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IDENTIFICATION OF CONTINUOUS TIME SYSTEMS

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Memorandum No. UCB/ERL M86/60

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# USE OF SPECIFIC PRIOR INFORMATION IN THE ADAPTIVE IDENTIFICATION OF CONTINUOUS TIME SYSTEMS

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## ABSTRACT

In this paper we modify an adaptive identification scheme given in [1]. The identification method assumes that real valued parameters appearing as linear multipliers of known polynomials are the only unknown components in the partially known transfer function. Thus, this scheme makes use of specific prior information about the system to more efficiently identify its partially known transfer function. We prove that the proposed scheme *always* identifies an "equivalent" transfer function provided the input is rich enough, and we specify conditions on the known polynomials which determine when and when not the unknown parameters are uniquely identifiable.

## Introduction

In order to implement any of the current controller design methodologies, e.g. pole-placement, LQ, etc., or if one wishes to simply analyze the behavior of a continuous time dynamical system, one must have an accurate model of the plant or system to be studied. Such models are usually generated by studying, analyzing and modelling the physics of the system. Then, once the structure of the model has been determined, one must fill in the various parameters of the model with exact physical data (e.g. the spring constants of all springs or the resistance values of all resistors.) However, depending upon the complexity of the specific system it may or may not be practical to measure the physical constants of the system. Thus, in the cases where measurement data is not available, one would like to determine the values of those unknown constants from the input-output properties of the system in question. Furthermore, frequently not every parameter of the system is unknown. Consequently, one would, for the sake of efficiency, want to use all prior knowledge of the system in the identification process. In [1] a scheme is proposed and studied which takes advantage of specific prior knowledge about a given partially known system. Specifically, [1], in part, deals with the identification of linear continuous time

systems whose transfer function is of the form: (for the 3 unknown case)

$$H(s, K) = \frac{n_0(s) + k_1 n_1(s) + k_2 n_2(s) + k_3 n_3(s) + k_1 k_2 n_4(s) + k_1 k_3 n_5(s) + k_2 k_3 n_6(s) + k_1 k_2 k_3 n_7(s)}{d_0(s) + k_1 d_1(s) + k_2 d_2(s) + k_3 d_3(s) + k_1 k_2 d_4(s) + k_1 k_3 d_5(s) + k_2 k_3 d_6(s) + k_1 k_2 k_3 d_7(s)}$$

In  $H(s, K)$  the only unknown pieces are the elements of  $K$  --  $k_1, k_2$  and  $k_3$ . All of the  $n_i(s)$ 's and  $d_i(s)$ 's are known polynomials. Thus, although it is not proven, when  $K$  has fewer than  $2n+1$  elements,  $n$  being the order of  $H(s, K)$ , it is intuitively suggestive that it is more efficient to identify only  $K$  and not the  $2n+1$  coefficients needed when the entire transfer function is assumed to be unknown. The scheme for identification given in [1] identifies only  $K$  and therefore completely determines  $H(s, K)$ . In this paper we employ the basic philosophy of [1] to identify transfer functions of the following form:

$$H(s, K) = \frac{n_0(s) + k_1 n_1(s) + \dots + k_p n_p(s)}{d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s)}$$

Notice that the  $H(s, K)$  we identify does not include elements with "multilinear" appearance of the  $k$ 's so it is an easier problem than that tackled in [1]. However, in [1] it is assumed that when the real value of  $K$  is stuck into  $H(s, K)$ , the numerator and denominator are coprime. Furthermore, it is assumed that there exist no constants  $c_1, \dots, c_p$  such that  $\sum_{i=1}^p c_i n_i(s) = \sum_{i=1}^p c_i d_i(s) \equiv 0$ . When

dealing with the identification of unknown systems, it seems particularly advantageous to make as few arbitrary assumptions about the structure and behavior of the of the unknown system as possible since the system is exactly that -- unknown. Therefore, in this paper we do not impose the just given assumptions but rather we analyze the behavior of the identification scheme when those assumptions are and are not violated. What we prove is that provided the support of the spectral measure of the input is not concentrated on  $n < r+1$  points where  $r := \max \deg(n_i d_j - n_j d_i) \quad i, j = 1, \dots, p$  one will *always* identify an equivalent transfer function -- a transfer function identical to the real one modulo pole-zero cancellations and multiplication by 1 in the form of  $a/a$  for  $a \in \mathbb{R}$ . Furthermore, we give conditions on the known polynomials  $n_0(s), \dots, n_p(s), d_0(s), \dots, d_p(s)$  which determine if the unknown parameters are not uniquely determined and we give conditions on the polynomials  $n_0(s), \dots, n_p(s), d_0(s), \dots, d_p(s)$  which determine if the unknown parameters are uniquely determined. Thus, when the unknown parameters are uniquely determined the transfer function identified is the exact transfer function of the real system and when the unknown parameters are not uniquely determined the identified transfer function is equivalent to the



real one. Furthermore, the method of analysis used to prove the above results differs from that used in [1]. In this paper we employ the ideas of Generalized Harmonic Analysis developed in [5] to prove the above assertions about parameter convergence. This method of analysis is a very general approach which can handle almost all types of input signals and is, unlike [1], not restricted to the case when the input is composed of simple sinusoids.

This paper is split up into seven sections. The first section defines explicitly the problem under taken and the assumptions made about the system being identified. The second section defines what is meant by parametric uniqueness and identifiability. The third section describes the identification process while the fourth section connects the ideas of input sufficient richness and parameter convergence. The fifth section talks more about identifiability and parametric uniqueness while the sixth section describes what types of systems the given identification scheme could be used on and gives examples of the identification process. The seventh and final section lists some comments about the identification process presented and gives suggestions for further reasearch.

### Statement of the Problem and Assumptions

This paper deals with the identification of all continuous time systems whose input-output transfer function is of the form:

$$H(s, K) = \frac{n_0(s) + k_1 n_1(s) + \dots + k_p n_p(s)}{d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s)} = \frac{\hat{y}(s)}{u(s)} \quad (1)$$

Where  $K \in \mathbb{R}^p$  is defined as  $K := [k_1 \ \dots \ k_p]^T$  and one assumes the following:

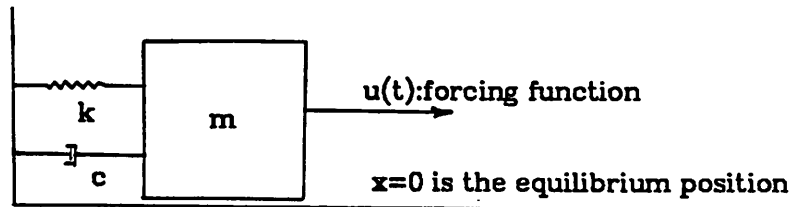
- A1) The unknown plant is strictly stable -- all its poles are located in the open left half of the complex plane.
- A2)  $n_0(s), n_1(s), \dots, n_p(s), d_0(s), d_1(s), \dots, d_p(s)$  are known polynomials in  $s \in \mathbb{C}$  with real coefficients.
- A3)  $k_1, k_2, \dots, k_p$  are real numbers representing the unknown parameters of the system.
- A4) There exists at least one K, call it  $K^* := [k_1^* \ k_2^* \ \dots \ k_p^*]^T$ , such that  $H(s, K^*)$  possesses the same input-output properties as the real system for every input.  $K^*$  is in some sense meant to represent the actual values of the unknown parameters and will be referred to as the "physically meaningful" value of K.

Note that the structure of  $H(s,K)$  is in fact very general. One has simply required that the  $n$ 's and the  $d$ 's be polynomials in  $s$  which means they can be zero, a constant or any  $n$ -th order polynomial. Note also that any transfer function which is completely unknown, except for the order of its denominator, may be put in to the same form as eqn.(1) with the appropriate choices of the  $n_i$ 's and the  $d_i$ 's. Specifically,  $H(s,K)$  for a completely unknown plant will be of the form:

$$H(s,K) = \frac{k_1 + k_2s + \dots + k_{p+1}s^p}{k_{p+2} + k_{p+3}s + \dots + k_{2p+1}s^{p-1} + s^p} \quad (2)$$

As mentioned earlier,  $H(s,K)$  given in eqn.(1) has a particular structure so as to separate known and unknown information about the system. The following two examples illustrate when one might have partial information about a system.

Consider the following simple dynamical system:



The dynamic equation governing the above system when  $m$  is the mass of the block,  $k$  is the spring constant of a linear spring and  $c$  is the damping coefficient of a velocity proportional viscous dashpot is:

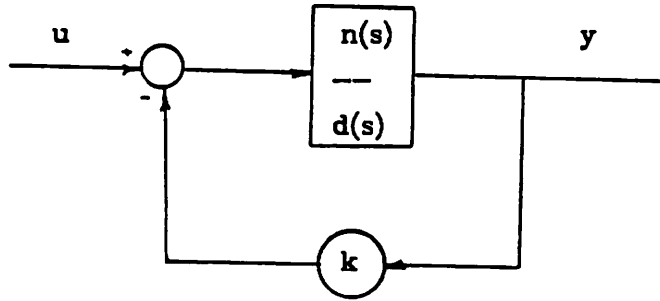
$$m\ddot{x} + c\dot{x} + kx = u(t) \quad (3)$$

The Laplace-domain transfer function of this system is:

$$\frac{\hat{x}(s)}{u(s)} = \frac{1}{ms^2 + cs + k} \quad (4)$$

Hence, if one knows the mass  $m$  of the block, one need only identify  $c$  and  $k$  — two parameters which could be difficult to measure directly.

From more of a controls point of view consider the following feedback control system:



Where  $n(s)$  and  $d(s)$  are both known polynomials in  $s$  and  $k$  – the feedback gain – is unknown. The overall transfer function of the above system is:

$$\frac{y(s)}{u(s)} = \frac{n(s)}{d(s) + kn(s)} \quad (5)$$

Thus, all that need be found to explicitly determine the transfer function between  $y$  and  $u$  is the value of  $k$ .

