vec{E}$. This can then be put in the form of an electrostatic Poynting theorem, which will identify the energy density. The crucial assumption in the derivation is that $E(\omega)$ is large only in a narrow enough region about ω_0 that $\epsilon(\omega)$ can be approximated as $\epsilon(\omega_0) + (\omega - \omega_0)\partial\epsilon/\partial\omega$.

The result is:

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega_0}(\omega_0\epsilon_r(\omega_0)) \cdot \frac{\hat{E}^2}{4}\right] = -\omega_0\epsilon_i(\omega_0)\frac{\hat{E}^2}{2} - \frac{1}{2}\,\mathrm{Re}\,\hat{\jmath}_{ext} \cdot \hat{E} \tag{32}$$

or substituting $\epsilon_i = \sigma_r/\omega_0$,

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega_0}(\omega_0\epsilon_r(\omega_0)) \cdot \frac{\hat{E}^2}{4}\right] = -\frac{1}{2}\,\mathrm{Re}\,(\hat{\jmath}_{int} + \hat{\jmath}_{ext}) \cdot \hat{E} \tag{33}$$

The identification of the quantity in brackets as the energy density is now obvious. Some confusion may arise as to the meaning of $\hat{\jmath}_{ext}$ since the original idea was to impose the electric field inconsistently. The total current however, must be consistent with the electric field. Since not all the current was accounted for by the internal current of the dielectric, an external current had to be included. This external current can be thought of as being the physical driver of the wave in the dielectric.

As with the Poynting theorem, the energy includes both the kinetic (internal) and the electrostatic parts (including the electrostatic part representing $\frac{1}{2}\rho_{ext}\phi$), so the wave kinetic energy is:

$$W = \left[\frac{\partial}{\partial\omega_0}(\omega_0\epsilon_r) - \epsilon_0\right] \cdot \frac{\hat{E}^2}{4}$$

The non-wave kinetic energy (due to energy absorption by Landau damping) can be computed by integrating the energy loss term $-\omega_0\epsilon_i(\omega_0)\frac{\hat{E}^2}{2}$ in the conservation law. Thus by this method, the kinetic wave energy and the absorbed kinetic energy have separate identities, and they can both be found. Note that the perturbation methods made no distinction between these energies.

A similar derivation could have been done using a Laplace transform, allowing the problem to be an initial value problem, but the non-wave kinetic energy at $t = 0$ would had to have been specified. This is not a major problem for this approximation, but for $\hat{E}(t)$ more rapidly varying, as it is in the perturbation methods, it is a serious problem. Note that no assumptions have been made about ϵ aside from its being independent of the field amplitude, but the important assumption that $E(t) \to 0$ as $t \to \pm\infty$ has been made.

Langmuir Energy by Linear Dielectric Function

This method can now be applied to the energy of a Langmuir wave. The dielectric function for the Langmuir wave is:

$$\epsilon(k,\omega) = 1 + \frac{1}{k^2\lambda_D^2}[1 + \frac{\omega}{kv_t}N(\frac{\omega}{kv_t})] \tag{34}$$

where $\lambda_D^2 = k_B T/ne^2$ (n = number density) and

$$\begin{aligned} N(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-\frac{\zeta^2}{2}}}{\zeta - z}\,d\zeta \\ &= \frac{1}{\sqrt{2}}Z(\frac{z}{\sqrt{2}}) \end{aligned} \tag{35}$$

Z being the standard Plasma Dispersion Function (as defined by Fried and Conte [2]). The contour of integration is taken below the pole at $\zeta = z$. (This normalization eliminates some annoying factors of $\sqrt{2}$.) ϵ_0 is taken to be 1 to simplify the algebra.

Applying formula (28), one finds after using $N'(z) = -[1 + zN(z)]$ that

$$W_{wave} = \frac{1}{k^2\lambda_D^2}[1 - z^2 + z(2 - z^2)\,\mathrm{Re}\,N(z)] \cdot \frac{\hat{E}^2}{4} \tag{36}$$

where $z = \omega/kv_t$. This represents the *wave* kinetic energy, i.e., the energy which will vanish when the electric field vanishes. There is an additional part of the energy due to the Landau damping of the wave, or in terms of the linear formula, the resistivity. Since the model is kinetic and collisionless, this energy must show up in the distribution function. This additional energy is accounted for by the source term in the energy conservation law (33):

$$\begin{aligned} W_{Landau} &= \int_{-\infty}^{t}\omega\,\mathrm{Im}\,\epsilon(\omega)\frac{\hat{E}^2(t')}{2}\,dt' \\ &= \frac{1}{k^2\lambda_D^2}\frac{\omega^2}{kv_t}\sqrt{\frac{\pi}{2}}\exp\left(-\frac{\omega^2}{2k^2v_t^2}\right)\int_{-\infty}^{t}\frac{\hat{E}^2(t')}{2}\,dt' \end{aligned} \tag{37}$$

for ω real (as assumed). Up until this point, no assumptions have been made about $\hat{E}$. For the sake of comparison with the results of the other two methods, $\hat{E} = E_1 \exp(\gamma t)$ will now be imposed, where E_1 is a constant, and γ is a positive growth rate. Equation (37) can now be integrated, giving

$$\int_{-\infty}^{t} \frac{\hat{E}^2(t')}{2}\,dt' = \frac{1}{\gamma}\frac{\hat{E}^2(t)}{4} \tag{38}$$

and so

$$W_{Landau} = \frac{1}{k^2\lambda_D^2}\sqrt{\frac{\pi}{2}}\frac{kv_t}{\gamma}z^2 e^{-\frac{z^2}{2}} \cdot \frac{\hat{E}^2(t)}{4} \tag{39}$$

This result clearly separates the wave kinetic energy from the kinetic energy due to damping, but it does assume that the electric field amplitude $\hat{E}$ varies slowly with respect to ω.

