LARGE DEVIATIONS IN DISSIPATIVE DYNAMICS: AN OPTIMAL CONTROL APPROACH

by

Efthimios Kappos

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TITLE PAGE

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ABSTRACT

In this research, a class of dissipative dynamics is specified for which the global large deviation problem can be completely solved. This is done using an optimal control approach and a relation between the stability structure of the dynamics and the control setting.

For systems in this class, there exist global Lyapunov functions that are strictly decreasing along any trajectory of the system, except on its α- and ω-limit sets.

These Lyapunov functions can be used to solve the optimal control problem that is associated to the stochastic small-noise problem. One finds that a controllability assumption relating the Lyapunov functions to the control dynamics is useful.

Finally, this leads to new qualitative results on the behavior of the large deviation exit paths and to some suggestions for obtaining computed solutions. A result of this kind is that exit from a region of attraction is from the 'nearest' saddle.

Shankar Sastry,

Committee Chairman.

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CONTENTS

CHAPTER 1: INTRODUCTION AND DYNAMICAL SETTING

The calculation of asymptotic estimates in large deviation theory leads to a variational setting that is best understood using optimal control. An example is the exit problem from an attracting domain.

In this first chapter, we specify a general class of dynamical systems that is relevant to large deviation theory and which can be well understood qualitatively. These systems are dissipative, structurally stable and simple to classify using orbit (Smale) diagrams. They are closely related to Morse-Smale vector fields satisfying the no-cycle condition.

1.1. Introduction

1.1.1. Background and Motivation

The aim of the present research is to work out a rigorous framework for large deviations in a global setting. It expands and generalizes the treatment of large deviation paths in the context of the theory of *small-noise perturbations of dynamical systems* , as presented in the monograph of Wentzell and Freidlin [W-F]. As motivation, we give here an overview of the kinds of problems this theory treats, from the viewpoint of the present work.

A nonlinear dynamical system:

$$\dot{x}_t = b(x_t) \tag{1}$$

is perturbed by the addition of small noise:

$$dx_t = b(x_t)dt + \varepsilon\sigma(x_t)dw_t \tag{2}$$

As $\varepsilon \to 0$ with probability tending to 1 the perturbed system trajectories approach those of the unperturbed system over any finite time interval. Of interest, however, are the trajectories of (2) that are far from those of (1); these are events whose probabilities go to 0, but among which we can distinguish some that are overwhelmingly more likely than the others. The estimation of probabilistic quantities associated with these special rare events, which are called *large deviations* , involves a variational setting closely related to the dynamics of the perturbed system.

The difficulty of moving along a specified trajectory γ is measured by the amount of *control action* required to steer the state of the following controlled system along γ :

$$\dot{x}_t = b(x_t) + \sigma(x_t)u_t \tag{3}$$

where the control cost is measured by the functional:

$$J(u) = \frac{1}{2}\int_0^T |u_s|^2 ds \tag{4}$$

and $x_{[0,T]} = \gamma$.

This connection between the controlled system (3)-(4) and the small-noise setting is new. In particular, it makes it reasonable to believe that only the *accessible trajectories* of (3)-(4) are of interest as large deviations of the stochastic system (2) (ie. those that the system (3) can be steered to, using a control u of finite cost, according to (4)). Now it becomes plausible that the non-singularity of σ for all x is not a necessary condition for the development of a theory.

In the literature of large deviations, σ is usually assumed uniformly of rank n. Only in Azencott [Az] is this requirement dropped, even though the action functional is still written in the calculus-of-variations way: given a path γ on $[0,T]$,

$$J(\gamma)=\frac{1}{2}\int_0^T ||\dot{\gamma}_t-b(\gamma_t)||_{(\sigma\sigma^*(\gamma_t))^{-1}}dt$$

The use of the action functional (4) has the additional advantage that it allows us to pose and solve the large deviation estimates as *optimal control problems* . The key to the connection between (1)-(2) and the control problem (3)-(4) is surprisingly simple:

If we start with the system:

$$dx_t=b(x_t)dt+\sigma(x_t)dw_t\ ,$$

(σ not necessarily of full rank) and we use a measure transformation:

$$M_t(\omega)=\exp[-\frac{1}{2}\int_0^t |u(x_s)|^2ds+\int_0^t u(x_s)dw_s]$$

the new drift term is:

$$b(x)+\sigma(x)u(x)$$

which we recognize as the vector field of the controlled system (3). The new stochastic differential equation is:

$$dy_t=\dot{y}_tdt+\sigma(y_t)d\tilde{w}_t$$

where:

$$\dot{y}_t=b(y_t)+\sigma(y_t)u(y_t)$$

All this works with $\sigma(x)$ not full rank, but another condition is needed: namely, that $u(x)$ must be defined on some open region of state space (see chapter 4 for exact statement). In the control setting, this means that u is a *feedback law* and that we have a *field* of control paths, ie. an open set, each point of which has a controlled path going through it. Thus each control law u gives us a measure transformation. The u corresponding to the *optimal feedback law* gives us the appropriate estimation of the large deviation asymptotics.

The goal of this research is to make rigorous the above heuristic discussion.

The simplest case where the optimal control problem yields a feedback law is when the drift vector field is the gradient of a function. Since our aim is to generalize this result to as wide a class of dynamics as possible, we give now a brief review of the known results in this case.

1.1.2. Gradient Flows and Optimal Control

In this section we give an account of Morse theory from the point of view of optimal control. Thus, from the beginning, we state the results in terms of the gradient vector field that a Morse function defines. We also explain how a Morse function solves certain minimum-energy control problems and point out that it defines optimal control *fields* i.e. unique optimal trajectories through each point of the state space. None of these results is new, but the author has not seen them collected together in the form presented here.

Let h be a Morse function on $\mathbf{R}^n$ such that:

(i) h has a finite number of critical points.

(ii) h is bounded below : $h(x)\geq 0$.

(iii) $h^{-1}([0,a])$ is compact for all $a\in\mathbf{R}$.

The gradient of h , dh , defines a vector field ∇h on $\mathbf{R}^n$ through the standard inner product in $\mathbf{R}^n$ by:

$$<\nabla h,v>(x)=dh(v(x)),v(x)\in T_x\mathbf{R}^n$$

We call $-\nabla h$ the *gradient vector field* (the reason for the sign change is that we want the flow to be towards a minimum of h). The flow of $-\nabla h$ is the *gradient flow* $\eta(t,x)$. To simplify matters we also assume:

(iv) $-\nabla h$ is a complete vector field *.

The results of Morse theory are now easily restated in terms of the vector field $-\nabla h$ and its flow:

Theorem: (Morse Theory for Gradient Flows)

Let h be a Morse function on $\mathbf{R}^n$ *satisfying the assumptions (i)-(iv). Let* $-\nabla h$ *be its gradient vector field. Then:*

* This can be translated into a requirement on h , eg. that h be smooth with globally Lipschitz derivatives.

(a) *the gradient flow in a neighborhood of a critical point of index k is locally conjugate (i.e. dynamically equivalent via a diffeomorphism) to the simple linear flow:*

$$\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}$$

(b) *the level sets* $h^{-1}(a)$, $a\in\mathbf{R}$ *are compact Lyapunov surfaces for the gradient vector field.*

(c) *if there are no critical values of h in* $[a-\varepsilon,a+\varepsilon]$, $a\in\mathbf{R}$, $\varepsilon>0$, *then* $h^{-1}(a-\varepsilon)$ *is diffeomorphic to* $h^{-1}(a+\varepsilon)$.

(d) *the homotopy of the Lyapunov surface* $h^{-1}(c+\varepsilon)$ *where c is a critical value of h and* $\varepsilon>0$ *is small is that of* $h^{-1}(c-\varepsilon)$ *with a k-cell attached, where k is the index of the isolated critical point p :* $h(p)=c$.

The above theorem is simply a restatement of the basic properties of a Morse function and we therefore omit the proof (see Milnor [Mi]).

Note that $h^{-1}(c)$, c a critical value, is *not* a manifold. If however, we consider, for each critical point $\bar{c}$ of index 0 its region of attraction $R_{\bar{c}}$, then $h^{-1}(c)\cap R_{\bar{c}}$ *is* a manifold: in fact it is a bounded hypersurface of $\mathbf{R}^n$, when non-empty (see Fig.1.1).

Now consider the *controlled dynamical system :*

$$\dot{x} = -\nabla h(x)+u \quad , x,u\in\mathbf{R}^n \tag{1}$$

and the 'control-energy' cost functional: for each $u\in C^0([0,T];\mathbf{R}^n)$:

$$S(u)=\frac{1}{2}\int_0^T |u_s|^2 ds \tag{2}$$

It is well-known that for the class of controlled dynamics given by (1) with the cost functional (2), explicit optimal feedback control solutions can be given.

Thus we can claim:

Minimum-Energy Control of Gradient Flows:

Let the points x_0 and x_1 be connected by a trajectory of the system:

$$\dot{y} = \nabla h(x)$$

then the given trajectory is the optimal path of going from x_0 to x_1 for the cost functional S.

Furthermore, the optimal cost of going from x_0 to x_1 is $2(h(x_1)-h(x_0))$ *and the optimal feedback law is:*

$$u^*(x) = 2\nabla h(x)$$

These facts are checked easily using the usual sufficiency conditions of optimal control. Since we shall essentially repeat these results -in a much more general context- in chapter 3, we omit their demonstration. We, however, note some properties of this control set-up:

Remarks:

1) The *optimal exit path* from the region of attraction $R_{\bar{c}}$ of an asymptotically stable equlibrium $\bar{c}$ occurs on a trajectory that tends to c_s , the saddle equilibrium of $-\nabla h$ with the lowest value of the Morse function that lies on $\overline{R_{\bar{c}}}$ (no ties occur, since h is a Morse function).

2) The exit from $R_{\bar{c}}$ starting from $\bar{c}$, takes an infinite amount of time since $-\nabla h$ goes to zero near the saddle c_s *.

3) The *isocost* surfaces starting from $\bar{c}$ are precisely the level sets of h , i.e. the Lyapunov surfaces of the flow. Moreover, the optimal trajectories are everywhere *perpendicular* to these Lyapunov surfaces.

In the next section we outline how the present work is motivated by and generalizes the above results.

* The optimal path is thus unsatisfactory from a control point-of-view. This singular behavior is resolved by modifying the cost functional S to include a cost proportional to the time a trajectory takes to be traversed, i.e. using the cost S' where:

$$S'(u) = S(u) + \varepsilon T$$

with ε small. Then, as $\varepsilon \to 0$, the optimal trajectories for the functional S' will converge to those of S .

1.1.3. Outline of Research

On the basis of the discussion of gradient flows in the previous section, let us see in what directions we can generalize the results there and what constraints we must place on the dynamical systems we hope to treat.

Of fundamental importance is to generalize the notion of an isolated, non-degenerate equilibrium point. The first generalization is thus to allow for a wider class of attractors. The natural concept that makes an attractor isolated and non-degenerate is that of *asymptotic stability* . Furthermore, we allow not only equilibrium points, but general compact manifolds which we call *attracting sets* . As in the case of Morse-Smale vector fields, the non-attracting ω- and α-limit sets are required to be *hyperbolic equilibria or closed orbits* .

To get a complete qualitative picture for the class of dynamics to be defined, we impose a global condition of *dissipativeness.* This allows *orbit diagrams* to be drawn for the flow, assuming a no-cycle condition is satisfied. The class of dynamics described above is specified in detail in section 2.

In chapter 2, we turn to the generalization of the Morse functions: these are the (strict) *Lyapunov functions* for the flow. In contrast to the Morse function giving a gradient flow, a Lyapunov function is *not unique* for a given flow. It is one of the main contributions of this research that the precise degree of non-uniqueness of Lyapunov functions is found. This leads naturally to a complete classification of all the possible Lyapunov functions of a given vector field and, as a result, of different classes of optimal control problems that share the same state dynamics (uncontrolled dynamics).

This is the subject of chapter 3. The basic connection between optimal control problems and global Lyapunov function theory is the result that any Lyapunov function is the optimal cost-functional of some control problem. Conversely, if a control problem has a smooth feedback-law solution, then its associated optimal cost funcional is well-defined and is a Lyapunov function for the state dynamics.

In chapter 4 we turn to large deviation theory. We give the rigorous foundation of the connection between the stochastic system (2) of section 1.1 and the corresponding optimal control system (3)-(4).

The machinery of the previous two chapters is then applicable. It leads to a new look at global large deviation results, especially when the Markov chain setting of Wentzell and Freidlin is appropriate.

1.2. A Class of Dynamical Systems

1.2.1. Attracting Sets

Let K be a compact, connected, q-dimensional submanifold of $\mathbf{R}^n$, $q<n$. Define the distance function to K:

$$d(x,K)=\inf_{y\in K}|x-y|$$

Since K is compact, d is well-defined and continuous. Now define, for $\varepsilon>0$, the set:

$$N_\varepsilon^K=\{x\in \mathbf{R}^n | d(x,K)<\varepsilon\}$$

For ε small enough, we can take this to be an ε- *tubular neighborhood* of K in $\mathbf{R}^n$.

More precisely, a *tubular neighborhood* of the submanifold K of $\mathbf{R}^n$ is a pair (f,B) , where $\mathrm{B}=(p,E,K)$ is a vector bundle over K and f is an embedding of E in $\mathbf{R}^n$ such that:

(a) f restricted to K is the identity map

(b) $f(E)$ is an open neighborhood of K in $\mathbf{R}^n$.

We shall, by abuse of notation, refer to $f(E)$ as the tubular neighborhood of K (see Hirsch [Hi] for details).

Note that the fibre over any $x\in K$ can be taken to be the normal space to $T_xK = \mathrm{N}_xK$ which, in $\mathbf{R}^n$, is identified with $(T_xK)^\perp$ (see the proof of Theorem 5.1 in [Hi]). In this case, (f,B) is called a *normal tubular neighborhood* (n.t.n.) of K and we can take $f(E)$ to be the set N_ε^K, for some small ε.

We use these neighborhoods as the open sets in the definition of an attracting set.

Definition 1:

The manifold K is an **attracting set** *if: given a normal tubular neighborhood N_δ^K, $\delta>0$ of K, we can find a $\varepsilon>0$ and a n.t.n. N_ε^K such that:*

(i) for all $x\in N_\varepsilon^K$, $\phi_t x\in N_\delta^K$ for all $t\geq 0$.

(ii) for all $x \in N_\varepsilon^K$, $d(x,K) \to 0$ *as* $t \to 0$.

Call $A(b)$ the union of the attracting sets of the vector field b: $A(b)=\bigcup K$.

Remarks:

1) If $K=\{x\}$, we are back to the definition of an asymptotically stable equilibrium. If K is a limit cycle (the only possible 1-dimensional compact,connected manifold), then we have defined an asymptotically stable limit cycle.

2) In higher dimensions, when $q \geq 0$, the attracting set contains more than one orbit of b (an example is a two-dimensional torus). We make no assumptions on the behavior of b on K. In particular, the flow of b may be conservative when restricted to K. However, we require K to be of dimension strictly less than that of the state space, as should be the case for dissipative systems (see Birkhoff [B]).

3) If $K=\{x\}$ is a hyperbolic attractor, then it is certainly an attracting set, by the Grobman-Hartman theorem.

Definition 2:

The **region of attraction** R_K *of the attracting set K is defined as the set of points* $x \in \mathbf{R}^n$ *satisfying:*

(i) $d(\phi_t x, K) \to 0$ *as* $t \to +\infty$.

(ii) $x \notin K$.

Note that for all $x \in R_K$, $b(x) \neq 0$. We have:

Lemma 1:

If $x \in R_K$, *the orbit of* x : $\{\phi_t x \ , \ t \in \mathbf{R}\}$ *belongs to* R_K. *Also,* R_K *is an open set and there is a normal tubular neighborhood of K contained in it.*

Proof:

If $x \in R_K$ and $y=\phi_{t'}x$ for some t',

$$d(\phi_t y, K) = d(\phi_{t+t'} x, K)$$

and the first part follows. Now let N_ε^K, N_δ^K be as in definition 2.1. If $x \in R_K$, there is a time $T>0$ such that $\phi_T x \in N_\varepsilon^K$. Since ϕ_T is a diffeomorphism, we can find a neighborhood U of x such that $\phi_T U \subset N_\varepsilon^K$. It is now clear that for all $y \in U$, $y \in R_K$ and hence R_K is open.

Finally, it is obvious that the normal tubular neighborhood N_ε^K is contained in R_K. ■

1.2.2. The Class of Flows Considered

First we recall some definitions (see [P-D] for details) (throughout, b(x) is a complete vector field and $\phi_t x$ is its flow).

Definition 3:

A point $x \in \mathbf{R}^n$ is said to be **wandering** *if there is a neighborhood V of x and a time T>0 such that: $\phi_t x \cap V = \emptyset$ for $|t|>T$. Otherwise, the point is called* **nonwandering** . *Write $\Omega(b)$ for the set of nonwandering points of the vector field b, for all points outside the set of attracting sets* A(*b*).

Definition 4:

Call a set a **critical element** *if it is a zero of b or a closed orbit (a zero will also be called an equilibrium). A point p is an* α **-limit point** *of the point x if there is a sequence of times $t_k \to -\infty$ such that $\phi_{t_k} x \to p$. It is an* ω **-limit point** *of x if the same holds for a sequence of times t_k going to $+\infty$. If for the point x, $|\phi_t x| \to +\infty$ as $t \to -\infty$ $(+\infty)$ we say ∞ is an α-limit point (ω-limit point) of x. Call $L_\alpha(b)$ and $L_\omega(b)$ the set of α- and ω-limit points of b, for x outside* A(*b*) *and belonging to a bounded orbit.*

Note that if $p \in L_\alpha \cup L_\omega$, then p is a non-wandering point. Thus, for a general flow, $\Omega(b) \supset L_\alpha(b) \cup L_\omega(b)$. Also note that $A(b) \subset L_\omega(b)$.

Definiton 5:

An equilibrium is called **hyperbolic** *if its linearization has no eigenvalues with zero real part. A closed orbit is* **hyperbolic** *if the linearization of its local Poincare map has no eigenvalues of modulus one.*

The stable manifold of a hyperbolic critical element σ is denoted by $W^s(\sigma)$. The unstable manifold of σ by $W^u(\sigma)$. For the existence and properties of the stable and unstable manifolds of critical elements refer to eg. Hirsch et.al.[HPS].

Definiton 6:

The vector field b on $\mathbf{R}^n$ *is a* **dissipative Morse-Smale** *vector field if:*

(i) there is a finite number of attracting sets and a finite number of critical elements, all hyperbolic.

(ii) (a) the set of ω*-limit points in* $\mathbf{R}^n - \mathrm{A}(b)$ *is equal to* $L_\omega(b)$.

(ii) (b) the set of α*-limit points in* $\mathbf{R}^n - \mathrm{A}(b)$ *is equal to* $L_\alpha(b) \cup \{\infty\}$.

(iii) $\Omega(b) = L_\omega(b) \cup L_\alpha(b) \cup \{\infty\}$.

(iv) if σ_1 *and* σ_2 *are critical elements, then* $W^u(\sigma_1)$ *and* $W^s(\sigma_2)$ *are transversal.*

Remark:

It is a consequence of condition (ii) (a) that every *future* trajectory $\{\phi_t x\}_{t\geq 0}$ of a dissipative Morse-Smale vector field is bounded. Something stronger is actually true: almost all points in $\mathbf{R}^n$ have their ω-limit sets in A(b). This is because the stable manifold of a critical element that is not an attractor has dimension strictly less than n and hence has Lebesgue measure 0 in $\mathbf{R}^n$. The *past* trajectories $\{\phi_t x\}_{t\leq 0}$ of b, however, generically go to ∞ or to a critical element of index n.

Definition 7:

Given critical elements or attracting sets σ_1 , σ_2 *introduce the* **relation** $<$ *by:* $\sigma_1 < \sigma_2$ *if there is an orbit* γ *that has* σ_1 *as its* ω*-limit point and* σ_2 *as its* α*-limit set.*

A set of critical elements satisfies the **no-cycle condition** *if we cannot find distinct critical elements* $\sigma_{i_1}, \sigma_{i_2}, .., \sigma_{i_m}$ *such that:* $\sigma_{i_1} < \sigma_{i_2} < .. < \sigma_{i_m} < \sigma_{i_1}$.

Now construct the **orbit diagram** of a dissipative Morse-Smale vector field that satisfies the no-cycle condition as follows:

Distinguish (n+1) levels, according to the index k of a critical element (i.e. the dimension of its unstable manifold). Thus $0<k<n$. The attracting sets are at the index-0 level. List all the critical elements and attracting sets at each level. Include the repellor at ∞ at the index-n level. Call these the **nodes** of the orbit diagram.

Connect node σ to node τ by an arrow pointing from σ to τ if $\tau<\sigma$.

Lemma 2:

$\tau<\sigma$ if and only if index τ < index σ and there is an orbit connecting τ and σ.

Proof:

This follows from transversality, by keeping track of the dimensions of the stable and unstable manifolds of the critical elements. ■

Note that the above lemma shows there are no *homoclinic orbits* (i.e. $\sigma<\sigma$ does not happen).

We are now in a position to define the class of dynamical systems for which the global Lyapunov theory of chapter 2 will be developed.

The Class D($\mathbf{R}^n$):

A complete vector field b on $\mathbf{R}^n$ belongs to the class D($\mathbf{R}^n$) *if:*

(a) b is a dissipative Morse-Smale vector field and

(b) the no-cycle condition is satisfied.

From now on, the vector fields under consideration will be assumed to be in D($\mathbf{R}^n$).

