AN ELEMENTARY PROOF OF KHARITONOV'S STABILITY THEOREM WITH EXTENSIONS

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R. J. Minnichelli, J. J. Anagnost, and C. A. Desoer

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ABSTRACT

This paper gives an elementary proof of Kharitonov's Theorem using simple complex plane geometry. Kharitonov's Theorem is a stability result for classes of polynomials defined by letting each coefficient vary independently in an arbitrary interval. The result states that the whole class is Hurwitz if and only if four special, well-defined polynomials are Hurwitz.

The paper also gives elementary proofs of two previously known extensions: for polynomials of degree less than six, the requirement is reduced to fewer than four polynomials; and the theorem is generalized to polynomials with complex coefficients. Finally, we apply Kharitonov's Theorem and the generalized stability theorem to find sufficient (but conservative) conditions for a class of polynomials to be U-Hurwitz for certain sets U of "undesirable" polynomial zero locations in the complex plane.

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1. INTRODUCTION

In 1978 V. L. Kharitonov [1] published a stability theorem for classes of polynomials defined by letting each coefficient vary independently in a specified (but arbitrary) interval. This remarkable result states that the whole class of polynomials is Hurwitz if and only if *four* special, well-defined polynomials are Hurwitz. Kharitonov also provided a generalization of this theorem for polynomials with complex coefficients [2].

Unfortunately, these results remained largely unknown or unappreciated for several years, in part due to the complicated induction argument of the original proof. Recently, however, the result has been considerably simplified, applied and extended (e.g. [3-9]). The simplifications in [3] and [4] involve using the behavior of this class of polynomials on the imaginary axis of the complex plane. In [3], the class is partitioned into line segments and the behavior of each segment is considered. In [4], the image of the whole class (when evaluated at a point on the imaginary axis) is shown to be a rectangle (in the complex plane) which can be analyzed all at once.

In this paper we extend the simplifications in [4] to prove Kharitinov's Theorem using only simple complex plane geometry and eliminating the use of the Hermite-Bieler Theorem (the "interlacing property"). Kharitonov's Theorem is sufficiently important to the engineering community to motivate every effort to increase its accessibility.

In Sections 5 and 6 we continue to use the complex plane geometry arguments to provide elementary derivations of two extensions of Kharitonov's Theorem. For polynomials of degree 3, 4 or 5, we show that Kharitonov's criterion is equivalent to checking 1, 2 or 3 polynomials, respectively. This result was first published by Anderson, Jury and Mansour [5]. Then we prove the generalization to polynomials with complex coefficients, due to Kharitonov [2] and elucidated by Bose and Shi [6].

In Section 7, we apply both the original theorem and the generalized theorem to produce sufficient conditions for the classes of polynomials to be U-Hurwitz for special forbidden regions U of polynomial zero locations. These conditions can be used to guarantee specified performance of linear systems in terms of damping ratio and settling time constraints. The conditions are conservative.

2. NOTATION

A polynomial with real coefficients is said to be Hurwitz if and only if all of its zeroes lie in $\mathring{\mathbb{C}}_-$ (i.e. $\forall z \in \mathbb{C}\ p(z)=0 \Rightarrow \mathrm{Re}\{z\}<0$). A set of polynomials is said to be Hurwitz if and only if every member is Hurwitz.

Fix $n \geq 1$ and $\underline{a}, \overline{a} \in \mathbb{R}^n$, $\underline{a}_k \leq \overline{a}_k$ $k=0,\ldots,n-1$. We define the set N to be the set of monic n^{th} degree polynomials of the form

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 \tag{1}$$

for all $a_0,\ldots,a_{n-1}$ such that $\underline{a}_k \leq a_k \leq \overline{a}_k$, $k=0,\ldots,n-1$. Then for all $\omega \in \mathbb{R}$ we define $H(\omega) := \{p(j\omega) : p \in N\} \subset \mathbb{C}$; i.e. $H(\omega)$ is the image of N under the evaluation map at $s=j\omega$ (where the evaluation map at s, $e_s(\cdot)$, maps polynomials into the complex plane and is given by $e_s(p) := p(s)$).

Next we define the polynomials (where $\underline{a}_n = \overline{a}_n := 1$):

$$g_1(s) := \underline{a}_0 + \overline{a}_2 s^2 + \underline{a}_4 s^4 + \cdots = \sum_{\substack{k=0\\ even}}^{n} j^k \cdot \min\{j^k \underline{a}_k, j^k \overline{a}_k\} \cdot s^k$$

$$g_2(s) := \overline{a}_0 + \underline{a}_2 s^2 + \overline{a}_4 s^4 + \cdots = \sum_{\substack{k=0\\ even}}^{n} j^k \cdot \max\{j^k \underline{a}_k, j^k \overline{a}_k\} \cdot s^k$$

$$h_1(s) := \underline{a}_1 s + \bar{a}_3 s^3 + \underline{a}_5 s^5 + \cdots = \sum_{\substack{k=1 \\ odd}}^{n} j^{k-1} \cdot \min\{j^{k-1}\underline{a}_k, j^{k-1}\bar{a}_k\} \cdot s^k$$

$$h_2(s) := \bar{a}_1 s + \underline{a}_3 s^3 + \bar{a}_5 s^5 + \cdots = \sum_{\substack{k=0 \\ odd}}^{n} j^{k-1} \cdot \max\{j^{k-1}\underline{a}_k, j^{k-1}\bar{a}_k\} \cdot s^k$$

Finally, we define the Kharitonov polynomials:

$$k_{11}(s) := g_1(s) + h_1(s)$$
$$k_{12}(s) := g_1(s) + h_2(s)$$
$$k_{21}(s) := g_2(s) + h_1(s)$$
$$k_{22}(s) := g_2(s) + h_2(s)$$

