ADAPTIVE CONTROL OF LINEARIZABLE SYSTEMS

by

S. S. Sastry and Alberto Isidori

Memorandum No. UCB/ERL M87/53

9 June 1987

(Revised March 1, 1988)

ADAPTIVE CONTROL OF LINEARIZABLE SYSTEMS

by

S. S. Sastry and Alberto Isidori

Memorandum No. UCB/ERL M87/53

9 June 1987

(Revised March 1, 1988)

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Title Page

ADAPTIVE CONTROL OF LINEARIZABLE SYSTEMS

by

S. S. Sastry and Alberto Isidori

Memorandum No. UCB/ERL M87/53

9 June 1987

(Revised March 1, 1988)

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Adaptive Control of Linearizable Systems

S. S. Sastry*

Electronics Research Laboratory
University of California
Berkeley, CA 94720 U.S.A.

Alberto Isidori†

Dipartimento di Informatica e Sistemistica
Universita di Roma, "La Sapienza"
18 Via Eudossiana
00184 Rome, Italy

Abstract

In this paper we give some initial results on the adaptive control of "minimum-phase" nonlinear systems which are exactly input-output linearizable by state feedback. Parameter adaptation is used as a technique to robustify the exact cancellation of nonlinear terms which is called for in the linearization technique. We review the applications of the techniques to the adaptive control of robot manipulators. Only the continuous time case is discussed in this paper--extensions to the discrete time and sampled data case are not obvious.

*Research supported in part by NASA under grant NAG2-243 and the U. S. Army Research Office under grant DAAG29-85-K-0072. This research is based largely on discussions with Dr. D.Normand-Cyrot of the Laboratoire des Signaux et Systemes, Gif Sur Yvette, France. I would like to thank her for these as well as her hospitality, during my visit to LSS in March 1987.

†Sponsored in part by a McKay Visiting Professorship held at the University of California, Berkeley, CA 94720.

1. Introduction

It is well-known that, under rather mild assumptions, the input-output response of a nonlinear system can be rendered linear by means of state feedback. This was implicitly or explicitly pointed out in several papers dealing with study of noninteracting control of nonlinear systems, like those of Porter [1970], Singh and Rugh [1972], Freund [1975] and Isidori, Krener, Gori-Giorgi and Monaco [1981]. Independently, a substantially identical synthesis technique was successfully implemented in some relevant practical applications, like the control of flight dynamics (Meyer and Cicolani [1980]) and the control of rigid-link robot manipulators via the so called "computed torque" method, to mention a few. Parallel to these developments, beginning with the work of Brockett [1978], several authors studied the problem of when the differential equation relating and input to the state can be rendered linear via state feedback and coordinates transformation. The problem was completely solved by Jakubczyk and Respondek [1980] and, independently, by Hunt, Su and Meyer [1983]. The former design technique is often referred to as exact input-output linearization, while the latter one as exact state-space linearization. The bridge between the two techniques lies in the fact that the design of a state-space linearizing control is equivalent to the design of "output" functions for which input-output linearization is possible. The theory is now well developed and understood (see for instance, expository surveys in Isidori [1985], Isidori [1986] and Claude [1986]) for the continuous time case. For the discrete time and sampled data versions of the theory, see Monaco and Normand-Cyrot [1986], Monaco, Normand-Cyrot and Stornelli [1986] and Jakubczyk [1987]. The class of systems is described (in the continuous time case) by

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x) u_i$$
$$y_1 = h_1(x)$$
$$\vdots$$
$$y_p = h_p(x)$$

with $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^p$ and f, g_i, h_j smooth functions.

A number of applications of these techniques have been made: their chief drawback however appears to arise from the fact that they rely on an exact cancellation of non-linear terms in order to get linear input-output behavior. Consequently, if there are errors or uncertainty in the model of the non-linear terms, the cancellation is no longer exact. In this paper we suggest the use of parameter adaptive control to help robustify i.e., make asymptotically exact the cancellation of nonlinear terms when the uncertainty in the non-linear terms is parametric. Some other attempts in this regard have been made by Marino and Nicosia [1986], Nicosia and Tomei [1984], using a combination of high gain, sliding modes and adaptation. Some previous work in this spirit is in Nam and Arapostathis [1986]. Our development is, we believe, considerately more general and straightforward than theirs (specifically, no error augmentation and stronger stability theorems) and was in turn motivated by our work in the adaptive control of a specific class of linearizable systems--rigid link robot manipulators (see Craig, Hsu, and Sastry [1987] for details), including implementation of the scheme on an industrial robot arm.

We would also like to mention the work done in parallel by Taylor, Kokotovic and Marino [1987] on the adaptive control of *fully-state* linearizable single input, single-output systems. While our scheme specializes to their scheme in the instance that the system is state (rather than input output) linearizable their paper also considers the effect of parasitic dynamics on the adaptation scheme. Taylor, Kokotovic and Marino prove the robustness of their scheme to parasitics; we have however not undertaken such a study here.

The paper is organized as follows: we give a brief review of input-output linearization theory for continuous time systems along with the concept of a minimum phase non-linear system as developed in Byrnes and Isidori [1984], in Section 2. We discuss the adaptive version of this control strategy in Section 3 along with its applications to the adaptive control of robot manipulators. In Section 4, we collect a few comments about the discrete time and sampled data cases along with some future directions.

2. Review of Exact Linearization Techniques

2.1. Basic Theory

A large class of non-linear control systems can be made to have linear input output behavior through a choice of non-linear state feedback control laws. We review the theory here in order to fix notation. Consider the single-input single output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.1}$$

with $x \in \mathbf{R}^n$; f, g, h smooth. Differentiating y with respect to time, one obtains

$$\dot{y} = L_f h + L_g h\, u \tag{2.2}$$

Here $L_f h$, $L_g h$ stand for the Lie derivatives of h w.r.t f, g respectively. If $(L_g h)(x) \neq 0 \; \forall\, x \in \mathbf{R}^n$, then the control law of the form $\alpha(x) + \beta(x)v$, namely

$$u = + \frac{1}{L_g h}(-L_f h + v)$$

yields the linear system

$$\dot{y} = v \tag{2.3}$$

In the instance that $L_g h(x) \equiv 0$, one differentiates (2.2) further to obtain

$$\ddot{y} = L_f^2 h + (L_g L_f h) u \tag{2.4}$$

In (2.4) above $L_f^2 h$ stands for $L_f(L_f h)$ and $L_g L_f h$ stands for $L_g(L_f h)$. As before, if $L_g L_f h \neq 0 \ \forall \ x \in \mathbf{R}^n$, the law

$$u = + \frac{1}{L_g L_f h} (-L_f^2 h + v)$$

linearizes the system (2.4) to yield

$$\ddot{y} = v \ .$$

More generally; if γ is the smallest integer such that $L_g L_f^i h \equiv 0$ for i = 0, ..., γ-2 and $L_g L_f^{\gamma-1} h(x) \neq 0 \ \forall x \in \mathbf{R}^n$ then the control law

$$u = \frac{1}{L_g L_f^{\gamma-1} h} (-L_f^{\gamma} h + v) \tag{2.5}$$

yields

$$y^{(\gamma)} = v \ . \tag{2.6}$$

The theory is considerably more complicated if $L_g L_f^{\gamma-1} h = 0$ for some values of x. We do not discuss this case here. For the multi-input multi-output case, consider the p-input, p-output nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x) u_1 + \cdots + g_p(x) u_p \\ y_1 &= h_1(x) \\ &\vdots \\ y_p &= h_p(x) \end{aligned} \tag{2.7}$$

Here $x \in \mathbf{R}^n$, $u \in \mathbf{R}^p$, $y \in \mathbf{R}^p$ and f, g_i, h_j are assumed smooth. Now, differentiate the outputs y_j with respect to time to get

$$\dot{y}_j = L_f h_j + \sum_{i=1}^{p} (L_{g_i} h_j) u_i \tag{2.8}$$

In (2.8) $L_f h_j$ stands for the Lie derivative of h_j with respect to f, similarly $L_{g_i} h_j$. Note that if each of the $(L_{g_i} h_j)(x) \equiv 0$, then the inputs do not appear in (2.8). Define γ_j to be the smallest integer such that at least one of the inputs appears in $y_j^{(\gamma_j)}$, i.e.,

$$y_j^{(\gamma_j)} = L_f^{\gamma_j} h_j + \sum_{i=1}^{p} L_{g_i}(L_f^{\gamma_j - 1} h_j) u_i \tag{2.9}$$

with at least one of the $L_{g_i}(L_f^{\gamma_j-1} h_j) \neq 0 \ \forall x$. Define the $p \times p$ matrix $A(x)$ as

$$A(x) = \begin{bmatrix} L_{g_1}(L_f^{\gamma_1 - 1} h_1) & \cdots & L_{g_p}(L_f^{\gamma_1 - 1} h_1) \\ \cdots & & \vdots \\ L_{g_1}(L_f^{\gamma_p - 1} h_p) & \cdots & L_{g_p}(L_f^{\gamma_p - 1} h_p) \end{bmatrix} \tag{2.10}$$