To compare this answer with the result of the perturbation methods, it is necessary to consider the perturbation result with set f equal to a Maxwellian, and take the limit as $\gamma \to 0$, since the derivation of formula (28) assumed that the variation of $\hat{E}$ was slow. (Note that it is not enough to set $\gamma = 0$, since the damping energy is proportional to $1/\gamma$, and would become infinite.) When the limit $\gamma \to 0$ is taken, equations (36) and (39) are indeed recovered. This agreement gives great confidence that the correct method and results have been obtained.

Theory for the Onset of Trapping in Langmuir Waves

The non-linear phenomenon which is responsible for the breakdown of linear theory is expected to be wave trapping. Wave trapping is conceptually fairly simple. The electric field of a wave sets up potential wells in the frame in which the wave is stationary. This potential alters the particle trajectories (see Figure 1). Particles which are on closed orbits are said to be trapped by the wave. These closed orbit are very different from the second order perturbed orbits used in the previous section, since the perturbed orbits do not reverse themselves except perhaps briefly. However, the orbits are very close for times which are short compared to the time it takes a particle to travel around its trapping orbit. Thus, one would expect that linear theory would be accurate as long as the trapped particles have not traveled a significant fraction of the way around their orbits. In physical terms, one might expect that $\omega_{tr}t \ll \pi$ for linear theory to hold, where ω_{tr} is the frequency of oscillation of the most deeply trapped (most nearly simple harmonic) orbits. (The choice of π in the above formula is convenient, but a bit arbitrary — $\pi/2$ might just as well have been used.)

This concept can be extended to the present case of time varying field amplitude by defining the cumulative trapping phase shift as

$$\Delta\Phi_{tr} = \int \omega_{tr}(t)\,dt \tag{40}$$

where

$$\omega_{tr}(t) = \sqrt{\frac{q}{m}k\hat{E}(t)} \tag{41}$$

is the trapping frequency. Substituting $E(t) = E_0 e^{-\gamma|t|}e^{i(kz-\omega t)}$, which is the form which will be assumed for the simulations, and integrating,

$$\Delta\Phi_{tr} = \frac{4}{\gamma}\sqrt{\frac{q}{m}kE_0} \tag{42}$$

(Note the notation change from Section B where the perturbing field amplitude was denoted as E_1 rather than E_0 — this is *only* a notation change.) This shows that the larger the damping rate, the smaller the effect of trapping, and the better linear theory applies. Since the dependence on E_0 is only a square root, a large damping rate compensate up for a considerable field strength.

Another important trapping parameter is the trapping velocity width, which is the range of velocities about the wave velocity for which there are trapped particles. In figure 1, for instance, the trapping velocity width is the difference between the center of the diagram and the top of the separatrix (the curve which separates the trapped from the untrapped orbits — it is the curve which meets itself at an angle at the edge of the diagram). The formula for this quantity is

$$\Delta v_{tr} = 2\sqrt{\frac{q}{m}\frac{\hat{E}}{k}} \tag{43}$$

The quantity which will be most useful in interpreting the simulation results will be the maximum trapping width, which is

$$\Delta v_{tr} = 2\sqrt{\frac{q}{m}\frac{E_0}{k}} \tag{44}$$

$\Delta v_{tr}/v_t$ is a measure of what fraction of the particles are trapped, and might be expected also to be a measure of when non-linearity is important, and in the non-self-consistent case, this is partially true. The significant deviations from the initial distribution are roughly confined to the trapping region, so if the trapping region is small, the distribution function will seem to be only mildly affected by trapping.

Now consider the meanings of Δv_{tr} and $\Delta\Phi_{tr}$ in the self-consistent case. The damping rate of a wave depends on the shape of the distribution function near the $\omega - kv$ resonance. The width in velocity of this resonance is determined by the damping rate γ, *not* the maximum trapping width Δv_{tr}. The shape (specifically the slope) of the distribution function will change *only* when the trapped particles have gone a significant way along their orbits; i.e., when $\Delta\Phi_{tr}$ becomes significant. The maximum trapping width Δv_{tr} plays no role in this, and so the only parameter relevant to non-linearity in self-consistent waves should be the cumulative trapping phase change $\Delta\Phi_{tr}$.

Numerical Particle Simulations of Langmuir Waves

To test the second order linear theory which was worked out in section B, some simulations were done on the problem of electrostatic waves in an unmagnetized plasma. The expected second order perturbed distribution function was worked out earlier, and the simulation itself is fairly cheap.