Remark: $\{k_{11}(\cdot), k_{12}(\cdot), k_{21}(\cdot), k_{22}(\cdot)\} \subset N$, and $\forall \omega \in \mathbb{R}$, $g_1(j\omega)$ and $g_2(j\omega)$ are purely real, while $h_1(j\omega)$ and $h_2(j\omega)$ are purely imaginary. Furthermore, $\forall \omega \geq 0$, we have

$$\mathrm{Re}\{g_1(j\omega)\} \leq \mathrm{Re}\{p(j\omega)\} \leq \mathrm{Re}\{g_2(j\omega)\} \qquad \forall p(\cdot) \in N \tag{2}$$

$$\mathrm{Im}\{h_1(j\omega)\} \leq \mathrm{Im}\{p(j\omega)\} \leq \mathrm{Im}\{h_2(j\omega)\} \qquad \forall p(\cdot) \in N \tag{3}$$

($\forall\ \omega \leq 0$ we switch h_1 and h_2 in Equation 3). Thus we see that $\forall \omega \in \mathbb{R}$, $H(\omega)$ is a level rectangle (i.e. the sides are parallel to the real and imaginary axes) with corners $k_{11}(j\omega)$, $k_{12}(j\omega)$, $k_{21}(j\omega)$, $k_{22}(j\omega)$, as shown in Figure 1.[1] This picture of $H(\omega)$ as a rectangle in $\mathbb{C}$ was developed by Dasgupta [4].

3. ELEMENTARY PROPERTIES OF HURWITZ POLYNOMIALS

Property 1: If a monic polynomial $p(\cdot)$ is Hurwitz, then all of its coefficients are positive.

Proof: The property is clear for $n=1$ and $n=2$ (since $(s+\nu+j\mu)(s+\nu-j\mu) = s^2+2\nu s+(\nu^2+\mu^2)$). In general, we can write $p(\cdot)$ as the product of k monomials and l binomials (corresponding to real zeroes and complex conjugate pairs, respectively)

$$p(s) = \prod_{i=1}^{k}(s+\gamma_i)\prod_{i=1}^{l}(s^2+\alpha_i s+\beta_i)$$

with $\gamma_i, \alpha_i, \beta_i$ all positive. Thus each coefficient of $p(\cdot)$ is a sum of products of positive numbers.

Property 2: If $p(\cdot)$ is Hurwitz with degree $n \geq 1$ then $arg(p(j\omega))$ is a continuous and strictly increasing function of ω.

Proof: $p(\cdot)$ Hurwitz $\Rightarrow p(s) = \prod_{i=1}^{n}(s-z_i)$ with $z_i = \alpha_i + j\beta_i$ and $\alpha_i < 0$ $(i=1, \ldots, n)$. So

$$arg(p(j\omega)) = \sum_{i=1}^{n} arg(j\omega + |\alpha_i| - j\beta_i) = \sum_{i=1}^{n} arctan\left(\frac{\omega-\beta_i}{|\alpha_i|}\right)$$

and the summands are all continuous and strictly increasing functions of ω.

[1] Actually, we have only shown that $H(\omega)$ is contained in the rectangle defined by the four Kharitonov polynomials, and, indeed, that is all we require in the sequel. However, it is clear that by taking convex combinations of the four Kharitonov polynomials, we remain in the set N and cover the rectangle. Thus we continue to refer to $H(\omega)$ as a rectangle.

Figure 2 illustrates the proof of Property 2; if all the zeroes of $p(s)$ are in the open left half plane, then the angle contribution of each zero increases as s moves up along the imaginary axis. Figure 3 shows the result; as ω goes from 0 to $+\infty$, $p(j\omega)$ starts on the positive real axis and smoothly circles strictly counterclockwise around the origin $n\pi/2$ radians before going to infinity.

If $arg(p(j\omega))$ is continuous in ω (i.e. if $p(j\omega)\neq 0\ \forall\omega$), we can define

$$arg_{net}(p) := \lim_{\omega\to\infty}\{arg(p(j\omega)) - arg(p(0))\}$$

where the right hand side of the equation denotes the *net total angle*, counting encirclements, subtended by $p(j\omega)$ as ω goes from 0 to $+\infty$. Consideration of the s-plane of Figure 2 reveals that each zero in $\mathbb{C}_-$ contributes $\pi/2$ to $arg_{net}(p)$ while each zero in $\mathbb{C}_+$ contributes $-\pi/2$ (pairwise for complex conjugates, of course). Thus we have the following characterization of Hurwitz polynomials.

Property 3: An n^{th} degree polynomial $p(\cdot)$ is Hurwitz if and only if $arg_{net}(p)$ is well defined (i.e. $p(j\omega)\neq 0\ \forall\omega$) and equal to $n\pi/2$.

4. KHARITONOV'S THEOREM

Lemma: If the Kharitonov polynomials $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}$ are Hurwitz then $0\notin H(\omega)\ \forall\omega\in\mathbb{R}$.

Proof: For $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}$ to be Hurwitz we must have $\underline{a}_i>0\ \forall i$ (Property 1). Clearly $0\notin H(0)=[\underline{a}_0,\bar{a}_0]$. Since $H(\cdot)$ is "continuous" (i.e. the four corners vary continuously with ω), if $0\in H(\omega)$ for some $\omega>0$ then 0 must be on the *boundary* of $H(\hat{\omega})$ for some $\hat{\omega}\leq\omega$. Since no corner may pass through zero (the corners are Hurwitz), we must have an edge containing zero in its *interior*. Without loss of generality we assume it's the "bottom" edge. Then, as illustrated in Figure 4, $k_{11}(j\hat{\omega})$ is on the negative real axis and $k_{21}(j\hat{\omega})$ is on the positive real axis. Property 2 implies that for $\delta\omega>0$ sufficiently small we have $k_{11}(j(\hat{\omega}+\delta\omega))$ in the open third quadrant and $k_{21}(j(\hat{\omega}+\delta\omega))$ in the open first quadrant. Since $\text{Im}\{k_{11}(j\omega)\}=\text{Im}\{k_{21}(j\omega)\}=\text{Im}\{h_1(j\omega)\}$, this is clearly not possible.

Remark: We see that the whole rectangle $H(\omega)$ must travel counterclockwise through a total angle of $n\pi/2$, always completely entering one quadrant before crossing into the next.

Theorem: The class of polynomials N is Hurwitz if and only if $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}$ is Hurwitz.

