LINEAR STABLE UNITY-FEEDBACK SYSTEM: NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY UNDER NONLINEAR PLANT PERTURBATIONS

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Abstract

We consider a linear (not necessarily time-invariant) stable unity-feedback system, where the plant and the compensator have normalized right-coprime factorizations; we study two cases of *nonlinear* plant perturbations (*additive* and *feedback*), with four subcases resulting from : 1) allowing exogenous input to ΔP or not , 2) allowing the observation of the output of ΔP or not. The plant perturbation ΔP is *not* required to be stable. Using the factorization approach we obtain *necessary* and *sufficient* conditions for all cases in terms of two pairs of nonlinear pseudo-state maps. Simple physical considerations explain the form of these necessary and sufficient conditions. Finally, we obtain the characterization of all perturbations ΔP for which the perturbed system remains stable.

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Introduction

Robust stability of feedback systems under *unstructured* perturbations of the plant model has been studied extensively. In the nonlinear case, the small gain theorem [Zam.1, Des.1] gives a *sufficiency* condition for robust stability of a stable system under nonlinear *stable* additive perturbations. *Sufficient* robust stability conditions were also obtained in [Åst.1, Cru.1, Des.3, Fra.1, Owe.1, Pos.1, San.1]. In the linear time-invariant case, *necessary and sufficient* conditions for robust stability for a *certain* class of possibly *unstable* plant perturbations have been obtained in [Doy.1 and references therein, Che.1]; for a *general* class of possibly unstable perturbations, the factorization approach yields *necessary and sufficient* conditions for robust stability of the feedback system under *fractional* perturbations of the subsystems [Che.2]. Furthermore, necessary and sufficient conditions for the *existence* of a controller for plants with additive or multiplicative uncertainty are given in [Vid.1].

For linear time-invariant stable unity-feedback systems with *nonlinear additive* plant perturbations, necessary and sufficient conditions have been obtained in two cases: **i)** the additive perturbation has an independent input, hence unmodelled dynamics which is not coupled to the nominal plant inputs can be taken into account [Bha.1], **ii)** the perturbed plant is considered as a one-input one-output plant [Hua.1] (see also [Hua.2] for the linear time-invariant additive perturbation case).

In this paper we consider a linear (not necessarily time-invariant) stable unity-feedback system, where the plant and the compensator have normalized right-coprime factorizations; we study two cases of *nonlinear* plant perturbations (*additive* and *feedback*), with four subcases resulting from : **1)** allowing exogenous input to ΔP or not , **2)** allowing the observation of the output of ΔP or not. The plant perturbation ΔP is *not* required to be stable. Using the factorization approach we obtain *necessary* and *sufficient* conditions for all cases in terms of two pairs of nonlinear pseudo-state maps. Simple physical considerations explain the form of these necessary and sufficient conditions. Finally, we obtain the characterization of all perturbations ΔP for which the perturbed system remains stable.

Notation: (e.g. [Wil.1, Saf.1, Des.1]) Let $\mathcal{T} \subset \mathbb{R}$ and let $\mathcal{V}$ be a normed vector space. Let $\zeta := \{ F \mid F : \mathcal{T} \to \mathcal{V} \}$ be the vector space of $\mathcal{V}$–valued functions on $\mathcal{T}$. For any $T \in \mathcal{T}$, the projection map $\Pi_T : \zeta \to \zeta$ is defined by $\Pi_T F(t) := \begin{cases} F(t) & t \le T ,\ t \in \mathcal{T} \\ \theta_\zeta & t > T ,\ t \in \mathcal{T} , \end{cases}$ where θ_ζ is the zero element in ζ. Let $\Lambda \subset \zeta$ be a normed vector space which is closed under the family of projection maps $\{ \Pi_T \}_{T \in \mathcal{T}}$. For any $F \in \Lambda$, let the norm $\| \Pi_{(\cdot)} F \| : \mathcal{T} \to \mathbb{R}_+$ be a nondecreasing function. The *extended space* Λ_e is defined by

$$\Lambda_e := \{ F \in \zeta \mid \forall T \in \mathcal{T},\ \Pi_T F \in \Lambda \} .$$

A map $F : \Lambda_e \to \Lambda_e$ is said to be *causal* iff for all $T \in \mathcal{T}$, Π_T commutes with $\Pi_T F$; equivalently, $\Pi_T F = \Pi_T F \Pi_T$.

A *feedback system* is said to be *well-posed* iff for all allowed inputs, all of the signals in the system are (uniquely) determined by causal maps.

In the following we will be considering a number of function spaces closely related to Λ_e. The superscript i and the superscript o refer to "input" and "output", respectively. Let Λ_e^i and Λ_e^o be extended function spaces analogous to Λ_e except that their functions take values in the normed spaces $\mathcal{V}^i$ and $\mathcal{V}^o$, respectively; the associated projections Π_T are redefined accordingly.

