ANALYSIS OF ADAPTIVE IDENTIFIERS IN THE PRESENCE OF UNMODELLED DYNAMICS: AVERAGING AND TUNED PARAMETERS

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ABSTRACT

Adaptive identifiers are designed with the assumption that the order of the plant is known. In this paper we analyze the behavior of a standard identifier when the plant contains additional dynamics, called unmodelled dynamics, which invalidate the known order assumption. The first result of our analysis is an input richness condition which does not depend on the order of the unmodelled dynamics to guarantee persistency of excitation of the regressor. Then we show that the PE condition leads to a BIBO stability property for the identifier. We use the method of averaging to formally define the notion of tuned parameters as the equilibrium of the identifier averaged system. It is shown that the tuned parameters always exist and that the actual parameters converge to some neighborhood of the tuned parameters. From the definition of the tuned parameters, we derive an explicit expression to calculate them and interpret them as the fixed parameter values which minimize the mean squared output error.

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1. Introduction

An adaptive identifier is a deterministic method of identifying recursively the parameters of an unknown system. An earlier analysis of this problem was performed by Lion 1967 and later extensively studied by Luders & Narendra 1973, 1974a, 1974b, Narendra & Valavani 1976, Kreisselmeier 1977, Goodwin & Sin 1984 and Anderson et al 1986. In this paper we will be concerned exclusively with the adaptive system structure presented and developed by Kreisselmeier 1977 and Narendra and Valavani 1979. In particular, we will be interested in analyzing and interpreting the behavior of such systems when the real plant contains some amount of unmodelled dynamics.

The existence of unmodelled dynamics is inevitable in any real system. Such perturbations exist for a variety of reasons. For example, it may be that there is incomplete physical knowledge about the system, or, some effects may be small and deliberatly neglected by the designer. For some systems, it is by choice that "unmodelled dynamics" are created. For example, to simplify a later design process, it is sometimes beneficial to have a lower order yet representative model of the system. For what ever reason, unmodelled dynamics will have to be dealt with in any identification/modelling process.

Under the idealized assumption of no unmodelled dynamics in the unknown plant, the persistency of excitation of the regressor is an important property which ensures identitfier stability and proper parameter convergence. In the presence of unmodelled dynamics, we will show that a persistently exciting (PE) regressor is equally important. Therefore, in this paper we determine necessary and sufficient conditions on the richness of the input in terms of the order of the identifier (not the unknown plant) to ensure that the regressor is almost always PE. Then we show how persistency of excitation ensures the boundedness of all identifier signals.

Since exact model matching between the identifier and the higher order plant can never occur, we utilize the notion of "tuned parameters" to analyze the behavior of the identifier parameters. The idea of tuned parameters is used extensively in Kosut & Friedlander 1985, Riedle & Kokotovic 1986 and

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Anderson et al 1986 to analyze the behavior of adaptive systems. In our analysis, since understanding and interpreting the behavior of an established identifier structure is the chief goal, we chose to utilize a specific definition of the tuned parameters suggested by the above mentioned papers. The definition we use is arrived at through the use of averaging and basically says that the tuned parameters are the equilibrium point of the averaged identifier equation. With this definition we

(1) explicitly specify how to calculate the tuned parameters,

(2) determine in what sense the identifier parameters converge to the tuned parameters,

(3) interpret the meaning of the tuned parameters as the parameter values which minimize the mean squared output error and

(4) show that the least squares algorithm asymptotic parameter value is equal to the tuned parameter value resulting from the gradient algorithm.

The format of this paper is as follows. The first section reviews the adaptive identifier. Then, we present some well known facts concerning identifier stability and parameter convergence when there are no unmodelled dynamics present. The update laws considered are the gradient and the least squares update. The next sections present an analysis of the same identifier when there exists unmodelled dynamics. Within that context the first subsection will be concerned with the input richness condition to ensure regressor persistency of excitation mentioned above. Then, we relate the PE condition to identifier stability. Following that is a section introducing the main concepts of averaging and its application to the identifier for defining the tuned parameters. In the remaining sections we present the results which relate the identifier parameters to the tuned parameters.

2. The Adaptive Identifier Under Ideal Conditions

2.1. The Adaptive Identifier Structure

To orient the reader and establish notation we present a brief overview of the adaptive identifier of Kreisselmeier 1977 and Narendra & Valavani 1976.

The adaptive identifier is a deterministic method for recursive identification of a rational function representation of a finite-dimensional, linear, time-invariant (FDLTI) plant. Its block structure is given in fig. 1 below.

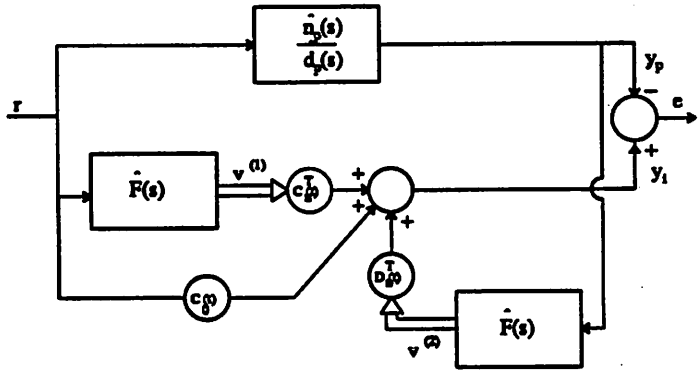

Figure 1: Block Structure of the Adaptive Identifier

To invoke this identification scheme we require 4 basic assumptions.

Identifier Assumptions

(1I) The unknown plant is a Finite Dimensional Linear Time Invariant (FDLTI) system of known order n with a proper transfer function $\hat{g}_p(s) = \dfrac{\hat{n}_p(s)}{\hat{d}_p(s)}$.

(2I) The unknown plant is stable.

(3I) $\hat{n}_p(s)$ and $\hat{d}_p(s)$ are coprime.

(4I) The input, r, and its derivative, $\dot{r}$, are bounded.

In fig. 1 the identifier parameters are $c_0(t) \in \mathbb{R}, C_n(t) \in \mathbb{R}^n$ and $D_n(t) \in \mathbb{R}^n$. The filter $\hat{F}(s)$ is a controllable canonical realization of a monic, n^{th} order, hurwitz polynomial, $\hat{\lambda}(s)$, chosen by the designer. Thus, if Λ, b is in controllable canonical form and $\det(sI-\Lambda)^{-1} = \hat{\lambda}(s)$ then

$$(sI-\Lambda)^{-1}b = \frac{1}{\hat{\lambda}(s)} \begin{bmatrix} 1 \\ s \\ . \\ . \\ s^{n-1} \end{bmatrix} =: \hat{F}(s) \quad \text{and} \quad \hat{V}^{(1)}(s) = \hat{F}(s)\hat{r}(s) \quad \text{and} \quad \hat{V}^{(2)} = \hat{F}(s)\hat{y}_p(s)$$

We define in mixed notation the time varying polynomials $C\,p_n(s)$ and $D\,p_n(s)$ as:

$$C\,p_n(s) := C_n^T(t) \begin{bmatrix} 1 \\ s \\ . \\ . \\ s^{n-1} \end{bmatrix} = c_1(t)+c_2(t)s+ \ldots +c_n(t)s^{n-1}$$

and

$$D\,p_n(s) := D_n^T(t) \begin{bmatrix} 1 \\ s \\ . \\ . \\ s^{n-1} \end{bmatrix} = d_1(t)+d_2(t)s+ \ldots +d_n(t)s^{n-1}$$

It is convenient to define the following quantities and important to note the resulting relationships.

$$\textbf{The Parameters} = \theta(t) := \begin{bmatrix} c_0(t) \\ C_n(t) \\ D_n(t) \end{bmatrix} \in \mathbb{R}^{2n+1} \quad \text{and} \quad \textbf{The Regressor} = W(t) := \begin{bmatrix} r(t) \\ V^{(1)}(t) \\ V^{(2)}(t) \end{bmatrix} \in \mathbb{R}^{2n+1}$$

then

$$e = y_i - y_p = \theta^T W - y_p$$

Also,

$$\textbf{The Regressor Transfer Function} = \hat{Q}(s) := \begin{bmatrix} 1 \\ \hat{F}(s) \\ \hat{F}(s)\hat{g}_p(s) \end{bmatrix}$$

thus,

$$\hat{W}(s) = \hat{Q}(s)\hat{r}(s)$$

Fact 2.1.1:

There exist unique values $c_0^* \in \mathbb{R}$, $C_n^* \in \mathbb{R}^n$ and $D_n^* \in \mathbb{R}^n$ such that;

$$\frac{\hat{n}_p(s)}{\hat{d}_p(s)} \equiv \frac{c_0^* \hat{\lambda}(s) + C^* p_n(s)}{\hat{\lambda}(s) - D^* p_n(s)} \qquad 2.1.2$$

The above fact says that the unknown plant can be uniquely parameterized in terms of W so that there exists a parallel structure between the unknown plant and the identifier. This parallel structure is illustrated by figure 2.

