BICOPRIME FACTORIZATIONS OF THE PLANT AND THEIR RELATION TO RIGHT- AND LEFT-COPRIME FACTORIZATIONS

by

C. A. Desoer and A. N. Gündeş

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ABSTRACT

In a general algebraic framework, starting with a bicoprime factorization $P = N_{pr} D^{-1} N_{pl}$, we obtain a left-coprime factorization, a right-coprime factorization and the generalized Bezout identities associated with the pairs (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$. We express the set of all H-stabilizing compensators for P in the unity-feedback configuration $S(P, C)$ in terms of (N_{pr}, D, N_{pl}) and the elements of the Bezout identity. The state-space representation $P = C(sI - A)^{-1}B$ is included as an example.

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In a general algebraic framework, starting with a bicoprime factorization $P = N_{pr} D^{-1} N_{pl}$, we obtain a left-coprime factorization, a right-coprime factorization and the generalized Bezout identities associated with the pairs (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$. We express the set of all $\mathcal{H}$–stabilizing compensators for P in the unity-feedback configuration $S(P, C)$ in terms of (N_{pr}, D, N_{pl}) and the elements of the Bezout identity. The state-space representation $P = C(sI - A)^{-1}B$ is included as an example.

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INTRODUCTION

The set of all H–stabilizing compensators and achievable performance for a given plant P has been of great interest in the analysis and synthesis of linear time-invariant MIMO systems. H–stabilizing compensators were first characterized in [You.1] for continuous-time and discrete-time lumped systems. An algebraic approach that incuded distributed as well as lumped continuous-time and discrete-time systems was given in [Des.1]. Algebraic formulations were used by many researchers; for a detailed review of the factorization approach and related topics until 1985 see [Vid.1] and the references therein.

So far the parametrization of all H–stabilizing compensators has been based on a right-coprime factorization ($P = N_p D_p^{-1}$) or a left-coprime factorization ($P = \tilde{D}_p^{-1} \tilde{N}_p$) of the plant [Des.2,3,4,Vid.1,2,Net.1]. In some cases however, a bicoprime factorization ($P = N_{pr} D^{-1} N_{pl}$) is all that is available; a perfect example of this situation is the state-space representation. In [Net.2], constant state feedback and output injection were used to go from a state-space representation to a right-coprime fraction representation (r.c.f.r.) and a left-coprime fraction representation (l.c.f.r.).

The problem studied in this paper is finding the class of all H–stabilizing compensators directly from a bicoprime fraction representation (b.c.f.r.) of P. We also show that a r.c.f.r. and a l.c.f.r. can be obtained directly from a b.c.f.r. (N_{pr}, D, N_{pl}) and from the generalized Bezout identities associated with this b.c.f.r. The system we consider is the unity-feedback system $S(P, C)$ shown in figure 1; note that the compensator is a one-degree-of-freedom compensator. This system is simpler than the two-input two-output MIMO plant and compensator system considered in [Des.4,Net.1].

We use the following symbols and abbreviations:

I/O	input-output
MIMO	multiinput-multioutput
a := b	a is defined as b

det$A$	the determinant of matrix A
$\mathcal{M}(H)$	the set of matrices with elements in H .

I. ALGEBRAIC BACKGROUND

1.1. Notation [Vid.1, Lan.1]:

H is a principal ring (i.e., an entire ring in which every ideal is principal).

$J \subset H$ is the group of units of H .

$I \subset H$ is a multiplicative subsystem, $0 \notin I$, $1 \in I$.

$G = H / I := \{ n / d \ : n \in H , d \in I \}$ is the ring of fractions of H associated with I .

G_s (Jacobson radical of the ring G) $:= \{ x \in G_s : (1 + xy)^{-1} \in G$ for all $y \in G \}$.

Note that (i) I = the set of units of G which are in H. (ii) Let $A \in \mathcal{M}(H)$, $B \in \mathcal{M}(G)$, then a) $A^{-1} \in \mathcal{M}(H)$ iff det$A \in J$ and b) $B^{-1} \in \mathcal{M}(G)$ iff det$B \in I$. (iii) Let $Y \in \mathcal{M}(G_s)$, $X, Z \in \mathcal{M}(G)$, then $XY, YZ \in \mathcal{M}(G_s)$ and $(I + XY)^{-1}$, $(I + YZ)^{-1} \in \mathcal{M}(G)$. (iv) Let $a, b \in H$, then $ab \in J$ iff a and $b \in J$. (v) Let $c, d \in H$. Then $cd \in I$ iff c and $d \in I$ [Des.4].

1.2. Example (Rational functions in s) : Let $\mathcal{U} \supset \mathbb{C}_+$ be a closed subset of $\mathbb{C}$, symmetric about the real axis, and let $\mathbb{C} \setminus \mathcal{U}$ be nonempty. Define $\bar{\mathcal{U}} := \mathcal{U} \cup \{ \infty \}$. The ring of proper scalar rational functions (with real coefficients) which are analytic in $\mathcal{U}$ is a principal ring; we denote it by $R_{\mathcal{U}}(s)$. Let $H = R_{\mathcal{U}}(s)$. Then $f \in J$ implies that f has neither poles nor zeros in $\bar{\mathcal{U}}$. I is the multiplicative subset of $R_{\mathcal{U}}(s)$ such that $f \in I$ implies $f(\infty)$ = a nonzero constant in $\mathbb{R}$; equivalently, $I \subset R_{\mathcal{U}}(s)$ is the set of proper, but not strictly proper, real rational functions which are analytic in $\mathcal{U}$. Then $R_{\mathcal{U}}(s) / I$ is the ring of proper rational functions $\mathbb{R}_p(s)$. The set of strictly proper rational functions $\mathbb{R}_{sp}(s)$ is the Jacobson radical of the ring $\mathbb{R}_p(s)$.