A causal map $H : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e$ is said to be *S-stable* iff there exists a continuous nondecreasing function $\phi_H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall (u_1, u_2) \in \Lambda^o \times \Lambda^i ,\ \| H(u_1, u_2) \| \le \phi_H(\| u_1 \| + \| u_2 \|).$$

An S-stable map need not be continuous. Note that the composition and the sum of S-stable maps are S-stable.

A well-posed (nonlinear) *feedback system* is called *S-stable* iff, for all allowed inputs, all of the signals in the feedback system are determined by causal S-stable maps.

A causal (nonlinear) map $P : \Lambda_e^i \to \Lambda_e^o$ is said to have a *right factorization* $(N_p , D_p ; X_p)$ iff there exist causal S-stable maps N_p , D_p, such that

(i) $D_p : X_p \subset \Lambda_e^i \to \Lambda_e^i$ is bijective and has a causal inverse, and (ii) $N_p : X_p \to \Lambda_e^o$, with $N_p[X_p] = P[\Lambda_e^i]$,

and (iii) $P = N_p D_p^{-1}$ [Vid.2, Ham.1].

X_p is called the *factorization space* of the right factorization $(N_p , D_p ; X_p)$ [Ham.1].

$(N_p , D_p ; X_p)$ is said to be a *normalized right-coprime factorization* of $P : \Lambda_e^i \to \Lambda_e^o$ iff

(i) $(N_p , D_p ; X_p)$ is a right factorization of P ,

and (ii) there exist causal S-stable maps $U_p : \Lambda_e^o \to X_p$ and $V_p : \Lambda_e^i \to X_p$ such that $U_p N_p + V_p D_p = I_{X_p}$, where I_{X_p} denotes the identity map on X_p .

Note that any causal *S-stable* map $P : \Lambda_e^i \to \Lambda_e^o$ has a normalized right-coprime factorization, namely $(P , I_{\Lambda_e^i} ; \Lambda_e^i)$.

1. Assumption : Consider the well-posed *linear* unity-feedback system $S(P, C)$ in Figure 1 : the plant and the compensator are given by causal *linear* (not necessarily time-invariant) maps $P : \Lambda_e^i \to \Lambda_e^o$ and $C : \Lambda_e^o \to \Lambda_e^i$ which have *normalized* right-coprime factorizations $(N_{pr}, D_{pr}; \Lambda_e^i)$ and $(N_{cr}, D_{cr}; \Lambda_e^o)$, respectively; N_{pr}, D_{pr}, N_{cr} and D_{cr} are *linear* maps (see for example [Man.1] for the continuous-time linear time-varying case).

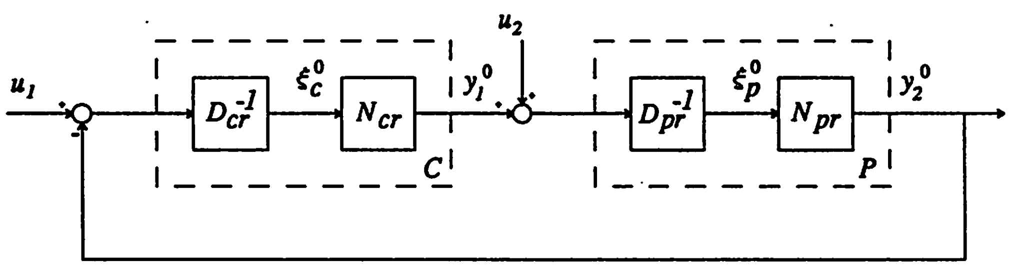

Figure 1 The feedback system $S(P, C)$

2. Lemma : Let Assumption 1 hold. From Figure 1, we obtain the causal S-stable linear map M, defined by *

$$M : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i \ , \ M : \begin{bmatrix} \xi_c^0 \\ \xi_p^0 \end{bmatrix} \mapsto \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} D_{cr} & N_{pr} \\ -N_{cr} & D_{pr} \end{bmatrix} \begin{bmatrix} \xi_c^0 \\ \xi_p^0 \end{bmatrix} . \qquad (1)$$

Then the system $S(P, C)$ is S-stable if and only if the bijective map M in (1) has a causal S-

*Equation (1) is written using matrix notation: the first equation states that $u_1 = D_{cr}(\xi_c^0) + N_{pr}(\xi_p^0)$.

stable inverse.

Proof : By Assumption 1, P and C have normalized right-coprime factorizations. Hence the well-posed system $S(P, C)$ is S-stable if and only if the pseudo-state maps $H_{\xi_c^0}^0 : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \xi_c^0$ and $H_{\xi_p^0}^0 : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \xi_p^0$ are S-stable: the sufficiency follows by Figure 1 and the S-stability of N_{pr}, D_{pr}, N_{cr} and D_{cr}; the necessity follows by the fact that the maps C and P have normalized right-coprime factorizations. Writing the summing node equations in Figure 1, we obtain (1); hence the system $S(P, C)$ is S-stable if and only if $M^{-1} = \begin{bmatrix} H_{\xi_c^0}^0 \\ H_{\xi_p^0}^0 \end{bmatrix}$ is S-stable.