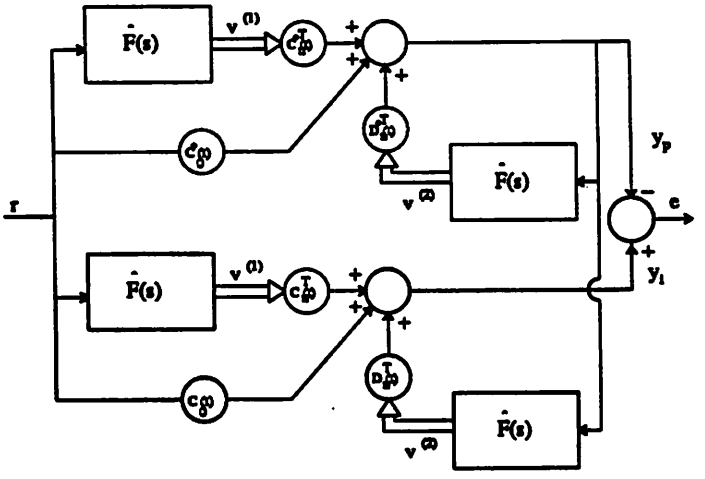

Figure 2: Block Structure of the Adaptive Identifier With Reparameterized Plant

The error equation can then be written as $e = (\theta(t) - \theta^*)^T W(t) = \phi^T W(t)$ where $\phi \in \mathbb{R}^{2n+1}$ is implicitly defined as the parameter error.

2.2. Update Laws

The two update laws we will be concerned with are the gradient algorithm and the least squares algorithm.

The gradient algorithm is given as

$$\dot{\theta} = -\varepsilon e W \qquad 2.2.1$$

while the least squares algorithm is given as

$$\dot{\theta} = -\varepsilon P W e \qquad \dot{P} = -\varepsilon P W W^T P \qquad P(0) = I \qquad 2.2.2$$

In both cases the parameter ε is a strictly positive real number known as the adaptation gain and the initial condition for θ is arbitrary but can be chosen to take into account any prior knowledge of the plant

parameters.

2.3. Identifier Stability and Parameter Convergence

The following fact is well known.

Fact 2.3.1:

For either the gradient or the least squares algorithms, the output error $e \in L_2 \cap L_\infty$, $e \to 0$ as $t \to \infty$ and $\theta, \dot{\theta} \in L_\infty$.

Though the above says that $e \to 0$, this does not necessarily mean that $\theta \to \theta^*$. To guarantee that it does, a condition on the regressor is required. The following definition is essential to the presentation of that condition.

Definition 2.3.2: Persistency of Excitation

A vector $W: \mathbb{R}_+ \to \mathbb{R}^{2n+1}$ is referred to as **Persistently Exciting (PE)** if and only if there exist constants $\alpha_1, \alpha_2, \delta > 0$ such that

$$\alpha_1 I \leq \int_{t_0}^{t_0+\delta} W(\tau)W^T(\tau)\, d\tau \leq \alpha_2 I \qquad \text{for all } t_0 \geq 0$$

The differential equations governing the behavior of the previously defined parameter errors are

$$\dot{\phi} = -\epsilon W W^T \phi \quad \text{(using the gradient algorithm)} \qquad 2.3.3$$

and

$$\dot{\phi} = -\epsilon P W W^T \phi \quad \text{(using the least squares update)} \qquad 2.3.4$$

In either case we simply have a linear, time varying system for which there exist specific conditions on W to determine when such systems are stable.

Fact 2.3.5:

Eqn. 2.3.3 is globally exponentially stable if and only if W is PE.

Fact 2.3.6:

Eqn. 2.3.4 is stable, $\lim_{t \to \infty} \phi(t) = 0$, if and only if W is PE. Furthermore, for t large enough ϕ is of the order of $\frac{1}{t}$.

2.4. Generalized Harmonic Analysis

It is useful to relate input richness to the persistency of excitation of the regressor. To do this we utilize the technique of Generalized Harmonic Analysis as given in Boyd & Sastry 1984. The following definitions and facts are fundamental to this subject.

Definition 2.4.1: Autocovariance (Stationarity)

A function $u:\mathbb{R}_+\to\mathbb{R}^n$ is said to have autocovariance $R_u(t)\in\mathbb{R}^{n\times n}$ if

$$\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T}u(\tau)u^T(t+\tau)d\tau=R_u(t)$$

with limit uniform in s. Furthermore, a time signal which possesses an autocovariance is referred to as stationary.

Definition 2.4.2: Cross-Correlation

The two functions $u:\mathbb{R}_+\to\mathbb{R}^n$ and $y:\mathbb{R}_+\to\mathbb{R}^m$ are said to have cross-correlation $R_{yu}(t)\in\mathbb{R}^{m\times n}$ if

$$\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T}y(\tau)u^T(t+\tau)d\tau=R_{yu}(t)$$

with limit uniform in s.

Fact and Definition 2.4.3: Spectral Measure

R_u is the Fourier transform of a positive semi-definite matrix, $S_u(d\omega)$, of bounded measures, that is;

$$R_u(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}S_u(d\omega)$$

and

$$\int_{-\infty}^{\infty}S_u(d\omega)<\infty$$

The corresponding inverse transform is;

$$S_u(d\omega)=\int_{-\infty}^{\infty}e^{-j\omega\tau}R_u(\tau)d\tau$$

$S_u(d\omega)$ is referred to as the Spectral Measure of u.

Given the above definitions and facts, we present the following main fact which establishes necessary and sufficient conditions on the input to guarantee that W is PE.

Fact 2.4.4:

If r has an autocovariance, then W is PE if and only if the support of the spectral measure of r contains at least 2n+1 points.

Noting that a bounded input and stable plant produce the upper bound of the PE condition and combining facts 2.3.5 and 2.4.4 (gradient algorithm) or 2.3.6 and 2.4.4 (least squares algorithm) results in the desired relationship between input spectral content (richness) and proper parameter convergence.

2.5. Interpretation of the Update Laws

In this section we give useful interpretations of the update laws.

The gradient algorithm causes ϕ to move along the gradient of e^2 as shown by the following.

$$\frac{\partial(e^2)}{\partial\phi} = \frac{\partial(\phi^T W)^2}{\partial\phi} = 2eW$$

Hence, when W is PE, ϕ is driven to zero along the "direction of steepest descent" of e^2.

On the other hand, the least squares algorithm yields a parameter value which asymptotically approaches the solution to a least squares optimization problem. To see this note the following definition.

Definition 2.5.1:

Assuming that the input, r, is stationary and W is PE, we define $\theta_{LS}(t):\mathbb{R}_+\rightarrow\mathbb{R}^{2n+1}$ for $t \geq \delta$ as

$$\theta_{LS}(t) := \text{Arg}\min_{\bar{\theta}} \frac{1}{t}\int_0^t (W^T(\tau)\bar{\theta} - y_p(\tau))^2\, d\tau$$

From our above definition of θ_{LS} we can easily calculate that

$$\theta_{LS}(t) = \left[\int_0^t W(\tau)W^T(\tau)\, d\tau\right]^{-1}\left[\int_0^t W(\tau)y_p(\tau)\, d\tau\right] \qquad t \geq \delta \qquad 2.5.2$$

where the above inverse exists because W was assumed to be PE over intervals of size δ. Now denote

$$P_{LS}(t) := \left[\varepsilon\int_0^t W(\tau)W(\tau)^T\, d\tau\right]^{-1}$$

so that for $t \geq \delta$

$$P_{LS}^{-1}(t) = P_{LS}^{-1}(\delta) + \varepsilon \int_{\delta}^{t} W(\tau) W^T(\tau)\, d\tau$$

Then eqn. 2.5.2 can be expressed as

$$\theta_{LS}(t) = P_{LS}(t) \left[\varepsilon \int_{0}^{t} W(\tau) y_p(\tau)\, d\tau \right] \qquad t \geq \delta$$

Hence, θ_{LS} and P_{LS} satisfy the differential equations

$$\dot{\theta}_{LS}(t) = -P_{LS}(t) W(t) W^T(t) P_{LS}(t) \left[\varepsilon \int_{0}^{t} W(\tau) y_p(\tau)\, d\tau \right] + P_{LS}(t) \left[\varepsilon W(t) y_p(t) \right]$$

$$= -\varepsilon P_{LS}(t) W(t) (W^T(t) \theta_{LS}(t) - y_p(t))$$

$$\dot{P}_{LS}(t) = -\varepsilon P_{LS}(t) W(t) W^T(t) P_{LS}(t)$$

Note that θ_{LS} and P_{LS} satisfy the same differential equations that θ and P satisfy in the least squares algorithm. The only difference is their respective values at $t = \delta$. Thus, the following fact easily follows.

Fact 2.5.3:

There exists a positive constant m such that for $t \geq \delta$

$$\| \theta(t) - \theta_{LS}(t) \| \leq \frac{m}{t} \qquad 2.5.4$$

In the above sense the θ described by the least squares update "essentially follows" $\theta_{LS}(t)$ -- the parameter value which, if fixed from 0 to t, would minimize the mean squared output error up to time t. Furthermore, the following fact is also true.

Fact 2.5.5:

If r is stationary and W is PE then $\lim_{t \to \infty} \theta_{LS}(t) = \lim_{t \to \infty} \theta(t) = R_W(0)^{-1} R_{y_p W}(0)$.

Remark:

Note that the above facts and interpretation of the least squares update only depend on the persistancy of excitation of W, not on its structure. Thus, the same conclusions hold true when there exist unmodelled dynamics as long as W is PE.

