LINEAR MAGNETIZED PLASMA RESPONSE TO AN OBLIQUE ELECTROSTATIC WAVE

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Abstract

The linear response of a spatially periodic magnetized Vlasov plasma distribution function is computed to second order in the electric field. The results for a specific electric field are then compared with the results of computer simulation for different amplitudes of the electric field. Both trapping and resonant heating are observed, and both appear to contribute (for the chosen parameters) to limiting the validity of linear theory at larger electric field amplitudes.

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Introduction

The second order linear response of an unmagnetized Vlasov plasma has been worked out [1]. It is the purpose of this report to treat the magnetized case. This extension is not difficult conceptually (provided that the concepts of [1] are understood), but involves quite a bit more algebra than is encountered in the unmagnetized case.

Also presented here is simulation work showing that the limits of linear theory are imposed by both trapping (as in the unmagnetized case), and by perpendicular heating. Rough estimates are provided for the wave amplitudes at which each of these phenomena become important.

Second Order Distribution Function for Oblique Electrostatic Waves

In order to compare the details of linear theory with simulations, the second order perturbed distribution function $f_2(v_\parallel, v_\perp)$ is needed for oblique electrostatic waves. It is computed in essentially the same way in which f_2 was calculated for unmagnetized waves in [1]. As before, the model is 1-d periodic with no collisions, and an imposed electric field. Now, however, there is a magnetic field at an angle θ to k, so k must be broken up into a $k_\parallel$ parallel to the magnetic field, and a $k_\perp$ perpendicular to the magnetic field. All three components of the velocity must now be considered. For the purposes of analysis, the components of velocity will be put in cylindrical coordinates, i. e., $v_\parallel$, $v_\perp$, and ϕ. f_2 contains all the information necessary for the computation of linear kinetic energy and mean velocity. The second order perturbed distribution function also shows how the kinetic energy is distributed between parallel and perpendicular components.

For convenience, the coordinates for the derivation will be chosen with $\vec{k}$ in the $\hat{x}$ direction with the magnetic field at angle θ to the $\hat{x}$ direction in the x-z plane. The phase angle ϕ will be defined so that when the magnetic field is in the $\hat{z}$ direction, ϕ is the usual angle in the x-y plane with $\phi = 0$ when the *velocity* is along the $\hat{x}$ dirction, and $\phi = \pi/2$ when the velocity is in the $\hat{y}$ direction (ϕ is 90° out of phase with the phase with respect to position).

The result is obtained by the method of characteristics, just as in the case of unmagnetized waves, although now the characteristics are helices rather than straight lines. The final result is

$$\bar{f}_2(v_\parallel, v_\perp) = \frac{\hat{E}^2}{4} \cdot \frac{q^2}{k^2 m^2} \sum_n \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \cdot$$

$$\left[\frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega_r - n\omega_c - k_\parallel v_\parallel)^2 + \gamma^2} \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) f_0(v_\parallel, v_\perp) \right] \tag{1}$$

where the bar denotes averaging over both position and phase. (See Appendix A for the complete calculation.)

Note that as in the case of the unmagnetized wave, the operator on f_0 is like a diffusion operator. In fact, one could make a quasilinear diffusion equation from the equation for $d\bar{f}_2/dt$ by substituting $\bar{f}$ for $\bar{f}_2$ and f_0. The form of the operator implies that all else being equal, the diffusion in the $v_\perp$ direction is greater for larger n.

Since f_2 is now a function of two variables ($v_\parallel$ and $v_\perp$), it will not be so useful a diagnostic in particle simulations due to the noise inherent in trying to fill a two-dimensional space with only 130,000 particles. Scalar quantities such as the mean parallel velocity, and the parallel and perpendicular kinetic energies (the mean perpendicular velocity is, of course zero in a magnetized plasma), are much better suited to be used as diagnostics. Fortunately, they are easy to compute from the distribution function. Assuming a single-temperature Maxwellian distribution for f_0, and letting $\mu = k_\perp v_t/\omega_c$, $u = v_\parallel/v_t$, $z = \omega/k_\parallel v_t$, $w = \omega_c/k_\parallel v_t$ and $\alpha = \gamma/k_\parallel v_t$ the results are

$$\bar{v}_\parallel = \frac{1}{nmv_t} \cdot \frac{\hat{E}^2}{4} \cdot \frac{1}{k^2\lambda_D^2} \sum_n I_n(\mu^2) e^{-\mu^2} \int \frac{(u+nw)}{(u+nw-z)^2+\alpha^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \tag{2}$$

$$\mathcal{E}_\parallel = \frac{\hat{E}^2}{4} \cdot \frac{1}{k^2\lambda_D^2} \sum_n I_n(\mu^2) e^{-\mu^2} \frac{1}{\sqrt{2\pi}} \int \frac{u(u+nw)}{(u+nw-z)^2+\alpha^2} e^{-\frac{u^2}{2}}\, du \tag{3}$$

and

$$\mathcal{E}_\perp = \frac{\hat{E}^2}{4} \frac{1}{k^2\lambda_D^2} \sum_n I_n(\mu^2) e^{-\mu^2} \int \frac{nw(u+nw)}{(u-z+nw)^2+\alpha^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \tag{4}$$

These integrals are well-behaved, and can be evaluated numerically with relative ease.

As with unmagnetized waves, a formula can be derived for $\bar{f}_2(t \to \infty)$. The derivation follows that of $\bar{f}_2$. The result is

$$\bar{f}_2 \to \frac{1}{4} \frac{q^2}{k^2 m^2} \sum_n \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \cdot$$

$$\left[J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) |\tilde{E}(\omega - n\omega_c - k_\parallel v_\parallel)|^2 \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) f_0(v_\parallel, v_\perp) \right] \tag{5}$$

where $\tilde{E}$ is the Fourier transform of the electric field envelope $\hat{E}$.

This last expression is much better suited to numerical comparison, as the particle distribution function is difficult to obtain accurately at any given instant. As with the instantaneous formulae for $\bar{v}_{\parallel}$, $\mathcal{E}_{\parallel}$, and $\mathcal{E}_{\perp}$, the integrals for these quantities as $t \to \infty$ are relatively easy to evaluate numerically for the electric field envelope which will be used in the simulations.

It is again worth noting the resemblance of this result to those of quasi-linear theory (see, for instance, Kennel and Engelmann [2]).

Theory for Trapping and Perpendicular Heating Due to Oblique Electrostatic Waves

Trapping can occur in waves in a magnetized plasma, such as oblique electrostatic waves, as well as in unmagnetized waves. Another phenomenon, perpendicular heating, is unique to the magnetized case, and must also be considered. It is the primary goal of this research to find the field strength at which trapping and perpendicular heating become important. First trapping will be analyzed, then perpendicular heating.

