STABILIZATION AND ROBUSTNESS OF THE NONLINEAR UNITY-FEEDBACK SYSTEM: FACTORIZATION APPROACH

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Abstract

This paper is a self-contained discussion of a right factorization approach in the stability analysis of the nonlinear continuous-time or discrete-time, time-invariant or time-varying, well-posed unity-feedback system $S_1(P, C)$. We show that a well-posed stable feedback system $S_1(P, C)$ implies that P and C have right factorizations. In the case where C is stable, P has a normalized right-coprime factorization. The factorization approach is used in stabilization and simultaneous stabilization results.

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Introduction

The unity-feedback configuration $S_1(P, C)$ (see Fig.1) has received considerable attention for very good engineering reasons. For P and C linear time-invariant, the factorization technique [Cal.1, Vid.2, Des.5,6, Net.1 and references therein] has been extremely successful in resolving many problems : finding all stabilizing compensators, robustness, sensitivity minimization, tracking, bounds on performance, etc.. In [Isi.1], nonlinear control system problems are studied in the state-space using differential-geometric approach: this approach is a flourishing area of research [see recent Proceedings CDC, MTNS 1987].

Recently Hammer [Ham.1,2,3] has developed a theory of factorizations for nonlinear time-invariant *discrete-time* systems. For a plant with a right-coprime factorization, a stabilizing configuration is proposed. A class of systems with right-coprime factorizations is also introduced. In a more general set-up Vidyasagar [Vid.1] has also proposed a stabilizing configuration for a plant with a right-coprime factorization (see Fig.4 with $u_2 \equiv u_3 \equiv u_4 \equiv 0$). A right factorization of a nonlinear time-varying continuous-time system has been recently obtained [Des.7].

In Section 1 we show that if a nonlinear (time-invariant or time-varying, continuous-time or discrete-time) $S_1(P, C)$ is well-posed and stable then the plant P and the compensator C have right factorizations. In Theorem 1.9 we show that if either P or C has a normalized right-coprime factorization, then the well-posed $S_1(P, C)$ is stable if and only if the pseudo-state map is stable.

In Section 2 we show that all plants which are stabilizable by incrementally stable compensators have normalized right-coprime factorizations.

In Section 3 the factorization approach is used in simultaneous stabilization results.

Section 1

1.1 Notation

Let $\mathcal{T} \subset \mathbb{R}$ and let $\mathcal{V}$ be a normed vector space. Let

$$\zeta := \{ F \mid F : \mathcal{T} \to \mathcal{V} \}$$

be the vector space of $\mathcal{V}$–valued functions on $\mathcal{T}$. For any $T \in \mathcal{T}$, the projection map $\Pi_T : \zeta \to \zeta$ is defined by

$$\Pi_T F(t) := \begin{cases} F(t) & t \le T ,\ t \in \mathcal{T} \\ \theta_\zeta & t > T ,\ t \in \mathcal{T} , \end{cases}$$

where θ_ζ is the zero element in ζ. Let $\Lambda \subset \zeta$ be a normed vector space which is closed under the family of projection maps $\{ \Pi_T \}_{T \in \mathcal{T}}$. For any $F \in \Lambda$, let the norm $\| \Pi_{(\cdot)} F \| : \mathcal{T} \to \mathbb{R}_+$ be a nondecreasing function. The extended space Λ_e is defined by

$$\Lambda_e := \{ F \in \zeta \mid \forall T \in \mathcal{T},\ \Pi_T F \in \Lambda \} \quad .$$

A map $F : \Lambda_e \to \Lambda_e$ is said to be causal if and only if for all $T \in \mathcal{T}$, Π_T commutes with $\Pi_T F$; equivalently,

$$\Pi_T F = \Pi_T F \Pi_T \quad .$$

A feedback system is said to be well-posed if and only if for all possible inputs, all of the signals in the feedback system are determined by causal maps.

The Unity-Feedback System $S_1(P,C)$

Consider the unity-feedback system $S_1(P, C)$ in Figure 1 : the plant and the compensator are given by causal *maps* $P : \Lambda_{ie} \to \Lambda_{oe}$ and $C : \Lambda_{oe} \to \Lambda_{ie}$, respectively. Λ_{ie} and Λ_{oe} are input and output extended spaces.

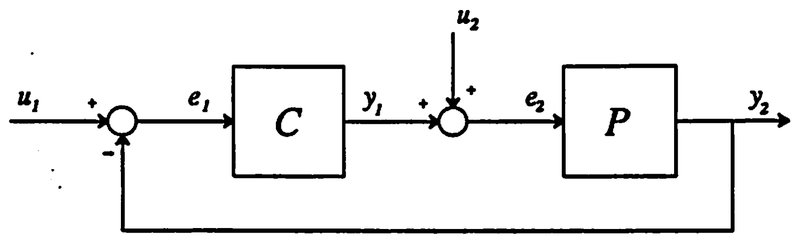

Figure 1 The feedback system $S_1(P, C)$

1.2 Definition : (Well-posed $S_1(P,C)$)

The feedback system $S_1(P, C)$ is said to be *well-posed* iff there exists a causal map H_{e_1}, such that

$$H_{e_1} : \Lambda_{oe} \times \Lambda_{ie} \rightarrow \Lambda_{oe} \quad , \quad H_{e_1} : (u_1, u_2) \mapsto e_1 \quad . \tag{1.1}$$

□

If $S_1(P, C)$ is well-posed then for all inputs (u_1, u_2), the signals e_1, e_2, y_1, y_2 are uniquely defined by the causal maps $H_{e_1}, H_{e_2}, H_{y_1}, H_{y_2}$, respectively.

Now we introduce a bounded-input bounded-output stability notion.

1.3 Definition : (Stable Map)

A causal map $H : \Lambda_{oe} \times \Lambda_{ie} \rightarrow \Lambda_{oe}$ is said to be *stable* iff there exists a continuous nondecreasing function $\phi_H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\forall (u_1, u_2) \in \Lambda_o \times \Lambda_i \ , \quad \| H(u_1, u_2) \| \leq \phi_H(\| u_1 \| + \| u_2 \|) \quad . \tag{1.2}$$

□

A stable map need not be continuous. Note that the composition and the sum of stable maps are stable.