3. Analysis of the Adaptive Identifier in the Presence of Unmodelled Dynamics

3.1. Introduction

In this section of the paper we analyze the behavior of the previously introduced identifier when the unknown plant contains unmodelled dynamics. To specifically define what we mean by unmodelled dynamics subsection 3.2 lists the new assumptions which we will use in the rest our analysis.

In our analysis of the unperturbed identifier, we saw that persistency of excitation was a crucial property which guaranteed identifier stability and proper parameter convergence. Roughly speaking, when there exists unmodelled dynamics in the unknown plant, the PE condition will play an equivalent role. Hence, it is important that we establish conditions which determine when the regressor is PE. Like the unperturbed case our analysis will result in a richness condition on the input. Unlike the unperturbed case, however, such a condition will not always yield a PE regressor but instead will almost always yield a PE regressor. After determining what generates a PE regressor, we establish another result which is the relationship between persistency of excitation and identifier stability. Then, since there do not exist parameter values, which result in exact model matching, as defined by fact 2.1.1, we use the method of averaging to define a set of tuned parameters. The definition we use says that the tuned parameters are the equilibrium point of the averaged identifier dynamic equation. This definition is found in Kosut & Friedlander 1985, Riedle & Kokotovic 1986 and Anderson et al 1986. In our analysis we apply this tuned parameter definition to the case when the gradient update is used and show that the tuned parameters always exist and can be explicitly calculated. Furthermore, we show that the tuned parameters are the parameter values which minimize the mean squared output error and we show that the actual parameters converge to some neighborhood of the tuned parameters.

3.2. New Identifier Assumptions

In order to perform our analysis it is necessary to make the following new assumptions which replace assumptions (1I)-(4I).

(1U) The unknown plant is a FDLTI system of unknown order N (which could be very large) with transfer function

$$\hat{g}(s)=\hat{g}_p(s)\frac{\hat{n}_u(s)}{\hat{d}_u(s)}=\frac{\hat{n}_p(s)}{\hat{d}_p(s)}\cdot\frac{\hat{n}_u(s)}{\hat{d}_u(s)}=\frac{\hat{n}(s)}{\hat{d}(s)} \qquad 3.2.1$$

(2U) The nominal part, $\hat{g}_p(s)=\dfrac{\hat{n}_p(s)}{\hat{d}_p(s)}$, is an unknown proper transfer function of known order $n \le N$.

(3U) $\hat{g}(s)$ is stable.

(4U) $\hat{n}(s)$ and $\hat{d}(s)$ are coprime.

(5U) The input, r, and its derivative, $\dot{r}$, are bounded.

Remarks:

(1) The unmodelled part may represent high frequency dynamics, almost pole-zero cancellations or any other affect not taken into account by the nominal plant.

(2) Let $\Delta\hat{g}_1(s) = \dfrac{\hat{n}_u(s)}{\hat{d}_u(s)} - 1$ and $\Delta\hat{g}_2(s) = \hat{g}_p(s)\,\Delta\hat{g}_1(s)$, then the plant $\hat{g}(s)$ may be written as

$$\hat{g}(s) = \hat{g}_p(s)(1+\Delta\hat{g}_1(s))$$

or

$$\hat{g}(s) = \hat{g}_p(s) + \Delta\hat{g}_2(s)$$

which give the standard representations of multiplicative and additive unmodelled dynamics of Doyle & Stein 1981.

(3) Notice that no assumption has been made about the properness of the unmodelled part $\dfrac{\hat{n}_u(s)}{\hat{d}_u(s)}$, hence, it is possible that;

$$|\Delta\hat{g}_1(j\omega)| \to \infty \quad as \quad |\omega| \to \infty$$

3.3. Persistency of Excitation in the Prescence of Unmodelled Dynamics

Under the assumptions of no unmodeled dynamics and the input $r(t)$ being stationary, fact 2.4.4 says that the condition for the regressor, $W(t)$, to be PE is equivalent to saying that the support of the spectral measure of the input, r, contain at least $2n+1$ frequencies. In the proof of this fact, Boyd & Sastry use the coprimeness of $\hat{n}_p(s)$ and $\hat{d}_p(s)$ and the fact that they know the exact order of the unknown plant to basically show that $\hat{Q}(s)$ -- the transfer function between r and W -- has a structure such that any set of 2n+1 frequencies, $\omega_1, \ldots, \omega_{2n+1}$, give 2n+1 linearly independent vectors $\hat{Q}(j\omega_1), \ldots, \hat{Q}(j\omega_{2n+1})$. Therefore $R_W(0)$ is positive definite and hence W is PE.

When there exist unmodelled dynamics however, the situation is more complicated. The reason for this is that $\hat{Q}(s)$ depends on the unmodelled part $-\dfrac{\hat{n}_u(s)}{\hat{d}_u(s)}$. Hence, it is not necessarily true that the same input conditions always yield a regressor which is PE. On the other hand, it is easy to see that a necessary condition for $W(t)$ to be PE, no matter what, is that the spectral content of the input be supported by at

least $2n+1$ points since $W(t)$, and thus $\hat{Q}(s)$, are themselves $2n+1$ dimensional. However, it is not obvious what the sufficiency condition is. One might guess that a sufficient condition is that the spectral content of the input be supported by at least 2N+1 (where N is the order of the plant $\hat{g}(s)$) points. The problem, however, is that either N is very large or unknown. Therefore, in order to establish a useable condition which relates input spectral content to regressor persistency of excitation, we will give up the conclusion that 'such a condition always yields a PE regressor' and replace it with 'such a condition almost always yields a PE regressor'. To do this we must look at the structure of $\hat{Q}(s)$ to show that almost every set of $2n+1$ frequencies create the necessary linearly independent vectors. To this end, we present the following lemmas and theorems which determine necessary and sufficient conditions on r in terms of n -- the order of the identifier -- to ensure that W is almost always PE.

The first step to establishing the desired input richness condition is to show that there exists at least one set of 2n+1 frequencies which make the 2n+1 vectors $\hat{Q}(j\omega_1),\ldots,\hat{Q}(j\omega_{2n+1})$ linearly independent. The following lemma does that.

Lemma 3.3.1

for each $l \in \mathbb{Z}$, $1 \le l \le n$, there exists a set of $2l+1$ frequencies, $(j\omega_{i_1},\ldots,j\omega_{i_{2l+1}})$, such that the regressor transfer function $\hat{Q}(s)$ generates $2l+1$ linearly independent vectors $\hat{Q}(j\omega_{i_1}),\ldots,\hat{Q}(j\omega_{i_{2l+1}})$.

Proof:

Clearly, when $N=n$ there are no unmodelled dynamics and proof of this lemma reduces to the original case covered by fact 2.4.4. Thus, in this proof we assume $N > n$.

Chose any 2N+1 frequencies, $(j\omega_1,\ldots,j\omega_{2N+1})$, (N is the order of the plant $\hat{g}(s)$). Then, using arguments similar to those used in the "Main Theorem" of Boyd & Sastry 1984 the following matrix, $M_{2N}(j\omega_1,\ldots,j\omega_{2N+1})$ has full row rank.

$$M_{2N}(j\omega_1,\ldots,j\omega_{2N+1}) = \begin{bmatrix} \hat{d}(j\omega_1) & & \hat{d}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{N-1}\hat{d}(j\omega_1) & \cdots & (j\omega_{2N+1})^{N-1}\hat{d}(j\omega_{2N+1}) \\ \hat{n}(j\omega_1) & & \hat{n}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{N-1}\hat{n}(j\omega_1) & & (j\omega_{2N+1})^{N-1}\hat{n}(j\omega_{2N+1}) \end{bmatrix} \qquad 3.3.2$$

Through repositioning and elimination of some rows of the previous matrix, we construct the following matrix which also has full row rank.

$$M_{2n+1}(\omega_1,\ldots,\omega_{2N+1}) = \begin{bmatrix} \hat{d}(j\omega_1)(j\omega_1)^n & & \hat{d}(j\omega_{2N+1})(j\omega_{2N+1})^n \\ \hat{d}(j\omega_1) & \cdots & \hat{d}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{d}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{d}(j\omega_{2N+1}) \\ \hat{n}(j\omega_1) & & \hat{n}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{n}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{n}(j\omega_{2N+1}) \end{bmatrix} \qquad 3.3.3$$

Thus, if $\hat{\lambda}(s) = \lambda_0+\lambda_1 s+\ldots+\lambda_n s^n$ then all of the following matracies are full row rank.

$$\overline{M}_{2n+1}(\omega_1,\ldots,\omega_{2N+1}) = \begin{bmatrix} \hat{d}(j\omega_1)\hat{\lambda}(j\omega_1) & & \hat{d}(j\omega_{2N+1})\hat{\lambda}(j\omega_{2N+1}) \\ \hat{d}(j\omega_1) & \cdots & \hat{d}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{d}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{d}(j\omega_{2N+1}) \\ \hat{n}(j\omega_1) & & \hat{n}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{n}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{n}(j\omega_{2N+1}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_n & \lambda_0 & \cdots & \lambda_{n-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} (j\omega_1)^n\hat{d}(j\omega_1) & & (j\omega_{2N+1})^n\hat{d}(j\omega_{2N+1}) \\ \hat{d}(j\omega_1) & \cdots & \hat{d}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{d}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{d}(j\omega_{2N+1}) \\ \hat{n}(j\omega_1) & & \hat{n}(j\omega_{2N+1}) \\ \vdots & & \vdots \\ (j\omega_1)^{n-1}\hat{n}(j\omega_1) & & (j\omega_{2N+1})^{n-1}\hat{n}(j\omega_{2N+1}) \end{bmatrix} \qquad 3.3.4$$