Then equations (2.9) may be written as

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_p^{(\gamma_p)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \tag{2.11}$$

If $A(x) \in \mathbf{R}^{p \times p}$ is bounded away from singularity, the state feedback control law

$$u = -A(x)^{-1} \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + A(x)^{-1} v \tag{2.12}$$

yields the closed loop decoupled, linear system

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_p^{(\gamma_p)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \tag{2.13}$$

Once linearization has been achieved, any further control objective such as model matching, pole placement, tracking may be easily met. The feedback law (2.12) is referred to as a *static-state feedback linearizing control law.*

If $A(x)$ defined in (2.10) is singular, linearization may still be achieved using dynamic state feedback. To keep the notation from proliferating we review the methods in the case that $p = 2$ (two inputs, two outputs). Suppose that $A(x)$ has rank 1 for all x. Using elementary column operations we may compress $A(x)$ to one column

$$A(x)T(x) = \begin{bmatrix} \tilde{a}_{11}(x) & 0 \\ \tilde{a}_{21}(x) & 0 \end{bmatrix}$$

with $T(x) \in \mathbb{R}^{2\times 2}$ a non-singular matrix. Now defining the new inputs $w = T^{-1}(x)u$, (2.11) reads

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ y_2^{(\gamma_2)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ L_f^{\gamma_2} h_2 \end{bmatrix} + \begin{bmatrix} \tilde{a}_{11}(x) \\ \tilde{a}_{21}(x) \end{bmatrix} w_1 \tag{2.14}$$

Also (2.7) now reads

$$\dot{x} = f(x) + \tilde{g}_1(x)w_1 + \tilde{g}_2(x)w_2 \tag{2.15}$$

where

$$\begin{bmatrix} \tilde{g}_1 & \tilde{g}_2 \end{bmatrix} = \begin{bmatrix} g_1 & g_2 \end{bmatrix} T$$

Differentiating the equations in (2.14) and using (2.15), we get

$$\begin{bmatrix} y_1^{\gamma_1+1} \\ y_2^{\gamma_2+1} \end{bmatrix} = \begin{bmatrix} L_f^{(\gamma_1+1)} h_1 + L_{\tilde{g}_1} L_f^{\gamma_1} h_1 w_1 + L_f \tilde{a}_{11} w_1 + L_{\tilde{g}_1} \tilde{a}_{11} w_1^2 \\ L_f^{(\gamma_2+1)} h_2 + L_{\tilde{g}_1} L_f^{\gamma_2} h_2 w_1 + L_f \tilde{a}_{21} w_1 + L_{\tilde{g}_1} \tilde{a}_{21} w_1^2 \end{bmatrix} \qquad c(x,w_1)$$

$$+ \begin{bmatrix} \tilde{a}_{11} & L_{\tilde{g}_2} L_f^{\gamma_1} h_1 + L_{\tilde{g}_2} \tilde{a}_{11} w_1 \\ \tilde{a}_{21} & L_{\tilde{g}_2} L_f^{\gamma_2} h_2 + L_{\tilde{g}_2} \tilde{a}_{21} w_1 \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ w_2 \end{bmatrix}$$

Note the appearance of the control term $\dot{w}_1$. Specifying $\dot{w}_1$ is equivalent to the placement of an integrator before w_1. Defining the coefficient matrix of $\dot{w}_1$, w_2 to be

$$B(x,w_1) := \begin{bmatrix} \tilde{a}_{11} & L_{\tilde{g}_2} L_f^{\gamma_1} h_1 + L_{\tilde{g}_2} \tilde{a}_{11} w_1 \\ \tilde{a}_{21} & L_{\tilde{g}_2} L_f^{\gamma_2} h_2 + L_{\tilde{g}_2} \tilde{a}_{21} w_1 \end{bmatrix} \tag{2.17}$$

we see that if $B(x,w_1)$ is bounded away from singularity then the control law

$$\begin{bmatrix} \dot{w}_1 \\ w_2 \end{bmatrix} = -B^{-1}(x,w_1)\, c(x,w_1) + B^{-1}(x,w_1) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2.18}$$

yields the linearized system

$$\begin{bmatrix} y_1^{(\gamma_1+1)} \\ y_2^{(\gamma_2+1)} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2.19}$$

The control law (2.18) is a *dynamic state feedback linearizing* decoupling control law. In the instance that $B(x,w_1)$ is singular, the foregoing procedure may be repeated on $B(x,w_1)$. The procedure ends in finitely many steps provided that the system is right invertible (for details, see Descusse and Moog [1985]).

2.2 Minimum Phase Non-linear Systems

We briefly review the definitions of minimum phase non-linear systems due to Isidori and co-workers (Byrnes and Isidori [1984], Isidori and Moog [1986]).

2.2.1. The Single-Input Single-Output Case

The theory is much simpler for the single input, single-output case. We recall the following definition:

Def. The system (2.1) is said to have *strong relative degree* γ if

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{\gamma-2} h(x) \equiv 0$$

and

$$L_g L_f^{\gamma-1} h(x) \neq 0 \quad \forall\, x \in \mathbf{R}^n.$$

Thus the system (2.1) is said to have strong relative degree γ if at each $x \in \mathbf{R}^n$ the output y needs to be differentiated γ times before terms involving the input appear on the right hand side as in (2.5), (2.6) above.

If a system has strong relative degree γ, it is easy to verify that at each $x^0 \in \mathbf{R}^n$ there exists a neighborhood U^0 of x^0 such that the mapping

$$T : U^0 \rightarrow \mathbf{R}^n$$

defined as

$$\begin{aligned} T_1(x) &= z_{11} = h(x) \\ &\vdots \\ T_2(x) &= z_{12} = L_f h(x) \\ \\ T_\gamma(x) &= z_{1\gamma} = L_f^{\gamma-1} h(x) \end{aligned}$$

with

$$dT_i(x)\, g(x) = 0 \quad \text{for} \quad i = \gamma+1, \ldots, n \ .$$

is a diffeomorphism onto its image. If we set $z_2 = (T_{\gamma+1}, \cdots, T_n)^T$, it follows that the equations (2.1) may be written in the *normal form* as

$$
\begin{aligned}
\dot{z}_{11} &= z_{12} \\
&\vdots \\
\dot{z}_{1\gamma-1} &= z_{1\gamma} \\
\dot{z}_{1\gamma} &= f_1(z_1,z_2) + g_1(z_1,z_2)u \\
\dot{z}_2 &= \psi(z_1,z_2)
\end{aligned}
\tag{2.20}
$$

$$
y = z_{11} \tag{2.21}
$$

In equation (2.20) above $f_1(z_1,z_2)$ represents $L_f^{\gamma}h(x)$ and $g_1(z_1,z_2)$ represents $L_g L_f^{\gamma-1}h(x)$. Now if x = 0 is an equilibrium point of the undriven system (i.e., $f(0) = 0$) and $h(0) = 0$, then the dynamics

$$
\dot{z}_2 = \psi(0,z_2) \tag{2.22}
$$

are referred to as the *zero-dynamics.*

Remark The dynamics are referred to as the zero dynamics since they are the dynamics which are made unobservable by state feedback. It might help the reader to note that the linearizing state feedback law is the nonlinear equivalent of placing some of the closed loop poles at the zeros of the system, thereby rendering them unobservable.

Note that the subset

$$
L = \{x \in U^0 : h(x) = L_f h(x) = \cdots = L_f^{\gamma-1}h(x) = 0\} = \{x \in U^0 : z_1 = 0\}
$$

can be made invariant by choosing

$$
u = \frac{1}{g_1(z_1,z_2)}\,(-f_1(z_1,z_2) + v) \ . \tag{2.23}
$$

The dynamics of (2.22) are the dynamics on this subspace. The nonlinear system (2.1) is said to be *minimum phase* if the zero-dynamics are asymptotically stable.

