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Stability of Nonlinear Systems with Three Time Scales*

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Abstract

We study the asymptotic stability of a singularly perturbed nonlinear time-invariant system $\mathcal{S}_{\varepsilon\nu}$, which has three vastly different time scales. The system $\mathcal{S}_{\varepsilon\nu}$ is approximated by three simpler systems over different time intervals. We give a straightforward proof of the fact that the asymptotic stability of $\mathcal{S}_{\varepsilon\nu}$ is guaranteed when the equilibrium points of the three simpler systems are exponentially stable and when the parameters ε and ν are sufficiently small.

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I. Introduction

We study the stability of a physical system $\mathcal{P}$ which is roughly modeled by a (mathematical) nonlinear time-invariant system $\mathcal{S}_M$ (to be called the medium system): we say roughly to indicate that some aspects of the dynamics of the physical system $\mathcal{P}$ have been neglected. For example, in electrical circuits, $\mathcal{S}_M$, is obtained from $\mathcal{P}$ by first neglecting stray capacitors and stray inductors, and second by approximating large capacitors by constant voltage sources and large inductors by constant current sources. Let $\mathcal{S}_{\varepsilon\nu}$ denote the (mathematical) system which includes these two classes of additional elements: the small additional elements are viewed as proportional to ε — so ε is *positive* and *small* — and the large ones are viewed as proportional to $1/\nu$ — so ν is *positive* and *small*. Roughly speaking by setting ε and ν equal to zero, $\mathcal{S}_{\varepsilon\nu}$ reduces to $\mathcal{S}_M$: $\mathcal{S}_{\varepsilon\nu}$ is obtained from $\mathcal{S}_M$ by singular perturbation (see, e.g., [Tih.1], [O'Ma.1]; for extensive list of references on singular perturbation see [Kok.1], [Sak.1]).

Systems such as $\mathcal{S}_{\varepsilon\nu}$ can be approximated by three simpler systems for different time scales. The simpler systems are:

i) the ***fast system***, denoted by $\mathcal{S}_F$, whose solution is a good approximation to the solution of the system $\mathcal{S}_{\varepsilon\nu}$ for the fast time scale;

ii) the ***medium system***, denoted by $\mathcal{S}_M$, which is a good approximation to the system $\mathcal{S}_{\varepsilon\nu}$ for the mid-range time scale;

iii) the ***slow system***, denoted by $\mathcal{S}_S$, which approximates the system $\mathcal{S}_{\varepsilon\nu}$ for the slow time scale.

The solutions of the simpler systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ approximate the solution of the system $\mathcal{S}_{\varepsilon\nu}$ for different time scales; stability of the system $\mathcal{S}_{\varepsilon\nu}$ can be guaranteed when the stability of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ is established.

Stability of singularly perturbed systems which appear in many areas of engineering and physics have been studied extensively; for example, Hoppensteadt [Hop.1] studied singularly perturbed nonlinear time-varying systems on $\mathbb{R}_+$, where there is one small parameter ε in the system equation. Later, Hoppensteadt [Hop.2] extended his results to the case of nonlinear time-varying systems represented by ordinary differential equations where there are several small parameters multiplying the derivatives. Habets [Hab.1], Chow [Cho.1], Grujic [Gru.1], Saberi & Khalil [Sab.1,2] studied stability of nonlinear singularly perturbed systems with one small parameter. Khalil [Kha.1] studied stability of a class of nonlinear multiparameter singularly perturbed systems, assuming that the mutual ratios of the parameters are bounded from below and above by positive constants.

Decomposition of autonomous linear systems with multiple time scales described by $A(\varepsilon)$, where $A(\cdot)$ is analytic in ε is carried out in [Cod.1]; this decomposition is extended to the nonlinear case in [Sil.1]. The input-output description of linear systems with multiple time scales described by $A(\varepsilon)$, $B(\varepsilon)$, and $C(\varepsilon)$ is given in [Sil.2].

The stability problem under consideration here is a generalization of the stability problem treated in [Des.1] in which a *linear* time-invariant system with three time scales (the linear version of the system $\mathcal{S}_{\varepsilon\nu}$) has been considered. We choose a parametrization of the time scales of the system $\mathcal{S}_{\varepsilon\nu}$ which can represent many physical situations. We give a geometric interpretation of the approximating systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$, and give an easy to follow proof of the fact that the stability of $\mathcal{S}_{\varepsilon\nu}$ follows from the stability of $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ when ε and ν are small.

II. Problem Statement

II.1. System $\mathcal{S}_{\varepsilon\nu}$

Let $s_{\varepsilon\nu}$ denote the state vector corresponding to the system $\mathcal{S}_{\varepsilon\nu}$, and let $s_{\varepsilon\nu}$ be decomposed as $s_{\varepsilon\nu} := (x, y, z)$. The evolution of the system $\mathcal{S}_{\varepsilon\nu}$ starting at $t = 0$ from (x^0, y^0, z^0) is governed by the following equations:

$$\mathcal{S}_{\varepsilon\nu}: \begin{cases} \dot{x} = f(x,y,z), & x(0) = x^0, \; x(t) \in B_x \subset \mathbb{R}^l, \\ \varepsilon\dot{y} = g(x,y,z), & y(0) = y^0, \; y(t) \in B_y \subset \mathbb{R}^m, \\ \dot{z} = \nu h(x,y,z), & z(0) = z^0, \; z(t) \in B_z \subset \mathbb{R}^n, \end{cases} \tag{2.1}$$

for all $t \in \mathbb{R}_+$. In (2.1), $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, and $\nu \in (0, \nu_0]$, $\nu_0 \ll 1$, and B_x, B_y, and B_z are given balls centered at the origin of $\mathbb{R}^l$, $\mathbb{R}^m$, and $\mathbb{R}^n$, respectively.

Intuitively, since ε is very small, whenever g is nonzero, the velocity $\dot{y}$ is very large: so the component y of the state vector $s_{\varepsilon\nu}$ is associated with the fast time scale; since ν is very small, whenever h is nonzero, the velocity $\dot{z}$ is very small: so the component z of $s_{\varepsilon\nu}$ is associated with the slow time scale; the component x of $s_{\varepsilon\nu}$ corresponds to the mid-range time scale.