By picking off the appropriate rows of $\overline{M}_{2n+1}$ and eliminating the columns which are linearly dependent, one can construct a $2l+1$ dimensional submatrix

$$\overline{M}_{2l+1}(\omega_{i_1},\ldots,\omega_{i_{2l+1}}) = \begin{bmatrix} \hat{d}(j\omega_{i_1})\hat{\lambda}(j\omega_{i_1}) & & \hat{d}(j\omega_{i_{2l+1}})\hat{\lambda}(j\omega_{i_{2l+1}}) \\ \hat{d}(j\omega_{i_1}) & \cdots & \hat{d}(j\omega_{i_{2l+1}}) \\ \vdots & & \vdots \\ (j\omega_{i_1})^{l-1}\hat{d}(j\omega_{i_1}) & & (j\omega_{i_{2l+1}})^{l-1}\hat{d}(j\omega_{i_{2l+1}}) \\ \hat{n}(j\omega_{i_1}) & & \hat{n}(j\omega_{i_{2l+1}}) \\ \vdots & & \vdots \\ (j\omega_{i_1})^{l-1}\hat{n}(j\omega_{i_1}) & & (j\omega_{i_{2l+1}})^{l-1}\hat{n}(j\omega_{i_{2l+1}}) \end{bmatrix} \qquad 3.3.5$$

which is nonsingular and $(j\omega_{i_1},\ldots,j\omega_{i_{2n+1}}) \in (j\omega_1,\ldots,j\omega_{2N+1})$. Using this fact and noting the following relationship gives the desired result.

$$\left[\hat{Q}(j\omega_{i_1}),\ldots,\hat{Q}(j\omega_{i_{2n+1}})\right] = \begin{bmatrix} \hat{d}(j\omega_{i_1})\hat{\lambda}(j\omega_{i_1}) & \cdots & \hat{d}(j\omega_{i_{2n+1}})\hat{\lambda}(j\omega_{i_{2n+1}}) \\ \hat{d}(j\omega_{i_1}) & \cdots & \hat{d}(j\omega_{i_{2n+1}}) \\ \vdots & \cdots & \vdots \\ (j\omega_{i_1})^{l-1}\hat{d}(j\omega_{i_1}) & \cdots & (j\omega_{i_{2n+1}})^{l-1}\hat{d}(j\omega_{i_{2n+1}}) \\ \hat{n}(j\omega_{i_1}) & \cdots & \hat{n}(j\omega_{i_{2n+1}}) \\ \vdots & \cdots & \vdots \\ (j\omega_{i_1})^{l-1}\hat{n}(j\omega_{i_1}) & \cdots & (j\omega_{i_{2n+1}})^{l-1}\hat{n}(j\omega_{i_{2n+1}}) \end{bmatrix}$$

$$\cdot \operatorname{diag}\left[\frac{1}{\hat{\lambda}(j\omega_{i_1})\hat{d}(j\omega_{i_1})},\ldots,\frac{1}{\hat{\lambda}(j\omega_{i_{2n+1}})\hat{d}(j\omega_{i_{2n+1}})}\right] \qquad 3.3.6$$

■

Given the above lemma, we present the following theorem.

Theorem 3.3.7: Persistency of Excitation of W(t) in the Presence of Unmodelled Dynamics

If r is stationary there exists a nowhere dense measure zero subset V in $\mathbb{R}^{2n+1}$ such that W is PE if and only if the spectral measure of r is supported by at least 2n+1 points which form a 2n+1 tuple not contained in V.

Proof:

⇒

Define a map $f : \mathbf{R}^{2n+1} \to C$ (*or* $\mathbf{R}^2$) by

$$f(\omega_1, \ldots, \omega_{2n+1}) := \det \begin{bmatrix} \hat{d}(j\omega_1)\hat{\lambda}(j\omega_1) & \cdots & \hat{d}(j\omega_{2n+1})\hat{\lambda}(j\omega_{2n+1}) \\ \hat{d}(j\omega_1) & \cdots & \hat{d}(j\omega_{2n+1}) \\ \vdots & \cdots & \vdots \\ (j\omega_1)^{n-1}\hat{d}(j\omega_1) & \cdots & (j\omega_{2n+1})^{n-1}\hat{d}(j\omega_{2n+1}) \\ \hat{n}(j\omega_1) & \cdots & \hat{n}(j\omega_{2n+1}) \\ \vdots & \cdots & \vdots \\ (j\omega_1)^{n-1}\hat{n}(j\omega_1) & \cdots & (j\omega_{2n+1})^{n-1}\hat{n}(j\omega_{2n+1}) \end{bmatrix}$$

$$= \det M_{2n+1}(\omega_1, \ldots, \omega_{2n+1}) \qquad 3.3.8$$

f is not identically zero by lemma 3.3.1 and it is analytic. Define the set V by

$$V := f^{-1}(0)$$

From lemma 1.8 p.288 of Hirsch & Smale 1979, V is measure zero.

Now let $\tilde{\omega}_1, \ldots, \tilde{\omega}_{2n+1}$ be the 2n+1 points in the support of the spectral measure of r such that $(\tilde{\omega}_1, \ldots, \tilde{\omega}_{2n+1}) \notin V$. It follows from eqn. 3.3.6 and the definition of V that the set of vectors $\hat{Q}(j\omega_1), \ldots, \hat{Q}(j\omega_{2n+1})$ are linearly independent. Now assume that W is not PE. Then theorem 3.2 of Boyd & Sastry 1984 says there exists a $C \in \mathbf{R}^{2n+1}$ such that $C \neq 0$ and

$$C^T R_W(0) C = 0$$

Hence,

$$\int_{-\infty}^{\infty} C^T \hat{Q}(j\omega) S_r(d\omega) \hat{Q}^*(j\omega) C = 0$$

Since r is a scalar, we get that

$$\int_{-\infty}^{\infty} |\hat{Q}^*(j\omega) C|^2 S_r(d\omega) = 0$$

$S_r(d\omega)$ is nonnegative so the above says that

$$\hat{Q}^*(j\omega) C = 0 \quad \text{for all } \omega \text{ in the support of the spectral measure of } r$$

This is a contradiction to the linear independence of $Q(j\bar{\omega}_1),\ldots,Q(j\bar{\omega}_{2n+1})$. The proof that V is a nowhere dense subset is trivial, so it is omitted here.

← omitted

Remarks:

(1) Main theorem 3.3.7 is very useful and important. It says that almost every input whose spectral measure is supported by at least 2n+1 points generates a PE regressor.

(2) The input spectral content condition of theorem 3.3.7 depends only on the size of the identifier so that the unmodelled dynamics can be any finite order for the result to hold.

(3) When the identifier is made strictly proper ($c_0 = 0$), from the proof of lemma 3.3.1, it can be seen that every input whose spectral measure is supported by at least 2N+1 points generates a PE regressor.

0.1. Generic Stability of the Identifier

When we attempt to identify an unknown plant whose order may be larger than assumed, the first thing we want to be sure of is that the identifier remains stable -- all identifier signals remain bounded. Since the plant is assumed to be stable, it poses no problem. As for the identifier itself, all of its dynamics can be summed up in one equation which we refer to as the identifier dynamic equation and which is as follows.

$$\dot{\theta} = -\varepsilon W W^T \theta + \varepsilon W y_p \quad \text{(gradient algorithm)}$$

$$\dot{\theta} = -\varepsilon P W W^T \theta + \varepsilon P W y_p \quad \text{(least squares update)}$$

Thus, regardless of the unmodelled dynamics present in the plant, both equations are linear time varying systems whose input output stability properties are determined by the absolute integrability of the product of their state transition matricies and the forcing terms. Hence, the combination of theorem 3.3.7 and facts 2.3.5 and 2.3.6 give what we refer to as the identifier generic stability property.

Theorem 3.4.1: Identifier Generic Stability Property

Almost every r (in the sense of theorem 3.3.7) whose spectral measure is supported by at least 2n+1 points yields y_p, y_i, θ, ϕ, W and e which are bounded.

Proof:

Follows from theorem 3.3.7, facts 2.3.5 and 2.3.6 and theorem 8.8 of C.T. Chen 1984.

Remark:

Though the above theorem does not say how large or small the bounds on the given signals are, it does draw a distinction between situations when there exist signals which wander off to ∞ and when there exist signals which simply become very large.

3.5. Introduction to Averaging Analysis

The method of averaging will play a key role in both our analysis of the identifier in the presence of unmodelled dynamics and in establishing a definition for the "Tuned Parameters". Therefore, in this section we present the pertinent ideas of averaging used in our analysis.