Remark Note that the previous analysis identifies the normal form (2.20)-(2.21) and the zero-dynamics (2.22) only *locally,* around any point x^0 of $\mathbf{R}^n$. Recent work of Byrnes and Isidori [1987b], has identified necessary and sufficient condition for the existence of a globally defined normal form. They have shown that a global version of the notion of zero dynamics is that of a dynamical system evolving on the smooth submanifold of $\mathbf{R}^n$:

$$L = \{x \in \mathbf{R}^n : h(x) = L_f h(x) = \cdots = L_f^{\gamma-1} h(x) = 0\}$$

and hereby defined by the vector field:

$$\bar{f}(x) = f(x) - \frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)} g(x) \qquad x \in L$$

(note that this is a vector field of L because $\bar{f}(x)$ is tangent to L). If L is connected and the zero dynamics is globally asymptotically stable (i.e., if the system is *globally minimum phase), then the normal forms are globally* defined if and only if the vector fields:

$$\bar{g}(x), ad_{\bar{f}}\bar{g}(x), \cdots, ad_{\bar{f}}^{\gamma-1}\bar{g}(x)$$

are complete, where:

$$\bar{g}(x) = \frac{1}{L_g L_f^{\gamma-1} h(x)} g(x), \quad \bar{f}(x) = f(x) - \frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)} g(x)$$

Note that this condition can be guaranteed by requiring that the vector fields in question are globally Lipschitz continuous, for example. In this paper we systematically assume global minimum phase property and the existence of globally defined normal forms.

An interesting application of the notion of normal form and minimum phase property is the following one. Assume the control v in (2.23) is chosen so that $y(t)$ tracks $y_M(t)$, i.e.,

$$\gamma = y_M^{(\gamma)} + \alpha_1(y_M^{(\gamma-1)} - y^{(\gamma-1)}) \cdots + \alpha_\gamma(y_M - y) \tag{2.24}$$

with $\alpha_1, \cdots, \alpha_\gamma$ chosen so that $s^\gamma + \alpha_1 s^{\gamma-1} + \cdots + \alpha_\gamma$ is a Hurwitz polynomial. It is easy to see that this control results in asymptotic tracking and bounded state z_1 (or equivalently $y, \dot{y}, \cdots, y^{\gamma-1}$) provided $y_M, \dot{y}_M, \cdots, y_M^{(\gamma-1)}$ are bounded.

Proposition 2.1. (Bounded Tracking in Minimum Phase Systems)

Assume that the new dynamics of the nonlinear system (2.1) or equivalently (2.20), (2.21) as defined in (2.22) are *exponentially-stable*. Further assume that $\psi(z_1,z_2)$ in (2.20) is Lipschitz in z_1, z_2. Then the control law (2.24) results in bounded tracking (i.e., $x \in \mathbf{R}^n$ bounded and $y(t) \rightarrow y_M(t)$), provided that $y_M, \dot{y}_M, \cdots, y_M^{(\gamma-1)}$ are bounded.

Proof. From the foregoing discussion it only remains to show that z_2 is bounded. We accomplish this using a converse theorem of Lyapunov (see Hahn [1967]). This proof technique has also been used in Bodson and Sastry [1984].

First since (2.22) is exponentially stable and ψ is Lipschitz in z_2 a converse Lyapunov theorem implies that $\exists\, V(z_2)$ such that

$$\begin{aligned} \alpha_1 |z_2|^2 &\le V(z_2) \le \alpha_2 |z_2|^2 \\ \frac{dV}{dz_2}\cdot\psi(0,z_2) &\le -\alpha_3 |z_2|^2 \\ \left|\frac{dV}{dz_2}\right| &\le \alpha_4 |z_2| \end{aligned} \tag{2.25}$$

Now the control law (2.24) yields bounded z_1, i.e.,

$$|z_1(t)| \le K \quad \forall t \tag{2.26}$$

Using (2.25) in (2.20) yields

$$\begin{aligned} \dot{V}(t) = \frac{dV}{dz_2}\psi(z_1, z_2) &\le -\alpha_3 |z_2|^2 + \frac{dV}{dz_2}\left(\psi(z_1,z_2) - \psi(0,z_2)\right) \\ &\le -\alpha_3 |z_2|^2 + \alpha_4 KL\, |z_2| \end{aligned} \tag{2.27}$$

with L representing the Lipschitz constant of $\psi(z_1,z_2)$ w.r.t. z_1. It is now easy to see that

$$\dot{V} \le 0 \ \text{ for } \ |z_2| \ge \frac{\alpha_4 KL}{\alpha_3}$$

Using this along with the bounds in (2.25) it is easy to establish that z_2 is bounded.

Remarks: (1) Proposition (2.1) establishes that a bounded input to the exponentially stable, unobservable dynamics yields a bounded state trajectory z_2.

(2) The hypothesis of Proposition (2.1) calls for a strong form of stability--exponential stability; in fact, counter examples to the Proposition exist if the zero-dynamics are not exponentially stable, for example, if some of the eigenvalues of $\frac{\partial}{\partial z_2}\psi(0,z_2)$ lie on the $j\omega$-axis.

(3) The hypotheses of Proposition (2.1) can, however, be weakened substantially by requiring only that all trajectories of (2.22) are eventually attracted to a compact set, for instance, by requiring that

$$z_2^T \psi(0,z_2) \le -\alpha |z_2|^2 \text{ for } |z_2| \ge R \tag{2.28}$$

Condition (2.28) can be thought of as being an attractivity condition--a simple contradiction argument involving

$$\frac{d}{dt}|z_2|^2 = z_2^T \psi(0,z_2)$$

should convince the reader that (2.28) guarantees that all trajectories of (2.22) eventually enter a ball of radius R. With condition (2.28) replacing the exponential stability hypotheses of Proposition (2.1) and the Lipschitz dependencies as before we see that the proof goes through. Condition (2.28) itself can be restated in a form reminiscent of (2.28) involving a more general Lyapunov function $V(z_z)$ with the weakening that

$$\frac{dV}{dz_2}\cdot\psi(0,z_2) \le -\alpha_3 |z_2|^2 \text{ only for } |z_2| \ge R\ . \tag{2.29}$$

Thus bounded tracking only requires that the conditions (2.25) only hold outside a ball of radius R. We refer to this condition as *exponential boundedness of the zero dynamics.*

2.2.2. The Multi-Input Multi-Output Case

Definitions of minimum phase for the square multi-input, multi-output nonlinear systems parallel the development on the SISO case above only if the matrix $A(x)$ defined in (2.10) is nonsingular for all $x \in \mathbb{R}^n$. In this case, locally around any point x^0 of $\mathbb{R}^n$, a diffeomorphism $(z_1, z_2) = T(x)$ can be defined, with:

$$z_1^T = (h_1, L_f h_1, \cdots, L_f^{\gamma_1 - 1} h_1, h_2, \cdots, L_f^{\gamma_2 - 1} h_2, \cdots, h_p, \cdots, L_f^{\gamma_p - 1} h_p)$$

In these coordinates the equations (2.1) read

$$\begin{aligned}
\dot{z}_{11} &= z_{12} \\
&\vdots \\
\dot{z}_{1\nu_1} &= f_1(z_1, z_2) + g_1(z_1, z_2)u \\
\dot{z}_{1\nu_1+1} &= z_{1\nu_1+2} \\
&\vdots \\
\dot{z}_{1m} &= f_m(z_1, z_2) + g_m(z_1, z_2)u \\
\dot{z}_2 &= \psi_1(z_1, z_2) + \psi_2(z_1, z_2)u
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
y_1 &= z_{11} \\
y_2 &= z_{1\nu_1+1} \\
&\vdots \\
y_m &= z_{1(m-\nu_p+1)}
\end{aligned} \tag{2.31}$$

In equation (2.28) above, $f_1(z_1, z_2)$ stands for $L_f^{\gamma_1} h_1(x)$, $g_1(z_1, z_2)$ for the first row of $A(x)$, $\cdots$ in the (z_1, z_2) coordinates. Note that the z_2 variables are driven by the input. Consequently a change is needed in the definition of zero-dynamics. Let $u^*(z_1, z_2)$ be the linearizing control, i.e.,

$$u^*(z_1, z_2) = -\begin{bmatrix} g_1(z_1, z_2) \\ \vdots \\ g_m(z_1, z_2) \end{bmatrix}^{-1} \begin{bmatrix} f_1(z_1, z_2) \\ \vdots \\ f_m(z_1, z_2) \end{bmatrix} \tag{2.32}$$