II.2. Circuit Example

As an example consider a nonlinear time-invariant circuit, $C_{\varepsilon\nu}$, (see Fig. 1a) which contains fast, usual, and slow elements. The fast element is a (linear) stray capacitor of ε farads, where $0 < \varepsilon \ll 1$; the usual element is a linear inductor of 1 henry; the slow element is a large voltage-controlled capacitor with $C(v_c) = \nu^{-1}(1 + \alpha_c v_c)^{-1}$, where $0 < \nu \ll 1$ is a constant in $(\text{farads})^{-1}$, $\alpha_c > 0$ is a constant in $(\text{volt})^{-1}$, and v_c is the voltage across the capacitor.

The nonlinear resistor r has the characteristic $i_r = \frac{1}{R}\varphi(v_r)$, where R is a constant in ohms, $\varphi(\cdot)$ is strictly increasing with $\varphi'(0) = 1$; here i_r and v_r are the current through and the voltage across the resistor, respectively.

The equations of the circuit $C_{\varepsilon\nu}$ are

$$C_{\varepsilon\nu}: \begin{cases} \dot{x} = -y + z, & x(0) = x^0, \ x(t) \in \mathbb{R}, \\ \varepsilon \dot{y} = x - \frac{1}{R}\varphi(y) & y(0) = y^0, \ y(t) \in \mathbb{R}, \\ \dot{z} = -\nu x(1 + \alpha_c z), & z(0) = z^0, \ z(t) > -\frac{1}{\alpha_c}, \end{cases} \tag{2.2}$$

for all $t \in \mathbb{R}_+$.

II.3. Assumptions on f, g, and h

We make the following assumptions:

A1) f, g, and h are C^1 with respect to their arguments on the ball $B := B_x \times B_y \times B_z$.

By (A1), the system $\mathcal{S}_{\varepsilon\nu}$ has a unique local solution $t \mapsto s_{\varepsilon\nu}(t)$ in B, and the solution is a C^1 function of the initial conditions $(x^0, y^0, z^0) \in B$ (see, e.g., [Die.1, Theorem 10.8.2]). Furthermore, we assume that for all (x^0, y^0, z^0) in some smaller ball $B^0 := B_x^0 \times B_y^0 \times B_z^0 \subseteq B$, the solution of the system $\mathcal{S}_{\varepsilon\nu}$ stays inside B; the balls $B_x^0 \subseteq B_x$, $B_y^0 \subseteq B_y$, and $B_z^0 \subseteq B_z$ are centered at the origin of $\mathbb{R}^l$, $\mathbb{R}^m$, and $\mathbb{R}^n$, respectively.

A2) $D_2 g(x,y,z)$ is nonsingular, $\forall \ (x,y,z) \in B$.

By (A2), and the implicit function theorem (see, e.g., [Die.1, Theorem 10.2.1])

$$\mathcal{M}_{xz} := \{(x,y,z) \in B : g(x,y,z) = \vartheta_m\}, \tag{2.3}$$

is *locally* an $(l+n)$-dimensional C^1 manifold in B (see, e.g., [Boo.1]). In addition, we *assume* that the following holds over B:

$$\begin{aligned} (x,y,z) \in \mathcal{M}_{xz} &\Leftrightarrow g(x,y,z) = \vartheta_m, \\ &\Leftrightarrow y = f_1(x,z), \end{aligned} \tag{2.4}$$

where f_1 is a C^1 function on $B_x \times B_z$. In other words, we assume that the pair

(x,z) is a global parametrization of the manifold $\mathcal{M}_{xz}$ on $B_x \times B_z$.

A3) The Jacobian of (f,g) with respect to (x,y), $\frac{\partial(f,g)}{\partial(x,y)}$, is nonsingular, $\forall\ (x,y,z) \in B$.

By (A3), and the implicit function theorem

$$\mathcal{M}_z := \{(x,y,z) \in B : f(x,y,z) = \vartheta_l,\ g(x,y,z) = \vartheta_m\} \subset \mathcal{M}_{xz}, \qquad (2.5)$$

is *locally* an n-dimensional C^1 manifold in B. In addition, we *assume* that the following holds over B:

$$(x,y,z) \in \mathcal{M}_z \Leftrightarrow f(x,y,z) = \vartheta_l,\ g(x,y,z) = \vartheta_m,$$
$$\Leftrightarrow x = h_1(z),\ y = h_2(z), \qquad (2.6)$$

where h_1 and h_2 are C^1 functions on B_z: thus z is a global parametrization of the manifold $\mathcal{M}_z$ on B_z.

A4) $\vartheta_{\varepsilon\nu} := (\vartheta_l, \vartheta_m, \vartheta_n)$ is the *unique* equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$ in B, i.e., for $(x,y,z) \in B$,

$$f(x,y,z) = \vartheta_l, \qquad (2.7a)$$
$$g(x,y,z) = \vartheta_m, \qquad (2.7b)$$
$$h(x,y,z) = \vartheta_n, \qquad (2.7c)$$

iff $x = \vartheta_l$, $y = \vartheta_m$, and $z = \vartheta_n$.

By (A4), we have

$$f_1(\vartheta_l, \vartheta_n) = \vartheta_m, \qquad (2.8a)$$
$$h_1(\vartheta_n) = \vartheta_l,\ h_2(\vartheta_n) = \vartheta_m. \qquad (2.8b)$$

Clearly, if $\vartheta_{\varepsilon\nu}$ is an asymptotically stable equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$, then its solution $t \mapsto s_{\varepsilon\nu}(t)$ starting at $t = 0$ from a point in B^0 tends to $\vartheta_{\varepsilon\nu}$ as $t \to \infty$.

In what follows, we will show that the asymptotic stability of this equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$ is guaranteed when the equilibrium points of three simpler systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$, to be introduced below, are exponentially stable.