Proof: The "only if" is immediate since $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}\subset N$. So suppose $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}$ is Hurwitz, and $p(\cdot)\in N$. The Lemma implies that $p(j\omega)\neq 0\ \forall\omega$, so $arg_{net}(p)$ is well defined. Furthermore, $p(j\omega)\in H(\omega)\ \forall\omega$, so $arg_{net}(p)=n\pi/2$. Property 3 implies that $p(\cdot)$ has n zeroes in $\mathbb{C}_-$ and is Hurwitz.

Remark: The theorem can be proven without refering to the net angle property and moving rectangle argument by using the continuity of zeroes of polynomials and the fact that N is pathwise connected (it is a parallelepiped in the space of polynomial coefficients). If $\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}$ were Hurwitz and $p(\cdot)\in N$ were not Hurwitz, then on any path in N connecting $k_{11}(\cdot)$ to $p(\cdot)$ there would be a polynomial, $\hat{p}(\cdot)$, with a zero *on* the imaginary axis, say at $s=j\hat{\omega}$. This implies that $\hat{p}(j\hat{\omega})=0$, which is forbidden by the Lemma.

Remark: We can clearly extend Kharitonov's Theorem to any set S of polynomials satisfying

$$\{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\}\subset S\subset N$$

since

$$S \text{ Hurwitz} \Rightarrow \{k_{11}(\cdot),k_{12}(\cdot),k_{21}(\cdot),k_{22}(\cdot)\} \text{ Hurwitz} \Rightarrow N \text{ Hurwitz} \Rightarrow S \text{ Hurwitz}$$

where the first and third implications follow from containment and the second implication follows from Kharitonov's Theorem. For example, we can have $a_i \in A_i \subset \mathbb{R}$ where each A_i is bounded (but not necessarily an interval) and contains $\underline{a}_i := inf\, A_i$ and $\overline{a}_i := sup\, A_i$.

5. SIMPLIFICATIONS FOR DEGREE LESS THAN 6

Anderson, Jury and Mansour [5] demonstrated that the Kharitonov condition could be reduced for polynomials of low degree. For polynomials of degree 3, 4 or 5, the Kharitonov test is equivalent to checking 1, 2 or 3 of the Kharitonov polynomials, respectively. For degree greater than 5, all four polynomials must be checked.

We demonstrate the results pictorially, relying heavily on Dasgupta's picture of $H(\omega)$ as a rectangle in the complex plane with corners given by the Kharitonov polynomials. Again, we use only simple complex plane geometry (in contrast to the detailed calculations of [5]). In the process of proving these results, we also develop a stronger sense of the roles played by the Kharitonov polynomials to force $H(\omega)$ to circle counterclockwise around the origin through a total angle of $n\pi/2$ radians as ω goes from zero to infinity.

We begin by noting that the angles of all monic polynomials of degree n converge to $n\pi/2\,(mod\ 2\pi)$ as $\omega \to +\infty$ (since $p(j\omega)=(j\omega)^n[1+O(1/\omega)]$). So for any choice of $\underline{a}_k$ and $\overline{a}_k$, $k=0,\ldots,n-1$, the rectangle $H(\omega)$ will go to infinity at an asymptotic angle of $n\pi/2\,(mod\ 2\pi)$. The only question is whether or not it will circle the origin $n/4$ times (or $n\pi/2$ radians) counterclockwise, without intersecting the origin, in the process.

$n=3$: Assume $\underline{a}_0>0$;[2] then N is Hurwitz if and only if $k_{21}(\cdot)$ is Hurwitz.

Proof: The "only if" is immediate since $k_{21}(\cdot)\in N$. So we assume that $k_{21}(\cdot)$ is Hurwitz, and argue pictorially, refering to Figure 5. $k_{21}(j\omega)$ will always be the "lower right corner" of the rectangle $H(\omega)$ (for $\omega>0$), so as $k_{21}(j\omega)$ travels away from the positive real axis, through the first quadrant and then the second, it essentially "pushes" $H(\omega)$ off the positive real axis ($H(0)$ lies in the positive real axis because of our assumption that $\underline{a}_0>0$), into the open first quadrant and then the second, forcing $H(\omega)$ to completely enter the second quadrant before crossing the real axis into the lower half plane. Once $k_{21}(j\omega)$ has crossed into the second quadrant, it can never cross the imaginary axis again (since $arg(k_{21}(j\omega))$ must increase monotonically and approach $3\pi/2$). Since $k_{21}(j\omega)$ lies on the "right" edge of $H(\omega)$, $H(\omega)$ cannot enter the right half plane again. We have shown that $H(\omega)$ must travel counterclockwise around the origin, through the first quadrant and completely into second, and then remain in the open left half plane as it goes to infinity at an asymptotic angle of $3\pi/2$. Thus we see that $arg_{net}(p)=3\pi/2\ \forall p\in N$, and conclude that N is Hurwitz.

$n=4$: Assume $\underline{a}_0>0$; then N is Hurwitz if and only if $\{k_{21}(\cdot),k_{22}(\cdot)\}$ is Hurwitz.

Proof: The "only if" is immediate since $\{k_{21}(\cdot),k_{22}(\cdot)\}\subset N$. So we assume that $k_{21}(\cdot)$ and $k_{22}(\cdot)$ are both Hurwitz. As with the case for $n=3$, $k_{21}(j\omega)$ pushes $H(\omega)$ off the positive real axis into the open first quadrant, then completely into the second quadrant before $H(\omega)$ can cross into the lower half plane. Once $H(\omega)$ is in the second quadrant,

[2] Of course, there is no loss of generality in assuming that $\underline{a}_i>0$ for any or all i. If any $\underline{a}_i<0$, then Property 1 of Section 3 tells us that at least one Kharitonov polynomial will not be Hurwitz, thus N will not be Hurwitz.

we see that $k_{22}(j\omega)$--the "upper right corner"--now pushes $H(\omega)$ completely into the third quadrant before $H(\omega)$ can cross into the right half plane. Once $k_{22}(j\omega)$ enters the open third quadrant, it can never cross the real axis again, thus $H(\omega)$ must remain in the open lower half plane as it travels to infinity at an asymptotic angle of 2π (or 0). We have shown that $H(\omega)$ must travel counterclockwise around the origin through a net angle of 2π, and we conclude that N is Hurwitz.

n=5: N is Hurwitz if and only if $\{k_{12}(\cdot), k_{21}(\cdot), k_{22}(\cdot)\}$ is Hurwitz.