The simulation model follows the calculation model. It is periodic, collisionless, and the electric field is imposed. The imposed electric field is an advantage in simulation, as it yields minimum noise and higher speed, and allows any shape for the temporal modulation of the electric field. The electric field was chosen to be $E_0 \exp(-\gamma|t|)\cos(kx - \omega t)$ (see Figure 2). This shape of the envelope of the electric field amplitude allows both of the calculations from section B to be tested; specifically, the calculation for an exponentially growing wave can be tested at $t \leq 0$, and the calculation for a complete wave pulse as $t \rightarrow \infty$ can be tested at the end of the simulation. The phase velocity, ω/k, was chosen as v_t, so that the resonant electrons would be at the maximum slope of the distribution function. This makes the effects of Landau damping most pronounced. The growth/damping rate γ was chosen as $0.25kv_t$, so that the resulting second order perturbed distribution function would have a width of $0.25v_t$ which is both wide enough to be observable and narrow enough to be realistic. The dimensionless maximum electric field amplitude, qE_0/kmv_t^2, was given the three different values 0.025, 0.05, and 0.1.

The results are shown in Figures 3 and 4. Figure 3 shows plots of $f(v, t \rightarrow \infty)$ averaged over x, for the three values of E_0 as observed, and plots of $f_2(v, t \rightarrow \infty)/f_0(v)$ both theoretical and observed (the smooth lines are theoretical). Figure 4 shows the observed values of the average velocity (which, of course, represents current drive in one dimension) and second order kinetic energy, both at $t = 0$ (not to be confused with the beginning of the simulation) as per equation 26, and as $t \rightarrow \infty$, versus the maximum field amplitude. The theoretical values, as computed by numerical integration, are represented by horizontal lines. These plots show both the correctness of the theory at linear amplitudes, and the onset of trapping effects. These effects include the deviation of the distribution function from the profile predicted by linear theory, particularly within the maximum trapping width (Figure 3), and the decrease in the relative energy and momentum given to the distribution (Figure 4). The maximum trapping width is also plotted to indicate the region over which significant deviation from linear theory might be expected. The simulation shows good agreement for quite large $\Delta v_{tr}/v_t$ (~ 0.45 for $E_0 = 0.05$), reinforcing the conclusion that it is the cumulative trapping phase shift which limits the effects of trapping rather than the trapping width.

The values of $\Delta\Phi_{tr}$ for the three runs shown are 2.53, 3.58, and 5.06 respectively. Comparing these values with Figure 3, $\Delta\Phi_{tr} \sim \pi$ seems to be roughly the value at which deviation from linear theory becomes large, as expected. This is convincing evidence that trapping is indeed the effect which causes deviations from linear theory, and that the simple theory of the previous section is accurate in predicting when such deviations will occur.

Conclusions and Future Work

The basic conclusions to be drawn from this work are that linear theory works well in the unmagnetized plasma, that trapping theory accurately predicts the breakdown of linear theory, and that simulation is capable of verifying both linear theory and the onset of trapping.

Future work will include the application of this technique to waves in a magnetized plasma. This case is not as well understood, and should exhibit some interesting phenomena due to the presence of many trapping resonances rather that just one, as in the unmagnetized case.

APPENDICES

Appendix A: Derivation of Second Order Perturbed Distribution Function for Langmuir Waves

The calculation of the second order distribution function begins by expanding the Vlasov equation in orders of the electric field:

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = 0 \tag{a1}$$

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = -\frac{q}{m}E\frac{\partial f_0}{\partial v} \tag{a2}$$

$$\frac{\partial f_2}{\partial t} + v\frac{\partial f_2}{\partial x} = -\frac{q}{m}E\frac{\partial f_1}{\partial v} \tag{a3}$$

Using the method of characteristics, in which $x(t) = x(t_0) + v \cdot (t - t_0)$ and $v(t) = \text{constant}$,

$$\frac{df_1}{dt} = -\frac{q}{m}E(x(t))\frac{\partial f_0}{\partial v} \tag{a4}$$

So assuming f_0 is independent of x,

$$\begin{aligned} f_1 &= \int_{-\infty}^{t} \frac{q}{m}\hat{E}\frac{\partial f_0}{\partial v} e^{-i(\omega t' - kx(t) - kv(t'-t) + i\gamma t')}\, dt' \\ &= -i\frac{q}{m}\hat{E}\frac{\partial f_0/\partial v}{\omega - kv + i\gamma} e^{-i(\omega t - kx + i\gamma t)} \end{aligned} \tag{a5}$$

Similarly,

$$\begin{aligned} \frac{df_2}{dt} &= -\frac{q}{m}E(t)\frac{\partial f_1}{\partial v} \\ &= \frac{q^2}{m^2}\hat{E}^2 \,\text{Re}\left[e^{-i(\omega t - kx(t) + i\gamma t)}\right] \text{Re}\left[ie^{-i(\omega t - kx(t) + i\gamma t)}\frac{d}{dv}\left(\frac{\partial f_0/\partial v}{\omega - kv + i\gamma}\right)\right] \end{aligned} \tag{a6}$$

$$\begin{aligned} \frac{d\bar{f}_2}{dt} &= \frac{q^2}{m^2}\frac{\hat{E}^2}{2}\,\text{Re}\left[ie^{2\gamma t}\frac{\partial}{\partial v}\left(\frac{\partial f_0/\partial v}{\omega - kv + i\gamma}\right)\right] \\ &= \frac{q^2}{m^2}\frac{\hat{E}^2}{2}\gamma e^{2\gamma t}\frac{\partial}{\partial v}\left(\frac{\partial f_0/\partial v}{(\omega - kv)^2 + \gamma^2}\right) \end{aligned} \tag{a7}$$

so

$$\bar{f}_2(v) = \frac{\hat{E}^2}{4}\frac{q^2}{m^2}\frac{\partial}{\partial v}\left[\frac{\partial f_0/\partial v}{(\omega - kv)^2 + \gamma^2}\right] \tag{a8}$$