1.4 Definition : (Stable $S_1(P,C)$)

A well-posed $S_1(P, C)$ *is* called *stable* iff there exist causal stable maps H_{e_1} , H_{e_2}, such that

$$H_{e_1} : \Lambda_{oe} \times \Lambda_{ie} \rightarrow \Lambda_{oe} \quad , \quad H_{e_1} : (u_1, u_2) \mapsto e_1 \tag{1.3a}$$

and

$$H_{e_2} : \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{ie} \quad , \quad H_{e_2} : (u_1 , u_2) \mapsto e_2 \quad . \tag{1.3b}$$

□

A well-posed $S_1(P, C)$ is stable if and only if all signals e_1, e_2, y_1, y_2 are uniquely determined by the causal stable maps $H_{e_1}, H_{e_2}, H_{y_1}, H_{y_2}$, respectively. In the linear case, the maps in eqns. (1.3a,b) are linear over the product space ; $H_{e_1}(u_1,u_2) = H_{e_1u_1}(u_1) + H_{e_1u_2}(u_2)$, $H_{e_2}(u_1,u_2) = H_{e_2u_1}(u_1) + H_{e_2u_2}(u_2)$. Hence one has to check the stability of four maps [Vid.2].

In the linear time-invariant case, the factorization approach is a major tool in the stability analysis of $S_1(P, C)$. Since the stable-factor factorization relies on transformation techniques, these tools cannot be readily extended to the stability analysis of a *nonlinear* $S_1(P, C)$. Following the linear theory as a guide and Vidyasagar [Vid.1] and Hammer [Ham.1,2], we introduce *right factorization* concepts for *nonlinear systems* in terms of stable maps and set theory.

1.5 Definition : (Right Factorization)

A causal map $P : \Lambda_{ie} \to \Lambda_{oe}$ is said to have a *right factorization* (N_p , D_p , X_p) if and only if there exist causal stable maps N_p , D_p , such that

(i) $D_p : X_p \subset \Lambda_{ie} \to \Lambda_{ie}$ is bijective and has a causal inverse ,

and (ii) $N_p : X_p \to \Lambda_{oe}$, with $N_p[X_p] = P[\Lambda_{ie}]$,

and (iii) $P = N_p D_p^{-1}$.

X_p is called the *factorization space* of the right factorization (N_p , D_p , X_p).

□

1.6 Theorem : (A necessary condition for stable $S_1(P,C)$)

Let P and C be causal maps such that $S_1(P, C)$ is well-posed and stable. Then the maps P and C have right factorizations.

Comment : Note that this is a generalization of the well-known result in the linear time-invariant case [Des.1 p.85].

Proof of Theorem 1.6 : By assumption, $H_{e_1}, H_{e_2}, H_{y_1}, H_{y_2}$ are causal stable maps. Let

$$D_c := H_{e_1}|_{u_2\equiv 0} = (I+PC)^{-1} \quad \text{and}$$

$$D_p := H_{e_2}|_{u_1\equiv 0} = (I+CP)^{-1} \quad .$$

$D_c : X_c := \Lambda_{oe} \to \Lambda_{oe}$ and $D_p : X_p := \Lambda_{ie} \to \Lambda_{ie}$ are causal stable bijective maps with causal inverses. Let

$$N_c := H_{y_1}|_{u_2\equiv 0} = C(I+PC)^{-1} \quad \text{and}$$

$$N_p := H_{y_2}|_{u_1\equiv 0} = P(I+CP)^{-1} \quad .$$

$N_c : X_c \to \Lambda_{ie}$, $N_p : X_p \to \Lambda_{oe}$ are causal stable maps and $N_c[X_c] = C[\Lambda_{oe}]$, $N_p[X_p] = P[\Lambda_{ie}]$ by construction. By calculation, $P = N_p D_p^{-1}$ and $C = N_c D_c^{-1}$; therefore (N_p , D_p , X_p) is a right factorization of P and (N_c , D_c , X_c) is a right factorization of C .

□

By Theorem 1.6 , any causal plant P which can be stabilized by $S_1(P, C)$ necessarily has a right factorization. However, this result is not only due to the configuration of $S_1(P, C)$. The idea in Theorem 1.6 can be generalized to feedback systems other than $S_1(P, C)$. As an illustration, consider the following example.

1.7 Example : Consider the system $S_3(P, V, U, M-D)$ in Figure 2 , where $P : \Lambda_{ie} \to \Lambda_{oe}$, $M-D : \Lambda_{ie} \to \Lambda_{ie}$, and V , U are causal maps over the appropriate extended spaces. Let the system $S_3(P, V, U, M-D)$ be well-posed, that is there exists a causal map H

$$H : \Lambda_{ie} \times \Lambda_{ie} \times \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{ie} \times \Lambda_{oe} \times \Lambda_{ie} \times \Lambda_{ie} \ , \ H : (u_1, u_2, u_3, u_4) \mapsto (e_1, y_3, y_v, e_4) \ .$$

Suppose also that the causal maps

$$H_{e_2} : \Lambda_{ie} \times \Lambda_{ie} \times \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{ie} \ , \ H_{e_2} : (u_1, u_2, u_3, u_4) \mapsto e_2$$

and $H_{y_2} : \Lambda_{ie} \times \Lambda_{ie} \times \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{oe}$, $H_{y_2} : (u_1, u_2, u_3, u_4) \mapsto y_2$

are stable. Then the plant P has a right factorization.

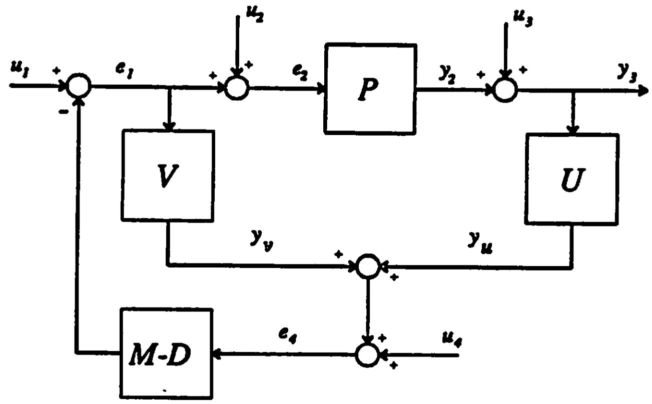

Figure 2 The feedback system $S_3(P, V, U, M-D)$

For $u_1 \equiv u_3 \equiv u_4 \equiv 0$, any $e_2 \in \Lambda_{ie}$ uniquely determines $y_u = UPe_2 \in \Lambda_{ie}$. By well-posedness, y_u determines e_1 uniquely. Then $u_2 = e_2 - e_1 \in \Lambda_{ie}$ is uniquely defined. Hence $H_{e_2}|_{u_1 \equiv u_3 \equiv u_4 \equiv 0} : \Lambda_{ie} \to \Lambda_{ie}$ is bijective. Since $H_{y_2}|_{u_1 \equiv u_3 \equiv u_4 \equiv 0} = P\, H_{e_2}|_{u_1 \equiv u_3 \equiv u_4 \equiv 0}$,

$$(H_{y_2}|_{u_1 \equiv u_3 \equiv u_4 \equiv 0}, H_{e_2}|_{u_1 \equiv u_3 \equiv u_4 \equiv 0}, \Lambda_{ie})$$

is a right factorization of P.