Using this control in the equations for z_2 and assuming as before that $0 \in \mathbf{R}^n$ is the equilibrium point of the undriven system (i.e., $f(0) = 0$) and $h_1(0) = \cdots = h_m(0) = 0$ we see that the subspace $\{(0, z_2)\} \subset \mathbf{R}^n$ is an invariant subspace and the zero dynamics are the dynamics of

$$\begin{aligned}
\dot{z}_2 &= \psi_1(0, z_2) + \psi_2(0, z_2)\, u^*(0, z_2) \\
&:= \psi(0, z_2)\,.
\end{aligned} \tag{2.33}$$

If $A(x)$ is nonsingular for all $x \in \mathbf{R}^n$, the global notion of zero dynamics and the conditions for the existence of normal forms are still similar to those illustrated in the SISO case. When global normal forms exist, then Proposition 2.1 can be easily verified to hold for tracking with bounded state variables if (2.33) is exponentially stable (and Lipschitz in z_1, z_2). Also the same remarks as those made after Proposition 2.1 hold for the case that the zero-dynamics are exponentially attractive.

If $A(x)$ is singular, the definition of zero dynamics is more subtle. As a matter of fact, there are different and non equivalent ways to extend the concept of "transmission zero," as pointed out in Isidori and Moog [1986], depending on which linear definition one chooses to generalize: (1) the dynamics associated with the subsystem that becomes unobservable when a certain state feedback is implemented (to the extent of maximizing unobservability), (2) the internal dynamics consistent with the constraint that the output is zero, (3) the dynamics of the inverse system. In particular, the notion that is behind (2) has some interesting featutes that render it particularly suitable for the design of stabilizing feedback (see Byrnes and Isidori [1987a]) and to study asymptotic tracking.

3. Adaptive Control of Linearizable Systems

In practical implementations of exactly linearizing control laws, the chief drawback is that they are based on exact cancellation of non-linear terms. If there is any uncertainty in the knowledge of the non-linear functions f and g, the cancellation is not exact and the resulting input-output equation is not linear. We suggest the use of parameter adaptive control to get asymptotically exact cancellation. The following simple example makes our philosophy clear.

3.1. The SISO Relative Degree One Case

Consider a SISO system of the form (2.1) with $L_g h(x) \neq 0$ (Relative Degree One). Further let $f(x)$ and $g(x)$ have the form

$$f(x) = \sum_{i=1}^{n_1} \theta_i^1 f_i(x) \tag{3.1}$$

$$g(x) = \sum_{j=1}^{n_2} \theta_j^2 g_j(x) \tag{3.2}$$

with θ_i^1, $i = 1, \cdots, n_1$; θ_j^2, $j = 1, \cdots, n_2$ unknown parameters and the $f_i(x)$, $g_j(x)$ known functions. At time t, our estimates of the functions f and g are respectively

$$\hat{f}(x) = \sum_{i=1}^{n_1} \hat{\theta}_i^1(t) f_i(x) \tag{3.3}$$

$$\hat{g}(x) = \sum_{j=1}^{n_2} \hat{\theta}_j^2(t) g_j(x) \tag{3.4}$$

with the $\hat{\theta}_i^1$, $\hat{\theta}_j^2$ standing for the estimates of the parameters θ_i^1, θ_j^2 respectively at time t. Consequently the control law u is replaced by

$$u = \frac{1}{\widehat{L_g h}} (-\widehat{L_f h} + v) \tag{3.5}$$

and $\widehat{L_g h}$, $\widehat{L_f h}$ are the estimates of $L_g h$, $L_f h$ respectively based on (3.3), (3.4), i.e.,

$$\widehat{L_f h} = \sum_{i=1}^{n_1} \hat{\theta}_i^1(t) L_{f_i} h \tag{3.6}$$

$$\widehat{L_g h} = \sum_{j=1}^{n_2} \hat{\theta}_j^2(t) L_{g_j} h \tag{3.7}$$

If we define $\theta \in \mathbf{R}^{n_1+n_2}$ to be the "true" parameter vector $(\theta^{1T}, \theta^{2T})^T$, $\hat{\theta} \in \mathbf{R}^{n_1+n_2}$ the parameter estimate and $\phi = \theta - \hat{\theta}$ the parameter error, then using u of (3.5) in equation (2.2) yields after some calculation.

$$\dot{y} = v + \phi^{1T} w_1 + \phi^{2T} w_2 \tag{3.8}$$

with

$$w_1 \in \mathbf{R}^{n_1} := \begin{bmatrix} L_{f_1} h \\ \vdots \\ L_{f_{n_1}} h \end{bmatrix} \tag{3.9}$$

and

$$w_2 \in \mathbf{R}^{n_2} := \begin{bmatrix} L_{g_1} h \\ \vdots \\ L_{g_{n_2}} h \end{bmatrix} \frac{(-\widehat{L_f h} + v)}{\widehat{L_g h}} \tag{3.10}$$

The control law used for tracking is

$$v = \dot{y}_M + \alpha(y_M - y)$$

and yields the following error equation relating $y - y_M := e$ to the parameter error $\phi = (\phi^{1T}, \phi^{2T})^T$:

$$\dot{e} + \alpha e = \phi^T w \quad . \tag{3.11}$$

$w \in \mathbf{R}^{n_1+n_2}$ is defined to be the concatenation of w_1, w_2. Now, it is easy to state the following theorem.

Theorem 3.1. (Adaptive Tracking)

Consider a minimum phase non-linear system of the form (2.1) with the assumptions on f, g as given in (3.3), (3.4). Define the control law

$$u = \frac{1}{\widehat{L_g h}} (-\widehat{L_f h} + \dot{y}_M + \alpha(y_M - y)) \tag{3.12}$$

Assume that $\widehat{L_g h}$ as defined by (3.7) is bounded away from zero. Then, if y_M is bounded the parameter update law

$$\dot{\phi} = -(y - y_M) w \tag{3.13}$$

yields bounded $y(t)$ asymptotically converging to $y_M(t)$. Further all state variables $x(t)$ of (2.1) are bounded.

Proof. The control law (3.12) yield the error equation

$$\dot{e} + \alpha e = \phi^T w \tag{3.11}$$

along with the update law

$$\dot{\phi} = -ew \quad . \tag{3.13}$$

The Lyapunov function $v(e, \phi) = \frac{1}{2} e^2 + \frac{1}{2} \phi^T \phi$ is decreasing along trajectories of (3.11),

$\dot{v}(e,\phi) = -\alpha e^2 \leq 0$; thereby establishing bounded e, ϕ. Also $\int_0^\infty e^2 dt < \infty$. However, to establish that $e \rightarrow 0$ as $t \rightarrow \infty$ we need to verify that e is uniformly continuous (or alternately that $\dot{e}$ is bounded). This in turn needs w, a continuous function of x (well defined since $L_g h$ is bounded away from zero) to be bounded. Now, note that bounded e, bounded y_M => y is bounded. From this and the minimum phase assumption (cf. Proposition 2.1) it follows that x is bounded. Hence w is bounded and e is uniformly continuous and so tends to zero as $t \rightarrow \infty$.

Remarks: (1) Prior bounds on the parameters θ^2 are frequently sufficient to guarantee that $L_g h$ is bounded away from zero. Several standard techniques exist in the literature for this purpose. [See Sastry and Bodson 1988]

(2) Theorem 3.1 makes no statement about parameter convergence. As is standard in the literature one can conclude from (3.11), (3.13) that e, ϕ both converge exponentially to zero if w is sufficiently rich, i.e., $\exists\ \alpha_1\ \alpha_2,\ \delta > 0$ such that

$$\alpha_1 I \geq \int_s^{s+\delta} w\, w^T dt \geq \alpha_2 I \tag{3.14}$$

The condition (3.14) is impossible to verify explicitly ahead of time since w is a function of x.

