LINEAR TIME-INVARIANT CONTROLLER DESIGN FOR TWO-CHANNEL DECENTRALIZED CONTROL SYSTEMS

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C. A. Desoer and A. N. Gündeş
Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley CA 94720 USA

Abstract

A two-channel linear time-invariant system is analyzed using a factorization approach. The set of all decentralized stabilizing controllers is given in terms of a right-coprime factorization of the plant.

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INTRODUCTION

In the control of large scale systems, a restriction on the controller structure is often required. These sytems have several local control stations; each local controller observes only local outputs and controls only local inputs. Such decentralized control of a large system results in a block-diagonal controller matrix structure.

In [Wan.1] it is shown that stabilization of the given plant using a linear time-invariant decentralized controller scheme is possible if and only if the plant has no unstable *fixed modes* (natural frequencies of the plant which remain as closed-loop natural frequencies irrespective of the decentralized controller used). There is a large number of papers on fixed modes and their implications on transmission zeros (see, for example, [Cor.1], [Dav.1], [Fes.1], [Lin.1], [Tar.1], [Xie.1]). Some fixed mode characterizations use pole placement methods as in [Bra.1]. Most of the work on decentralized controller synthesis is based on state-space techniques (see, for example, [Guc.1]).

In this paper we consider a simple linear time-invariant two-channel system, where the plant has two inputs and two outputs; control of input one uses output one and control of input two uses output two only. We use the fixed-mode characterization in [And.1] but use a right-coprime factorization of the plant. Using a factorization approach we give the set of all stabilizing decentralized controllers for our two-channel system and show that these controllers are not parametrized by a free matrix as in the case of full (unstructured) output feedback.

The paper is organized as follows: Section I has the description and analysis of the system, stability definitions and theorems. The set of all stabilizing decentralized controllers is given in theorem 2.4 of section II and is followed by an algorithm to construct stabilizing controllers for a given two-channel system based on *any* right-coprime factorization of the plant. In section III we use this algorithm in two examples.

We use the following symbols and abbreviations:

l.t-i. linear time-invariant

I/O	input-output
w.l.o.g.	without loss of generality
a := b	a is defined as b
e.r.o.s	elementary row operations
e.c.o.s	elementary column operations
r.c.(l.c.)	right (left)-coprime
c.f.r.	coprime fraction representation
det$A$	the determinant of matrix A
$\mathcal{M}(R_u)$	the set of matrices with elements in R_u.

SECTION I

Analysis

1.1. Notation [Lan. 1, p.71-77], [Vid.1, Appendix A, B]:

$\mathcal{U} \supset \mathbb{C}_+$ is a closed subset of $\mathbb{C}$, symmetric about the real axis, and $\mathbb{C} \setminus \mathcal{U}$ is nonempty. $\bar{\mathcal{U}} := \mathcal{U} \cup \{ \infty \}$.

R_u is the ring of proper scalar rational functions (with real coefficients) which are analytic in $\mathcal{U}$.

$\mathcal{J}$ is the group of units of R_u ; equivalently, $f \in \mathcal{J}$ has neither poles nor zeros in $\bar{\mathcal{U}}$.

$\mathcal{I}$ is the multiplicative subset of elements of R_u such that $f \in \mathcal{I}$ implies $f(\infty)$ = a nonzero constant in $\mathbb{C}$; equivalently, $\mathcal{I} \subset R_u$ is the set of proper but not strictly proper rational functions which are analytic in $\mathcal{U}$.

Then $R_u / \mathcal{I} := \{ n / d \ : n \in R_u , d \in \mathcal{I} \}$ is the ring of fractions of R_u associated with $\mathcal{I}$; this ring is the ring of proper rational functions $\mathbb{R}_p(s)$.

The set of strictly proper rational functions $\mathbb{R}_{sp}(s)$ is the Jacobson radical of the ring $\mathbb{R}_p(s)$; $f \in \mathbb{R}_{sp}(s)$ goes to 0 as s goes to ∞.

Note that (i) $\mathcal{I}$ = the set of units of $\mathbb{R}_p(s)$ which are in R_u. (ii) Let $A \in \mathcal{M}(R_u)$, $B \in \mathcal{M}(\mathbb{R}_p(s))$. Then a) $A^{-1} \in \mathcal{M}(R_u)$ iff $\det A \in \mathcal{J}$ and b) $B^{-1} \in \mathcal{M}(\mathbb{R}_p(s))$ iff $\det B \in \mathcal{I}$. (iii) Let $Y \in \mathcal{M}(\mathbb{R}_{sp}(s))$, $X, Z \in \mathcal{M}(\mathbb{R}_p(s))$. Then $XY, YZ \in \mathcal{M}(\mathbb{R}_{sp}(s))$, and $(I + XY)^{-1}$, $(I + YZ)^{-1} \in \mathcal{M}(\mathbb{R}_p(s))$. (iv) Let $a, b \in R_u$. Then $ab \in \mathcal{J}$ iff a and $b \in \mathcal{J}$. (v) Let $c, d \in R_u$. Then $cd \in \mathcal{I}$ iff c and $d \in \mathcal{I}$.

1.2. Definitions (Coprime Factorizations in R_u):

(i) The pair $(N, D) \in \mathcal{M}(R_u)$ is called **right-coprime (r.c.)** iff there exist $U, V \in \mathcal{M}(R_u)$ such that

$$UN + VD = I \tag{1.1}$$

(ii) The pair $(N, D) \in \mathcal{M}(R_u)$ is called a **right-fraction representation (r.f.r.)** of $P \in \mathcal{M}(\mathbb{R}_p(s))$ iff

$$D \text{ is square}, \ \det D \in \mathcal{I} \ \text{ and } \ P = ND^{-1} \tag{1.2}$$

(iii) The pair $(N, D) \in \mathcal{M}(R_u)$ is called a **right-coprime-fraction representation (r.c.f.r.)** of $P \in \mathcal{M}(\mathbb{R}_p(s))$ iff (N, D) is a r.f.r. of P and (N, D) is r.c.