□

Now we introduce the right-coprime factorization concept for causal nonlinear maps.

1.8 Definition : (Normalized Right-Coprime Factorization)

(N_p, D_p, X_p) is said to be a *normalized right-coprime factorization* of $P : \Lambda_{ie} \to \Lambda_{oe}$ iff

(i) (N_p, D_p, X_p) is a right factorization of P

and (ii) there exist causal stable maps $U_p : \Lambda_{oe} \to X_p$ and $V_p : \Lambda_{ie} \to X_p$ such that

$$U_p N_p + V_p D_p = I_{X_p} , \qquad (1.4)$$

where I_{X_p} denotes the identity map on X_p.

□

A similar definition is stated in [Vid.1]. In [Ham.1], the right-coprimeness notion is introduced for *discrete-time* systems, where condition (ii) of Definition 1.8 is interpereted as : given any "unimodular" M, there exist causal stable maps U_p and V_p such that $U_p N_p + V_p D_p = M$. Note that this statement implies eqn.(1.4). However the converse need not be true since nonlinear maps are not necessarily right distributive.

The following theorem gives a necessary and sufficient condition for stability of a well-posed $S_1(P, C)$ provided that either P or C have normalized right-coprime factorizations. Since the roles of P and C can be interchanged, we state only the case where C has a normalized right-coprime factorization. Clearly, any causal stable map $C : \Lambda_{oe} \rightarrow \Lambda_{ie}$ has a normalized right-coprime factorization, namely $(C, I_{\Lambda_{oe}}, \Lambda_{oe})$.

1.9 Theorem : (**Necessary and sufficient condition for stable $S_1(P,C)$**)

Let $C : \Lambda_{oe} \rightarrow \Lambda_{ie}$ have a normalized right-coprime factorization (N_c, D_c, X_c) and $S_1(P, C)$ be well-posed. Let $\xi_c \in X_c$ be defined as in Figure 3. Then

$S_1(P, C)$ is stable if and only if there exists a causal stable map $H_{\xi_c} : (u_1, u_2) \mapsto \xi_c$.

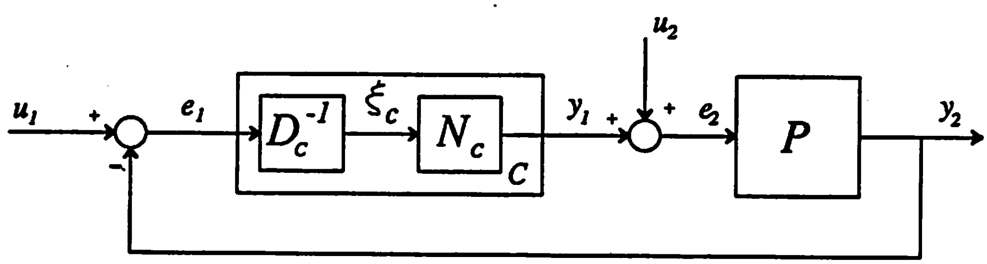

Figure 3 $S_1(P, N_c D_c^{-1})$

Proof :

" if "

By assumption, there exists a causal and stable map $H_{\xi_c} : (u_1, u_2) \mapsto \xi_c$. Since (N_c, D_c, X_c) is a right factorization of C,

$$H_{e_1}(u_1, u_2) := D_c H_{\xi_c}(u_1, u_2) \text{ and}$$

$$H_{e_2}(u_1, u_2) := u_2 + N_c H_{\xi_c}(u_1, u_2)$$

are causal stable maps. By Definition 1.4, $S_1(P, C)$ is stable.

" only if "

By assumption, H_{e_1}, H_{e_2} are causal, stable maps. Since (N_c, D_c, X_c) is a normalized right-coprime factorization of C, for any $(e_1, y_1) \in \Lambda_{oe} X \Lambda_{ie}$ such that $y_1 = Pe_1 = N_c D_c^{-1} e_1$, there exists a unique $\xi_c \in X_c$ such that

$$D_c \xi_c = e_1 \tag{1.5a}$$

$$N_c \xi_c = y_1 \ , \tag{1.5b}$$

and there exist causal stable maps $U_c : \Lambda_{oe} \to X_c$, $V_c : \Lambda_{ie} \to X_c$ such that

$$U_c N_c + V_c D_c = I_{X_c} \ . \tag{1.6}$$

From equations (1.5a,b) and (1.6) we get

$$\begin{aligned} \xi_c &= U_c N_c \xi_c + V_c D_c \xi_c \\ &= U_c H_{y_1}(u_1, u_2) + V_c H_{e_1}(u_1, u_2) \\ &= U_c(H_{e_2}(u_1, u_2) - u_2) + V_c H_{e_1}(u_1, u_2) \\ &=: H_{\xi_c}(u_1, u_2) \end{aligned} \tag{1.7}$$

The map H_{ξ_c} defined in equation (1.7) is causal and stable.

□

The idea in Theorem 1.9 can be generalized to well-posed feedback systems other than $S_1(P, C)$. Clearly, the necessity part requires only that the well-posed feedback system has causal stable maps H_{e_1} and H_{y_1}. Note that the stability assumption on such a feedback configuration can be a stronger requirement. The sufficiency condition is clearly a property of the feedback configuration. It holds for $S_1(P, C)$, but it may also hold for other well-posed systems. To illustrate the idea we give the following example. The example deals with the causal plant P. Hence Theorem 1.9 should be interpreted after suitable subscript interchanges : $c \leftrightarrow p$, $1 \leftrightarrow 2$ and $o \leftrightarrow i$.