The method of averaging is an approximation technique used to analyze the behavior of a nonautonomous system of differential equations and is extensively studied on a generic level in Hale 1980 and Sanders & Verhulst 1985. In such an analysis a time independent, averaged system is defined and used to approximate the original time varying system. The validity of this technique depends on the proper choice of an adjustable parameter used to slow down the rate at which the state evolves. This parameter must be chosen small enough so that the averaged system can capture the basic behavior of the original system.

3.6. Definition of the Averaged System

In general, the systems one can analyze with the method of averaging are those which can be represented by a differential equation of the form;

$$\dot{x} = \varepsilon f(t,x,\varepsilon) \qquad x(0) = x_0 \tag{3.6.1}$$

where $x \in \mathbb{R}^n$, $t \geq 0$, $0 < \varepsilon \leq \varepsilon_0$, f is piecewise continuous in t and $f(t,0,0) = 0$. The method of averaging as applied to such systems has been extensively studied by Hale 1980, Sanders & Verhulst 1985, Fu, Bodson & Sastry 1985, Riedle & Kokotovic 1985, Bodson et al 1986 and Bodson 1986. For our purposes we will be interested in generalizing the results of Fu, Bodson & Sastry 1985, Bodson et al 1986 and Bodson 1986 so that equation 3.6.1 includes a separate forcing term. The system form we are interested in is:

$$\dot{x} = \varepsilon f(t,x,\varepsilon) + \varepsilon g(t) \qquad x(0) = x_0 \tag{3.6.2}$$

where again $x \in \mathbb{R}^n$, $t \geq 0$, $0 < \varepsilon \leq \varepsilon_0$, f and g are piecewise continuous in t and $f(t,0,0) = 0$. In the above system, ε plays the role of the adjustable parameter and is used to slow down the dynamics of the original system.

We now define what is meant by the average value of a function

Definition 3.6.3: The Average Value of a Function and the Convergence Function

The function $f(t,x,0)$ is said to have average value $f_{av}(x)$ if there exists a continuous function $\gamma(T):\mathbb{R}_+ \to \mathbb{R}_+$ which is strictly decreasing and such that $\gamma(T) \to 0$ as $T \to \infty$ with

$$\left\| \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau,x,0)\, d\tau - f_{av}(x) \right\| \leq \gamma(T) \tag{3.6.4}$$

for all $t_0 \geq 0$, $T \geq 0$ and $x \in B_h(\bar{x}) := \{x \in \mathbb{R}^n \mid |x-\bar{x}| \leq h\}$. The function $\gamma(T)$ is referred to as the

convergence function.

Note that if one defines a new function, $d(t,x)$, as

$$d(t,x) := f(t,x,0) - f_{av}(x) \qquad 3.6.5$$

then $f(t,x,0)$ has mean value $f_{av}(x)$ if and only if $d(t,x)$ has zero mean value.

With the above definition, and assuming f_{av} and g_{av} exist, we define the average system as

$$\dot{x}_{av} = \varepsilon f_{av}(x_{av}) + \varepsilon g_{av} \qquad x_{av}(0) = x_0 \qquad 3.6.6$$

where

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau,x,0)\, d\tau \quad \text{and} \quad g_{av} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} g(\tau)\, d\tau \qquad 3.6.7$$

with the limits uniform in t_0 and x.

The following definition will be useful and is taken from Hahn 1967.

Definition 3.6.8: Class K Function

A function, $\alpha(\varepsilon):\mathbb{R}_+ \to \mathbb{R}_+$, belongs to class K ($\alpha(\varepsilon) \in K$) if it is continuous, strictly increasing and $\alpha(0)=0$.

3.7. Averaging Theorems

Before stating our main averaging theorem, we first make some assumptions about the nature of f, f_{av}, g and g_{av}. Let $B_h(\bar{x})$ be the closed ball of radius h centered at $\bar{x}$ and let $\varepsilon_0 > 0$.

(A1) $\bar{x}$ is an equilibrium point of 3.6.6, i.e.

$$0 = \varepsilon f_{av}(\bar{x}) + \varepsilon g_{av}$$

and $f_{av}(x)$ is Lipschitz in x, i.e. there exists an $l_{av} \geq 0$ such that;

$$\|f_{av}(x_1) - f_{av}(x_2)\| \leq l_{av} \|x_1 - x_2\|$$

for all $x_1, x_2 \in B_h(\bar{x})$.

(A2) $f(t,x,\varepsilon)$ is Lipschitz in x, i.e. there exists an $l_1 \geq 0$ such that;

$$\|f(t,x_1,\varepsilon) - f(t,x_2,\varepsilon)\| \leq l_1 \|x_1 - x_2\|$$

for all $t \geq 0$ and $x_1, x_2 \in B_h(\bar{x})$ and $0 \leq \varepsilon \leq \varepsilon_0$.

(A3) $f(t,x,\varepsilon)$ is Lipschitz in ε linearly in x, i.e. for some $l_2 \geq 0$;

$$\|f(t,x,\varepsilon_1) - f(t,x,\varepsilon_2)\| \leq l_2 \|x\| \, |\varepsilon_1 - \varepsilon_2|$$

for all $t \geq 0, x \in B_h(\bar{x})$ and $0 \leq \varepsilon_1, \varepsilon_2 \leq \varepsilon_0$.

(A4) The function $d(t,x) := f(t,x,0) - f_{av}(x)$ is piecewise continuous in t, has a bounded and continuous first partial derivative with respect to x and $d(t,0) = 0$. Moreover, $d(t,x)$ has zero mean value with convergence function $\gamma_1(t)//x//$ and $\frac{\partial d(t,x)}{\partial x}$ has zero mean value with convergence function $\gamma_1(t)$.

(A5) $g(t)$ possesses the average value g_{av}, is piecewise continuous in t and is a uniformly bounded function of t, i.e. for some $g_b > 0$

$$//g(t)// \leq g_b \qquad \text{for all } t \geq 0$$

Remarks:

(1) In assumption (A4) and (A5) we assume that f and g possess average values f_{av} and g_{av} respectively. Not every function possesses an average value. However, Hale 1980 proves that all periodic or almost periodic functions do. We do not restrict ourselves to thoses cases but simply consider the existence of an average value as a condition for the following theorem.

(2) If f is linear in x and ε, i.e. $f(t,x,\varepsilon) = \varepsilon A(t)x$, then (A4) simply assumes that $A(t)$ possesses an average value.

We now present our main averaging theorem which will be useful in our analysis of the identifier.

Theorem 3.7.1: Averaged System Approximation Theorem

If: the original and averaged systems:

$$\dot{x} = \varepsilon f(t,x,\varepsilon) + \varepsilon g(t) \quad x(0) = x_0 \qquad 3.7.2$$

$$\dot{x}_{av} = \varepsilon f_{av}(x_{av}) + \varepsilon g_{av} \quad x_{av}(0) = x_0 \qquad 3.7.3$$

satisfy assumptions (A1)-(A5), $\bar{x}$ is an exponentially stable equilibrium point of eqn.3.7.3 and the initial condition $x_0 \in B_h(\bar{x})$ is sufficiently close to the equilibrium point $\bar{x}$

Then: there exists an $\varepsilon_1 > 0$ and a class K function $\eta(\varepsilon) \in$ K such that for all $\varepsilon < \varepsilon_1$, $//x(t)//$ is a bounded function of t for all $t \geq 0$ and;

$$\lim_{t \to \infty} //x(t) - \bar{x}// \leq \eta(\varepsilon)$$

Proof: See appendix A.

In words, the above theorem says that as $t \to \infty$ and for ε small enough, the solution to the original system converges to a ball of radius $\eta(\varepsilon) \in K$ centered at the equilibrium point of the averaged system and as $\varepsilon \to 0$ the radius of this ball goes to 0.

3.8. Use of Averaging to Define the Tuned System

As mentioned previously, we will use the method of averaging to define the tuned parameters of an adaptive identifier when the gradient algorithm is used. Recall that in section 3.4 the important dynamics of the identifier were captured in a single equation referred to as the identifier dynamic equation which we restate here.

$$\dot{\theta} = -\varepsilon W(t)W(t)^T\theta(t) + \varepsilon W(t)y_p(t) \qquad 3.8.1$$

It is this equation we apply averaging to to obtain the following tuned parameter definition.

Definition 3.8.2: Tuned Parameters and the Identifier Averaged Equation

If the input, r, has a an autocovariance then the tuned parameters, which we denote by θ_T^*, are the equilibrium points of the following averaged equation which we refer to as the identifier averaged system.

$$\dot{\theta}_{av} = -\varepsilon R_w(0)\theta_{av} + \varepsilon R_{y_p w}(0) \qquad 3.8.3$$

where;

$$R_w(0) := \lim_{T\to\infty} \frac{1}{T} \int_{t_o}^{t_o+T} W(\tau)W(\tau)^T \, d\tau \quad \text{and} \quad R_{y_p w}(0) := \lim_{T\to\infty} \frac{1}{T} \int_{t_o}^{t_o+T} y_p(\tau)W(\tau) \, d\tau \qquad 3.8.4$$

Note that the above definitions are expressed in terms of the autocovariance of the regressor vector and the cross-correlation between the regressor vector and the output of the plant. Also, since the regressor vector is generated by the input and a designer chosen filter, F, the autocovariance and cross-correlation terms mentioned above are dependent on that choice of input and filter. Hence, the above defined quantities are both input and filter choice dependent. (Of course we only consider real valued inputs so that R_W and $R_{y_p W}$ are themselves real valued.)