(3) It has recently become popular in the literature to not use adaptation (e.g., Marino [1988]) but to replace the control law of (3.12) by the "sliding mode" control law

$$u = \frac{1}{\widehat{L_g h}} (-\widehat{L_f h} + \dot{y}_M + k\ sgn(y_M - y)) \tag{3.15}$$

The error equation (3.11) is then replaced by one of the form

$$\dot{e} + k\, sgn\, e = d(t) \tag{3.16}$$

where $d(t)$ is a mismatch term which may be easily bounded using bounds on f_i, g_j and the ϕ_i's above. It is then easy to see that if $k > \sup_t |d(t)|$ then the error e goes to zero (in fact in finite time). This philosophy is not at odds with adaptation as described in Theorem 3.1 above. We feel that it can be used quite gainfully when the parameter error $\phi(t)$ is small. If however ϕ is large, the gain k is large resulting in unacceptable chatter, large control activity and other such undesirable behavior. Adaptation offers a less traumatic scheme of parameter tuning in this instance.

(4) It is important to note that here and in what follows the parameter update laws require knowledge of the state variables. This in turn is necessitated by the state-feedback linearization methodology.

3.2. **Extensions to Higher Relative Degree SISO Systems**

We first consider the extensions of the results of the previous section to SISO systems with relative degree γ, i.e., $L_g h = L_g L_f h = \cdots = L_g L_f^{\gamma-2} h \equiv 0$ with $L_g L_f^{\gamma-1} h \neq 0$. The non-adaptive linearizing control law then is of the form

$$u = + \frac{1}{L_g L_f^{\gamma-1} h} (-L_f^{\gamma} h + v) \tag{3.17}$$

If f an g are not completely known but of the form (3.1), (3.2), we need to replace $L_f^{\gamma} h$ and $L_g L_f^{\gamma-1} h$ by their estimates. We define these as follows

$$\widehat{L_f^{\gamma} h} := L_{\hat{f}}^{\gamma} h \tag{3.18}$$

$$\widehat{L_g L_f^{\gamma-1} h} := L_{\hat{g}} L_{\hat{f}}^{\gamma-1} h \tag{3.19}$$

For $\gamma \geq 2$, equations (3.18), (3.19) are not linear in the unknown parameters θ_i. For example,

$$L_f^2 h = \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \frac{\partial}{\partial x} \left[\frac{\partial h}{\partial x} f_j \right] f_i \theta_i^1 \theta_j^1 \tag{3.20}$$

and

$$L_g L_f h = \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} \frac{\partial}{\partial x} \left[\frac{\partial h}{\partial x} f_j \right] g_i \theta_i^2 \theta_j^1 \ . \tag{3.21}$$

and so on. The development of the preceding section could easily be repeated if we defined each of the parameter products to be a new parameter in which case the $\theta_i^1 \theta_j^1$ and $\theta_i^2 \theta_j^1$ of (3.20) and (3.21) are parameters. Let $\Theta \in \mathbf{R}^k$ be the k-(large !) dimensional vector of parameters θ_i^1, θ_j^2, $\theta_i^1\ \theta_j^2$, $\theta_i^1\ \theta_j^1$, ... Thus for example if $\gamma = 3$ then Θ contains θ_i^1, θ_j^2, $\theta_i^1\theta_j^1$, $\theta_i^1\theta_j^1\theta_k^1$, $\theta_i^1\theta_j^2$, $\theta_i^1\theta_j^1\theta_k^2$. Now for the purposes of tracking the control law to be implemented is

$$v = y_M^{(\gamma)} + \alpha_1(y_M^{(\gamma-1)} - y^{(\gamma-1)}) + \cdots + \alpha_\gamma(y_M - y)$$

where $\dot{y} = L_f h$, $\ddot{y} = L_f^2 h$, etc. are state feedback terms. In the absence of precise information about $L_f h$, $L_f^2 h$, ... etc the tracking law to be implemented is

$$\hat{v} = y_M^{(\gamma)} + \alpha_1(y_M^{(\gamma-1)} - \widehat{L_f^{\gamma-1}h}) + \cdots + \alpha_\gamma(y_M - y)\ . \tag{3.22}$$

The overall adaptive control law then is

$$u = \frac{1}{\widehat{L_g L_f^{\gamma-1}h}}\,(-\widehat{L_f^{\gamma}h} + \hat{v})\ . \tag{3.23}$$

Using this yields the error equation, (with $\Phi := \Theta - \hat{\Theta}$ representing the parameter error)

$$e^{(\gamma)} + \alpha_1 e^{(\gamma-1)} + \cdots + \alpha_\gamma e = \Phi^T W_1 + \Phi^T W_2 \tag{3.24}$$

The two terms on the right hand side arise respectively from the mismatch between the ideal linearizing law and the actual linearizing law and the mismatch between the ideal tracking control v and the actual tracking control $\hat{v}$. For definiteness, consider the case that $\gamma = 2$ and $n_1 = n_2 = 1$. Then with $\Theta^T = [\theta^1, \theta^2, (\theta^1)^2, \theta^1\theta^2]$ we get

$$W_1^T = \left[0 \;\vdots\; 0 \;\vdots\; L_{f_1}^2 h \;\vdots\; L_{g_1}L_{f_1}h \frac{(-\widehat{L_f^2 h} + \hat{v})}{\widehat{L_g L_f h}} \right] \tag{3.25}$$

and

$$W_2^T = [\alpha_1 L_{f_1} h \;\vdots\; 0 \;\vdots\; 0\ 0]\ . \tag{3.26}$$

Note that W_1 and W_2 can be added to get a new regressor vector W. It is of interest to note that θ^2 cannot be explicitly identified in this case since the regressor multiplying it is zero. Also note that W is a function of both x, y_M $\hat{\theta}$. Note that terms involving only θ^2 or even any products of the θ^2 are absent and so may be dropped from the vector Θ.

We keep the form (3.24) of the error equation. Note that $s^{\gamma} + \alpha_1 s^{\gamma-1} + \cdots + \alpha_{\gamma}$ is Hurwitz by choice of tracking control. Now for the purposes of adaptation we need a signal of the form

$$e_1 = \beta_1 e^{(\gamma-1)} + \cdots + \beta_{\gamma} e \tag{3.27}$$

with the transfer function

$$\beta_1 s^{\gamma-1} + \cdots + \beta_{\gamma} | s^{\gamma} + \alpha_1 s^{\gamma-1} + \cdots + \alpha_{\gamma} \tag{3.28}$$

strictly positive real. Indeed if such a signal e_1 were measurable the basic tracking theorem would follow immediately from arguments similar to the linear arguments. The difficulty with constructing the signal (3.27) is that $\dot{e}$, $e^{(2)}$, $\cdots$ $e^{(\gamma-1)}$ are not measurable since

$$\dot{e} = \dot{y}_M - L_f h$$
$$\ddot{e} = \ddot{y}_M - L_f^2 h$$

and so on with the $L_f^i h$ not explicitly available.

Motivated by the linear case where a so-called *augmented error* scheme is necessitated we define the augmented error. Some notation is needed at this point. Define

$$M(s) = \frac{1}{s^{\gamma}+\alpha_1 s^{\gamma-1}+\cdots+\alpha_{\gamma}} . \tag{3.29}$$

Then the equation (3.29) may be written as

$$e = M(s) \cdot \Phi^T W \tag{3.30}$$

with the convention (standard in the adaptive control literature) that the hybrid notation (3.30) refers to the convolution between the inverse Laplace transform of $M(s)$ and $\Phi^T W$. Also the exponentially decaying initial condition terms are dropped since they do not alter the stability proof (for these points and a review of linear adaptive control we refer our forthcoming book Sastry and Bodson [1988], or the survey paper Sastry [1984]). Define the polynomial

$$L(s) = M^{-1}(s) \tag{3.31}$$

and the augmented error

$$e_1 = e + (\hat{\Theta}^T L^{-1}(s)W - L^{-1}(s)\hat{\Theta}^T W) \tag{3.32}$$

Note that the last two terms are not equal and refer respectively to each component of W being filtered by $L^{-1}(s)$ before being multiplied by $\hat{\Theta}^T$ and filtering of $\hat{\Theta}^T W$ by $L^{-1}(s)$. If $\hat{\Theta}$ were indeed constant they would be identical. This observation enables us to rewrite

$$e_1 = e + (\Phi^{T} L^{-1}(s)W - L^{-1}(s)\Phi^{T} W) . \tag{3.33}$$

Note that e_1 in the form (3.32) can be obtained from measurable signals unlike (3.33), since Φ is not available. However (3.33) is critical to our analysis, since we may use (3.30) in (3.33) to get

$$e_1 = \Phi^T L^{-1}(s)W . \tag{3.34}$$

For convenience we will denote

$$\xi := L^{-1}(s)W \in \mathbf{R}^k . \tag{3.35}$$

Error equation (3.34) is key to the choice of the identification algorithm. Here is one choice of parameter update law:

$$\dot{\hat{\Theta}} = \dot{\Phi} = \frac{-e_1 \xi}{1+\xi^T \xi} \tag{3.36}$$

(3.36) is referred to as a *normalized gradient-type algorithm* (unlike the unnormalized update law (3.13)). Some properties of $\dot{\Phi}$, e_1 follow immediately with no assumptions on the boundedness of ξ. In what follows we will use the following notation