□

Let Assumption 1 hold and let $S(P, C)$ be S-stable. Then the map M defined in (1) has a causal S-stable inverse M^{-1}. This inverse map is linear and is given by

$$M^{-1} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i \ , \ M^{-1} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} \xi_c^0 \\ \xi_p^0 \end{bmatrix} = \begin{bmatrix} D_{pl} & -N_{pl} \\ N_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{2}$$

where N_{cl}, D_{cl}, N_{pl} and D_{pl} denote the four submaps of M^{-1}; they are causal *linear* S-stable maps.

3. Comment : The causal S-stable maps denoted by D_{pl} and D_{cl} in (2) are bijective with causal inverses $D_{pl}^{-1} : \Lambda_e^o \to \Lambda_e^o$ and $D_{cl}^{-1} : \Lambda_e^i \to \Lambda_e^i$, respectively (indeed, using (3b) and (3c) below, $D_{pl}^{-1} = (I + PC)D_{cr}$, $D_{cl}^{-1} = (I + CP)D_{pr}$). From the equation

$$M^{-1}M = I_{\Lambda_e^o \times \Lambda_e^i} , \tag{3a}$$

we have "left factorizations" $D_{pl}^{-1}N_{pl}$ and $D_{cl}^{-1}N_{cl}$ of P and C, respectively; in fact these are "left-coprime factorizations" since

$$D_{pl}D_{cr} + N_{pl}N_{cr} = I_{\Lambda_e^o} \tag{3b}$$

$$D_{cl}D_{pr} + N_{cl}N_{pr} = I_{\Lambda_e^i} . \tag{3c}$$

□

Let $\Delta P : \Lambda_e^i \to \Lambda_e^o$ ($\Delta P : \Lambda_e^o \to \Lambda_e^i$) be any causal *nonlinear* map such that the feedback system $S((P, \Delta P)_{22}, C)$ in Figure 2 ($\hat{S}((P, \Delta P)_{22}, C)$ in Figure 3) is well-posed.

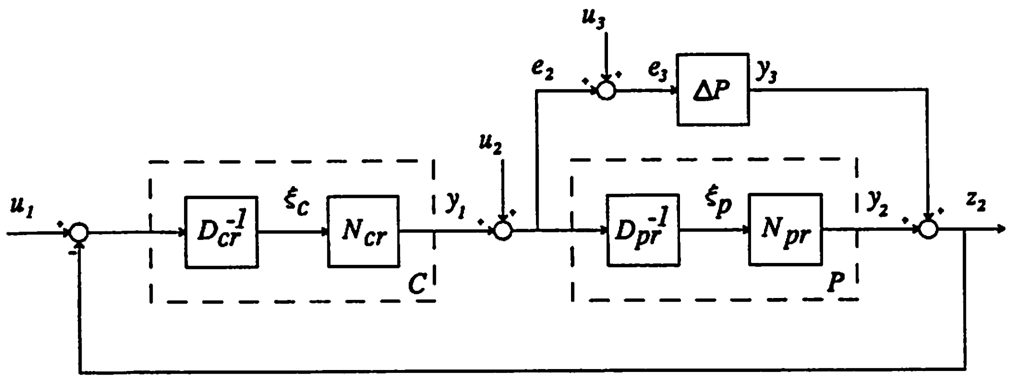

Figure 2 The feedback system $S((P, \Delta P)_{22}, C)$

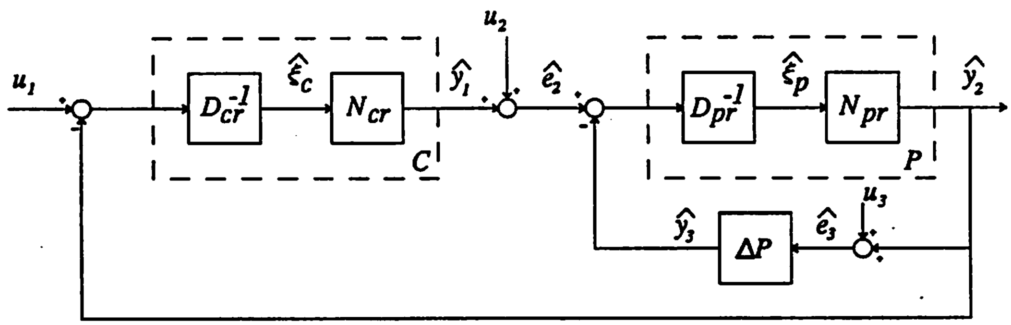

Figure 3 The feedback system $\hat{S}((P, \Delta P)_{22}, C)$

We consider four perturbation cases :

i) $S((P, \Delta P)_{22}, C)$ $(\hat{S}((P, \Delta P)_{22}, C))$: This perturbed system is obtained from $S(P, C)$ by replacing P with a nonlinear perturbed version $(P, \Delta P)_{22}$ which has *two* inputs (e_2, u_3) $((\hat{e}_2, u_3))$ and *two* observed outputs (y_2, y_3) $((\hat{y}_2, \hat{y}_3))$.