III. Approximations to $\mathcal{S}_{\varepsilon\nu}$

III.1. Systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$

In the following we introduce the fast, medium, and slow systems denoted by $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$, respectively.

System $\mathcal{S}_F$: In order to study the behavior of the system $\mathcal{S}_{\varepsilon\nu}$ for t small, we let $t = \varepsilon\tau$ in (2.1), and then let $\varepsilon = 0$; we obtain

$$\frac{dx}{d\tau} = \vartheta_l \iff x(t) = x^0 = \text{const.}, \tag{3.1a}$$

$$\mathcal{S}_F : \frac{dy}{d\tau} = g(x^0, y, z^0), \; y(0) = y^0, \tag{3.1b}$$

$$\frac{dz}{d\tau} = \vartheta_n \iff z(t) = z^0 = \text{const.} \tag{3.1c}$$

We define $\mathcal{S}_F$ to be the system represented by the differential equation (3.1b).

We think of the system $\mathcal{S}_F$ as the fast-time scale approximation to the system $\mathcal{S}_{\varepsilon\nu}$ valid for small t.

The fast circuit, C_F, corresponding to the circuit $C_{\varepsilon\nu}$ is given in Fig. 1b, and is obtained by open-circuiting the inductor in $C_{\varepsilon\nu}$. The equation of the circuit C_F is

$$C_F : \frac{dy}{d\tau} = -\frac{1}{R}\varphi(y), \; y(0) = y^0. \tag{3.2}$$

Since g (by (A1)) is C^1, the system $\mathcal{S}_F$ has a unique local solution in B_y, and the solution is a C^1 function of $(x^0, y^0, z^0) \in B^0$, where x^0 and z^0 are viewed as parameters. (For C^1 dependence of the solution on parameters see, e.g. [Die.1, Theorem 10.7.4].)

System $\mathcal{S}_M$: If in (2.1) we want to consider the case $\varepsilon \to 0$ and $\nu \to 0$, it seems natural to consider

$$\dot{x} = f(x,y,z^0),\ x(0) = x^0, \tag{3.3a}$$

$$g(x,y,z^0) = \vartheta_m, \tag{3.3b}$$

$$\dot{z} = \vartheta_n \iff z(t) = z^0 = \text{const.} \tag{3.3c}$$

By (2.4), we can replace the algebraic equation (3.3b) by

$$y = f_1(x,z^0). \tag{3.4}$$

Substituting y from (3.4) into (3.3a) we define $\mathcal{S}_M$ to be the system represented by the following equations:

$$\mathcal{S}_M : \begin{cases} \dot{x} = f(x,\ f_1(x,z^0),\ z^0),\ x(0) = x^0, & (3.5a) \\ y = f_1(x,z^0). & (3.5b) \end{cases}$$

We think of the system $\mathcal{S}_M$ as the mid-range time scale approximation to the system $\mathcal{S}_{\varepsilon\nu}$.

The medium circuit, C_M, corresponding to the circuit $C_{\varepsilon\nu}$ is given in Fig. 1c, and is obtained by open-circuiting the stray capacitor and short-circuiting the large capacitor in $C_{\varepsilon\nu}$. Let $\varphi_1(\cdot)$ be the inverse function of $\frac{1}{R}\varphi(\cdot)$, then the equations of the circuit C_M are

$$C_M : \begin{cases} \dot{x} = -\varphi_1(x), x(0) = x^0, & (3.6a) \\ y = \varphi_1(x). & (3.6b) \end{cases}$$

Since f (by (A1)) and f_1 are C^1, the system $\mathcal{S}_M$ has a unique local solution in $B_x \times B_y$, and the solution is a C^1 function of $(x^0,\ z^0) \in B_x^0 \times B_z^0$, where z^0 is viewed as a parameter. By (3.3b) the trajectory corresponding to the solution of the system $\mathcal{S}_M$ lies on the manifold $\mathcal{M}_{xz}$, $\forall\ (x^0,\ z^0) \in B_x^0 \times B_z^0$.

System $\mathcal{S}_S$: In order to study the behavior of the system $\mathcal{S}_{\varepsilon\nu}$ for t large, we let $t = \frac{1}{\nu}t'$ in (2.1), and then let $\nu = 0$; we obtain

$$f(x,y,z) = \vartheta_l, \tag{3.7a}$$

$$g(x,y,z) = \vartheta_m, \tag{3.7b}$$

$$\frac{dz}{dt'} = h(x,y,z),\ z(0) = z^0. \tag{3.7c}$$

By (2.6), we can replace the algebraic equations (3.7a) and (3.7b) by

$$x = h_1(z), \tag{3.8a}$$

$$y = h_2(z), \tag{3.8b}$$

Substituting x and y from (3.8a) and (3.8b), respectively, into (3.7c) we define $\mathcal{S}_S$ to be the system represented by the following equations:

$$\mathcal{S}_S : \begin{cases} x = h_1(z), & \text{(3.9a)} \\ y = h_2(z), & \text{(3.9b)} \\ \dfrac{dz}{dt'} = h(h_1(z),\ h_2(z),\ z),\ z(0) = z^0. & \text{(3.9c)} \end{cases}$$

We think of the system $\mathcal{S}_S$ as the slow-time-scale approximation to the system $\mathcal{S}_{\varepsilon\nu}$ valid for large t.

The slow circuit, C_S, corresponding to the circuit $C_{\varepsilon\nu}$ is given in Fig. 1d, and is obtained by open-circuiting the stray capacitor and short-circuiting the inductor in $C_{\varepsilon\nu}$. The equations of the circuit C_S are

$$C_S : \begin{cases} x = \dfrac{1}{R}\varphi(z), & \text{(3.10a)} \\ y = z, & \text{(3.10b)} \\ \dfrac{dz}{dt'} = -\dfrac{1}{R}\varphi(z)(1 + \alpha_c z),\ \ z(0) = z^0. & \text{(3.10c)} \end{cases}$$

Since h (by (A1)), h_1, and h_2 are C^1, the system $\mathcal{S}_S$ has a unique local solution in B, and the solution is a C^1 function of $z^0 \in B_z^0$. By (3.7a) and (3.7b) the trajectory corresponding to the solution of $\mathcal{S}_S$ lies on the manifold $\mathcal{M}_z$, $\forall$ $(x^0, y^0, z^0) \in B^0$.