From the given definitions we obtain the following lemma.

Lemma 3.8.5: Tuned Parameter Formula, Existence and Uniqueness

The tuned parameters

1) always exist

2) satisfy the following equation

$$R_W(0)\theta_T^* = R_{y,W}(0) \qquad 3.8.6$$

3) are unique if and only if W is PE.

From the above definition and lemma several interesting things result:

(1) The tuned parameters can be easily calculated using eqn. 3.8.6 which, of course, depends on the plant, input and filter, F.

(2) When W is not PE the tuned parameters constitute an entire subspace.

(3) From fact 2.5.5, when the input is PE, the asymptotic parameter values from the least squares update equal the tuned parameters from the gradient update. Hence, the gradient update tuned parameters also minimize the mean squared output error -- a result similar to that obtained by Riedle & Kokotovic 1986.

3.9. Relationship between the Identifier Parameters and The Tuned Parameters

To obtain a relationship between the identifier parameters and the just defined tuned parameters, we use the results of the averaging section. Since eqn. 3.8.1 is a linear equation, we make use of its linearity to relax the requirement on the initial conditions and obtain the following theorem.

Theorem 3.9.1: Tuned Parameter Approximation Theorem

If the input is stationary and W is PE then there exists an $\varepsilon_1 \geq 0$ and a class K function $\eta(\varepsilon) \in K$ such that for all $\varepsilon \leq \varepsilon_1$

$$\lim_{t\to\infty} \|\theta(t) - \theta_T^*\| \leq \eta(\varepsilon)$$

Proof: See appendix B.

Remarks:

(1) Like theorem 3.7.1 the above theorem says that the identifier parameters converge to a ball of radius $\eta(\varepsilon) \in K$ centered at the tuned parameters. Hence, as ε goes to zero and t goes to infinity, the identified parameter values converge to the tuned parameters.

(2) Unlike theorem 3.7.1, the above theorem is a global result in the sense that it is valid for any initial condition for θ. Such an improvement was achievable because of the linearity of eqn. 3.8.1.

4. Example Section

4.1. Introduction and Definition of Example System

In this section we present a simple example which illustrates the conclusions of the previous section. Specifically, we take a nominal system, $\frac{1}{s+1}$ with unmodelled dynamics, $\frac{10}{s+10}$ and perform a first order identification on the resulting system. In our identification procedure we assumed that the unknown plant is strictly proper so that there are only two unknowns in our first order identification. A schematic of the resulting identifier structure is shown below.

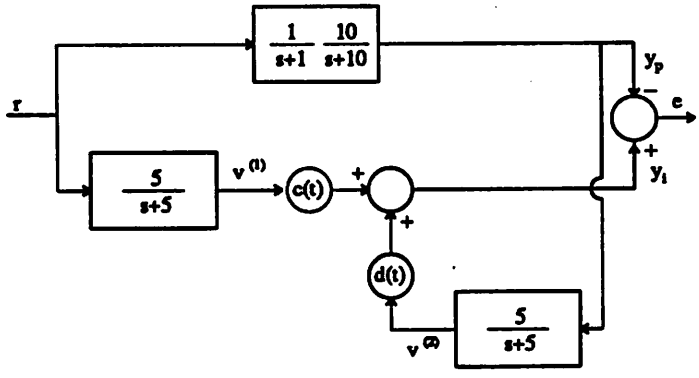

Figure 3: Block Structure of Identifier Example

The identification is performed using the least squares update and the gradient update. Furthermore, the gradient update averaged system was calculated and its response is also included. Simulation results are given using two different adaptation gains. The first has $\varepsilon = 0.1$ with those results given in figure 4 while the second has $\varepsilon = 1.0$ with those results given in figure 5. In both cases the input is 2+2sin(t)+2sin(5t).

To calculate the gradient update averaged system one must calculate $R_W(0)$ and $R_{y_p W}(0)$. In the case of inputs with discrete spectra, calculation of those quantities reduces to a summation as follows

$$R_W(0) = \sum_{i=1}^{m} \hat{\mathcal{Q}}(j\omega_i)\hat{\mathcal{Q}}^*(j\omega_i)S_r(\omega_i)$$

and

$$R_{y_p W}(0) = \sum_{i=1}^{m} \hat{\mathcal{G}}^*(j\omega_i)\hat{\mathcal{Q}}(j\omega_i)S_r(\omega_i)$$

where $S_r(\omega)$ is the spectral magnitude of r (a scalar) at frequency ω and superscript * denotes complex conjugate transpose. For this example, with the given input, $R_W(0)$ and $R_{y_p W}(0)$ turn out to be

$$R_W(0) = \begin{bmatrix} 6.923 & 4.811 \\ 4.811 & 4.983 \end{bmatrix} \qquad R_{y_p W}(0) = \begin{bmatrix} 5.189 \\ 4.983 \end{bmatrix}$$

The results on the following page illustrate how the identifier parameters "converge" to the tuned parameters. Furthermore, though the parameters from the least squares update haven't converged to the tuned parameters from the gradient update, they eventually will.

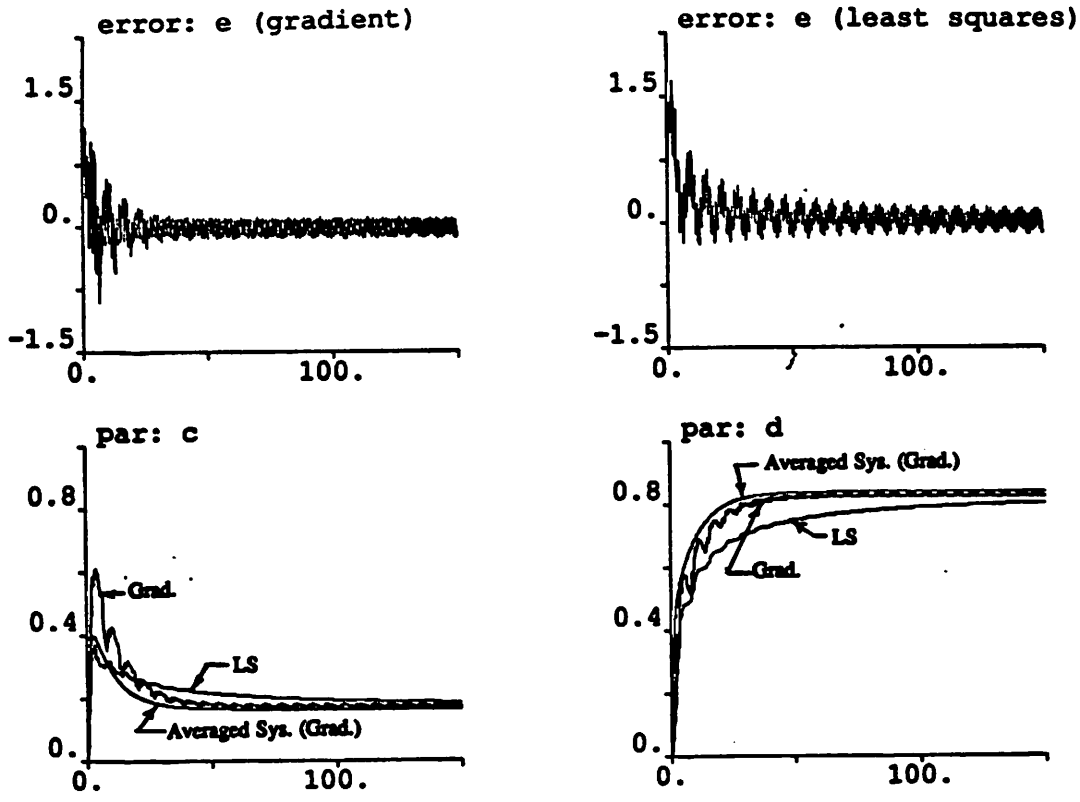

Figure 4: Strictly proper first order identification with input 2+2sin(t)+2sin(5t) and update gain -- $\varepsilon = 0.1$.