Discussion of the Class D($\mathbf{R}^n$):

D($\mathbf{R}^n$) is a general enough class of dynamics to include many familiar examples of interest: for example, the Josephson junction models away from bifurcation, multi-machine models in power systems etc. The sense in which a vector field in D($\mathbf{R}^n$) is dissipative is consistent with previous attempts to formalize this notion (see G.Birkhoff [B] and, more recently, J.Willems [W]).

As Birkhoff requires ([B],pp.31-32), asymptotically as $t\to\infty$, the motion of a dissipative system takes place close to bounded sets of dimension lower than that of the state space. On such a set, the motion is assumed conservative. In our formulation, the attracting sets are compact manifolds of dimension strictly less than n.

According to Willems, dissipativeness has to do with the existence of a function which decreases with the forward (in time) evolution of the system dynamics. This is captured in our theory by the concept of a global Lyapunov function. Our theory goes further than that in specifying exactly how many such functions we can find and how they are related.

The members of $\mathbf{D}(\mathbf{R}^n)$ are *structurally stable* (see Palis and DeMelo [P-D] for a definition of structural stability). In fact, if we compactify the state space using the point at infinity, we see that the class we defined is very similar to the Morse-Smale vector fields, except for the existence of the more general class of asymptotic attractors.

In Figure 1.2 we give some simple examples of two-dimensional vector fields of class $\mathbf{D}(\mathbf{R}^n)$ and their orbit diagrams.

CHAPTER 2: GLOBAL LYAPUNOV THEORY

The fundamental property of the dissipative dynamical systems we defined in chapter 1 is that we can find hypersurfaces, called Lyapunov Surfaces, that are globally transverse to the flow. In a single region of attraction, we can find Lyapunov surfaces intersecting all orbits in that region.

Moving Lyapunov surfaces along the flow gives Lyapunov Functions. Using the orbit diagram of the vector field, we can then classify all Lyapunov functions according to the order in which they sweep past the saddle critical elements of the flow.

The above construction generalizes in a natural way the results of Morse theory, where the dynamical system is obtained from a gradient vector field.

2.1. Introduction

A Lyapunov function for a dynamical system is a generalization of the energy function of a classical mechanical system. It is a *scalar function* V on the state space with the property that it is non-increasing on trajectories of the system, i.e. $\frac{dV}{dt} \leq 0$. When the system is dissipative, the Lyapunov function is strictly decreasing, $\frac{dV}{dt} < 0$.

In system theory, Lyapunov functions are usually local; in this case the existence of a local Lyapunov function is equivalent to the local stability of an equilibrium. Only if the equilibrium is globally stable is the Lyapunov function also defined globally.

In general, each stable equilibrium is stable in a region of state-space that we call its *region of attraction* . This is the natural domain of definition of a Lyapunov function and in power systems, Lyapunov functions have been used to estimate the region of attraction of an equilibrium.

In the mathematical theory of dynamical systems, on the other hand, global Lyapunov functions are considered and have proved very useful in the study of global dynamics for certain classes of flows (see F.W.Wilson [Wi1],[Wi2], Franks [Fr] and Pugh and Shub [P-S]). In particular, just as Morse theory helps to describe the topological structure of manifolds, global Lyapunov functions are used to describe homological properties of dynamical systems, as in Franks.

Our research was developed with a motivation very different from the above. It generalizes global Lyapunov theory in two directions: first, the attracting sets are allowed to be arbitrary compact manifolds. Second, and more importantly, the class of Lyapunov functions we consider is more general: roughly speaking, the level sets of Lyapunov functions considered before tend, asymptotically, to the boundary of the regions of attraction of the critical elements. In our research, these level sets are transverse to the boundary and hence are not contained in a single region of attraction. As we saw in sections 1.1.2 and 1.1.3, this is more natural since it generalizes directly Morse theory and the gradient flow results.

This chapter is structured as follows: section 2.2 describes Lyapunov surfaces. These are seen to be global analogs of the transverse neighborhoods to the flow obtained from the flow-box theorem (see section 2.2.1). Section 2.3 derives a botany of Lyapunov functions. The most direct kind that follows from the Lyapunov surface concept is a Lyapunov function defined in the region of attraction of a single attracting set. This is the type of Lyapunov function familiar from system theory and power systems.

Using the results of section 2.2.3 on Lyapunov surfaces for saddle critical elements, we are led to global Lyapunov functions whose level sets propagate across saddles in a prescribed order. This allows the classification scheme of section 2.3.2.

2.2. Existence of Lyapunov Surfaces

2.2.1. Basic Definitions and Properties

The region of attraction of an attracting set K is foliated in an obvious way by the 1-dimensional orbits of the vector field b: each x in R_K belongs to a unique orbit and the vector field is non-singular in all of R_K.

The global Lyapunov theory we are developing gives foliations dual to the above: there are foliations of R_K with leaves which are (n-1)-dimensional submanifolds (hypersurfaces), which we call Lyapunov surfaces and which are transverse to the 1-dimensional foliation by the orbits of b.

The construction of Lyapunov surfaces has two steps: first, relying on the flow-box theorem, we get local Lyapunov surfaces. Then, the properties of the attracting set K are used to patch together the local surfaces to obtain a Lyapunov surface intersecting all orbits of b in R_K.

Definition 1:

A **Lyapunov surface** S *for the vector field b is a hypersurface of* $\mathbf{R}^n$, *bounded as a subset of* $\mathbf{R}^n$, *with the property that at all points* $x \in S$*:*

$$\langle b(x)\rangle + T_x S = T_x \mathbf{R}^n$$

(i.e. S *is transverse to the flow of b) and such that each orbit of b intersects* S *at most once.*

We shall be interested in Lyapunov surfaces that are contained in R_K, for $K \in A(b)$. In this case we have:

Definition 2:

A Lyapunov surface S *is* **complete for the attracting set** *K if* S *is contained in* R_K *and if all orbits of b in* R_K *intersect* S *(exactly once).*

A Lyapunov surface may have more than one connected component. Each component of S has an orientation induced on its normal bundle NS by the flow: we say that the vector $n(x) \in N_x S$ points *inwards* if $b(x) \cdot n(x) > 0$.

Every non-singular point x of the vector field b has a Lyapunov surface locally: just find a hypersurface whose tangent space at x is transverse to $\langle b(x) \rangle$; then a neighborhood of x on the hypersurface is transverse to the vector field by openness of transversality.

We shall need the above result in a stronger form, where we simultaneously rectify the vector field around x. This is given by the flow-box theorem (for a proof, see Arnol'd [Ar],p.227):

Theorem 1 (Flow-Box):

Let b be a C^r vector field ($r \geq 1$). Let $x \in \mathbf{R}^n$ be such that $b(x) \neq 0$. Then there is a C^r-diffeomorphism ψ mapping a neighborhood U of x onto an open ball $B_\delta(0) \subset \mathbf{R}^n$, with $\psi(x)=0$ and the vector field b to the constant vector field:

$$\psi_* b(\psi^{-1}(y)) = e_1(y) \quad , y \in B_\delta(0)$$

where $\{e_1, \ldots, e_n\}$ is a standard Euclidean basis for $T_0\mathbf{R}^n$.

Corollary:

Let $b(x) \neq 0$; then there exists a Lyapunov surface containing x.

Proof:

Let $\overline{V} = \{y \in B_\delta(0), y_1 = 0\}$ where $B_\delta(0)$ is as above. Then $V = \psi^{-1}(\overline{V})$ is the desired Lyapunov surface. ■

A basic property of Lyapunov surfaces is that they can be moved along the flow in an arbitrary manner, while remaining transverse to the flow. We can use the diffeomorphisms ϕ_t of the flow or general maps that move each point on the Lyapunov surface by a variable amount along its orbit.

We summarize the above in the two basic lemmas:

Lemma 1:

Let S *be a Lyapunov surface and* $t\in\mathbf{R}$ *be given. Then the image of* S *under* ϕ_t *is again a Lyapunov surface.*

Lemma 2:

Let a *be a smooth, real-valued function on* S. *Define the map* χ *from* S *to* R_K *by:*

$$\chi(x)=\phi(a(x),x)$$

Then the image of S *under* χ *is again a Lyapunov surface.*

Proof of Lemma 1:

Let i be the embedding map of S. Then $\phi_t \circ i$ is the embedding map of $\phi_t(S)$ and it is clear that $\phi_t(S)$ is a hypersurface, since ϕ_t is a diffeomorphism.

To prove transversality, choose a basis $\{e_1(x),...,e_{n-1}(x)\}$ for T_xS , viewed as a subspace of $T_x\mathbf{R}^n$, for $x\in S$. Since $b(x)$ is trasverse to S , $\{e_1(x),...,e_{n-1}(x),b(x)\}$ is a basis for $T_x\mathbf{R}^n$. The vectors $\{(T_x\phi_t)(e_1(x)),...,(T_x\phi_t)(e_{n-1}(x)),(T_x\phi_t)(b(x))\}$ are linearly independent since $T_x\phi_t$ is an isomorphism. Furthermore, we have the identity:

$$(T_x\phi_t)b(x)=b(\phi_t x)$$

It suffices to show that $(T_x\phi_t)(e_i(x))\in T_{\phi_t(x)}\phi_t(S)$. Since:

$$T_p(\phi_t \circ i)=T_{i(p)}\phi_t \circ T_p i \ , \ p\in\mathbf{R}^{n-1} \ , \ i(p)=x\in S$$

if $e_i(x)=(T_p i)v$ for some $v\in T_p\mathbf{R}^{n-1}$, we have:

$$T_p(\phi_t \circ i)(v)=T_{i(p)}\phi_t \circ T_p i(v)=$$

$$=T_{i(p)}\phi_t \circ e_i(x)=(T_x\phi_t)(e_i(x))$$

and so $(T_x\phi_t)(e_i(x))$ is in $T_p(\phi_t \circ i)\mathbf{R}^{n-1}=T_{\phi_t(i(p))}\phi_t(S)$. ∎

Proof of Lemma 2:

We distinguish two cases:

$$\text{(i) } \chi(x)\neq 0 \text{ and (ii) } \chi(x)=0.$$

Case (i): Pick a neighborhood U of x in S such that χ is non-zero on U. Rescale the vector field b locally:

$$b'(z)=\chi(y)b(z)\ ,\ y\in U\ ,\ z\in \text{orbit of } y$$

It is easily seen that the map defined in the lemma is the map $\phi'_1|_U$ where ϕ' is the flow of b'. By Lemma 1, $\phi'_1 U$ is transverse to b' and therefore to b.

Case (ii): Pick a neighborhood U of x and a time τ such that $\tau > \sup_{x\in U}\chi(x)$. $\phi_\tau U$ is tranverse to b at x, by Theorem 1. If we define $\chi'(z)=\chi(\phi_{-\tau}z)-\tau$ for $z\in\phi_\tau U$, then $\chi'(z)\neq 0$ and $\phi(\chi(x),x)=\phi(\chi'(\phi_\tau x),\phi_\tau x)$. We are now back to case (i). ■

2.2.2. Complete Lyapunov Surfaces for Attracting Sets

Consider the equivalence relation ~ in $\mathbf{R}^n$:

$$x \sim y \Leftrightarrow \exists\, t \in \mathbf{R} \text{ such that } y = \phi_t x$$

In the region of attraction R_K of an attracting set K, this equivalence will yield a quotient space Q that is a compact manifold and is diffeomorphic to any complete Lyapunov surface for K. As a result, R_K is diffeomorphic to $Q \times \mathbf{R}$ and Lyapunov functions are easily obtained from functions on $Q \times \mathbf{R}$.

Theorem 2: (Existence of Complete Lyapunov Surfaces for Attracting Sets)

Let K be an attracting set for the vector field $b \in \mathbf{D}(\mathbf{R}^n)$. Let an open neighborhood of K be given in R_K.

Then there is a complete Lyapunov surface for K in that neighborhood.

Corollary:

The quotient space Q of R_K under the equivalence relation ~ is a compact manifold. Moreover, R_K is diffeomorphic to the product space $Q \times \mathbf{R}$.

Proof:

There is no loss in generality in assuming that the open neighborhood of K is an ε–normal tubular neighborhood N_ε^K, such that $\overline{N_\varepsilon^K} \subset R_K$. The set $\partial N_\varepsilon^K = \{x : d(x,K) = \varepsilon\}$ is closed and bounded and hence compact. It is in fact a manifold, diffeomorphic to $K \times S^{n-q-1}$ (Where q is the dimension of K).

The equivalence relation ~ in R_K yields the quotient space $Q := R_K/\sim$,which has a manifold structure. This is because each element of Q is an orbit γ of b in R_K: $\gamma = \{\phi_t x, t \in \mathbf{R}\}$ and, by the flow-box theorem, we can find a neighborhood U of x in R_K (since R_K is open) mapped onto $B_\delta(0)$. The equivalence relation in $B_\delta(0)$ is simple: it yields the quotient space $\overline{V}$ of Corollary 2.1 and hence a coordinate map β for a neighborhood of γ (see Fig.2.1) (since $\pi \circ \psi^{-1}$ maps $\overline{V}$ to $\pi(V) = \pi \circ \psi^{-1}(V)$ diffeomorphically and hence the inverse β of $\pi \circ \psi^{-1}$ exists).

The canonical map $\pi:R_K\rightarrow Q$, sending each point to its orbit in Q is locally *onto* since, when transformed via ψ to a flow-box, it can be written as:

$$\pi=\psi^{-1}\circ\pi'\circ\psi$$

where π' is the canonical projection in $\mathbf{R}^n$, taking $(y_1,y_2,..,y_n)$ to $(y_2,..,y_n)$ and is clearly onto.

The equivalence relation ~ is called a *regular* equivalence and Q the *quotient manifold* (see Abraham et.al.[Ab],p173).

Next, we want to show that Q is compact. First, we claim that every orbit of b in R_K intersects ∂N_ε^K. This is because R_K, and therefore $\overline{N_\varepsilon^K}$, contains no α–limit points of b. Suppose, otherwise, that there is an orbit $\gamma=\{\phi_t x, t\in\mathbf{R}\}$ that is contained in N_ε^K. Define $\xi_i:=\phi_{-i}x$, $i\geq 0$; the sequence $\{\xi_i\}_{i=1}^{\infty}$ has a limit point in $\overline{N_\varepsilon^K}$ because $\overline{N_\varepsilon^K}$ is compact. This is a contradiction, since $\overline{N_\varepsilon^K}$ contains no α–limit points. All points of R^K tend to K as $t\rightarrow+\infty$ and they are eventually in $\overline{N_\varepsilon^K}$. On the other hand, the orbits of all points of $\overline{N_\varepsilon^K}$ intersect ∂N_ε^K. Thus, all orbits in R_K hit ∂N_ε^K.

If we restrict the canonical projection map to ∂N_ε^K we get, by the above, that $\pi(\partial N_\varepsilon^K)=Q$. Since ∂N_ε^K is a smooth manifold and π a smooth map, we get that the image of the continuous map $\pi|_{\partial N_\varepsilon^K}$ is compact, because ∂N_ε^K is. Thus Q is compact (see Munkres [Mu], p.167).

An open cover of Q is obtained as follows: first, find another neighborhood $N_{\overline{\varepsilon}}^K$ such that $\phi_t(N_{\overline{\varepsilon}}^K)\subset N_\varepsilon^K$ for all $t\geq 0$. For all points $x\in N_{\overline{\varepsilon}}^K$, find a neighborhood U_x in $N_{\overline{\varepsilon}}^K K$ and a local Lyapunov surface V_x as in Cor.2 that is mapped diffeomorphically to $\pi(V_x)$, a neighborhood of $\pi(x)$ in Q. The $\pi(V_x)$ cover Q:

$$Q=\bigcup_{x\in N_{\overline{\varepsilon}}^K}\pi(V_x)$$

Since Q is compact, we can find a finite subcover:

$$Q=\bigcup_{i=1}^{m}\pi(V_i)$$

where the V_i are neighborhoods of the points x_i , $i=1,..,m$.

This finite cover will be used to get a *global section* of the quotient $\pi: R_K \to Q$, i.e. a smooth map $s: Q \to R_K$ such that $\pi(s(x))=x$ on Q.

Note that we already have *local* sections (sections defined on open subsets of Q). These are obtained (see Fig.2.1) by mapping back to R_K, using the rectifying diffeomorphism ψ_i, the neighborhoods of the x_i:

$$s_i = \psi_i^{-1} \circ \beta_i$$

$$s_i: \pi(V_i) \to V_i \subset R_K$$

To get the global section, we need to patch together the local ones; this is accomplished using a *partition of unity* subordinate to the open sets $\{\pi(V_i), i=1,..,m\}$ and the following *transition maps :*

On a non-empty intersection $\pi(V_i) \cap \pi(V_j) \neq \emptyset$ the map:

$$\gamma_{kj}: s_j(\pi(V_j) \cap \pi(V_k)) \to \mathbf{R}$$

sending x to:

$$\gamma_{kj}(x) = \inf_{t \in \mathbf{R}} \{ |t| : \phi_t x \in V_k \}$$

is smooth and satisfies:

$$s_k(p) = \phi(\gamma_{kj}(s_j(p)), s_j(p)) \ , \ \ p \in \pi(V_j) \cap \pi(V_k)$$

When $j=k$, we have: $\gamma_{jj}(p)=0 \ \forall j$. On triple intersections we have the consistency condition: $\gamma_{li}(s_i(p)) = \gamma_{lk}(s_k(p)) + \gamma_{ki}(s_i(p))$. *

Let $\{\alpha_i, i=1,..,m\}$ be the functions that define the partition of unity (i.e. $supp \alpha_i \subset \pi(V_i)$ and $\sum_{i=1}^{m} \alpha_i(p) = 1, \forall p \in Q$). The expression for the global section s can now be given on each open set $\pi(V_i)$:

* proved using the group property of the flow:

$$s_l = \phi(\gamma_{lk}(s_k), s_k) = \phi(\gamma_{lk}(s_k), \phi(\gamma_{ki}(s_i), s_i)) = \phi(\gamma_{lk}(s_k) + \gamma_{ki}(s_i), s_i)$$

and on the other hand $s_l = \phi(\gamma_{li}(s_i), s_i)$. Comparing, we get the desired result.

$$s(p)=\phi\left(\sum_{j=1,j\neq i}^{m}\alpha_j(p)\gamma_{ji}(s_i(p)),s_i(p)\right) \quad \text{for } p\in\pi(V_i)$$

We claim that this gives a consistently defined s on all of Q. To show this, take a non-empty intersection $\pi(V_i)\cap\pi(V_k)\neq\emptyset, i\neq k$. We have two expressions for $s(p)$ for points p in the intersection:

$$s(p)=\phi(\sum\alpha_j(p)\gamma_{ji}(s_i(p)),s_i(p))$$

$$s(p)=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),s_k(p)) \qquad \text{(b)}$$

We must show that the expressions (a),(b) for $s(p)$ are equal:

$$(b)=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),s_k(p))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p)),\phi(\gamma_{ki}(s_i(p)),s_i(p)))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{lk}(s_k(p))+\gamma_{ki}(s_i(p)),s_i(p))=$$

substituting for γ_{lk} using the consistency relation:

$$=\phi(\sum\alpha_l(p)[\gamma_{li}(s_i(p))-\gamma_{ki}(s_i(p))]+\gamma_{ki}(s_i(p)),s_i(p))=$$

$$=\phi(\sum\alpha_l(p)\gamma_{li}(s_i(p))+(-\sum\alpha_l(p)+1)\gamma_{ki}(s_i(p)),s_i(p))=$$

and the result follows from the partition of unity:

$$=\phi(\sum\alpha_l(p)\gamma_{li}(s_i(p)),s_i(p))=(a)$$

It is clear that s is smooth and one-to-one. It remains to show that the image of Q under s is *transverse* to the flow. We know the local sections are transverse to b by construction of the local Lyapunov surfaces V_i. On each V_i, s modifies the local section s_i using the flow ϕ and a smooth function f that moves each point of $s_i(\pi(V_i))$ along the orbit. Here:

$$f(s_i(p))=\sum_j\alpha_j(p)\gamma_{ji}(s_i(p))$$

Therefore, Lemma 2 applies and the modified section is also transverse to the flow. □

Proof of Corollary:

It has been proved that Q is a compact manifold. To show R_K is diffeomorphic to $Q \times \mathbf{R}$, assume given a global section s, constructed as in the previous proof. Then, the flow ϕ gives the required diffeomorphism $\overline{\phi}$; we have:

$$\overline{\phi}: Q \times \mathbf{R} \rightarrow R_K$$

$$(p,t) \mapsto \phi(t, s(p))$$

and $\overline{\phi}$ is obviously smooth.

It is onto, since, for any $y \in R_K$, we can find a $t' \in \mathbf{R}$ such that $\phi_{t'} y = x' \in S$; then $(-t', p')$ goes to y, where $s(p') = x'$.

It is one-to-one since if $\overline{\phi}(t,p) = \overline{\phi}(t', p')$ we must have $p = p'$ since s is one-to-one and orbits do not intersect. Then $t = t'$, by uniqueness of solutions to the flow. □

2.2.3 Lyapunov Surfaces of Critical Elements

Before we can define global Lyapunov functions, we must discuss Lyapunov surfaces for critical elements.

A critical element σ is a **repellor** if its stable manifold is trivial. By reversing the flow direction, a repellor becomes an attracting set. Thus we have:

Lemma 3:

If σ is a repellor, there is a complete Lyapunov surface for σ in any given neighborhood of it.