1.3. Definitions (Coprime Factorizations in H):

(i) The pair (N_p, D_p) , where $N_p , D_p \in \mathcal{m}(H)$, is called *right-coprime* (r.c.) iff there exist $U_p , V_p \in \mathcal{m}(H)$ such that

$$V_p D_p + U_p N_p = I \tag{1.1}$$

(ii) The pair (N_p, D_p) is called a *right-fraction representation* (r.f.r.) of $P \in \mathcal{m}(G)$ iff

$$D_p \text{ is square, } \det D_p \in I \text{ and } P = N_p D_p^{-1} \tag{1.2}$$

(iii) The pair (N_p, D_p) is called a *right-coprime-fraction representation* (r.c.f.r.) of $P \in \mathcal{m}(G)$ iff (N_p, D_p) is a r.f.r. of P and (N_p, D_p) is r.c.

The definitions of *left-coprime* (l.c.), *left-fraction representation* (l.f.r.) and *left-coprime-fraction representation* (l.c.f.r.) are duals of (i), (ii), and (iii), respectively [Vid.1, Net.1, Des.4].

(iv) The triple (N_{pr}, D, N_{pl}), $N_{pr}, D, N_{pl} \in \mathcal{m}(H)$ is called a *bicoprime-fraction representation* (b.c.f.r.) of $P \in \mathcal{m}(G)$ iff the pair (N_{pr}, D) is *right-coprime* , the pair (D , N_{pl}) is *left-coprime*, $\det D \in I$ and $P = N_{pr} D^{-1} N_{pl}$.

□

Note that every $P \in \mathcal{m}(G)$ has a r.c.f.r. (N_p, D_p) , a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$, and a b.c.f.r. (N_{pr}, D, N_{pl}) in H because H is a principal ring [Vid.1].

II. MAIN RESULTS

Consider the system $S(P, C)$ in figure 1. We analyze this system with (i) a r.c.f.r. of P and a l.c.f.r. of C , (ii) a l.c.f.r. of P and a r.c.f.r. of C , (iii) a b.c.f.r. of P and a l.c.f.r. of C , (iv) a b.c.f.r. of P and a r.c.f.r. of C . The first two analyses give us the well-known set $S(P)$ of all H–stabilizing compensators in terms of familiar r.c.f.r. and l.c.f.r. of P [Vid.1,2,Des.2].

2.1. Assumptions:

(A) $P \in G_s^{n_o x n_i}$. Let (N_p, D_p) be a r.c.f.r., $(\tilde{D}_p, \tilde{N}_p)$ be a l.c.f.r., (N_{pr}, D, N_{pl}) be a b.c.f.r. of P , where $N_p \in H^{n_o x n_i}$, $D_p \in H^{n_i x n_i}$, $\tilde{D}_p \in H^{n_o x n_o}$, $\tilde{N}_p \in H^{n_o x n_i}$, $N_{pr} \in H^{n_o x n}$, $D \in H^{n x n}$, $N_{pl} \in H^{n x n_i}$.

(B) $C \in G^{n_i \times n_o}$. Let $(\tilde{D}_c, \tilde{N}_c)$ be a l.c.f.r. and (N_c, D_c) be a r.c.f.r. of C , where $\tilde{D}_c \in H^{n_i \times n_i}$, $\tilde{N}_c \in H^{n_i \times n_o}$, $N_c \in H^{n_i \times n_o}$, $D_c \in H^{n_o \times n_o}$.

If P satisfies assumption (A) we have the following *generalized Bezout identities*:

(1) For the r.c. pair (N_p, D_p) and the l.c. pair $(\tilde{D}_p, \tilde{N}_p)$, where $P = N_p D_p^{-1} = \tilde{D}_p^{-1} \tilde{N}_p$, there are matrices V_p , U_p , $\tilde{U}_p$, $\tilde{V}_p \in \mathcal{m}(H)$ such that

$$\begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} . \tag{2.1}$$

$((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ is called a *doubly-coprime factorization* of P.

(2) For the b.c.f.r. (N_{pr}, D, N_{pl}) we have two *generalized Bezout identities*: for the r.c. pair (N_{pr}, D), there are matrices V_{pr}, U_{pr}, $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, $\tilde{V} \in \mathcal{m}(H)$ such that

$$\begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N_{pr} & \tilde{V} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix} ; \tag{2.2}$$

for the l.c. pair (D, N_{pl}) there are matrices V_{pl}, U_{pl}, X, Y, U, $V \in \mathcal{m}(H)$ such that

$$\begin{bmatrix} D & -N_{pl} \\ U & V \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix} . \tag{2.3}$$

Each matrix in equations (2.1), (2.2), (2.3) is unimodular.

Let $y := \begin{bmatrix} y_m \\ y_m' \end{bmatrix}$, $u := \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}$; the map $H_{yu} : u \mapsto y$ is called the I/O map.

2.2. Definition (H-stability): The system $S(P, C)$ in figure 1 is said to be *H-stable* iff $H_{yu} \in \mathcal{m}(H)$.

2.3. Definition (H-stabilizing compensator): (1) C is called an *H-stabilizing compensator for P* iff $C \in G^{n_i \times n_o}$ satisfies assumption (B) *and* the system $S(P, C)$ is H-stable.