### Identifiability and Parametric Uniqueness

At this point however, it is important to point out that there exist two levels of identification of transfer functions which occur in the form of eqn.(1). One could in fact be performing the identification for the sole purpose of obtaining the exact value of one of the  $k$ 's which may have an important physical meaning, or, one could in fact be interested in finding any transfer function which possesses the same input-output properties as the system being identified with no real concern as to what each specific  $k$  might be. More specifically, given the structure of  $H(s, K)$ , there may exist situations when  $H(s, K^*) = H(s, K^* + C)$  for some nonzero  $C \in \mathbb{R}^p$ . Thus,  $H(s, K^*)$  and  $H(s, K^* + C)$  are equivalent but numerically not the same i.e. there exists more than one value of  $K$  which works namely  $K^*$  and  $K^* + C$ . As a simple illustration of the just described situation consider  $H(s, K)$  defined below:

$$H(s, K) = \frac{k_1(s+1) + k_2(s^2+2) + k_3(s^3+3s)}{k_1(s^3+1) + k_2(s) + k_3(s^2+1)}$$

Notice that  $H(s, 1, 2, 3)$  and  $H(s, 2, 4, 6)$  are, from an input-output stand point, the same transfer function. However, the specific  $K$ 's which generate them are quite different. Therefore, given a transfer function in the form of eqn.(1), under specific circumstances there may be many  $K$ 's which give the same transfer function from an input-output stand point. And, since this scheme for

identification relies solely on input-output measurements, when there exist many  $K$ 's which "work", it may be impossible to identify the "physically meaningful" value of  $K$ . At this point then, one proposes the following definitions which specify exactly what is meant by parametric uniqueness. One first however, recalls assumption (A4) which declares the existence of at least one set of  $k$ 's, called  $K^*$ , which make eqn.(1) equal to the transfer function of the system one is trying to identify. Note that the existence of  $K^*$  is in a sense implicit in the assumption that eqn.(1) is a legitimate representation of the system being studied. With that, one states the following definitions.

**Definition1:** A transfer function of the form given in eqn.(1) is referred to as *Parametrically Unique* if for each  $K \in \mathbb{R}^p$  there does not exist some  $\hat{K} \in \mathbb{R}^p$  such that  $\hat{K} \neq K$  and  $H(s, \hat{K}) = H(s, K)$ .

**Definition2:** A transfer function of the form given in eqn.(1) is referred to as *Parametrically Nonunique* if for some  $K \in \mathbb{R}^p$  there exists a  $\hat{K} \in \mathbb{R}^p$  such that  $\hat{K} \neq K$  and  $H(s, \hat{K}) = H(s, K)$ .

It will be shown that the values of  $n_0(s), \dots, n_p(s), d_0(s), \dots, d_p(s)$  will determine the parametric uniqueness of  $H(s, K)$ . Thus, before one can say anything about an identification process, one must first establish exactly when there exist many values of  $K$  which 'work' and specify how those quantities are characterized.

To start with assume that  $H(s)$  is the transfer function of the unknown linear time invariant plant which one is trying to identify and assume the appropriate modelling of that plant yields  $H(s, K)$  which is of the form of eqn.(1). In addition, let assumptions A1)-A4) hold. Thus  $H(s) = H(s, K^*)$ . One will therefore accept any  $H(s, \hat{K})$  such that  $H(s) = H(s, \hat{K})$  where  $H(s, \hat{K})$  could also be written as  $H(s, K^* + C)$ . Hence, it is necessary to characterize the set of all  $C \in \mathbb{R}^p$  such that  $H(s) = H(s, K^* + C)$ .

At this point it will be convenient to introduce the following notation.

$$\text{Let } C := [c_1 \dots c_p]^T \in \mathbb{R}^p \text{ and } 1C := [1 \ c_1 \ \dots \ c_p]^T \in \mathbb{R}^{p+1}.$$

$$\text{Let } 1K := [1 \ k_1 \ \dots \ k_p]^T \in \mathbb{R}^{p+1} \text{ and } 1K^* := [1 \ k_1^* \ \dots \ k_p^*]^T \in \mathbb{R}^{p+1}$$

$$\text{Let } N_0 := [n_0 \ n_1 \ \dots \ n_p]^T \in \mathbb{R}[s]^{p+1} \text{ and } N_1 := [n_1 \ \dots \ n_p]^T \in \mathbb{R}[s]^p.$$

$$\text{Let } D_0 := [d_0 \ d_1 \ \dots \ d_p]^T \in \mathbb{R}[s]^{p+1} \text{ and } D_1 := [d_1 \ \dots \ d_p]^T \in \mathbb{R}[s]^p.$$

Where  $\mathbb{R}[s]^p$  denotes the set of all  $p$ -dimensional vectors whose elements are composed of polynomials in the variable  $s \in \mathbb{C}$ .

The following theorem characterizes the set of K's which "work". One should be forewarned that this set of K's depends on  $K^*$  – the physically meaningful value of K. Thus, it is not a set which can be calculated a priori ( $K^*$  is unknown.) However, it will prove to be an important characterization.

**Theorem 1:** If  $X \subseteq \mathbb{R}^p$  is defined as the set of all  $C \in \mathbb{R}^p$  such that ;

$$H(s) = \frac{n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)}{d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)} = \frac{n_0(s) + (k_1^* + c_1) n_1(s) + \dots + (k_p^* + c_p) n_p(s)}{d_0(s) + (k_1^* + c_1) d_1(s) + \dots + (k_p^* + c_p) d_p(s)} \quad (6)$$

Then  $X = Q := N_L(IL(s)) \text{ Mod } B$  where  $IL(s) := [N_1 \ D_1][ -D_0 \ N_0 ]^T 1K^*$ ,  $N_L(\ )$  denotes the left null space of ( ) over the reals and  $B := \{C \in \mathbb{R}^p : [1 \ (K^* + C)^T][D_0 \ N_0] \equiv 0\}$ .

**Proof:** Let X be defined as stated above and assume  $C \in X$ . Eqn.(6) can thus be written as ;

$$\begin{aligned} & [n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)][d_0(s) + (k_1^* + c_1) d_1(s) + \dots + (k_p^* + c_p) d_p(s)] \\ & - [d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)][n_0(s) + (k_1^* + c_1) n_1(s) + \dots + (k_p^* + c_p) n_p(s)] \equiv 0 \end{aligned}$$

Which, when expanded, can be written as;

$$\begin{aligned} & [n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)][c_1 d_1(s) + \dots + c_p d_p(s)] \\ & - [d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)][c_1 n_1(s) + \dots + c_p n_p(s)] \equiv 0. \end{aligned}$$

And thus can be written as;

$$C^T [N_1 \ D_1][ -D_0 \ N_0 ]^T 1K^* \equiv 0 \text{ or } C^T IL(s) \equiv 0$$

In addition,  $C \notin B$  because  $H(s) \neq 0/0$ . Therefore,  $X \subset Q$ .

Let X be defined as stated above and assume that  $C \in Q$ . Therefore,

$$C^T [N_1 \ D_1][ -D_0 \ N_0 ]^T 1K^* \equiv 0$$

so one can write;

$$\begin{aligned} & [n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)][c_1 d_1(s) + \dots + c_p d_p(s)] \\ & - [d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)][c_1 n_1(s) + \dots + c_p n_p(s)] \equiv 0. \end{aligned}$$

Adding zero to the left side in the form of;

$$\begin{aligned} & [n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)][d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)] \\ & - [n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)][d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)] \end{aligned}$$

yields;

$$[n_0(s) + k_1 n_1(s) + \dots + k_p n_p(s)][d_0(s) + (k_1 + c_1)d_1(s) + \dots + (k_p + c_p)d_p(s)] \\ - [d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s)][n_0(s) + (k_1 + c_1)n_1(s) + \dots + (k_p + c_p)n_p(s)] \equiv 0.$$

So because  $C \notin B$  the above gives;

$$\frac{n_0(s) + k_1 n_1(s) + \dots + k_p n_p(s)}{d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s)} = \frac{n_0(s) + (k_1 + c_1)n_1(s) + \dots + (k_p + c_p)n_p(s)}{d_0(s) + (k_1 + c_1)d_1(s) + \dots + (k_p + c_p)d_p(s)}$$

Therefore,  $Q \subset X$ . ■

Review of the theorem just given specifying the set which describes all K's which "work" reveals that such a characterization is rather obscure. That is, the set X (or Q) is not necessarily a linear subspace and that makes both the definition of and the operation with X rather messy.

In the following claim one shall give necessary conditions for when a transfer function given in the form of eqn.(1) is parametrically unique.

**Claim1:** Let  $H(s, K)$  be given in the form of eqn.(1). Then, if there exists  $C \in \mathbb{R}^p$  such that  $C \neq 0$  and  $C^T [N_1 D_1] \equiv 0$  or if there exists  $1C \in \mathbb{R}^{p+1}$  such that  $1C^T [N_0 D_0] \equiv 0$  then  $H(s, K)$  is not parametrically unique.

**Proof;** Given later.

### The Identification Process

The above discussion dealt entirely with the structure of the partially known transfer function and how that structure determined whether or not there existed unique values of the parameters being identified. Such an aspect is very important to the success of this identification scheme and will be returned to later. One now, however, turns to the actual method used to identify the k's.