Proof: The "only if" is immediate since $\{k_{12}(\cdot), k_{21}(\cdot), k_{22}(\cdot)\} \subset N$. So assume that $k_{12}(\cdot)$, $k_{21}(\cdot)$, and $k_{22}(\cdot)$ are Hurwitz. As before, $k_{21}(j\omega)$ pushes $H(\omega)$ off the real axis, through the first quadrant and completely into the second; $k_{22}(j\omega)$ pushes $H(\omega)$ from the second quadrant completely into the third; and $k_{12}(j\omega)$ pushes $H(\omega)$ from the third quadrant completely into the fourth. Once $k_{12}(j\omega)$ enters the fourth quadrant, it cannot cross the imaginary axis again, and since it is on the "left edge" of $H(\omega)$, $H(\omega)$ must remain in the open right half plane as it goes to infinity at an asymptotic angle of $5\pi/2$ (or $\pi/2$). We have shown that $H(\omega)$ circles the origin counterclockwise through a total angle of $5\pi/2$, and so $arg_{net}(p) = 5\pi/2 \ \forall p \in N$. We conclude that N is Hurwitz.

6. EXTENSION TO POLYNOMIALS WITH COMPLEX COEFFICIENTS

In 1978 Kharitonov [2] published a generalization of the theorem of Section 4 for classes of polynomials with *complex* coefficients, defined by letting the real and imaginary parts of each coefficient vary independently in arbitrary intervals. Such a class of polynomials is Hurwitz if and only if *eight* special, well-defined polynomials are Hurwitz. (No English translation of [2] is known to the authors, and our brief references to [2] are paraphrased from references made in [6].) The proof of this generalization was recently simplified by Bose and Shi [6], although their derivation was still complex, requiring considerable notation.

We use Dasgupta's rectangle concept and simple complex plane geometry to present an elementary derivation of the generalization, completely analogous to the derivation in Sections 2-4 of the original theorem.

A prime motivation for this result for control engineers is provided in Section 7, where we establish sufficient tests for the whole class N to have U-Hurwitz properties which insure specified settling times and/or damping ratios of linear systems. Other motivations are given in [6].

We begin by defining the set N^* of polynomials of the form

$$p(s) = s^n + (\alpha_{n-1} + j\beta_{n-1})s^{n-1} + \cdots + (\alpha_0 + j\beta_0)$$

where $\underline{\alpha}_k \le \alpha_k \le \bar{\alpha}_k$ and $\underline{\beta}_k \le \beta_k \le \bar{\beta}_k$, $k=0,\ldots,n-1$. For all $\omega \in \mathbb{R}$ we define $H^*(\omega) := \{p(j\omega) : p \in N^*\}$, the image of N^* under the evaluation map at $s=j\omega$. Our development of the theorem will follow two parallel lines of reasoning, one for $\omega \ge 0$ and one for $\omega \le 0$. The following polynomials will be used when considering $\omega \ge 0$:

$$g_1^+(s) := \underline{\alpha}_0 + j\bar{\beta}_1 s + \bar{\alpha}_2 s^2 + j\underline{\beta}_3 s^3 + \underline{\alpha}_4 s^4 + \cdots$$

$$g_2^+(s) := \bar{\alpha}_0 + j\underline{\beta}_1 s + \underline{\alpha}_2 s^2 + j\bar{\beta}_3 s^3 + \bar{\alpha}_4 s^4 + \cdots$$

$$h_1^+(s) := j\underline{\beta}_0 + \underline{\alpha}_1 s + j\bar{\beta}_2 s^2 + \bar{\alpha}_3 s^3 + j\underline{\beta}_4 s^4 + \cdots$$

$$h_2^+(s) := j\bar{\beta}_0 + \bar{\alpha}_1 s + j\underline{\beta}_2 s^2 + \underline{\alpha}_3 s^3 + j\bar{\beta}_4 s^4 + \cdots$$

$$k_{11}^+(s) := g_1^+(s) + h_1^+(s)$$

$$k_{12}^+(s) := g_1^+(s) + h_2^+(s)$$

$$k_{21}^{+}(s) := g_2^{+}(s) + h_1^{+}(s)$$

$$k_{22}^{+}(s) := g_2^{+}(s) + h_2^{+}(s)$$

Remark: $\{k_{11}^{+}(\cdot), k_{12}^{+}(\cdot), k_{21}^{+}(\cdot), k_{22}^{+}(\cdot)\} \subset N^*$, and $\forall \omega \in \mathbb{R}$, $g_1^{+}(j\omega)$ and $g_2^{+}(j\omega)$ are purely real while $h_1^{+}(j\omega)$ and $h_2^{+}(j\omega)$ are purely imaginary. Furthermore, $\forall p(\cdot) \in N^*$ we have

$$\mathrm{Re}\{g_1^{+}(j\omega)\} \le \mathrm{Re}\{p(j\omega)\} \le \mathrm{Re}\{g_2^{+}(j\omega)\} \qquad \forall \omega \ge 0$$

$$\mathrm{Im}\{h_1^{+}(j\omega)\} \le \mathrm{Im}\{p(j\omega)\} \le \mathrm{Im}\{h_2^{+}(j\omega)\} \qquad \forall \omega \ge 0$$

Thus we see that $\forall \omega \ge 0$, $H^*(\omega)$ is a <u>level</u> <u>rectangle</u> (i.e. the sides are parallel to the real and imaginary axes) with corners $k_{11}^{+}(j\omega)$, $k_{12}^{+}(j\omega)$, $k_{21}^{+}(j\omega)$, and $k_{22}^{+}(j\omega)$.