(2) The set

$$\mathrm{S}(P) := \{ \, C : C \ \textit{H-stabilizes} \ P \, \} \tag{2.4}$$

is called the *set of all H*–stabilizing compensators for P .

2.4. Analysis : Case (1) Let $P = N_p D_p^{-1}$ and let $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_p, D_p) is r.c. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see figure 2). $S(P, C)$ is then described by equations (2.5)-(2.6).

$$\left[\tilde{D}_c D_p + \tilde{N}_c N_p\right]\xi_p = \left[\tilde{D}_c \;\vdots\; \tilde{N}_c\right]\begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}, \qquad (2.5)$$

$$\begin{bmatrix} N_p \\ \cdots \\ D_p \end{bmatrix}\xi_p = \begin{bmatrix} y_m \\ \cdots \\ y_m' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cdots & \cdots \\ I_{n_i} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}. \qquad (2.6)$$

$S(P, C)$ is H–stable if and only if $\left[\tilde{D}_c D_p + \tilde{N}_c N_p\right] \in \mathfrak{m}(H)$ is unimodular [Vid.1,2,Des.4]. It is well-known (see for example [Vid.1,Des.2,4,Net.1] that the set $\mathrm{S}(P)$ of all H–stabilizing compensators is given by

$$\mathrm{S}(P) = \{\ (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p) \ :\ Q \in H^{n_i x n_o}\ \}, \qquad (2.7)$$

where V_p , U_p , $\tilde{N}_p$, $\tilde{D}_p$ are as in equation (2.1).

Case (2) Now let $P = \tilde{D}_p^{-1}\tilde{N}_p$, $C = N_c D_c^{-1}$, where $(\tilde{D}_p, \tilde{N}_p)$ is l.c. and (N_c, D_c) is r.c. (see figure 3). $S(P, C)$ is then described by equations (2.8)-(2.9).

$$\left[\tilde{D}_p D_c + \tilde{N}_p N_c\right]\xi_c = \left[\tilde{N}_p \;\vdots\; \tilde{D}_p\right]\begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}, \qquad (2.8)$$

$$\begin{bmatrix} -D_c \\ \cdots \\ N_c \end{bmatrix}\xi_c = \begin{bmatrix} y_m \\ \cdots \\ y_m' \end{bmatrix} + \begin{bmatrix} 0 & -I_{n_o} \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}. \qquad (2.9)$$

$S(P, C)$ is H–stable if and only if $\left[\tilde{D}_p D_c + \tilde{N}_p N_c\right] \in \mathfrak{m}(H)$ is unimodular (which is equivalent to $\left[\tilde{D}_c D_p + \tilde{N}_c N_p\right] \in \mathfrak{m}(H)$ is unimodular). The set $\mathrm{S}(P)$ of all H–stabilizing compensators is given by

$$\mathrm{S}(P) = \{\ (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1} \ :\ Q \in H^{n_i x n_o}\ \}, \qquad (2.10)$$

where $\tilde{U}$, $\tilde{V}$, N_p , D_p are as in equation (2.1).

Case (3) Now let $P = N_{pr} D^{-1} N_{pl}$ and let $C = \tilde{D}_c^{-1} \tilde{N}_c$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see figure 4). $S(P, C)$ is then described by equations (2.11)-(2.12).

$$\begin{bmatrix} D & \vdots & -N_{pl} \\ \cdots & & \cdots \\ \tilde{N}_c N_{pr} & \vdots & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_m' \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \tilde{N}_c \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}, \tag{2.11}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_i} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_m' \end{bmatrix} = \begin{bmatrix} y_m \\ \cdots \\ y_m' \end{bmatrix}. \tag{2.12}$$

Equations (2.11)-(2.12) are of the form

$$D_H \xi = N_L u$$
$$N_R \xi = y$$

where (N_R, D_H) is a r.c. pair and (D_H, N_L) is a l.c. pair, $N_R, D_H, N_L \in \mathcal{M}(H)$. The system $S(P, C)$ is H–stable if and only if $D_H^{-1} \in \mathcal{M}(H)$; equivalently, $S(P, C)$ is H–stable if and only if

$$D_H = \begin{bmatrix} D & \vdots & -N_{pl} \\ \cdots & & \cdots \\ \tilde{N}_c N_{pr} & \vdots & \tilde{D}_c \end{bmatrix} \text{ is unimodular}. \tag{2.13}$$

Let $R := \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix}$; by equation (2.3), $R \in \mathcal{M}(H)$ is unimodular. Post-multiply D_H by R:

$$D_H R = \begin{bmatrix} I_n & 0 \\ \tilde{N}_c N_{pr} V_{pl} - \tilde{D}_c U_{pl} & \tilde{N}_c N_{pr} X + \tilde{D}_c Y \end{bmatrix}. \tag{2.14}$$

But D_H is unimodular if and only if $D_H R$ is unimodular; hence (2.13) holds if and only if

$$\tilde{N}_c N_{pr} X + \tilde{D}_c Y =: D_{HR} \text{ is unimodular}. \tag{2.15}$$

The set $S(P)$ of all H–stabilizing compensators is then the set of all $\tilde{D}_c^{-1} \tilde{N}_c$ such that equation (2.15) is satisfied.