The basic concept behind trapping is the same in the magnetized case as in the unmagnetized. The mechanism for a magnetized plasma is, however, somewhat more complicated. Instead of a single resonant velocity, there are an infinity of them, one corresponding to each harmonic of the cyclotron frequency. All these resonant velocities satisfy $\omega - k_{\parallel} v_{\parallel} - n\omega_c = 0$. In the rest frames of particles at each of these resonant velocities the wave has the same apparent $\vec{k}$, but an apparent frequency of $n\omega_c$ where n is the harmonic number (in the non-relativistic approximation, a boost parallel to the magnetic field does not alter the fields). To understand the trapping in the parallel direction, it is necessary to average the force on a trapped particle over the short time scale (the gyromotion). From

$$\frac{d}{dt}\vec{v} = \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{6}$$

the equation in the parallel direction is obtained by dotting with the unit vector in the direction of the magnetic field, $\hat{b}$:

$$\begin{aligned} \dot{v}_{\parallel} &= \frac{q}{m}\vec{E} \cdot \hat{b} \\ &= \frac{q}{m}E(\vec{x}, t)cos\theta \end{aligned} \tag{7}$$

From this point on, the coordinates will be chosen so that $\vec{B}$ is in the $\hat{z}$ direction, and the electric field is in the x-z plane. Thus, $\vec{k}\cdot\vec{x} = k_\parallel z(t) + k_\perp x(t)$. Since we are at the resonant velocity looking at particles which are at or near resonance, $z(t)$ (the z coordinate of the particle in the resonance frame) is slowly varying (i.e., $v_\parallel$ is small) since it is parallel to the magnetic field, and only $x(t)$ and $y(t)$ will be rapidly varying. Only $x(t)$ is of interest, and setting $x(t) = X(t) + \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi)$ where $X(t)$ is the guiding center position and therefore slowly varying, gives

$$\begin{aligned} E(t) &= \hat{E}e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\ &= \hat{E}e^{i(k_\parallel z + k_\perp X + \frac{k_\perp v_\perp}{\omega_c}\sin(\omega_c t+\phi) - n\omega_c t)} \\ &= \hat{E}e^{i(k_\parallel z + k_\perp X + n\phi)}e^{i[\frac{k_\perp v_\perp}{\omega_c}\sin(\omega_c t+\phi) - n(\omega_c t+\phi)]} \end{aligned} \tag{8}$$

Averaging this over a gyroperiod (and taking the real part) gives

$$\bar{E} = \hat{E}\cos(k_\parallel z + k_\perp X + n\phi)J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) \tag{9}$$

Thus,

$$\frac{d}{dt}\bar{v}_\parallel = \frac{q}{m}\hat{E}\cos\theta\cos(k_\parallel z + k_\perp X + n\phi)J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) \tag{10}$$

and for the most deeply trapped particles,

$$\frac{d^2}{dt^2}z' = -\left|\frac{q}{m}\hat{E}\cos\theta k_\parallel J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right| z' \tag{11}$$

where z' is the z-component relative to the bottom of the trapping well. This is the same formula as for one dimension, except for the factors of $\cos\theta$ and J_n. These factors are easy to understand in physical terms. The $\cos\theta$ factor exists because of the angle between the electric field and the direction of motion. It is the component of the electric field along the parallel direction instead of the entire electric field. The second factor results from the averaging of the electric field over the particle orbit, which may be comparable to a wavelength in radius.

The same quantities Δv_{tr} and $\Delta\Phi_{tr}$ (the maximum trapping width and the cumulative trapping phase shift) can be extended directly from the one dimensional case to this equation for $v_\parallel$. By direct analogy,

$$\Delta v_{tr} = 2\sqrt{\left|\frac{q}{m}\frac{E_0\cos\theta}{k_\parallel}J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right|} \tag{12}$$

and

$$\Delta\Phi_{tr} = \frac{4}{\gamma}\sqrt{\frac{q}{m}k_\parallel E_0\cos\theta J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)} \tag{13}$$

As in the one dimensional case, if either Δv_{t_r} or $\Delta\Phi_{t_r}$ is small enough, the distribution will *seem* to follow the linear theory closely, but in the self-consistent case, $\Delta\Phi_{t_r}$ should be the only important parameter.

As mentioned in the beginning of this section, the magnetic field introduces a new effect: resonant perpendicular heating (or cooling). The derivation is similar to the derivation for the trapping, beginning with the same equation of motion (6), but dotting it with $\vec{v}_\perp$ instead of $\hat{b}$. The result is

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}v_\perp^2\right) &= \frac{q}{m}\vec{E}\cdot\vec{v}_\perp \\ &= \frac{q}{m}\hat{E}\sin\theta v_\perp e^{i(k_\parallel z + k_\perp X + n\phi)}\cos(\omega_c t+\phi)e^{i[\frac{k_\perp v_\perp}{\omega_c}\sin(\omega_c t+\phi)-n(\omega_c t+\phi)]} \qquad (14)\end{aligned}$$

Averaging again over a gyroperiod gives

$$\frac{d}{dt}\left(\frac{1}{2}\overline{v_\perp^2}\right) = \frac{q}{m}\hat{E}\sin\theta\cos(k_\parallel z + k_\perp X + n\phi)\frac{n\omega_c}{k_\perp}J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) \qquad (15)$$

This formula has some interesting ramifications. For particles which are not trapped, z will vary fairly rapidly, and the average of the right-hand side will be close to zero. Particles which are trapped, however, have a limit to the z coordinate, and so the right-hand side may have a substantial average. This formula is also interesting in the extreme non-linear case because of the J_n factor. Since the argument of the cosine is nearly constant for given particles, it will either gain or lose energy monotonically until the value of $v_\perp$ approaches a value such that $J_n(k_\perp v_\perp/\omega_c)$ vanishes (except, of course, for the case $n = 0$). Since the trapping in $v_\parallel$ also has a factor of J_n in it (Equation (11)), the effect of the resonance in the perpendicular direction is to reduce the effects of trapping in the parallel direction, except for the important case of $n = 0$. Since the effect of this equation is to reduce the effects of trapping, yet it relies on trapping for its effect, it is somewhat self-limiting. The following derivation of a condition for linearity does not take this into account, and so should be quite conservative. At first inspection, it might also seem that the perpendicular damping would not disappear when the angle of propagation is perpendicular to the direction of the magnetic field. It is assumed, however, in deriving (15) that the particle being followed is in resonance with the wave, and since $k_\parallel$ is zero for perpendicular propagation, resonance requires that $\omega - n\omega_c = 0$. This is never true for Bernstein modes which are propagating perpendicular to the magnetic field, so the mode will remain undamped as theory predicts.

The perpendicular resonance can be said to have become non-linear when the perpendicular velocity has changed by enough to significantly alter the value of $J_n(k_\perp v_\perp/\omega_c)$, or if the particle picks up perpendicular energy comparable to the thermal energy, i.e. the distribution function is significantly altered on a scale large enough to affect the bulk plasma, and thereby the dispersion relation. For a rule of thumb, the factor of $\cos(k_\parallel z + k_\perp X + n\phi)$ can be omited, and the factor of J_n can be replaced by 1/2 (since the J_0 term does not contribute), leaving

$$\frac{d}{dt}\left(v_\perp^2\right) \sim \frac{q}{m}\hat{E}\sin\theta \cdot \frac{n\omega_c}{k_\perp} \tag{16}$$

The solution, plugging in $\hat{E} = E_0 \exp(\gamma t)$ is,

$$\Delta(v_\perp^2) \sim \frac{1}{\gamma}\frac{qE_0}{m}\sin\theta \cdot \frac{n\omega_c}{k_\perp} \tag{17}$$

or

$$\Delta\left[\left(\frac{k_\perp v_\perp}{\omega_c}\right)^2\right] \sim n\frac{qk_\perp E_0 \sin\theta}{\gamma m \omega_c} \tag{18}$$

For linear theory to hold, this quantity should be much less than 1, but it may hold reasonably well for larger values since all that is really required is that $\Delta v_\perp \ll \omega_c/k_\perp$ which is a weaker condition than (18). It should also be noted that the higher values of n are not as important, so for the rule of thumb, n can be omitted from this formula.

The second condition, that the change in energy be much less than the thermal energy, is derived trivially from the same formula:

$$\Delta\left[\left(\frac{v_\perp}{v_t}\right)^2\right] \sim n\frac{q\omega_c E_0 \sin\theta}{\gamma m k_\perp v_t^2} \tag{19}$$

Again, n may be omitted for a rule of thumb. The final result is that if

$$\frac{qE_0}{m}\frac{\omega_c}{\gamma k} \ll \min\left(\frac{\omega_c^2}{k_\perp^2}, v_t^2\right) \tag{20}$$

and the trapping condition $\Delta\Phi_{tr} \ll \pi$ is satisfied, then linear theory will be accurate.