Suppose that in case i) we observe only z_2 $(\hat{y}_2)$; then we obtain

ii) $S((P, \Delta P)_{21}, C)$ $(\hat{S}((P, \Delta P)_{21}, C))$: the perturbation $(P, \Delta P)_{21}$ has *two* inputs (e_2, u_3) $((\hat{e}_2, u_3))$ and *one* observed output z_2 $(\hat{y}_2)$.

Suppose that we set $u_3 \equiv 0$ in case i), then we obtain

iii) $S((P, \Delta P)_{12}, C)$ $(\hat{S}((P, \Delta P)_{12}, C))$: the perturbation $(P, \Delta P)_{12}$ has *one* input e_2 $(\hat{e}_2)$ and *two* observed outputs (y_2, y_3) $((\hat{y}_2, \hat{y}_3))$.

Suppose that in case i), we set $u_3 \equiv 0$ and observe only z_2 $(\hat{y}_2)$; then we obtain

iv) $S(P + \Delta P, C)$ $(\hat{S}((P, \Delta P)_{11}, C))$: the perturbation $P + \Delta P$ $((P, \Delta P)_{11})$ has *one* input e_2 ($\hat{e}_2$) and *one* observed output z_2 ($\hat{y}_2$).

Note that for $i, j = 1, 2$, the $(i+1)$-input system $S((P, \Delta P)_{ij}, C)$ ($\hat{S}((P, \Delta P)_{ij}, C)$) is S-stable iff the $j+1$ outputs (i.e. j outputs of $(P, \Delta P)_{ij}$ and y_1 ($\hat{y}_1$)) are determined by causal S-stable maps.

4. Theorem : **(Necessary and Sufficient Condition for Robustness)** Let Assumption 1 hold. Let the *linear* system $S(P, C)$ be S-stable. Then for any causal *nonlinear* map $\Delta P : \Lambda_e^i \to \Lambda_e^o$ ($\Delta P : \Lambda_e^o \to \Lambda_e^i$),

i) the well-posed $S((P, \Delta P)_{22}, C)$ ($\hat{S}((P, \Delta P)_{22}, C)$) is S-stable if and only if $\Delta P (I + N_{cr} D_{pl} \Delta P)^{-1}$ ($\Delta P (I + N_{pr} D_{cl} \Delta P)^{-1}$) is S-stable.

ii) the well-posed $S((P, \Delta P)_{21}, C)$ ($\hat{S}((P, \Delta P)_{21}, C)$) is S-stable if and only if $D_{pl} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1}$ ($N_{pl} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1}$) is S-stable.

iii) the well-posed $S((P, \Delta P)_{12}, C)$ ($\hat{S}((P, \Delta P)_{12}, C)$) is S-stable if and only if $\Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} D_{pr}$ ($\Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} N_{pr}$) is S-stable.

iv) the well-posed $S(P + \Delta P, C)$ ($\hat{S}((P, \Delta P)_{11}, C)$) is S-stable if and only if $D_{pl} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} D_{pr}$ ($N_{pl} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} N_{pr}$) is S-stable.

5. Comment : We offer the following explanation on the forms of the necessary and sufficient conditions for $S((P, \Delta P)_{ij}, C), i, j = 1, 2$. Similar explanations apply for $\hat{S}((P, \Delta P)_{ij}, C), i, j = 1, 2$.

1) The effect of *not* observing y_3:

Since y_3 is *not* observed, instead of considering the system $S((P, \Delta P)_{22}, C)$ (i.e. the S-stability of the map $(u_1, u_2, u_3) \mapsto (y_1, y_2, y_3)$), we consider the system $S((P, \Delta P)_{21}, C)$ (i.e. the S-stability of the map $(u_1, u_2, u_3) \mapsto (y_1, z_2)$). Using the "left factorization" of P mentioned in Comment 3, we redraw the latter system as in Figure 4.