(i) β is a *generic* $L_2 \cap L_\infty$ function which *goes to zero as to* $t \to \infty$

(ii) γ is a *generic* $L_2 \cap L_\infty$ function

(iii) K is a (large) *bound*

(iv) $\| z \|_t$ will refer to the norm $\sup_{\tau \le t} |z(t)|$, the *truncated* L_∞ *norm.*

Proposition 3.2 (Properties of the Identifier)

Consider the error equation

$$e_1 = \Phi^T \xi \tag{3.35}$$

with the update law

$$\dot{\Phi} = \frac{-e_1 \xi}{1+\xi^T \xi} \tag{3.36}$$

Then

$$\begin{aligned} &\Phi \in L_\infty, \dot{\Phi} \in L_2 \cap L_\infty \text{ and} \\ &|\Phi^T \xi(t)| \le \gamma(1+ \| \xi \|_t) \quad \forall t \,. \end{aligned} \tag{3.37}$$

Proof: Consider the Lyapunov function

$$V(\Phi) = \Phi^T \Phi$$

Then we have

$$\dot{V} = \frac{-2e_1^2}{1+\xi^T \xi} \le 0 \tag{3.38}$$

so that we get that $\Phi \in L_\infty$. since $\int \dot{V} dt < \infty$, we also have that $e_1/(1+\xi^T \xi)^{1/2} \in L_2$. Further since

$$\dot{\Phi} = \frac{-\xi \xi^T \Phi}{1+\xi^T \xi} \tag{3.39}$$

we have that $\dot{\Phi} \in L_\infty$. Also since

$$|\dot{\Phi}|^2 = \frac{e_1^2}{1+\xi^T\xi} \cdot \frac{\xi^T\xi}{1+\xi^T\xi} \tag{3.40}$$

we see that $\dot{\Phi} \in L_2$ (the first term in (3.40) is integrable see (3.38) above and the second bounded). Finally, define

$$\gamma = \frac{e_1}{\sqrt{1+\xi^T\xi}} \cdot \frac{\sqrt{1+\xi^T\xi}}{1+\|\xi\|_t} .$$

The first term is in $L_2 \cap L_\infty$ and the second bounded. Thus γ is indeed in $L_2 \cap L_\infty$. Hence (3.37) follows.

■

Remarks: The conclusions of Proposition (3.2) are generic identifier properties. Other identifiers such as the normalized least squares identifier also yield these properties...for details see [Sastry and Bodson [1988].

We are now ready to state and prove the main theorem:

Theorem (3.3) (Basic Tracking theorem for SISO for Relative Degree Greater than 1)

Consider the control law of (3.22), (3.23) applied to an exponentially minimum phase nonlinear system with parameter uncertainty as given in (3.1), (3.2).

If $y_M, \dot{y}_M, \cdots, y_M^{(\gamma-1)}$ are bounded, $\widehat{L_g L_f^{(\gamma-1)} h}$ is bounded away from zero, $f, g, h, L_f^k h$, $L_g L_f^k h$ are Lipschitz continuous functions of x, and $W(x, \hat{\theta})$ has bounded derivatives in $x, \hat{\theta}$ *then* the parameter update law

$$\dot{\Phi} = \frac{-e\,\xi}{1+\xi^T\xi} \tag{3.36}$$

with

$$\xi = L^{-1}(s)W \tag{3.35}$$

yields bounded tracking (i.e., x is bounded and $y \to y_M$ as $t \to \infty$).

Proof: The proof uses three technical lemmas proven in Sastry-Bodson [1988] and summarized in the

Appendix.

Step 1. Bound on the Error Augmentation

By the Swapping Lemma (Lemma 3 of the Appendix) with L^{-1} (playing the role of H) = $c(sI-A)^{-1}b$.

$$\Phi^T L^{-1} W - L^{-1}\Phi^T W = -c(sI-A)^{-1}\{[(sI-A)^{-1}bW^T]\dot{\Phi}\}\ . \tag{3.41}$$

Using the fact that $\dot{\Phi} \in L_2$ and that $(sI-A)^{-1}b$ is stable (since L^{-1} is stable) we get

$$|[(sI-A)^{-1}bW^T]\dot{\Phi}| \le \gamma \| W \|_t + \gamma\ . \tag{3.42}$$

Now using Lemma 2 of the appendix and the fact that $c(sI-A)^{-1}$ is strictly proper we get

$$|\Phi^T L^{-1} W - L^{-1}\Phi^T W| \le \beta \| W \|_t + \beta\ . \tag{3.43}$$

Step 2. Regularity of W, $\Phi^T W$

Note that the differential equation for

$$z_1 = (y, \dot{y}, \cdots, y^{(\gamma-1)})^T \text{ is}$$

$$z_1 = M(s)\begin{bmatrix} 1 \\ \vdots \\ s^{\gamma-1} \end{bmatrix}\Phi^T W + \begin{bmatrix} y_M \\ \vdots \\ y_M^{(\gamma-1)} \end{bmatrix} \tag{3.44}$$

Since Φ is bounded and $y_M, \cdots, y_M^{(\gamma-1)}$ are bounded by hypothesis and $s^k M(s)$ are all proper stable transfer functions we have that

$$\| z_1 \|_t \le K \| W \|_t + K\ . \tag{3.45}$$

Using (3.45) in the exponentially minimum phase zero dynamics

$$\dot{z}_2 = \psi(z_1, z_2) \tag{3.46}$$

we get

$$\| z_2 \|_t \;\leq\; K \, \| W \|_t + K \, . \tag{3.47}$$

Combining (3.45), (3.46) and noting that x is a diffeomorphism of z_1, z_2 we see that

$$\| x \|_t \;\leq\; K \, \| W \|_t + K$$

and

$$\| \dot{x} \|_t \;\leq\; K \, \| W \|_t + K \, . \tag{3.48}$$

Using the facts that $\| \partial W / \partial x \|$ and $\| \partial W / \partial \hat{\theta} \|$ are bounded and (3.48) we get

$$\| \dot{W} \|_t \;\leq\; K \, \| W \|_t + K \, . \tag{3.49}$$

thus W is regular => $\xi = L^{-1} W$ is regular as well (since L^{-1} is stable). A similar calculation yields. $\Phi^T W$ to be regular as well. For consider,

$$\frac{d}{dt} (\Phi^T W) = \dot{\Phi}^T W + \Phi^T \dot{W} \, . \tag{3.50}$$

Using (3.49) and $\Phi , \dot{\Phi} \in L_\infty$ we get

$$\| \frac{d}{dt} (\Phi^T W) \|_t \;\leq\; K \, \| W \|_t + K \, . \tag{3.51}$$

But from equations (3.44), (3.47)

$$\| x \|_t \;\leq\; K \, \| \Phi^T W \|_t + K \tag{3.52}$$

so that

$$\| W \| \;\leq\; K \, \| \Phi^T W \|_t + K \, . \tag{3.53}$$

Combining (3.53) with (3.51) yields the regularity of $\Phi^T W$. From the regularity of $\xi, \Phi^T W$ one can establish that $\dfrac{\Phi^T \xi}{1+ \| \xi \|_t}$ has bounded derivative and so is uniformly continuous. Since by (3.37)

$$\frac{|\Phi^T \xi(t)|}{1+\|\xi\|_t} \in L_1 \cap L_\infty$$

we see that it infact goes to zero as $t \to \infty$. (A uniformly continuous L_1 function tends to zero as $t \to \infty$.) Thus

$$|\Phi^T \xi(t)| \;\le\; \beta(1+\|\xi\|_t)\,. \tag{3.54}$$

Step 3. Final Estimates

$$e = e_1 + \Phi^T L^{-1} W - L^{-1}\Phi^T W$$

is the equation relating the true output error to the augmented error. Using (3.43) we get