Appendix B: Derivation of Linear Energy by Perturbed Particle Velocities

The calculation by perturbed velocities begins by expanding the equation

$$\frac{dv}{dt} = \frac{q}{m}E(x) \tag{b1}$$

in orders of the electric field. The expansion is

$$\frac{dv_0}{dt} = 0 \tag{b2}$$

$$\frac{dv_1}{dt} = \frac{q}{m}E(x_0) \tag{b3}$$

$$\frac{dv_2}{dt} = \frac{q}{m}(x_1\frac{\partial}{\partial x_0})E(x_0) \tag{b4}$$

Setting $E = E_1 \exp(ikx - i\omega t + \gamma t)$ turns the $x_1 \cdot \partial/\partial x$ into ikx_1. Some care must be taken in the interpretation of these formulae, as the velocities are taken to be complex (for now), but x_0 and x_1 must be purely real.

These equations can now be solved directly:

$$v_0 = \text{constant} \tag{b5}$$

$$x_0 = x_i + v_0 t \tag{b6}$$

$$v_1 = i\frac{q}{m}E_1\frac{e^{ikx_i - i(\omega - kv_0 + i\gamma)t}}{\omega - kv_0 + i\gamma} \tag{b7}$$

$$x_1 = -\frac{q}{m}E_1\frac{e^{ikx_i - i(\omega - kv_0 + i\gamma)t}}{(\omega - kv_0 + i\gamma)^2} \tag{b8}$$

In solving for the second order perturbed velocity, one needs to keep two things in mind: only the non-spatially dependent part of v_2 is needed, and that it is the real parts of the first order quantities that appear in the equation for v_2. Thus the formula

$$\overline{\operatorname{Re} A \operatorname{Re} B} = \frac{1}{2}\operatorname{Re}\overline{AB^*}$$

when A and B are both of the form $\hat{A}e^{ikx}$, can be used. The differential equation becomes

$$\begin{aligned}\frac{d\bar{v}_2}{dt} &= \frac{1}{2}\frac{q^2}{m^2}kE_1^2e^{2\gamma t}\operatorname{Re}\frac{i}{(\omega - kv_0 + i\gamma)^2} \\ &= \frac{q^2}{m^2}k\frac{E_1^2}{2}\frac{2\gamma(\omega - kv_0)}{[(\omega - kv_0)^2 + \gamma^2]^2}e^{2\gamma t}\end{aligned} \tag{b9}$$

The solution is

$$\bar{v}_2 = \frac{E_1^2}{4}\frac{q^2}{m^2}\frac{2k(\omega - kv_0)}{[(\omega - kv_0)^2 + \gamma^2]^2}e^{2\gamma t}$$

$$= \frac{E_1^2}{4}\frac{q^2}{m^2}e^{2\gamma t}\frac{d}{dv_0}\left[\frac{1}{(\omega - kv_0)^2 + \gamma^2}\right] \tag{b10}$$

Now all that remains is to find $\frac{1}{2}v_1^2 + v_0 v_2$ and average it over the distribution function. Using the same rule for the real part of a product,

$$v_1^2 = \frac{E_1^2}{2}\frac{q^2}{m^2}e^{2\gamma t}\frac{1}{(\omega - kv_0)^2 + \gamma^2} \tag{b11}$$

so

$$\frac{1}{2}v_1^2 + v_0 v_2 = \frac{E_1^2}{4}\frac{q^2}{m^2}e^{2\gamma t}\frac{d}{dv_0}\left[\frac{v_0}{(\omega - kv_0)^2 + \gamma^2}\right] \tag{b12}$$

after some simple manipulation. Averaging over the distribution function and integrating by parts gives a total kinetic energy

$$W = -\frac{\hat{E}^2}{4}\frac{q^2}{m}\int \frac{v\frac{\partial f}{\partial v}}{(\omega - kv)^2 + \gamma^2}\,dv \tag{b13}$$

References

[1] Krall and Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill Book Co., 1973), page 392

[2] Fried and Conte, *The Plasma Dispersion Function* (Academic Press New York,1961)