Note that these formulae depend on E_0/γ rather than on $\sqrt{E_0}/\gamma$ as with the parallel trapping. Thus for small damping rates and small field amplitudes, the perpendicular heating will cause non-linear effects before trapping can set in.

Numerical Particle Simulations of Oblique Electrostatic Wave

The same sort of simulation as was done for unmagnetized waves in [1] can also be done for oblique electrostatic waves. As with the unmagnetized waves, a large number of particles is necessary; but, unlike the unmagnetized simulation, the distribution function is in *two* variables – $v_\parallel$ and $v_\perp$. As before, the field amplitude at which non-linear effects set in is of interest. The situation is not so simple as for the unmagnetized wave, and so simulation should be a useful adjunct to theory.

Again, the simulation model follows the theoretical model (as described in the previous section. It is 1-d (with all three velocity components, however), collisionless, and periodic, with a neutralizing background charge density representing immobile ions. The initial distribution function f_0 is Maxwellian, and the imposed electric field is again $E = E_0 \exp(-\gamma|t|) \exp(i(kx - \omega t))$.

The distribution function cannot be as easily plotted and interpreted in this case as it was in the case of unmagnetized waves, because it is a function of both $v_\parallel$ and $v_\perp$. This problem is compounded by the fact that the number of particles is not enough to smoothly fill the two-dimensional velocity space. (Two examples of contour plots of $f(v_\parallel, v_\perp)$ will be shown, but, as will be apparent, they are not as informative as the distribution function plots were for the unmagnetized wave.) The mean parallel velocity (which, of course, represents direct current drive), the parallel kinetic energy, and the perpendicular kinetic energy (which can generate current indirectly when collisions are present) are, however, easily calculated and just as informative as were the mean velocity and kinetic energy for the unmagnetized wave. Each of these can be calculated from the theory (through through the numerical evaluation of some well-behaved integrals) and from the simulation both as the imposed wave is growing exponentially, and as $t \to \infty$.

The parameters chosen for these runs are: $k_\parallel v_t/\omega_c = 1$, $k_\perp v_t/\omega_c = 2.4$ (so $kv_t/\omega_c = 2.6$ and $\theta \approx 67°$), $\omega/\omega_c = 1.5$, $\gamma/\omega_c = .25$. Five runs were made at differing values of E_0. These values were chosen such that $qE_0/mv_t\omega_c$ would be equal to 0.125, 0.25, 0.5, 1, and 2 (note that this is a slightly different normalization from that of the unmagnetized wave simulations — but that the two can be compared through the formula for $\Delta\Phi_{tr}$). As with the unmagnetized simulations, the parameters were chosen for ease of simulation rather than for realism, and bear no resemblance to any self-consistent wave.

A least squares fit to a growing exponential with exponent γ was made on the last e-folding in E of the growing part of each the three diagnostics. The purpose of these curves was to obtain a

more accurate value for the diagnostics (which they did) and to provide an estimate of the error due to noise which could be used as an estimate of the error of the diagnostics as $t \rightarrow \infty$ (which they did not). The standard deviations of the diagnostics about the fit exponential curve should be a fairly good measure of the error of any given point on the diagnostic curves. The instantaneous value of the diagnostics is actually plotted in Figure 1, rather than the value obtained from the least squares fit, for two reasons: first, the assumption under which it was deemed appropriate in the first place — that a fixed level of noise was superimposed on a basically correct curve — appears to be false, and second, the correct curve is expected to deviate from a growing exponential, and thus there would be a systematic error in using the least square value.

Figure 1 shows the values of the diagnostics obtained from simulation (appropriately normalized) and the theoretical linear values. All the diagnostics converge toward the linear values quite well as the field amplitude decreases, except for the parallel energy, which seems to show either an oscillation with field amplitude, or fairly strong noise. Since another run with a slightly modified particle loader (bit-reversed in the x-direction instead of the direction parallel to the magnetic field — see the technical note at the end of this section) yielded values which deviated with similar amplitude but without sign of oscillation, it seems likely that the error is due to noise caused by some small systematic error in the loading scheme. This is initially somewhat surprising, given the large number of particles in the system, but quite reasonable when it is remembered that only a small fraction of those particles are resonant with the wave. Note also that the perpendicular kinetic energy decreases with increasing field amplitude. Since perpendicular heating can drive a current once the effects of collisions are taken into account, this loss of efficiency is important. More will be said about the results shown in Figure 1.

Figure 2 shows the parallel kinetic energy as a function of time, along with the exponential curve which was fit to the parallel kinetic energy using least squares, for the lowest amplitude excitation over the last e-folding of the electric field before $t = 0$. The estimated error from this least squares fit is ~3%, which is in fair agreement with the deviation of this diagnostic from theory at $t = 0$ (~6%), but far smaller than the deviation from theory at $t = \infty$. That some low frequency noise is present is plainly visible. That it is *only* low frequency and that it exists at all in what should be a uniform system is more evidence for the noise being the result of systematic error.

Figure 3 shows contour plots of $f(v_{\parallel}, v_{\perp})$ before the simulation and at $t = \infty$ for the highest excitation amplitude, clearly showing the effects of the wave. Some current drive effect is also clearly

visible.

When Figure 1 is compared with the results of the unmagnetized simulation, it is seen that, as with the unmagnetized simulation, the linear theory holds up to a certain field amplitude, then the results start diverging. Applying Formula (13) to the simulation, it is found (setting $J_n \sim .5$) that $\Delta\Phi_{t_r}$ ranges from 1.67 to 6.67, and that comparable deviation is found at comparable values of $\Delta\Phi_{t_r}$. This is a strong confirmation of the validity of Formula (13).

The perpendicular heating condition (20) can aslo be evaluated. The values of the left-hand side range from 0.19 to 3.08, and the values of the right-hand side (which should both be much larger than the value of the left-hand side for linearity to be guaranteed) are 0.17 and 1.0. The good agreement of the simulation with linear theory indicates that either an error was made in the derivation of the condition (20), or that the many approximations made in deriving (20) were very conservative.

A mildly surprising feature of the simulation is that the parallel energy *increases* with increasing non-linearity. This is easily explained when it is realized that the distribution function is actually made *steeper* at some resonances according to linear theory. When trapping sets in, these places do not steepen as linear theory predicts, and may even flatten out instead of steepening. Thus these resonances contribute more heating than expected. This effect does not contribute to the current drive, as the resonances which ought to steepen the distribution function are primarily at velocities opposed to the direction of current drive.

Technical note:

Because of the high noise level associated with random loading in this number of phase space dimensions, the particles were loaded with some care. The velocities in the *parallel* direction (rather than the x direction) were loaded in bit-reversed order, the phase of the perpendicular velocity was chosen in 3-reversed order, and the magnitude of the perpendicular velocity was chosen in 5-reversed order. This arrangement should introduce the minimum noise in the parallel direction. (The bit-reversed loader has fewer problems with recurrence than the 3- and 5- reversed loaders.) See Birdsall and Langdon [3] for details on bit-reversing.