Consider a transfer function given in the form of eqn.(1) and multiply it out to get it into the following form.

$$\hat{y}(s)[d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s)] = \hat{u}(s)[n_0(s) + k_1 n_1(s) + \dots + k_p n_p(s)] \quad (9)$$

where  $\hat{y}(s)$  and  $\hat{u}(s)$  are the Laplace transforms of the output and input of the system respectively. Collecting unknown terms on the right hand side and

dividing both sides by  $\Lambda(s)$  – a polynomial whose order is equal to the maximum order of  $d_0(s), \dots, d_p(s)$  and whose zeros are located in the open left half of the complex plane – one gets:

$$\frac{\hat{y}(s)d_0(s) - \hat{u}(s)n_0(s)}{\Lambda(s)} = k_1 \frac{\hat{u}(s)n_1(s) - \hat{y}(s)d_1(s)}{\Lambda(s)} + \dots + k_p \frac{\hat{u}(s)n_p(s) - \hat{y}(s)d_p(s)}{\Lambda(s)} \quad (10)$$

Note that one has effectively extracted the unknown quantities from the original transfer function and made them multipliers of known signals – signals which are simply filtered versions of the input and the output of the system. To tidy things up a bit let:

$$\begin{aligned} \hat{w}_0(s) &:= \frac{\hat{y}(s)d_0(s) - \hat{u}(s)n_0(s)}{\Lambda(s)} \\ \hat{w}_1(s) &:= \frac{\hat{u}(s)n_1(s) - \hat{y}(s)d_1(s)}{\Lambda(s)} \\ &\vdots \\ \hat{w}_p(s) &:= \frac{\hat{u}(s)n_p(s) - \hat{y}(s)d_p(s)}{\Lambda(s)} \end{aligned} \quad (11)$$

Thus eqn.(10) becomes;

$$\hat{w}_0(s) = k_1 \hat{w}_1(s) + \dots + k_p \hat{w}_p(s) \quad (12)$$

Taking the Laplace inverse of eqn.(12) yields:

$$w_0(t) = k_1 w_1(t) + \dots + k_p w_p(t) \quad (13)$$

And again recall that  $w_0(t), \dots, w_p(t)$  are generated from the input and output of the system only. In eqn.(13) replace  $k_1, \dots, k_p$  by  $k_1^\circ, \dots, k_p^\circ$  and let  $\vartheta_1(t), \dots, \vartheta_p(t)$  be the time varying parameters corresponding to  $k_1^\circ, \dots, k_p^\circ$  respectively. Now consider the following signal:

$$\tilde{w}_0(t) := w_1(t)\vartheta_1(t) + \dots + w_p(t)\vartheta_p(t) \quad (14)$$

Subtract eqn.(14) from the modified eqn.(13) to get the error equation;

$$e(t) := w_0(t) - \tilde{w}_0(t) = (k_1^\circ - \vartheta_1(t))w_1(t) + \dots + (k_p^\circ - \vartheta_p(t))w_p(t) \quad (15)$$

which will be used to update the parameters  $\vartheta_1(t), \dots, \vartheta_p(t)$ . Let:

$$\varphi_1(t) := k_1^\circ - \vartheta_1(t)$$

$$\varphi_2(t) := k_2^\circ - \vartheta_2(t)$$

$$\varphi_p(t) := k_p^* - \vartheta_p(t)$$

Thus, the  $\varphi$ 's represent parameter errors, and with the above definition for the  $\varphi$ 's, eqn(15) becomes;

$$e(t) = w_1(t)\varphi_1(t) + \dots + w_p(t)\varphi_p(t) \quad (16)$$

If one lets  $W(t) := [w_1(t) \dots w_p(t)]^T$ ,  $\Theta(t) := [\vartheta_1(t) \dots \vartheta_p(t)]^T$  and  $\Phi(t) := [\varphi_1(t) \dots \varphi_p(t)]^T$  then equation (16) becomes;

$$e(t) = W(t)^T \Phi(t) \quad (17)$$

At this point one must establish an algorithm for updating  $\Phi(t)$  (and thus  $\Theta(t)$ ) which will make  $\Phi(t)$  go to zero as  $t$  goes to  $\infty$ . One common update scheme, the projection algorithm, defines the dynamic behavior of  $\Phi(t)$  as;

$$\dot{\Phi}(t) = -\gamma e(t) W(t) \quad (18)$$

where  $\gamma > 0$  is the update gain.

With the above defined update scheme one uses a simple Lyapunov argument to prove the following claim about system error behavior.

**Theorem2:** If the input,  $u(t)$ , to the plant and update system is bounded and has a bounded derivative and the plant is stable then the error signal,  $e(t)$ , defined by eqn.(15) will decay to zero as  $t$  goes to  $\infty$ .

**Proof:** Consider the Lyapunov function  $V(\Phi) = \frac{1}{2} \Phi^T \Phi \geq 0$ . Therefore,  $\dot{V}(\Phi) = \Phi^T \dot{\Phi}$ . By eqns. (17) and (18) one gets  $\dot{V}(\Phi) = -\gamma e^2(t) \leq 0$ . Thus,  $V(\Phi)$  is bounded which means that  $\Phi$  is bounded. Furthermore, since the unknown transfer function is stable and the input is bounded,  $W$  is bounded.  $W$  and  $\Phi$  bounded implies that  $e$  is bounded and  $e$  and  $W$  bounded implies that  $\dot{\Phi}$  is bounded. Note that  $W$  is composed of filtered versions of the input and output of the unknown system. Thus,  $W$  is just linear combinations of the unknown plant states and the input. Therefore,  $\dot{W}$  is also composed of linear combinations of the unknown plant states, the input and the time derivative of the input. Since, the unknown plant is stable



and the input and its time derivative are bounded,  $\dot{W}$  is bounded.  $\dot{W}$  and  $\dot{\Phi}$  bounded imply that  $W$  and  $\Phi$  are uniformly continuous which implies that  $\dot{V}(\Phi)$  is uniformly continuous. Now note that since  $V(\Phi)$  is bounded, one gets that  $|V(\Phi(t))| = |\int_0^t \dot{V} dt + V(\Phi(0))| \leq M < \infty$  for all  $t$ . Thus, since  $\dot{V}$  is uniformly continuous,  $\dot{V}(\Phi)$  goes to 0 as  $t$  goes to  $\infty$  which means that  $e$  goes to 0 as  $t$  goes to  $\infty$ . ■

Therefore, given any bounded input whose derivative is also bounded one will always drive  $e(t)$  to zero. However, as discussed in [2],  $e(t)$  going to zero in no way implies that the parameter errors go to zero. One needs a condition on the spectral content of  $u(t)$  to guarantee parameter convergence, and it is that aspect one shall consider next.

### Sufficient Richness and Parameter Convergence

As with all adaptive identification schemes, parameter convergence in this scheme will rely on a condition on the spectral content of the input,  $u(t)$ , to guarantee that the signal  $W(t)$  is persistently exciting. It has been shown in [3] and [4] that in order for parameter convergence to be assured in identification schemes similar to that which is defined by eqns. (11),(17) and (18), one must be sure that the signal  $W(t)$  is persistently exciting. Then [5] showed that in order for a signal like  $W(t)$  to be persistently exciting, the input must be sufficiently rich – contain enough spectral lines. How many is enough in this case will be determined. But first recall that previously it was shown that due to the structure of  $H(s,K)$ , there may be many  $K$ 's which work. Thus, it will be shown that parameter convergence in this case will mean that the updated parameters converge to the set  $K^* + N_L(IL(s))$  where  $N_L(IL(s))$  was defined previously. Whether  $N_L(IL(s))$  contains no elements or many elements is in fact a separate issue dictated by the structure of  $H(s,K)$ .

From this point on the definitions and terminology are the same as that used in [5] except for quantities previously defined in this paper. In particular one will repeatedly rely on the concepts of autocovariance and the spectral measure of a time function. Thus, we review the definitions of those terms and give two lemmas; the first tells how the spectral measure of the input to a stable system is passed to the output and the second makes the connection between the ideas of autocovariance and persistent excitation. The definitions and lemmas are from [5].

**Definition4;** A function  $u:\mathbb{R}_+ \rightarrow \mathbb{R}^n$  is said to have *autocovariance*  $R_u(\tau) \in \mathbb{R}^{n \times n}$  if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_s^{s+T} u(t)u(t+\tau)^T dt = R_u(\tau)$$

with the limit uniform in  $s$ .

**Definition5;** The *spectral measure* of  $u:\mathbb{R}_+ \rightarrow \mathbb{R}^n$ ,  $S_u(\nu)$ , is simply the Fourier transform of the autocovariance of  $u(t)$ . Specifically,

$$S_u(\nu) = \int_{-\infty}^{\infty} e^{-i\nu\tau} R_u(\tau) d\tau$$

As pointed out in [5], "if  $u$  is scalar valued, then  $S_u$  is just a positive bounded measure;  $2S_u([\omega_0, \omega_1])$  can then be interpreted as the average energy contained in  $u$  in the frequency band  $[\omega_0, \omega_1]$ ."

**Lemma2:** Suppose  $u:\mathbb{R}_+ \rightarrow \mathbb{R}^n$  has autocovariance  $R_u(\tau)$ , its spectral measure is  $S_u$  and  $h$  is an  $m \times n$  matrix of bounded measures. Then  $y = h * u$  has an autocovariance  $R_y$  and its spectral measure is given by:

$$S_y(d\nu) = H(j\nu)S_u(d\nu)H(j\nu)^*$$

In particular,

$$R_y(0) = \int H(j\nu)S_u(d\nu)H(j\nu)^*$$

where  $H(j\nu)$  is the Fourier transform of  $h$  and superscript  $*$  denotes complex conjugate transpose.

**Lemma3:** Suppose  $w$  has autocovariance  $R_w(\tau)$ . Then there exist positive constants  $\alpha$  and  $\delta$  such that for all  $s \geq 0$

$$\alpha I \leq \int_s^{s+\delta} ww^T dt$$

(i.e.  $w$  is persistently exciting.) if and only if  $R_w(\tau)$  is positive definite.

Keeping in mind the above definitions and lemmas, one presents the following main theorem determining when parameter convergence results.

**Theorem3:** Assume that one is attempting to identify a plant whose transfer function,  $H(s, K)$ , is of the form given in eqn.(1). Let assumptions A1) - A4) hold,

and assume the input,  $u(t)$ , has an autocovariance  $R_u(\tau)$  and  $\dot{u}(t)$  is bounded. Let the method of identification be that as defined by equations (11), (17) and (18). Also, let  $r := \max \deg(n_i d_j - d_i n_j) \quad i, j = 0, 1, \dots, p$  and let  $X \subset \mathbb{R}^p$  be defined as the set of all  $C \in \mathbb{R}^p$  such that;

$$\frac{n_0(s) + k_1^* n_1(s) + \dots + k_p^* n_p(s)}{d_0(s) + k_1^* d_1(s) + \dots + k_p^* d_p(s)} = \frac{n_0(s) + (k_1^* + c_1) n_1(s) + \dots + (k_p^* + c_p) n_p(s)}{d_0(s) + (k_1^* + c_1) d_1(s) + \dots + (k_p^* + c_p) d_p(s)}$$

Let  $B := \{C \in \mathbb{R}^p : [1 \ (K^* + C)^T][N_0 \ D_0] \equiv 0\}$ . Then, if the support of the spectral measure of  $u(t)$  is not concentrated on  $n < r + 1$  points, the parameters  $[\vartheta_1(t), \dots, \vartheta_p(t)]$  will converge to the set defined by  $K^* + (X \cup B)$  as  $t$  goes to  $\infty$ .