Case (4) Finally let $P = N_{pr} D^{-1} N_{pl}$ and let $C = N_c D_c^{-1}$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. and (N_c, D_c) is l.c. (see figure 5). $S(P, C)$ is then described by equations (2.16)-(2.17).

$$\begin{bmatrix} D & \vdots & -N_{pl}N_c \\ \cdots & & \cdots \\ N_{pr} & \vdots & D_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_c \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_o} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}, \tag{2.16}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & N_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_c \end{bmatrix} = \begin{bmatrix} y_m \\ \cdots \\ y_m' \end{bmatrix}. \tag{2.17}$$

Following similar steps as in case (3) of the analysis, we conclude that $S(P, C)$ is H–stable if and only if

$$\hat{D}_H := \begin{bmatrix} D & \vdots & -N_{pl}N_c \\ \cdots & & \cdots \\ N_{pr} & \vdots & D_c \end{bmatrix} \text{ is unimodular}. \tag{2.18}$$

Let $L := \begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix}$; by equation (2.2), $L \in \mathcal{m}(H)$ is unimodular. Pre-multiply $\hat{D}_H$ by L:

$$L\hat{D}_H = \begin{bmatrix} I_n & -V_{pr}N_{pl}N_c + U_{pr}D_c \\ 0 & \tilde{X} N_{pl}N_c + \tilde{Y} D_c \end{bmatrix}. \tag{2.19}$$

But $\hat{D}_H$ is unimodular if and only if $L\hat{D}_H$ is unimodular; hence the set $S(P)$ of all H–stabilizing compensators is then the set of all $N_c D_c^{-1}$ such that

$$\tilde{X} N_{pl}N_c + \tilde{Y} D_c =: \hat{D}_{HL} \text{ is unimodular}. \tag{2.20}$$

2.5. Proposition: Let $P \in \mathcal{m}(G_s)$; let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P; hence, equations (2.2)-(2.3) hold. Then

$$(N_p, D_p) := (N_{pr}X, Y) \text{ is a r.c.f.r. of } P ; \tag{2.21}$$

$$(\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \tilde{X} N_{pl}) \text{ is a l.c.f.r. of } P , \tag{2.22}$$

where $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{m}(H)$ are defined in equations (2.2)-(2.3).

Proof of proposition 2.5: By assumption, $P = N_{pr} D^{-1} N_{pl}$, and equations (2.2)-(2.3) hold.

Clearly $N_{pr}X$, Y , $\tilde{Y}$, $\tilde{X}N_{pl} \in \mathcal{M}(H)$. We must show that $(N_{pr}X , Y)$ is a r.c. pair with $\det Y \in I$ and that $(\tilde{Y} , \tilde{X}N_{pl})$ is a l.c. pair with $\det\tilde{Y} \in I$.

Now $P \in \mathcal{M}(G_S)$. Post-multiply P by Y ; then using $N_{pl}Y = DX$ from the Bezout equation (2.3), we obtain

$$PY = N_{pr} D^{-1}N_{pl}Y = N_{pr}X \in \mathcal{M}(G_S) . \tag{2.23}$$

Now pre-multiply P by $\tilde{Y}$; then using $\tilde{Y}N_{pr} = \tilde{X}D$ from the Bezout equation (2.2) we obtain

$$\tilde{Y}P = \tilde{Y}N_{pr} D^{-1}N_{pl} = \tilde{X}N_{pl} \in \mathcal{M}(G_S) . \tag{2.24}$$

Using equations (2.2)-(2.3) we now obtain a generalized Bezout identity for $(N_{pr}X , Y)$ and $(\tilde{Y} , \tilde{X}N_{pl})$:

$$\begin{bmatrix} V + UV_{pr}N_{pl} & UU_{pr} \\ -\tilde{X}N_{pl} & \tilde{Y} \end{bmatrix} \begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X & \tilde{V} + N_{pr}V_{pl}\tilde{U} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} . \tag{2.25}$$

Note the similarity between equations (2.1) and (2.25). Each of the three matrices in equation (2.25) has elements in H and hence is unimodular. From now on we refer to the matrices on the left-hand side as M and $\hat{M}$, respectively; equation (2.25) then reads

$$M\hat{M} = I_{n_o+n_i} . \tag{2.25a}$$

By equation (2.25), $(N_{pr}X , Y)$ is a r.c. pair and $(\tilde{Y} , \tilde{X}N_{pl})$ is a l.c. pair; more specifically, if $(N_{pr}X , Y) =: (N_p , D_p)$ and $(\tilde{Y} , \tilde{X}N_{pl}) =: (\tilde{D}_p , \tilde{N}_p)$, then

$$V_pD_p + U_pN_p = I_{n_i} , \quad \tilde{N}_p\tilde{U}_p + \tilde{D}_p\tilde{V}_p = I_{n_o} , \tag{2.26}$$

where

$$V_p := V + UV_{pr}N_{pl} , \; U_p := UU_{pr} , \; \tilde{U}_p := U_{pl}\tilde{U} , \; \tilde{V}_p = \tilde{V} + N_{pr}V_{pl}\tilde{U} . \tag{2.27}$$

Since $N_p := N_{pr}X \in \mathcal{M}(G_S)$ and $U_pN_p := UU_{pr}N_{pr}X \in \mathcal{M}(G_S)$, equation (2.26) implies that $\det(V_pD_p) = \det(I_{n_i} - U_pN_p) \in I$ and hence, $\det V_p \in I$ and $\det D_p := \det Y \in I$. Similarly, since $\tilde{N}_p := \tilde{X}N_{pl} \in \mathcal{M}(G_S)$ from equation (2.24), equation (2.26) implies that $\det(\tilde{D}_p\tilde{V}_p) = \det(I_{n_o} - \tilde{N}_p\tilde{U}_p) \in I$ and hence, $\det\tilde{V}_p \in I$ and $\det\tilde{D}_p := \det\tilde{Y} \in I$.