Summary and Conclusions

Linear theory has been used to compute the second order perturbed distribution function for an oblique electrostatic electron wave both for an exponentially growing wave (Equation (1)) and for a wave pulse of arbitrary shape (Formula (5)). The results are: for the growing wave

$$\bar{f}_2(v_\parallel, v_\perp) = \frac{\hat{E}^2}{4} \cdot \frac{q^2}{k^2 m^2} \sum_n \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \cdot \left[\frac{J_n^2 \left(\frac{k_\perp v_\perp}{\omega_c} \right)}{(\omega_r - n\omega_c - k_\parallel v_\parallel)^2 + \gamma^2} \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) f_0(v_\parallel, v_\perp) \right] \tag{21}$$

and for the wave pulse

$$\bar{f}_2 \to \frac{1}{4} \frac{q^2}{k^2 m^2} \sum_n \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \cdot \left[J_n^2 \left(\frac{k_\perp v_\perp}{\omega_c} \right) |\tilde{E}(\omega - n\omega_c - k_\parallel v_\parallel)|^2 \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp} \frac{\partial}{\partial v_\perp} \right) f_0(v_\parallel, v_\perp) \right] \tag{22}$$

This linear theory has been tested via particle simulation on its predictions regarding several second order quantities. The electric field varied in time, though it was uniform in space. The results were in agreement with theory, although the tests have not been self-consistent in the sense that the electric field was imposed rather than solved for using Poisson's equation. The onset of non-linearity was observed, and the threshold level fit the predictions of a simple trapping model. This model yields two parameters which predict the behavior of the plasma in the presence of oblique electrostatic waves: the cumulative trapping phase shift

$$\Delta\Phi_{tr} = \frac{4}{\gamma} \sqrt{\left| \frac{q}{m} k_\parallel E_0 \cos\theta J_n \left(\frac{k_\perp v_\perp}{\omega_c} \right) \right|} \tag{23}$$

and the perpendicular energy change

$$\Delta(v_\perp^2) \sim \frac{1}{\gamma} \left| \frac{qE_0}{m} \right| \cdot \frac{n\omega_c}{k} \tag{24}$$

If both of these parameters are small, the behavior should be linear, and indeed the results are consistent with this prediction.

It is fair to say that the one dimensional effects causing non-linearity are now understood. Unfortunately, these conclusions cannot be blithely applied to three dimensions. While the cumulative trapping phase shift and perpendicular energy change are functions only of the electric field strength and its temporal envelope in one dimension, in three dimensions, the geometry becomes important, and can considerably alter both parameters.

Appendix A: Second Order Perturbed Distribution Function for Oblique Electrostatic Wave

For convenience, the coordinates for the derivation will be chosen with $\vec{k}$ in the $\hat{x}$ direction with the magnetic field at angle θ to the $\hat{x}$ direction in the x-z plane. The phase angle ϕ will be defined so that when the magnetic field is in the $\hat{z}$ direction, ϕ is the usual angle in the x-y plane with $\phi = 0$ when the *velocity* is along the $\hat{x}$ dirction, and $\phi = \pi/2$ when the velocity is in the $\hat{y}$ direction (ϕ is 90° out of phase with the phase with respect to position). The unperturbed orbit in such a coordinate system is such that

$$v_x(t') = v_\perp \cos(\omega_c(t'-t) + \phi(t)) \sin\theta + v_\parallel \cos\theta \tag{a1}$$

$$x(t') = x(t) + \frac{v_\perp}{\omega_c}\left[\sin\left(\omega_c(t'-t) + \phi(t)\right) - \sin\phi(t)\right]\sin\theta + v_\parallel(t'-t)\cos\theta \tag{a2}$$

$$\phi(t') = \phi(t) + \omega_c(t'-t) \tag{a3}$$

with $v_\parallel$ and $v_\perp$ constants (recall that $k_\parallel \equiv k\cos\theta$ and $k_\perp \equiv k\sin\theta$).

The first equation to be solved is

$$\frac{df_1}{dt} = -\frac{q}{m}\vec{E}\cdot\frac{\partial f_0}{\partial \vec{v}} \tag{a4}$$

with

$$\vec{E} = E\hat{k}e^{i(kx-\omega t)} \tag{a5}$$

In cylindrical velocity coordinates, the differential operator becomes

$$\hat{k}\cdot\frac{\partial}{\partial\vec{v}} = \frac{1}{k}\left[k_\parallel\frac{\partial}{\partial v_\parallel} + k_\perp\cos\phi\frac{\partial}{\partial v_\perp} - k_\perp\sin\phi\frac{1}{v_\perp}\frac{\partial}{\partial\phi}\right] \tag{a6}$$

Plugging this operator into the equation for f_1 and assuming that f_0 does not depend on ϕ, yields

$$\frac{df_1}{dt} = -\frac{q}{m}\frac{E}{k}\left[k_\parallel\frac{\partial}{\partial v_\parallel} + \cos\phi k_\perp\frac{\partial}{\partial v_\perp}\right] f_0\cdot e^{i(kx-\omega t)} \tag{a7}$$

Plugging in $x(t')$ and $\phi(t')$ and solving by the method of characteristics,

$$\begin{aligned} f_1 = -\frac{q}{m}\frac{E}{k}e^{i(kx-\omega t)}\cdot\Bigg\{ & k_\parallel\frac{\partial f_0}{\partial v_\parallel}\int_{-\infty}^{t} e^{i\{\frac{k_\perp v_\perp}{\omega_c}[\sin(\phi+\omega_c(t'-t))-\sin\phi]+k_\parallel v_\parallel(t'-t)-\omega(t'-t)\}}\,dt' \\ & + k_\perp\frac{\partial f_0}{\partial v_\perp}\int_{-\infty}^{t}\cos\left(\phi+\omega_c(t'-t)\right)e^{i\{\frac{k_\perp v_\perp}{\omega_c}[\sin(\phi+\omega_c(t'-t))-\sin\phi]+k_\parallel v_\parallel(t'-t)-\omega(t'-t)\}}\,dt'\Bigg\} \\ = -\frac{q}{m}\frac{E}{k}e^{i(kx-\omega t)}e^{-i\frac{k_\perp v_\perp}{\omega_c}\sin\phi}\times & \end{aligned}$$

$$\left\{k_\parallel \frac{\partial f_0}{\partial v_\parallel} \int_0^\infty e^{i[k_\perp v_\perp \sin(\phi-\omega_c\tau)-k_\parallel v_\parallel \tau+\omega\tau]}\, d\tau \right.$$

$$\left. + k_\perp \frac{\partial f_0}{\partial v_\perp} \int_0^\infty \cos(\phi-\omega_c\tau) e^{i[k_\perp v_\perp \sin(\phi-\omega_c\tau)-k_\parallel v_\parallel \tau+\omega\tau]}\, d\tau \right\}$$

$$= -\frac{q}{m}\frac{E}{k} e^{i(kx-\omega t)} e^{-i\frac{k_\perp v_\perp}{\omega_c}\sin\phi} \times$$

$$\sum_n \left[k_\parallel \frac{\partial f_0}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right] J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{in\phi} \int_0^\infty e^{i(\omega-n\omega_c-k_\parallel v_\parallel)\tau}\, d\tau$$

$$= -i\frac{q}{m}\frac{E}{k} e^{i(kx-\omega t)} e^{-i\frac{k_\perp v_\perp}{\omega_c}\sin\phi} \sum_n \frac{J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{in\phi}}{\omega - n\omega_c - k_\parallel v_\parallel} \left[k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right] f_0 \qquad (a8)$$

Note that it was necessary to assume $\mathrm{Im}\,\omega > 0$, i.e. a growing wave. It will also be useful when calculating f_2 to write f_1 as

$$f_1 = -i\frac{q}{m}\frac{E}{k} e^{i(kx-\omega t)}$$

$$\sum_n \sum_m \frac{J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) J_m\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{i(n-m)\phi}}{\omega - n\omega_c - k_\parallel v_\parallel} \left[k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right] f_0 \qquad (a9)$$

Note that the first order perturbed distribution function is a function of the velocity phase angle, which the zero order distribution function is not.

The equation for the second order perturbed distribution function is

$$\frac{df_2}{dt} = -\frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial\vec{v}} f_1$$

$$= -\frac{q}{m} E\,\mathrm{Re}\, e^{i(kx-\omega t)} \cdot \mathrm{Re}\left(\hat{k}\cdot\frac{\partial}{\partial\vec{v}}\right) f_1 \qquad (a10)$$

The real part of all quantities being previously implied.