The definitions of left-coprime (l.c.), left-fraction representation (l.f.r.) and left-coprime-fraction representation (l.c.f.r.) are duals of (i), (ii), and (iii), respectively [Vid.1, Net.1, Des.1].

■

Note that (i) every $P \in \mathcal{M}(\mathbb{R}_p(s))$ has a r.c.f.r. $(N, D) \in \mathcal{M}(R_u)$ and a l.c.f.r. $(\tilde{D}, \tilde{N}) \in \mathcal{M}(R_u)$ because R_u is a principal ring [Vid.1]. (ii) Let (N, D) be a r.c.f.r. of $P \in \mathcal{M}(\mathbb{R}_p(s))$. Then (X, Y) is a r.c.f.r. of P iff $(X, Y) = (NR, DR)$ for some unimodular $R \in \mathcal{M}(R_u)$.

1.3. Assumptions: Consider the 2-channel decentralized control system $S(P, C_d)$ shown in figure 1.

(A) Let P and C_d have no hidden $\mathcal{U}$-unstable modes so that they can be specified by their I/O representations.

(B) Let $P \in \mathbb{R}_p(s)^{2x2}$ be a 2-channel plant. (1.3)

Let (N, D) be a r.c.f.r. of P, where (1.4a)

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \ D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, N_1, N_2, D_1, D_2 \in R_u^{1x2}, \tag{1.4b}$$

$$\text{with} \quad rank \begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix} \geq 1, \quad \text{for all } s \in \bar{\mathcal{U}} \tag{1.5a}$$

$$\text{and} \quad rank \begin{bmatrix} D_2(s) \\ N_2(s) \end{bmatrix} \geq 1, \quad \text{for all } s \in \bar{\mathcal{U}}. \tag{1.5b}$$

(C) Let $C_d = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \in \mathbb{R}_p(s)^{2x2}$ be a decentralized compensator. (1.6)

Let $(\tilde{D}', \tilde{N}')$ be a l.c.f.r. of C_d, where (1.7a)

$$\tilde{D}' = \begin{bmatrix} \tilde{d}'_1 & 0 \\ 0 & \tilde{d}'_2 \end{bmatrix}, \tilde{N}' = \begin{bmatrix} \tilde{n}'_1 & 0 \\ 0 & \tilde{n}'_2 \end{bmatrix}, \tilde{n}'_1, \tilde{n}'_2 \in R_u, \tilde{d}'_1, \tilde{d}'_2 \in \dot{\mathcal{I}}. \qquad (1.7b)$$

■

Note that by definition 1.2, $\det \tilde{D}' \in \dot{\mathcal{I}}$; equivalently, $\tilde{d}'_1 \in \dot{\mathcal{I}}$ and $\tilde{d}'_2 \in \dot{\mathcal{I}}$. Therefore, by equations (1.7 a-b), $(\tilde{d}'_1, \tilde{n}'_1)$ is a coprime-fraction-representation (c.f.r.) of c_1 and $(\tilde{d}'_2, \tilde{n}'_2)$ is a c.f.r. of c_2.

Assumption (A) holds throughout this paper. The plants under consideration satisfy assumption (B) except in some cases, where we require in addition that the plant be strictly proper.

Comments : 1) Assumptions (1.4 a-b) imply that [Cal.1, Vid.1]

$$rank \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = rank \begin{bmatrix} D_1(s) \\ D_2(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = 2 \quad \text{for all } s \in \bar{\mathcal{U}}. \qquad (1.8)$$

2) Equation (1.8) with assumptions (1.5 a-b) implies that the plant P has *no decentralized fixed modes* in $\bar{\mathcal{U}}$ [And.1].

3) Assumption (1.5 a) implies that the "Smith Form" of $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ has at least one "1" in its two diagonal entries. The second diagonal entry can be zero or some other element of R_u. If $rank \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}(s) = 2$ for all $s \in \bar{\mathcal{U}}$, then the second entry is also equal to 1 ; equivalently, the matrix $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ is unimodular.

If the plant P is strictly proper and assumption (1.5 a) holds, then $rank \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}(s)$ is always 1 at $s = \infty$ since $N_1 \in \mathbb{R}_{sp}(s)^{1x2}$; therefore the second diagonal entry in the Smith Form cannot be 1 since $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ is not unimodular.

Similar comments can be made about assumption (1.5 b).

■

Using the representations of P and C_d as in assumptions (1.3)-(1.7) we redraw the decentralized control system as in figure 2.

Let $y := \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y'_1 \\ y'_2 \end{bmatrix}$, $u := \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u'_1 \\ u'_2 \end{bmatrix}$, $\xi := \begin{bmatrix} y'_1 \\ y'_2 \\ \cdots \\ \xi_p \end{bmatrix}$. Then $S(P, C_d)$ is described by equations (1.11)-(1.12) below.

$$\begin{bmatrix} 1 & 0 & \vdots & -D_1 \\ 0 & 1 & \vdots & -D_2 \\ \tilde{d}'_1 & 0 & \vdots & \tilde{n}'_1 N_1 \\ 0 & \tilde{d}'_2 & \vdots & \tilde{n}'_2 N_2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ \cdots \\ \xi_p \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \tilde{n}'_1 & 0 \\ 0 & 0 & 0 & \tilde{n}'_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_1' \\ u_2' \end{bmatrix} \tag{1.11}$$

$$\begin{bmatrix} 0 & 0 & \vdots & N_1 \\ 0 & 0 & \vdots & N_2 \\ 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ \cdots \\ \xi_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_1' \\ y_2' \end{bmatrix} \tag{1.12}$$

Using obvious notations we write equations (1.11)-(1.12) in the form

$$D_H \xi = N_L u \tag{1.13}$$
$$N_R \xi = y \tag{1.14}$$

By inspection (or by e.r.o.'s and e.c.o.'s in $R_{\mathcal{U}}$), (N_R , D_H) is r.c. and (D_H , N_L) is l.c. Let $H_{yu} : u \mapsto y$. If $\det D_H \in \mathring{\mathcal{I}}$ (equivalently, if the system $S(P, C_d)$ is well-posed), then

$$H_{yu} = N_R D_H^{-1} N_L \in \mathcal{M}(\mathbb{R}_p(s)) \tag{1.15}$$

In terms of P and C_d, H_{yu} is given by:

$$H_{yu} = \begin{bmatrix} P(I + C_d)^{-1} & P(I + C_d P)^{-1} C_d \\ (I + C_d P)^{-1} - I & (I + C_d P)^{-1} C_d \end{bmatrix} \tag{1.16}$$

1.5. Definition **($\mathcal{U}$–stability) :** $S(P, C_d)$ is called $\mathcal{U}$–stable if and only if $H_{yu} \in \mathcal{M}(R_u)$.