**Remark:** Notice that the above theorem and theorem 1 basically state that the parameters in this identification scheme will converge to the set  $K^* + (X \cup B) = K^* + N_L(IL(s))$  – the set which contains the set of all  $K$ 's which work in addition to the  $K$ 's which would predict  $H(s, K)$  to be  $0/0$ . Parameter convergence to the set of all  $K$ 's which work is an acceptable solution because one has identified an equivalent transfer function and in some cases that is the best one can do. Parameter convergence to a value of  $K$  making  $H(s, K) = 0/0$  is, however, an unacceptable but unavoidable solution. To determine if such solutions exist one simply checks to see if  $[N_0 \ D_0]$  has a left null space over the reals.

**Proof:** Let  $X$  and  $r$  be defined as above and assume the support of the spectral measure of  $u(t)$  is not concentrated on  $n < r + 1$  points. Recall eqn.(18) which defines the dynamic behavior of the parameter error –  $\Phi(t)$ .

$$\dot{\Phi}(t) = -\gamma e(t) W(t) \tag{18}$$

By eqn.(17) the above equation becomes;

$$\dot{\Phi}(t) = -\gamma W(t) W(t)^T \Phi(t) \tag{19}$$

By theorem 1 in [4] eqn.(19) defines an exponentially stable system if and only if  $W(t)$  is a bounded persistently exciting signal i.e. there exists positive constants  $\alpha_1, \alpha_2$  and  $\delta$  such that for all  $s \geq 0$ ;

$$\alpha_1 I \leq \int_s^{s+\delta} W W^T dt \leq \alpha_2 I$$

Note that in this case the boundedness of  $W(t)$  comes from the fact that the input is bounded and the unknown system is stable. Thus,  $\alpha_2$  exists. Lemma 3 gives  $\alpha_1$  if and only if  $R_w(0)$  is positive definite. Thus, one must show that the

spectral measure of  $u(t)$  filters through to  $W(t)$  in such a way that  $R_w(0)$  is positive definite.

To calculate  $R_w(0)$  one first calculates the transfer function between  $\hat{u}(s)$  and  $\hat{W}(s)$  as given below;

$$L(s) := \frac{1}{\Lambda(s)(d_0(s) + k_1 d_1(s) + \dots + k_p d_p(s))} [N_1 \ D_1] [-D_0 \ N_0]^T 1K^* \quad (20)$$

Therefore, one has that;

$$\hat{W}(s) = L(s)\hat{u}(s)$$

where hatted quantities are Laplace transforms of the corresponding time domain signals. Let  $S_u(d\nu)$  denote the spectral measure of  $u$  over the interval  $d\nu$ . Then, by lemma 1,  $W$  has spectral measure:

$$S_w(d\nu) = L(j\nu)S_u(d\nu)L(j\nu)^*$$

and autocovariance at  $\tau=0$ :

$$R_w(0) = \int L(j\nu)S_u(d\nu)L(j\nu)^* \quad (21)$$

At this point one states and proves a necessary claim.

**Claim:**  $N_L(IL(s)) = N(R_w(0))$ ;  $N_L(\cdot)$  denotes the left null space of  $(\cdot)$  over the reals and  $N(\cdot)$  denotes the usual null space of  $(\cdot)$  over the reals. Recall that  $IL(s) := [N_0 \ D_0] [-D_0 \ N_0]^T 1K^*$ .

**Proof:** Assume nontrivial  $C \in \mathbb{R}^p$  is an element of  $N_L(IL(s))$ . Then  $IL(s)^T C \equiv 0$  so  $L(s)^T C \equiv 0$ . From Eqn.(21) it follows that  $R_w(0)C = \int L(j\nu)S_u(d\nu)L(j\nu)^* C \equiv 0$ . Thus,  $C$  is an element of  $N(R_w(0))$  so  $N_L(IL(s)) \subset N(R_w(0))$ .

Assume now that nontrivial  $C \in \mathbb{R}^p$  is an element of  $N(R_w(0))$ . Thus,  $R_w(0)C \equiv 0$  so  $C^T R_w(0)C \equiv 0$ . From eqn.(21) it follows that (noting that  $S_u(d\nu)$  is scalar valued)  $0 = C^T R_w(0)C = \int |L(j\nu)^* C|^2 S_u(d\nu)$ . Therefore, since  $\nu$  is continuous in  $|L(j\nu)^* C|^2$ ,  $L(j\nu)^* C$  must be zero for all  $\nu$  in the support of  $S_u(d\nu)$ . Recall that the support of  $S_u(d\nu)$  contains  $\tau+1$  or more points. Thus, the numerator polynomial of  $L(j\nu)^* C$ , namely  $IL(j\nu)^* C$ , must be zero at all of those points. However,  $IL(j\nu)^* C$  is at most an  $\tau^{\text{th}}$  order polynomial. Hence, it must be identically zero. Thus,  $IL(s)^T C \equiv 0$  so  $C$  is an element of  $N_L(IL(s))$  and  $N(R_w(0)) \subset N_L(IL(s))$ . ■

Note that the above claim merely establishes the fact that if  $IL(s)$  has a left null space,  $W(t)$  will never be persistently exciting, and if it doesn't, one needs at least  $\tau+1$  points in the support of  $S_u(d\nu)$  to guarantee that  $W(t)$  is persistently exciting.

Since  $IL(s)$  is composed of  $n_i(s)$ 's and  $d_i(s)$ 's and there are no constraints on those quantities,  $IL(s)$  could very well have a left null space. Thus, in general, one can not guarantee that  $W(t)$  will always be, strictly speaking, persistently exciting. However, using the Partial Convergence Theorem given in [5] p.9 one can say something useful (the main point of this proof) about parameter convergence. Therefore, one states the Partial Convergence Theorem of [5] and applies it to this problem.

**Partial Convergence Theorem:** Suppose that  $\dot{u}$  is bounded. Then

$$\lim_{t \rightarrow \infty} R_w(0)\Phi(t) = 0$$

Thus, in this case, the above theorem says that  $\Phi(t)$ , the parameter error, converges to  $N(R_w(0))$  which, by the above claim, equals  $N_L(IL(s))$ . Since  $\Phi(t) = K^* - \Theta(t)$ , one has that  $\Theta(t)$ , the parameters, converge to the space  $K^* + N_L(IL(s))$  which, by theorem 1, is the same space as  $K^* + (X \cup B)$ . Note that as pointed out in [5], the above Partial Convergence Theorem says that the distance between  $\Theta(t)$  and  $K^* + (X \cup B)$  goes to 0 and not that  $\Theta(t)$  goes to some  $\Theta(\infty) \in K^* + (X \cup B)$ . ■

**Remark;** Note that even though [5] deals with an MRAC scheme, the theorems and analysis borrowed from [5] and used here still apply. There are however, some differences in the structure of equivalent signals and vectors. In particular, the form of the transfer function being identified here is more general than that dealt with in [5] since in this system the quantities  $n_0(s), \dots, n_p(s), d_0(s), \dots, d_p(s)$  are allowed to be any polynomials whereas corresponding terms in [5] are much more structured and of the form  $s^p, s^{p-1}, \dots, 1, s^p, s^{p-1}, \dots, 1$ . Therefore, with that added structure one never needs to worry about the existence of  $K$ 's such that  $s^p + s^{p-1}k_1 + \dots + k_p = 0$  since there are none. Furthermore, it is the added generality of allowing  $n_0(s), \dots, n_p(s), d_0(s), \dots, d_p(s)$  to be any polynomials that in a certain sense eliminates the idea of always having only one value of  $K$  which works. Under this identification scheme when there is more than one value of  $K$  which works one has, in a sense, over parameterized the system and that fact is borne out by the behavior of the system. That is, the dynamics can only achieve the real number of degrees of freedom – the number of linearly independent columns achievable in  $R_w(0)$  – and that may be less than the number of parameters to be identified.

Furthermore, recall that parameter convergence to a set rather than a point arises do to the fact that  $W(t)$  is not, strictly speaking, persistently exciting. In this case the use of a signal whose spectral measure is supported by more than  $r+1$  points will not increase the degree of persistent excitation of  $W(t)$  because the lack of persistent excitation is a result of the structure of  $H(s,K)$ . Specifically, as long as the spectral measure of  $u(t)$  is supported at  $r+1$  or more points, the degree of persistent excitation of  $W(t)$  is determined by the existence of a left null space of  $IL(s)$  — a factor which is not affected by the input having *more* than  $r+1$  points in the support of its spectral measure. So, what the above theorem really says is that if one puts in an input whose spectral measure is supported by  $r+1$  or more points then, after a sufficient amount of time, the system will have identified the transfer function but not necessarily the  $K^*$  value. In addition, since convergence is defined in some cases as convergence to a particular set, it may be that the updated parameters  $\vartheta_1(t) \cdots \vartheta_p(t)$  never settle on one value. However, at any time  $t$  the values of  $\vartheta_1(t), \dots, \vartheta_p(t)$  read off and used as  $k_1, \dots, k_p$  in  $H(s,K)$  will in fact give the right transfer function.

So, to summarize things to this point it is evident that sufficient richness of the input signal is the only criterion for ensuring "parameter convergence". And, as one saw, parameter convergence means basically that after a sufficient amount of time the real plant and the identified transfer function are equivalent. However, that in no way implies that the resulting identified parameters are the so called physically meaningful ones or that the parameters even converge at all. If one wants to identify the physically meaningful values of the parameters and there are many values for  $K$  which work, then more information about the parameters must be known. For example, to know that a particular physically meaningful parameter is never zero may allow one to factor that parameter out reducing the number of unknowns in such a way that they are now unique. Therefore, at this point, one turns to analyzing exactly when  $H(s,K)$  is parametrically unique.

#### **More on Identifiability and Parametric Uniqueness**

Recall the matrix  $IL(s)$  (where one has dropped the  $*$  from  $K^*$ ):

$$IL(s) := [N_1 \ D_1] [-D_0 \ N_0]^T 1K$$

and recall the fact that the time varying parameters converge to the set  $K^* + N_L(IL(s))$ .