Before integrating the equation, it is best to average over x, as the spatial structure of f_2 is of no interest. It is also assumed at this point that k is real. Now

$$\frac{d\bar{f}_2}{dt} = -\frac{q}{m}\frac{E}{2}\hat{k}\cdot\frac{\partial}{\partial\vec{v}}\,\mathrm{Re}\left[f_1 e^{-i(kx-\omega^* t)}\right]$$

$$= -i\frac{q}{m}\frac{E}{2k} e^{2\gamma t}\left\{\left(\hat{k}\cdot\frac{\partial}{\partial\vec{v}}\right)\sum_n\sum_m \frac{J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) J_m\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{i(n-m)\phi}}{\omega - n\omega_c - k_\parallel v_\parallel}\right.$$

$$\left. \left[k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right] f_0 \right\} \qquad (a11)$$

where $\gamma = \mathrm{Im}\,\omega$. The velocity phase angle ϕ can also be integrated over now since the phase structure of f_2 is of no value in computing the quantities of interest. (One must actually be careful

to denote $\phi(t)$ as $\phi(t_0)+\omega_c(t-t_0)$ and average over $\phi(t_0)$, but the end result is the same.) The differential operator can be rewritten as

$$\vec{k}\cdot\frac{\partial}{\partial\vec{v}} = k_\parallel\frac{\partial}{\partial v_\parallel} + k_\perp\cos\phi\frac{\partial}{\partial v_\perp} - \frac{k_\perp\sin\phi}{v_\perp}\frac{\partial}{\partial\phi}$$
$$= k_\parallel\frac{\partial}{\partial v_\parallel} + \frac{k_\perp}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\cos\phi - \frac{k_\perp}{v_\perp}\frac{\partial}{\partial\phi}\sin\phi \qquad (a12)$$

Integrating over ϕ will eliminate the $\partial/\partial\phi$ term so that only two integrals must be calculated:

$$\int_0^{2\pi} e^{i(n-m)\phi}\frac{d\phi}{2\pi} = \delta_{n,m} \qquad (a13)$$

and

$$\int_0^{2\pi}\cos\phi e^{i(n-m)\phi}\frac{d\phi}{2\pi} = \frac{1}{2}(\delta_{n,m-1}+\delta_{n,m+1}) \qquad (a14)$$

Using these, the sums over m can be evaluated:

$$\sum_m J_m\delta_{n,m} = J_n \qquad (a15)$$

$$\sum_m J_m(z)\frac{1}{2}(\delta_{n,m-1}+\delta_{n,m+1}) = \frac{J_{n+1}(z)+J_{n-1}(z)}{2}$$
$$= \frac{n}{z}J_n(z) \qquad (a16)$$

Plugging these in,

$$\frac{d\bar{f}_2}{dt} = i\frac{E^2}{2}\cdot\frac{q^2}{k^2m^2}\sum_n\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)\cdot$$
$$\left[\frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{\omega-n\omega_c-k_\parallel v_\parallel}\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)f_0\right]\times e^{2\gamma t} \qquad (a17)$$

Now taking the real part,

$$\frac{d\bar{f}_2}{dt} = \frac{E^2}{4}\cdot\frac{q^2}{k^2m^2}\sum_n\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)\cdot$$
$$\left[\frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega_r-n\omega_c-k_\parallel v_\parallel)^2+\gamma^2}\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)f_0\right]\times 2\gamma e^{2\gamma t} \qquad (a18)$$

and so

$$\bar{f}_2(v_\parallel,v_\perp) = \frac{E^2}{4}\cdot\frac{q^2}{k^2m^2}\sum_n\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)\cdot$$
$$\left[\frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega_r-n\omega_c-k_\parallel v_\parallel)^2+\gamma^2}\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)f_0(v_\parallel,v_\perp)\right] \qquad (a19)$$

The kinetic energy of the wave can now be calculated in order to compare the result with that of the energy derived from the linear dispersion relation. The kinetic energy can be separated into parallel and perpendicular components. Starting with the parallel component,

$$\mathcal{E}_{\parallel} = \iint \frac{1}{2} m v_{\parallel}^2 \bar{f}_2 \, dv_{\parallel} \, v_{\perp} dv_{\perp} \tag{a20}$$

Breaking up $\bar{f}_2$ into two terms, the $\partial/\partial v_{\perp}$ term can be integrated immediately, yielding zero since $nJ_n(0) = 0$ for all n. Integrating the $\partial/\partial v_{\parallel}$ term by parts and substituting

$$\left(k_{\parallel} \frac{\partial}{\partial v_{\parallel}} + \frac{n\omega_c}{v_{\perp}} \frac{\partial}{\partial v_{\perp}}\right) f_0 = -\frac{1}{v_t^2}(k_{\parallel} v_{\parallel} + n\omega_c) f_0 \tag{a21}$$

yields

$$\mathcal{E}_{\parallel} = \frac{E^2}{4} \cdot \frac{1}{k^2 \lambda_D^2} \sum_n \iint \frac{k_{\parallel} v_{\parallel} (k_{\parallel} v_{\parallel} + n\omega_c)}{(\omega - n\omega_c - k_{\parallel} v_{\parallel})^2 + \gamma^2} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_c}\right) f_0 \, dv_{\parallel} \, v_{\perp} dv_{\perp} \tag{a22}$$

Setting

$$f_0 = \frac{1}{\sqrt{2\pi}^3 v_t^3} e^{-\frac{v_{\parallel}^2 + v_{\perp}^2}{2v_t^2}} \tag{a23}$$

separates the integrals:

$$\mathcal{E}_{\parallel} = \frac{E^2}{4} \cdot \frac{1}{k^2 \lambda_D^2} \sum_n \int J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_c}\right) \frac{1}{v_t^2} e^{-\frac{v_{\perp}^2}{2v_t^2}} v_{\perp} dv_{\perp} \cdot$$

$$\int \frac{k_{\parallel} v_{\parallel} (k_{\parallel} v_{\parallel} + n\omega_c)}{(\omega - n\omega_c - k_{\parallel} v_{\parallel})^2 + \gamma^2} \frac{1}{\sqrt{2\pi} v_t} e^{-\frac{v_{\parallel}^2}{2v_t^2}} dv_{\parallel} \tag{a24}$$

Setting $\mu = k_{\perp} v_t / \omega_c$ and $y = v_{\perp}/v_t$ reduces the first integral to

$$\int J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_c}\right) \frac{1}{v_t^2} e^{-\frac{v_{\perp}^2}{2v_t^2}} v_{\perp} dv_{\perp} = \int J_n^2(\mu y) e^{-\frac{y^2}{2}} y dy$$

$$= I_n(\mu^2) e^{-\mu^2} \tag{a25}$$

The algebra for the second integral can be simplified if some new symbols are introduced. Let $u = v_{\parallel}/v_t$, $z = \omega/k_{\parallel} v_t$, $w = \omega_c/k_{\parallel} v_t$ and $\alpha = \gamma/k_{\parallel} v_t$ then the integral becomes

$$\frac{1}{\sqrt{2\pi}} \int \frac{u(u+nw)}{(u+nw-z)^2 + \alpha^2} e^{-\frac{u^2}{2}} du$$

$$= 1 + \frac{1}{\sqrt{2\pi}} \int \frac{(2z - nw)(u - z + nw) + z(z - nw) - \alpha^2}{(u - z + nw)^2 + \alpha^2} e^{-\frac{u^2}{2}} du$$

$$= 1 + (nw - 2z) \operatorname{Re} \frac{1}{\sqrt{2\pi}} \int \frac{e^{-\frac{u^2}{2}}}{u - z + nw - i\alpha} du$$

$$+ \frac{z(z - nw) - \alpha^2}{\alpha} \operatorname{Im} \frac{1}{\sqrt{2\pi}} \int \frac{e^{-\frac{u^2}{2}}}{u - z + nw - i\alpha} du$$

$$= 1 + (nw - 2z) \operatorname{Re} N(z - nw + i\alpha) + \frac{z(z - nw) - \alpha^2}{\alpha} \operatorname{Im} N(z - nw + i\alpha) \tag{a26}$$