For future reference, note that by (2.4) and (2.6) for all $z \in B_z$, we have

$$h_2(z) = f_1(h_1(z), z). \tag{3.11}$$

III.2. Geometric Interpretation

By (2.4), for fixed $x^0 \in B_x^0$, $z^0 \in B_z^0$ and any $y^0 \in B_y^0$, $y_F^e := f_1(x^0, z^0)$ is the unique equilibrium point of the system $\mathcal{S}_F$, and the point $(x^0, y_F^e, z^0) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ lies on the manifold $\mathcal{M}_{xz}$. By (2.6), for fixed $z^0 \in B_z^0$ and any $x^0 \in B_x^0$, and hence $y^0 = f_1(x^0, z^0)$, $(x_M^e, y_M^e) := (h_1(z^0), h_2(z^0))$ is the unique equilibrium point of the system $\mathcal{S}_M$, and the point $(x_M^e, y_M^e, z^0) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ lies on the manifold $\mathcal{M}_z$. Clearly, $(\vartheta_l, \vartheta_m, \vartheta_n)$ is the unique equilibrium point of the system $\mathcal{S}_S$.

To help visualize the evolution of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$, let $l = m = n = 1$ (e.g., the circuit example), and let the equilibrium points of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ be exponentially stable; then we have the situation depicted on Fig. 2. Starting at $t = 0$ from a given initial point $(x^0, y^0, z^0) \in B^0$, the trajectory corresponding to the solution of the fast system $\mathcal{S}_F$ (depicted by the solid vertical line ① in Fig. 2) moves down to the point $(x^0, f_1(x^0, z^0), z^0) \in \mathcal{M}_{xz}$. Given $(x^0, z^0) \in B_x^0 \times B_z^0$, the trajectory corresponding to the solution of the system $\mathcal{S}_M$ (depicted by the solid line ② in Fig. 2) starts at the point $(x^0, f_1(x^0, z^0), z^0) \in \mathcal{M}_{xz}$ and while lying in the manifold $\mathcal{M}_{xz}$ with $z(t) = z^0$, converges to the point $(h_1(z^0), h_2(z^0), z^0) \in \mathcal{M}_z$. Given $z^0 \in B_z^0$, the trajectory corresponding to the solution of the slow system $\mathcal{S}_S$ (depicted by the solid line ③ in Fig. 2) starts at the point $(h_1(z^0), h_2(z^0), z^0) \in \mathcal{M}_z$ and while lying in the manifold $\mathcal{M}_z$, drifts to the origin.

For ε and ν sufficiently small, it turns out that the solution of the system $\mathcal{S}_{\varepsilon\nu}$ (the trajectory corresponding to this solution is shown by the dashed line in Fig. 2) is approximated by the solutions of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ described above.

IV. Stability Considerations

IV.1. Physical Interpretation

We start with an intuitive discussion: assuming that the equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$, $\vartheta_{\varepsilon\nu} = (\vartheta_l, \vartheta_m, \vartheta_n)$ is asymptotically stable, and $\varepsilon_0 \ll 1$, $\nu_0 \ll 1$, we would expect that the solutions of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ are good approximations to the solution of the system $\mathcal{S}_{\varepsilon\nu}$ over appropriate time intervals. In fact, starting at $t = 0$ from an initial point $(x^0, y^0, z^0) \in B^0$ off the manifold $\mathcal{M}_{xz}$, the solution of the system $\mathcal{S}_{\varepsilon\nu}$ is close to the solution of the fast system, $\mathcal{S}_F$, over a short time interval. Refer to Fig. 2 and consider the trajectory corresponding to the solution of the system $\mathcal{S}_{\varepsilon\nu}$ (shown by the dashed line), which is close to the (vertical) trajectory (labeled ① in Fig. 2) corresponding to the solution of the system $\mathcal{S}_F$. Next, for (x, y, z) close to the manifold $\mathcal{M}_{xz}$, the solution of the system $\mathcal{S}_{\varepsilon\nu}$ will be close to the solution of the medium system, $\mathcal{S}_M$ (whose trajectory is labeled ② in Fig. 2), provided that $\forall\ (x, y, z) \in B^0$, the equilibrium point of the system $\mathcal{S}_F$, $y_F^e = f_1(x, z)$ is (exponentially) stable, i.e., the manifold $\mathcal{M}_{xz}$ is (exponentially) stable in B^0. Finally, as t becomes very large, the solution of the system $\mathcal{S}_{\varepsilon\nu}$ will be close to that of the slow system, $\mathcal{S}_S$ (whose trajectory is labeled ③ in Fig. 2), provided that $\forall\ (x, z) \in B_x^0 \times B_z^0$, the equilibrium point of the system $\mathcal{S}_M$, $(x_M^e, y_M^e) = (h_1(z), h_2(z))$ is (exponentially) stable. For large t, the solution of the systems $\mathcal{S}_{\varepsilon\nu}$ and $\mathcal{S}_S$ remain close and converge to $\vartheta_{\varepsilon\nu} = (\vartheta_l, \vartheta_m, \vartheta_n)$ as $t \to \infty$, provided that the equilibrium point of the system $\mathcal{S}_S$, $(\vartheta_l, \vartheta_m, \vartheta_n)$ is (exponentially) stable.

Thus we expect that, if the equilibrium points of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ have sufficiently strong stability properties, and if $\varepsilon > 0$ and $\nu > 0$ are sufficiently small, then the system $\mathcal{S}_{\varepsilon\nu}$ is asymptotically stable.