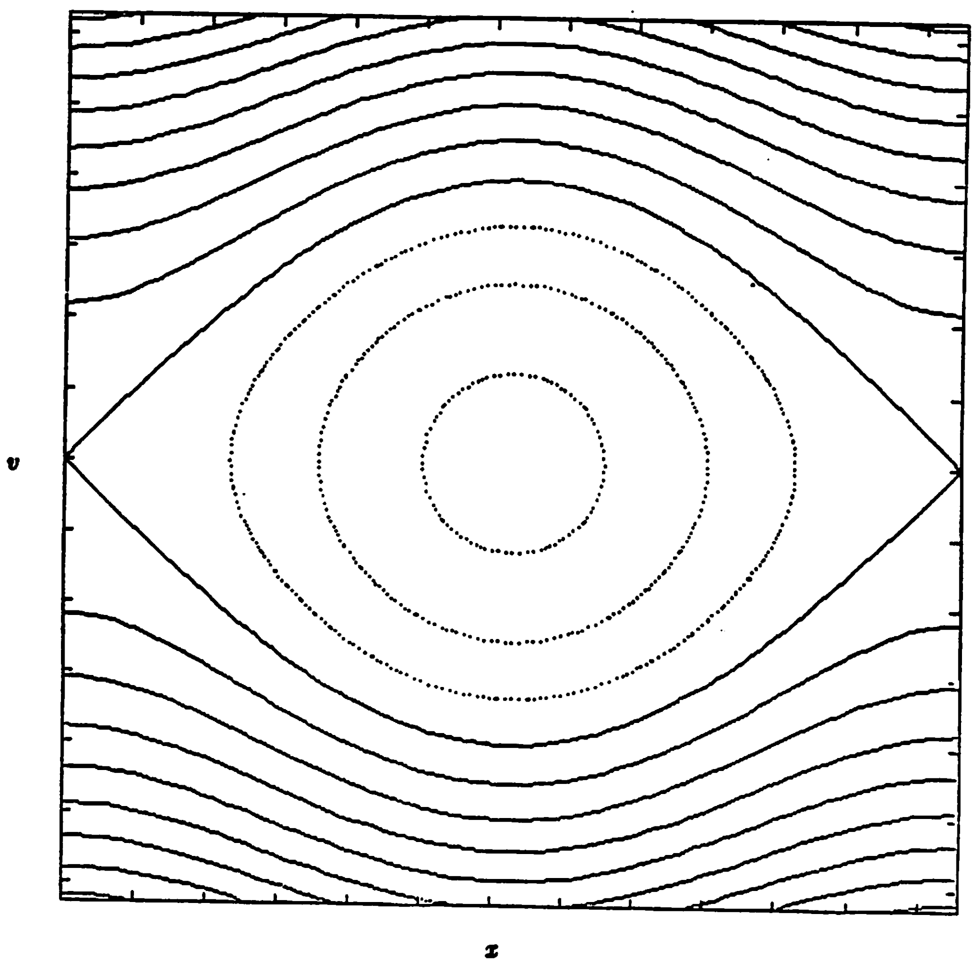


Figure 1. Orbits in a stationary sinusoidal potential. Orbits indicated by dotted lines are trapped. (Scale was chosen for convenience only.)

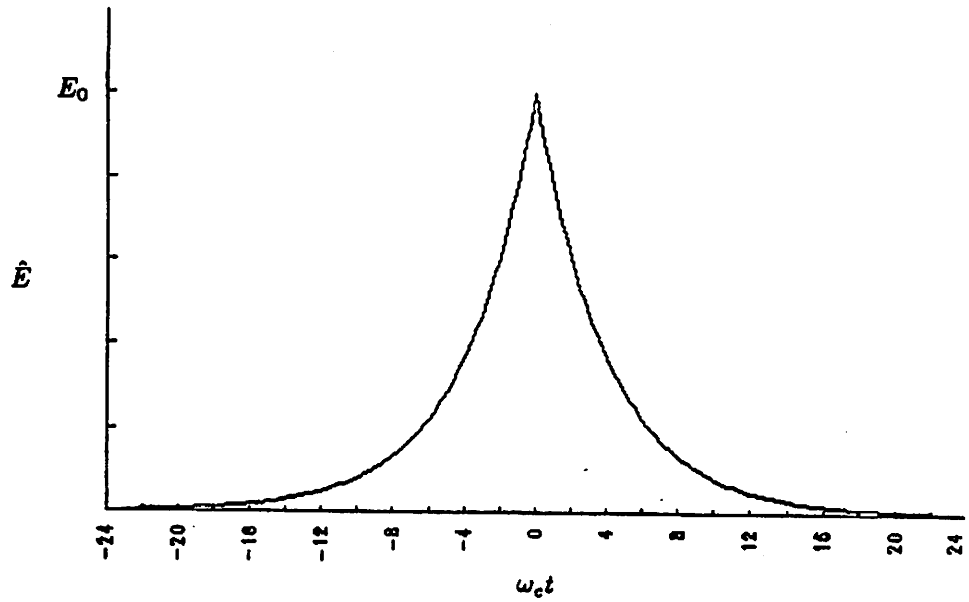
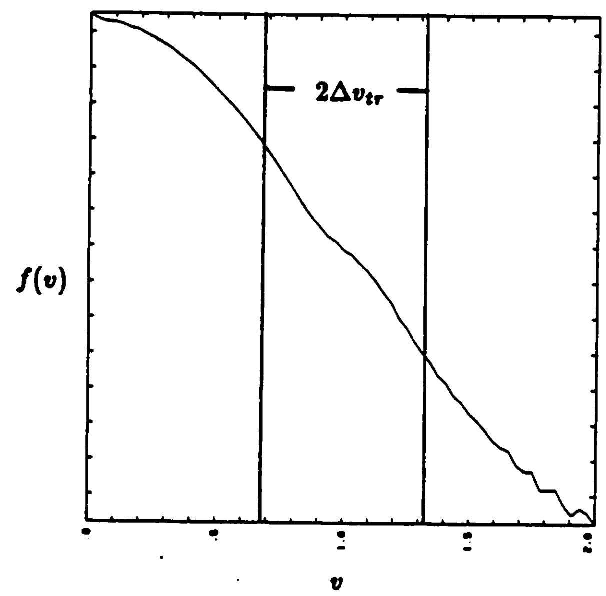