A critical element σ is called a **saddle** if it has non-trivial stable and unstable manifolds. We distinguish **saddle equilibria** and **saddle orbits** . A saddle's region of attraction is its stable manifold, which is not an open subset of $\mathbf{R}^n$; thus there is no concept of complete Lyapunov surfaces for saddles. However, the 'future' orbits of points close to the stable manifold pass close to the unstable manifold of σ. We are therefore interested in Lyapunov surfaces in neighborhoods of the stable and unstable manifolds. We give separately the cases of σ being an equilibrium and a closed orbit.

Saddle Equilibria:

Let σ be a hyperbolic equilibrium of index $k, 0<k<n$. $W^u(\sigma)$, its unstable manifold, is k-dimensional and $W^s(\sigma)$, the stable manifold, is (n-k)-dimensional. The two manifolds are invariant under b and intersect transversely at σ:

$$T_\sigma W^u(\sigma) \oplus T_\sigma W^s(\sigma) = T_\sigma \mathbf{R}^n$$

As we saw in the previous section, on $W^s(\sigma)$ we can find a complete Lyapunov surface $S^s(\sigma)$ for σ as an attracting set on $W^s(\sigma)$. Similarly, we can find a complete Lyapunov surface $S^u(\sigma)$ on $W^u(\sigma)$, since σ is a repellor on $W^u(\sigma)$.

Consider the normal bundle $NW^s(\sigma)$ and its restriction to S^s, $NW^s(\sigma)|_{S^s}$. We can find a tubular neighborhood of S^s in $NW^s(\sigma)|_{S^s}$ of size ε; this means that all points on the neighborhood are a distance less than ε from S^s. This neighborhood $N_\varepsilon(S^s)$ can be made transverse to the flow, since S^s is transverse and it is also compact (since transversality holds on a neighborhood of any given point, we cover S^s with finitely many such neighborhoods and make ε small enough so that $N_\varepsilon(S^s)$ is inside the union of these neighborhoods). A similar construction gives a transverse neighborhood of S^u, $N_{\varepsilon'}(S^u)$ in $NW^u(\sigma)|_{S^u}$. The two neighborhoods are then Lyapunov surfaces for b. Note, finally, that we can assume that they are disjoint, by making them sufficiently small.

Saddle Orbits:

Let σ be a hyperbolic closed orbit. It has a k-dimensional unstable manifold $W^u(\sigma)$, $(0<k<n-1)$ and a $(n-k-1)$-dimensional stable manifold $W^s(\sigma)$. We have the splitting: $T_x\mathbf{R}^n = T_xW^u(\sigma)+T_xW^s(\sigma)$ where $T_xW^u(\sigma)\cap T_xW^s(\sigma) = \langle b(x)\rangle$.

The construction above for equilibria extends to the present case with minor changes.

2.3 Lyapunov Functions

We are now ready to define Lyapunov functions globally. First, we find Lyapunov functions defined in a region of attraction of an attracting set K. This is a simple application of the existence theory of section 2.2. Then, Lyapunov functions on the whole of $\mathbf{R}^n$ are obtained using the propagation of Lyapunov surfaces past saddle critical elements which is given in section 2.3.2.

2.3.1. Lyapunov Functions from Complete Lyapunov Surfaces

Corollary 2.1 establishes that, for an attracting set K, its region of attraction R_K is diffeomorphic to $Q\times\mathbf{R}$, where Q is the quotient manifold $R_K/\sim$ under the regular equivalence of belonging to the same orbit. The diffeomorphism $\overline{\phi}:Q\times\mathbf{R}\rightarrow R_K$ can then be used to map functions on $Q\times\mathbf{R}$ to Lyapunov functions on R_K.

Definition 3:

A continuous function V defined on an open subset $U\subset\mathbf{R}^n$ is a **Lyapunov function** *for the vector field $b\in\mathbf{D}(\mathbf{R}^n)$ if:*

(i) V is constant on each attracting set and on each critical element,

(ii) V is smooth on $U-\Omega(b)$ and

(iii) $dV(b)<0$ everywhere on $U-\Omega(b)$ (alternatively, if $y=\phi_t x, t>0$, then $V(y)<V(x)$).

Fix an attracting set K. Consider the following class of functions $\mathbf{L}(K)$ on $Q_K\times\mathbf{R}$:

$$\mathrm{a}\in\mathbf{L}(K) \quad \textit{iff}$$

(i) a is smooth and maps $Q_K\times\mathbf{R}\rightarrow\mathbf{R}$: $(p,\tau)\rightarrow \mathrm{a}(p,\tau)$.

(ii) for every $p\in Q_K$,$\mathrm{a}(p,.)$:$\mathbf{R}\rightarrow\mathbf{R}$ is a diffeomorphism and let us assume that $\frac{\partial \mathrm{a}}{\partial\tau}(p,\tau)>0$.

Also consider a smooth strictly monotone increasing function v:$\mathbf{R}\rightarrow\mathbf{R}_+$.

Theorem 3:

To every $\mathbf{a} \in \mathbf{L}(K)$ *and* v *as above, there corresponds a Lyapunov function* V *defined on* R_K*. Conversely, if* V *is a Lyapunov function on* R_K*, we can find functions* $\mathbf{a}$ *of class* $\mathbf{L}(K)$ *and* v *as above such that* V *is obtained from them.*

Proof:

For each $\tau \in \mathbf{R}$, $\mathbf{a}(.,\tau)$ is a smooth function from Q_K to $\mathbf{R}$ and as τ varies, we foliate $Q_K \times \mathbf{R}$ with the graphs of the functions $\mathbf{a}(.,\tau)$. The map:

$$Q_K \times \mathbf{R} \to Q_K \times \mathbf{R}$$

$$(p,\tau) \mapsto (p, \mathbf{a}(p,\tau))$$

is a diffeomorphism which we call $\overline{\mathbf{a}}$.

We also have the diffeomorphism $\overline{\phi}$ mapping $Q_K \times \mathbf{R}$ to R_K. Now simply define, for $x \in R_K$:

$$V(x) = v \circ \pi_\tau \circ \overline{\mathbf{a}}^{-1} \circ \overline{\phi}^{-1}(x)$$

where π_τ is the projection $(p,\tau) \to \tau$. V is obviously a smooth function from R_K to R_+.

To check that it is a Lyapunov function, note that the level sets $\{V = constant\}$ are the images under $\overline{\phi}$ of $\mathbf{a}(.,\tau)$ for each τ. These, as we can see from Lemma 2.2, are transverse to the flow and hence are complete Lyapunov surfaces. □

2.3.2 Classification of Global Lyapunov Functions

In section 2.2.2 Lyapunov surfaces were obtained for saddle critical elements in neighborhoods of their stable and unstable manifolds. The flow of the vector field can be used to propagate the Lyapunov surfaces near the stable manifold to the Lyapunov surface near the unstable one. With this technique we can then discuss global Lyapunov functions whose level sets intersect more than one region of attraction and are defined globally, on the whole of $\mathbf{R}^n$.

Propagation of Lyapunov Surfaces past a Saddle:

The flow close to the unstable manifold of a saddle σ passes near the saddle and then stays close to the stable manifold of σ (see Fig.2.2). This allows us to define a diffeomorphism between the "punctured" Lyapunov surfaces $N(S^s)-S^s$ and $N'(S^u)-S^u$. The Lyapunov surfaces S^s and S^u have to be removed since W^s and W^u are invariant under the flow.

Theorem 4:

Let σ be a saddle critical element. We can find punctured Lyapunov surfaces $N(S^s)-S^s$ and $N'(S^u)-S^u$ and a smooth positive real function $a:N(S^s)-S^s \to \mathbf{R}_+$ such that $N'(S^u)-S^u$ is the diffeomorphic image of $N(S^s)-S^s$ under the map:

$$x \to \phi(a(x),x)$$

Proof:

We do separately the cases of σ an equilibrium and a closed orbit.

Case 1: Saddle Equilibrium:

Let σ have index $k, 0<k<n$. It is known ([P-D]) that there is a homeomorphism h that takes a neighborhood U of σ to an open set of $\mathbf{R}^n$, sending σ to O and on which the push-forward h^*b of the vector field b is the simple linear vector field:

$$\dot{\xi}=\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}\xi$$

The general solution of the above linear equation is:

$$\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}=\begin{bmatrix}e^t\xi_1(0)\\ e^t\xi_2(0)\end{bmatrix}\ ,\ \xi_1(0)\in \mathbf{R}^k\ ,\ \xi_2(0)\in \mathbf{R}^{n-k}$$

We can assume that the neighborhoods of S^s and S^u found in section 2.2.2, $N_\varepsilon(S^s)$ and $N_{\varepsilon'}(S^u)$ are in U and do not intersect. Also suppose that the image under h of U is a ball around 0, $h(U)=B_\eta(0)$, $\eta>0$.

If we have an initial condition close to S^s, i.e. $\xi_2(0)$ is close to S^s and $|\xi_1(0)|$ is non-zero and small relative to $|\xi_2(0)|$, then we can find a time $T>0$ such that $e^{-T}\xi_2(0)$ is arbitrarily smalll and $e^T\xi_1(0)$ is larger than η.

Thus it is clear that if $\xi_1(0)$ is sufficiently small, the orbit of $\begin{bmatrix}\xi_1(0)\\ \xi_2(0)\end{bmatrix}$ intersects $h(N_{\varepsilon'}(S^u)-S^u)$. It also follows that for the linear vector field, a small punctured neighborhood of $h(S^u)$ is mapped by the flow onto a small neighborhood of $h(S^s)$. Thus $h(N(S^u)-S^u)$ is mapped by the homeomorphism h onto a neighborhood $h(N'(S^s)-S^s)$ of $h(S^u)$.

Having established that there is a continuous 1-1 and onto map between the two neighborhoods, we can map back using h and get a diffeomorphism between them in $\mathbf{R}^n$, by the flow diffeomorphism $\phi(t,x)$.

Case 2: Saddle Closed Orbit:

Let σ have index k , $0<k<n-1$.

The Poincare map P_Σ on a local Lyapunov surface Σ of a point $x\in\sigma$ is locally equivalent to one of the following maps (see Theorem 5.5 of [P-D],p.72):

$$A_k^1=\begin{bmatrix}B_+ & 0\\ 0 & C_+\end{bmatrix}\ ,\ A_k^2=\begin{bmatrix}B_- & 0\\ 0 & C_+\end{bmatrix}$$

$$A_k^3=\begin{bmatrix}B_+ & 0\\ 0 & C_-\end{bmatrix}\ ,\ A_k^4=\begin{bmatrix}B_- & 0\\ 0 & C_-\end{bmatrix}$$

where $B_+=\frac{1}{2}I_k$ where I_k is the k-dimensional identity matrix, $C_+=2I_{n-1-k}$ and B_- and C_- differ from B_+ and C_- only in that their (1,1) elements have a negative sign. Thus, $A_k^i\in \mathbf{R}^{(n-1)\times(n-1)}$, i=1,2,3,4.

The solution of the discrete system:

$$\xi_{j+1}=A^i_k\xi_j$$

is $\xi_j=(A^i_k)^j\xi_0$. In particular, there are contracting and expanding directions as in the case if the linear flow above. And just as in case 1, a point close to the stable manifold of σ on Σ gets mapped close to the unstable manifold of σ on Σ (see Fig.2.3). Reasoning as in the previous case, we get a diffeomorphism between the sets $\Sigma\cap(N(S^s)-S^s)$ and $\Sigma\cap(N'(S^u)-S^u)$. □

Remark:

In Morse theory, one is interested in describing how the maps given above for saddle elements contribute to the relative cohomology of the level sets of Morse functions. Although the homological aspects of this transformations of Lyapunov surfaces around saddles is interesting and may lead to a deeeper understanding of optimal control for dynamics of class $\mathbf{D}(\mathbf{R}^n)$ (see chapter 3), we shall leave the topic for future research.

Classification of Global Lyapunov Functions:

First, we motivate the basis for our classification method by an example. Consider a vector field in the plane with the phase portrait of Fig.2.5(a) (its orbit diagram is shown in Fig.2.5(b)). There are three attractors and two saddles. In Figs.2.5(c) and (d) we show two possible global Lyapunov functions by plotting a few of its Lyapunov surfaces. In chapter 3 we shall see how for some optimal control problems (and related large deviation problems) a Lyapunov function is the optimal cost functional starting from a given attractor. It is easy to see that in case (c) optimal exit is from saddle s_2 while in case (d) it is from saddle s_1, since it is less costly to take those paths (one should view the Lyapunov surfaces as isocost surfaces).

We would like a classification scheme that differentiates between these two types of Lyapunov function on the basis of the qualitative feature that exit paths go through different critical elements. The aim is then to generalize this to all vector fields of class $\mathbf{D}(\mathbf{R}^n)$.

In order to get different kinds of Lyapunov functions than those obtained in section 2.2 we need to abandon complete Lyapunov surfaces. In particular, we will use Lyapunov surfaces that are global sections of Q_K but with the unstable manifolds of some critical elements removed (Fig.2.4).

A global Lyapunov function is always increasing as we move on a stable manifold from the ω-limit set to the α-limit set of the manifold. The converse holds for an unstable manifold. As we move on a chain of manifolds starting from an attractor, we want to define a Lyapunov function consistently. To do this, we have imposed the no-cycle condition in Chapter 1. The best way to book-keep the different possible global Lyapunov functions is to look at the *orbit diagram* of b. There, we can classify all the global Lyapunov functions by specifying an order on the set of critical elements of each index level. We now make precise the above ideas.

Definition 4:

Let σ be a critical element of index k , $k>0$ such that $W^u(\sigma)\cap R_K\neq\emptyset$, where R_K is the region of attraction of some $K\in A(b)$. We call σ a **k-ancestor** *of K.*

Similarly, if σ_1 and σ_2 are critical elements with $k_1<k_2$ and are such that $W^u(\sigma_2)\cap W^s(\sigma_1)\neq\emptyset$, we call σ_2 a **k-ancestor** *of σ_1 and σ_1 a* **k-descendant** *of σ_2.*

Let the orbit diagram of the vector field $b\in D(\mathbf{R}^n)$ have $l_0>0$ attracting sets and $l_k\geq0$ critical elements of index k ($1\leq k\leq n$) (all numbers are finite). In numbering critical elements and attractors, we shall use superscripts to denote *index* and subscripts for *ordering*. Let $m_i^{k_1k_2}$ be the number of k_2-ancestors of the critical element (or attractor) $\sigma_i^{k_1}$ of index k_1.

Define the sets:

$$N=\{1,2,..,n\}$$

$$c^k=\{\sigma_1^k,\sigma_2^k,..,\sigma_{l_k}^k\}\ ,\ k\in N$$

$$v^{k_2}(\sigma_i^{k_1})=\{\sigma_1^{k_1k_2},\sigma_2^{k_1k_2},..,\sigma_{m_i^{k_1k_2}}^{k_1k_2}\}\ ,\ k_1,k_2\in N\ ,\ k_2>k_1\ ,\ \sigma_i^{k_1}\in c^{k_1}$$

$$\nu(\sigma_i^{k_1})=\bigcup_{k>k_1}\nu^k(\sigma_i^{k_1})$$

c^k is the set of critical elements of b of index k while $\nu_i^{k_i k}$ counts the k-ancestors of σ_{k_i}. Finally, $\nu_i^{k_i}$ is the set of all ancestors of $\sigma_i^{k_i}$. Note that:

$$l_k=\sum_{k_1=0}^{k-1}\sum_{i=1}^{l_{k_1}} m_i^{k_1 k} \quad , k>0$$

Since an orbit diagram, considered as a set with the order relation < is not even a partially-ordered set, we shall need to proceed inductively. This will require examining generalized orbit diagrams, where the nodes are not single attracting sets or critical elements but groups of them. The price we pay in added complexity of the relevant statements is offset by the sharpness of the results. To see this we first remark that a given global Lyapunov function orders *all* members of $\Omega(b)=L_\alpha(b)\cup L_\omega(b)$:

Remark:

Suppose a global Lyapunov function V is given for the vector field $b\in \mathbf{D}(\mathbf{R}^n)$. *Then the members of* $\Omega(b)$ *(attracting sets and critical elements) and hence the sets* c^k *and* $\nu(\sigma)$ *for all* $\sigma\in\Omega(b)$ *are ordered in a unique way by the Lyapunov function: given any two* σ_i , $\sigma_j\in\Omega(b)$, $\sigma_i<(\leq)\sigma_j$ *if and only if* $V(\sigma_i)<(\leq)V(\sigma_j)$.

We now arrive at the general scheme by giving a number of steps that will lead to the desired classification theorems.

Step 1: Identify *clusters* $\{C_i\}_i$ of nodes. These are subsets such that if $\sigma\in C_i$ of index k, then there is a path down from σ to K, different from itself, passing nodes of the same cluster and each C_i is connected as a graph.

Step 2: For each attractor, look at the set $\nu(K)$, $K\in A(b)$. *Discard* any node σ that is such that there is a chain of nodes (other than itself) joining σ to K. (This is because a Lyapunov function strictly increases as we move up a branch of the orbit diagram; e.g. in Fig.2.6 we cannot ever have

$V(\sigma^1) > V(\sigma^2)$.)

Step 3: Call the nodes in $v(K)$ remaining after the discarding process in each cluster a *proper-ancestor set (p.a. set).* Fix orderings on the proper-ancestor sets of all attractors that is *consistent.* This means that if C_1 and C_2 are proper-ancestor sets of two attractors such that $C_1 \cap C_2 \neq \emptyset$, then the intersection the two orderings coincide.

Step 4: After discarding elements of each cluster and ordering *select* the first node for each p.a. set of each attractor. This is the first saddle critical element to be swept by Lyapunov surfaces starting from each R_K. For every attractor look at all attractors that adjoin it by these selected saddle elements. Order the attractors by looking at the order of the selected first elements (there may be more than one first attractors). We have the Lemma:

Lemma 4:

> *Consider the set* $K_\sigma = (\bigcup K_i) \cup W^u(\sigma) \cup \sigma$ *composed of a saddle critical element that is ordered first according to steps 1-4 above, its unstable manifold and attractors that are connected to* σ *in the orbit diagram. Then* K_σ *is an attracting set (with boundary) and hence has Lyapunov functions defined in its region of attraction.*

The proof follows from the results on the propagation of Lyapunov surfaces past saddles. It consists of patching together the neighborhoods that yield a complete Lyapunov surface (see proof of existence theorem) except the ones that intersect $W^u(\sigma)$ with a local Lyapunov surface $N(W^u(\sigma)) - W^u(\sigma)$.

Step 5: Consider the sets K_σ (by convention, $K_\emptyset$ means that attractor K was not ordered first in step 4. There is a *generalized orbit diagram* associated with this new set of attractors. Its nodes are the sets K_σ and the remaining critical elements and the branches are formed in the obvious way: connect node σ_i to node σ_j if there was at least one arrow connecting them in the original orbit diagram.

Step 6: Repeat steps 1-4 for the new diagram.

Step 7: Apply step 5 again and continue with step 6 until all critical elements are taken care of.

Example: In Fig.2.7, we give the steps of this procedure until we exhaust the critical elements of b.

The previous discussion has proved the following fundamental theorem on the classification of global Lyapunov functions:

Theorem 5:

Fix an ordering of the proper-ancestor sets.

Then any two global Lyapunov functions with the same ordering yield the same qualitative behavior of exit paths, i.e. exit from each attractror goes through the same sequence of saddle critical elements for both Lyapunov functions.

Conversely, there exists a global Lyapunov function such that the ordering it induces on the proper-ancestor sets according to its value on the elements of $\Omega(b)$ *coincides with the given one.*

CHAPTER 3: OPTIMAL CONTROL USING LYAPUNOV FUNCTIONS

Large deviation theory leads to certain nonlinear optimal control problems with a quadratic cost-of-control functional and control objective of going against the uncontrolled flow away from an attractor and exiting from its region of attraction.

It turns out that these optimal control problems are sometimes solved by a Lyapunov function: the cost of control is given by the difference in the Lyapunov function values. The condition under which this happens is a form of controllability that relates the control system dynamics to the Lyapunov functions of the uncontrolled dynamics. Along any orbit, this condition is satisfied for the general control system that has at least two controls.

Using Lyapunov functions to solve optimal control problems yields qualitative insight into the geometry of the optimal control paths; for example, exit from a region of attraction occurs from the 'closest' saddle critical element. This insight can be exploited to obtain solutions to specific problems: we do this for the case of the Josephson junction.

3.1. Introduction

In this chapter, we pose and solve optimization problems for a special class of control dynamics and a quadratic cost-of-control functional. It is important to point out that, even though the dynamics considered are nonlinear, we are far from treating the most general case. By thus restricting ourselves, however, we are able to develop a geometric framework that yields substantial results for this class of nonlinear systems.

The key to our treatment is the Hamiltonian approach that leads, in our case, to controls of a feedback form and to a sufficiency theory that is closely related to that of Boltyanskii [Bo] and Young [Y].

The aim of this chapter is to generalize the case of *gradient systems* with *non-singular control matrix* to the case of *dissipative dynamics* with possibly *singular control matrix*. What makes this generalization possible is a form of controllability that relates the control dynamics to Lyapunov functions.

In section 3.2, we present this form of controllability. We give local and global conditions and try to answer the question of which dynamics are controllable. The optimal control setup is given in section 3.3.1 and the main results of this chapter are stated and their significance briefly discussed. Their proof is given in section 3.3.2. Section 3.4 is devoted to applications of the theory. In section 3.4.1 it is shown how the saddle exits can be used as a guide in obtaining solutions to nonlinear optimal control problems and the Josephson junction is given as a simple example in section 3.4.2.