At this point we know that $Y^{-1} \in \mathcal{M}(G)$ and $\tilde{Y}^{-1} \in \mathcal{M}(G)$. Then equation (2.23) implies that

$$P = N_{pr} X Y^{-1}, \tag{2.28}$$

and similarly, equation (2.24) implies that

$$P = \tilde{Y}^{-1} \tilde{X} N_{pl}. \tag{2.29}$$

Finally, since equations (2.28) and (2.26) hold and since $\det Y \in \mathcal{I}$, $(N_{pr}X, Y) =: (N_p, D_p)$, with $N_p, D_p \in \mathcal{M}(H)$, is a r.c.f.r. of P. Since equations (2.29) and (2.26) hold and since $\det \tilde{Y} \in \mathcal{I}$, $(\tilde{Y}, \tilde{X} N_{pl}) =: (\tilde{D}_p, \tilde{N}_p)$, with $\tilde{D}_p, \tilde{N}_p \in \mathcal{M}(H)$, is a l.c.f.r. of P.

□

Comment: If $P \in \mathcal{M}(G)$ but not $\mathcal{M}(G_S)$, equations (2.21)-(2.22) still give a r.c.f.r. and a l.c.f.r. of P, respectively. The only difference in this case is in showing that $\det Y \in \mathcal{I}$ and $\det \tilde{Y} \in \mathcal{I}$:

Consider the Bezout equation (2.2) for the r.c. pair (N_{pr}, D). Since $P \in \mathcal{M}(G)$, $\det V_{pr}$ is not necessarily $\in \mathcal{I}$. Take $T \in \mathcal{M}(H)$ such that $\det(V_{pr} - T\tilde{X}) \in \mathcal{I}$ [Vid.1]. Rewrite equation (2.2):

$$\begin{bmatrix} V_{pr} - T\tilde{X} & U_{pr} + T\tilde{Y} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} - DT \\ N_{pr} & \tilde{V} - N_{pr}T \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.30}$$

Since $\det D \in \mathcal{I}$, from equation (2.30) we get $\det\left[(V_{pr} - T\tilde{X})D\right] = \det(I_n - (U_{pr} + T\tilde{Y})N_{pr}) = \det(I_{n_o} - N_{pr}(U_{pr} + T\tilde{Y})) = \det\left[(\tilde{V} - TN_{pr})\tilde{Y}\right] \in \mathcal{I}$; equivalently, $\det(\tilde{V} - TN_{pr}) \in \mathcal{I}$ *and* $\det \tilde{Y} \in \mathcal{I}$.

Similarly, consider the Bezout equation (2.3). Take $\hat{T} \in \mathcal{M}(H)$ such that $\det(V_{pl} - \hat{T}X) \in \mathcal{I}$. Rewrite equation (2.3):

$$\begin{bmatrix} D & -N_{pl} \\ U + \hat{T}D & V - \hat{T}N_{pl} \end{bmatrix} \begin{bmatrix} V_{pl} - X\hat{T} & X \\ -U_{pl} - Y\hat{T} & Y \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix}. \tag{2.31}$$

Since $\det D \in \mathcal{I}$, from equation (2.31) we get $\det\left[(V_{pl} - X\hat{T})D\right] = \det(I_n - N_{pl}(U_{pl} + Y\hat{T})) =$

$\det(I_{n_i} - (U_{pl} + \hat{T}Y)N_{pl}) \;\; = \det\left[Y(V - \hat{T}N_{pl})\right] \;\in I$; equivalently, $\det(V - \hat{T}N_{pl}) \in I$ *and* $\det Y \in I$.

□

2.6. Theorem (Set of all H–stabilizing compensators): Let $P \in \mathfrak{M}(G_S)$ and let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P, hence equations (2.2) and (2.3) hold. Then

$$S(P) = \{ \; (V + UV_{pr}N_{pl} - Q\tilde{X}N_{pl})^{-1}(UU_{pr} + Q\tilde{Y}) \; : \; Q \in \mathfrak{M}(H) \; \} \; ; \qquad (2.32)$$

equivalently,

$$S(P) = \{ \; (U_{pl}\tilde{U} + YQ)(\tilde{V} + N_{pr}V_{pl}\tilde{U} - N_{pr}XQ)^{-1} \; : \; Q \in \mathfrak{M}(H) \; \} \; ; \qquad (2.33)$$

where the matrices in equations (2.32)-(2.33) are as in the generalized Bezout equation (2.25).

Comment: By proposition 2.5 we know how to obtain a r.c.f.r. (N_p, D_p) and a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ from a b.c.f.r. (N_{pr}, D, N_{pl}) of $P \in \mathfrak{M}(G_S)$: with (N_p, D_p) as in equation (2.21), $(\tilde{D}_p, \tilde{N}_p)$ as in equation (2.22), and V_p, U_p, $\tilde{V}_p$, $\tilde{U}_p$ as in equation (2.27), the generalized Bezout equation (2.25) is the same as the Bezout equation (2.1). Furthermore, observe that equation (2.21) substituted into equation (2.15) implies

$$D_{HR} = \tilde{N}_c N_p + \tilde{D}_c D_p \;\; ; \qquad (2.34)$$

and hence, H–stability using analysis 2.4-case (3) is equivalent to establishing H–stability using case (1). Therefore it is no surprise that $S(P)$ in equation (2.32) is the same as $S(P)$ in equation (2.7), with equations (2.22) and (2.27) in mind. Similarly, equation (2.22) substituted into equation (2.20) implies

$$D_{HL} = \tilde{N}_p N_c + \tilde{D}_p D_c \; ; \qquad (2.35)$$

and hence, H–stability using analysis 2.4-case (4) is equivalent to case (2). Therefore, $S(P)$ in equation (2.33) is the same as $S(P)$ in equation (2.10), with equations (2.21) and (2.27) in mind.