The following polynomials will be used when considering $\omega \le 0$:

$$g_1^{-}(s) := \underline{\alpha}_0 + j\underline{\beta}_1 s + \bar{\alpha}_2 s^2 + j\bar{\beta}_3 s^3 + \underline{\alpha}_4 s^4 + \cdots$$

$$g_2^{-}(s) := \bar{\alpha}_0 + j\bar{\beta}_1 s + \underline{\alpha}_2 s^2 + j\underline{\beta}_3 s^3 + \bar{\alpha}_4 s^4 + \cdots$$

$$h_1^{-}(s) := j\underline{\beta}_0 + \bar{\alpha}_1 s + j\bar{\beta}_2 s^2 + \underline{\alpha}_3 s^3 + j\underline{\beta}_4 s^4 + \cdots$$

$$h_2^{-}(s) := j\bar{\beta}_0 + \underline{\alpha}_1 s + j\underline{\beta}_2 s^2 + \bar{\alpha}_3 s^3 + j\bar{\beta}_4 s^4 + \cdots$$

$$k_{11}^{-}(s) := g_1^{-}(s) + h_1^{-}(s)$$

$$k_{12}^{-}(s) := g_1^{-}(s) + h_2^{-}(s)$$

$$k_{21}^{-}(s) := g_2^{-}(s) + h_1^{-}(s)$$

$$k_{22}^{-}(s) := g_2^{-}(s) + h_2^{-}(s)$$

Remark: $\{k_{11}^{-}(\cdot), k_{12}^{-}(\cdot), k_{21}^{-}(\cdot), k_{22}^{-}(\cdot)\} \subset N^*$, and $\forall \omega \in \mathbb{R}$, $g_1^{-}(j\omega)$ and $g_2^{-}(j\omega)$ are purely real while $h_1^{-}(j\omega)$ and $h_2^{-}(j\omega)$ are purely imaginary. Furthermore, $\forall p(\cdot) \in N^*$ we have

$$\mathrm{Re}\{g_1^{-}(j\omega)\} \le \mathrm{Re}\{p(j\omega)\} \le \mathrm{Re}\{g_2^{-}(j\omega)\} \qquad \forall \omega \le 0$$

$$\mathrm{Im}\{h_1^{-}(j\omega)\} \le \mathrm{Im}\{p(j\omega)\} \le \mathrm{Im}\{h_2^{-}(j\omega)\} \qquad \forall \omega \le 0$$

Thus we see that $\forall \omega \le 0$, $H^*(\omega)$ is a <u>level</u> <u>rectangle</u> with corners $k_{11}^{-}(j\omega)$, $k_{12}^{-}(j\omega)$, $k_{21}^{-}(j\omega)$, and $k_{22}^{-}(j\omega)$.

Finally, we define the set K^* of Kharitonov polynomials:

$$K^* := \{k_{11}^{+}(\cdot), k_{12}^{+}(\cdot), k_{21}^{+}(\cdot), k_{22}^{+}(\cdot), k_{11}^{-}(\cdot), k_{12}^{-}(\cdot), k_{21}^{-}(\cdot), k_{22}^{-}(\cdot)\}.$$

Now we state the critical property of Hurwitz polynomials with complex coefficients.

Property 1: If $p(\cdot)$ is Hurwitz with degree $n \ge 1$ then $arg(p(j\omega))$ is a continuous and strictly increasing function of ω.

The proof is identical to that of Property 2 in Section 3, and Figures 2 and 3 still illustrate the property (with the modification that, in Figure 3, $p(0)$ need not be on the real axis and $p(j\omega)$ need not go through an angle of $n\pi/2$; the important point is that $p(j\omega)$ circles *strictly counter-clockwise* around the origin).

Lemma: If the eight Kharitonov polynomials in K^* are Hurwitz then $0 \notin H^*(\omega)$ $\forall \omega \in \mathbb{R}$.

Proof: First we note that, as ω goes to infinity, the rectangle $H^*(\omega)$ must travel to infinity at an uniform asymptotic angle of $n\pi/2 \ (mod\ 2\pi)$ (since $p \in N \Rightarrow p(j\omega) = (j\omega)^n [1 + O(1/\omega)]$). So $0 \notin H^*(\omega)$ for ω sufficiently large. Now suppose $0 \in H^*(\omega)$ for some $\omega \in \mathbb{R}$. Since $H^*(\omega)$ is continuous, 0 must be on the *boundary* of $H^*(\hat{\omega})$ for some $\hat{\omega} \ge \omega$. Since the corners are Hurwitz, 0 must be in the *interior* of an edge of $H^*(\hat{\omega})$; without loss of generality, we assume it's the "bottom" edge. If $\hat{\omega} \geq 0$, we conclude that $k_{11}^{+}(j\hat{\omega})$ is on the negative real axis while $k_{21}^{+}(j\hat{\omega})$ is on the positive real axis (refer to Figure 4). It follows from Property 1 that for $\delta\omega > 0$ sufficiently small, $k_{11}^{+}(j(\hat{\omega}+\delta\omega))$ is in the open third quadrant while $k_{21}^{+}(j(\hat{\omega}+\delta\omega))$ is in the open first quadrant. Since $\mathrm{Im}\{k_{11}^{+}(j\omega)\} = \mathrm{Im}\{k_{21}^{+}(j\omega)\} := \mathrm{Im}\{h_{1}^{+}(j\omega)\}\ \forall\omega\in\mathbb{R}$, this is clearly not possible. If $\hat{\omega} < 0$, we substitute $k_{11}^{-}(\cdot)$, $k_{21}^{-}(\cdot)$ and $h_{1}^{-}(\cdot)$ for $k_{11}^{+}(\cdot)$, $k_{21}^{+}(\cdot)$ and $h_{1}^{+}(\cdot)$ in the preceding argument and deduce the same contradiction.

Theorem: The class of polynomials N^* is Hurwitz if and only if K^* is Hurwitz.

Proof: The "only if" is immediate since $K^* \subset N^*$. So assume K^* is Hurwitz and note that N^* is pathwise connected (N^* is a parallelepiped in the space of polynomial coefficients). Since the zeroes of a complex polynomial depend continuously on its coefficients, if $p(\cdot)\in N^*$ were not Hurwitz, then on any path in N^* connecting $k_{11}^{+}(\cdot)$ to $p(\cdot)$ there would be a polynomial $\hat{p}(\cdot)\in N^*$ with a zero *on* the imaginary axis, say at $s = j\hat{\omega}$. So $\hat{p}(j\hat{\omega}) = 0$, which is forbidden by the Lemma.