3.2. Lyapunov Controllability

We are considering nonlinear control system dynamics of the form:

$$\dot{x}_t = b(x_t) + \sigma(x_t)u$$

$$= b(x_t) + \sum_{i=1}^{m} \sigma_i(x_t)u_i \qquad (1)$$

where $b \in D(\mathbf{R}^n)$ and $\sigma_i, i=1,..,m$ are smooth, complete vector fields on $\mathbf{R}^n$. We call σ the *control matrix* and the pair $\{b,\sigma\}$ the *control dynamics (CD)*.

We can make sense of the above equation in two ways, depending on the concept of control we are using:

(i) $u:[0,T]\to\mathbf{R}^m$ is a piecewise continuous function of t.

(ii) $u:\mathbf{R}^n\to\mathbf{R}^m$ is a piecewise continuous function of x.

We call controls of type (i) **open-loop controls** and denote them by $PC([0,T];\mathbf{R}^m)$ and those of type (ii) **feedback controls** and denote them by $PC(\mathbf{R}^n;\mathbf{R}^m)$. A more general setting requires the control u to lie in some subset of $\mathbf{R}^m$; thus, we have:

(iii) $u:[0,T]\to U$ is piecewise continuous and $U\subset\mathbf{R}^m$, U closed.

(iv) $u:[0,T]\to U(x)$ is piecewise continuous and $U(x)\subset\mathbf{R}^m\ \forall x\in\mathbf{R}^n$, $U(x)$ closed.

The differential equation (1) is well-defined in all these cases (almost everywhere).

Remark:

Most practical controllers use feedback control. However, modern optimal control formulations use open-loop controls, since this allows more general classes of controls functions (u can be a measure). By restricting ourselves to feedback controls we seem to be losing some of the recent theory, but in our case this is not relevant, since the control indicatrix will always be convex in the problems we shall treat.

We now turn to the study of the connections between the control system dynamics and global Lyapunov functions. Our definitions are motivated by the large deviation problems associated to the system (1).

Definition 1:

The control dynamics $\{b,\sigma\}$ *are called* **singular at** $x\in\mathbf{R}^m$ *if* $<\{\sigma_1(x),..,\sigma_m(x)\}>\neq T_x\mathbf{R}^n$ *(for example, if* $m<n$*). They are called* **singular** *if they are singular for some* $x\in\mathbf{R}^n$.

Definition 2:

The **indicatrix** $\mathbf{I}(x) \subset T_x\mathbf{R}^n$ *of the control dynamics* $\{b,\sigma\}$ *at a point* $x \in \mathbf{R}^n$ *is defined by:*

$$\mathbf{I}(x) = \{v \in T_x\mathbf{R}^n : v = b(x) + \sigma(x)u \ , \ u \in U(x)\}$$

$\mathbf{I}(x)$ is a closed set. All information pertaining to accessibility problems for the control dynamics (1) is contained in the field of indicatrices. For example, if $\mathbf{I}(x)$ contains a neighborhood of 0 for all $x \in \mathbf{R}^n$ then we can reach any desired point from any given starting point in finite time.

When we consider controls of type (ii), we have that $\mathbf{I}(x)$ is an affine subspace of $T_x\mathbf{R}^n$ and is a convex set.

Now suppose we are given a Lyapunov function V for the dynamics $b \in \mathbf{D}(\mathbf{R}^n)$. On the open set $\mathbf{R}^n - \Omega(b)$ the *gradient one-form* dV is non-zero.

Definition 3:

The control dynamics $\{b,\sigma\}$ *are said to be* **V-controllable at** $x \in \mathbf{R}^n - \Omega(b)$ *if for some* $i \in \{1,..,m\}$ *we have:*

$$dV(\sigma_i(x)) \neq 0$$

They are said to be **V-controllable** *if they are V-controllable at every* $x \in \mathbf{R}^n - \Omega(b)$.

We call the control dynamics **Lyapunov controllable** *if we can find a Lyapunov function V for which they are V-controllable.*

We have the following alternative global definition of V-controllability:

Lemma 1:

The control dynamics are V-controllable if and only if there exist smooth functions α_i , $i=1,..,m$ *from* $\mathbf{R}^n - \Omega(b)$ *to* $\mathbf{R}$ *such that the vector field:*

$$\mathbf{X} = \sum_{i=1}^{m} \alpha_i \sigma_i$$

satisfies: $dV(\mathbf{X})\neq 0$ *everywhere on* $\mathbf{R}^n-\Omega(b)$.

Of interest is also the converse question:

Given the control dynamics $\{b,\sigma\}$, *can we determine whether they are Lyapunov controllable?*

If $\{b,\sigma\}$ are **nonsingular** for all $x\in\mathbf{R}^n$, then they are Lyapunov controllable. The general answer is not known in the **singular** case. A partial answer is given by the following result:

Lemma 2:

Let α_i , $i=1,..,m$ *be smooth functions from* $\mathbf{R}^n-\Omega(b)$ *to* $\mathbf{R}$ *such that* $\mathbf{X}(x)=\sum_{i=1}^{m}\alpha_i(x)\sigma_i(x)\in\langle b(x)\rangle$ *for all* $x\in\mathbf{R}^n-\Omega(b)$.

Then the control dynamics are Lyapunov controllable.

Much stronger results hold for the case of controls of type (i) or (ii). They are *genericity* results and say that, in dimension $m>1$, the *typical* control dynamics are V-controllable if we restrict attention to a low-dimensional submanifold of $\mathbf{R}^n$.

Definition 4:

Given a k-dimensional submanifold M of $\mathbf{R}^n-\Omega(b)$ *and a Lyapunov function V, we say that* $\{b,\sigma\}$ *is V-***controllable on M** *if it is V-controllable at each* $x\in M$.

We now have the theorem:

Theorem 1:

Let $m\geq 2$ *and a Lyapunov function V be given.*

Given an (m-1)-dimensional submanifold M of $\mathbf{R}^n-\Omega(b)$, *control dynamics* $\{b,\sigma\}$ *and any neighborhood N of* $\{b,\sigma\}$ *in the space of control dynamics* *, *we can find control dynamics* $\{b,\sigma'\}$ *in N that are V-controllable on M.*

* The space of control dynamics is metrized in a natural way as a product space of smooth vector fields.

In particular, the set of trajectories of b in R_K that are V-controllable is dense in $Q=R_K/\sim$.

We can also make a genericity statement in the space of Lyapunov functions:

Theorem 2:

Let $m\geq 2$. Let a (m-1)-dimensional submanifold M of $\mathbf{R}^n-\Omega(b)$ be given.

Let control dynamics $\{b,\sigma\}$ be given such that σ_i, $i=1,..,m$ are non-vanishing vector fields on $\mathbf{R}^n-\Omega(b)$.

Given a Lyapunov function V on $\mathbf{R}^n-\Omega(b)$, we can find a Lyapunov function V' in any given neighborhood of V and control dynamics $\{b,\sigma'\}$ close to $\{b,\sigma\}$ such that $\{b,\sigma'\}$ are V'-controllable.

Remarks:

1) The genericity results above imply that, in the general case, the control vector fields $\sigma_1,\sigma_2,..,\sigma_m$ are almost everywhere non-zero.

2) One can extend the genericity stamenent in Theorem 1, always in the case $m\geq 2$, to controllability of the control dynamics $\{b,\sigma\}$ with respect to representatives of all the possible types of global Lyapunov functions of chapter 2.

The proofs of Theorems 1 and 2 are based on the *transversality theorem.* We give only the proof of Theorem 1, since the other proof is similar.

Proof of Theorem 1:

Since $\Omega(b)$ is a closed set, $D\doteq\mathbf{R}^n-\Omega(b)$ is open and is hence a manifold. On D, dV is a non-vanishing one-form. It defines a smooth, (n-1)-dimensional subbundle B of the (n-dimensional) tangent bundle TD of D by:

$$B_x=\{v\in T_xD:dV(v)=0\}$$

Now consider each σ_i as a section of the tangent bundle TD, i.e. $\sigma_i:D\rightarrow TD$. We state the elementary transversality theorem that we shall use (see, for example, Golubitsky and Guillemin [G-

G],p.56):

Theorem (Elementary Transversality Theorem):

Let X and Y be smooth manifolds and W a smooth submanifold of Y.

Then, the set of smooth mappings of X to Y which intersect W transversely is dense in $C^\infty(X,Y)$ *and if W is closed, then this set is also open.*

Theorem 3.1 follows from the transversality theorem above and the fact that the zero set of the typical m-vector function with components $dV(\sigma_i)$ has dimension (n-m). Thus, the intersection of this zero set with a typical m-dimensional submanifold is empty. The details are omitted. □

3.3. Optimal Control Using Lyapunov Functions

3.3.1. Optimal Control Setup and Basic Results

Consider the control dynamics $\{b,\sigma\}$ with controls of type (i) of section 3.2. Introduce, for a control function u of type (i) the cost functional:

$$J(u)=\frac{1}{2}\int_0^T |u(t)|^2 dt$$

We will be interested in control problems of the following form:

(i) Fix a normal tubular neighborhood $N_\varepsilon(K)\subset R_K$ of the attractor $K\in \mathrm{A}(b)$.

(ii) Fix a point $\bar{x}\in R_K - N_\varepsilon(K)$.

Assume there are control functions $u\in \bigcup_{T\in \mathbf{R}} PC([0,T];\mathbf{R}^m)$ such that if $\phi([0,T])$ is the corresponding solution of the control system dynamics $\dot{x}=b(x)+\sigma(x)u$, then $\phi([0,T])\subset R_K$, $\phi(0)\in \overline{N_\varepsilon(K)}$ and $\phi(T)=\bar{x}$. Such controls are called *admissible* and we denote them by $\mathrm{C}(K,N_\varepsilon(K),\bar{x})$.

Now the optimal control problem is:

Optimal Control Problem:

Assume $\mathrm{C}(K,N_\varepsilon(K),\bar{x})\neq\emptyset$.

Find $u^*\in \mathrm{C}(K,N_\varepsilon(K),\bar{x})$ *such that:*

$$J(u^*)=\inf_{u\in \mathrm{C}(K,N_\varepsilon(K),\bar{x})} J(u)$$

The dependence on the tubular neighborhood is avoided by defining the function of $\bar{x}$ only:

$$V(\bar{x})=\sup_{N_\varepsilon(K)} \inf_{u\in \mathrm{C}(K,N_\varepsilon(K),\bar{x})} J(u)$$

It is easy to see that the above function is well-defined. Note, however, that there may not be a control function $u\in \bigcup_{T\in \mathbf{R}} PC([0,T];\mathbf{R}^m)$ that provides the minimum (because it would have to be defined on an infinite time interval).

By finding the supremum over all normal tubular neighborhoods of K, we are essentially taking the limit as $\varepsilon \to 0$ (since for $\varepsilon' < \varepsilon$, $N_{\varepsilon'}(K) \subset N_\varepsilon(K)$).

Let us also consider the dependence of the optimal control problem on the terminal point $\bar{x}$.

Definition 5:

A point $x \in R_K$ is called **accessible** *from K if* $\bigcap_{N_\varepsilon(K)} C(K, N_\varepsilon(K), x) \neq \emptyset$.

The set of accessible points from K is denoted by $A(K)$ (note that $A(K) \subset R_K$).

We have the result that:

Lemma 3:

Under the Lyapunov controllability assumption, the set $A(K)$ is open.

Using the genericity results of the previous section, we can prove that all the saddle critical elements that form the proper-ancestor sets of the attractors of b are accessible from their attractors:

Lemma 4:

The elements of the proper-ancestor sets of each attractor K are, for the generic control dynamics $\{b,\sigma\}$, accessible from K in the sense that they are in $\overline{A(K)}$.

Using the above control setup, we are now ready to state the fundamental results relating Lyapunov controllability and nonlinear optimal control.

Theorem 3:

Let V_0 be a Lyapunov function for the dynamics $b \in \mathbf{D}(\mathbf{R}^n)$.

Let $\{b,\sigma\}$ be Lyapunov controllable with respect to V_0 and let the proper ancestor set of each K be accessible from K (alternatively, suppose $\{b,\sigma\}$ is generic in the sense of the above Lemma).

Fix an attracting set $K \in \mathbf{A}(b)$.

If:

(i) $\sigma(x)$ *is full rank for all* $x \in R_K$ *or*

(ii) each optimal control function $u(t)$ *is non-vanishing in* $A(K)$ *and the optimal flow in state space has no stationary points in* $A(K)$ *(i.e.* $\dot{x} \neq 0$*)*

Then:

The optimal control problem is solved in the set $A(K)$ *by a* **feedback control :**

$$u^*(x) = \sigma^T(x) p(x)$$

with $p(x) = \nabla V(x)$*, where* V *is a Lyapunov function for* b*.*

Moreover, for any $K \in \mathrm{A}(b)$ *such that condition (i) or (ii) above holds, the optimal exit path from* R_K *is along a trajectory* γ *such that:*

$$\lim_{t \to -\infty} \gamma(t) \in K$$

$$\lim_{t \to +\infty} \gamma(t) \in \sigma^*$$

where σ^* *is the saddle critical element that is 'closest' to* K *in the sense that* $V(\sigma^*) = \min_\sigma V(\sigma)$*.*

Finally, the optimal exit cost is equal to: $V(\sigma^*) - V(K)$*.*

Corollary:

For an attractor $K \in \mathrm{A}(b)$ *such that the conditions of theorem 3 are satisfied, the Lyapunov function* V *defined above satisfies the Hamilton-Jacobi equation:*

$$b^T(x) \nabla V(x) + \frac{1}{2} \nabla V^T(x) \sigma \sigma^T(x) \nabla V(x) = 0$$

in the set $A(K)$*.*

The proof of the above results is given in the next section.

3.3.2. Proof of Basic Results

The proof of Theorem 3 is divided into several steps. First, we use the V_0-controllability of the control dynamics to prove that the set $A(K)$ contains the interior -relative to the flow of b- of all the complete Lyapunov surfaces of V_0 and a neighborhood of every proper ancestor set of K.

Next, by standard existence theorems we prove that an optimizing control exists for all $x \in A(K)$. This is easy, since our cost functional J is convex.

At this point, we introduce the Hamitonian $H(x,p)$ related to the optimal control problem. We use it to show that on each optimal path $p \neq 0$. In the case $\sigma(x)$ full rank, this immediately gives that the function V defined by $\int p dx$ is smooth in R_K and is a Lyapunov function. In the more general case, we need the assumption (ii) of the theorem.

Proof of Theorem 3.3:

The aim of the proof is to construct the function V. It is this construction that is the new feature of this result. (remember that the traditional sufficiency theory suffers from an inability to show the existence of this function, which, if it exists and is smooth, then gives directly trajectories that are optimal - see Boltyanskii [Bo]).

It is easy to see that, for every point $x \in A(K)$ the set $C(K, N_\varepsilon(K), x)$ is non-empty for sufficiently small ε. Since our cost functional is convex, we get directly the existence of an optimal trajectory solving the optimal control problem.

This optimal trajectory satisfies the *Pontryagin maximum principle.* If we define the pre-Hamiltonian:

$$H_0(x,p,u) = p^T(b(x) + \sigma(x)u) - \frac{1}{2}|u|^2$$

and form the Hamiltonian by maximization:

$$H(x,p) = \sup_{u \in \mathbf{R}^m} H_0(x,p,u) = p^T b(x) + \frac{1}{2} p^T \sigma(x) \sigma^T(x) p$$

then, there is a solution of the system:

$$\dot{x}=\frac{\partial H}{\partial p}=b(x)+\sigma(x)\sigma^T(x)p$$

$$\dot{p}=-\frac{\partial H}{\partial x}=-\nabla b(x)p-p^T\frac{\partial}{\partial x}\sigma(x)\sigma^T(x)p$$

(where the last term is to be interpreted in the sense that for each component of p, p_i, the term in $\dot{p}_i$ is the following: $p^T\frac{\partial}{\partial x_i}\sigma(x)\sigma^T(x)p$) defined on an interval $[0,T]$, with initial condition $x(0)\in\overline{N_\varepsilon(K)}$ and terminal condition $x(T)=x$ and such that the optimal trajectory coincides with the x-trajectory of the above system. The optimal control is: $u^*(x,p)=\sigma^T(x)p$.

Next note that the function H is invariant under the flow of the above system:

$$\frac{dH}{dt}=\frac{\partial H}{\partial x}\dot{x}+\frac{\partial H}{\partial p}\dot{p}=\frac{\partial H}{\partial x}\,\frac{\partial H}{\partial p}-\frac{\partial H}{\partial p}\,\frac{\partial H}{\partial x}=0$$

Also note that in (x,p)-space, the plane $\{(x,p) : p=0\}$ corresponds to the trajectories of the uncontrolled system, which are also optimal for the quadratic cost functional, but are attracted to K instead of going against the flow as we desire. In any case, in (x,p)-space, H gives rise to a flow, and hence we can assume uniqueness and existence of solutions there. This means that on any optimal trajectory other than the uncontrolled ones, we have: $p\neq 0$ everywhere.

If σ is full rank, we can proceed to define the function V. If not, we need the condition (ii) of the theorem.

The assumption $\dot{x}\neq 0$ tells us that the definition of the quantity V by: $V(T)=\int_0^T p(t)\dot{x}(t)dt$ along any optimal path starting from a point on $N_\varepsilon(K)$ makes sense. Moreover, (see Arnol'd's book [Ar2]), there is a standard way of using the optimal flow to propagate an (n-1)-dimensional initial Lagrange submanifold to obtain an m-dimensional Lagrange submanifold that defines a *function of x* in (x,p)-space (see also Maslov [M]). Thus the function V is really a function of x: $V(x)=\int pdx$ and p is a function of x. In fact: $p(x)=\nabla V(x)$. To obtain a proper initial (n-1)-dimensional submanifold, we have to take the limit, as ε goes to zero of the normal tubular neighborhoods of K and set $p=0$. If we then substitute $p=\nabla V$ in the Hamiltonian and look at the level set $\{H(x,p)=0\}$, we obtain the Hamilton-Jacobi equation of the corollary.

Now from this equation we see that, since $u^*(x)=\sigma^T(x)p(x)$ is non-zero everywhere, the quadratic term in the equation is positive and hence the first term $b^T(x)\nabla V(x)$ is negative. This means the function V is a Lyapunov function in $A(K)$.

Finally, to get the saddle exit, we need to extend the construction above to the part of the boundary of R_K that is accessible. This is possible, since we assumed a neighborhood of every element of the proper ancestor set of K to be accessible. Then, the function V is defined at least on these neighborhoods and in addition one can show that no part of an invariant manifold of a critical element can be accessible from K, without the critical element itself being accessible. This gives the desired result. □

3.4. Applications

3.4.1. Algorithms for Solving Nonlinear Optimal Control Problems

The previous section tells us that, under the conditions of Theorem 3, to find the optimal exit path from a region of attraction R_K, we need only look at the saddle critical elements belonging to the proper ancestor set of K. This can result in a significant reduction of the work needed to compute the optimal control solution. Note, however, that, in general, we are still left with a two-point boundary problem.

We can systematize the search further by exploiting the Hamiltonian structure in phase-space and working *backwards* from the saddle critical elements. We do so for the case of a saddle equilibrium:

1) First, note that the optimal trajectory lies in the set $\{(x,p) : H(x,p)=0\}$.

2) **Linearization:** Since the optimal control functional V is a Lyapunov function for the vector field b, we have that near the saddle critical element $p=\nabla V \to 0$ as $x \to s$. Hence, since we also have that $b(x)\to 0$, we get that $\dot{x}\to 0$ for the optimal path.

Let us linearize the optimal flow in phase-space:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} b(x)+\sigma\sigma^T(x)p \\ -\nabla b(x)p+\dfrac{\partial}{\partial x}(p^T\sigma\sigma^T(x)p) \end{bmatrix}$$

near $(x,p)=(s,0)$; calling ξ and π the incremental variables:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \nabla b(s) & \sigma\sigma^T(s) \\ 0 & -\nabla b(s) \end{bmatrix} \begin{bmatrix} \xi \\ \pi \end{bmatrix}$$

The 2n eigenvalues at s are therefore the n real eigenvalues of $\nabla b(s)$ and their negatives: if s is of index k, $\nabla b(s)$ has k unstable eigenvalues and (n-k) stable ones. The optimal control introduces k stable eigenvalues equal and opposite in sign to the unstable ones of b and (n-k) unstable ones equal and opposite in sign to the stable ones of b.

We should look for the optimal path through the saddle s in a direction belonging to the **stable** *eigenspace of* $(s,0)$ *that also belongs to the set* $\{(x,p) : H(x,p)=0\}$ *and* **does not lie** *on the space spanned by the (n-k) stable eigenvectors of* $\nabla b(s)$.

The above suggests the following algorithm for finding an approximate solution to the optimal exit problem from R_K:

1) Select a saddle equilibrium s of the p.a. set of K.

2) Select a terminal condition for the optimal flow as follows:

Pick a direction:

$$e=\sum_{i=1}^{(n-k)} a_i\, e_i + \sum_{j=n-k+1}^{n} a_j f_j$$

where e_i are the eigenvectors at $(s,0)$ corresponding to the flow of b and f_j are the k additional eigenvectors arising from the optimal control paths that go against the flow of b, $\sum_{i=1}^{n} a_i=1$ but $\sum_{i=1}^{(n-k)} a_i<1$.

Choose ε small enough and pick:

$$\begin{bmatrix} x(0) \\ p(0) \end{bmatrix} = \varepsilon\, e$$

Propagate this terminal condition backwards in time.

3) If the orbit above diverges from K, choose a new initial condition until K is approached by the backward orbit. For that choice, compute the optimal cost as follows:

$$V(s)-V(k)=\int p dx$$

where $x(t)$ and $p(t)$ are the chosen trajectories in phase-space.

4) Repeat for all the saddle elements in the proper ancestor set of K. The optimal exit path from R_K is the one with the smallest cost.