Figure 2. Electric field envelope function



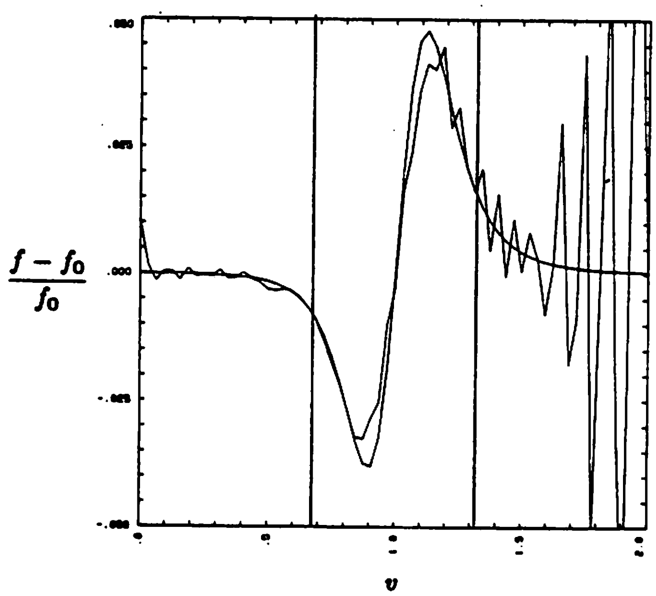


Figure 3a. Distribution function and perturbation for $E_0 = 0.025$

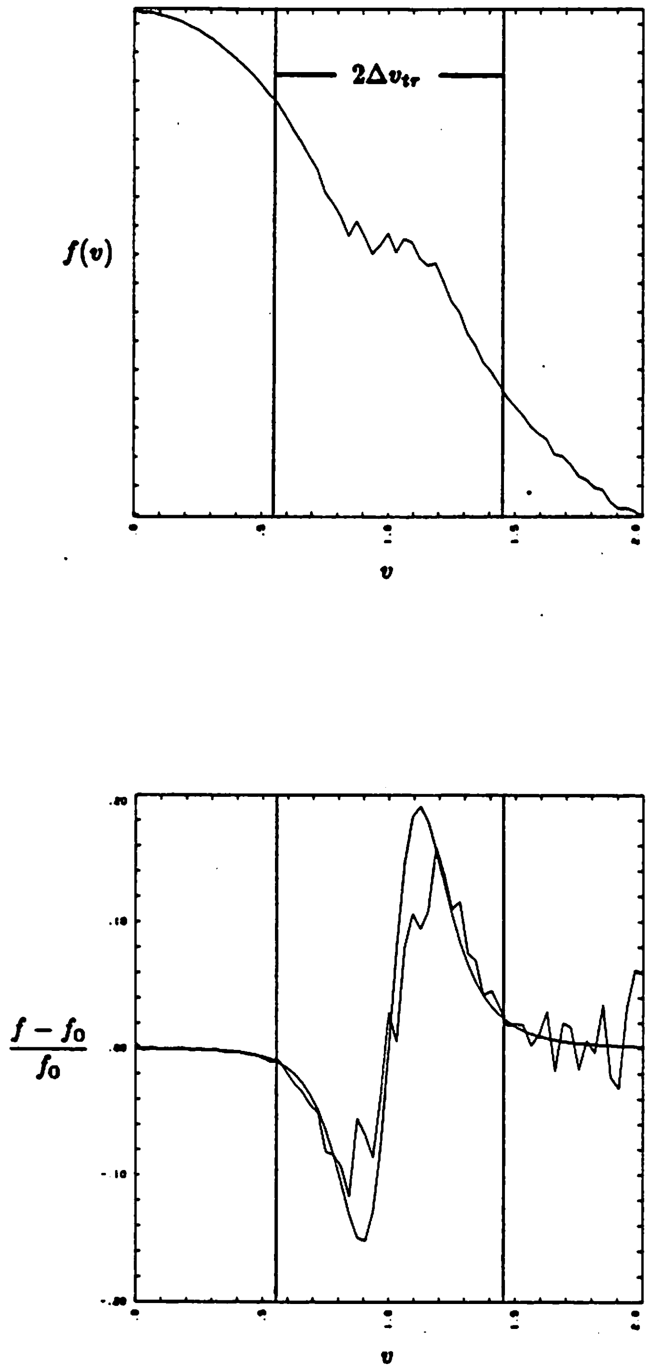