Taking the limit as $\alpha \to 0$ for comparison later with the results of the calculation using the linear dielectric function,

$$\begin{aligned}\operatorname{Im} N(z-nw+i\alpha) &\to \operatorname{Im}\left[N(z-nw)+i\alpha N'(z-nw)\right] \\ &\to \operatorname{Im}\left[N(z-nw)-i\alpha\big(1+(z-nw)N(z-nw)\big)\right] \\ &\to \operatorname{Im} N(z-nw)-\alpha\big(1+(z-nw)\operatorname{Re} N(z-nw)\big) \qquad (a27)\end{aligned}$$

(where $N(z) = 1/\sqrt{2}Z(z/\sqrt{2})$ and Z is the standard Plasma Dispersion Function) and

$$\begin{aligned}\mathcal{E}_\parallel \to \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2}\Big\{1-z(z-nw)+\left[2z-nw-z(z-nw)^2\right]N(z-nw)+ \\ \sqrt{\frac{\pi}{2}}\frac{z(z-nw)}{\alpha}e^{-\frac{1}{2}(z-nw)^2}\Big\} \qquad (a28)\end{aligned}$$

The perpendicular energy is evaluated in much the same way:

$$\begin{aligned}\mathcal{E}_\perp &= \iint \frac{1}{2}mv_\perp^2 f_2\, dv_\parallel\, v_\perp dv_\perp\, d\phi \\ &= -\frac{E^2}{4}\frac{\omega_p^2}{k^2}\sum_n \iint \frac{n\omega_c J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega-n\omega_c-k_\parallel v_\parallel)^2+\gamma^2}\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)2\pi f_0\, dv_\parallel\, v_\perp dv_\perp \\ &= \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2}\int \frac{nw(u+nw)}{(u-z+nw)^2+\alpha^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du \\ &= \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2}\left\{nw\operatorname{Re} N(z-nw+i\alpha)+\frac{znw}{\alpha}\operatorname{Im} N(z-nw+i\alpha)\right\} \qquad (a29) \\ &\to \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2} \\ &\quad\times\left\{-znw+\left[nw-znw(z-nw)\right]\operatorname{Re} N(z-nw)+\sqrt{\frac{\pi}{2}}\frac{znw}{\alpha}e^{-\frac{1}{2}(z-nw)^2}\right\} \qquad (a30)\end{aligned}$$

The total energy is thus

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_\parallel+\mathcal{E}_\perp \\ &= -\frac{E^2}{4}\frac{\omega_p^2}{k^2}\sum_n \iint \frac{(k_\parallel v_\parallel+n\omega_c)J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega-n\omega_c-k_\parallel v_\parallel)^2+\gamma^2}\left(k_\parallel\frac{\partial}{\partial v_\parallel}+\frac{n\omega_c}{v_\perp}\frac{\partial}{\partial v_\perp}\right)f_0\, dv_\parallel\, v_\perp dv_\perp\, d\phi \\ &= \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n \iint \frac{(k_\parallel v_\parallel+n\omega_c)^2 J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega-n\omega_c-k_\parallel v_\parallel)^2+\gamma^2}2\pi f_0\, dv_\parallel\, v_\perp dv_\perp \\ &= \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2}.\end{aligned}$$

$$\left\{1+2z\,\mathrm{Re}\,N(z-nw+i\alpha)+\frac{z^2-\alpha^2}{\alpha}\,\mathrm{Im}\,N(z-nw+i\alpha)\right\} \tag{a31}$$

$$\rightarrow \frac{E^2}{4}\frac{1}{k^2\lambda_D^2}\sum_n I_n(\mu^2)e^{-\mu^2}\cdot$$

$$\left\{1-z^2+[2z-z^2(z-nw)]\,\mathrm{Re}\,N(z-nw)+\frac{z^2}{\alpha}\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(z-nw)^2}\right\} \tag{a32}$$

To ensure that the calculation of f_2 is correct, the kinetic energy of a electrostatic wave will now be calculated using the dielectric function:

$$W_{wave}=\frac{\hat{E}^2}{4}\left[\frac{\partial}{\partial\omega}(\omega\,\mathrm{Re}\,\epsilon)-\epsilon_0\right] \tag{a33}$$

and

$$W_{Landau}=\omega\,\mathrm{Im}\,\epsilon\cdot\int^t\frac{\hat{E}(t')^2}{2}\,dt' \tag{a34}$$

where W_{wave} is the wave kinetic energy density and W_{Landau} is the energy which has been absorbed by the distribution function due to Landau damping. These expressions assume that all changes in the electric field amplitude are infinitesimally slow, and the result is only the total energy, rather than f_2 itself. The result will be a completely independent check, however, on the energy as derived from the second order perturbed distribution function. The result is

$$W_{wave}=\frac{\hat{E}^2}{4}\frac{1}{k^2\lambda_D^2}\left\{1-\frac{\omega^2}{k_\parallel^2v_t^2}+\sum_{n=-\infty}^{\infty}\left[2\frac{\omega}{k_\parallel v_t}-\frac{\omega^2}{k_\parallel^2v_t^2}\left(\frac{\omega-n\omega_c}{k_\parallel v_t}\right)\right]N\left(\frac{\omega-n\omega_c}{k_\parallel v_t}\right)\right.$$
$$\left.\times\,I_n\left(\frac{k_\perp^2v_t^2}{\omega_c^2}\right)\exp\left(-\frac{k_\perp^2v_t^2}{\omega_c^2}\right)\right\} \tag{a35}$$

where

$$N(z)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-\zeta^2/2}}{\zeta-z}\,d\zeta \tag{a36}$$
$$=\frac{1}{\sqrt{2}}Z\left(\frac{z}{\sqrt{2}}\right)$$

and Z is the standard Plasma Dispersion Function. The contour of integration goes under the pole at $\zeta=z$ as with the Z function. Use has also been made of the identities

$$N'(z)=-(1+zN(z)) \tag{a37}$$

and

$$\sum_n I_n(z)e^{-z}=1 \tag{a38}$$

To simplify the notation, let $z = \omega/k_\parallel v_t$, $w = \omega_c/k_\parallel v_t$, and $\mu = k_\perp v_t/\omega_c$, then

$$W_{wave} = \frac{\hat{E}^2}{4}\frac{1}{k^2\lambda_D^2}\left\{1 - z^2 + \sum_n [2z - z^2(z-nw)]N(z-nw)I_n(\mu^2)e^{-\mu^2}\right\} \qquad (a39)$$

The purely kinetic (Landau damping) part of the energy for an exponentially growing wave is

$$W_{Landau} = \frac{\hat{E}^2}{4}\cdot\frac{\omega}{\gamma}\,\mathrm{Im}\,\epsilon \qquad (a40)$$

where γ is the growth rate. For real argument,

$$\mathrm{Im}\,N(z) = \sqrt{\frac{\pi}{2}}e^{-\frac{z^2}{2}} \qquad (a41)$$

so, setting $\alpha = \gamma/k_\parallel v_t$,

$$\begin{aligned} W_{Landau} &= \frac{\hat{E}^2}{4}\cdot\frac{\omega}{\gamma}\cdot\frac{1}{k^2\lambda_D^2}\sum_n \sqrt{\frac{\pi}{2}}\frac{\omega}{k_\parallel v_t}e^{-\frac{1}{2}\left(\frac{\omega-n\omega_c}{k_\parallel v_t}\right)^2} I_n\left(\frac{k_\perp^2 v_t^2}{\omega_c^2}\right)e^{-\frac{k_\perp^2 v_t^2}{\omega_c^2}} \\ &= \frac{\hat{E}^2}{4}\frac{1}{k^2\lambda_D^2}\frac{z^2}{\alpha}\sum_n \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(z-nw)^2}I_n(\mu^2)e^{-\mu^2} \end{aligned} \qquad (a42)$$

Equations (a39) and (a42) sum to Equation (a32), thus verifying it.