1.6. Theorem **($\mathcal{U}$–stability) :** Let Assumptions (A), (B), (C) hold. Then $S(P, C_d)$ is $\mathcal{U}$–stable if and only if $\det D_H \in \dot{\mathcal{J}}$.

1.7. Comments : 1) Manipulating the expression for $\det D_H$ (see equation (1.11)) we see that theorem 1.6 can be established as $S(P, C_d)$ is $\mathcal{U}$–stable if and only if

$$\det D_H = \det(\tilde{D}'D + \tilde{N}'N) = \det\begin{bmatrix} \tilde{d}'_1 D_1 + \tilde{n}'_1 N_1 \\ \tilde{d}'_2 D_2 + \tilde{n}'_2 N_2 \end{bmatrix} \in \dot{\mathcal{J}} \tag{1.17}$$

$$\Leftrightarrow \begin{bmatrix} \tilde{d}'_1 & 0 & \vdots & \tilde{n}'_1 & 0 \\ 0 & \tilde{d}'_2 & \vdots & 0 & \tilde{n}'_2 \end{bmatrix} \begin{bmatrix} D \\ \cdots \\ N \end{bmatrix} = R \in R_u^{2x2}, \ R \text{ is unimodular} \tag{1.18}$$

$$\Leftrightarrow \begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} \begin{bmatrix} D \\ \cdots \\ N \end{bmatrix} = R \in R_u^{2x2}, \ R \text{ is unimodular.} \tag{1.19}$$

From equation (1.17) and the c.f.r.'s of P and C_d we get

$$\det D_H = \det\begin{bmatrix} \tilde{d}'_1 & 0 \\ 0 & \tilde{d}'_2 \end{bmatrix} \det(I + C_d P) \det\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \det\tilde{D}' \det(I + C_d P) \det D. \tag{1.20}$$

2) Equation (1.18) is equivalent to a Bezout Identity

$$V_d D + U_d N = R, \ R \in R_u^{2x2} \text{ is unimodular} \tag{1.21}$$

where V_d, $U_d \in R_u^{2x2}$ are *diagonal* matrices defined in an obvious manner by equation (1.18).

■

Proof of Theorem 1.6 : **(=>)** The map $H_{21} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ is given by $(I + C_d P)^{-1} - I$. Since $S(P, C_d)$ is $\mathcal{U}$–stable, $H_{21} \in \mathcal{M}(R_u)$. Therefore, $(I + C_d P)^{-1} \in \mathcal{M}(R_u)$ and hence,

$$\det[\,(I + C_d P)^{-1}\,] \in R_u. \tag{1.22}$$

From equation (1.20) we get

$$(\det D_H)^{-1} = (\det\tilde{D}')^{-1} \det(I + C_d P)^{-1} (\det D)^{-1}. \tag{1.23}$$

Since (N, D) is a r.c.f.r. of P and $(\tilde{D}', \tilde{N}')$ is a l.c.f.r. of C_d, $\det D \in \dot{\mathcal{I}}$ and $\det \tilde{D}' \in \dot{\mathcal{I}}$. By equation (1.11), $\det D_H \in R_u$; hence, using equations (1.22)-(1.23), $(\det D_H)^{-1} \in \mathbb{R}_p(s)$. Therefore, $\det D_H \in \dot{\mathcal{I}}$. Then equation (1.15) holds. Finally, $H_{yu} \in \mathcal{m}(R_u)$ implies that $\det D_H \in \dot{\mathcal{J}}$ since (N_R, D_H) is r.c. and (D_H, N_L) is l.c. [Vid.1].

(<=) $\det D_H \in \dot{\mathcal{J}}$ implies that $D_H^{-1} \in \mathcal{m}(R_u)$. Therefore, by equation (1.15), $H_{yu} = N_R D_H^{-1} N_L \in \mathcal{m}(R_u)$.

■

SECTION II

Synthesis

Let assumptions (A) and (B) hold.

2.1. Definition ($\mathcal{U}$–stabilizing decentralized compensator): C_d is called a $\mathcal{U}$–stabilizing decentralized compensator for P (equivalently, C_d *$\mathcal{U}$–stabilizes P*) iff (i) C_d satisfies assumption (C), *and* (ii) $S(P, C_d)$ in figures 1, 2 is $\mathcal{U}$–stable.

2.2. Definition (Set of all $\mathcal{U}$–stabilizing decentralized compensators):

$$S_d(P) := \{ \; C_d \; : \; C_d \; \mathcal{U}\text{–}stabilizes \; P \; \} \tag{2.1}$$

is called the set of *all* $\mathcal{U}$–stabilizing decentralized compensators for the given P.