1.10 Example : Consider the well-posed system $S_3(N_p D_p^{-1}, V_p, U_p, M - D_p)$ in Figure 4. Let the causal map $P : \Lambda_{ie} \to \Lambda_{oe}$ have a normalized right-coprime factorization (N_p, D_p, Λ_{ie}). Let $U_p : \Lambda_{oe} \to \Lambda_{ie}$ and $V_p : \Lambda_{ie} \to \Lambda_{ie}$ be causal stable maps such that equation (1.4) holds for $X_p = \Lambda_{ie}$. Then $S_3(N_p D_p^{-1}, V_p, U_p, M - D_p)$ is stable if and only if the causal map H_{ξ_p} is stable.

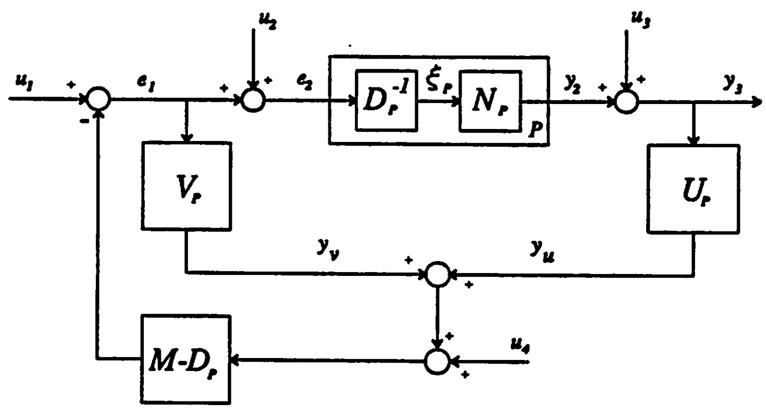

Figure 4 The feedback system $S_3(N_pD_p^{-1}, V_p, U_p, M-D_p)$

The feedback system is stable if and only if the causal maps $H_{e_1}, H_{y_2}, H_{y_v}$ and H_{y_u} are stable. If the maps H_{e_1} and H_{y_2} are stable, then the map

$$H_{\xi_p}(u_1, u_2, u_3, u_4) := U_pH_{y_2}(u_1, u_2, u_3, u_4) + V_p\,[H_{e_1}(u_1, u_2, u_3, u_4) + u_2]$$

is causal and stable. Conversely, if there exists a causal stable map H_{ξ_p} then the maps

$$H_{e_1}(u_1, u_2, u_3, u_4) := D_p\,H_{\xi_p}(u_1, u_2, u_3, u_4) - u_2$$

$$H_{y_2}(u_1, u_2, u_3, u_4) := N_p\,H_{\xi_p}(u_1, u_2, u_3, u_4)$$

$$H_{y_v}(u_1, u_2, u_3, u_4) := V_p\,H_{e_1}(u_1, u_2, u_3, u_4)$$

$$H_{y_u}(u_1, u_2, u_3, u_4) := U_p\,[H_{y_2}(u_1, u_2, u_3, u_4) + u_3]$$

are causal and stable since U_p and V_p are causal stable maps. Hence the system in Figure 4 is stable.

□

The Small Gain Theorem [Des.1, Zam.1] can be considered as a corollary to Theorem 1.9.

1.11 Corollary : (Small Gain Theorem)

Let P , C be causal stable maps with the property that for some nonnegative γ_p , γ_c , β_p , β_c , for all $x \in \mathbb{R}_+$, $\phi_P(x) := \gamma_p x + \beta_p$, $\phi_C(x) := \gamma_c x + \beta_c$ (see equation (1.2)) . Let

$S_1(P, C)$ be well-posed. Then $S_1(P, C)$ is stable if $\gamma_p \gamma_c < 1$.

Proof :

Since C is stable, $(C, I_{\Lambda_{oe}}, \Lambda_{oe})$ is a normalized right-coprime factorization of C. By setting $\xi_c := e_1$, a standard calculation shows that if $\gamma_p \gamma_c < 1$, then H_{ξ_c} is stable. By Theorem 1.9, we conclude that $S_1(P, C)$ is stable.

□

A generalization [Vid.1] of the unimodularity concept of linear time-invariant systems is introduced next.

1.12 Definition : (Unimodular Map)

A causal stable map $H : \Lambda_e \to \Lambda_e$ is called *unimodular* iff H is bijective and $H^{-1} : \Lambda_e \to \Lambda_e$ is causal and stable.

1.13 Corollary :

Let $S_1(P, C)$ be well-posed, stable with the maps P and C causal and stable. Then $(I + PC)$ is unimodular.

Comment : If either P or C is *not* stable, $(I + PC)$ need not be stable; however $(I + PC)^{-1}$ is always causal stable if $S_1(P, C)$ is stable.

Proof of Corollary 1.13 : By assumption, $(I + PC) : \Lambda_{oe} \to \Lambda_{oe}$ is causal and stable. Since $S_1(P, C)$ is well-posed, $(I + PC)$ is bijective and has a causal inverse. $(C, I_{\Lambda_{oe}}, \Lambda_{oe})$ is a normalized right-coprime factorization of C; hence $\xi_c := e_1$. The stability of $S_1(P, C)$ and Theorem 1.9 imply that

$$(I + PC)^{-1} = H_{e_1}|_{u_2=0} : u_1 \mapsto e_1$$

is stable.

□

Section 2

Theorem 1.9 and the two corollaries are due to the asumption that either C or P have normalized right-coprime factorizations. The following theorem establishes a class of systems, not necessarily stable, which have normalized right-coprime factorizations. First, we state an incremental stability definition [Saf.1, Des.1, Des.2] which generalizes an important property of stable *linear* maps : regardless of the input, the deviation at the output due to a bounded deviation in the input is bounded.

2.1 Definition : **(Incrementally Stable Map)**

A causal map $H : \Lambda_e \times \Lambda_{ie} \to \Lambda_{oe}$ is called *incrementally stable* iff H is stable *and* there exists a continuous nondecreasing function $\tilde{\phi}_H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall (v, u) \in \Lambda_e \times \Lambda_{ie} \ , \ \forall (\Delta v, \Delta u) \in \Lambda \times \Lambda_i \ ,$$
$$\| H(v+\Delta v, u+\Delta u) - H(v, u) \| \leq \tilde{\phi}_H(\| \Delta v \| + \| \Delta u \|) \ .$$

2.2 Theorem : **(Normalized right-coprime factorization and stable $S_1(P, C)$)**

Let $S_1(P, C)$ be well-posed. If P is incrementally stable, we have :

$S_1(P, C)$ is stable if and only if C has a normalized right-coprime factorization of the form $(N_c \ , I_{X_e} - PN_c \ , X_c \subset \Lambda_{oe})$ for some causal stable map N_c from Λ_{oe} into Λ_{ie} .