Although the discussion above justifies theorem 2.6, we now give a formal proof.

Proof of theorem 2.6: We only prove that the set $S(P)$ in equation (2.32) is the set of all H–stabilizing compensators; the proof of equation (2.33) is entirely similar.

If C is defined by the expression in equation (2.32) then C H–stabilizes P :

Let

$$C = \tilde{D}_c^{-1}\tilde{N}_c , \quad \tilde{D}_c = V + UV_{pr}N_{pl} - Q\tilde{X}\, N_{pl} , \quad \tilde{N}_c = UU_{pr} + Q\tilde{Y} . \tag{2.36}$$

We must show that (i) C satisfies assumption (B), i.e., $\tilde{D}_c$, $\tilde{N}_c \in \mathcal{M}(H)$ with $\det\tilde{D}_c \in I$ and the pair $(\tilde{D}_c, \tilde{N}_c)$ is l.c., and (ii) $S(P, C)$ is H–stable, i.e., equation (2.15) holds.

(i) From equation (2.36) clearly $\tilde{D}_c$, $\tilde{N}_c \in \mathcal{M}(H)$. Using the generalized Bezout equation (2.25) we obtain

$$\begin{aligned} D_{HR} &= \tilde{N}_c N_{pr} X + \tilde{D}_c Y \\ &= (UU_{pr} + Q\tilde{Y})N_{pr}X + (V + UV_{pr}N_{pl} - Q\tilde{X}\, N_{pl})Y = I_{n_i} . \end{aligned} \tag{2.37}$$

By equation (2.37) $(\tilde{D}_c, \tilde{N}_c)$ is a l.c. pair. In the proof of proposition 2.5 we showed that $N_{pr}X \in \mathcal{M}(G_S)$ (see equation (2.23)), and hence $\tilde{N}_c N_{pr} X \in \mathcal{M}(G_S)$. We conclude from equation (2.37) that $\det(\tilde{D}_c Y) = \det(I_{n_i} - \tilde{N}_c N_{pr} X) \in I$, therefore $\det\tilde{D}_c \in I$; consequently, $(\tilde{D}_c, \tilde{N}_c)$ is a l.c.f.r. of C.

(ii) From equation (2.37), $D_{HR} = I_{n_i}$. Therefore $S(P, C)$ is H–stable since equation (2.15) holds.

Any C that H–stabilizes P is an element of the set $S(P)$ defined by equation (2.32):
Let $C \in \mathcal{M}(G)$ H–stabilize P . Let $(\tilde{D}_c, \tilde{N}_c)$ be a l.c.f.r. of C . By assumption, $S(P, C)$ is H–stable; equivalently, by normalizing equation (2.15), $D_{HR} = I_{n_i}$. Then

$$\left[\tilde{D}_c \ \vdots\ \tilde{N}_c\right] \begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X & \tilde{V} + N_{pr}V_{pl}\tilde{U} \end{bmatrix} =: \left[I_{n_i} \ \vdots\ Q\right] , \tag{2.38}$$

where $Q := -\tilde{D}_c U_{pl}\tilde{U} + \tilde{N}_c(\tilde{V} + N_{pr}V_{pl}\tilde{U}) \in H^{n_i x n_o}$. Post-multiply both sides of equation (2.38) by the unimodular matrix M defined in equations (2.25)-(2.25a):

$$\left[\tilde{D}_c \ \vdots\ \tilde{N}_c\right] = \left[I_{n_i} \ \vdots\ Q\right] \begin{bmatrix} V + UV_{pr}N_{pl} & UU_{pr} \\ -\tilde{X}\, N_{pl} & \tilde{Y} \end{bmatrix} . \tag{2.39}$$

Clearly from equation (2.39), $C = \tilde{D}_c^{-1}\tilde{N}_c$ is in the set $S(P)$ in equation (2.32) for some

$Q \in H^{n_i x n_o}$ (in fact, there is a unique Q for each C ; we prove this in corollary 2.7).

□

2.7. Corollary: Let $C_1, C_2 \in \mathrm{S}(P)$; then $C_1 = C_2$ if and only if $Q_1 = Q_2$. Equivalently, the map $Q \mapsto C$, $Q \in \mathcal{M}(H)$, $C \in \mathrm{S}(P)$, is one-to-one.

Proof: Let $\mathrm{S}(P)$ be given as in equation (2.32); the proof for equation (2.33) is entirely similar.

Let $C_1 = \tilde{D}_{c1}^{-1}\tilde{N}_{c1}$, $C_2 = \tilde{D}_{c2}^{-1}\tilde{N}_{c2}$. By equation (2.38)

$$\left[\tilde{D}_{c1} \ \vdots \ \tilde{N}_{c1}\right]\hat{M} = \left[I_{n_i} \ \vdots \ Q_1\right] = \tilde{D}_{c1}\left[I_{n_i} \ \vdots \ C_1\right]\hat{M} \, , \tag{2.40}$$

and

$$\left[\tilde{D}_{c2} \ \vdots \ \tilde{N}_{c2}\right]\hat{M} = \left[I_{n_i} \ \vdots \ Q_2\right] = \tilde{D}_{c2}\left[I_{n_i} \ \vdots \ C_2\right]\hat{M} \, . \tag{2.41}$$

But $C_1 = C_2$ in equations (2.40)-(2.41) implies $\left[I_{n_i} \ \vdots \ C_1\right]\hat{M} = \tilde{D}_{c1}^{-1}\left[I_{n_i} \ \vdots \ Q_1\right] = \tilde{D}_{c2}^{-1}\left[I_{n_i} \ \vdots \ Q_2\right]$ and hence, $\tilde{D}_{c1} = \tilde{D}_{c2}$; consequently, $Q_1 = Q_2$.