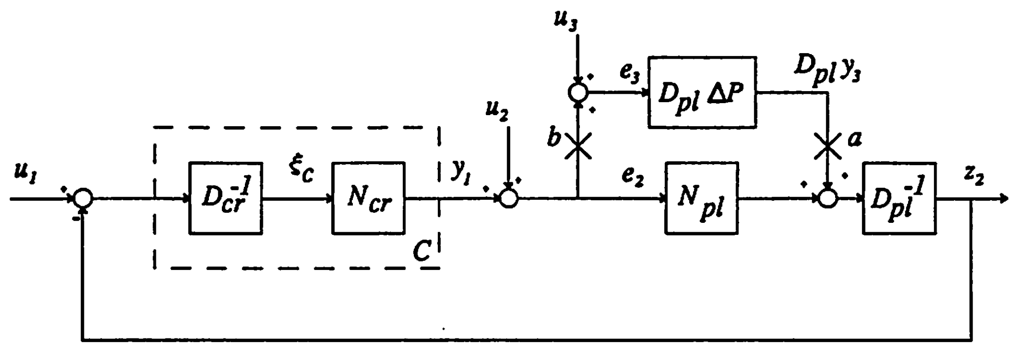

Figure 4 The feedback system $S((P, \Delta P)_{21}, C)$

Now view Figure 4 as a feedback system Σ consisting of the nonlinear, possibly unstable, subsystem $D_{pl}\Delta P$ closed in a feedback loop by the S-stable subsystem whose input is at a and output at b ; note that $b = -N_{cr}a + D_{pr}N_{cl}u_1 + D_{pr}D_{cl}u_2$. The resulting closed loop system Σ is S-stable if and only if $(D_{pl}\Delta P)(I + N_{cr}(D_{pl}\Delta P))^{-1}$ is S-stable [Des.2].

In conclusion, whenever we fail to observe y_3, the necessary and sufficient condition for S-stability has D_{pl} as an additional *left* factor.

2) The effect of setting $u_3 \equiv 0$:

By linearity and S-stability of $S(P, C)$, the map $y_3 \mapsto e_2$ (see Figure 2) is given by $e_2 = -N_{cr}D_{pl}y_3 + D_{pr}(N_{cl}u_1 + D_{cl}u_2)$. Now consider Figure 2 as a feedback system Σ consisting of the subsystem ΔP in a closed loop with the S-stable subsytem whose input is y_3 and output is e_2. Whenever $u_3 \equiv 0$, the inputs to this equivalent system Σ are in the range of D_{pr}, hence the necessary and sufficient condition for S-stability has D_{pr} as a *right* factor.

Proof of Theorem 4 : Since $S(P, C)$ is S-stable by assumption, the *linear* map M^{-1} given by (2) is causal S-stable. Writing the summing node equations in Figure 2 (Figure 3) in terms of ξ_c, ξ_p and e_3 ($\hat{\xi}_c$, $\hat{\xi}_p$ and $\hat{e}_3$), we obtain

$$M\begin{bmatrix}\xi_c\\ \xi_p\end{bmatrix} = \begin{bmatrix}u_1 - \Delta P e_3\\ u_2\end{bmatrix} \qquad \left(M\begin{bmatrix}\hat{\xi}_c\\ \hat{\xi}_p\end{bmatrix} = \begin{bmatrix}u_1\\ u_2 - \Delta P\hat{e}_3\end{bmatrix} \right) \tag{4a}$$

$$e_3 = u_2 + u_3 + N_{cr}\xi_c \ . \qquad \left(\hat{e}_3 = u_3 + N_{pr}\hat{\xi}_p \ . \right) \tag{4b}$$

By linearity of M^{-1} and equation (2) we obtain

$$\begin{bmatrix}\xi_c\\ \xi_p\end{bmatrix} = \begin{bmatrix}\xi_c^0\\ \xi_p^0\end{bmatrix} - \begin{bmatrix}D_{pl}\\ N_{cl}\end{bmatrix}\Delta P e_3 \ . \qquad \left(\begin{bmatrix}\hat{\xi}_c\\ \hat{\xi}_p\end{bmatrix} = \begin{bmatrix}\xi_c^0\\ \xi_p^0\end{bmatrix} - \begin{bmatrix}-N_{pl}\\ D_{cl}\end{bmatrix}\Delta P\hat{e}_3 \ . \right) \tag{5}$$

From (4b) and (5), e_3 ($\hat{e}_3$) is determined by

$$e_3 = u_2 + u_3 + N_{cr}\xi_c^0 - N_{cr}D_{pl}\Delta P e_3 . \qquad \left[\hat{e}_3 = u_3 + N_{pr}\xi_p^0 - N_{pr}D_{cl}\Delta P\hat{e}_3 . \right] \tag{6}$$

Substituting equation (2) in (6) and using the equalities $I - N_{cr}N_{pl} = D_{pr}D_{cl}$ and $N_{cr}D_{pl} = D_{pr}N_{cl}$ from the equation $MM^{-1} = I$, we obtain

$$e_3 = (I + N_{cr}D_{pl}\Delta P)^{-1}\left[D_{pr}\left[N_{cl} \;\; D_{cl} \right] \;\; I \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \tag{7a}$$

$$\left[\hat{e}_3 = (I + N_{pr}D_{cl}\Delta P)^{-1}\left[N_{pr}\left[N_{cl} \;\; D_{cl} \right] \;\; \cdot I \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \right] \tag{7b}$$

Substituting (7a) ((7b)) in (5), we obtain the pseudo-state map $\begin{bmatrix} H_{\xi_c} \\ H_{\xi_p} \end{bmatrix} : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \begin{bmatrix} \xi_c \\ \xi_p \end{bmatrix}$