Comment : Theorem 2.2 gives a parametrization of all stabilizing compensators C provided that the plant is incrementally stable. This theorem [Des.2] extends the Q-parametrization result of the linear case [Zam.2] . It is interesting to note that Theorem 2.2 motivates a normalized right-coprime factorization approach.

Proof of Theorem 2.2 :

" only if "

Since $S_1(P, C)$ is well-posed and stable, as in Theorem 1.6, the maps

$$D_c := H_{e_1}|_{u_2=0} = (I+PC)^{-1} : \Lambda_{oe} \to \Lambda_{oe} ,$$

$$N_c := H_{y_1}|_{u_2=0} = C(I+PC)^{-1} : \Lambda_{oe} \to \Lambda_{ie}$$

are causal stable maps. $(N_c , D_c , \Lambda_{oe})$ is a right factorization of C. For the causal stable maps $P : \Lambda_{ie} \to \Lambda_{oe}$ and $I_{\Lambda_{oe}}$ we get

$$PN_c + I_{\Lambda_{oe}} D_c = PC(I+PC)^{-1} + (I+PC)^{-1} = I_{\Lambda_{oe}} . \tag{2.1}$$

From equation (2.1), we conclude that $D_c = I - PN_c$ and that $(N_c , I - PN_c , \Lambda_{oe})$ is a normalized right-coprime factorization of C.

" if "

By assumption, $S_1(P, C)$ is well-posed and C has a normalized right-coprime factorization (N_c , D_c , X_c) with $PN_c + D_c = I_{X_c}$. By Theorem 1.9, it is sufficient to show that H_{ξ_c} is causal and stable. Writing the summing node equations in Figure 3, we get

$$D_c \xi_c = u_1 - P(N_c \xi_c + u_2) . \tag{2.2}$$

By well-posedness, for any input $(u_1, u_2) \in \Lambda_{oe} \times \Lambda_{ie}$, equation (2.2) determines ξ_c uniquely; indeed

$$\xi_c = H_{\xi_c}(u_1, u_2) = D_c^{-1} H_{e_1}(u_1, u_2) . \tag{2.3}$$

Equation (2.3) also shows that H_{ξ_c} is causal.

Adding $PN_c \xi_c$ to both sides in equation (2.2) and using the normalized right-coprime factorization of C, we get

$$(D_c + PN_c)\xi_c = \xi_c = u_1 + PN_c \xi_c - P(N_c \xi_c + u_2) . \tag{2.4}$$

Since P is incrementally stable, for some $\tilde{\phi}_P$

$$\forall (u_1, u_2) \in \Lambda_o \times \Lambda_i , \forall T \in \mathcal{T} , \| \Pi_T \xi_c \| \le \| u_1 \| + \| PN_c \xi_c - P(N_c \xi_c + u_2) \|$$

$$\le \| u_1 \| + \tilde{\phi}_P (\| u_2 \|) ,$$

hence H_{ξ_c} is stable. By Theorem 1.9, we conclude that $S_1(P, C)$ is stable.

□

Using the incremental stability argument, we show that the unimodularity condition in

Corollary 1.12 is also a sufficient condition for stability of $S_1(P, C)$.

2.3 Proposition :

Let $S_1(P, C)$ be well-posed. If P is incrementally stable and if C is stable, we have :

$S_1(P, C)$ is stable if and only if $(I + PC)$ is unimodular.

Comment : Note that the test for stability of a two-input two-output system reduces to that of a one-input one-output system.

Proof of Proposition 2.3 :

" only if "

Follows from Corollary 1.12.

" if "

By assumption, $(I + PC)$ is unimodular. Since C is stable, (C, I, Λ_{oe}) is a normalized right-coprime factorization of C ; hence $e_1 := \xi_c$ (see Figure 3). By Theorem 1.9, it suffices to show that the causal map H_{e_1} is stable. Writing the summing node equations in $S_1(P, C)$, we get

$$e_1 = u_1 - P(Ce_1 + u_2) \quad . \tag{2.5}$$

Adding PCe_1 to both sides in equation (2.5), we get

$$(I + PC)e_1 = u_1 + PCe_1 - P(Ce_1 + u_2) \quad . \tag{2.6}$$

By unimodularity of $(I + PC)$, $e_1 \in \Lambda_o$ if and only if $(I + PC)e_1 \in \Lambda_o$. From equation (2.6), by incremental stability of P, we get

$$\begin{gathered} \forall (u_1, u_2) \in \Lambda_o \times \Lambda_i \ , \ \forall T \in \mathcal{T} \\ \| \Pi_T (I + PC) e_1 \| \le \| u_1 \| + \tilde{\phi}_P (\| u_2 \|) \quad . \end{gathered} \tag{2.7}$$

By equation (2.7), the causal map $(u_1, u_2) \mapsto (I + PC)e_1$ is stable, hence H_{e_1} is stable.

□

With all the assumptions made in Example 1.10, provided that $u_2 \equiv u_3 \equiv u_4 \equiv 0$, it can be easily shown that the system $S_3(N_p D_p^{-1}, V_p, U_p, M-D_p)$ in Figure 4 is stable [Vid.1]. In the following example, incremental stability assumptions are made to give a stabilizing configuration for a plant with a normalized right-coprime factorization.

2.4 Example : Consider the feedback system in Figure 4 and let all of the assumptions in Example 1.10 hold. Suppose now that the causal maps D_p, U_p, V_p and M are incrementally stable. Then the well-posed feedback system in Figure 4 is stable.

It suffices to show that the causal map

$$H_{\xi_p} : \Lambda_{ie} \times \Lambda_{ie} \times \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{ie} \quad , \; H_{\xi_p} : (u_1, u_2, u_3, u_4) \mapsto \xi_p$$

is stable. Writing the summing node equations in Figure 4, we get

$$u_1 - (M - D_p)[\Gamma(u_1, u_2, u_3, u_4)] = D_p \xi_p - u_2 , \tag{2.8a}$$

where

$$\Gamma(u_1, u_2, u_3, u_4) := u_4 + U_p(u_3 + N_p \xi_p) + V_p(D_p \xi_p - u_2) . \tag{2.8b}$$

Adding $M\xi_p$ to both sides in equation (2.8a) gives

$$M\xi_p = u_1 + u_2 + \{ M\xi_p - M[\Gamma(u_1, u_2, u_3, u_4)]\} + \{ D_p[\Gamma(u_1, u_2, u_3, u_4)] - D_p \xi_p \} . \tag{2.9}$$

The incremental stability of U_p and V_p imply that the map $\Gamma - H_{\xi_p}$, namely

$$(\Gamma - H_{\xi_p})(u_1, u_2, u_3, u_4) = u_4 + \{U_p(N_p \xi_p + u_3) - U_p N_p \xi_p\} + \{V_p(D_p \xi_p - u_2) - V_p D_p \xi_p\} ,$$

is stable. Going back to equation (2.9), by unimodularity and incremental stability of M, the claim follows.