Remark: We could not use the $arg_{net}(\cdot)$ property (Property 3) of Section 3; since the complex zeroes are not symmetric about the real axis, we cannot conclude that they each contribute $\pi/2$ radians of phase as ω goes from zero to infinity. However, we can easily modify $arg_{net}(\cdot)$ to accommodate complex polynomials. If $p(j\omega)\neq 0\ \forall\omega\in\mathbb{R}$, we define $arg_{net}^*(p(\cdot))$ to be the asymptotic net change in phase of $p(j\omega)$ as ω goes from $-\infty$ to $+\infty$. We easily see that each zero in $\mathbb{C}_-$ contributes $+\pi$ to $arg_{net}^*(p(\cdot))$ while each zero in $\mathbb{C}_+$ contributes $-\pi$. Thus we have the following characterization for Hurwitz complex polynomials: an n^{th} degree polynomial $p(\cdot)$ is Hurwitz if and only if $arg_{net}^*(p(\cdot))$ is well-defined and equal to $n\pi$. Now suppose K is Hurwitz. Since $arg(p(j\omega)) \underset{\omega\to+\infty}{\to} n\pi/2\ (mod\ 2\pi)$ and $arg(p(j\omega)) \underset{\omega\to-\infty}{\to} -n\pi/2\ (mod\ 2\pi)$ $\forall p(\cdot)\in N^*$, the Lemma implies that $arg_{net}^*(\cdot)$ is *constant* on N^*. We conclude that N^* is Hurwitz.

7. APPLICATION TO *U*-HURWITZ POLYNOMIALS (PERFORMANCE ROBUSTNESS)

The use of desirable pole locations of linear time-invariant dynamical systems to specify system performance is quite common, and the relationship between system pole locations and the damping ratios and settling times of system modes is well understood by the control engineering community. Thus the most common specified domains for system poles are the type shown in Figure 7. An angular sector in the left half plane, symmetric about the real axis, specifies the damping ratio ($\zeta = \sin^{-1}\theta$, where θ is shown in Figure 7). A σ-half-plane (the region to the left of the vertical line $\{s : \mathrm{Re}\{s\} = -\sigma\}$ for $\sigma > 0$) specifies the settling time ($T_s \approx 4/\sigma$, where T_s is the "2% settling time"). Of course, the complement of a desirable region is an "undesirable"--or forbidden--region. In the case of multiple specifications, the forbidden region is the union of all the individual forbidden regions--we denote it by U (as shown in Figure 7). A polynomial with no zeroes in U is said to be U-Hurwitz.

We do not have a general necessary and sufficient condition for a polynomial class of the form N to be U-Hurwitz. However, for the types of U described above (specified damping ratio and/or settling time) we apply Kharitonov's original and generalized stability theorems to provide a sufficient condition. The condition for the settling time problem involves checking that four derived polynomials with real coefficients are Hurwitz (not U-Hurwitz). The condition for the damping ratio problem involves checking that eight derived polynomials with complex coefficients are Hurwitz. For combined specifications, of course, all twelve polynomials must be checked.

Problem 1. The Settling Time Problem: $U=\{s:\mathrm{Re}\{s\}\geq-\sigma\}$. Consider the complex plane transformation $w=s+\sigma$. For any polynomial $p(\cdot)$ define $\hat{p}(\cdot)$ by $\hat{p}(w)=p(w-\sigma)=p(s)$. If $p(\cdot)$ is of the form

$$p(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_0$$

then

$$\begin{aligned}\hat{p}(w)&=(w-\sigma)^n+a_{n-1}(w-\sigma)^{n-1}+\cdots+a_0\\&=w^n+[a_{n-1}-n\sigma]w^{n-1}+\cdots+[a_0-a_1\sigma+a_2\sigma^2-\cdots+(-\sigma)^n]\\&=w^n+b_{n-1}w^{n-1}+\cdots+b_0\end{aligned}$$

$$b_k:=\sum_{i=k}^{n}\begin{bmatrix} i\\ k\end{bmatrix}a_i(-\sigma)^{i-k}$$

where $a_n:=1$. Clearly $p(\cdot)$ is U-Hurwitz if and only if $\hat{p}(\cdot)$ is Hurwitz. Given independent intervals for a_k, $k=0,\ldots,n-1$, we can easily determine the corresponding range for b_k, $k=0,\ldots,n-1$; i.e. $\underline{b}_0=\underline{a}_0-\bar{a}_1\sigma+\underline{a}_2\sigma^2-\cdots$, $\bar{b}_0=\bar{a}_0-\underline{a}_1\sigma+\bar{a}_2\sigma^2-\cdots$, etc. Now we define $\hat{N}$ to be the set of polynomials with coefficients $b_k, k=0,\ldots,n-1$ satisfying $\underline{b}_k\leq b_k\leq\bar{b}_k$, and we consider the polynomial transformation $\hat{T}:(a_0\cdots a_{n-1})\mapsto(b_0\cdots b_{n-1})$. $\hat{N}$ is the smallest parallelepiped containing $\hat{T}(N)$. Note that $\hat{T}$ is affine, so that the size of $\hat{N}$ is proportional to the size of N. The four Kharitonov polynomials based on $\underline{b}_k$ and $\bar{b}_k$ are easily defined, and if these polynomials are Hurwitz, then $\hat{N}$ will be Hurwitz. This implies that $\hat{T}(N)\subset\hat{N}$ is Hurwitz, which implies that N is U-Hurwitz. Thus we have shown that N is guaranteed to be U-Hurwitz if four well-defined polynomials with real coefficients are Hurwitz.

The sufficient condition derived above is not a necessary condition. Since $\hat{T}(N)$ does not contain $\hat{N}$, we have no indication that $\hat{T}(N)$ should contain the four Kharitonov polynomials for $\hat{N}$. In fact, we note that (assuming $\sigma\neq0$) the choice of a_i's required to produce $\bar{b}_0$ (namely $\bar{a}_0$, $\underline{a}_1$, $\bar{a}_2$, $\underline{a}_3,\ldots$) will also produce $\bar{b}_2$ (and $\bar{b}_k$ $\forall$ k odd, and $\underline{b}_k$ $\forall$ k even). Assuming $n\geq3$ and all intervals $[\underline{a}_k,\bar{a}_k]$ are nontrivial, the pairs $(\underline{b}_0,\bar{b}_2)$ and $(\bar{b}_0,\underline{b}_2)$ cannot be in $\hat{T}(N)$. It follows that none of the Kharitonov polynomials for $\hat{N}$ can be in $\hat{T}(N)$. Thus we see that our sufficient condition is conservative.