2.3. Lemma: Let $\hat{d}_{21}$, $\hat{n}_{21}$, $\lambda_2 \in R_u$, $\lambda_2 \neq 0$, be such that

$$rank \begin{bmatrix} \hat{d}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix} (s) \geq 1 \quad \text{for all } s \in \bar{\mathcal{U}} . \tag{2.2}$$

Then there exists $r \in R_u$ such that

$$rank \begin{bmatrix} \hat{d}_{21} + r\hat{n}_{21} & \vdots & \lambda_2 \end{bmatrix} (s) = 1 \, , \quad \text{for all } s \in \bar{\mathcal{U}} . \tag{2.3}$$

Proof : Condition (2.3) clearly holds for all $s \in \bar{\mathcal{U}}$ such that $\lambda_2(s) \neq 0$. Consider $S_2 = \{ \; s \in \bar{\mathcal{U}} : \lambda_2(s) = 0 \; \}$. For $s_2 \in S_2$, by equation (2.2), $\hat{d}_{21}(s_2)$ and $\hat{n}_{21}(s_2)$ cannot *both* be zero. Hence choose $r(\cdot) \in R_u$ as follows: for $s_2 \in S_2$, if $\hat{d}_{21}(s_2) \neq 0$, choose $r(s_2) = 0$, else choose $r(s_2) \neq 0$; for $s \notin S_2$, choose $r(s)$ arbitrarily. Clearly any such $r(\cdot)$ satisfies condition (2.3).

■

2.4. Theorem (Class of all $\mathcal{U}$–stabilizing decentralized compensators):

Let $P \in \mathbb{R}_{sp}(s)^{2x2}$ satisfy assumptions (1.4 a-b) and (1.5 a-b). Then

(i) there are *unimodular* matrices $L_1, R_1, M_2, T \in \mathcal{R}_u^{2x2}$ such that

$$\begin{bmatrix} L_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & M_2 \end{bmatrix} \begin{bmatrix} D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2 \end{bmatrix} R_1 =: \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \cdots & \cdots \\ \hat{d}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathcal{R}_u, \lambda_2 \neq 0, \tag{2.6a}$$

$$\text{the pair } (\hat{d}_{21}, \lambda_2) \text{ is coprime ;} \tag{2.6b}$$

and for *all* $q_1 \in \mathcal{R}_u$,

$$\begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix} T \begin{bmatrix} \lambda_2 \\ -\lambda_1(\hat{d}_{21}+r\hat{n}_{21}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.6c}$$

(ii) the set of all $\mathcal{U}$–stabilizing decentralized compensators $S_d(P)$ is given by

$$S_d(P) = \{ C_d = \begin{bmatrix} \tilde{d}'_1 & 0 \\ 0 & \tilde{d}'_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{n}'_1 & 0 \\ 0 & \tilde{n}'_2 \end{bmatrix} : \begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} = \begin{bmatrix} [1 \;\; q_1\,] T L_1 & \vdots & 0 \\ 0 & \vdots & [1 \;\; q_2\,] M_2 \end{bmatrix},$$

$$q_1, q_2 \in \mathcal{R}_u \text{ such that } 1 + [\,1 \;\; q_1\,] T \begin{bmatrix} 0 \\ -\lambda_1 q_2 \hat{n}_{21} \end{bmatrix} \in \mathcal{J} \} . \tag{2.7}$$

■

2.5. Comments: 1) If assumptions (A), (B), (C) hold, then by theorem 1.6, C_d $\mathcal{U}$–stabilizes P if and only if equation (1.19) (equivalently, equation (1.18)) holds. We use equation (1.19) in theorem 2.4 for finding all $\mathcal{U}$–stabilizing decentralized compensators fo P.

2) Finding a $\mathcal{U}$–stabilizing decentralized controller C_d is equivalent to solving equation (1.21) for some *diagonal* $U_d \in \mathcal{R}_u$. Now by assumption (1.4 a), since (N, D) is r.c., there exist $V, U \in \mathcal{R}_u$ (not necessarily diagonal) such that

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{2.8}$$

where $(\tilde{D}, \tilde{N})$ is a l.c.f.r. of P.

It is well known (see for example [Vid.1, Net.1, Des.1]) that centralized (full-feedback) compensators that $\mathcal{U}$–stabilize P are given by

$$C = (V - Q\tilde{N})^{-1}(U + Q\tilde{D}) \qquad (2.9)$$

where $Q \in \mathcal{M}(R_u)$ is such that $\det(V - Q\tilde{N}) \in \mathcal{I}$ (in the case that $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$, $\det(V - Q\tilde{N}) \in \mathcal{I}$ for *all* $Q \in \mathcal{M}(R_u)$).

If the system $S(P, C_d)$ in figure 1 used a centralized controller C - i.e., a two-input two-output controller - then the class of all $\mathcal{U}$-stabilizing controllers would be parametrized by a 2x2 matrix in R_u as seen from equation (2.9). In the present case, C_d consists of two single-input single-output controllers, so one would expect a parametrization in terms of two scalar functions in R_u. Theorem 2.4 shows that this is indeed true with the two scalar functions q_1, q_2 chosen so that equation (2.7) holds.

■

Proof of Theprem 2.4 : (i) Let $L_1, R_1 \in R_u^{2x2}$ be unimodular matrices such that $L_1\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}R_1$ is the Smith form of $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$. By assumption (1.5 a), one of the diagonal entries (the smallest invariant factor) of this Smith form has to be equal to 1. Let λ_1 denote the second invariant factor; from comment 1.4.3, $\lambda_1 \in R_u$ is possibly 0 but not 1 because P is strictly proper.

Let $\begin{bmatrix} D_2 \\ N_2 \end{bmatrix}R_1 =: \begin{bmatrix} d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix}$. Since L_1, R_1 are unimodular, from equation (1.8) we get

$$rank(\begin{bmatrix} L_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I \end{bmatrix}\begin{bmatrix} D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2 \end{bmatrix}R_1\,)(s) = rank\begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \cdots & \cdots \\ d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix}(s) = 2\,,\ \text{for all } s \in \bar{\mathcal{U}}\,. \qquad (2.10)$$

Equation (2.10) implies that d_{22} and n_{22} cannot both be identically zero since $\lambda_1 \neq 1$. Therefore the pair (d_{22}, n_{22}) has a nonzero greatest common divisor (g.c.d.) denoted by $\lambda_2 \in R_u$. Hence there is a unimodular matrix $\hat{M}_2 \in R_u^{2x2}$ such that

$$\hat{M}_2\begin{bmatrix} D_2 \\ N_2 \end{bmatrix}R_1 = \hat{M}_2\begin{bmatrix} d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix} =: \begin{bmatrix} \hat{d}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix},\ \lambda_2 \in R_u\,,\ \lambda_2 \neq 0. \qquad (2.11)$$

The pair $(\hat{d}_{21}, \lambda_2)$ in equation (2.11) is not always coprime; but since $\hat{M}_2$, R_1 are unimodular, assumption (1.5 b) implies that the matrix in equation (2.11) has rank $\geq 1 \ \forall \ s \in \bar{\mathcal{U}}$. Then by lemma 2.3, there is $r \in R_u$ such that the pair $(\hat{d}_{21}+r\hat{n}_{21}, \lambda_2)$ is coprime and hence, assertion (2.6 b) is satisfied. Note that r is not unique.