IV.2. Three Observations

We will have to established relations between the dynamics of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ and the system $\mathcal{S}_{\varepsilon\nu}$. To accomplish this we will need some inequalities; in particular we will need to know about the vector field of some of these systems and their behavior near the manifolds defined above. The purpose of this section is to establish the inequalities (4.4), (4.7), (4.8), (4.9), and (4.12) given below.

i) Using (3.11), we note that $\forall\ (x,y,z) \in B$

$$\|y - h_2(z)\| \le \|y - f_1(x,z)\| + \|f_1(x,z) - f_1(h_1(z), z)\|. \tag{4.1}$$

Since f_1 is C^1 on $B_x \times B_z$, $\forall\ (x,z) \in B_x \times B_z$, we have (see, e.g., [Die.1, Theorem 8.5.4])

$$\|f_1(x,z) - f_1(h_1(z), z)\| \le K_1\|x - h_1(z)\|, \tag{4.2}$$

where

$$K_1 = \sup_{(x,z)\in B_x \times B_z} \|D_1 f_1(x,z)\|. \tag{4.3}$$

Using (4.2) in (4.1) we conclude that $\forall\ (x,y,z) \in B$

$$\|y - h_2(z)\| \le \|y - f_1(x,z)\| + K_1\|x - h_1(z)\|. \tag{4.4}$$

ii) By (A1), f is C^1 on B, hence there are positive constants d_1, d_2, and d_3 such that $\forall\ (x,y,z) \in B$

$$\|f(x,y,z)\| \le d_1\|x\| + d_2\|y\| + d_3\|z\|. \tag{4.5}$$

We can rewrite (4.5) as

$$\|f(x,y,z)\| \le d_1\|x - h_1(z)\| + d_1\|h_1(z)\| + d_2\|y - h_2(z)\| + d_2\|h_2(z)\| + d_3\|z\|. \tag{4.6}$$

Since h_2 is C^1 on B_z, there is a positive constant d_4 such that $\|h_2(z)\| \le d_4\|z\|$; using this inequality and (4.4) in (4.6), for some positive constant K_2, $\forall\ (x,y,z) \in B$, we have

$$\|f(x,y,z)\| \le K_2(\|z\| + \|x - h_1(z)\| + \|y - h_1(x,z)\|). \tag{4.7}$$

Similarly, since h is C^1 on B (by (A1)), for some positive constant K_3, $\forall\ (x,y,z) \in B$, we have

$$\|h(x,y,z)\| \le K_3(\|z\| + \|x - h_1(z)\| + \|y - f_1(x,z)\|). \tag{4.8}$$

iii) Since f is C^1 on B, $\forall\ (x,y,z) \in B$, we have

$$\|f(x,y,z) - f(x, f_1(x,z), z)\| \le K_4\|y - f_1(x,z)\|, \tag{4.9}$$

where

$$K_4 = \sup_{(x,y,z)\in B} \|D_2 f(x,y,z)\|. \tag{4.10}$$

Similarly, since h is C^1 on B, for some positive constant K_5, $\forall\ (x,y,z) \in B$, we have

$$\|h(x,y,z) - h(h_1(z), h_2(z), z)\| \le K_5(\|x - h_1(z)\| + \|y - h_2(z)\|). \tag{4.11}$$

Using (4.4) in (4.11) we have

$$\|h(x,y,z) - h(h_1(z), h_2(z), z)\| \le K_5[(1 + K_1)\|x - h_1(z)\| + \|y - f_1(x,z)\|]. \tag{4.12}$$

In (4.4), (4.7), (4.8), (4.9), and (4.12) we will replace K_1, K_2, K_3, K_4, and K_5 by $K := \max(K_1, K_2, K_3, K_4, K_5)$, in order to simplify the later computations.

IV.3. Three Assumptions on Stability

In the section IV.2, in order to simplify the later computations, we replaced the constants K_i, $i=1,\ldots,5$, by the single constant K defined above. In the following we will introduce three Lyapunov functions V_F, V_M, and V_S each with two bounding functions of the class $\mathcal{K}$ (see, e.g., [Hah.1]), and two constants which appear in the bounds of their derivatives (see, e.g., (4.16), (4.17a), and (4.17b)). As in the section above, without loss of generality, we replace each pair of the bounding functions by two bounding functions $a(\cdot)$ and $b(\cdot)$ belonging to the class $\mathcal{K}$, and each pair of the constants by two constants k and d.

A5) For all $(x^0, z^0) \in B_x^0 \times B_z^0$, $y_F^e = f_1(x^0, z^0)$, the equilibrium point of the system $\mathcal{S}_F$ is *exponentially* stable.

By (A5), there exist positive constants α_F and β_F such that $\forall\ (x^0, z^0) \in B_x^0 \times B_z^0$, $\forall\ y^0 \in B_y^0$, and $\forall\ \tau \geq 0$

$$\|\Phi_F(x^0, z^0; \tau, y^0) - f_1(x^0, z^0)\| \leq \alpha_F\, e^{-\beta_F \tau} \|y^0 - f_1(x^0, z^0)\|, \quad (4.13)$$

where $\tau \mapsto \Phi_F(x^0, z^0; \tau, y^0)$ denotes the solution of (3.1b) starting at $\Phi_F(x^0, z^0; 0, y^0) = y^0$ and depending on the parameters x^0 and z^0. Since g is C^1, by the converse theorem (see, e.g., [Hah.1, Theorem 56.1], [Hab.1, Lemma 2]), there exists a Lyapunov function $V_F : B^0 \to \mathbb{R}_+$, given as follows: choose $T_F = (2 \ln \alpha_F + \ln 2)/2\beta_F$, and define

$$V_F(x, y, z) = \int_0^{T_F} \|\Phi_F(x, z; \tau, y) - f_1(x, z)\|^2\, d\tau; \quad (4.14)$$

then

$$a(\|y - f_1(x,z)\|) \leq V_F(x,y,z) \leq b(\|y - f_1(x,z)\|), \quad (4.15)$$

where the functions $a(\cdot)$ and $b(\cdot)$ belong to the class $\mathcal{K}$; furthermore, there are positive constants k and d such that

$$\frac{\partial V_F(x,y,z)}{\partial y} g(x,y,z) \leq -k\|y - f_1(x,z)\|^2, \tag{4.16}$$

$$\left| \frac{\partial V_F(x,y,z)}{\partial x} \right| \leq d\|y - f_1(x,z)\|, \tag{4.17a}$$

$$\left| \frac{\partial V_F(x,y,z)}{\partial z} \right| \leq d\|y - f_1(x,z)\|. \tag{4.17b}$$

A6) For all $z^0 \in B_z^0$, $(x_M^e, y_M^e) = (h_1(z^0), h_2(z^0))$, the equilibrium point of the system $\mathcal{S}_M$ is *exponentially* stable.