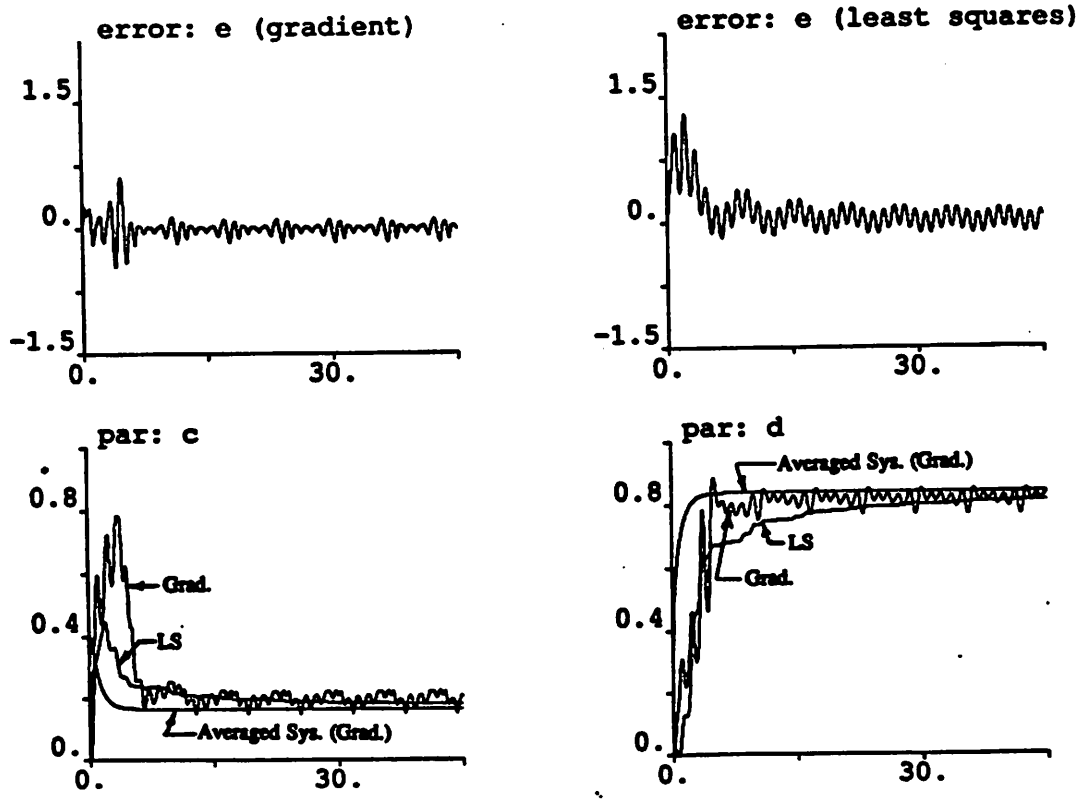

Figure 5: Strictly proper first order identification with input 2+2sin(t)+2sin(5t) and update gain -- $\varepsilon = 1.0$.

Conclusions

In this paper we reviewed the adaptive identifier of Kreisselmeier 1977 and Narendra & Valavani 1976 under the assumption of no unmodelled dynamics. Then, assuming the existence of FDLTI unmodelled dynamics, we further analyzed the behavior of the identifier. Results from that analysis include a necessary and sufficient input richness condition to ensure that the regressor is almost always PE. This condition basically says that almost any input whose spectral measure is supported by at least 2n+1 points will produce a PE regressor regardless of the order of the unmodelled dynamics. Then, we used the method of averaging to define the tuned parameters of the identifier as the equilibrium point of the averaged identifier dynamic equation. We presented further results which show that the tuned parameters are unique when the regressor is PE and that the identifier parameters converge to a ball centered at the tuned parameters and whose radius is a strictly decreasing function of the adaptation gain. The interpretation of the defined tuned parameters is that they are the fixed values of the identifier parameters which minimize the mean squared output error between the identifier and the unknown plant. Furthermore, it was pointed out that the tuned parameters from the gradient update are equal to the asymptotic parameter values from the least squares update. We concluded this paper with an example designed to illustrate the above described results.

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Appendix A

Proof of Theorem 3.7.1 (Nonlinear averaging result)

In this proof we follow the technique of Fu & Sastry 1987.

Using assumptions (A1)-(A4) it is shown that there exists a coordinate transformation of the form;

$$x = z + \varepsilon w_\varepsilon(t,z) \tag{a.1}$$

where $w_\varepsilon(t,z):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ is defined as;

$$w_\varepsilon(t,z) := \int_0^t (f(\tau,z,0)-f_{av}(z))e^{-\varepsilon(t-\tau)}\,d\tau$$

and satisfies;

$$\|\varepsilon w_\varepsilon(t,z)\| \le \zeta_1(\varepsilon)\|z\| \tag{a.2}$$

for all $z\in B_h$, $0 < \varepsilon \le \varepsilon'_1$ for some $\varepsilon'_1 > 0$ and some class K function $\zeta_1(\varepsilon)$.

Under such a transformation, the system described by eqn. 4.3.2 becomes;

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon(I + p_2(t,z,\varepsilon))g(t) \tag{a.3}$$

where $p_1(t,z,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^n$, $p_2(t,z,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^{n\times n}$ satisfy;

$$\|p_1(t,z,\varepsilon)\| \le \zeta_2(\varepsilon)\|z\| \tag{a.4}$$

and

$$\|p_2(t,z,\varepsilon)\| \le \zeta_3(\varepsilon) \tag{a.5}$$

for all $z\in B_h$, $0 < \varepsilon \le \varepsilon'_1$ and for some class K functions $\zeta_2(\varepsilon)$ and $\zeta_3(\varepsilon)$.

Define $\tilde{z} := z - \bar{x}$. From the assumptions, (a.3) can then be rewritten as;

$$\begin{aligned}\dot{\tilde{z}} &= \varepsilon(f_{av}(\tilde{z} + \bar{x})-f_{av}(\bar{x})) + \varepsilon p_1(t,z,\varepsilon) + \varepsilon p_2(t,z,\varepsilon)g(t) + \varepsilon(g(t)-g_{av}) \\ &= \varepsilon\frac{\partial f_{av}(z)}{\partial z}\Big|_{z=\bar{x}}\tilde{z} + \varepsilon p_1(t,\tilde{z} + \bar{x},\varepsilon) + \varepsilon p_2(t,\tilde{z} + \bar{x},\varepsilon)g(t) + \varepsilon(g(t)-g_{av}) + \varepsilon\ h.o.t.(\tilde{z})\end{aligned} \tag{a.6}$$

Now define $A := \dfrac{\partial f_{av}(z)}{\partial z}\Big|_{z=\bar{x}}$ which is a Hurwitz matrix since $\bar{x}$ is an exponentially stable equilibrium point of the averaged system -- eqn. 4.3.3.

Next, we claim that there exists a class K function $\zeta_4(\varepsilon)$ such that;

$$\|\varepsilon\int_0^t e^{\varepsilon A(t-\tau)}(g(\tau) - g_{av})\,d\tau\| \le \zeta_4(\varepsilon) \tag{a.7}$$

and its proof is shown below.

Define

$$u_\varepsilon(t) := \int_0^t e^{\varepsilon A(t-\tau)}(g(\tau)-g_{av})d\tau \tag{a.8}$$

thus

$$u_0(t) = \int_0^t (g(\tau)-g_{av})d\tau \tag{a.9}$$

By definition of g_{av} there exists a function $\gamma_g(t)$ which goes to 0 as t goes to ∞ and

$$\|u_0(t+t_0)-u_0(t_0)\| \le \gamma_g(t)t \tag{a.10}$$

for all t, $t_0 \ge 0$. Integrating (a.8) by parts yields

$$u_\varepsilon(t) = u_0(t) + \varepsilon A\int_0^t e^{\varepsilon A(t-\tau)}u_0(\tau)d\tau \tag{a.11}$$

Using the fact that

$$\varepsilon A\int_0^t e^{\varepsilon A(t-\tau)}u_0(t)d\tau = -u_0(t) + e^{\varepsilon At}u_0(t) \tag{a.12}$$

(a.11) can be rewritten as

$$u_\varepsilon(t) = e^{\varepsilon At}u_0(t) + \varepsilon A\int_0^t e^{\varepsilon A(t-\tau)}(u_0(\tau)-u_0(t))d\tau \tag{a.13}$$

Now, using (a.10) one gets

$$\|u_\varepsilon(t)\| \le me^{-\varepsilon\alpha t}\gamma_g(t)t + \varepsilon m\|A\|\int_0^t e^{-\varepsilon\alpha(t-\tau)}\gamma_g(t-\tau)\ (t-\tau)d\tau \tag{a.14}$$

where

$$\|e^{\varepsilon At}\| \le me^{-\varepsilon\alpha t} \tag{a.15}$$

Consequently,

$$\|\varepsilon u_\varepsilon(t)\| \le \sup_{t'\ge 0}\ \gamma_g(\frac{t'}{\varepsilon})\ mt'e^{-\alpha t'} + m\|A\|\int_0^{\infty}\gamma_g(\frac{\tau'}{\varepsilon})\tau'e^{-\alpha\tau'}d\tau' \tag{a.16}$$

Since $g(t)$ is bounded, there exists a $\beta > 0$ such that $\gamma_g(t) \le \beta$ for all $t \ge 0$. Note that for all $t' \ge 0$, $t'e^{-\alpha t'} \le \frac{e^{-1}}{\alpha}$ and $t'e^{-\alpha t'} \le t'$. Thus,

$$\|\varepsilon u_\varepsilon(t)\| \le \sup_{t'\in[0,\sqrt{\varepsilon}]}\ (\gamma_g(\frac{t'}{\varepsilon})\ mt'e^{-\alpha t'}) + \sup_{t'\ge\sqrt{\varepsilon}}\ (\gamma_g(\frac{t'}{\varepsilon}\ mt'e^{-\alpha t'}) \tag{a.17}$$

$$+ m\|A\|\int_0^{\sqrt{\varepsilon}}\gamma_g(\frac{\tau'}{\varepsilon})\ \tau'e^{-\alpha\tau'}d\tau + m\|A\|\int_{\sqrt{\varepsilon}}^{\infty}\gamma_g(\frac{\tau'}{\varepsilon})\ \tau'e^{-\alpha\tau'}d\tau'$$

This in turn implies that

$$\|\varepsilon u_\varepsilon(t)\| \le m\left[\beta\sqrt{\varepsilon} + \gamma_g(\frac{1}{\sqrt{\varepsilon}})\frac{e^{-1}}{\alpha}\right] + m\|A\|\left[\beta\frac{\varepsilon}{2} + \gamma_g(\frac{1}{\sqrt{\varepsilon}})(\frac{1}{\alpha^2}+\frac{\sqrt{\varepsilon}}{\alpha})e^{-\alpha\sqrt{\varepsilon}}\right]$$

$$=: \zeta_4(\varepsilon) \tag{a.18}$$

Which proves the claim.