□

Section 3

3.1 Theorem : **(Simultaneous Stabilization)**

Let $P(\cdot\ ,\ \cdot):\Lambda_e \times \Lambda_{ie} \rightarrow \Lambda_{oe}$ be causal and incrementally stable. For some fixed causal $C:\Lambda_{oe} \rightarrow \Lambda_{ie}$, let $S_1(P(v\ ,\ \cdot)\ ,C\ )$ be well-posed for all $v \in \Lambda$. Then

$S_1(P(v_0\ ,\ \cdot\ )\ ,C\ )$ is stable for some $v_0 \in \Lambda$ if and only if $S_1(P(v\ ,\ \cdot\ )\ ,C\ )$ is stable for all $v \in \Lambda$.

Comments :

a) In other words, if we have a family of incrementally stable plants $\{ P(v\ ,\ \cdot\ ) \}_{v \in \Lambda}$ and if one member is stabilized by some C , then the whole family is stabilized by that C .

b) The one-input one-output plant $P(v\ ,\ \cdot\ )$ can be considered as an input-output description for a fixed parameter v or for a fixed bounded auxiliary input v . In any case, $P(v\ ,\ \cdot\ )$ is assumed to be a complete description of the plant for any $v \in \Lambda$ [Bha.1].

Proof of Theorem 3.1 :

" if "

Obvious.

" only if "

By assumption, $S_1(P(v_0\ ,\ \cdot\ )\ ,C\ )$ is stable for some $v_0 \in \Lambda$. By Theorem 2.2, there exists a causal stable map N_c such that $(N_c\ ,I - P(v_0\ ,N_c(\cdot))\ ,\Lambda_{oe}\ )$ is a normalized right-coprime factorization of C . For any $v \in \Lambda$, consider $S_1(\ P(v\ ,\ \cdot\ )\ ,C\ )$ in Figure 5 :

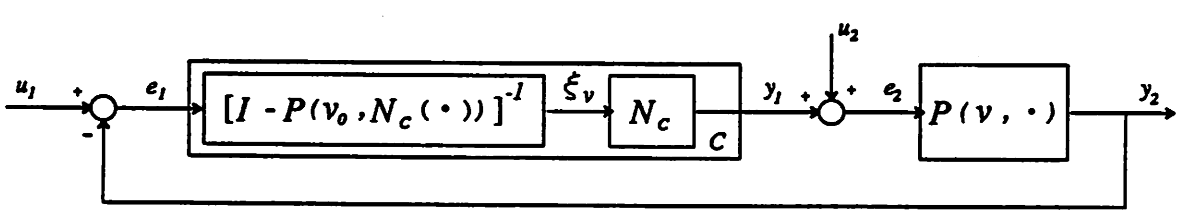

Figure 5 $S_1(P(v\ ,\ \cdot\ )\ ,C\ )$.

By Theorem 1.9 , it is sufficient to show that the causal map H_{ξ_v} associated with $S_1(P(v\,,\,\cdot\,)\,,C\,)$ is stable. The summing node equations in Figure 5 give

$$\xi_v = u_1 + P(v_0\,, N_c\xi_v) - P(v\,, N_c\xi_v + u_2)\ . \tag{3.1}$$

Since $P(\,\cdot\,,\,\cdot\,)$ is incrementally stable, equation (3.1) yields

$$\forall (u_1\,, u_2) \in \Lambda_o \times \Lambda_i\ ,\ \forall v \in \Lambda\ ,\ \forall T \in \mathcal{T}\ ,$$

$$\| \Pi_T \xi_v \| \le \| u_1 \| + \tilde{\phi}_P(\| v - v_0 \| + \| u_2 \|)\ .$$

Hence, H_{ξ_v} is stable for all $v \in \Lambda$. By Theorem 1.9 , we conclude that $S_1(P(v\,,\,\cdot\,), C\,)$ is stable for all $v \in \Lambda$.

□

In the case that $P(v,\cdot) := \hat{H}_{y_2}|_{u_2 = v} : (u_1, v) \mapsto y_2$ where $\hat{H}_{y_2}$ is the restricted input-output map of an incrementally stable $\hat{S}_1(\hat{P}, \hat{C})$ for some causal $\hat{P}$ and $\hat{C}$, the two-step stabilization results in [Ana.1, Des.3] become special cases of Theorem 3.1 .

The following theorem [Des.2] establishes a necessary and sufficient condition for simultaneous stabilization of two plants which need not be members of an incrementally stable family of plants. Our use of the factorization approach greatly simplifies the proof.

3.2 Theorem : (Simultaneous Stabilization)

Let $P_1 : \Lambda_{ie} \to \Lambda_{oe}$ be causal and incrementally stable. Let $S_1(P_1, C)$ be well-posed and stable. (Hence by Theorem 2.2 , C has a normalized right-coprime factorization $(N_c\,, I - P_1 N_c\,, \Lambda_{oe})$ for some causal stable $N_c : \Lambda_{oe} \to \Lambda_{ie}$). Let $P_2 : \Lambda_{ie} \to \Lambda_{oe}$ be any causal map such that $S_1(P_2\,, C)$ and $S_1(P_2 - P_1\,, N_C)$ are well-posed. Then

$S_1(P_2\,, C)$ is stable if and only if $S_1(P_2 - P_1, N_C)$ is stable.

Comment : Note that the causal map P_2 need not be stable: the perturbation replaces the incrementally stable P_1 by an arbitrary nonlinear P_2 which is only subject to $S_1(P_2 - P_1, N_C)$ be stable in order to have $S_1(P_2\,, C)$ stable.

Proof of Theorem 3.2 :

Consider Figures 6a and 6b .