$\left[\begin{bmatrix} \hat{H}_{\hat{\xi}_c} \\ \hat{H}_{\hat{\xi}_p} \end{bmatrix} : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \begin{bmatrix} \hat{\xi}_c \\ \hat{\xi}_p \end{bmatrix} \right]$, where

$$\begin{bmatrix} \xi_c \\ \xi_p \end{bmatrix} = \begin{bmatrix} \xi_c^0 \\ \xi_p^0 \end{bmatrix} - \begin{bmatrix} D_{pl} \\ N_{cl} \end{bmatrix} \Delta P (I + N_{cr}D_{pl}\Delta P)^{-1}\left[D_{pr}\left[N_{cl} \;\; D_{cl} \right] \;\; I \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \tag{8a}$$

$$\left[\begin{bmatrix} \hat{\xi}_c \\ \hat{\xi}_p \end{bmatrix} = \begin{bmatrix} \xi_c^0 \\ \xi_p^0 \end{bmatrix} - \begin{bmatrix} -N_{pl} \\ D_{cl} \end{bmatrix} \Delta P (I + N_{pr}D_{cl}\Delta P)^{-1}\left[N_{pr}\left[N_{cl} \;\; D_{cl} \right] \;\; I \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \right] \tag{8b}$$

Now we state the necessary and sufficient conditions for the four cases in terms of the pseudo-state maps given by (8a) ((8b)).

i) the well-posed $S((P, \Delta P)_{22}, C)$ ($\hat{S}((P, \Delta P)_{22}, C)$) is S-stable if and only if H_{ξ_c} and H_{ξ_p} ($\hat{H}_{\hat{\xi}_c}$ and $\hat{H}_{\hat{\xi}_p}$) are S-stable. The sufficiency follows from Figure 2 (Figure 3), and the S-stability of N_{pr}, D_{pr}, N_{cr} and D_{cr}. The necessity follows by the fact that C and P have normalized right-coprime factorizations. Using similar reasoning, we get the following :

ii) the well-posed $S((P, \Delta P)_{21}, C)$ ($\hat{S}((P, \Delta P)_{21}, C)$) is S-stable if and only if H_{ξ_c} ($\hat{H}_{\hat{\xi}_c}$) is S-stable.

iii) the well-posed $S((P,\Delta P)_{12}, C)$ $(\hat{S}((P,\Delta P)_{12}, C))$ is S-stable if and only if $H_{\xi_c}|_{u_3\equiv 0}$ and $H_{\xi_p}|_{u_3\equiv 0}$ $(\hat{H}_{\hat{\xi}_c}|_{u_3\equiv 0}$ and $\hat{H}_{\hat{\xi}_p}|_{u_3\equiv 0})$ are S-stable.

iv) the well-posed $S(P+\Delta P, C)$ $(\hat{S}((P,\Delta P)_{11}, C))$ is S-stable if and only if $H_{\xi_c}|_{u_3\equiv 0}$ $(\hat{H}_{\hat{\xi}_c}|_{u_3\equiv 0})$ is S-stable.

Using equation (8a) ((8b)), we consider the four cases just mentioned.

i) Equation (8a) ((8b)) shows that H_{ξ_c} and H_{ξ_p} $(\hat{H}_{\hat{\xi}_c}$ and $\hat{H}_{\hat{\xi}_p})$ are S-stable if and only if the map F_1 $(\hat{F}_1)$, where

$$F_1 := \begin{bmatrix} D_{pl} \\ N_{cl} \end{bmatrix} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} \begin{bmatrix} D_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} & I \end{bmatrix} \tag{9a}$$

$$\left[\quad \hat{F}_1 := \begin{bmatrix} -N_{pl} \\ D_{cl} \end{bmatrix} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} \begin{bmatrix} N_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} & I \end{bmatrix} \quad \right] \tag{9b}$$

is S-stable. Since D_{pl}, D_{cl}, N_{pl}, N_{cl}, N_{pr} and D_{pr} are S-stable maps, F_1 $(\hat{F}_1)$ is S-stable if $\Delta P (I + N_{cr} D_{pl} \Delta P)^{-1}$ $(\Delta P (I + N_{pr} D_{cl} \Delta P)^{-1})$ is S-stable. Conversely, by (1), (2) and equation (9a) ((9b)),

$$\Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} = \begin{bmatrix} D_{cr} & N_{pr} \end{bmatrix} F_1 \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

$$\left[\quad \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} = \begin{bmatrix} -N_{cr} & D_{pr} \end{bmatrix} \hat{F}_1 \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad \right]$$

is S-stable if F_1 $(\hat{F}_1)$ is S-stable. Hence case i) follows.