Problem 2. The Damping Ratio Problem: $U=\{s:-(\pi/2+\theta)\leq arg\ s\leq\pi/2+\theta\}$. We will also consider $U^+:=\{s:-\pi/2+\theta\leq arg\ s\leq\pi/2+\theta\}$ and $U^-:=\{s:-(\pi/2+\theta)\leq arg\ s\leq\pi/2-\theta\}$, the right half plane "tilted" counterclockwise by an angle θ and $-\theta$, respectively. Since the zeroes of a polynomial with real coefficients are symmetric with respect to the real axis, U-Hurwitz, U^+-Hurwitz and U^--Hurwitz are equivalent properties.

Now, considering U^+, we make the transformation $x=se^{-j\theta}$ and define $\tilde{p}(x):=p(xe^{j\theta})=p(s)$ and $\bar{p}(x):=e^{-jn\theta}\tilde{p}(x)$; i.e.

$$\begin{aligned}\tilde{p}(x)&=(xe^{j\theta})^n+a_{n-1}(xe^{j\theta})^{n-1}+\cdots+a_0\\&=e^{jn\theta}(x^n+a_{n-1}e^{-j\theta}x^{n-1}+\cdots+a_0e^{-jn\theta})\\&=:e^{jn\theta}[x^n+(\alpha_{n-1}+j\beta_{n-1})x^{n-1}+\cdots+(\alpha_0+j\beta_0)]\\&=e^{jn\theta}\bar{p}(x).\end{aligned}$$

So $p(\cdot)$ is U-Hurwitz if and only if $\bar{p}(\cdot)$ is Hurwitz (if and only if $\tilde{p}(\cdot)$ is Hurwitz). We consider the transformation $T^+:(a_0 \cdots a_{n-1}) \mapsto ((\alpha_0,\beta_0) \cdots (\alpha_{n-1},\beta_{n-1}))$. Under T^+, each coefficient (α_k,β_k) of $\hat{p}(\cdot)$ depends only on a_k; i.e. T^+ is "decoupled." Thus we abuse our own notation and write $(\alpha_0,\beta_0)=T^+(a_0), \ldots, (\alpha_{n-1},\beta_{n-1})=T^+(a_{n-1})$. Since T^+ is linear, we can easily find the extreme coefficients $\underline{\alpha}_k$, $\bar{\alpha}_k$, $\underline{\beta}_k$, and $\bar{\beta}_k$ from the real and imaginary parts of $e^{-jk\theta}\underline{a}_k$ and $e^{-jk\theta}\bar{a}_k$. We define the class N^+ of polynomials with complex coefficients to be all polynomials satisfying $\underline{\alpha}_k \le \alpha_k \le \bar{\alpha}_k$ and $\underline{\beta}_k \le \beta_k \le \bar{\beta}_k$ $k=0, \ldots, n-1$; N^+ is the smallest parallelepiped containing $T^+(N)$. The eight Kharitonov polynomials for N^+ are defined as in Section 6, and if these eight polynomials are Hurwitz then the generalized stability theorem implies that N^+ is Hurwitz. This implies that $T^+(N)$ is Hurwitz, which in turn implies that N is U-Hurwitz. Thus we have shown that N is guaranteed to be U-Hurwitz if eight well-defined polynomials with complex coefficients are Hurwitz.

The sufficient condition derived above is not a necessary condition. Although the coefficients in $T^+(N)$ vary independently, the real and imaginary parts of each coefficient are linearly dependent; i.e. $T^+([\underline{a}_k,\bar{a}_k])$ is either the line segment connecting $(\underline{\alpha}_k,\underline{\beta}_k)$ to $(\bar{\alpha}_k,\bar{\beta}_k)$ or the line segment connecting $(\underline{\alpha}_k,\bar{\beta}_k)$ to $(\bar{\alpha}_k,\underline{\beta}_k)$ (depending on whether $e^{jk\theta}$ is in an even or odd quadrant--$T^-([\underline{a}_k,\bar{a}_k])$ would be the other line segment). So $T^+(N)$ will not contain N^+; indeed, $T^+(N)$ is an n-dimensional linear slice of the $2n$-dimensional parallelepiped N^+. Thus we see that our sufficient condition is conservative.

Remark: It is clear that the derivation above can be extended to half planes defined by any line in $\mathbb{C}$ by considering transformations of the form $z=\sigma+se^{j\theta}$. The test would involve testing at most eight polynomials with complex coefficients. Combining such tests, we can then derive tests for any *desirable* region defined by a *polygon* in the s-plane with m sides. The test would involve testing $8m$ polynomials. Such polygons might be used, for instance, to define desirable regions in the unit circle for discrete systems.

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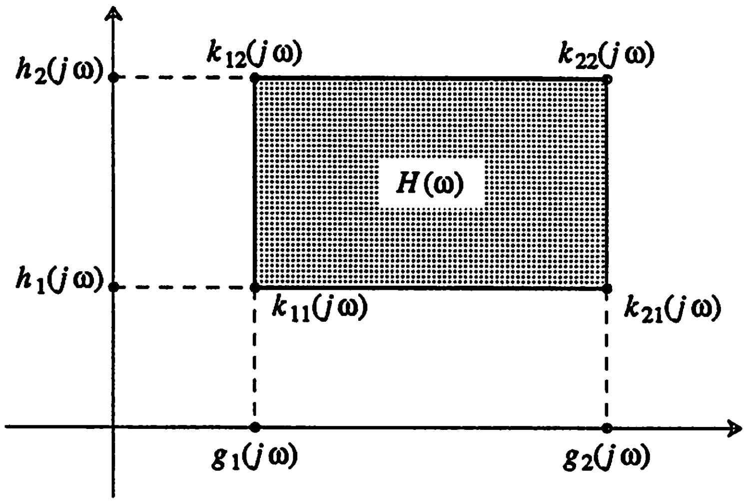

Figure 1:
Rectangular image of N at $s=j\omega$ ($\omega>0$).