We apply this technique in the next section for the case of the Josephson junction.

3.4.2. An Example: The Josephson Junction

The Josephson Junction Model:

The Josephson junction (see Fig.3.1) consists of two superconductors separated by a thin gap. To model the device, one uses the difference in the wave functions of the two super-conductors $\phi=\phi_1-\phi_2$. This leads to a damped sine-Gordon partial differential equation (see, for example, [L-H]).

The simplified model we shall use is obtained by discretization. It leads to a single-point junction and has a simple mechanical analogue: a damped pendulum subject to constant external torque. The electric circuit that corresponds to this model is given in Fig.3.2. Applying Kirchoff's current law at node 1, one gets:

$$C\frac{dv}{dt}+\frac{v}{R}+I_o\sin\phi=I_{dc} \tag{1}$$

where C and R are the junction capacitance and resistance, I_o is the threshold current, I_{dc} is the constant forcing current and the voltage v is related to the wave-function difference ϕ by:

$$v=\frac{h}{4\pi e}\,\frac{d\phi}{dt} \tag{2}$$

where h is Planck's constant.

Letting $I=\frac{I_{dc}}{I_o}$, $\omega_J^2=\frac{4\pi e I_o}{hC}$ and $G=\frac{1}{\omega_J RC}$, we can rewrite equation (1) in the normalized variable $\theta=\frac{\phi}{\omega_J}$:

$$\ddot{\theta}+G\dot{\theta}+\sin\theta=I \tag{3}$$

or, in state space form, with states: $x=\theta$, $y=\dot{\theta}$:

$$\dot{x}=y$$

$$\dot{y}=-Gy-\sin x+I \tag{4}$$

If we use the potential:

$$U(\theta)=-\cos\theta-I\theta$$

equation (4) becomes:

$$\dot{x}=y$$

$$\dot{x}=-Gy-U'(x) \tag{4a}$$

Note that the natural state space for (x,y) is not $\mathbf{R}^2$ but $S^1\times\mathbf{R}$ (a cylinder). The d.c. current driven Josephson junction model has been studied for a long time (as a damped forced pendulum) and exhaustively (Andronov [An], Lévi-Hoppensteadt [L-H] etc.).

The parameters G and I can be considered as bifurcation parameters. Different pairs of values lead to different phase portraits. We shall be interested in the range of G and I where a *stable equilibrium* E coexists with a *stable running orbit* S (a running orbit is a closed orbit in $S^1\times\mathbf{R}$ that wraps around the cylinder once, i.e. is not homotopically trivial). We summarize in the following result, taken from [L-H]:

Theorem: (Lévi,Hoppensteadt)

For $I<1$ there are two equilibrium points of the system (4) in $S^1\times\mathbf{R}$: $E=(\sin^{-1}I,0)$, an exponentially stable focus and $C=(\pi-\sin^{-1}I,0)$, a saddle. In addition, there is a $G_0(I)$ such that for $0<G<G_0(I)$ there is a unique exponentially stable running orbit S.

A typical phase portrait of (4) in the region of G and I above is given in Fig.3.3 (G=0.5 , I=0.7).

Remark:

A consequence of the above theorem is that for a single value of the current I there are two possible values of the voltage: a zero one corresponding to E and a non-zero one corresponding to S because, by relation (2), $\frac{d\phi}{dt}\neq 0$ on S.

Optimal Exit Problems:

When small noise perturbs the dynamics (4) of the junction, transitions between the regions of attraction of E and S are possible. This means that small noise can cause the junction to switch from a conducting ($v\neq 0$) to a non-conducting state ($v=0$). It is of interest to compute, asymptotically, the probabilities of these transitions and to establish, for given values of the bifurcation parameters G and I, whether the equilibrium E or the running state S is more stable.

Our introduction of control in the Josephson junction model is therefore dictated by the small-noise setting for it. Small noise enters the junction through the y-channel only:

$$\dot{x}=y$$

$$\dot{y}=-Gy-\sin x+I+Gn \tag{5}$$

with n being white noise. For the control problem associated to this small-noise setup, we consider instead of n a control action $u\in \mathbf{R}$. The cost functional is the quadratic cost-of-control functional of the optimal control set-up of section 3.3.1.

For the optimal exit from the region of attraction of E, solutions are known and use the energy function of the junction dynamics. In the physics literature, there have been many attempts to solve the problem of optimal exit from the region of attraction of S (see Matkowsky et.al.[Ma], Risken and Vollmer [R-V] and Ben-Jacob et.al.[B-J]). Most use formal asymptotic methods and approximate solutions to the control problem. These approximate solutions are either incorrect ([R-V]), or of limited value, since (see [B-J]) in the small G limit taken in order to obtain approximate solutions to the control problem, the running orbit S runs off to infinity and thus the problem of relative stability of E and S is non-existent (see Fig.3.4).

In this section, we give a complete treatment of the optimal exit problem for the two regions of attraction. We start by giving an explicit optimal feedback control solution to the problem of exit from R_E using the energy function of the junction dynamics, which is also a Lyapunov function for the system dynamics in R_E. We confirm that exit occurs from the saddle point C. Next, we see that the energy function cannot be of help in solving the exit problem from R_S since the energy is not constant on S. Rather, we use the algorithmic suggestions of the previous section to obtain the approximate optimal exit path using computer simulation. We arrive to the conclusion that, for the values of G and I chosen (G=0.5 , I=0.7), the equilibrium E is more stable than the runing orbit S. This is not at all obvious from the phase portrait of the system dynamics (Fig.3.3).

Exit from R_E:

The region of attraction R_E of the exponentially stable focus E is bounded by the two trajectories that form the stable manifold of the saddle equilibrium C (Fig.3.3).

Let us introduce the energy function for the Josephson junction:

$$T(x,y)=\frac{1}{2}y^2+U(x)$$

When there is no dissipation (G=0), the energy level sets of T coincide with trajectories of the system dynamics (in Fig.3.5 we give some of these level sets superimposed on the phase portrait of the dissipatibe system).

We claim that *in the case of the junction with dissipation ($G\neq 0$), the energy function is a Lyapunov function for the junction dynamics (4) and also solves the optimal exit problem for the region of attraction of E.*

Theorem 4:

The energy function $T(x,y)$ is a Lyapunov function for the Josephson junction dynamics (4) in the open set $R_E-\{(x,y):y=0\}$.

Furthermore, in R_E, T solves the Hamilton-Jacobi equation:

$$yT_x+T_x(-Gy-\sin x+I)+GT_y^2=0$$

Hence, the feedback control:

$$u^*(x,y)=2\ [\ 0\ G\ ]\begin{bmatrix}T_x\\T_y\end{bmatrix}=2Gy$$

defined in R_E is such that every trajectory of the resulting control system dynamics:

$\dot{x}=y$

$\dot{y}=Gy-\sin x+I$

is optimal for the cost functional: $J(u)=\int |u(t)|^2 dt$.

In particular, optimal exit from R_E is from the saddle equilibrium C.

In Fig.3.6 we show the optimal exit path from R_E.

Proof:

The partial derivatives of T are:

$$T_x=U'(x)=\sin x-I$$

and

$$T_y=y$$

Hence:

$$\frac{dT}{dt}=T_x\dot{x}+T_y\dot{y}=(\sin x-I)y+y(-Gy-\sin x+I)=-Gy^2$$

which is strictly less than zero in the set $R_E-\{y=0\}$ and so T is a Lyapunov function there.

One also easily checks that the Hamilton-Jacobi equation is satisfied:

$$y(\sin x-I)+y(-Gy-\sin x+I)+Gy^2=0$$

This equation is true everywhere in $\mathbf{R}^2$, but only in R_E does it produce optimal paths of the desired type: *going away from an attractor and exiting its region of attraction.*

The optimality of the trajectories of the control system in the theorem is a simple consequence of the Carathéodory theorem. Note that the trajectories of this system are the reflection about the x-axis of those of the uncontrolled dynamics (4), but with the direction reversed: the optimal vector field at the point (x,y) is related to the vector field of (4) as follows:

$\dot{x}_{opt}(x,y)=-\dot{x}(x,-y)$

$\dot{y}_{opt}(x,y)=\dot{y}(x,-y)$

Thus the trajectory that approaches the saddle C is the optimal exit path from R_E. □

Exit from R_S:

The energy function is of no use in the region of attraction of the running orbit S: it is not constant on S and the Hamilton-Jacobi equation yields no paths that go away from the attractor S and exit from R_S (see Fig.3.5). We thus need a more general theory, specifically the results of chapter 3.

We state the basic results for the exit problem from R_S:

Theorem 5:

The exit problem for the Josephson junction is solved in the accessible set $A(S)\subset R_S$ by the feedback control:

$$u^*(x)=2\,[\,0\ G\,]\begin{bmatrix}p_x(x,y)\\ p_y(x,y)\end{bmatrix}$$

where:

$$\begin{bmatrix}p_x\\ p_y\end{bmatrix}=\nabla V$$

with V a Lyapunov function for the uncontrolled dynamics (4) defined in A(S) and satisfying the Hamilton-Jacobi equation in A(S):

$$yp_x+p_y(-Gy-\sin x+I)+Gp_y^2=0$$

Optimal exit from R_S is from the saddle C.

Proof:

The set $A(S)$ is the open subset of R_S that contains all points of R_S above the x-axis and those points below that are to the left of the vertical line segment that is tangent to the stable manifold of C (shaded area in Fig.3.7). Accessibility is easily checked, since at every point (x,y) such that $y>0$, the indicatrix I(x,y) is the open half-plane $\{(v_x,v_y)\in T_{(x,y)}\mathbf{R}^2 : v_x>0\}$, at every point such that

$y<0$ it is the other half-plane and on the line $y=0$ the indicatrix is the line $\{v_x=0\}$. In particular, C is in $\overline{A(S)}$.

To apply theorem 3.3, we need only show that $p_y \neq 0$ in the set R_S. It would then follow that u^* is everywhere non-zero and hence, since $\dot{x}=y>0$ there, Theorem 3.3 is directly applicable.

To show that $p_y \neq 0$, we first form the pre-Hamiltonian:

$$H_0(x,y,p_x,p_y,u)=p_x y+p_y(-Gy-\sin x+I+Gu)-\frac{1}{2}u^2$$

Applying the maximum principle gives the Hamiltonian:

$$H(x,y,p_x,p_y)=\inf_u H_0(x,y,p_x,p_y,u)=p_x y+p_y(-Gy-\sin x+I)+Gp_y^2$$

The optimal control trajectories in (x,y,p_x,p_y) satisfy:

$$\dot{x}=\frac{\partial H}{\partial p_x}=y$$

$$\dot{y}=\frac{\partial H}{\partial p_y}=-Gy-\sin x+I+2Gp_y$$

$$\dot{p}_x=-\frac{\partial H}{\partial x}=\cos x p_y$$

$$\dot{p}_y=-\frac{\partial H}{\partial y}=-p_x+Gp_y$$

These trajectories lie on the set $\{H(x,y,p_x,p_y)=0\}$. In fact, the trajectories of the system dynamics with *no control* are also optimal. For these trajectories, $p_x=0$ and $p_y=0$. Hence, as in the proof of theorem 3.3, we see that one of the dual variables p_x and p_y is non-zero on every other optimal trajectory.

Now suppose $p_y=0$ on a point of an optimal trajectory that connects S to a point in R_S. This means that $p_x \neq 0$ there. But by the fact that the trajectory lies in the 0-set of the Hamiltonian, we deduce that the only way to satisfy $H=0$ is to have $y=0$. Thus we have found that only on the x-axis do we have possibly $p_y=0$. We will obtain a contradiction.

The trajectory containing this point has started from a neighborhood of S. Since we are assuming that the trajectory is contained in R_S, this implies that at that point in state-space, $\dot{y}>0$ and $\dot{x}=0$ (see phase portrait, Fig.3.3). Therefore, the trajectory must have crossed the x-axis going down at an earlier time. But on the half-plane $\{y>0\}$, $\dot{x}>0$ while on the half-plane $\{y<0\}$, $\dot{x}<0$. This means that the trajectory must have crossed the x-axis at a point to the right of the point where $p_y=0$ (see Fig.3.8). And after this point is passed, the trajectory must continue to the right and has either to cross S again or cross itself. In either case, optimality is easily contradicted (there cannot be any stationary points of the optimal flow, since $\dot{x}\neq 0$). Theorem 3.3 is applicable and gives the results desired. □

Computed Solutions:

It is not possible to obtain a solution to the optimal control problem in R_S analytically. But we saw in the previous section that theory gives us some help in searching for a computed solution. To see how this is applied to our example, it is first necessary to linearize the optimal (Hamiltonian) flow around the saddle C. In (x,p)-space the saddle equilibrium is at:

$$(x,y,p_x,p_y)=(\pi-\sin^{-1}I,0,0,0)$$

Calling ξ the variable of the linearized system, we have:

$$\dot{\xi}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1-I^2)^{1/2} & -G & 0 & 2G \\ 0 & 0 & 0 & (1-I^2)^{1/2} \\ 0 & 0 & -1 & G \end{bmatrix}\xi$$

At the saddle, the eigenvalues of the above linear system are:

$$s_{1,2}=-\frac{G}{2}\pm\frac{1}{2}[G^2+4(1-I^2)^{1/2}]^{1/2}$$

$$s_{3,4}=\frac{G}{2}\pm\frac{1}{2}[G^2+4(1-I^2)^{1/2}]^{1/2}$$

Note that s_1 and s_2 are the eigenvalues of the system without the control and $s_3=-s_2$ and $s_4=-s_1$ are eigenvalues introduced by the control action. For the range of G and I we are considering, all eigenvalues are real, with s_2 and s_4 being negative. Thus even in (x,p)-space, C is a saddle equilibrium.

We can compute the eigenvectors corresponding to these eigenvalues: for s_1 and s_2 they of course coincide with the directions of the stable and unstable manifold of C and they lie on the plane $\{(x,p) : p=0\}$:

$$u_1=\begin{bmatrix}1\\s_1\\0\\0\end{bmatrix} \; , \; u_2=\begin{bmatrix}1\\s_2\\0\\0\end{bmatrix}$$

For the other two eigenvalues, we find:

$$u_3=\begin{bmatrix}1\\s_3\\-(1-I^2)^{1/2}\\s_3\end{bmatrix} \; , \; u_4=\begin{bmatrix}1\\s_4\\-(1-I^2)^{1/2}\\s_4\end{bmatrix}$$

Given the relations between the four eigenvalues, we get the projections of these eigenvectors in state space shown in Fig.3.9.

Now let us see how the theory of section 3.4.1 is applied to this case: for the region of attraction R_E, the exit path (see Fig.3.6) reaches the saddle C in the direction of the stable eigenvalue introduced by the control, s_3. This is the direction that is the reflection about the x-axis of the unstable direction of the uncontrolled system, as expected. Thus, in this case, we have that the direction of the exit path is exactly obtained by the linearization of the optimal flow and coincides with the new stable direction introduced by the control.

In the case of the region of attraction of the running orbit S, R_S, things are a little more complicated: the exit direction is *not* simply the direction of the new stable eigenvalue, but lies, in (x,p)-space, in the span of the eigenvectors u_1 and u_3. The shooting method described in section 3.4.1 gives the approximate exit direction shown in Fig.3.10(a) and the exit path of Fig.10(b).

The optimal exit cost is calculated along the optimal exit path using the integral $\int p dx$. In the two cases, for the parameter values we used in our simulations, where G=0.5 and I=0.7, we obtained the costs:

$$V(C)-V(E)\cong 0.315$$

and

$$V(C)-V(S)\cong 0.0235$$

Thus, for these parameter values, it is much cheaper to exit from R_S than from R_E. As we see in chapter 4, for the large deviation problem, this means that the equilibrium E is more stable than the running orbit S.

CHAPTER 4: LARGE DEVIATIONS

In this chapter we apply the machinery of the previous chapters to large deviation problems in a global setting.

It is first necessary to prove for *singular diffusions* the basic estimates (upper and lower bounds) for the probability that the diffusion path follows a prescribed path in state space. This can be done by modifying existing proofs in two ways: First, the action functional is written in terms of the control action required to steer the associated control system along the chosen path. Second, the Girsanov transformation is applied assuming the existence of a feedback control law on an open set, instead of a single control time-function.

This approach is then applied to the *local* problem in large deviations: the rare excursions away from an attractor,but staying in its region of attraction. We clear up questions relating to the choice of the domain which we want the diffusion to exit from through the use of Lyapunov surfaces.

Finally, we address the global *large* deviation problem. The machinery of chapters 2 and 3 is directly applicable and so we are able to desribe the transition probabilities of exiting one region of attraction and moving close to a different attractor.

4.1. Introduction

In this chapter we put to work the optimal control machinery of chapter 3 in order to solve the large deviation problem for the diffusion process:

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \varepsilon\sigma(X_t^\varepsilon)dW_t \qquad (1)$$

We improve on the results of Wentzell and Freidlin [W-F] in the following ways:

1) **Saddle Exits:** Because our global dynamical setting is more specific than that in [W-F] (their compacta are our attracting sets, but they make no other assumption on the global dynamics-we have worked with the class $\mathbf{D}(\mathbf{R}^n)$), we are able to obtain stronger results on the qualitative behavior of exit paths. In particular, in the case of non-singular diffusions the large deviation paths exit a region of attraction from the 'nearest' saddle (in the sense of control). This is also the case for singular diffusions, as long as the conditions of Theorem 3.3 are satisfied by the optimal control.

2) **Singular Diffusions:** We claim that substantially the same results hold here as in the non-singular diffusion case, assuming the conditions of Theorem 3.3 are satisfied. This is the first occasion where the geometry of large deviation paths is elucidated for singular diffusions (analytically, singular diffusions have been treated by Azencott [Az]). The assumptions above seem minimal, in the sense that if they fail, very little can be said, in general, about the large deviation paths without first solving the optimal control problem.

3) **Algorithmic Implications:** As we saw in chapter 3, we can make use of qualitative information on saddle exits to solve the optimal control problem associated to the large deviation setup (or use it as a guide in obtaining computed solutions). In the absence of such information, it is very difficult to solve such nonlinear control problems and hence to describe the large deviation behavior of the diffusion process.

This last chapter has more the character of an outline. The optimal control set-up of chapter 3 is directly applicable to the case of non-singular diffusion. Thus we are able to state the results on saddle exits and the qualitative features of exit paths relevant to class $\mathbf{D}(\mathbf{R}^n)$ that improve on the results of

4.3. Large Deviation Exit Paths and Transitions Between Attractors

4.3.1. Local Results: Exit from a Domain

In this section we address the problem of large deviations away from an attractor, but staying in its region of attraction.

It is customary in large deviation accounts to fix an arbitrary domain $D \subset R_K$ that has a smooth boundary ∂D which is transverse to the flow of b. This suggests to us that good candidates for D are the interiors of Lyapunov surfaces relative to the flow of b. It is obvious that the large deviation results critically depend on the choice of this domain: a path that is optimal for one choice of D may not be optimal for another (see Fig.4.1).

In general (see Fig.4.2), a path that starts at a point $x \in R_K$ first visits a neighborhood of the attractor K. It stays there for a long time, occasionally deviating from it but coming back to it without leaving D. When a large deviation occurs it is most likely to do so from the point x^* on the boundary of D that minimizes the action functional V.

Before we state and prove the main result, we need to make some additional assumptions on the attractor K and the domain D:

Assumption 1:

On each attractor $K \in A(b)$, there is a trajectory that is dense in K, i.e. such that it crosses every neighborhood of each point in K.

Now in R_K we have available Lyapunov surfaces that we can use to define domains D. To do this, we consider, for each Lyapunov surface S the interior of S relative to the flow of b: $intS \triangleq \{x : x = \phi_t y \ , y \in S \ , t > 0\}$. Moreover, we know that, under the conditions of Theorem 3.3, the optimal cost functional V is a Lyapunov function for the dynamics b. We now assume the following relation between S and V: consider $V^* = \inf_{y \in S} V(y)$ (it exists because V is continuous and S is compact).

Assumption 2:

The Lyapunov surface $V^{-1}(V^*)$ intersects S in a single point $y^* \in S$.

The main result of this section can now be stated:

Conjecture 3:

Consider a point $x \in int V^{-1}(V^*)$.

Then, given any $\delta > 0$:

$$\lim_{\varepsilon \to 0} P_x(|X^\varepsilon_{\tau^\varepsilon}) - y^*| < \delta) = 1$$

where $\tau^\varepsilon = \inf_t (X^\varepsilon_t \in S)$.

(For the case of non-singular diffusion, this is Theorem 2.1, chapter 4,p.108 of [W-F]).

The proof of this conjecture in the case of non-singular diffusion is the same as that in [W-F] and will not be given. Here, we generalized the dynamical setting of the local problem by relating the existence of appropriate domains D to the existence of Lyapunov surfaces. Moreover, we have available enough Lyapunov surfaces to guarantee the uniqueness of the minimum. In the next section, (as in section 3.3), this becomes the key to generalizing the control problem from one with terminal points to one with terminal surface.

(2) above, but driven instead by white noise, which, mathematically, is badly behaved (is not even continuous) . Considering only paths of (2) makes sense because of the following basic result of Varadhan [V] that says that the support of the diffusion process coincides with the accessible set of the control system (2).

Theorem: (Varadhan) Support of a Diffusion Process

Fix $x \in \mathbf{R}^n$. *Then:*

$$supp\{X_t^\varepsilon(x), t \in [0,T], T \in \mathbf{R}\} = \overline{A(x)}$$

We now state the two basic estimates:

Conjecture 1: (Lower Bound)

Let $x \in \mathbf{R}^n$ *and u be a non-vanishing feedback control law defined on an open subset U of* $\mathbf{R}^n$ *such that* $\overline{U}$ *is compact.*

Denote $\phi_u = \phi_u([0,T],x)$ *the trajectory of the system* $\dot{x} = b(x) + \sigma(x)u(x)$ *starting at x. Let* $J(u) = \frac{1}{2}\int_0^T |u(t)|^2 dt$ *be the cost of control corresponding to* ϕ_u.