ii) Equation (8a) ((8b)) shows that H_{ξ_c} $(\hat{H}_{\hat{\xi}_c})$ is S-stable if and only if the map F_2 $(\hat{F}_2)$,

$$F_2 := D_{pl} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} \begin{bmatrix} D_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} & I \end{bmatrix}$$

$$\left[\quad \hat{F}_2 := N_{pl} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} \begin{bmatrix} N_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} & I \end{bmatrix} \quad \right]$$

is S-stable. F_2 $(\hat{F}_2)$ is S-stable if $D_{pl} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1}$ $(N_{pl} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1})$ is S-stable. Conversely, $F_2 \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$ $\left(\hat{F}_2 \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}\right)$ is S-stable if F_2 $(\hat{F}_2)$ is S-stable. Hence case ii) follows.

iii) Equation (8a) ((8b)) shows that $H_{\xi_e}|_{u_3\equiv 0}$ and $H_{\xi_p}|_{u_3\equiv 0}$ ($\hat{H}_{\xi_e}|_{u_3\equiv 0}$ and $\hat{H}_{\xi_p}|_{u_3\equiv 0}$) are S-stable if and only if the map F_3 ($\hat{F}_3$),

$$F_3 := \begin{bmatrix} D_{pl} \\ N_{cl} \end{bmatrix} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} D_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} \tag{10a}$$

$$\left(\hat{F}_3 := \begin{bmatrix} -N_{pl} \\ D_{cl} \end{bmatrix} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} N_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} \right) \tag{10b}$$

is S-stable. F_3 ($\hat{F}_3$) is S-stable if $\Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} D_{pr}$ ($\Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} N_{pr}$) is S-stable. Conversely, by (1), (2) and equation (10a) ((10b)), $\begin{bmatrix} D_{cr} & N_{pr} \end{bmatrix} F_3 \begin{bmatrix} N_{pr} \\ D_{pr} \end{bmatrix}$ ($\begin{bmatrix} -N_{cr} & D_{pr} \end{bmatrix} \hat{F}_3 \begin{bmatrix} N_{pr} \\ D_{pr} \end{bmatrix}$) is S-stable if F_3 ($\hat{F}_3$) is S-stable. Hence case iii) follows.

iv) Equation (8a) ((8b)) shows that $H_{\xi_e}|_{u_3\equiv 0}$ ($\hat{H}_{\xi_e}|_{u_3\equiv 0}$) is S-stable if and only if the map F_4 ($\hat{F}_4$)

$$F_4 := D_{pl} \Delta P (I + N_{cr} D_{pl} \Delta P)^{-1} D_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix}$$

$$\left(\hat{F}_4 := N_{pl} \Delta P (I + N_{pr} D_{cl} \Delta P)^{-1} N_{pr} \begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix} \right)$$

is S-stable. Case iv) follows by the fact that $\begin{bmatrix} N_{cl} & D_{cl} \end{bmatrix}$ is S-stable and has an S-stable right inverse, namely $\begin{bmatrix} N_{pr} \\ D_{pr} \end{bmatrix}$.

□

Solving for ΔP in the four (three) necessary and sufficient conditions in Proposition 4, we obtain a characterization of the set of all nonlinear perturbations ΔP for which the perturbed system remains S-stable (called addmissable perturbations) :

6. Corollary : (Characterization of admissable ΔP 's) Let Assumption 1 hold. Let the linear system $S(P, C)$ be S-stable. Then

i) the well-posed $S((P, \Delta P)_{22}, C)$ ($\hat{S}((P, \Delta P)_{22}, C)$) is S-stable if and only if $\Delta P = Q(I - N_{cr} D_{pl} Q)^{-1}$ ($\Delta P = Q(I - N_{pr} D_{cl} Q)^{-1}$) for some causal S-stable map Q.

ii) the well-posed $S((P, \Delta P)_{21}, C)$ ($\hat{S}((P, \Delta P)_{21}, C)$) is S-stable if and only if $\Delta P = D_{pl}^{-1} Q(I - N_{cr} Q)^{-1}$ ($N_{pl} \Delta P = Q(I - D_{cr} Q)^{-1}$) for some causal S-stable map Q.

iii) the well-posed $S((P,\Delta P)_{12}, C)$ ($\hat{S}((P,\Delta P)_{12}, C)$) is S-stable if and only if $\Delta P = Q(I - N_{cl}Q)^{-1}D_{pr}^{-1}$ ($\Delta P N_{pr} = Q(I - D_{cl}Q)^{-1}$) for some causal S-stable map Q.

iv) the well-posed $S(P+\Delta P, C)$ is S-stable if and only if $\Delta P = D_{pl}^{-1}Q(D_{pr} - N_{cr}Q)^{-1}$ for some causal S-stable Q.

□

Conclusion

From Corollary 6 i)-iii), we conclude that for ΔP to be an admissable perturbation, ΔP , $D_{pl}\Delta P$ and $\Delta P D_{pr}$ (ΔP , $N_{pl}\Delta P$ and $\Delta P N_{pr}$) must have the specific normalized right-coprime factorizations.