Let $M_2 := \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \hat{M}_2$. Then $M_2 \in R_u^{2x2}$ is unimodular, and hence equation (2.10) implies that

$$rank(\begin{bmatrix} L_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & M_2 \end{bmatrix} \begin{bmatrix} D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2 \end{bmatrix} R_1)(s) = rank \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \cdots & \cdots \\ \hat{d}_{21}+r\hat{n}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix} (s) = 2 \, , \ \text{for all} \ s \in \bar{\mathcal{U}}$$

and thus, the pair (λ_1, λ_2) is coprime. Therefore there is a unimodular $T \in R_u^{2x2}$ such that

$$T \begin{bmatrix} \lambda_2 \\ -\lambda_1(\hat{d}_{21}+r\hat{n}_{21}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and hence, for *all* $q_1 \in R_u$, equation (2.6 c) holds.

(ii) (a) First we show that $S_d(P)$ given in equation (2.7) is a subset of $S_d(P)$ defined in equation (2.1); equivalently, with $\tilde{d}'_1, \tilde{n}'_1, \tilde{d}'_2, \tilde{n}'_2$ as in equation (2.7), C_d $\mathcal{U}$–stabilizes $P \in \mathbb{R}_{sp}(s)^{2x2}$.

By theorem 1.6, $S(P, C_d)$ is $\mathcal{U}$–stable if and only if equation (1.19) holds. If $\tilde{d}'_1, \tilde{n}'_1, \tilde{d}'_2, \tilde{n}'_2$ are specified as in equation (2.7), then by equations (2.6 a-c) and (2.7)

$$\det(\begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} \begin{bmatrix} D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2 \end{bmatrix} R_1) = \det D_H \det R_1 = \det(\begin{bmatrix} [1 \ \ q_1]TL_1 & \vdots & 0 \\ 0 & \vdots & [1 \ \ q_2]M_2 \end{bmatrix} \begin{bmatrix} D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2 \end{bmatrix} R_1)$$

$$= [\,1 \ \ q_1\,]\,T \begin{bmatrix} \lambda_2 \\ -\lambda_1(\hat{d}_{21}+r\hat{n}_{21}+q_2\hat{n}_{21}) \end{bmatrix} = 1 + [\,1 \ \ q_1\,]\,T \begin{bmatrix} 0 \\ -\lambda_1 q_2 \hat{n}_{21} \end{bmatrix} \in \mathcal{J} \, . \qquad (2.12)$$

Since R_1 is unimodular, equation (2.12) implies that $\det D_H \in \mathcal{J}$.

From equation (2.7), $\tilde{d}'_1, \tilde{n}'_1, \tilde{d}'_2, \tilde{n}'_2$ are clearly in R_u since $L_1, M_2, T \in R_u^{2x2}$ and $q_1, q_2, r \in R_u$. $(\tilde{d}'_1, \tilde{n}'_1)$ is a coprime pair since L_1, R_1, T unimodular implies that $rank\begin{bmatrix}\tilde{d}'_1 & \vdots & \tilde{n}'_1\end{bmatrix} = \begin{bmatrix}1 & \vdots & q_1\end{bmatrix}TL_1 = 1 \ \forall \ s \in \bar{\mathcal{U}}$. Similarly, M_2 unimodular implies that the pair $(\tilde{d}'_2, \tilde{n}'_2)$ is coprime. Therefore $(\tilde{D}', \tilde{N}') = (\begin{bmatrix}\tilde{d}'_1 & 0 \\ 0 & \tilde{d}'_2\end{bmatrix}, \begin{bmatrix}\tilde{n}'_1 & 0 \\ 0 & \tilde{n}'_2\end{bmatrix})$ is a l.c.f.r. of C_d and hence, assumption (1.7 a) holds. Now since $\det D_H \in \mathcal{J}$, from equation (1.17),

$$\tilde{D}'D + \tilde{N}'N = R, \quad R \in R_u^{2x2} \text{ is unimodular.} \tag{2.13}$$

By assumption, $P \in \mathbb{R}_{sp}(s)^{2x2}$, hence $N \in \mathbb{R}_{sp}(s)^{2x2}$. Then by equation (2.13), $\det\tilde{D}' \in \mathcal{I}$ [Vid.1, Net.1, Des.1]; equivalently, $\tilde{d}'_1 \in \mathcal{I}$ and $\tilde{d}'_2 \in \mathcal{I}$. Therefore assumption (1.7 b) holds.

Since we have shown that $C_d \in S_d(P)$ given by equation (2.7) satisfies assumption (C) and that $S(P, C_d)$ is $\mathcal{U}$–stable, by definition 2.1, C_d $\mathcal{U}$–stabilizes P ; hence $C_d \in S_d(P)$.

(b) Second, we show that *any* $\mathcal{U}$–stabilizing decentralized compensator C_d is a member of $S_d(P)$ specified by equation (2.7) for some q_1, q_2 as in equation (2.7).