References

[1] Wm. S. Lawson, *Limits of Linear Response of a Vlasov Distribution*, U. C. Berkeley / Electronics Research Lab memo M86/44 (June 1986)

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[3] Birdsall and Langdon, *Plasma Physics via Computer Simulation* 1st ed. (McGraw-Hill Book Co., 1985)

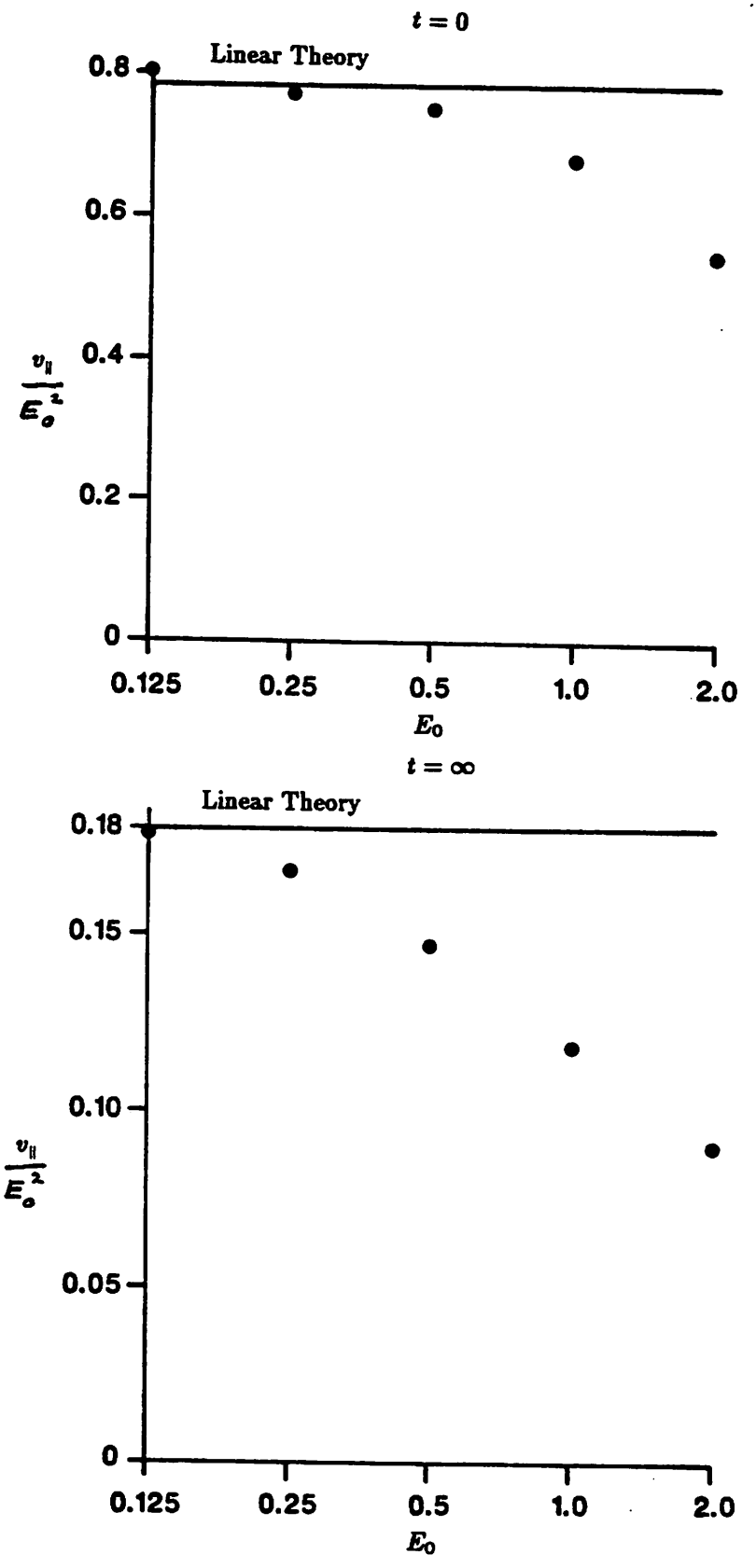
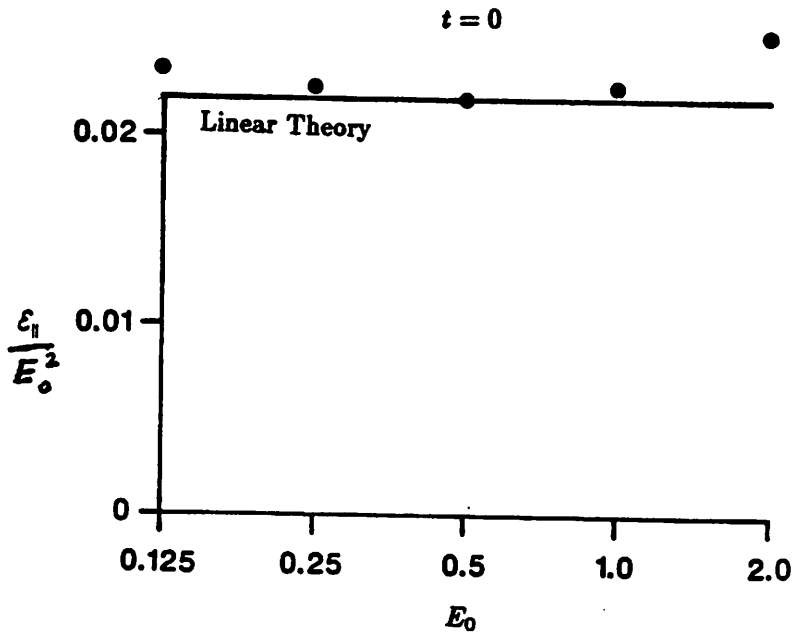