Note that, by (3.11), $\forall\ z^0 \in B_z^0$, $y_M^e = h_2(z^0) = f_1(h_1(z^0), z^0)$; hence (A6) is equivalent to the assumption that $\forall\ z^0 \in B_z^0$, $h_1(z_0)$ is exponentially stable equilibrium point of (3.5a); hence, there exist positive constants α_M and β_M such that $\forall\ z^0 \in B_z^0$, $\forall\ x^0 \in B_x^0$, and $\forall\ t \geq 0$

$$\|\Phi_M(z^0; t, x^0) - h_1(z^0)\| \leq \alpha_M e^{-\beta_M t} \|x^0 - h_1(z^0)\|, \tag{4.18}$$

where $t \mapsto \Phi_M(z^0; t, x^0)$ denotes the solution of (3.5a) starting at $\Phi_M(z^0; 0, x^0) = x^0$ and depending on the parameter z^0. Since f, and f_1 are C^1, by the converse theorem, there exists a Lyapunov function $V_M : B_x^0 \times B_z^0 \to \mathbb{R}_+$, given as follows: choose $T_M = (2 \ln \alpha_M + \ln 2)/2\beta_M$, and define

$$V_M(x,z) = \int_0^{T_M} \|\Phi_M(z; t, x) - h_1(z)\|^2 \, dt; \tag{4.19}$$

then

$$a(\|x - h_1(z)\|) \leq V_M(x,z) \leq b(\|x - h_1(z)\|), \tag{4.20}$$

and

$$\frac{\partial V_M(x,z)}{\partial x} f(x, f_1(x,z), z) \leq -k\|x - h_1(z)\|^2, \tag{4.21}$$

$$\left| \frac{\partial V_M(x,z)}{\partial x} \right| \leq d\|x - h_1(z)\|, \tag{4.22a}$$

$$\left|\frac{\partial V_M(x,z)}{\partial z}\right| \leq d\|x - h_1(z)\|. \tag{4.22b}$$

A7) The equilibrium point of the system $\mathcal{S}_S$, $(\vartheta_l, \vartheta_m, \vartheta_n)$ is *exponentially* stable.

Note that, by (2.8b), (A7) is equivalent to the assumption that ϑ_n is exponentially stable equilibrium point of (3.9c), i.e., there exist positive constants α_S and β_S such that $\forall\ z^0 \in B_z^0$, and $\forall\ t' \geq 0$

$$\|\Phi_S(t', z^0)\| \leq \alpha_S\, e^{-\beta_S t'}\|z^0\|, \tag{4.23}$$

where $t' \mapsto \Phi_S(t', z^0)$ denotes the solution of (3.9c) starting at $\Phi_S(0, z^0) = z^0$. Since h, h_1, and h_2 are C^1, by the converse theorem, there exists a Lyapunov function $V_S : B_z^0 \to \mathbb{R}_+$, given as follows: choose $T_S = (2\ln\alpha_S + \ln 2)/2\beta_S$, and define

$$V_S(z) = \int_0^{T_S} \|\Phi_S(t', z)\|^2\, dt'; \tag{4.24}$$

then

$$a(\|z\|) \leq V_S(z) \leq b(\|z\|), \tag{4.25}$$

and

$$\frac{\partial V_S(z)}{\partial z} h(h_1(z), h_2(z), z) \leq -k\|z\|^2, \tag{4.26}$$

$$\left|\frac{\partial V_S(z)}{\partial z}\right| \leq d\|z\|. \tag{4.27}$$

V. Stability of the System $\mathcal{S}_{\varepsilon\nu}$

V.1. Stability Result

We are prepared to prove that $\vartheta_{\varepsilon\nu} = (\vartheta_l, \vartheta_m, \vartheta_n)$ is the *asymptotically* stable equilibrium of the system $\mathcal{S}_{\varepsilon\nu}$.

Theorem: Let (A1)-(A7) hold, then there exist $0 < \varepsilon_0 \ll 1$ and $0 < \nu_0 \ll 1$ such that $\forall\ \varepsilon \in (0, \varepsilon_0]$, $\forall\ \nu \in (0, \nu_0]$, and $\forall\ (x^0, y^0, z^0) \in B^0$, $\vartheta_{\varepsilon\nu} = (\vartheta_l, \vartheta_m, \vartheta_n)$, the equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$ is asymptotically stable.

Proof: We define $V_{\varepsilon\nu} : B^0 \to \mathbb{R}_+$ by

$$V_{\varepsilon\nu}(x,y,z) := V_S(z) + V_M(x,z) + V_F(x,y,z), \tag{5.1}$$

and prove that $V_{\varepsilon\nu}$ is a Lyapunov function for the system $\mathcal{S}_{\varepsilon\nu}$ and $D_{[2.1]}\ V_{\varepsilon\nu} < 0$ along the trajectories corresponding to the solution of (2.1), provided that ε and ν are chosen sufficiently small. We compute $D_{[2.1]}\ V_{\varepsilon\nu}$.

$$\begin{aligned} D_{[2.1]}\ V_{\varepsilon\nu}(x,y,x) = \nu\ & \frac{\partial V_S(z)}{\partial z}[h(x,y,z) - h(h_1(z), h_2(z), z) \\ & \qquad + h(h_1(z), h_2(z), z)] \\ & + \frac{\partial V_M(x,z)}{\partial x}[f(x,y,z) - f(x, f_1(x,z), z) \\ & \qquad + f(x, f_1(x,z), z)] \\ & + \nu \frac{\partial V_M(x,z)}{\partial z} h(x,y,z) \\ & + \frac{\partial V_F(x,y,z)}{\partial x} f(x,y,z) \\ & + \frac{1}{\varepsilon}\frac{\partial V_F(x,y,z)}{\partial y} g(x,y,z) \\ & + \nu \frac{\partial V_F(x,y,z)}{\partial z} h(x,y,z) \end{aligned} \tag{5.2}$$