Let C_d $\mathcal{U}$–stabilize P ; by definition 2.1, C_d satisfies assumption (C) and $S(P, C_d)$ is $\mathcal{U}$–stable. Let $\tilde{d}'_1, \tilde{n}'_1, \tilde{d}'_2, \tilde{n}'_2$ be as in assumptions (1.7 a-b). Then by theorem 1.6, equation (1.19) holds and without loss of generality, we assume that the corresponding r.c.f.r. (N, D) of P is such that $R = I$ in equation (1.19). We rewrite equation (1.19) as:

$$\begin{bmatrix}\tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2\end{bmatrix}\begin{bmatrix}D_1 \\ N_1 \\ \cdots \\ D_2 \\ N_2\end{bmatrix} =: \begin{bmatrix}\tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2\end{bmatrix}\begin{bmatrix}d_{11} & d_{12} \\ n_{11} & n_{12} \\ \cdots & \cdots \\ d_{21} & d_{22} \\ n_{21} & n_{22}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \tag{2.14}$$

Equation (2.14) implies that the pairs (d_{11}, n_{11}) and (d_{22}, n_{22}) are coprime. Furthermore, $\begin{bmatrix}\tilde{d}'_1 & \tilde{n}'_1 \\ -n_{11} & d_{11}\end{bmatrix} =: L_1 \in R_u^{2x2}$ is unimodular and $\begin{bmatrix}\tilde{d}'_2 & \tilde{n}'_2 \\ -n_{22} & d_{22}\end{bmatrix} =: M_2 \in R_u^{2x2}$ is unimodular since

$\det L_1 = 1$ and $\det M_2 = 1$.

Let $\lambda_1 := -n_{11}d_{12} + d_{11}n_{12}$ and $\hat{n}_{21} := -n_{22}d_{21} + d_{22}n_{21}$. Then

$$\begin{bmatrix} L_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ M_2 & \vdots & M_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ n_{11} & n_{12} \\ \cdots & \cdots \\ d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \cdots & \cdots \\ 0 & 1 \\ \hat{n}_{21} & 0 \end{bmatrix} \tag{2.15}$$

Note that with $\lambda_2 = 1$, $r = 0$, and $\hat{d}_{21} = 0$, the right-hand side of equation (2.15) is of the same form as that of equation (2.6 a). Clearly, condition (2.6 b) holds since $\lambda_2 = 1$, and equation (2.6 c) holds with $T = I$ since $\hat{d}_{21} = 0$.

Since (N, D) is a r.c., it is possible to complete $\begin{bmatrix} D_1 \\ N_1 \\ D_2 \\ N_2 \end{bmatrix}$ into a unimodular matrix by adding two columns; although there are many ways of accomplishing this we choose a simple one. From equation (2.15), with $T = I$, it is easy to verify that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & -\lambda_1 & 0 \\ -\hat{n}_{21} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} TL_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & M_2 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & \vdots & L_1^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \vdots & 0 \\ n_{11} & n_{12} & \vdots & & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ d_{21} & d_{22} & \vdots & & \vdots & \\ n_{21} & n_{22} & \vdots & 0 & \vdots & M_2^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = I . \tag{2.16}$$

Then each of the three matrices in equation (2.16) are unimodular. By equation (2.14), with $q_1 := [\, \tilde{d}'_1 \;\; \tilde{n}'_1 \,] L_1^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$, $q_2 := [\, \tilde{d}'_2 \;\; \tilde{n}'_2 \,] M_2^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$,

$$\begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & \vdots & L_1^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \vdots & 0 \\ n_{11} & n_{12} & \vdots & & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ d_{21} & d_{22} & \vdots & & \vdots & \\ n_{21} & n_{22} & \vdots & 0 & \vdots & M_2^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & q_1 & 0 \\ 0 & 1 & \vdots & 0 & q_2 \end{bmatrix} . \tag{2.17}$$

With $q_1 = 0$, $q_2 = 0$, from equations (2.16)-(2.17) we get

$$\begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & q_1 & 0 \\ 0 & 1 & \vdots & 0 & q_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & -\lambda_1 & 0 \\ -\hat{n}_{21} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} TL_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & M_2 \end{bmatrix}$$

$$= \begin{bmatrix} [\,1 \;\; 0\,]TL_1 & \vdots & 0 \\ 0 & \vdots & [\,1 \;\; 0\,]M_2 \end{bmatrix}. \qquad (2.18)$$

Since $q_1 = 0$, $q_2 = 0$, we have $1 + [\,1 \;\; q_1\,]T \begin{bmatrix} 0 \\ -\lambda_1 q_2 \hat{n}_{21} \end{bmatrix} = 1$, where $T = I$. Equation (2.18) shows that $\tilde{d}'_1$, $\tilde{n}'_1$, $\tilde{d}'_2$, $\tilde{n}'_2$ satisfy equation (2.7). Hence, any $\mathcal{U}$–stabilizing decentralized compensator C_d is in the set $\mathcal{S}_d(P)$ of equation (2.7).

■

2.6. Remarks : 1) The proof of theorem 2.4 suggests the following algorithm for finding a $\mathcal{U}$–stabilizing decentralized compensator C_d for a given 2-channel strictly proper 2x2 plant P.

Algorithm :

Given: $P \in \mathbb{R}_{sp}(s)^{2x2}$, and a r.c.f.r. (N, D) of P such that assumptions (1.4 a-b), (1.5 a-b) hold.

Step 1 : Put $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ into the Smith form; equivalently, find unimodular L_1, $R_1 \in R_u^{2x2}$ such that $L_1 \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} R_1 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$, $\lambda_1 \in R_u$. Let $\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} R_1 =: \begin{bmatrix} d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix}$.

Step 2 : Put $\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} R_1$ into a triangular (Hermite-like) form; equivalently, find a unimodular $\hat{M}_2 \in R_u^{2x2}$ such that $\hat{M}_2 \begin{bmatrix} d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix} =: \begin{bmatrix} \hat{d}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix}$, $\lambda_2 \in R_u$, $\lambda_2 \neq 0$. Then find an $r \in R_u$ such that the pair $(\hat{d}_{21} + r\hat{n}_{21}, \lambda_2)$ is coprime. Let $M_2 := \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \hat{M}_2$.