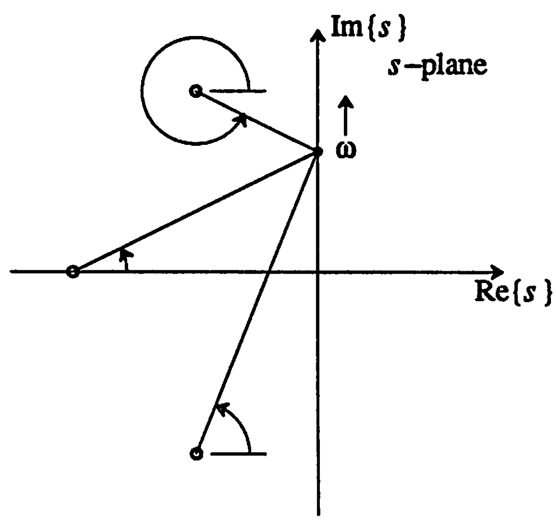

Figure 2:
Angle of $p(j\omega)$ for Hurwitz $p(\cdot)$.

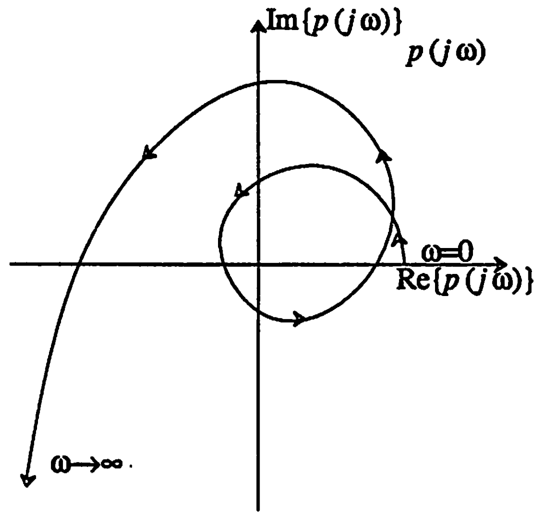

Figure 3:
Image of $p(j\omega)$ for Hurwitz $p(\cdot)$ (n=7).

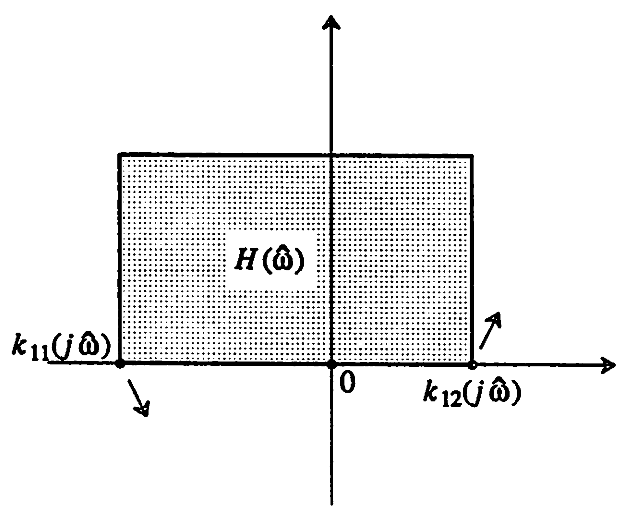

Figure 4:
Why 0 cannot enter $H(\omega)$.

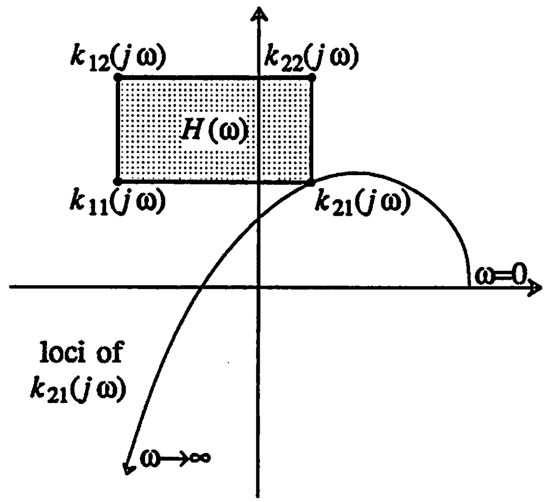

Figure 5:
Polynomials of degree $n=3$.

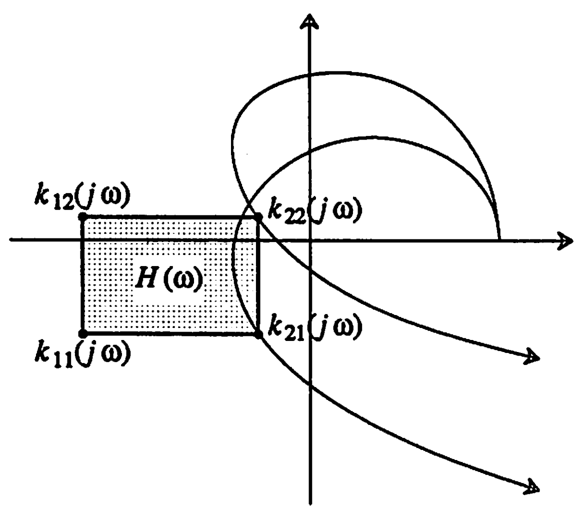

Figure 6:
Polynomials of degree $n=4$.

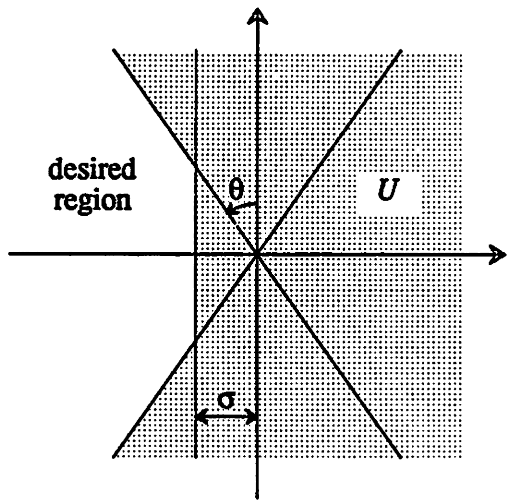

Figure 7:
Forbidden region U for systems with
specifications on settling time and damping ratio.