Now suppose C_1 is given by a l.c.f.r. $(\tilde{D}_{c1}, \tilde{N}_{c1})$ but C_2 is given by a r.c.f.r. (N_{c2}, D_{c2}). Then by equations (2.33) and (2.25),

$$M \begin{bmatrix} -N_{c2} \\ \cdots \\ D_{c2} \end{bmatrix} = \begin{bmatrix} -Q_2 \\ \cdots \\ I_{n_o} \end{bmatrix} . \tag{2.42}$$

Then multiplying equation (2.42) on the left by equation (2.40) and using equation (2.25a) we obtain

$$\left[\tilde{D}_{c1} \ \vdots \ \tilde{N}_{c1}\right]\hat{M}M \begin{bmatrix} -N_{c2} \\ \cdots \\ D_{c2} \end{bmatrix} = \left[I_{n_i} \ \vdots \ Q_1\right] \begin{bmatrix} -Q_2 \\ \cdots \\ I_{n_o} \end{bmatrix} . \tag{2.43}$$

But $C_1 = C_2$ implies that $\tilde{N}_{c1}D_{c2} = \tilde{D}_{c1}N_{c2}$. Therefore by equation (2.43),

$$\left[-\tilde{D}_{c1}N_{c2} + \tilde{N}_{c1}D_{c2}\right] = Q_1 - Q_2 = 0 \, .$$

We conclude that, for each $C \in \mathrm{S}(P)$ there is a unique $Q \in \mathcal{M}(H)$ such that C is a member of the set $\mathrm{S}(P)$ in equation (2.32) (equivalently, in equation (2.33)).

□

2.8. Example: Let $H = R_u(s)$ as in example 1.2. Let $P \in \mathbb{R}_{sp}(s)^{n_o \times n_i}$ be represented by its

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where (C, A, B) is stabilizable and detectable. Then $P = \frac{C}{s+a}\left[\frac{(sI-A)}{(s+a)}\right]^{-1} B$, where $-a \in \mathbb{C} \setminus \bar{\mathcal{U}}$. The pair $\left[\frac{C}{s+a}, \frac{(sI-A)}{(s+a)}\right]$ is r.c. in $R_u(s)$ and the pair $\left[\frac{(sI-A)}{(s+a)}, B\right]$ is l.c. in $R_u(s)$, and $\det\frac{(sI-A)}{(s+a)} \in I$. Therefore, $(N_{pr}, D, N_{pl}) = \left[\frac{C}{(s+a)}, \frac{(sI-A)}{(s+a)}, B\right]$ is a b.c.f.r. of P. Then $(N_p, D_p) = \left[\frac{C}{(s+a)}X, Y\right]$ is a r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p) = (\tilde{Y}, \tilde{X}B)$ is a l.c.f.r. of P.

□

III. CONCLUSIONS

Given a b.c.f.r. (N_{pr}, D, N_{pl}) for $P \in \mathcal{M}(G_s)$, we find the class of all H–stabilizing compensators; with $V, U, V_{pr}, U_{pr}, \tilde{X}, \tilde{Y}$ as in equation (2.25),

$$C = (\tilde{D}_c, \tilde{N}_c) = (V + UV_{pr}N_{pl} - Q\tilde{X}N_{pl})^{-1}(UU_{pr} + Q\tilde{Y}) \tag{3.1}$$

H–stabilizes P, where $Q \in \mathcal{M}(H)$ is a free parameter. If we design a two-degrees-of-freedom compensator $C = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}$ as in [Des.2,3], then $C = \tilde{D}_c^{-1}\begin{bmatrix} Q_{21} & \tilde{N}_c \end{bmatrix}$, where $Q_{21} \in \mathcal{M}(H)$, and $(\tilde{D}_c, \tilde{N}_c)$ is given by equation (3.1) above; in this case there are two free parameters.

From the given b.c.f.r. (N_{pr}, D, N_{pl}) we also obtain a r.c.f.r., a l.c.f.r. and the associated generalized Bezout identity. The methods used in this paper make it easier to establish some fundamental results in decentralized control theory (work in progress).

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Figure Captions:

Figure 1 The system $S(P, C)$.

Figure 2 $S(P, C)$ with $P = N_p D_p^{-1}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$.

Figure 3 $S(P, C)$ with $P = \tilde{D}_p^{-1}\tilde{N}_p$ and $C = N_c D_c^{-1}$.

Figure 4 $S(P, C)$ with $P = N_{pr} D^{-1} N_{pl}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$.

Figure 5 $S(P, C)$ with $P = N_{pr} D^{-1} N_{pl}$ and $C = N_c D_c^{-1}$.