Step 3' : Find a unimodular $T \in R_u 2x2$ such that $T \begin{bmatrix} \lambda_2 \\ -\lambda_1(\hat{d}_{21} + r\hat{n}_{21}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Step 4 : Then for all $q_1 , q_2 \in R_u$ such that

$$1 + [\ 1 \quad q_1\ ]\, T \begin{bmatrix} 0 \\ -\lambda_1 q_2 \hat{n}_{21} \end{bmatrix} = 1\ , \tag{2.19}$$

a $\mathcal{U}$–stabilizing decentralized compensator is given by

$$C_d = \begin{bmatrix} \tilde{d}'_1 & 0 \\ 0 & \tilde{n}'_1 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{n}'_1 & 0 \\ 0 & \tilde{n}'_2 \end{bmatrix}, \text{ where}$$

$$\begin{bmatrix} \tilde{d}'_1 & \tilde{n}'_1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{d}'_2 & \tilde{n}'_2 \end{bmatrix} = \begin{bmatrix} [\ 1 \quad q_1\ ]TL_1 & \vdots & 0 \\ 0 & \vdots & [\ 1 \quad q_2\ ]M_2 \end{bmatrix}. \tag{2.20}$$

Note that if q_2 is chosen as zero, then equation (2.16) holds for all $q_1 \in R_u$.

2) If $P \in \mathbb{R}_p(s)^{2x2}$ but not strictly proper, then the algorithm above needs slight modifications. If P is proper, then from comment 1.4.3, λ_1 in equation (2.6 a) may be equal to 1 ; in that case, d_{22} , n_{22} in equation (2.10) may both be identically zero. If $d_{22} = n_{22} = 0$ then in equation (2.11) $\lambda_2 = 0$ and by assumption (1.5 b), $rank \begin{bmatrix} d_{21} \\ n_{21} \end{bmatrix}(s) = 1$ for all $s \in \bar{\mathcal{U}}$; equivalently, (d_{21} , n_{21}) is a coprime pair.

Another consequence of P proper is that equation (2.13) no longer guarantees that $\tilde{d}'_1$ and $\tilde{d}'_2$ are in $\mathcal{I}$. Therefore an additional restriction on $q_1 , q_2 \in R_u$ is needed.

Considering these two differences for proper but not strictly proper plants, we modify the algorithm as follows:

Step 2 : (i) Case 1 : $d_{22} = n_{22} = 0$. Find a unimodular $\hat{M}_2 \in R_u^{2x2}$ such that

$$\hat{M}_2 \begin{bmatrix} d_{21} & d_{22} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{d}_{21} & \lambda_2 \\ \hat{n}_{21} & 0 \end{bmatrix}.$$

Choose $r \in R_u$ such that $[\ 1 \quad r\ ]\hat{M}_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathcal{I}$. Let M_2 be the same as in the algorithm.

(ii) Case 2 : at least one of d_{22} , n_{22} is not identically equal to zero. Find $\hat{M}_2$ as in the algorithm, but choose $r \in R_u$ such that the pair $(\hat{d}_{21} + r\hat{n}_{21})$ is coprime *and* $[\,1 \;\; r\,]\hat{M}_2\begin{bmatrix}1\\0\end{bmatrix} \in i$. Let M_2 be defined the same way as in the algorithm.

Step 4 : For all $q_1 \in R_u$ such that $[\,1 \;\; q_1\,]TL_1\begin{bmatrix}1\\0\end{bmatrix} \in i$, for all $q_2 \in R_u$ such that $[\,1 \;\; q_2\,]M_2\begin{bmatrix}1\\0\end{bmatrix} \in i$ with, in addition, q_1, q_2 such that equation (2.19) is satisfied, C_d is given by equation (2.20).

■

SECTION III

Examples and Conclusions

In examples 3.1 and 3.2 we follow the algorithm in remark 2.6.1 to find a $\mathcal{U}$–stabilizing decentralized compensator for a given strictly proper plant.

3.1. Example :

$$\text{Given } P = \begin{bmatrix} 0 & \frac{1}{s-2} \\ \frac{1}{s-1} & \frac{-(s+1)}{(s-1)(s-2)} \end{bmatrix} \in \mathbb{R}_{sp}(s)^{2x2} \tag{3.1}$$

and a r.c.f.r. (N, D) of P, where

$$N = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \frac{s-1}{s+1} \\ \frac{s-2}{s+1} & 0 \end{bmatrix}. \tag{3.2}$$

Step 1 : With $L_1 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{s+1} & 1 \end{bmatrix}$, $R_1 = \begin{bmatrix} 1 & \frac{-(s-1)}{s+1} \\ 0 & 1 \end{bmatrix}$, (3.3)

we get $\lambda_1 = \frac{-(s-1)}{(s+1)^2}$. Then $\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} R_1 = \begin{bmatrix} \frac{s-2}{s+1} & \frac{-(s-1)(s-2)}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix}$.

Step 2 : Choose $r = 0$. Then $M_2 = \begin{bmatrix} -1 & \frac{5s-1}{s+1} \\ \frac{1}{s+1} & \frac{(s-1)(s-2)}{(s+1)^2} \end{bmatrix}$, (3.5)

and the pair $(\hat{d}_{21}, \lambda_2) = (\frac{-(s-2)}{s+1}, 1)$ is coprime. Note that in this case $(\hat{d}_{21} + r\hat{n}_{21}, \lambda_2)$ is coprime for all $r \in R_u$.

Step 3 : $T = \begin{bmatrix} 1 & 0 \\ \frac{(s-1)(s-2)}{(s+1)^3} & 1 \end{bmatrix}$. (3.6)

Step 4 : $\tilde{d}'_1, \tilde{n}'_1, \tilde{d}'_2, \tilde{n}'_2$ of C_d are given by equation (2.20) where T is given by equation (3.6), M_2 is given by equation (3.5) and L_1 is given by equation (3.3), and $q_1, q_2 \in R_u$ satisfy equation (2.19).