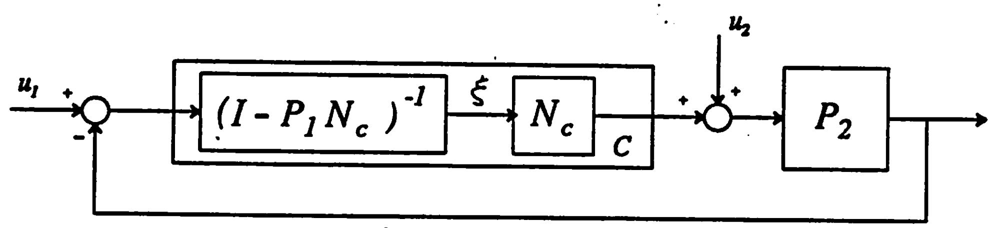

Figure 6a $S_1(P_2, C)$.

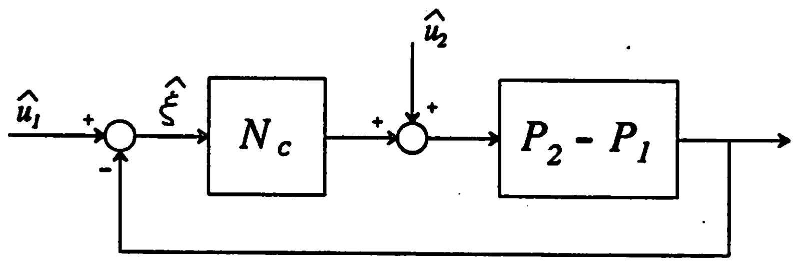

Figure 6b $S_1(P_2-P_1, N_c)$.

By Theorem 1.9 , it is enough to show that

$$H_\xi \text{ is causal, stable if and only if } \hat{H}_{\hat{\xi}} \text{ is causal, stable} \ .$$

" only if "

By assumption, $S_1(P_2-P_1, N_c)$ is well-posed; hence the map $\hat{H}_{\hat{\xi}}$ is causal. Writing the summing node equations in Figure 6b ; we get

$$\hat{\xi} = \hat{u}_1 - (P_2-P_1)(N_c\hat{\xi}+\hat{u}_2) \ . \tag{3.2}$$

Subtracting $P_1N_c\hat{\xi}$ from both sides of equation (3.2), we get

$$(I-P_1N_c)\hat{\xi} = \hat{u}_1 + P_1(N_c\hat{\xi}+\hat{u}_2) - P_1N_c\hat{\xi} - P_2(N_c\hat{\xi}+\hat{u}_2) \ . \tag{3.3}$$

Let

$$\hat{F} : \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{oe} \ , \ \hat{F} : (\hat{u}_1, \hat{u}_2) \mapsto \hat{u}_1 + P_1(N_c\hat{H}_{\hat{\xi}}(\hat{u}_1, \hat{u}_2)+\hat{u}_2) - P_1N_c\hat{H}_{\hat{\xi}}(\hat{u}_1, \hat{u}_2) \ .$$

$\hat{F}$ is causal and stable since by incremental stability of P_1 ,

$$\forall(\hat{u}_1, \hat{u}_2) \in \Lambda_o \times \Lambda_i \ , \ \forall T \in \tau \quad \| \Pi_T \hat{F}(\hat{u}_1, \hat{u}_2) \| \le \| \hat{u}_1 \| + \tilde{\phi}_{P_1}(\| \hat{u}_2 \|) \ .$$

Then equation (3.3) can be rewritten as

$$(I - P_1 N_c)\hat{\xi} = \hat{F}(\hat{u}_1, \hat{u}_2) - P_2(N_c\hat{\xi} + \hat{u}_2) \ . \tag{3.4}$$

Comparing the summing node equations of Figure 6a with equation (3.4) and by well-posedness of $S_1(P_2, C)$, we get

$$\hat{H}_{\hat{\xi}}(\hat{u}_1, \hat{u}_2) = H_{\xi}(\hat{F}(\hat{u}_1, \hat{u}_2), \hat{u}_2) \ . \tag{3.5}$$

Since $\hat{F}$ and H_{ξ} are stable maps, $H_{\hat{\xi}}$ is stable.

" if "

By assumption, $S_1(P_2, C)$ is well-posed; hence H_{ξ} is causal. Writing the summing node equations in Figure 6a , we get

$$\xi = u_1 + P_1 N_c \xi - P_2(N_c \xi + u_2) \ . \tag{3.6}$$

Let

$$F : \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{oe} \ , \ F : (u_1, u_2) \mapsto u_1 + P_1 N_c H_{\xi}(u_1, u_2) - P_1(N_c H_{\xi}(u_1, u_2) + u_2) \tag{3.7}$$

F is causal and stable since P_1 is incrementally stable. Adding and subtracting $P_1(N_c\xi + u_2)$ on the right hand side of equation (3.6) and using equation (3.7) , we get

$$\begin{aligned} \xi &= u_1 + P_1 N_c \xi - P_1(N_c \xi + u_2) - (P_2 - P_1)(N_c \xi + u_2) \\ &= F(u_1, u_2) - (P_2 - P_1)(N_c \xi + u_2) \ . \end{aligned} \tag{3.8}$$

Comparing equations (3.2) with (3.8) and by well-posedness of $S_1(P_2 - P_1, N_c)$, we get

$$H_{\xi}(u_1, u_2) = \hat{H}_{\hat{\xi}}(F(u_1, u_2), u_2) \ . \tag{3.9}$$

Since F and $\hat{H}_{\hat{\xi}}$ are stable maps, H_{ξ} is stable.

□

Consider the feedback system $S_2(\Delta P, N_{cr}, D_{pl})$ shown in Figure 7, where $\Delta P : \Lambda_{ie} \to \Lambda_{oe}$ is causal; $N_{cr} : \Lambda_{oe} \to \Lambda_{ie}$ and $D_{pl} : \Lambda_{oe} \to \Lambda_{oe}$ are causal *stable* maps (not necessarily linear). We assume that $S_2(\Delta P, N_{cr}, D_{pl})$ is well-posed.