Suppose for some $t \ge 0$, $\|\bar{z}\| \le h' < h$, then we could bound the higher order terms by some class K function i.e. $\|h.o.t.(\bar{z})\| \le \delta_1(h')\|\bar{z}\|$. In that case the bound on $\|\bar{z}\|$ can be expressed as;

$$\|\bar{z}(t')\| \le \|e^{\varepsilon A t'}\bar{z}(0)\| + \|\int_0^{t'}\varepsilon e^{\varepsilon A(t'-\tau)}\left[p_1(\tau,\bar{z}+\bar{x},\varepsilon) + p_2(\tau,\bar{z}+\bar{x},\varepsilon)g(\tau) + h.o.t.(\bar{z})\right]d\tau\|$$

$$+ \|\int_0^{t'}\varepsilon e^{\varepsilon A(t-\tau)}(g(\tau)-g_{av})d\tau\|$$

$$\le me^{-\varepsilon\alpha t'}\|\bar{z}(0)\| + \varepsilon\int_0^{t'}me^{-\varepsilon\alpha(t'-\tau)}\left[\zeta_2(\varepsilon)(\|\bar{z}\|+|\bar{x}\|)+\zeta_3(\varepsilon)g_b+\delta_1(h')\|\bar{z}\|\right]d\tau + \zeta_4(\varepsilon) \tag{a.19}$$

Now let $v(t)$ be defined as;

$$v(t) := \|\bar{z}(t)\|e^{\varepsilon\alpha t}$$

Then, using (a.19) we have;

$$v(t) \le mv(0) + \left[\frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha} + \frac{m\zeta_3(\varepsilon)g_b}{\alpha} + \zeta_4(\varepsilon)\right]e^{\varepsilon\alpha t} + \varepsilon\int_0^t m(\zeta_2(\varepsilon) + \delta_1(h'))v(\tau)\,d\tau \tag{a.20}$$

Applying the generalized Bellman-Gronwell Lemma to eqn.(a.20) we get;

$$v(t) \le \left[mv(0) + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha} + \frac{m\zeta_3(\varepsilon)g_b}{\alpha} + \zeta_4(\varepsilon)\right]e^{\varepsilon m(\zeta_2(\varepsilon)+\delta(h'))t}$$

$$+ \frac{\alpha}{\alpha-m(\zeta_2(\varepsilon)+\delta_1(h'))}\left[\zeta_4(\varepsilon) + \frac{mg_b\zeta_3(\varepsilon)}{\alpha} + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha}\right](e^{\varepsilon\alpha t} - e^{\varepsilon m(\zeta_2(\varepsilon)+\delta_1(h'))t}) \tag{a.21}$$

Hence, by definition of $v(t)$ and assuming that ε'_1 and h' are small enough such that $m(\zeta_2(\varepsilon'_1)+\delta_1(h')) < \alpha$ (a.21) can be converted to;

$$\|\tilde{z}(t')\| \le \left[m\|\tilde{z}(0)\| + \zeta_4(\varepsilon) + \frac{mg_b\zeta_3(\varepsilon)}{\alpha} + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha}\right] e^{-\varepsilon\left[\alpha-m(\zeta_2(\varepsilon)+\delta_1(h'))\right]t'}$$

$$+ \frac{\alpha}{\alpha-m(\zeta_2(\varepsilon)+\delta_1(h'))}\left[\zeta_4(\varepsilon) + \frac{mg_b\zeta_3(\varepsilon)}{\alpha_b} + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha}\right]$$

$$=: L_a(t') + \bar{\eta}(\varepsilon) \tag{a.22}$$

for all $t' \in [0,t]$ where $L_a(t') \le L_a(0)$ for all $t' \ge 0$ and $L_a(t') \to 0$ as $t' \to \infty$.

Consequently, if $\|\tilde{z}(0)\|$ is sufficiently small and there exists an $\varepsilon_1 > 0$ with $\varepsilon_1 \le \varepsilon'_1$ and such that

$$L_a(0) + \bar{\eta}(\varepsilon) \le h' \tag{a.23}$$

for all $\varepsilon \le \varepsilon_1$, then the bound on $\|\tilde{z}(t')\|$ in (a.22) is actually valid for all $t \ge 0$. To see this, consider a proof by contradiction. Assume that $\tilde{z}$ jumps out of the h' ball at time t. Thus, $\|\tilde{z}\| \ge h'$ at time t. However, eqn.(a.22) says that $\|\tilde{z}(t')\| \le L_a(t') + \bar{\eta}(\varepsilon)$ for all $t' \in [0,t]$ and, because $L_a(t')$ is nonzero and strictly decreasing, $L_a(t') + \bar{\eta}(\varepsilon) < L_a(0) + \bar{\eta}(\varepsilon) \le h'$. Hence, we have a contradiction so that if $\tilde{z}$ stays in the h' ball for any length of time, it always stays in the h' ball. Therefore, eqn.(a.22) is valid for all $t \ge 0$ and

$$\lim_{t\to\infty}\|\tilde{z}(t)\| \le \bar{\eta}(\varepsilon) \tag{a.24}$$

Combining this result with the fact that $\|x(t) - \bar{x}\| \le \|\tilde{z}(t)\| + \|x(t) - z(t)\|$ and $\|x(t) - z(t)\| \le \zeta_1(\varepsilon)\|z(t)\|$ yields our final result;

$$\lim_{t\to\infty}\|x(t) - \bar{x}\| \le \eta(\varepsilon)$$

Appendix B

Proof of Theorem 3.9.1 (Linear averaging result)

The proof of this theorem is almost identical to the one for the nonlinear case. Therefore, it is not fully written out here. Instead, we point out the differences between the two and explain the resulting consequences.

For the sake of generality we assume our equation to be of the form;

$$\dot{x}(t) = -\varepsilon B(t)x(t) + \varepsilon g(t) \tag{b.25}$$

where $B(t):\mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $x(t):\mathbb{R}_+ \to \mathbb{R}^n$ and $g(t):\mathbb{R}_+ \to \mathbb{R}^n$. The resulting averaged equation is;

$$\dot{x}_{av}(t) = -\varepsilon B_{av}x_{av}(t) + \varepsilon g_{av} \tag{b.26}$$

$-B_{av}$ is assumed to have all of its eigenvalues in the open left half of the complex plane and $\bar{x}$ is assumed to be an exponentially stable equilibrium point of (b.26). With these equations the same transformation is used to define the variable z except that the resulting differential equation defining the evolution of z is given as;

$$\dot{z}(t) = -\varepsilon B_{av}z(t) + \varepsilon^2 p_1(t,\varepsilon)z(t) + \varepsilon(I + \varepsilon p_2(t,\varepsilon))g(t) \tag{b.27}$$

and

$$\|\varepsilon p_1(t,\varepsilon)\| \le \zeta_2(\varepsilon) \quad \text{and} \quad \|\varepsilon p_2(t,\varepsilon)\| \le \zeta_3(\varepsilon)$$

Now, defining $\tilde{z}$ as in the nonlinear case results in the following differential equation for $\tilde{z}$;

$$\dot{\tilde{z}}(t) = -\varepsilon B_{av}\tilde{z} + \varepsilon^2 p_1(t,\varepsilon)\,(z + \bar{x}) + \varepsilon^2 p_2(t,\varepsilon)g(t) + \varepsilon(g(t) - g_{av}) \tag{b.28}$$

Notice that in this equation there are no higher order terms to deal with. Therefore, we don't have to worry about bounding them inside some B_h ball. Hence, the bound on $\tilde{z}$ given for the nonlinear case is valid for the linear case except that $\delta_1(h')$ term is zero and the result is global in $\tilde{z}$. Thus, for the linear case the bound on $\tilde{z}$ is;

$$\|\tilde{z}(t)\| \le \left[m\|\tilde{z}(0)\| + \zeta_4(\varepsilon) + \frac{mg_b\zeta_3(\varepsilon)}{\alpha} + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha}\right] e^{-\varepsilon(\alpha - m\zeta_2(\varepsilon))t}$$

$$+ \frac{\alpha}{\alpha - m\zeta_2(\varepsilon)}\left[\zeta_4(\varepsilon) + \frac{mg_b\zeta_3(\varepsilon)}{\alpha} + \frac{m\zeta_2(\varepsilon)\|\bar{x}\|}{\alpha}\right]$$

$$:= L_a(t') + \bar{\eta}(\varepsilon) \tag{b.29}$$

The restriction that ε be small enough such that $m\zeta_2(\varepsilon) < \alpha$ is still in force. However, there is no restriction on the size of $\|\tilde{z}(0)\|$ or a further restriction on ε like the one of eqn.(a.23) since there is no h' to worry about. The rest of the proof follows that given in the nonlinear case.