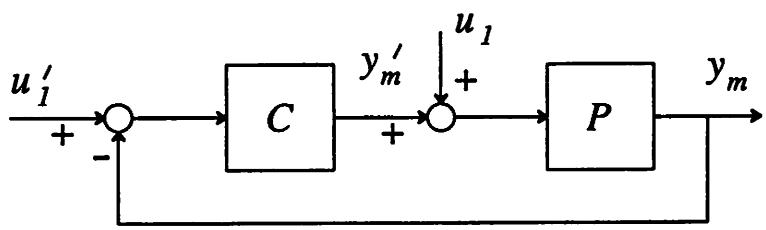

Figure 1

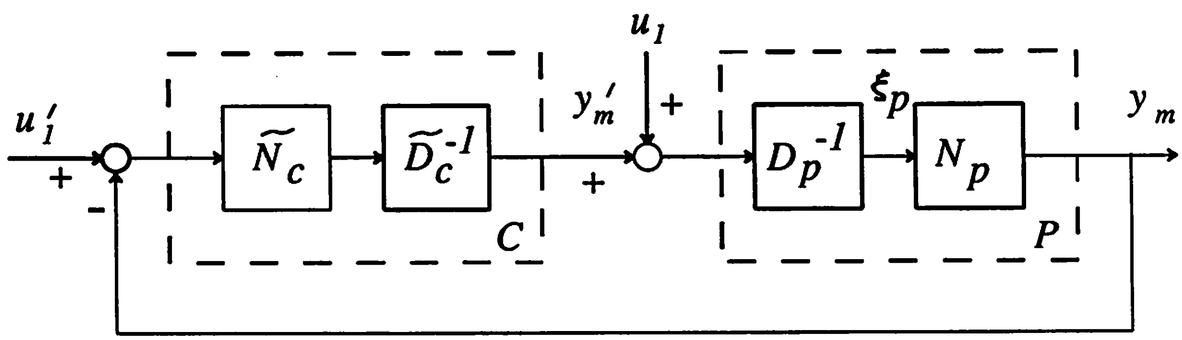

Figure 2

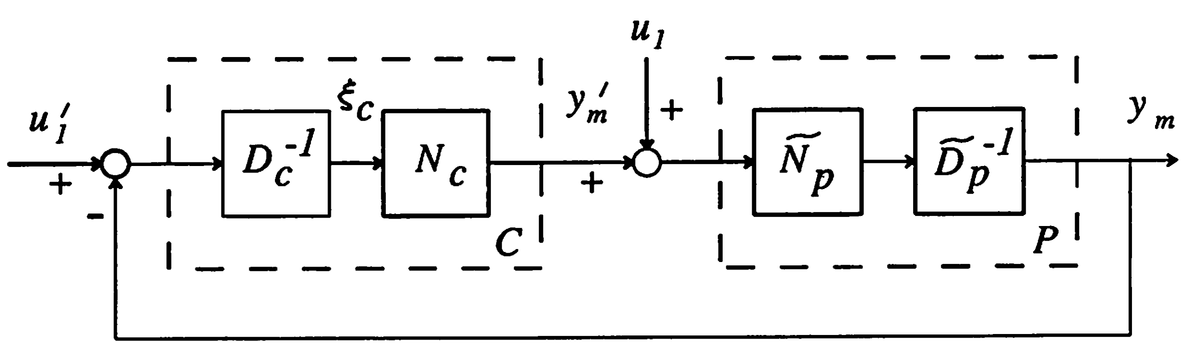

Figure 3

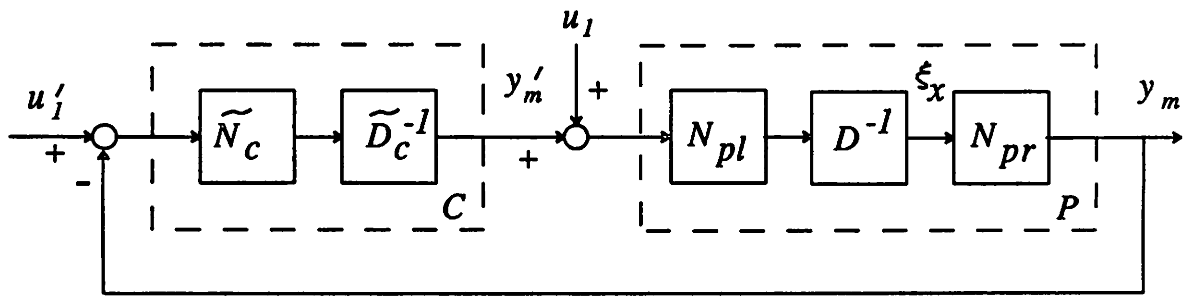

Figure 4

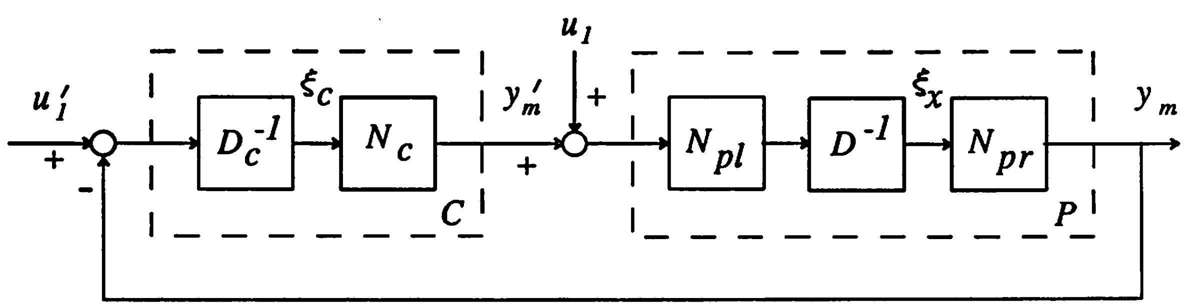

Figure 5