From theorems 1 and 3 it was shown that the set  $K^* + N_L(IL(s))$  serves the following two purposes;

- 1) it defines the set of  $K$ 's (modulo those which make  $H(s, K) = 0/0$ ) which work -- determines the uniqueness of  $K^*$  -- and
- 2) it defines the set to which the time varying identification parameters converge.

Thus, to determine the uniqueness of  $K^*$  one asks the question: Does there exist a  $C \in \mathbb{R}^p$  and a vector  $1K \in \mathbb{R}^{p+1}$  used in  $IL(s)$  such that  $C^T IL(s) \equiv 0$ ? If the answer is no then  $H(s, K)$  is parametrically unique. Conversely, if the answer is yes, then  $H(s, K)$  is parametrically nonunique. But, if one could show (by having slight knowledge of  $1K^*$ ) that  $1K^*$  was not in the set of  $1K$ 's making  $C^T IL(s) \equiv 0$  then, although  $1K^*$  is not unique, it would be the value identified by the identification scheme -- it is identifiable. Thus, the aspects of parametric uniqueness and identifiability are duly separated.

At this point then one wishes to study the matrix  $IL(s)$ . And, in particular, determine for what values of  $K$  it has a left null space. As a first analysis of  $IL(s)$  one determines when such a matrix has a left null space for all values of  $K$ . As it turns out, Claim 1 above gives such necessary conditions. Thus, Claim 1 is restated and proven here.

**Claim1;** Let  $H(s, K)$  be given in the form of eqn.(1). Then, if there exists  $C \in \mathbb{R}^p$  such that  $C \neq 0$  and  $C^T [N_1 D_1] \equiv 0$  or if there exists  $1C \in \mathbb{R}^{p+1}$  such that  $1C^T [N_0 D_0] \equiv 0$  then  $H(s, K)$  is not parametrically unique (i.e. it will be shown that  $IL(s)$  has a non-trivial left null space regardless of the value of  $1K^*$ .)

**Proof;** Assume  $H(s, K)$  is of the form given in eqn.(1) and choose any  $1K \in \mathbb{R}^p$  such that  $1K^T [N_0 D_0] \neq 0$  to be used in  $IL(s)$ . (The just given condition is imposed on  $1K$  since we know such a condition is true of  $1K^*$ .) Also, assume there exists  $C \in \mathbb{R}^p$  such that  $C \neq 0$  and  $C^T [N_1 D_1] \equiv 0$ . Then, obviously,  $C^T IL(s) \equiv 0$ . Now assume there exists a  $1C \in \mathbb{R}^{p+1}$  such that and  $1C^T [N_0 D_0] \equiv 0$ . Thus, note that  $1C^T [N_0 D_0] [-D_0 N_0]^T 1K \equiv 0$  and  $1K^T [N_0 D_0] [-D_0 N_0]^T 1K \equiv 0$ . Subtracting these two quantities gives  $(K - C)^T IL(s) \equiv 0$ . Thus, note that  $1C \neq 1K$  so  $C \neq K$  and  $IL(s)$  has a non-trivial left null space.

So, what the above theorem really says is that in order for  $H(s,K)$  to be parametrically unique,  $[N_0 D_0]$  must not have a left null space over the reals. If it does, one can never identify the unknown quantities in  $H(s,K)$  uniquely. However, by theorem 3 one can always identify an equivalent transfer function for  $H(s,K)$ .

Thus far one has given only necessary conditions for parametric uniqueness and identifiability. In order to study the situation further, one must express  $IL(s)$  in a different form.

By calculation it is easy to check the following.

$$IL(s) = \Gamma(K)P(s)$$

where  $\Gamma(K) \in \mathbb{R}^{p \times q}$ ,  $P(s) \in \mathbb{R}[s]^q$ ,  $q := \sum_{i=1}^{i=p} i$ .

$$\Gamma(K) := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & k_2 & k_3 & \dots & k_p & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & -k_1 & 0 & \dots & 0 & k_3 & k_4 & \dots & k_p & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & -k_1 & \dots & 0 & -k_2 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \dots & 0 & 0 & -k_2 & \dots & \vdots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & -k_1 & 0 & 0 & \dots & -k_2 & 0 & \dots & k_p \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & -k_1 & 0 & \dots & 0 & -k_2 & \dots & -k_{p-1} \end{bmatrix}$$

and

$$P(s) := \begin{bmatrix} -(d_0(s)n_1(s) - d_1(s)n_0(s)) \\ -(d_0(s)n_2(s) - d_2(s)n_0(s)) \\ \vdots \\ -(d_0(s)n_p(s) - d_p(s)n_0(s)) \\ d_1(s)n_2(s) - d_2(s)n_1(s) \\ d_1(s)n_3(s) - d_3(s)n_1(s) \\ \vdots \\ d_1(s)n_p(s) - d_p(s)n_1(s) \\ d_2(s)n_3(s) - d_3(s)n_2(s) \\ d_2(s)n_4(s) - d_4(s)n_2(s) \\ \vdots \\ d_2(s)n_p(s) - d_p(s)n_2(s) \\ d_3(s)n_4(s) - d_4(s)n_3(s) \\ \vdots \\ d_3(s)n_p(s) - d_p(s)n_3(s) \\ \vdots \\ d_{p-1}(s)n_p(s) - d_p(s)n_{p-1}(s) \end{bmatrix}$$

Thus, the question still stands: Does there exist a  $C \in \mathbb{R}^p$  and a  $K \in \mathbb{R}^p$  such that  $C^T \Gamma(K)P(s) \equiv 0$ ? In this form the unknown  $K$  forms a matrix while  $C$  is still a

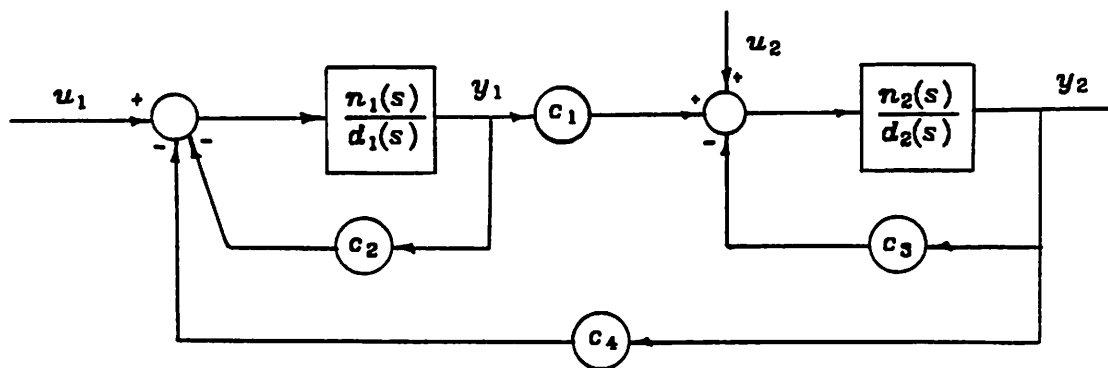


vector. Notice then that for  $C^T\Gamma(K)P(s)$  to be identically zero two things must happen.  $P(s)$  must have a left null space over the reals and there must exist a  $C$  and a  $K$  such that  $C^T\Gamma(K)$  is an element of that left null space. Furthermore, examination of  $\Gamma(K)$  and  $C$  reveals that  $C^T\Gamma(K)$  can only equal a vector whose first  $p$  terms are not all zero. Thus, one has a sufficient condition for both parametric uniqueness and identifiability namely :  $H(s,K)$  is parametrically unique and therefore  $K^*$  is identifiable if the only elements in the left null space of  $P(s)$  have zeros in the first  $p$  places. Beyond this, characterizing the sets of  $K$ 's and  $C$ 's which make  $C^T\Gamma(K)$  lie in the left null space of  $P(s)$  must be dealt with on a case by case basis.

**Examples §**

In this section one wishes to illustrate two different types of systems whose transfer functions are of the form given in eqn.(1) and therefore would be eligible for the identification scheme presented in this paper.

The first class of systems presented is the general class of interconnected systems whose innerconnection gains are unknown. An example of such a system is given below.



In the above system the interconnection gains are unknown and the interconnected blocks are known. The matrix of transfer functions between  $[u_1 \ u_2]^T$  and  $[y_1 \ y_2]^T$  for the above system is:

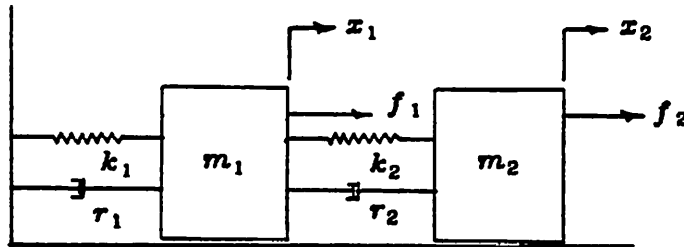
$$\frac{\begin{bmatrix} n_1(s)d_2(s)+c_3n_1(s)n_2(s) & -c_4n_1(s)n_2(s) \\ c_1n_1(s)n_2(s) & d_1(s)n_2(s)+c_2n_1(s)n_2(s) \end{bmatrix}}{d_1(s)d_2(s)+c_2n_1(s)d_2(s)+c_3n_2(s)d_1(s)+(c_1c_4+c_2c_3)n_1(s)n_2(s)}}$$

Notice that as long as none of the  $c$ 's are zero, the  $u_2, y_1$  or the  $u_1, y_2$  input-output pair could be used to identify the unknown  $c$ 's by the method presented in this paper. (In the identification procedure simply make one of the unknown parameters equal to  $c_1c_4+c_2c_3$ .) If the  $u_1, y_1$  or the  $u_2, y_2$  input-output pair is used, only three of the four parameters are identifiable. When one of the  $c$ 's is zero, one can still use the presented identification scheme except that care must be taken in choosing the proper input-output pair for the identification.