For the diffusion process:

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t$$

with $X_0^\varepsilon = x$ *we have that for* $\delta > 0$, $\gamma > 0$ *there is an* $\varepsilon_0 > 0$ *such that:*

$$P(\sup_{t \in [0,T]} |X_t^\varepsilon - \phi_u(t)| < \delta) \geq \exp[-\frac{J(u)+\gamma}{\varepsilon^2}]$$

for $0 < \varepsilon < \varepsilon_0$.

Conjecture 2: (Upper Bound)

Let $s>0$. *Let:*

$$\Phi(s)=\{\phi_u : \phi_u(0)=x\ ,\ J(u)\leq s\}$$

Then, for any $\delta>0$, $\gamma>0$ *and* $s_0>0$, *there is an* $\varepsilon_0>0$ *such that:*

$$P(\sup_{t\in[0,T]} |X_t^\varepsilon-\Phi(s)|\geq\delta)\leq\exp[-\frac{s-\gamma}{\varepsilon^2}]$$

These estimates are proved in Wentzell and Freidlin [W-F] using an action functional that is, in the case of non-singular diffusions, the same as our cost function J. We hope to prove these conjectures using the Girsanov theorem that employs the feedback control u to change the measure (since we bounded the control on the set $\overline{U}$ we are satisfying the conditions in Ikeda and Watanabe for the Girsanov transformation).

Wentzell and Freidlin. The extension of these results to the singular case, however, (but with the conditions of theorem 3.3 satisfied) is left as a conjecture.

Section 4.2 forms the basis of our treatment of the large deviation problem. It gives the two basic estimates (lower and upper bounds) that desribe the probability that a diffusion path stays close to a chosen path of the associated control system. The influence of chapter 3 is evident in that the action funtional is written in the control sense and the Girsanov transformation (see Girsanov [Gi], Ikeda and Watanabe [I-W]) uses feedback laws instead of a single control time-function. Section 4.3 introduces the *stability* structure of the dynamics of class $\mathbf{D}(\mathbf{R}^n)$ to deal with the local problem of large deviations away from an attractor, but staying in its region of attraction.

Finally, section 4.4 gives the global picture. The rare transitions between regions of attraction are dealt using the global saddle exit results of chapter 3. Section 4.5 summarizes the results.

4.2. Basic Estimates

In [W-F], the two basic estimates (Theorems 2.1 and 2.2, pp.74-77) give bounds on the probability that the diffusion will, for a finite time, stay close to a predescribed path in state space. Large deviation theory says that, even though any path other than that of the drift dynamics b will have asymptotically zero probability of being followed, one can distinguish between these paths: some will be more likely than others. The key is the two *exponential* estimates with exponent depending on $\frac{1}{\varepsilon^2}$ and the action functional V of the selected path. As $\varepsilon \to 0$, these exponential estimates lead to the selection of those paths with the minimal action: they become overwhelmingly more likely to be followed. The principle of this selection process is the following:

Suppose that we have k independent events $E_i, i=1,..,k$, each with probability equal to $e^{-\frac{V_i}{\varepsilon^2}}$, where the V_i are some positive numbers with a unique minimum V^* corresponding to event E^*. Then, as $\varepsilon \to 0$, the conditional probabilities $P(E_i | \bigcup_j E_j)$ will all tend to zero except the one corresponding to V^*. This is because we have:

$$P(E_i | \bigcup_j E_j) = \frac{e^{-\frac{V_i}{\varepsilon^2}}}{\sum_j e^{-\frac{V_j}{\varepsilon^2}}}$$

which tends to zero. When $E_i = E^*$ then the conditional probability tends to one. Thus, as $\varepsilon \to 0$ event E^* is overwhelmingly more likely to occur, assuming one of the events E_i occured.

The generalization of this principle to the infinite dimensional space of paths is called the *Laplace method* and is the basis of large deviation theory.

In our treatment, we consider only paths which are trajectories of *the control system associated to the diffusion:*

$$\dot{x} = b(x) + \sigma(x) u \tag{2}$$

for $u \in \bigcup_T PC([0,T])$. Note that, in the engineering sense, the stochastic diffusion process is the same as

4.3.2. Global Results: Transitions Between Attractors

We want to globalize the picture of section 4.3.1. This means that we want to consider all the attractors of b and large deviations that take us from a neighborhood of one attractor to a neighborhood of another.

As far as the optimal control setup is concerned, we are now considering the following setup:

Global Optimal Control Setup:

Fix attractors $K_1, K_2 \in A(\mathbf{R}^n)$.

Assume that the set of common ancestors of K_1 *and* K_2 *is non-empty.*

Find the optimal cost of reaching the set $B=\overline{R_{K_1}} \cap \overline{R_{K_2}}$ *starting from a neighborhood* $N(K_1)$ *of* K_1.

From the theory of chapter 3, we have on each region of attraction R_K a feedback control solution:

$$u_K^*(x)=\sigma^T(x)p_K(x)$$

where:

$$p_K(x)=\nabla V_K(x)$$

(we used the subscript K on the function V to indicate that each function is defined only on a region of attraction). The functions V_K are Lyapunov functions for b on R_K.

Call $\sigma^*_{K_1K_2}$ the saddle critical element belonging to the set B that minimises V_K in B.

$$V(\sigma^*_{K_1K_2})=\inf_{x\in B} V(x)$$

We have the result:

Conjecture 4:

Fix $x \in R_{K_1}$ *and* $\delta > 0$.

Then:

$$\lim_{\varepsilon \to 0} P_x(|X^{\varepsilon}_{\tau_{K_1}} - \sigma^*_{K_1 K_2}| < \delta) = 1$$

where: $\tau^{\varepsilon}_{K_1} = \inf_t \{X^{\varepsilon}_t \in \overline{R_{K_2}}\}$.

Furthermore:

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log E_x\{\tau_{K_1}\} = V(\sigma^*_{K_1 K_2})$$

In the case of non-singular diffusion, this conjecture is essentially Lemma 2.1, p.171 of [W-F].

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4.4. Conclusions and Directions for Further Research

The approach of this thesis is, we believe, the most appropriate one for attacking global large deviation problems for diffusions.

In particular, *stability* in the sense of the vector field belonging to the class $D(R^n)$ (see chapter 2) seems essential, if the small noise problems are not to become hopelessly complicated (see, for example the work by Katok and Kiefer [K-K])). In addition (see chapter 3), it is correct to focus only on paths that are going away from attractors as likely large deviation paths. The concept of Lyapunov controllability arises naturally from this approach.

The fact that Lyapunov functions solve optimal control problems associated to the large deviation setting is not surprising, but is pleasant, since it allows substantial geometrical information to be extracted on the exit paths. The conditions of theorem 3.3 seem minimal for this generalization of gradient vector fields to work. From an optimal control point-of-view, we also have arrived at a nice sufficiency theory that is more geometrical than that of Boltyanskii [B]. The important novelty of our work is that the function V that yields the optimal control paths is shown to exist and is constructed using the geometry of the space (x,p).

One of the directions further research should take is obvious:

1) *Singular Diffusions:* Complete the treatment of singular diffusions. From an analytic viewpoint, the work of Azencott [Az] seems to deal satisfactorily with the singular case and takes into account the fact that only the paths that are the outputs of the control system associated to the diffusion are relevant for the small-noise asymptotics. Yet, his approach falls short of recognising that the action functional has a control interpretation and also, since the global geometric framework is not elucidated as in our chapter 1, his large deviation results are limited. This seems to be the reason why he requires hypoellipticity (a condition of controllability in all directions) instead of Lyapunov controllability, which only takes into account paths that move away from attractors. It is hoped that results can be obtained even in the absence of hypoellipticity.

2) *Applications:* A number of applications, in addition to the Josephson junction, are likely to be solved using the methods of this work. It is to be understood that new results will be obtained by employing techniques that are fundamentally qualitative in nature, as in our treatment of the Josephson junction.

Possible areas of application are: power system stability problems, transitions between metastable states in physics (multiple equilibria, tunneling effects etc.) and problems in stochastic global optimisation.

3) *Singularities in Optimal Control:* The present research motivates a new attack on the nonlinear optimal control problem, from the vantage point of modern dynamical system theory. The concept of chattering controls and the analytical questions of existence and necessary conditions have absorbed most recent work in optimal control theory. However, geometry has a lot more to contribute to the qualitative undestanding of optimal control, in addition to the insight it has provided to the question of controllability and observability. The direction which this research is suggesting is the application of geometry and dynamical systems to typical qualitative features of optimal control paths (eg. saddle exits). This is largely possible today because of our much greater understanding of the geometry of symplectic spaces and singularities of their Lagrange submanifolds.

The theory of singularities can play a role, at least in low dimensions, in making clear what singular behavior to expect in the typical or generic case in optimal control. A first effort in this direction is the work of J.Ortmans [Or].

4) *Bifurcations in the Presence of Small Noise:* Another area, closely related to the one above is the area of bifurcations in dynamical systems and the effect of small noise on them. A first approach to this problem is by Sastry [S]. Again here it seems that a clear specification of the class of dynamics we hope to treat is essential. A tough problem in this setup arises from the fact that we have simultaneously two changing parameters to deal with: the bifurcation parameters and the small epsilon that yields the large deviation behavior. Extreme care should be exercised in handling the limiting changes in these two variables.

FIGURE CAPTIONS

Fig.1.1: A double well potential and its gradient flow.

Fig.1.2: Phase-space flows and orbit diagrams for two examples of dissipative dynamics in the plane.

Fig.2.1: Mapping a neighborhood of x on Q by the projection π and on $\mathbf{R}^n$ by the flow-box diffeomorphism ψ.

Fig.2.2: Flow near a saddle equilibrium: $W^u(\sigma)$ and $W^s(\sigma)$ are the stable and unstable manifolds, S^u and S^s are Lyapunov surfaces on them and the flow maps the punctured neighborhood of S^u to the punctured neighborhood of S^s.

Fig.2.3: Flow near a saddle limit cycle: on the Poincaré surface Σ, a point close to the unstable manifold $W^u(\sigma)$ is mapped to a point close to the stable manifold $W^s(\sigma)$.k

Fig.2.4: Propagation of a complete Lyapunov surface past a saddle: the Lyapunov surface is no longer complete: it misses the unstable manifold of σ.

Fig.2.5: A two-dimensional example of a vector field of class $\mathbf{D}(\mathbf{R}^n)$ that has two different Lyapunov functions: (a) gives the phase-portrait, (b) the orbit diagram and (c) and (d) the two Lyapunov functions (by showing some of their Lyapunov surfaces).

Fig.2.6: Part of an orbit diagram that shows that, if σ_1 is at a higher index level than σ_2, then $V(\sigma_1)$ must be greater than $V(\sigma_2)$.

Fig.2.7: Demonstration of the selection procedure among proper ancestor sets and the iterative collapsing of orbit diagrams: here we ordered σ_2 before σ_1 and we obtained the Lyapunov function of Fig.2.5(c).

Fig.3.1: The Josephson junction: A cryogenic device consisting of two semiconductors separated by a thin gap.

Fig.3.2: The Josephson junction circuit model (current-driven case).

Fig.3.3: Phase-portrait of the Josephson junction for the parameter values: G=0.5 and I=0.7. E: Stable equilibrium, S: Stable running orbit, C: Saddle equilibrium.

Fig.3.4: The effect of letting $G \to 0$: The running orbit runs off to infinity.

Fig.3.5: Level sets of the energy function $T(x,y)$, superimposed on the phase-portrait of the Josephson junction.

Fig.3.6: Optimal exit path from R_E and level set of the energy function touching the saddle C.

Fig.3.7: Accessible set of the running orbit (shaded area).

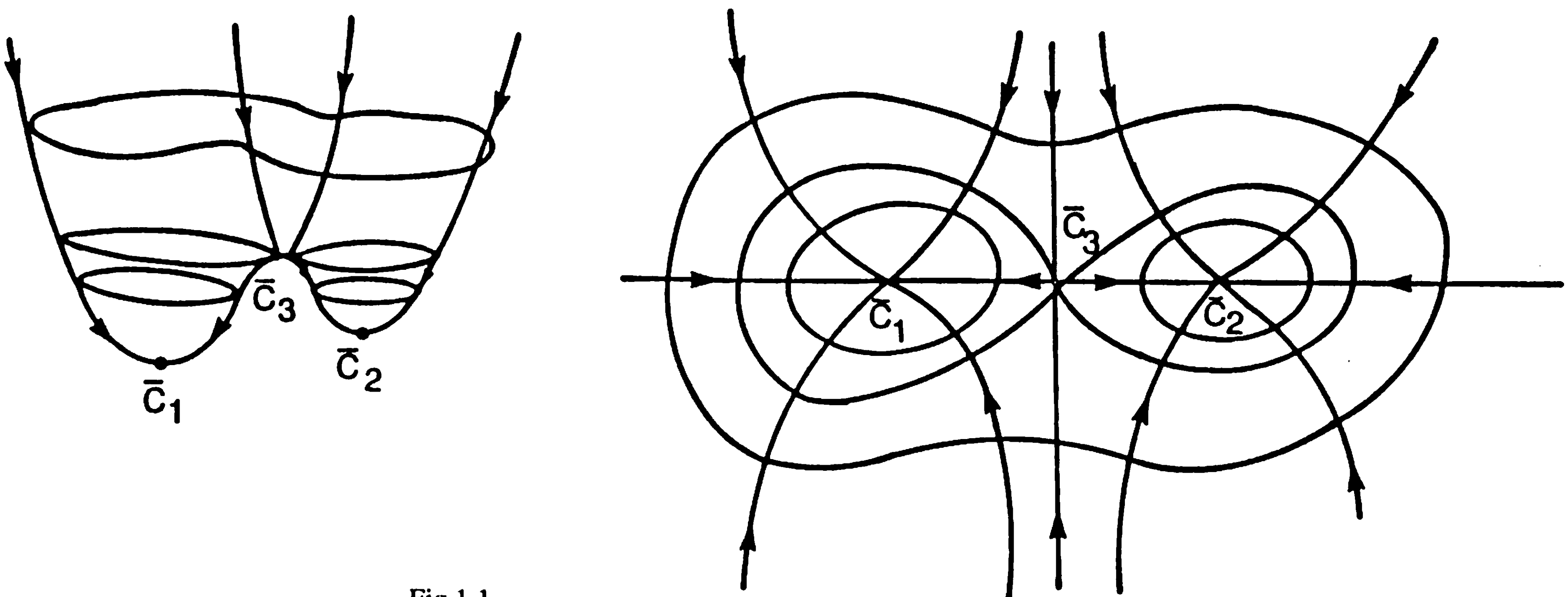
Fig.3.8: An upward crossing of the x-axis by an optimal path. The shaded area must contain an equilibrium of the optimal flow.

Fig.3.9: Linearization around C: The stable and unstable directions in state-space. $W^s(C)$ and $W^u(C)$ are the stable and unstable manifolds of the system with no control.

Fig.3.10: (a) The exit direction from R_S and (b) The exit path from R_S.

Fig.4.1: Dependence of the exit path on the domain D.

Fig.4.2: Typical large deviation path starting from a point in R_K, the region of attraction of the attractor K.



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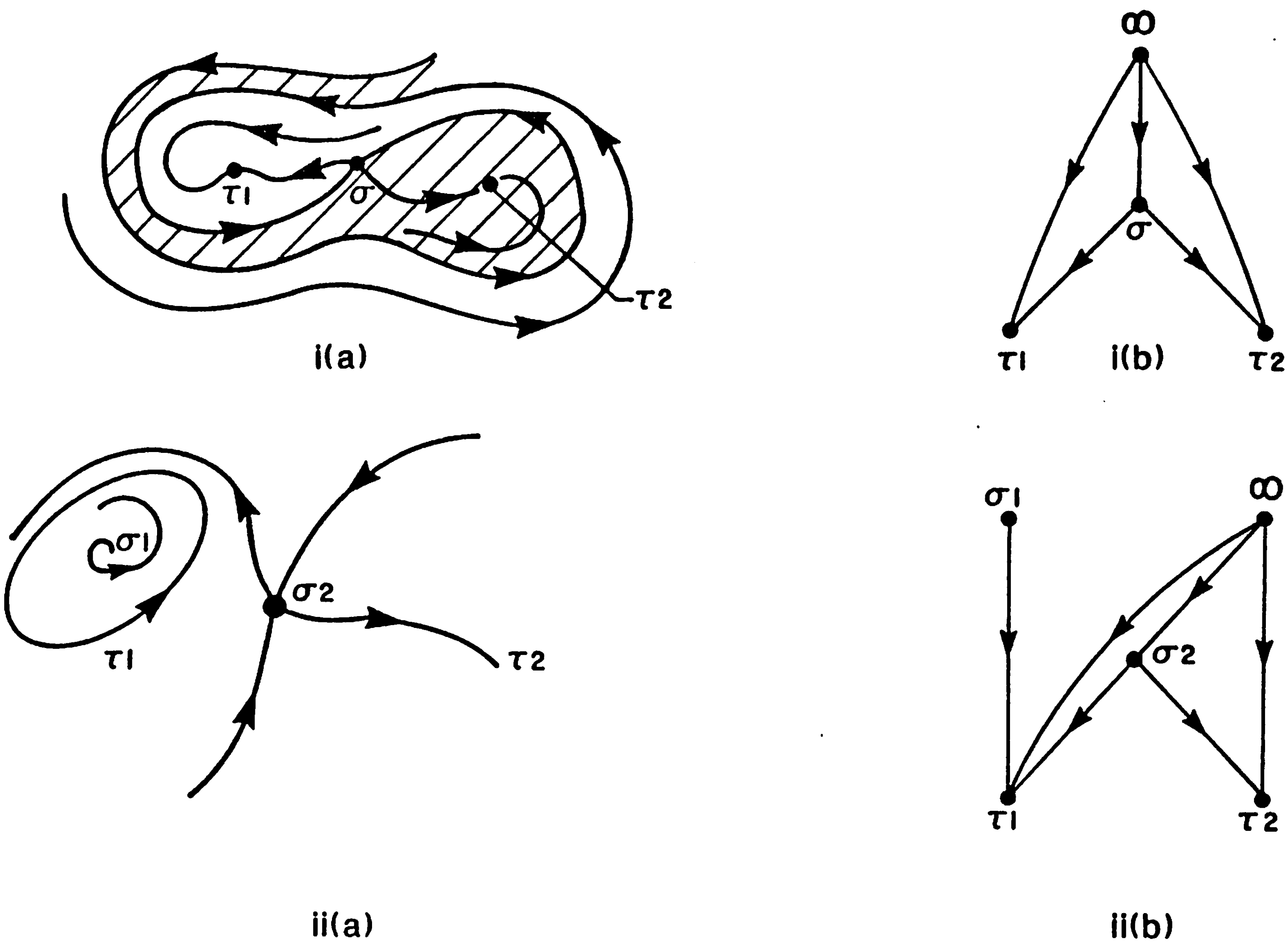

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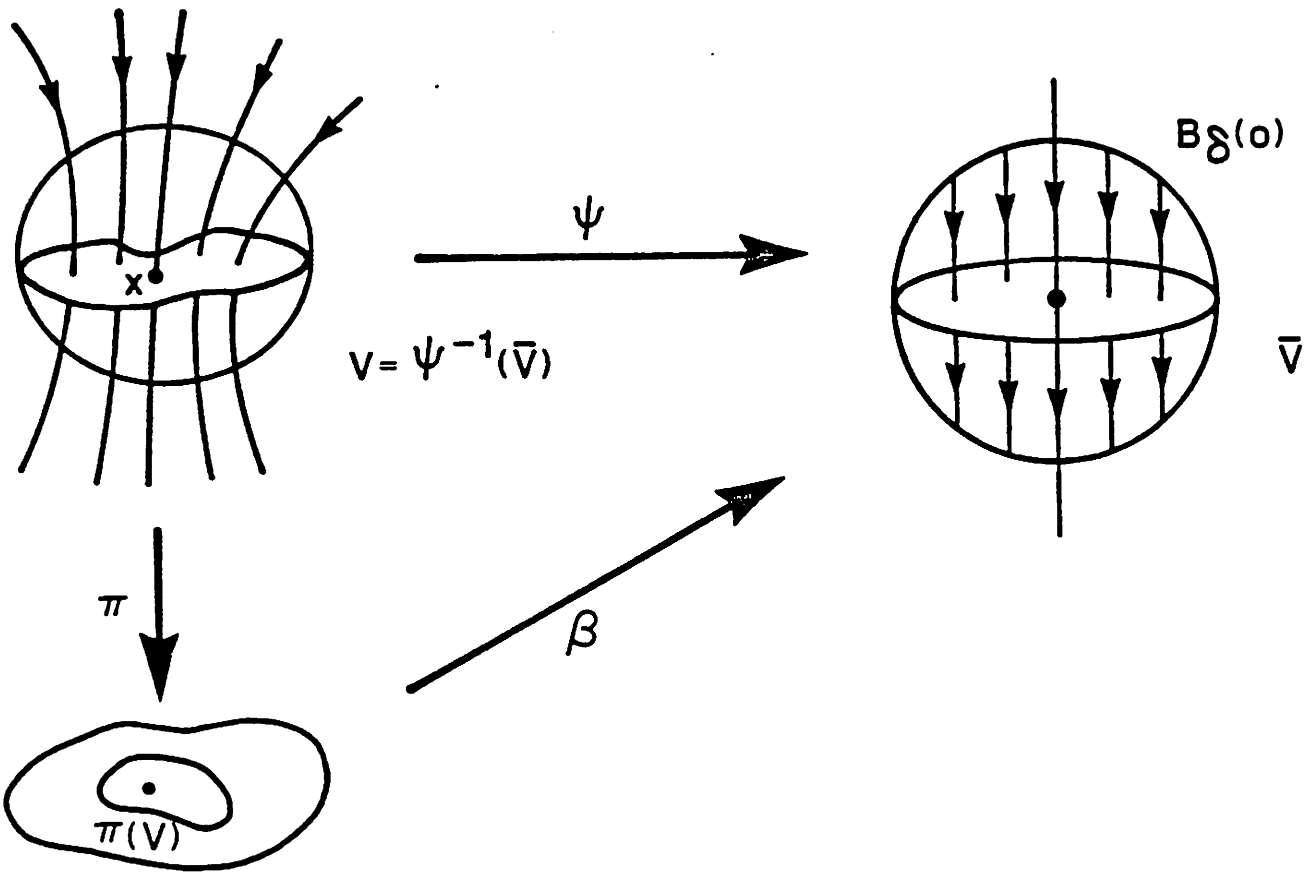