Figure 3b. Distribution function and perturbation for $E_0 = 0.05$

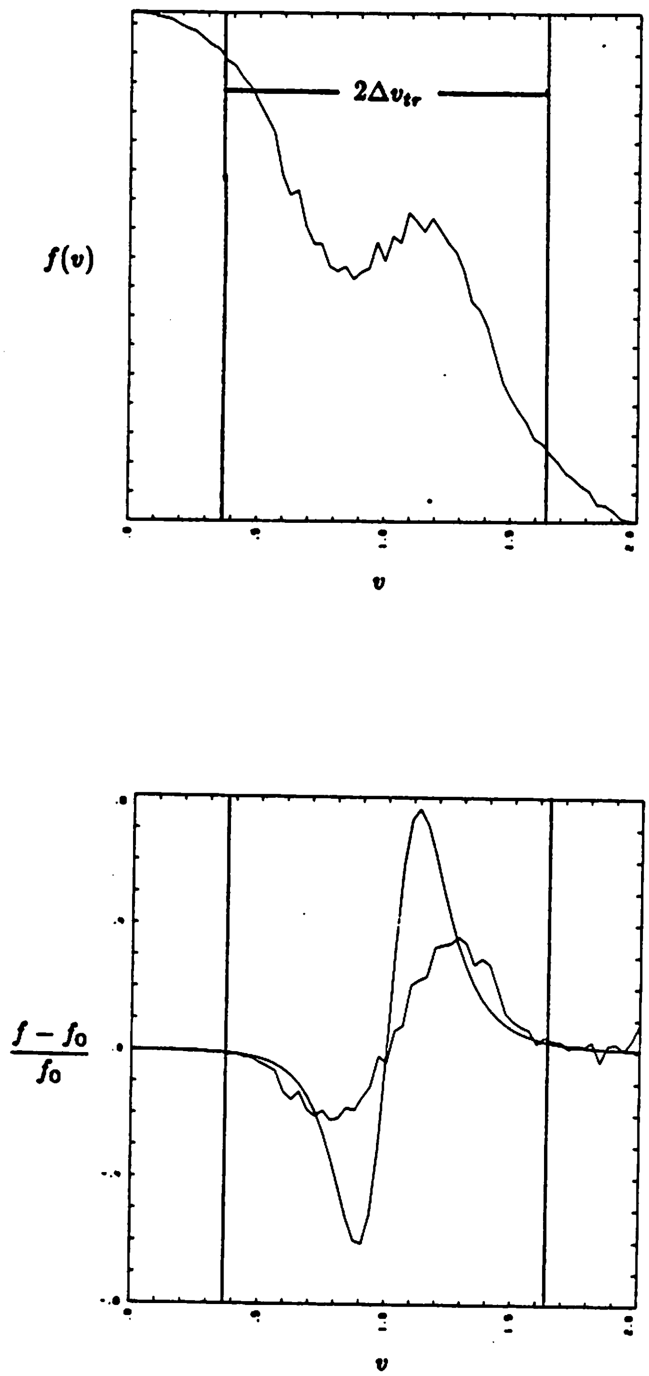


Figure 3c. Distribution function and perturbation for $E_0 = 0.1$