From more of a physical point of view, any spring-mass-damper system or its electrical analog will generate system transfer functions of the form given in

§ All simulations presented in this section were done using Simnon - An Interactive Simulation Program For Nonlinear Systems, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

eqn.(1) when there exist unknown system parameters like spring constants, resistor values etc.. Typically, such systems will produce transfer functions where the unknown parameters appear in the "multilinear" form. To convert such problems to ones this identification scheme can handle, one simply defines a new parameter as the multilinear parameter. As an example, consider the following simple system:



Now assume that  $m_2=2, r_2=1, k_2=3$  and everything else is unknown. Then, making a forcing function on either mass the input and the position of either mass the output produces a matrix of transfer functions for the above system as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\begin{bmatrix} 2s^2+s+3 & s+3 \\ s+3 & (s+3)+k_1+m_1s^2+r_1s \end{bmatrix}}{(2s^3+6s^2)+k_1(2s^2+s+3)+m_1(2s^4+s^3+3s^2)+r_1(2s^3+s^2+3s)} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}$$

Thus, in this case, no multilinear terms appear. Also, one can use any input-output pair to identify the unknown parameters. To illustrate the identification process one uses the 2,2 entry of the above matrix of transfer functions to identify the unknowns  $m_1, r_1$  and  $k_1$ . Thus, letting  $k_1=k, k_2=m_1$  and  $k_3=r_1$  gives:

$$\begin{aligned} n_0 &= s+3 & d_0 &= 2s^3+6s^2 \\ n_1 &= 1 & d_1 &= 2s^2+s+3 \\ n_2 &= s^2 & d_2 &= 2s^4+s^3+3s^2 \\ n_3 &= s & d_3 &= 2s^3+s^2+3s \end{aligned}$$

Note that neither  $N_0$  nor  $D_0$  has a left null space so that the necessary conditions for parametric uniqueness are met. Now form the vector  $P(s)$ .

$$P(s) = \begin{bmatrix} -(d_0n_1-d_1n_0) \\ -(d_0n_2-d_2n_0) \\ -(d_0n_3-d_3n_0) \\ d_1n_2-d_2n_1 \\ d_1n_3-d_3n_1 \\ d_2n_3-d_3n_2 \end{bmatrix} = \begin{bmatrix} s^2+6s+9 \\ s^4+6s^3+9s^2 \\ s^3+6s^2+9s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that every element in the left null space of  $P(s)$  has the first 3 terms

equal to zero. Thus, the sufficient condition for parametric uniqueness is met so that no matter what  $m_1, \tau_1$  or  $k_1$  is, they can be uniquely identified.

Below are simulation results of the successful identification of  $m_1, \tau_1$  and  $k_1$  when the real values of those parameters are 3, 2 and 2 respectively.

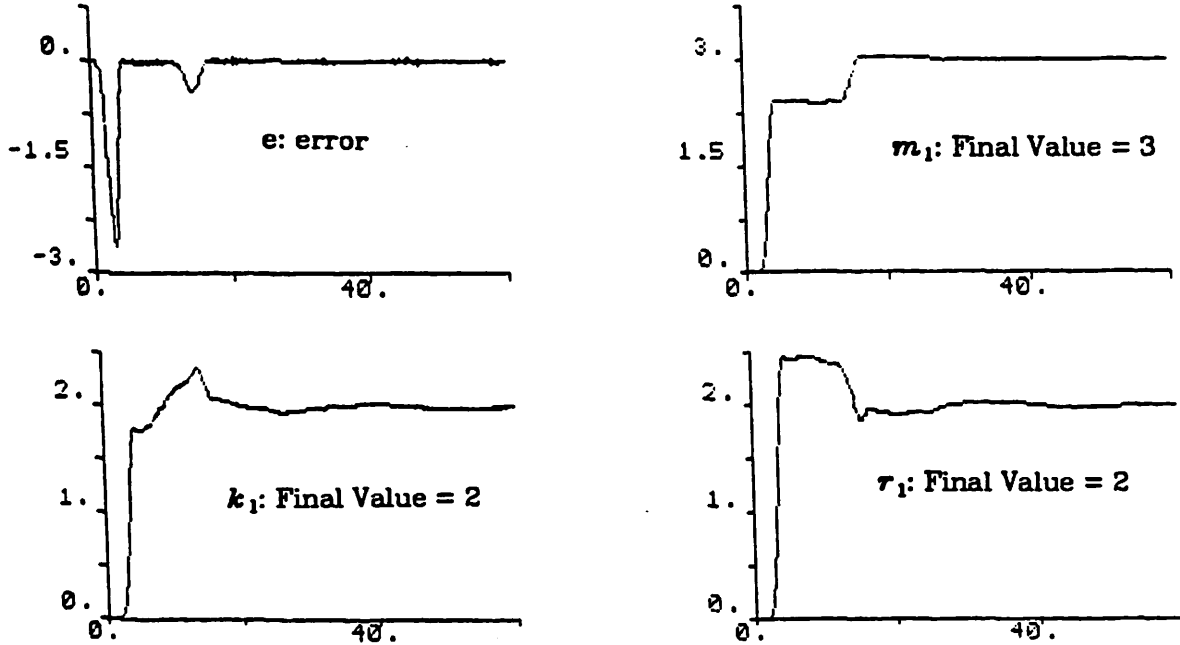


Fig. 1: Simulation results for the identification of the given spring-mass-damper system.  $u(t)=2+2\sin(.5t)+2\sin(t)$ , all initial conditions were set to 0 and the update gain for all parameters was 1. The values of the input parameters were  $m_1=3, \tau_1=2$  and  $k_1=2$ .

As illustrated by the above discussion, it is clear that the presented identification scheme has application. To present further this identification procedure and to emphasize the ideas of parametric uniqueness and identifiability, one presents several more simple examples.

**Example 1**

Consider the following partially known transfer function.

$$H(s, K) = \frac{6s + s^3 + k_1 s^2 + k_2 (s^2 + 5s + 6)}{4s + s^3 + k_1 s^2 + k_2 (s^2 + 5s + 4)} \quad (23)$$

Then

$$\begin{aligned} n_0 &= s^3 + 6s & d_0 &= s^3 + 4s \\ n_1 &= s^2 & d_1 &= s^2 \\ n_2 &= s^2 + 5s + 6 & d_2 &= s^2 + 5s + 4 \end{aligned}$$

Note that neither  $[N_1 D_1]$  nor  $[N_0 D_0]$  has a left null space over the reals — both necessary conditions for parametric uniqueness and identifiability are satisfied. Now calculate  $P(s)$ .

$$P(s) = \begin{bmatrix} -(d_0(s)n_1(s) - d_1(s)n_0(s)) \\ -(d_0(s)n_2(s) - d_2(s)n_0(s)) \\ d_1(s)n_2(s) - d_2(s)n_1(s) \end{bmatrix} = \begin{bmatrix} 2s^3 \\ 10s^2 \\ 2s^2 \end{bmatrix}$$

The only element in the left null space of  $P(s)$  is  $[0 \ -1 \ 5]$ . So any element in the left null space of  $P(s)$  is of the form  $[0 \ -\alpha \ 5\alpha]$ ,  $\alpha \in \mathbb{R}$ . Does there exist a  $K \in \mathbb{R}^2$  and a  $C \in \mathbb{R}^2$  such that  $C^T \Gamma(K) = [0 \ -\alpha \ 5\alpha]$ ? Well,

$$C^T \Gamma(K) = [c_1 \ c_2] \begin{bmatrix} 1 & 0 & k_2 \\ 0 & 1 & -k_1 \end{bmatrix} = [c_1 \ c_2 \ c_1 k_2 - c_2 k_1]$$

Thus,

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -\alpha \\ k_1 &= 5 \\ k_2 &= \text{anything} \end{aligned}$$

does it. Therefore,  $H(s, K)$  is not parametrically unique. However, if  $k_1 \neq 5$  then  $k_1$  and  $k_2$  are identifiable. Plugging  $k_1 = 5$  into  $H(s, K)$  gives;

$$H(s, 5, k_2) = \frac{(s+2)(s+3)(s+k_2)}{(s+4)(s+1)(s+k_2)}$$

Thus, for  $k_1 = 5$  a pole-zero cancellation results and the value of  $k_2$  does not matter. However, as mentioned, if  $k_1 \neq 5$  then no pole-zero cancellation results and  $k_1$  and  $k_2$  are identifiable.

Below are the results of two identification simulations. The first run uses the scheme presented in this paper to identify  $H(s, K)$  given in eqn.(23) with  $k_1 = 5$  and  $k_2 = \text{unspecified}$ . The second run uses the same identification scheme on  $H(s, K)$  except that here  $k_1 = 2$  and  $k_2 = 1$ .

Results of Run 1

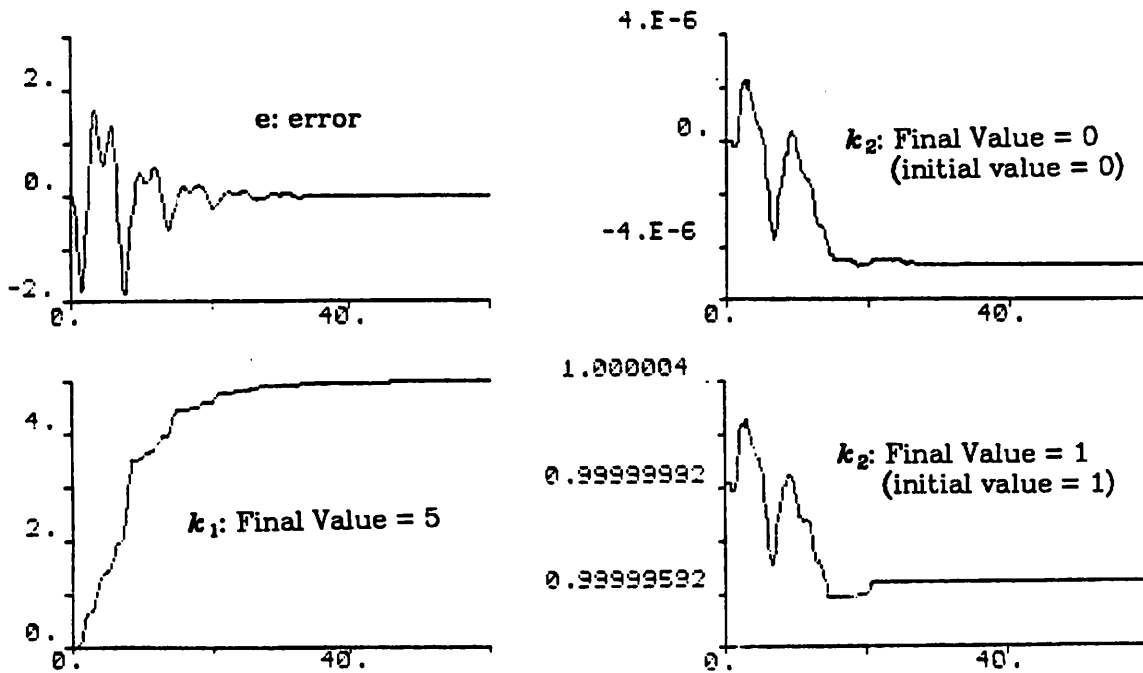


Fig. 2: Simulation results for the identification of eqn.(23).  $u(t)=4\sin(t)+4\sin(2t)$ , all initial conditions were set to 0 and the update gain for all parameters was 1. The values of the input parameters were  $k_1=5$  and  $k_2=unspecified$ .