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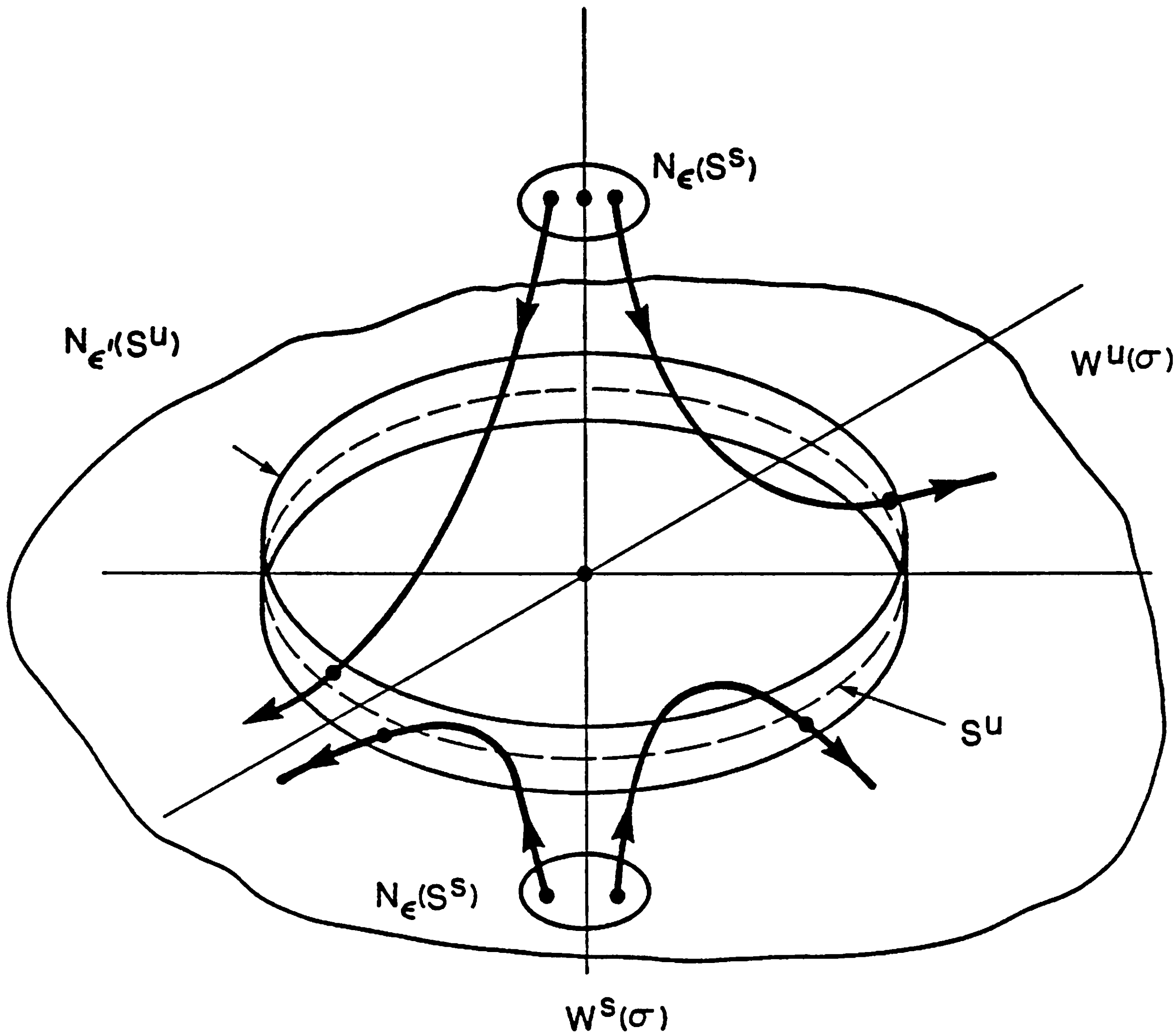

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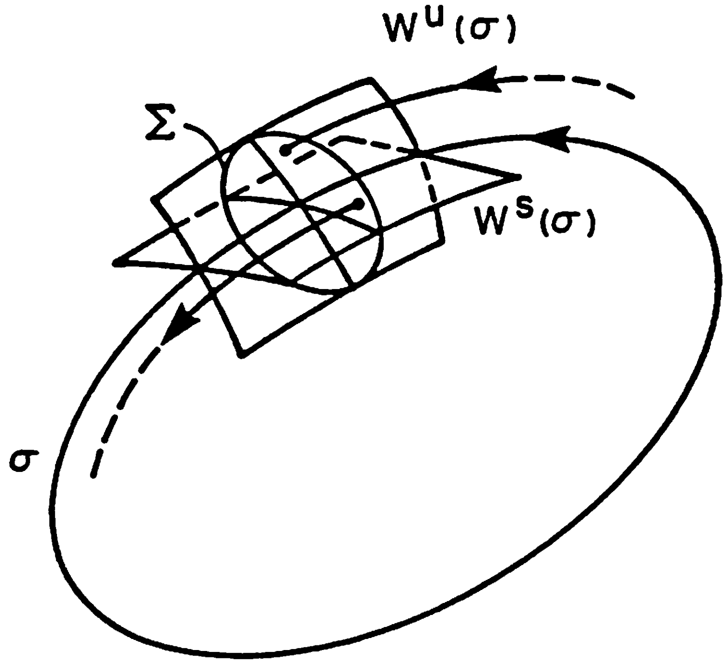

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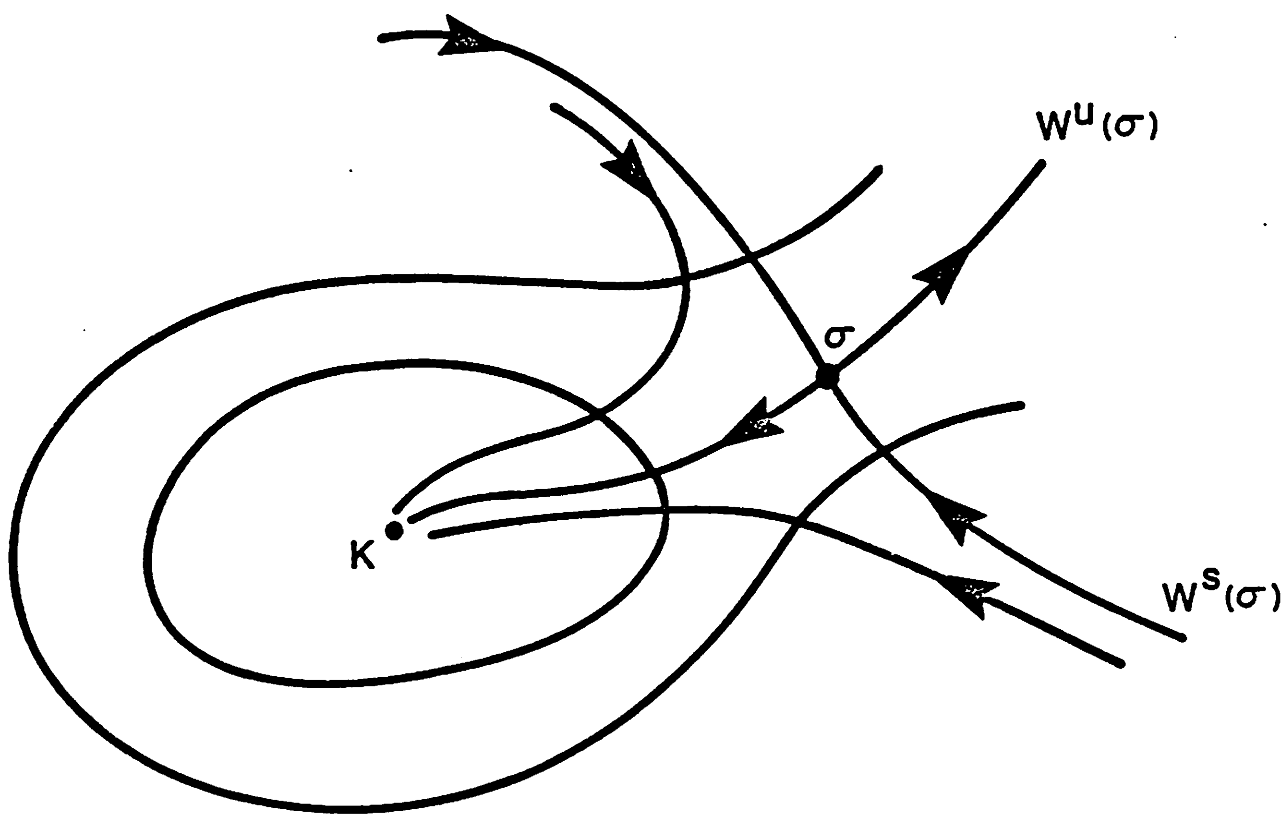

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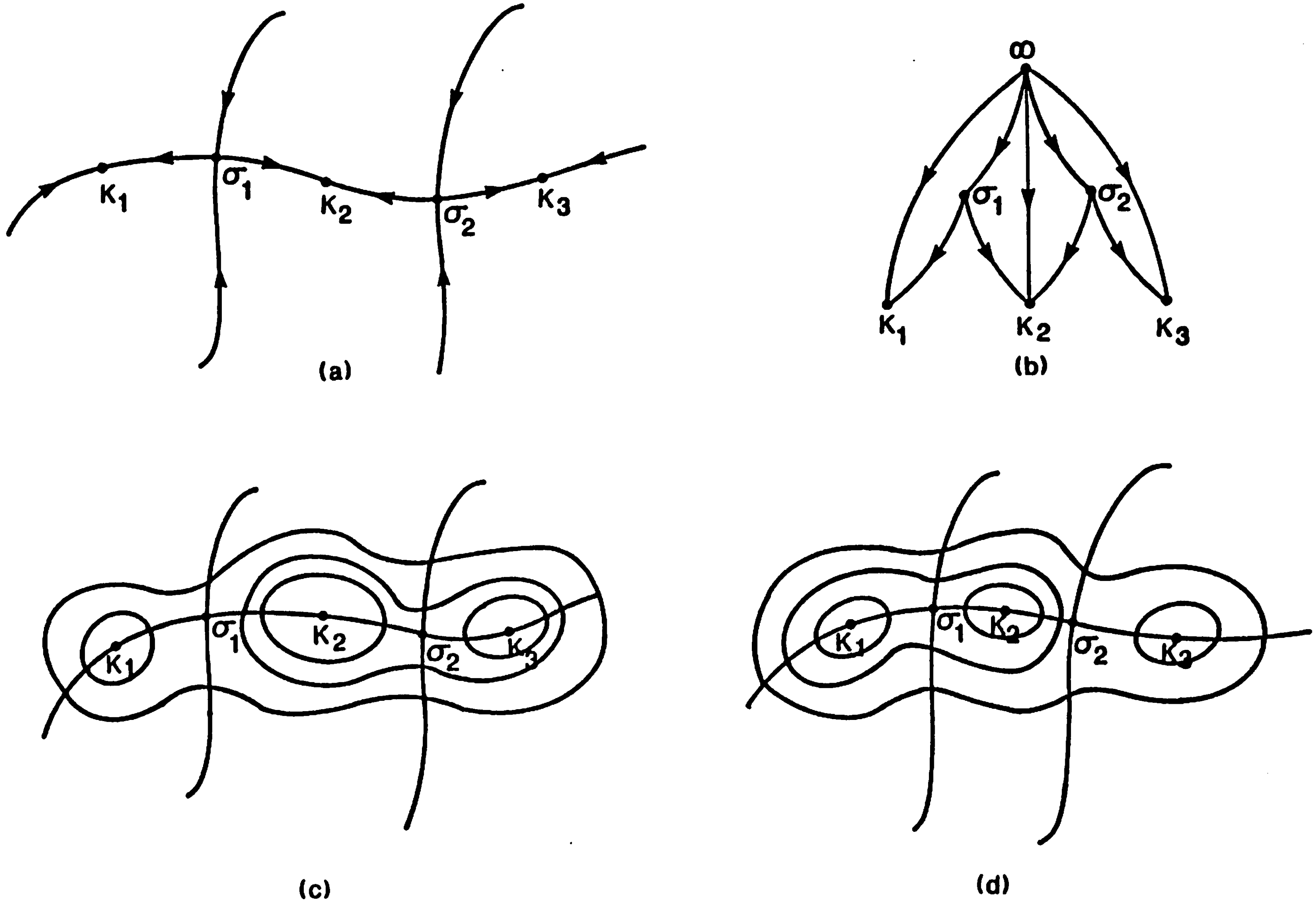

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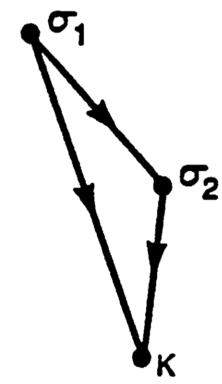

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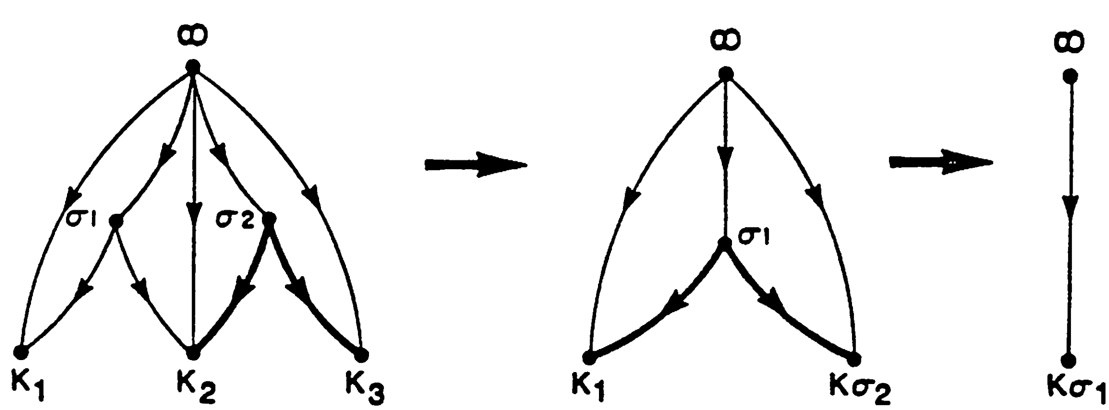

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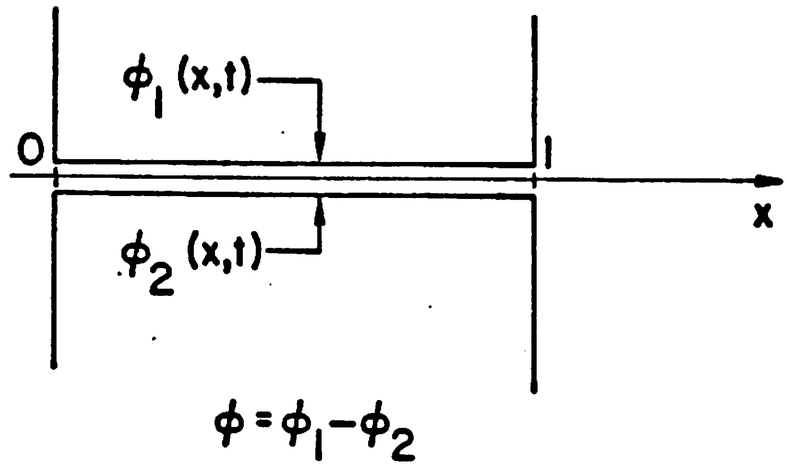

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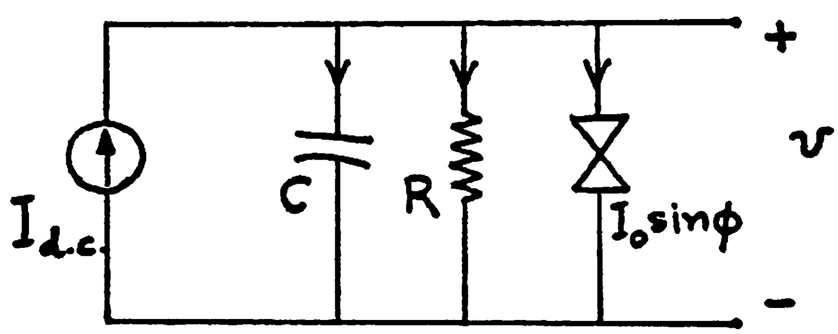

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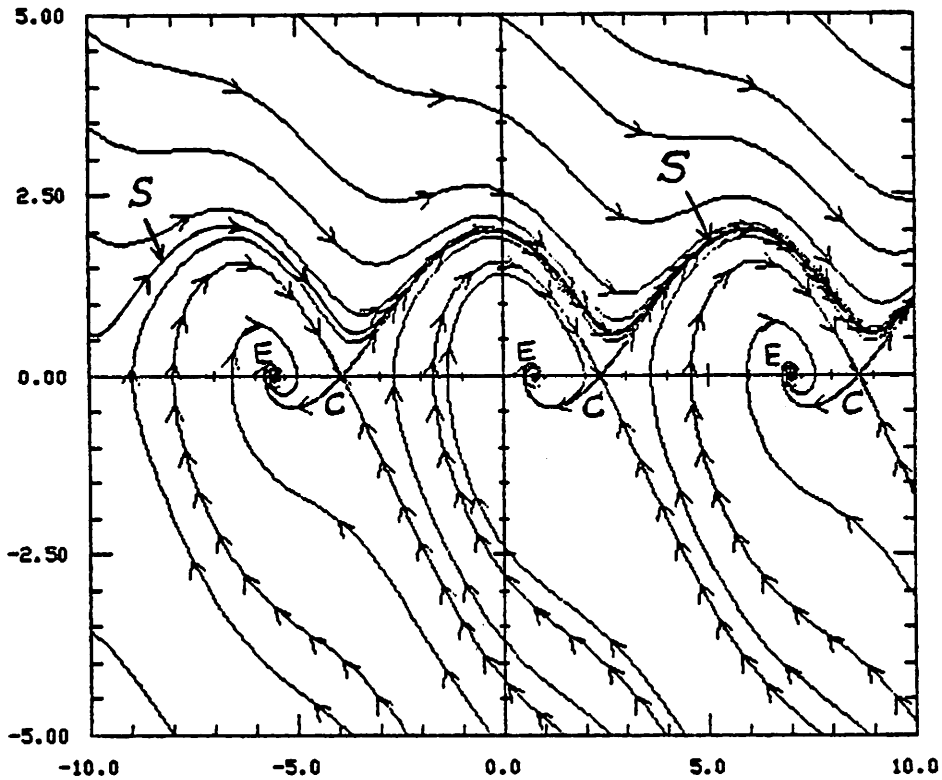