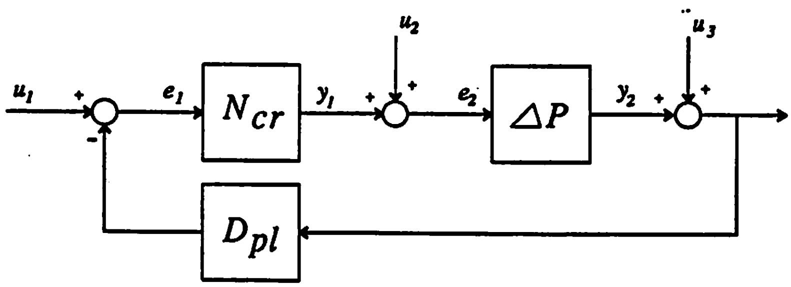

Figure 7 The feedback system $S_2(\Delta P, N_{cr}, D_{pl})$

3.3 Lemma :

The well-posed system $S_2(\Delta P, N_{cr}, D_{pl})$ (Fig. 7) is stable if and only if the causal map $H^2{}_{y_2} : (u_1, u_2, u_3) \mapsto y_2$ is stable.

Proof :

Immediate, since N_{cr}, D_{pl} are stable maps.

□

Theorem 3.4 below motivated by [Des.4], states conditions under which a *linear* compensator C stabilizes the one-input one-output plant $P + \Delta P$ provided the *linear* system $S_1(P, C)$ is stable.

3.4 Theorem : (Robustness of Stable Linear S_1(P , C) under Nonlinear Perturbation)

Let $P : \Lambda_{ie} \to \Lambda_{oe}$ and $C : \Lambda_{oe} \to \Lambda_{ie}$ be causal *linear* maps where $P = D_{pl}^{-1} N_{pl}$, $C = N_{cr} D_{cr}^{-1}$ and $N_{pl} N_{cr} + D_{pl} D_{cr} = I$. $N_{pl} : \Lambda_{ie} \to \Lambda_{oe}$, $N_{cr} : \Lambda_{oe} \to \Lambda_{ie}$ are *linear* causal and stable. $D_{pl} : \Lambda_{oe} \to \Lambda_{oe}$, $D_{cr} : \Lambda_{oe} \to \Lambda_{oe}$ are causal *linear* bijective stable with causal inverses. Let $\Delta P : \Lambda_{ie} \to \Lambda_{oe}$ be a causal map and let $S_2(\Delta P, N_{cr}, D_{pl})$, $S_1(P + \Delta P, C)$ be well-posed. Then

$S_1(P + \Delta P, C)$ is stable if $S_2(\Delta P, N_{cr}, D_{pl})$ is stable.

Comment : Note that P and C are linear; ΔP need *not* be linear; $P, C, \Delta P$ are not required to be stable.

Proof of Theorem 3.4 :

By assumption, C has a normalized right-coprime factorization $(N_{cr}, D_{cr}, \Lambda_{oe})$. Then by Theorem 1.9, $S_1(P+\Delta P, C)$ is stable if and only if the causal map H_ξ defined by equation (3.10) is stable :

$$D_{cr}\xi - u_1 + PN_{cr}\xi + Pu_2 + \Delta P(N_{cr}\xi + u_2) = 0 \ , \tag{3.10}$$

where we used the linearity of P. Note that equation (3.10) is of the form $F(\xi)=0$ for fixed u_1 and u_2. For any map G with the property that $G(p)=0$ if and only if $p=0$, the solutions of $GF(\xi)=0$ and $F(\xi)=0$ are identical. The map G need not be bijective, however if G is chosen to be linear then it must be bijective. Choosing $G=D_{pl}$ and using linearity, we get an equivalent equation for equation (3.10) :

$$D_{pl}D_{cr}\xi = D_{pl}u_1 - N_{pl}N_{cr}\xi - N_{pl}u_2 - D_{pl}\Delta P(N_{cr}\xi + u_2) \ . \tag{3.11}$$

Since $N_{pl}N_{cr} + D_{pl}D_{cr} = I$, we conclude that $S_1(P+\Delta P, C)$ is stable if and only if the H_ξ map defined by the equation (3.12) is stable :

$$\xi = H_\xi(u_1, u_2) = D_{pl}u_1 - N_{pl}u_2 - D_{pl}\Delta P(N_{cr}\xi + u_2) \ . \tag{3.12}$$

Let $F : \Lambda_{oe} \times \Lambda_{ie} \to \Lambda_{oe}$, $F : (u_1, u_2) \mapsto D_{pl}u_1 - N_{pl}u_2$. Clearly F is causal and stable. By assumption $S_2(\Delta P, N_{cr}, D_{pl})$ is stable, hence $H^2{}_{e_1}|_{u_3=0}$ is a stable map. A simple comparison of equation (3.12) and the summing node equations in Figure 7, yields

$$H_\xi(u_1, u_2) = H^2{}_{e_1}(F(u_1, u_2), u_2, 0) \ . \tag{3.13}$$

Since F and $H^2{}_{e_1}$ are stable, so is H_ξ and we conclude that $S_1(P+\Delta P, C)$ is stable.

□

3.5 Fact : The converse to Theorem 3.4 holds if P is stable. The proof follows by Theorem 3.2 since P is incrementally stable by linearity and C has a normalized right-coprime factorization; hence D_{pl} can be chosen to be the identity map, and the third input u_3 can be taken care of in the first input u_1.

3.6 Fact : In the case where ΔP is stable, $S_2(\Delta P, N_{cr}, D_{pl})$ restricted to the inputs $(F(u_1, u_2) + D_{pl}u_3, u_2, u_3)$ is stable if $S_1(P+\Delta P, C)$ is stable. The proof follows by the stability of ΔP and the fact that

$$H^2_{e_1}(F(u_1, u_2) + D_{pl}u_3, u_2, u_3) = H_\xi(u_1, u_2) \quad .$$

□

Conclusion

In the linear time-invariant case, right and normalized right-coprime factorizations exist as a property of the causal map, regardless of the feedback configuration the map is in. When generalizing these concepts to nonlinear causal maps, we show the existence of these properties as a result of the specific feedback configuration the plant is in. Although the unity-feedback system $S_1(P, C)$ is of main interest, some of the results can be extended to other feedback systems as illustrated by examples.

If either the plant or the compensator has a normalized right-coprime factorization, the stability of a well-posed $S_1(P, C)$ is equivalent to the stability of one causal pseudo-state map. This result is the main step used in all of the simultaneous stabilization proofs in Section 3.

Since the composition of maps is only left distributive over addition, a left factorization definition, although possible, does not bring an alternative solution to the stabilization problem as in the linear time-invariant $S_1(P, C)$ case. However, as emphasized by Hammer [Ham.1,2,3], left factorization can be used also to enumarate solutions U, V of the normalized right-coprime factorization requirement $UN + VD = I$, where (N, D, X) is a normalized right-coprime factorization of the map in question.

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