Examination of the above results shows that the identification scheme successfully identified  $k_1$ . However, since  $k_2$  is lost in the pole-zero cancellation, it can go to any value -- which seems to depend on its initial condition.

**Results of Run 2**

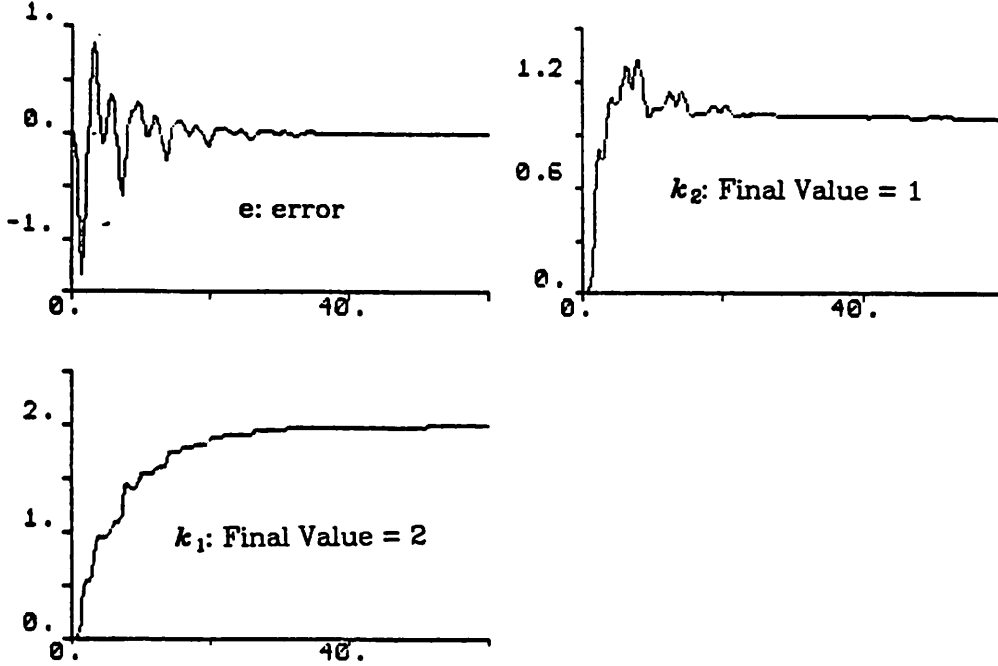


Fig. 3: Simulation results for the identification of eqn.(23).  $u(t)=4\sin(t)+4\sin(2t)$ , all initial conditions were set to 0 and the update gain for all parameters was 1. The values of the input parameters were  $k_1=2$  and  $k_2=1$ .

Examination of the above results shows that the scheme was successful in identifying both parameters as predicted.

**Example 2**

Consider the partially known transfer function given below.

$$H(s,K) = \frac{3+k_1(s+2)+k_2s^2+k_3(s+2)}{(s^3+2s+3)+k_1(s^2+s)+k_3(s^2+s)} \quad (24)$$

In this case;

$$\begin{aligned} n_0 &= 3 & d_0 &= s^3+2s+3 \\ n_1 &= s+2 & d_1 &= s^2+s \\ n_2 &= s^2 & d_2 &= 0 \\ n_3 &= s+2 & d_3 &= s^2+s \end{aligned}$$

Thus,  $[N_1 D_1]$  has a left null space over the reals making  $H(s,K)$  parametrically nonunique by Claim 1. Note that eqn.(24) can be written as given below. In this form the nonuniqueness of the parameters is more obvious.

$$H(s,K) = \frac{3 + (k_1 + k_3)(s+2) + k_2 s^2}{(s^2 + 2s + 3) + (k_1 + k_3)(s^2 + s)}$$

A simulation was performed to illustrate the above predicted nonuniqueness of acceptable parameter values. In that simulation  $k_1=2, k_2=5$  and  $k_3=3$ . The results of the simulation are given below.

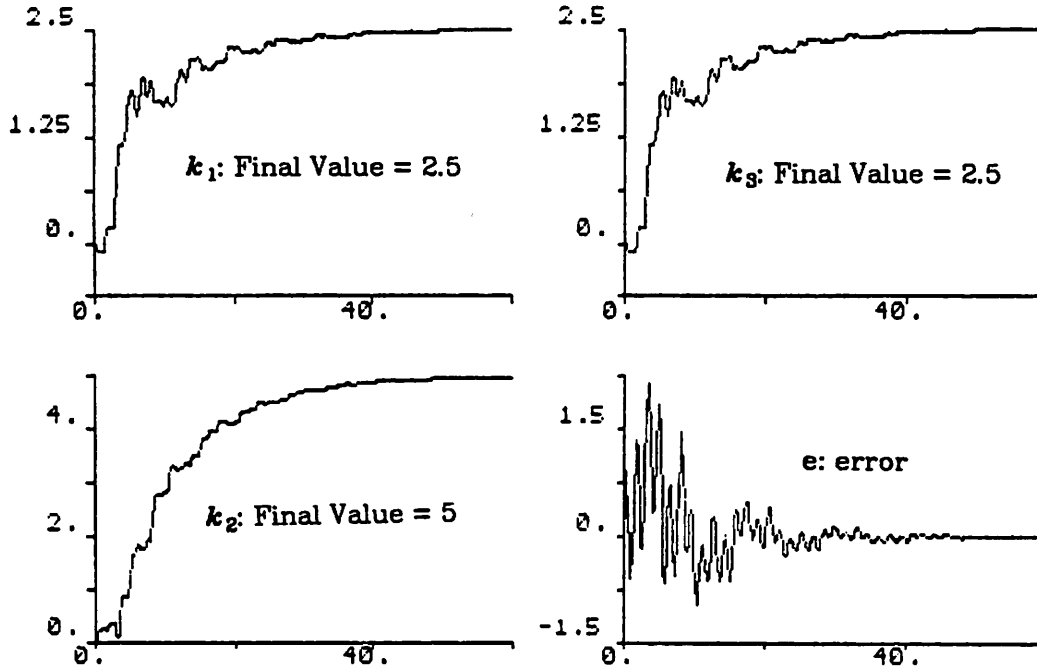


Fig. 4: Simulation results for the identification of eqn.(24).  $u(t)=2\sin(.5t)+2\sin(1.5t)+2\sin(4t)$ , all initial conditions were set to 0 and the update gain for all parameters was 2. The values of the input parameters were  $k_1=2, k_2=5$  and  $k_3=3$ .

Examination of the above simulation results reveals that the parameters did not converge to their input values. However, the values they did converge to namely  $k_1=2.5, k_2=5$  and  $k_3=2.5$  do yield an equivalent transfer function when plugged into  $H(s,K)$  given in eqn.(24).

### Example 3

Consider yet another partially known transfer function as given below.

$$H(s,K) = \frac{(s+1) + k_1(s+1) + k_2 s^2}{(s^2+1) + k_1(s^2+1) + k_3(s^3+11s)} \quad (25)$$

For this transfer function;



$$\begin{aligned} n_0 &= s+1 & d_0 &= s^2+1 \\ n_1 &= s+1 & d_1 &= s^2+1 \\ n_2 &= s^2 & d_2 &= 0 \\ n_3 &= 0 & d_3 &= s^3+11s \end{aligned}$$

Thus,  $[N_0 D_0]$  has a left null space over the reals so that again  $H(s,K)$  is not parametrically unique. Rewriting eqn.(25) in the following form illustrates the parametric nonuniqueness of the parameters.

$$H(s,K) = \frac{(1+k_1)(s+1)+k_2s^2}{(1+k_1)(s^2+1)+k_3(s^3+11s)}$$

A simulation was also run for this system with  $k_1=5, k_2=1$  and  $k_3=7$ . The results of that simulation are given below.

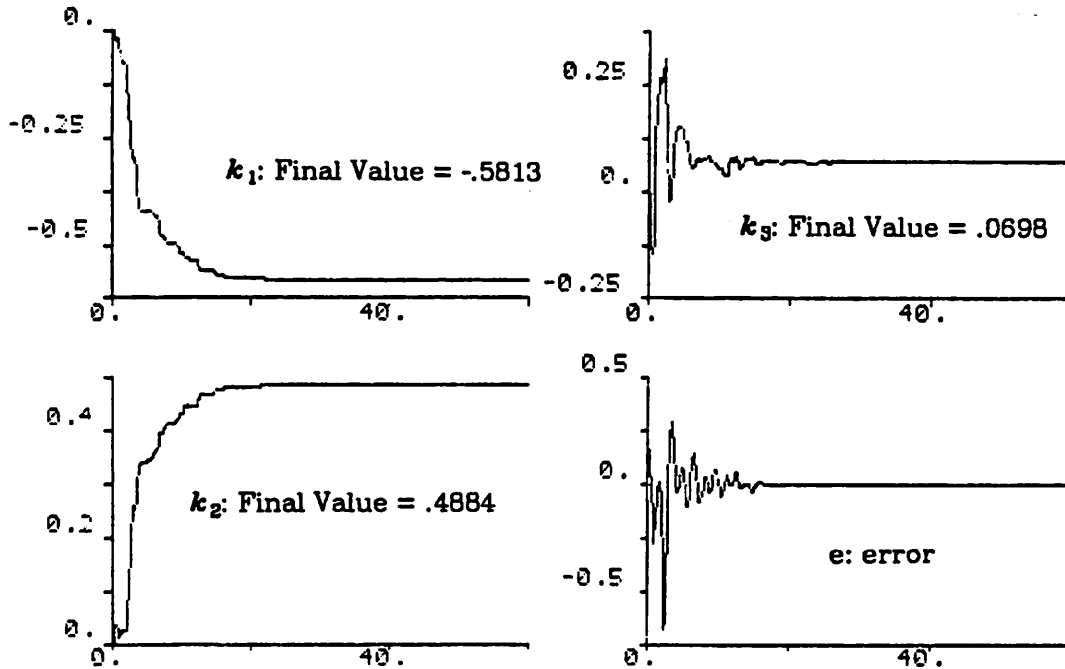


Fig. 5: Simulation results for the identification of eqn.(25).  $u(t)=2\sin(.5t)+2\sin(t)+2\sin(2t)$ , all initial conditions were set to 0 and the update gain for all parameters was 1. The values of the input parameters were  $k_1=5, k_2=1$  and  $k_3=7$ .