Figure 1a. Average parallel velocity versus E_0 at $t = 0$ and $t = \infty$



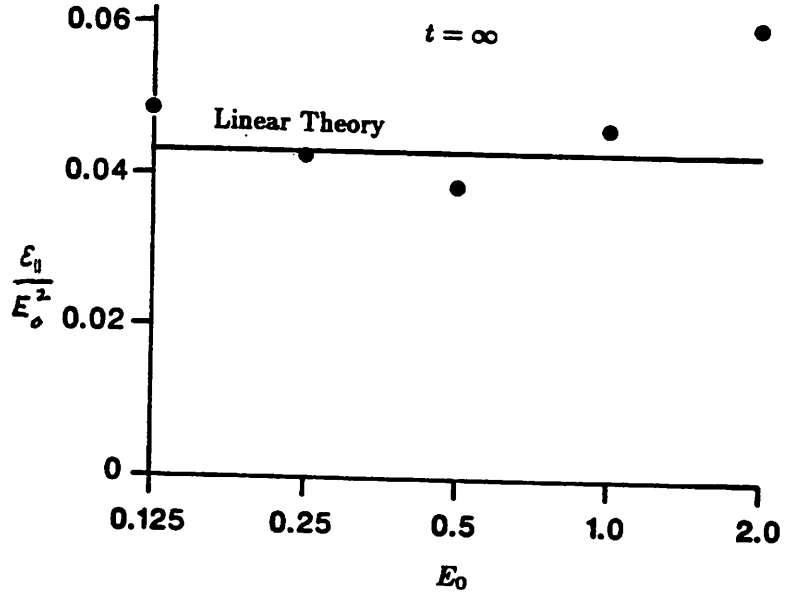


Figure 1b. Parallel energy density versus E_0 at $t = 0$ and $t = \infty$

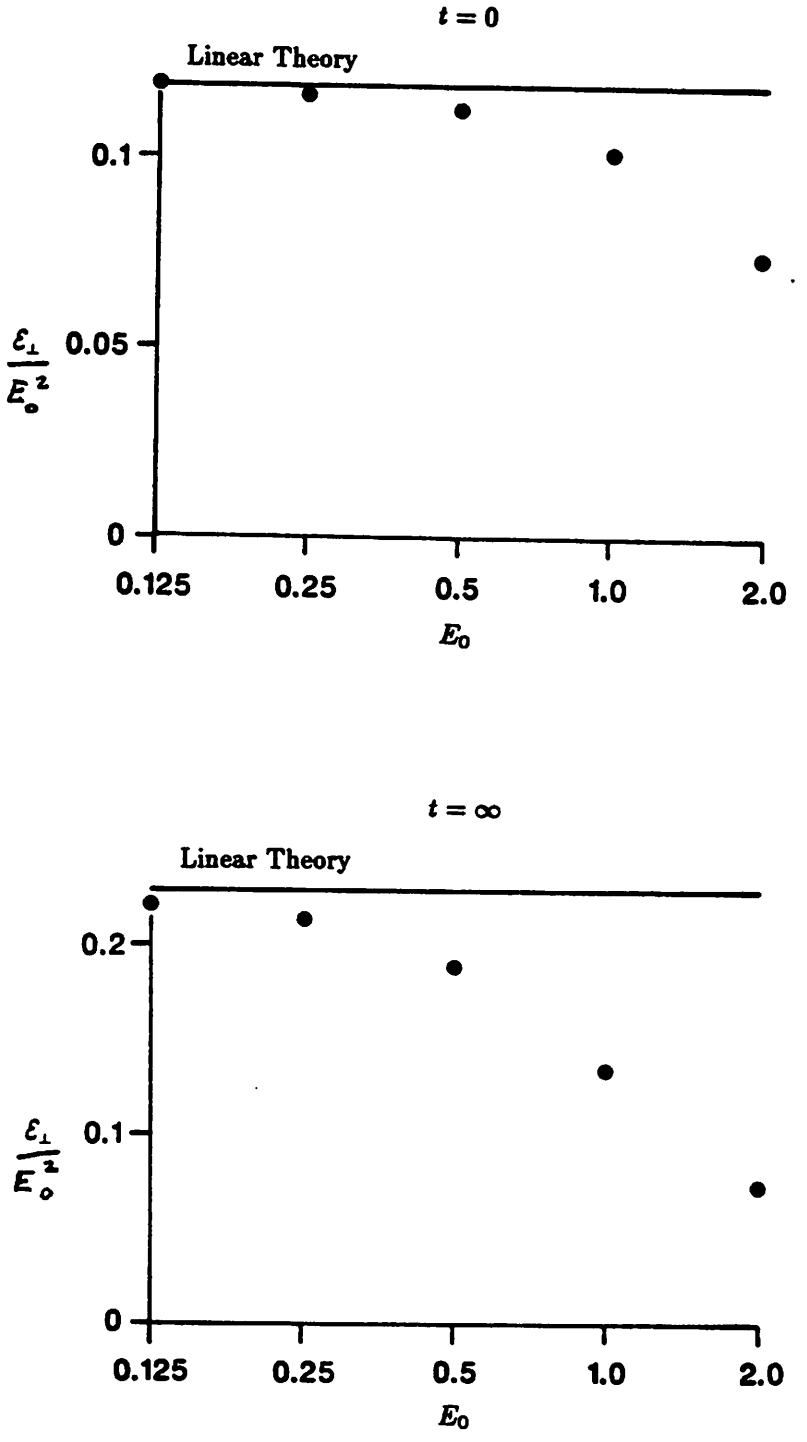


Figure 1c. Perpendicular energy density versus E_0 at $t = 0$ and $t = \infty$

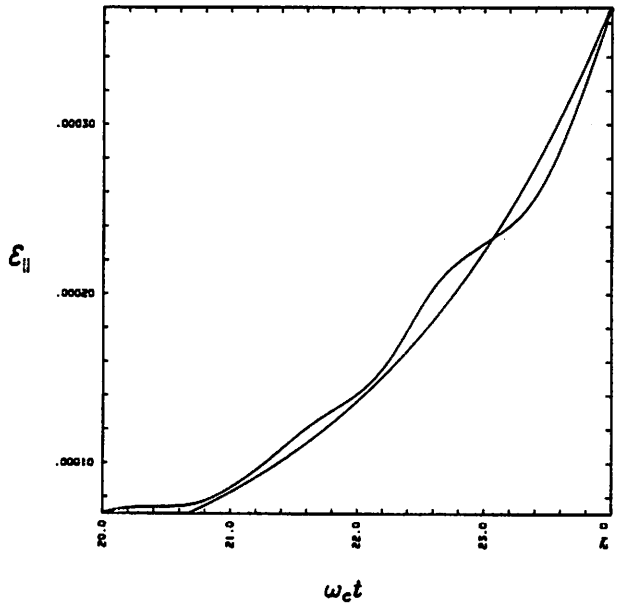
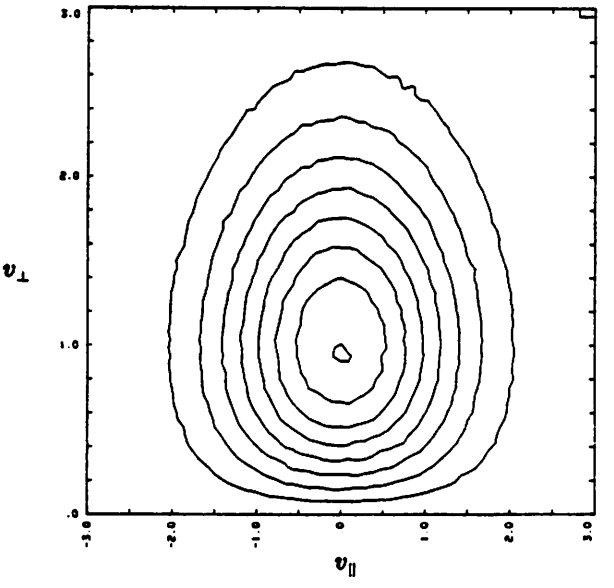


Figure 2. $\mathcal{E}_{\|}$ versus time at lowest excitation level ($E_0 = 0.125$). ($\omega_c t = 24$ here corresponds to $t = 0$ in the theory.)



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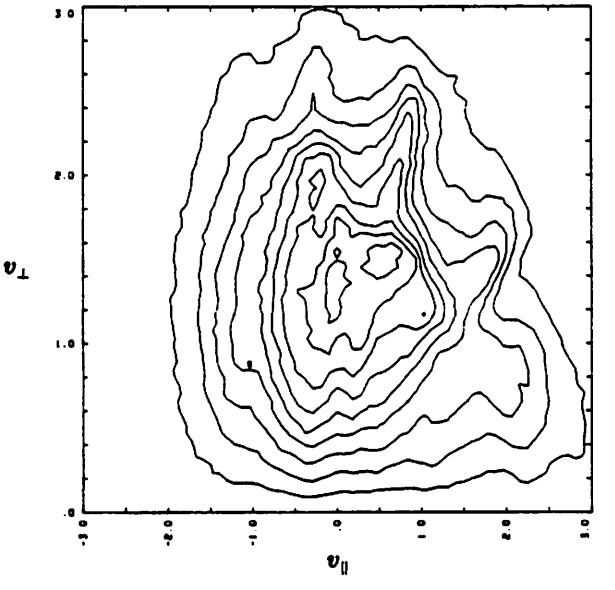

Final

Figure 3. Plots of $v_\perp f(v_\parallel, v_\perp)$ before and after perturbation. Excitation level was $E_0 = 2$.