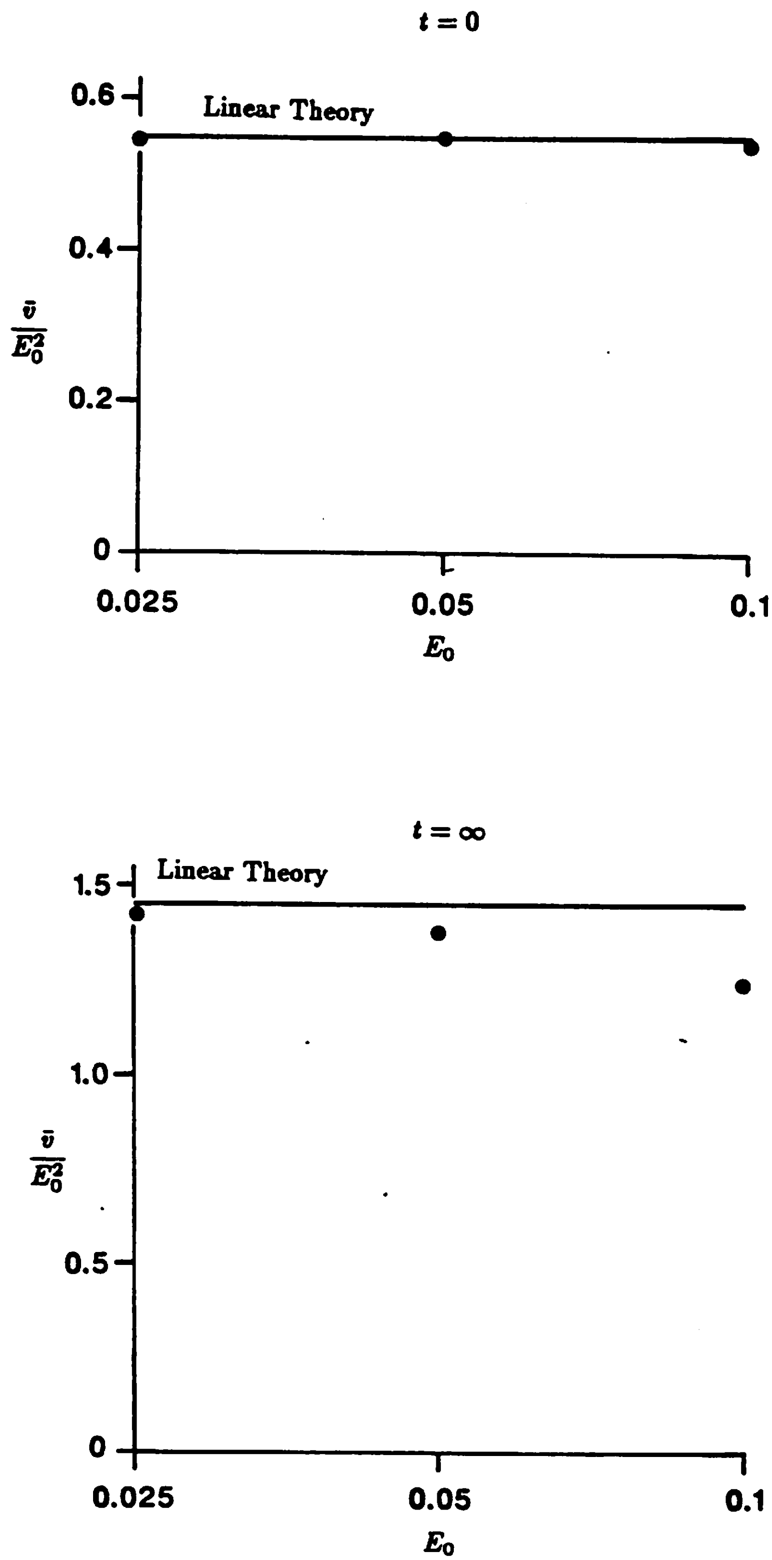


Figure 4a. Average velocity versus E_0 at $t = 0$ and $t = \infty$

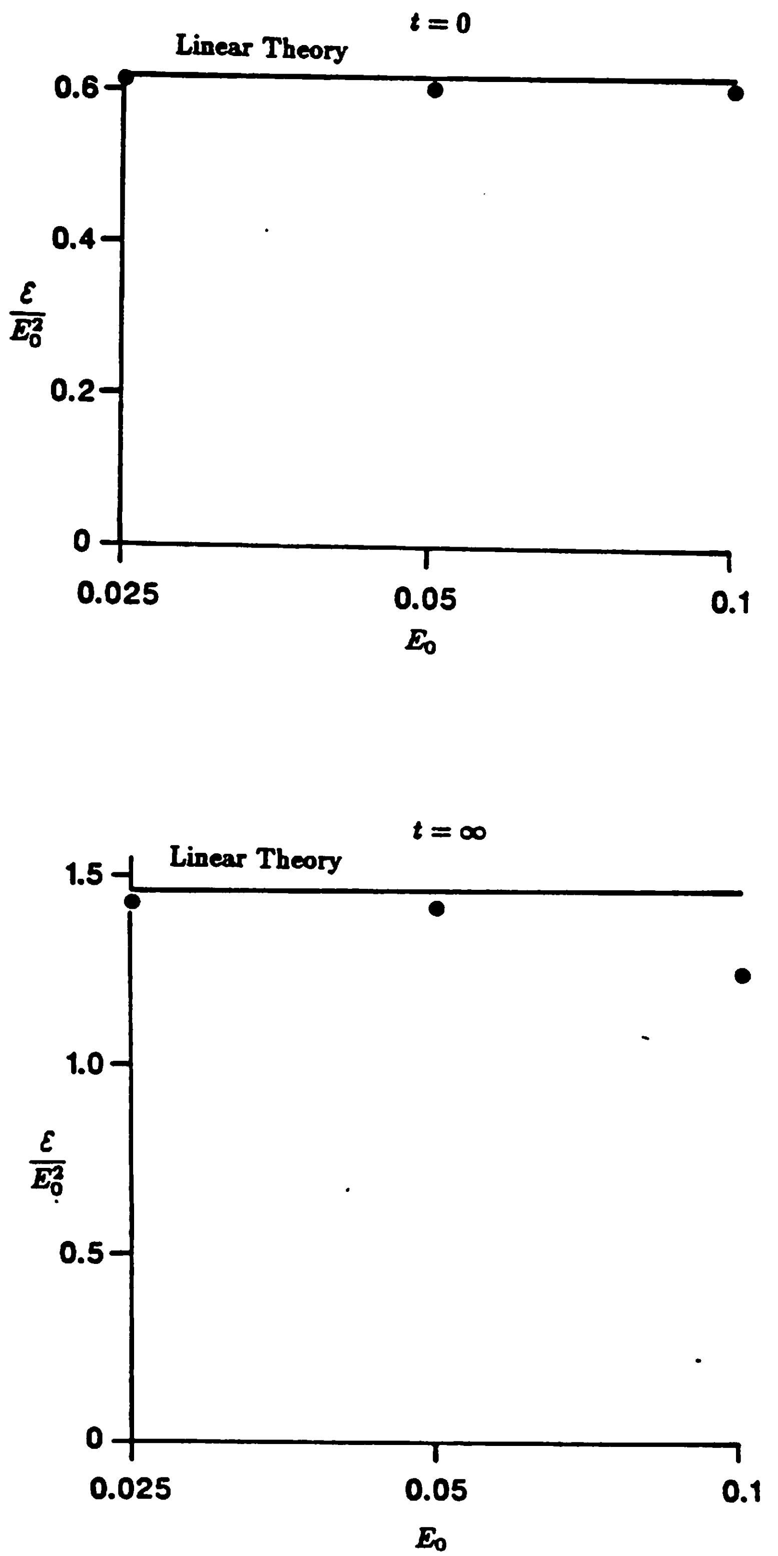


Figure 4b. Energy density versus E_0 at $t = 0$ and $t = \infty$