Using inequalities (4.11), (4.26), and (4.27) in the first term on the right hand side of (5.2), inequalities (4.9), (4.21), and (4.22a) in the second term, inequalities (4.8) and (4.22b) in the third term, inequalities (4.7) and (4.17d) in the fourth term, inequality (4.16) in the fifth term, and inequalities (4.8) and (4.17b) in the sixth term, we obtain

$$D_{[2.1]}\, V_{\varepsilon\nu}(x,y,z) \le -v^T A\, v, \tag{5.3}$$

where $v^T = [\|z\|, \|x - h_1(z)\|, \|y - f_1(x,z)\|]$, and

$$A = \begin{bmatrix} \nu k & -\frac{1}{2}\nu dK(2+K) & -\frac{1}{2}(2\nu+1)dK \\ -\frac{1}{2}\nu dK(2+K) & k - \nu dK & -(\nu+1)dK \\ -\frac{1}{2}(2\nu+1)dK & -(\nu+1)dK & \frac{1}{\varepsilon} - (\nu+1)dK \end{bmatrix}. \tag{5.4}$$

For appropriate values of ε and ν, the matrix A will be positive definite, for example for any ε and ν satisfying

$$\nu < \frac{k/dK}{1 + dK(2+K)^2/4k} =: \nu^0, \tag{5.5}$$

$$\varepsilon < \frac{N_\varepsilon}{D_\varepsilon} =: \varepsilon^0, \tag{5.6}$$

where

$$N_\varepsilon = k\nu(k - \nu dK) - \nu^2 d^2 K^2 (2+K)^2/4, \tag{5.7a}$$

$$\begin{aligned} D_\varepsilon = k(2\nu+1)^2 d^2 K^2/4 + k^2\nu(\nu+1)dK \\ + \nu(\nu+1)(2\nu+1)d^3K^3/2 \\ + k\nu(\nu+1)^2 d^2 K^2, \end{aligned} \tag{5.7b}$$

the matrix A is positive definite, and $D_{[2.1]}\, V_{\varepsilon\nu} < 0$; hence $\vartheta_{\varepsilon\nu}$ is the asymptotically stable equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$. ∎

V.2. *Conclusions*

We studied the asymptotic stability of the equilibrium point of a singularly perturbed time-invariant nonlinear system $\mathcal{S}_{\varepsilon\nu}$. Due to the specific model of the system $\mathcal{S}_{\varepsilon\nu}$, we can approximate it over different time scales by three simpler systems, $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$. We specified the manifolds $\mathcal{M}_{xz}$ and $\mathcal{M}_z$ which contain solutions of the systems $\mathcal{S}_M$ and $\mathcal{S}_S$, respectively; this gives the picture in Fig. 2 when $l = m = n = 1$. Using an argument similar to that suggested by Habets

[Hab.1] we gave a straightforward proof of the fact that the asymptotic stability of the equilibrium point of the system $\mathcal{S}_{\varepsilon\nu}$ is guaranteed when the exponential stability of the equilibrium points of the systems $\mathcal{S}_F$, $\mathcal{S}_M$, and $\mathcal{S}_S$ is established and when ε and ν are sufficiently small. We gave estimates of ε^0 and ν^0, where $\forall\ \varepsilon \in (0, \varepsilon^0]$ and $\forall\ \nu \in (0, \nu^0]$ the stability result holds. The values we gave for ε^0 and ν^0 are not the best possible estimates, because in the inequalities we systematically used as small a number of positive contants as possible: as a benefit, the details of the proof were quite straightforward.

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Figure Captions

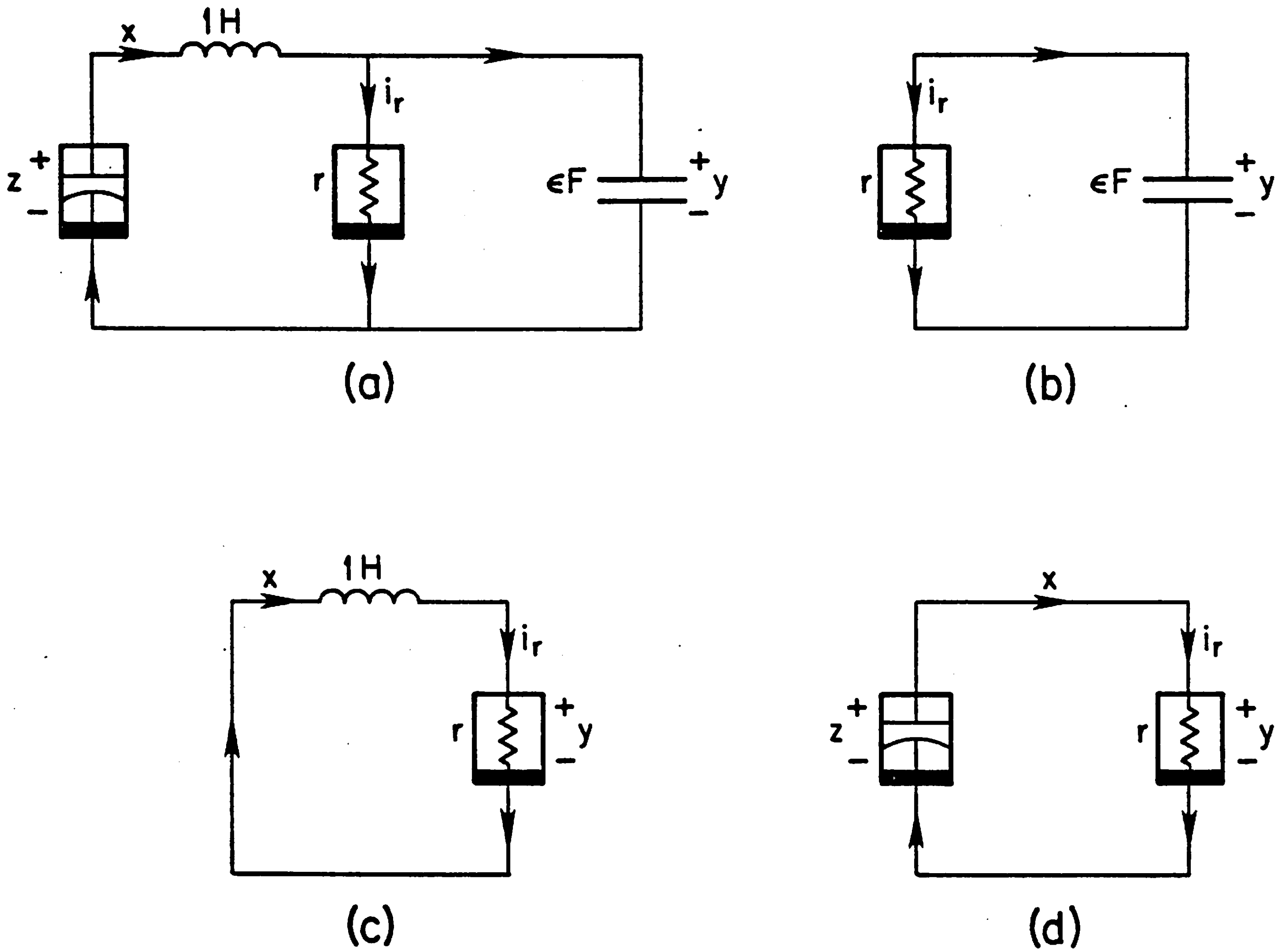
Fig.1.(a) The normalized nonlinear time-invariant circuit $C_{\varepsilon\nu}$ includes a stray capacitor of ε farads, an inductor of 1 henry, and a large nonlinear voltage-controlled capacitor.