Examination of the above results shows that again the parameters do not converge to their input values but do converge to values that, when plugged into eqn.(25), yield a transfer function which is equivalent to the input transfer function. (In this case one needs to multiply the identified transfer function by 14.33/14.33 to get exactly the same transfer function as input.)

**Example 4**

Finally, consider the following slightly higher order partially known transfer function.

$$H(s,K) = \frac{s + k_1 s^5 + k_3 (s^2 + 2) + k_4}{(57s^2 + 1) + k_1 s^6 + k_2 (5s^4 + 1) + k_4 (4.5s^5 + 29s^3 + 14.5s)} \quad (26)$$

For this transfer function;

$$\begin{aligned} n_0 &= s & d_0 &= 57s^2 + 1 \\ n_1 &= s^5 & d_1 &= s^6 \\ n_2 &= 0 & d_2 &= 5s^4 + 1 \\ n_3 &= s^2 + 2 & d_3 &= 0 \\ n_4 &= 1 & d_4 &= 4.5s^5 + 29s^3 + 14.5s \end{aligned}$$

Thus, neither  $[N_0 D_0]$  nor  $[N_1 D_1]$  has a left null space over the reals so that all necessary conditions for parametric uniqueness and identifiability are satisfied. Now check  $P(s)$ .

$$P(s) = \begin{bmatrix} -(d_0 n_1 - d_1 n_0) \\ -(d_0 n_2 - d_2 n_0) \\ -(d_0 n_3 - d_3 n_0) \\ -(d_0 n_4 - d_4 n_0) \\ d_1 n_2 - d_2 n_1 \\ d_1 n_3 - d_3 n_1 \\ d_1 n_4 - d_4 n_1 \\ d_2 n_3 - d_3 n_2 \\ d_2 n_4 - d_4 n_2 \\ d_3 n_4 - d_4 n_3 \end{bmatrix} = \begin{bmatrix} -56s^7 - s^5 \\ 5s^5 + s \\ -57s^4 - 115s^2 - 2 \\ 4.5s^6 + 29s^4 - 42.5s^2 - 1 \\ -5s^9 - s^5 \\ s^8 + 2s^6 \\ -4.5s^{10} - 29s^8 - 13.5s^6 \\ 5s^6 + 10s^4 + s^2 + 2 \\ 5s^4 + 1 \\ -4.5s^7 - 38s^5 - 72.5s^3 - 29s \end{bmatrix}$$

From  $P(s)$  the coefficient matrix  $P_c$  is formed and is given below.

$$P_c = \begin{bmatrix} 0 & 0 & 0 & -56 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -57 & 0 & -115 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4.5 & 0 & 29 & 0 & -42.5 & 0 & -1 \\ 0 & -5 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.5 & 0 & -29 & 0 & -13.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 10 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4.5 & 0 & -38 & 0 & -72.5 & 0 & -29 & 0 \end{bmatrix}$$

It can easily be shown that  $P_c$  has no left null space which implies that  $P(s)$  has no left null space over the reals. Thus,  $H(s,K)$  is parametrically unique and any value for  $K$  in  $H(s,K)$  is uniquely identifiable. On the next page are the simulation results when  $k_1=1, k_2=5, k_3=4$  and  $k_4=2$ .

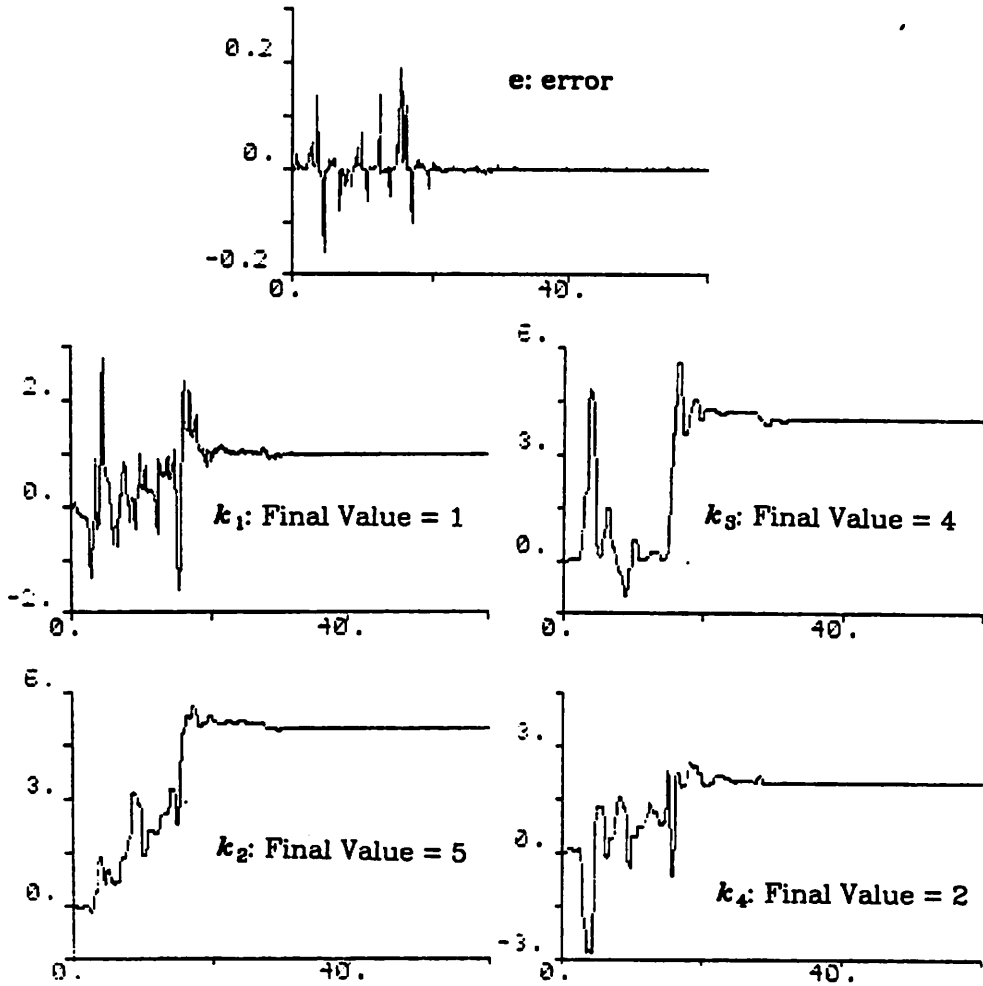


Fig. 6: Simulation results for the identification of eqn.(26).  $u(t)=2+2\sin(.5t)+2\sin(t)+2\sin(1.5t)+2\sin(2t)+2\sin(4t)$ , all initial conditions were set to 0 and the update gain for all parameters was 1000. The values of the input parameters were  $k_1=1, k_2=5, k_3=4$  and  $k_4=2$ .

Examination of the above simulation results shows that the identifier was indeed successful in identifying the exact input values for  $k_1, k_2, k_3$  and  $k_4$ .

## Conclusion

In this paper one has presented a method similar to a scheme given in [1] which utilizes specific prior knowledge of an unknown continuous time linear system to deterministically identify the transfer function of that system. It was shown that an equivalent transfer function (i.e. one which is identical to the unknown one modulo multiplication by 1 in the form of  $a/a$  where  $a \in \mathbb{R}$  and possibly pole-zero cancellations) can always be identified provided the support of the spectral measure of the input is not concentrated on  $n < r + 1$  points where  $r := \max \deg(n_i d_j - n_j d_i) \quad i, j = 1, \dots, p$ . Furthermore, conditions on the structure of the prior information were given which determine when and when not unique values of the unknown parameters of the system are identifiable.

The advantages to the given identification scheme lie in the fact that one does not have to identify every coefficient in the transfer function just the unknown parameters. Thus, the amount of computation necessary to perform the identification is in some cases significantly reduced (e.g. in example 4 only 4 parameters were identified as opposed to the necessary 13 when the transfer function is assumed to be completely unknown). In addition, it is intuitively argued that because fewer parameters are being identified, the real time convergence rates for those parameters is improved.

Note that in all of our analysis it was assumed that there exist at least one set of parameter values which made the assumed form of the transfer function used by the identifier equal to the transfer function of the real system. However, when there exist unmodelled dynamics in the system, it is not true that such a set of parameter values exist. Therefore, now that one knows the system behavior for the ideal case, more analysis will have to be done to determine the behavior of this identification scheme when used to identify a system in the presence of unmodelled dynamics. One way to perform such a robustness study would be to make the  $n_i(s)$ 's and  $d_i(s)$ 's proper, stable rational functions rather than polynomials by dividing the numerator and denominator of  $H(s, K)$  by  $\Lambda(s)$  — a Hurwitz polynomial of sufficient order. Then, well defined norms such as the sup norm over  $j\omega$  could be used to measure uncertainty in each  $n_i(s)$  or  $d_i(s)$  which could then be related back to the total uncertainty about the system. Thus, once the system uncertainty is expressed in a familiar measure, such as the previously mentioned sup norm, *existing* robustness results could be applied.

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