$$|e| \;\le\; |e_1| + \beta\,\|W\|_t + \beta\,.$$

Using (3.53) we get

$$|e| \;\le\; |e_1| + \beta\,\|\Phi^T W\|_t + \beta\,. \tag{3.55}$$

Apply the BOBI lemma (Lemma 1) of the Appendix to

$$e = M(s)\,\Phi^T W$$

along with the established regularity of $\Phi^T W$ to get

$$\|\Phi^T W\|_t \;\le\; K|e| + K\,. \tag{3.56}$$

Using (3.56) in (3.55) we get

$$|e| < |e_1| + \beta\,\|e\|_t + \beta\,. \tag{3.57}$$

Using (3.54) for $e_1 = \Phi^T \xi$ we get

$$|e| \;\le\; \beta\,\|e\|_t + \beta + \beta\,\|\xi\|_t \tag{3.58}$$

ξ is related to W by stable filtering. Hence

$$\| \xi \|_t \leq K \| W \|_t + K . \tag{3.59}$$

Using the estimate (3.53) followed by (3.56) in (3.55) we see that (3.58) may be written as

$$| e | \leq \beta \| e \|_t + \beta . \tag{3.60}$$

Since $\beta \rightarrow 0$ as $t \rightarrow \infty$ we see from (3.60) that e goes to zero as $t \rightarrow \infty$. This in turn can be verified to yield bounded W, x, etc.

■

Remarks

(1) The parameter update law appears not to take into account *prior parameter* information such as the mutual existence of θ_i, θ_j, $\theta_i\theta_j$ and so on. It is important, however, to note that the best estimate of $\theta_i\theta_j$ in the transient period may not be $\hat{\theta}_i\hat{\theta}_j$. If, however, the parameters are close to their correct values such information is useful. But, since parameter convergence is not guaranteed in the Proof of Theorem (3.3), it may not be a good idea to constrain the estimate of $\theta_i\theta_j$ to be close to $\hat{\theta}_i\hat{\theta}_j$.

The preceding remarks are not designed, however, to ameliorate completely one's concerns that the number of parameters increases very rapidly with γ.

(2) In several problems it turns out that $L_f^\gamma h$ and $L_g L_f^{\gamma-1} h$ depend linearly on some unknown parameters. It is then clear that the development of the previous theorem can be carried through.

(3) Thus far we have only assumed parameter uncertainty in f and g; but not in h. It is not hard to see that if h depends linearly on unknown parameters then we can mimic the aforementioned procedure quite easily.

(4) Parameter convergence can be guaranteed in Theorem (3.3) above if W is sufficiently rich in the sense stated after Theorem (3.1).

(5) If some of the $\theta_i s$ are known they can be replaced in the algorithm by these true values and adaptation for them turned off.

(6) The parameter update law is a state feedback law as before.

3.3. Adaptive Control of MIMO Systems Decouplable by Static State Feedback

From the preceding discussion it is easy to see how the linearizing, decoupling static state feedback control law for square systems (minimum phase) can be made adaptive--by replacing the control law of (2.12) by

$$u = + \hat{A}(x)^{-1}\left[- \begin{bmatrix} \widehat{L_f^{\gamma_1} h_1} \\ \vdots \\ \widehat{L_f^{\gamma_p} h_p} \end{bmatrix} + \hat{v} \right] \tag{3.61}$$

Note that if $A(x)$ is invertible then the linearizing control law is also the decoupling control law. Thus if $A(x)$ and the $L_f^{\gamma_i} h_i$ depend linearly on certain unknown parameters the schemes of the previous sections (those of section (3.1) if $\gamma_1 = \gamma_2 = \cdots = \gamma_p = 1$ and those of section 3.2 in other cases) can be readily adapted. The details are more notationally cumbersome than insightful. Instead we will illustrate our theory on a important class of such systems which partially motivate this present work (see Craig, Hsu, and Sastry [1987])--the adaptive control of rigid link robot manipulators. We sketch only a few of the details of the application relevant to our present context; the interested reader is referred to the paper referenced above. The example (unfortunately) has no zero dynamics. If $q \in \mathbb{R}^n$ represents the joint angles of a rigid link robot manipulator its dynamics may be written as

$$M(q)\ddot{q} + C(q,\dot{q}) = u \tag{3.62}$$

In (3.62) $M(q) \in \mathbb{R}^{n \times n}$ is the positive definite inertia matrix, $C(q,\dot{q})$ represent the Coriolis, gravitational, and friction terms and $u \in \mathbb{R}^n$ represents the control input to the joint motors (torques). In applications $M(q)$ and $C(q,\dot{q})$ are not known exactly but depend linearly on some unknown parameters (such as payloads, frictional coefficients, ...) i.e.,

$$\begin{aligned} M(q) &= \sum_{i=1}^{n_1} \theta_i^2 M_i(q) \\ C(q,\dot{q}) &= \sum_{j=1}^{n_2} \theta_j^1 C_j(q_1\dot{q}) \end{aligned} \tag{3.63}$$

Writing the equations (3.62) in state space form with $x^T = (q^T, \dot{q}^T)$ and $y = q$; we see that the system is decouplable in the sense of Section 2 with $\gamma_1 = \cdots = \gamma_n = 2$,

$$A(x) = M^{-1}(q)$$

and

$$\begin{bmatrix} L_f^{\gamma_1} h_1 \\ L_f^{\gamma_n} h_n \end{bmatrix} = -M^{-1}(q) C(q, \dot{q}) \tag{3.64}$$

with decoupling control law given by

$$u = C(q, \dot{q}) + M(q) v \tag{3.65}$$

Note that the quantities in equation (3.64) depend on a complicated fashion on the unknown parameters θ^1, θ^2 while the equation (3.65) depends on them *linearly*. For the sake of tracking v is chosen to be

$$v = \ddot{q}_M + \alpha_1(\dot{q}_M - \dot{q}) + \alpha_2(q_M - q) \tag{3.66}$$

and the overall control law (3.65), (3.66) is referred to as the *computed torque* scheme. To make this adaptive the law (3.65) is replaced by

$$u = \hat{C}(q, \dot{q}) + \hat{M}(q) v \ . \tag{3.67}$$

resulting in the error equation where $e = q_M - q$

$$\ddot{e} + \alpha_1 \dot{e} + \alpha_2 e = \hat{M}^{-1}(q) \sum_{j=1}^{n_1} \phi_j^1 C_j(q, \dot{q}) + \hat{M}^{-1}(q) \sum_{i=1}^{n_2} M_i(q) \ddot{q} \phi_i^2 \tag{3.68}$$

This may be abbreviated as

$$\ddot{e} + \alpha_1 \dot{e} + \alpha_2 e = W \Phi \tag{3.69}$$

where $W \in \mathbb{R}^{n \times (n_1 + n_2)}$ is a function of q, $\dot{q}$, and $\ddot{q}$ and Φ is the parameter error vector. The parameter update law

$$\dot{\Phi} = -W^T E_1 \tag{3.70}$$

where $E_1 = \dot{e} + \alpha_3 e$ is chosen so that $\frac{s+\alpha_3}{s^2+\alpha_1 s+\alpha_2}$ is strictly positive real can be shown to yield bounded tracking. The *error augmentation of Section 3.2 is not necessary in* this application since both $y, \dot{y}$ $(q, \dot{q})$ are available as states so that $L_f h_i$ do not have to be estimated. Note that the system is minimum phase--there are infact no zero dynamics at all. It is however unfortunate that the signal W is a function of $\ddot{q}$ -- this, however, is caused by the form of the equations. As in the previous cases it is important to keep $\hat{M}(q)$ from becoming singular, using prior parameter bounds.

We end this section with two remarks:

(1) Adaptive control of square multivariable non-linear systems decouplable by static state feedback is straightforward...it is however important to have the linearizing control depend linearly on parameters.

(2) Adaptive control of nonlinear systems not decouplable by static state feedback is not easy or obvious...some of the reasons are also discussed in the next section.

4. Concluding Remarks

We have presented some initial results on the use of parameter adaptive control for obtaining asymptotically exact cancellation in linearizing control laws for a class of continuous time systems decouplable by static state feedback. The extension to continuous time systems not decouplable by static state feedback is not as obvious for two reasons:

(i) The different matrices involved in the development of the control laws in this case, namely $T(x)$, $c(x, w_1)$, $B(x, w_1)$ in equations (2.14), (2.16), (2.17) depend in extremely complicated fashion on the unknown parameters.