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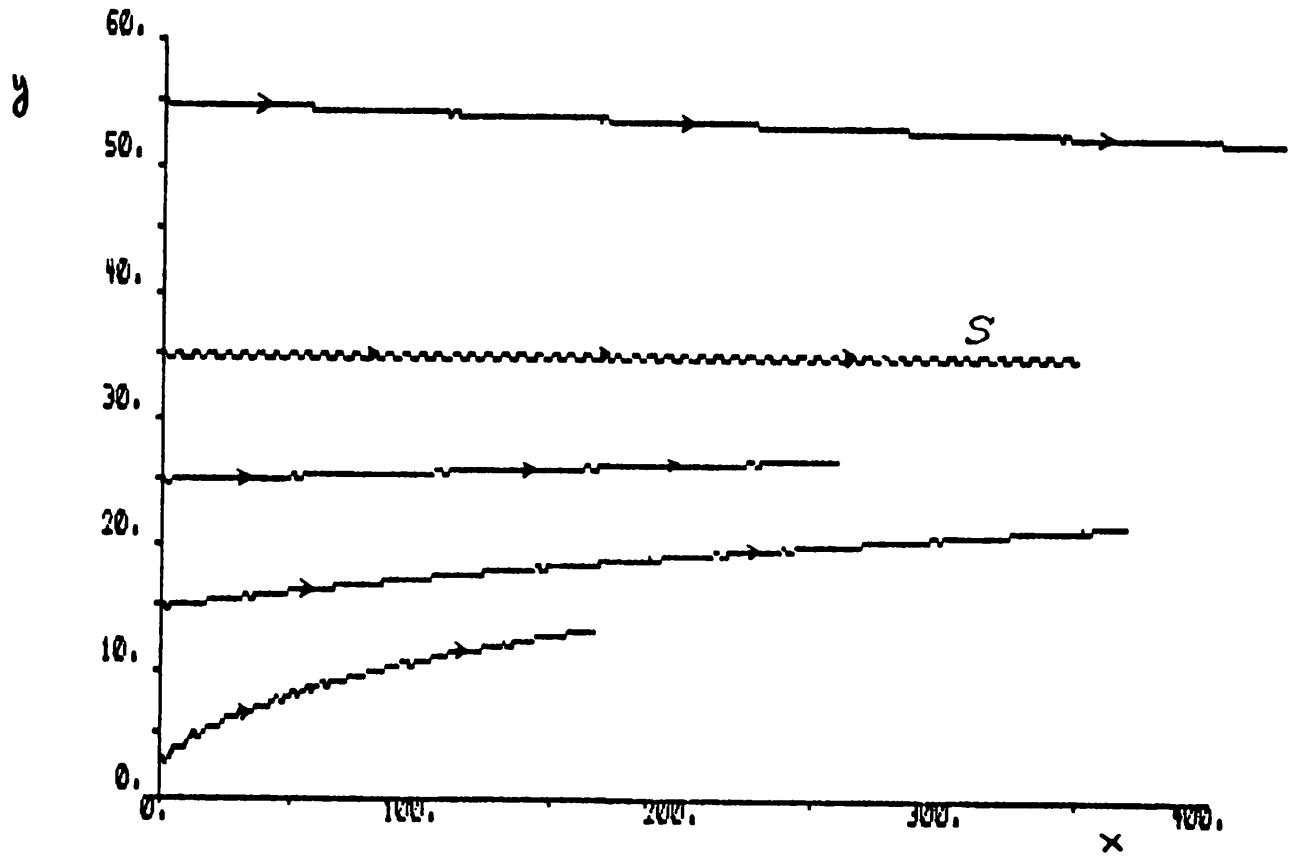

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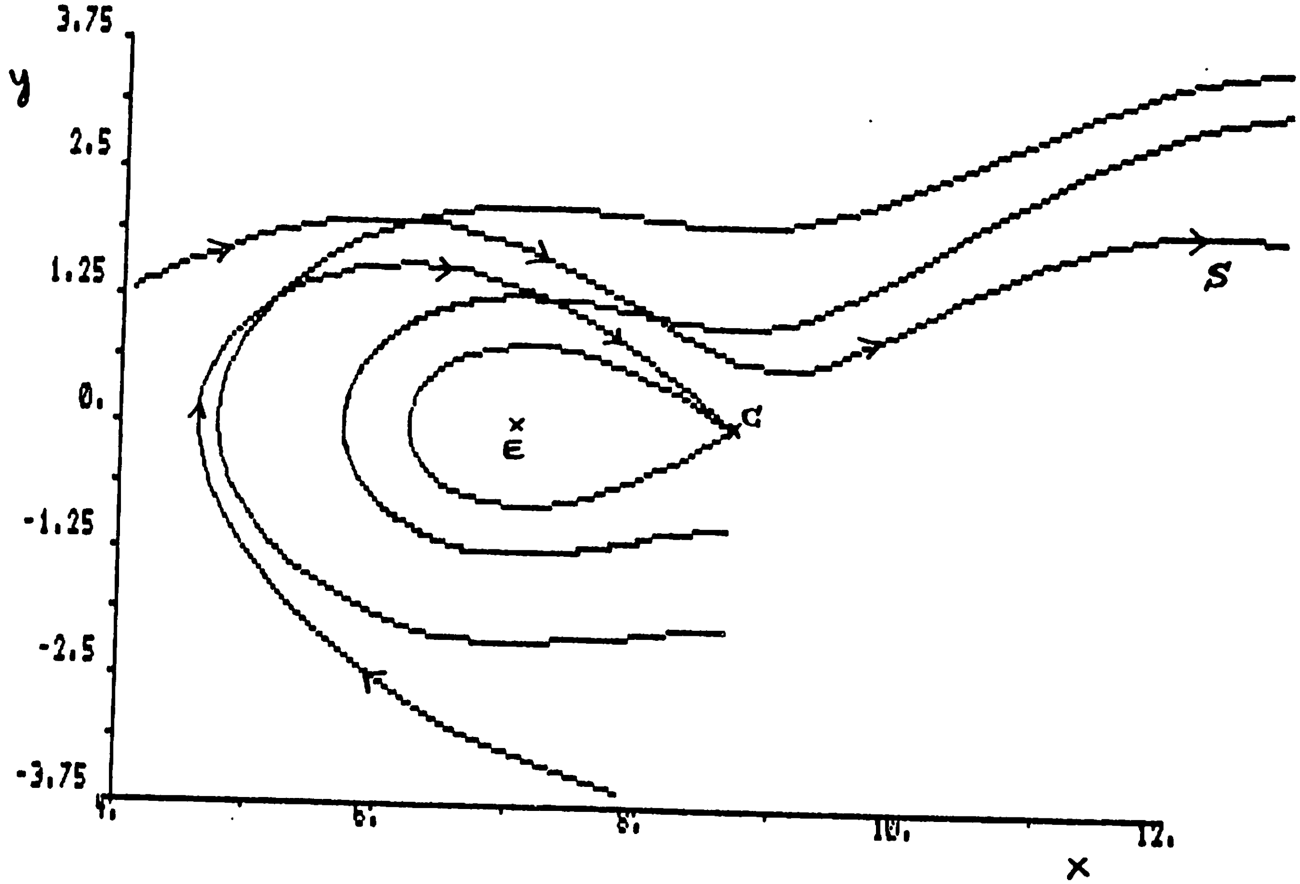

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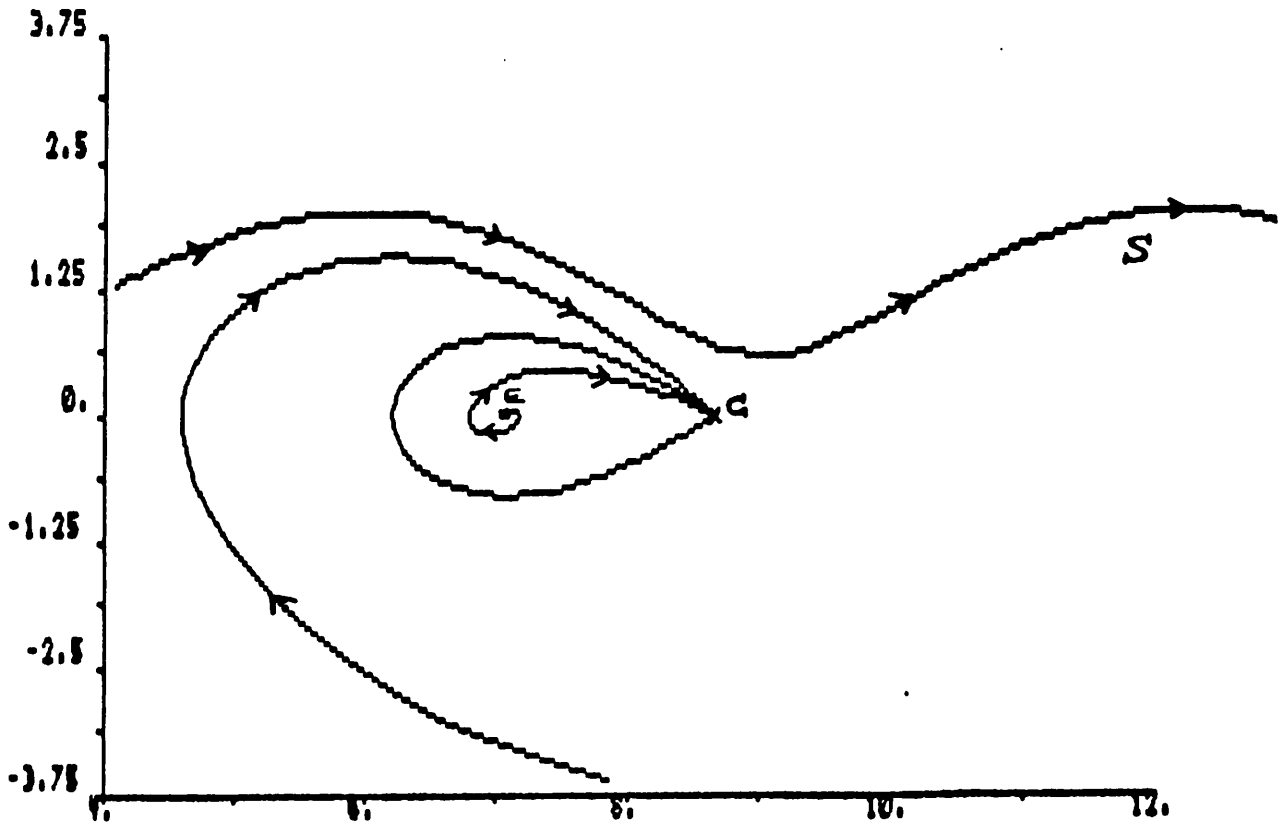

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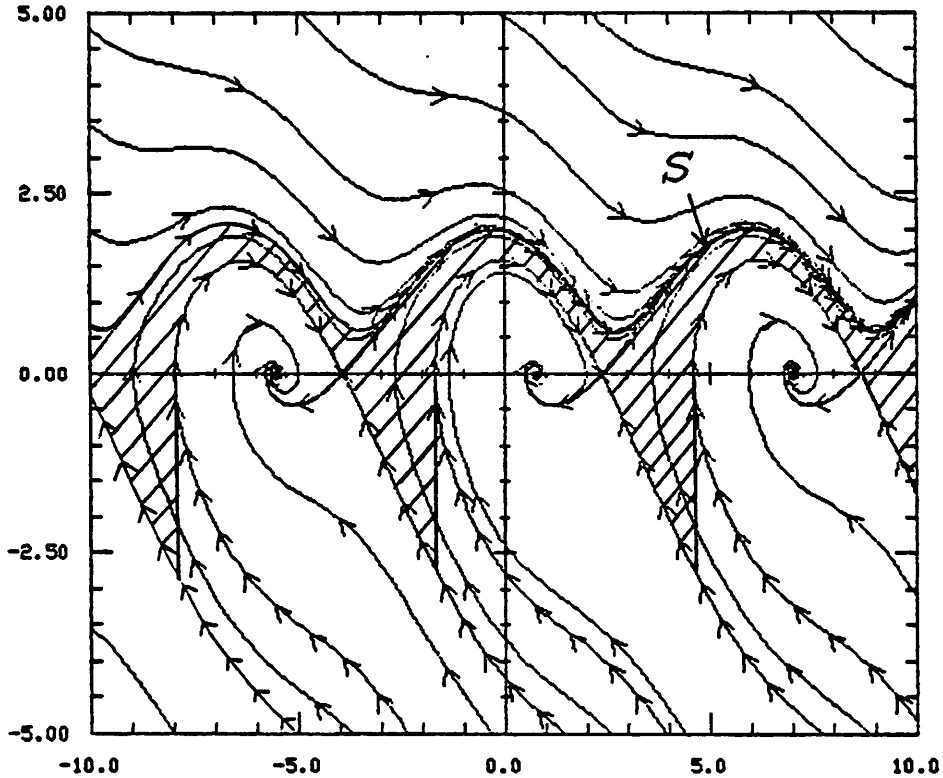

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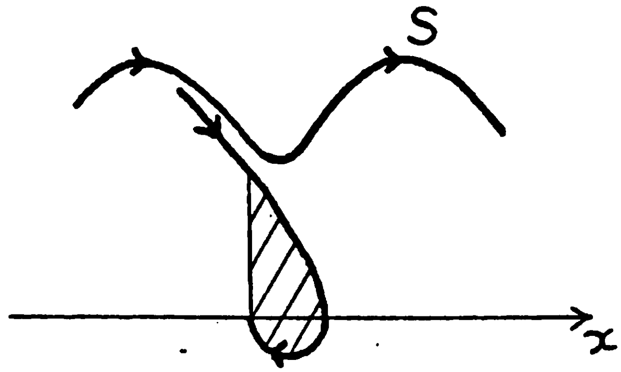

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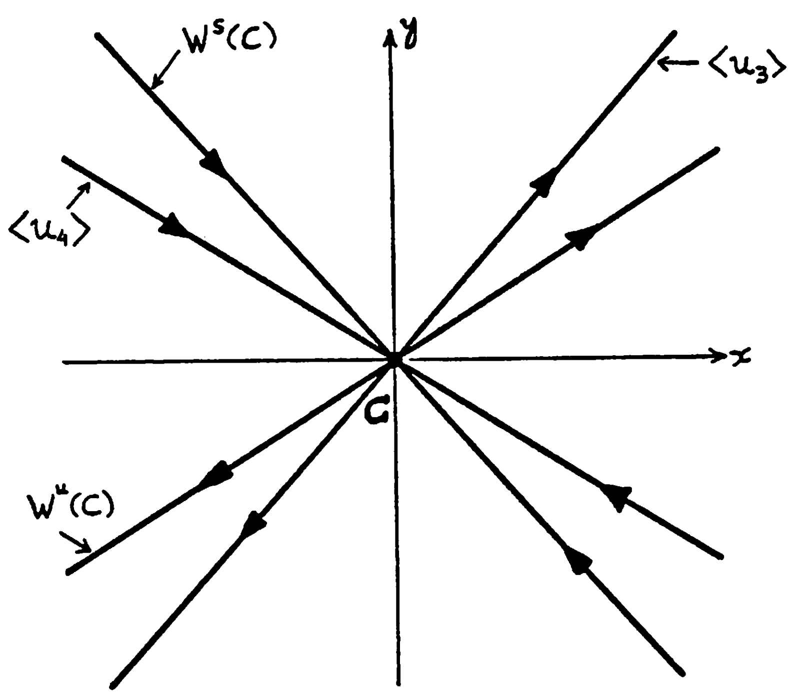

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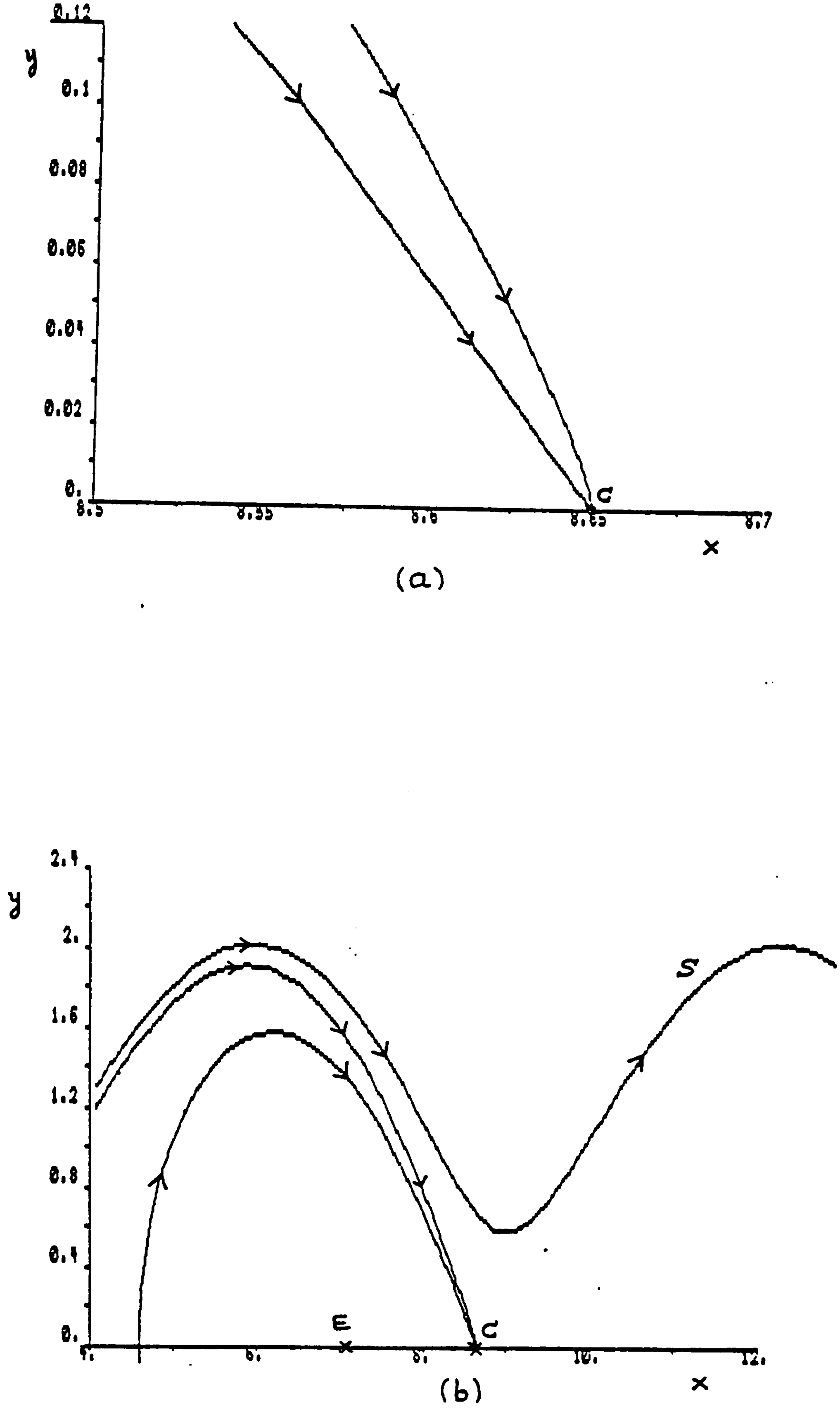

0.12
y
0.1
0.08
0.06
0.04
0.02
0.
8.5
8.55
8.6
8.65
8.7
C
x
(a)
2.4
y
2.
1.6
1.2
0.8
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8.
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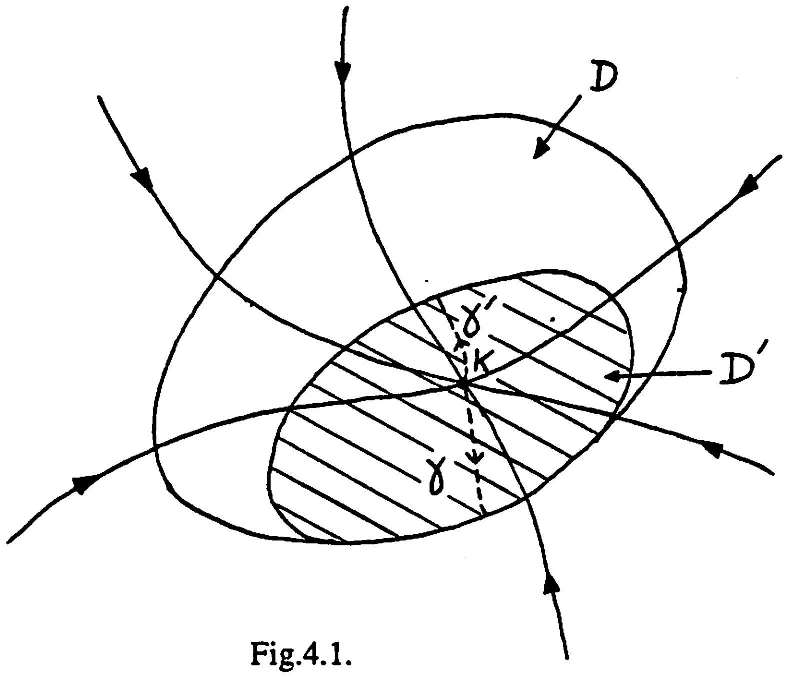
12.
E
C
S
x
(b)

Fig.3.10.



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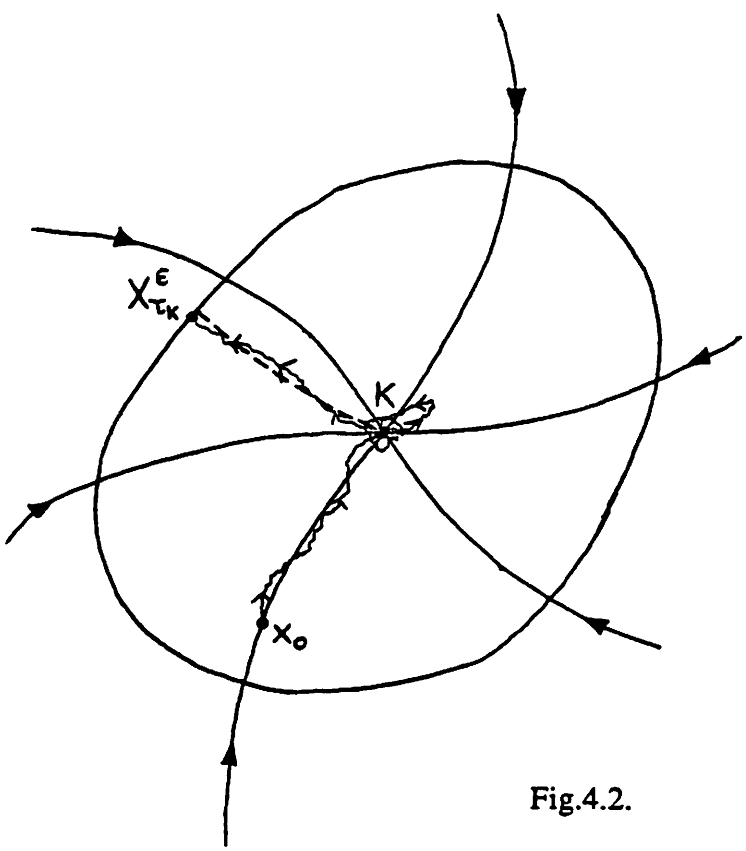

Fig.4.2.