In the case that the plant P has right- and left-coprime factorizations (see [Vid.3] for the linear time-invariant case, [Man.1] for the continuous-time linear time-varying case), the set of all stabilizing compensators C for the nominal plant is given in terms of the plant factorizations and a free linear stable parameter; hence the factors N_{cl} , N_{cr} , D_{cl} and D_{cr} of C in Corollary 6 would also depend on this free linear stable parameter.

References

[Åst.1] K. J. Åström, "*Robustness of a Design Method Based on Assignment of Poles and Zeros*", IEEE Transactions on Automatic Control, vol. AC-25 no. 3, pp. 588-591, June 1980.

[Bha.1] A. Bhaya and C. A. Desoer, "*Robust Stability under Additive Perturbations*", IEEE Transactions on Automatic Control, vol. AC-30 no. 12, pp. 1233-1234, December 1985.

[Che.1] M. J. Chen and C. A. Desoer, "*Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems*", International Journal of Control, vol.35, pp. 255-267, February 1982.

[Che.2] M. J. Chen and C. A. Desoer, "*Algebraic Theory for Robust Stability of Interconnected Systems: Necessary and Sufficient Conditions*", IEEE Transactions on

Automatic Control, vol. AC-29 no. 6, pp. 511-519, June 1984.

[Cru.1] J. B. Cruz, J. S. Freudenberg and D. P. Looze, "*A Relationship Between Sensitivity and Stability of Multivariable Feedback Systems*", IEEE Transactions on Automatic Control, vol. AC-26 no. 1, pp. 66-74, February 1981.

[Des.1] C. A. Desoer, M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, 1975.

[Des.2] C. A. Desoer and W. S. Chan, "*The Feedback Interconnection of Lumped Linear Time-Invariant Systems*", Journal of Franklin Institute, vol.300, pp. 335-351, November - December 1975.

[Des.3] C. A. Desoer, F. M. Callier and W. S. Chan, "*Robustness of Stability Conditions for Linear Time-Invariant Feedback Systems*", IEEE Transactions on Automatic Control, vol. AC-22 no. 4, pp. 586-590, August 1977.

[Doy.1] J. C. Doyle and G. Stein, "*Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis*", IEEE Transactions on Automatic Control, vol. AC-26 no. 1, pp. 4-16, February 1981.

[Fra.1] B. A. Francis, "*On Robustness of the Stability of Feedback Systems*", IEEE Transactions on Automatic Control, vol. AC-25 no. 4, pp. 817-818, August 1980.

[Ham.1] J. Hammer, "*Nonlinear Systems, Stabilization and Coprimeness*", International Journal of Control, vol. 42 no. 1, pp. 1-20, 1985.

[Hua.1] Q. Huang and R. W. Liu, "*Robustness Stability for Nonlinear Perturbation*", pp. 514-517, Asilomar Conference 1985.

[Hua.2] Q. Huang, R. W. Liu, "*A Necessary and Sufficient Condition for Stability of a Perturbed System*", IEEE Transactions on Automatic Control, vol. AC-32 no. 4, pp. 337-340, April 1987.

[Man.1] V. Manousiouthakis, "*On Time-Varying Control*", Proceedings of ACC Minneapolis, pp. 2050-2053, 1987.

[Owe.1] D. H. Owens and A. Chotai, "*Robust Stability of Multivariable Feedback Systems with Respect to Linear and Nonlinear Feedback Perturbations*", IEEE Transactions on Automatic Control, vol. AC-27 no. 1, pp. 254-256, February 1982.

[Pos.1] I. Postlethwaite, J. M. Edmunds and A. G. J. MacFarlane, "*Principal Gains and Principal Phases in the Analysis of Linear Multivariable Feedback Systems*", IEEE Transactions on Automatic Control, vol. AC-26 no. 1, pp. 32-46, February 1981.

[Saf.1] M. G. Safonov, *Stability and Robustness of Multivariable Feedback Systems*, MIT Press, 1980.

[San.1] N. R. Sandell, "*Robust Stability of Systems with Application to Singular Perturbations*", Automatica, vol.15, pp. 467-470, July 1979.

[Vid.1] M. Vidyasagar and H. Kimura, "*Robust Controllers for Uncertain Linear Multivariable Systems*", Automatica, vol.22 no. 1, pp. 85-94, 1986.

[Vid.2] M. Vidyasagar, N. Viswanadham, "*Stabilization of Linear and Nonlinear Dynamical Systems using an Observer-Controller Configuration*", Systems and Control Letters, vol. 1 no. 2 , pp. 87-91, August 1981.

[Vid.3] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

[Wil.1] J. C. Willems, *The Analysis of Feedback Systems*, MIT Press, 1971.

[Zam.1] G. Zames, "*On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part I : Conditions Derived Using Concepts of Loop Gain, Conicity and Positivity*", IEEE Transactions on Automatic Control, vol. AC-11 no. 2, pp. 228-238, April 1966.