■

3.2. Example :

$$\text{Given } P = \begin{bmatrix} \frac{2(s^3-7s^2+6s+2)}{(s+1)^2(s-2)(s-3)} & \frac{-(s-1)}{(s-2)(s-3)} \\ \frac{1}{s-3} & \frac{1}{s-3} \end{bmatrix} \in \mathbb{R}_{sp}(s)^{2x2} \tag{3.7}$$

and a r.c.f.r. (N, D) of P, where

$$N = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{2s-1}{(s+1)^2} & \frac{3s^2-4s-1}{(s+1)^3} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \frac{s-1}{s+1} \\ \frac{s^2-9s+2}{(s+1)^2} & \frac{2(s^3-7s^2+6s+2)}{(s+1)^3} \end{bmatrix}. \tag{3.8}$$

Step 1: With $L_1 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{s+1} & 1 \end{bmatrix}$, $R_1 = \begin{bmatrix} 1 & \frac{-(s-1)}{s+1} \\ 0 & 1 \end{bmatrix}$ we get, $\lambda_1 = \frac{-(s-1)}{(s+1)^2}$. Then

$$\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} R_1 = \begin{bmatrix} \frac{s^2-9s+2}{(s+1)^2} & \frac{(s-2)(s-3)}{(s+1)^2} \\ \frac{2s-1}{(s+1)^2} & \frac{s-2}{(s+1)^2} \end{bmatrix}.$$

Step 2: $\hat{M}_2 = \begin{bmatrix} 1 & 4 \\ \frac{-1}{s+1} & \frac{s-3}{s+1} \end{bmatrix}$. Then $\hat{d}_{21} = \frac{-(s-2)}{s+1}$, $\hat{n}_{21} = \frac{1}{s+1}$, $\lambda_2 = \frac{s-2}{s+1}$. Now we can choose any $r \in R_u$ such that $\hat{d}_{21}+r\hat{n}_{21}$ has no zero at 2 (where λ_2 has a zero). Choose for example $r = -1$. Then $\hat{d}_{21}+ r\hat{n}_{21} = \frac{-(s-1)}{s+1}$ is coprime with λ_2.

Step 4 : For *all* $q_1, q_2 \in R_u$ such that equation (2.19) is satisfied,

$$[\tilde{d}'_1 \;\; \tilde{n}'_1] = [\,1 \;\; q_1\,] \begin{bmatrix} \frac{s^2+5s+40}{(s+1)^2} & -27 \\ \frac{-(s-3)}{(s+1)^3} & \frac{s-2}{s+1} \end{bmatrix} \text{ and } [\tilde{d}'_2 \;\; \tilde{n}'_2] = [\,1 \;\; q_2\,] \begin{bmatrix} \frac{s+2}{s+1} & \frac{3s+7}{s+1} \\ \frac{-1}{s+1} & \frac{s-3}{s+1} \end{bmatrix}.$$

Conclusions

Theorem 2.4 gives the set of all decentralized controllers which stabilize a two-channel system, where each channel has a single input and a single output. This class is given in terms of two scalar rational functions, which are chosen according to equation (2.7); therefore the parametrization is in terms of one free parameter although the system has two local controllers.

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Figure Captions

Figure 1: The system $S(P, C_d)$

Figure 2: The system $S(P, C_d)$ after factorization

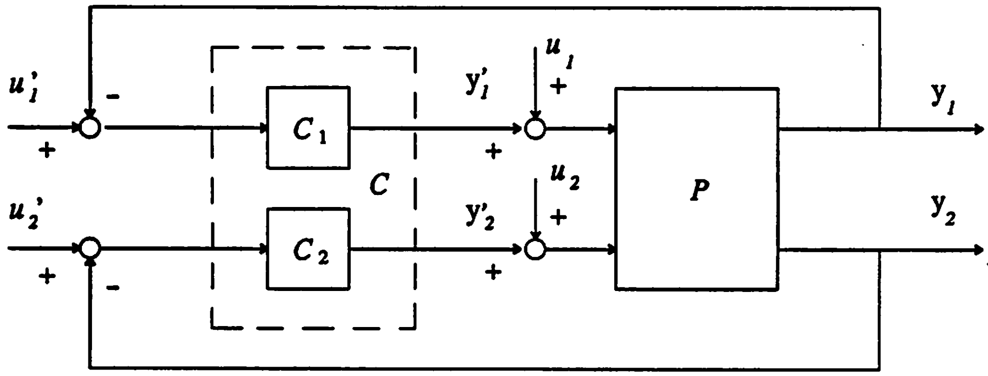

Figure 1

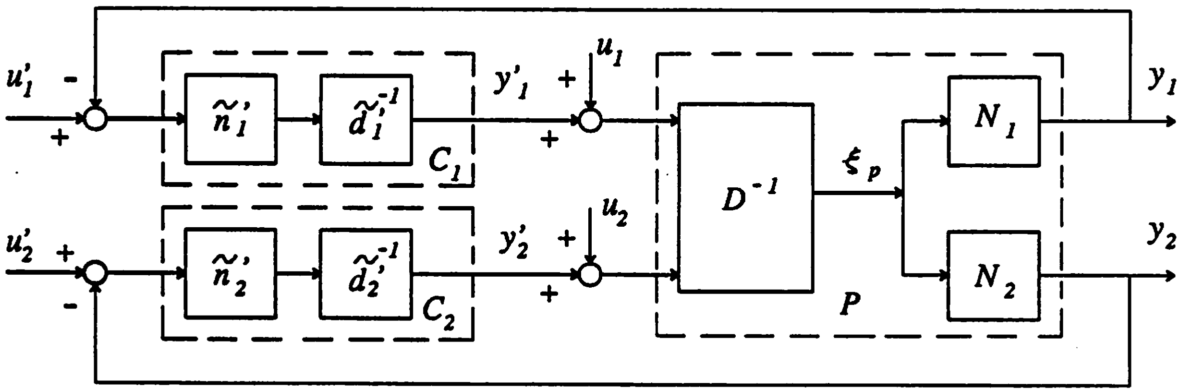

Figure 2