(b) The fast circuit, C_F, includes only the stray capacitor.

(c) The medium circuit, C_M, includes only the 1-henry inductor.

(d) The slow circuit, C_S, includes only the large capacitor.

Fig.2. Trajectories and equilibrium points of $\mathcal{S}_F$, $\mathcal{S}_M$, $\mathcal{S}_S$, and $\mathcal{S}_{\varepsilon\nu}$.


x
1H
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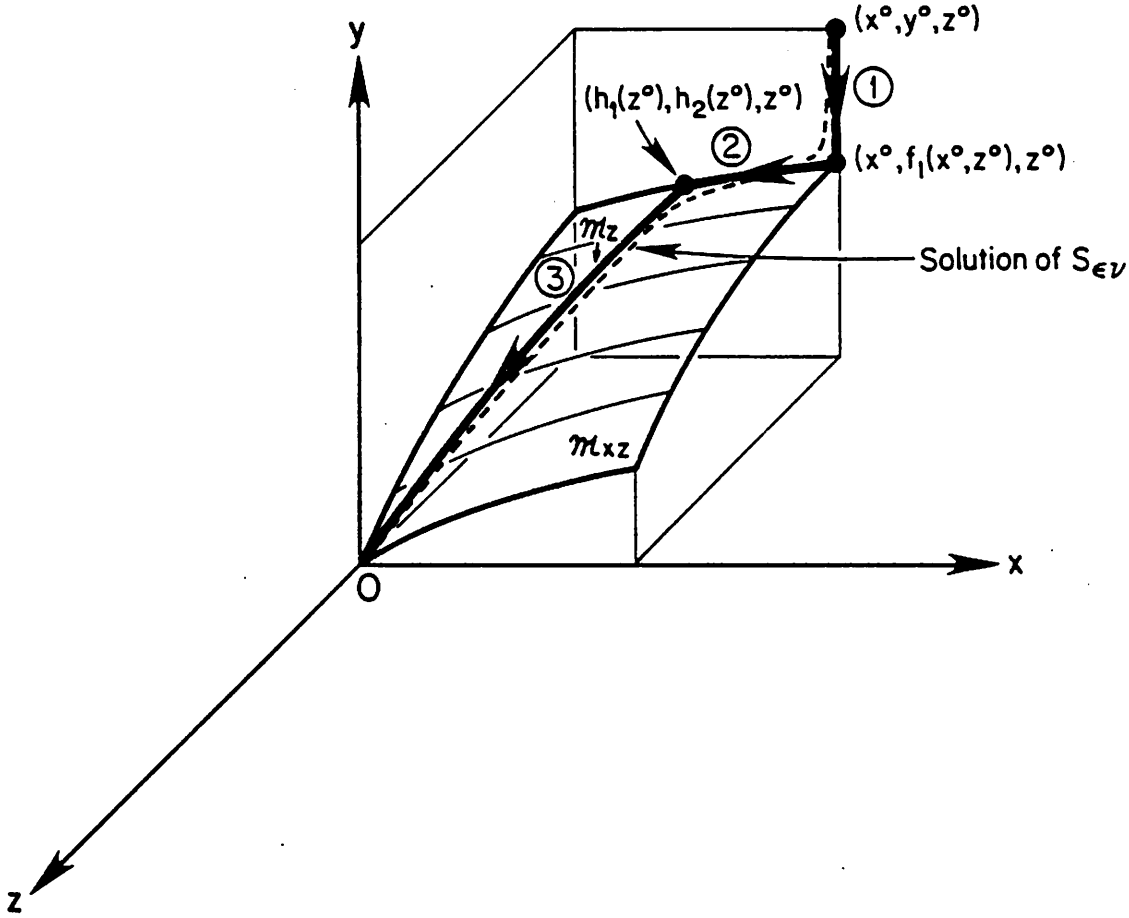
y
(x^o, y^o, z^o)
①
$(h_1(z^o), h_2(z^o), z^o)$
②
$(x^o, f_1(x^o, z^o), z^o)$
$\mathcal{M}_z$
③
Solution of $S_{\epsilon\nu}$
$\mathcal{M}_{xz}$
O
x
z