(ii) While the "true" $A(x)$ may have rank $q < p$, its estimate $\hat{A}(x)$ during the course of adaptation may well be full rank in which case the procedure of Section 2.1 cannot be followed.

The discrete time and sampled data case are also not obvious for similar reasons:

(i) The non-adaptive theory, as discussed in Monaco, Normand Cyrot and Stornelli (1986) is fairly complicated since

$$y_{k+1} = h \circ (f(x_k) + g(x_k)u_k) \tag{4.1}$$

is not linear in u_k in the discrete time case and a formal series for (4.1) in u_k needs to be obtained (and inverted!) for the linearization. Consequently the parametric dependence of the control law is complex.

(ii) The notions of zero-dynamics are not as yet completely developed. Further, even in the linear case, the zeros of a sampled system can be outside the unit disc even when the continuous time system is minimum phase and the sampling is fast enough (Astrom, Hagander, Sternby [1985]).

Thus, we feel that the present contribution is only a first step in the development of a comprehensive theory of adaptive control for linearizable systems.

Appendix Technical Lemmas

We state three important lemmas which are proven in Sastry-Bodson [1988]. The notation γ, β, K from the text is widely used in the appendix.

Lemma 1 (BOBI Stability)

Let $y = H(s)u$ be the output of a proper, *minimum phase* linear system with input u. If u, $\dot{u} \in L_{\infty e}$ and u is regular i.e.,

$$\| \dot{u} \|_t \leq K \| u \|_t + K$$

Then

$$\| u \|_t \leq K \| y \|_t + K .$$

Remark: If the input is regular and the plant is minimum phase then bounded system output implies bounded system input.

Lemma 2

Let $y = H(s)u$ be the output of a proper stable system $H(s)$ driven by u.

If $\quad \| u \|_t \leq \gamma \| q \|_t + \gamma$

Then $\quad \| y \|_t \leq \gamma \| q \|_t + \gamma$

If, in addition H is *strictly proper*

$$|y|_t \leq \beta \| q \|_t + \beta .$$

Remark: This is a slight generalization of several standard results; note that if H is strictly proper we get a $L_2 \cap L_\infty$ function which goes to zero as the bound.

Lemma 3 (Swapping Lemma)

If $H(s) = c(sI-A)^{-1}b + d$ is the minimal realization of a proper transfer function then

$$H(s)\,(W^T\Phi) - (H(s)W^T)\Phi = -\,c(sI-A)^{-1}\{[(sI-A)^{-1}bW^T]\dot{\Phi}\}$$

■

References

1. K. J. Astrom, P. Hagander, and J. Sternby (1985), "Zeros of Sampled Systems," *Automatic*, vol. 20, pp. 31–38.

2. M. Bodson and S. Sastry (1984), "Small Signal I/O Stability of Nonlinear Control Systems," unpublished, Memo No. UCB/ERL M84/70, University of California, Berkeley .

3. C. Byrnes and A. Isidori (1984), "A Frequency Domain Philosophy for Nonlinear Systems with Applications to Stabilization and Adaptive Control," in *Proc. of the IEEE Conf. on Decision and Control*, Las Vegas, pp. 1569–1573.

4. D. Claude (1986), "Everything You Always Wanted to Know About Linearization," in *Algebraic and Geometric Methods in Nonlinear Control Theory*, ed. Fliess & Hazewinkel Riedel, Dordrecht.

5. J. Craig, P. Hsu, and S. Sastry (1987), "Adaptive Control of Mechanical Manipulators," to appear in the *Int. J. of Robotics Research*, vol. 6, Summer 1987.

6. J. Descusse and C. H. Moog (1985), "Decoupling with Dynamic Compensation for Strongly Invertible Affine Nonlinear Systems," *Int. J. of Control*, No. 42, pp. 1387–1398.

7. W. Hahn (1967), *Stability of Motion*, Springer-Verlag, Berlin.

8. A. Isidori (1986), "Control of Nonlinear Systems via Dynamic State Feedback," in *Algebraic and Geometric Methods in Nonlinear Control Theory*, Reidel, Dordrecht.

9. A. Isidori and C. H. Moog, "On the Nonlinear Equivalent of the Notion of Transmission Zeros," to appear in *Modeling and Adaptive Control*, ed. C. Byrnes and A. Kurszanski, Springer-Verlag, Lecture Notes in Information and Control.

10. G. Meyer and L. Cicolani (1980), 'Applications of Nonlinear System Inverses to Automatic Flight Control Design – System Concepts and Flight Evaluations," AGARDo-

graph 251 on Theory and Applications of Optional Control in Aerospace Systems, P. Kent (Ed.), NATO 1980.

11. S. Monaco and D. Normand-Cyrot (1986), "Nonlinear Systems in Discrete Time," in *Algebraic and Geometric Methods in Nonlinear Control Theory*, ed. Fliess & Hazewinkel, Riedel, Dordrecht.

12. S. Monaco, D. Normand-Cyrot, and S. Stornelli (1986), "On the Linearizing Feedback in Nonlinear Sampled Data Control Schemes," in *Proc. of the IEEE Conf. on Decision and Control*, Athens, pp. 2056–2060.

13. K. Nam and A. Arapostathis, "A Model Reference Adaptive Control Scheme for Pure Feedback Nonlinear Systems," preprint, University of Texas (Austin), December 1986.

14. R. Marino and S. Nicosia (1985), "Singular Perturbation Techniques in the Adaptive Control of Elastic Robots," *Proc. of the IFAC-SYROCO Workshop*, Barcelona, Nov. 1985, pp. 11-16.

15. S. Nicosia and P. Tomei (1984), "Model Reference Adaptive Control for Rigid Robots," *Automatica*, vol. 20, pp. 635-644.

16. S. N. Singh and W. J. Rugh (1972), "Decoupling in a Class of Nonlinear Systems by State Variable Feedback," *Trans. ASME J. Dyn. Syst. Measur. and Control* vol. 94, pp. 323-324.

17. E. Freund (1975), "The Structure of Decoupled Nonlinear Systems," *Int. J. of Control* vol 21, ppo. 651-654.

18. A. Isidori, A. J. Krener, C. Gori Giorgi, S. Monaco (1981), "Nonlinear Decoupling via Feedback: A Differential Geometric Approach," *IEEE Trans. Auto. Conrol*, vol. AC-26, pp. 331-345.

19. S. S. Sastry and M. Bodson, "Adaptive Control: Stability, Convergence and Robust-

ness,'' to appear Prentice Hall (1988).

20. D. Taylor, P. Kokotovic, R. Marino (1987), ''Nonlinear Adaptive Control with Unmodelled Dynamics,'' Univ. of Illinois, Preprint, Dec. 1987.

21. S. S. Sastry (1984), Model Reference Adaptive Control: Stability and Rameper Convergence and Robustness, *IMA J. of Math. Control and Info.*, vol. 1, pp. 27-66.

22. W. A. Porter (1970), Diagonalization and Inverses for Nonlinear Systems, *Int. J. Control,* vol. 21, pp. 67-76.

23. R. W. Brockett (1978), Feedback Invariants for Nonlinear Systems, 6th IFAC Congress (Helsinki), pp. 1115-1120.

24. B. Jakubczyk, W. Respondek (1980), On the linearization of Control Ssytems, *Bull Acad. Polonaise Sci. Ser. Sci. Math.*, vol. 28, pp. 517-522.

25. L. R. Hunt, R. Su, G. Meyer (1983), Design for Multi-Input Nonlinear Systems, in "Differential Geometric Control Theory," R. W. Brockett, R. S. Millman and H. Sussmann, (eds.) Brikhauser, pp. 268-298.

26. A. Isidori (1985), Nonlinear Control Ssytems: an Introduction, Lecture Notes in Control and Information Sciences, vol. 72, Springer Verlag.

27. B. Jakubczyk (1987), Feedback Linearization of Discrete Time Systems, *Systems and Control Lett.*, vol. 9, pp. 411-416.

28. C. Byrnes, A. Isidori (1987a), Local Stabilization of Minimum Phase Nonlinear Systems, *Systems and Control Lett.*, vol. 10, to appear.

29. C. Byrnes, A. Isidori (1987b), Analysis and Design of Nonlinear Feedback Systems. Part I: Zero Dynamics and Global Normal Forms, preprint.

30. R. Marino (1988), Sliding Mode Control of Nonlinear Systems, *Systems and Control Letters*, vol. 10, to appear.