NORMAL FORMS FOR NONLINEAR VECTOR FIELDS—PART I: THEORY AND ALGORITHM

by

Leon O. Chua and Hiroshi Kokubu

Memorandum No. UCB/ERL M87/81

September 1987

NORMAL FORMS FOR NONLINEAR VECTOR FIELDS—PART I: THEORY AND ALGORITHM

by

L. O. Chua and H. Kokubu

Memorandum No. UCB/ERL M87/81

September 1987

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NORMAL FORMS FOR NONLINEAR VECTOR FIELDS—PART I: THEORY AND ALGORITHM

by

L. O. Chua and H. Kokubu

Memorandum No. UCB/ERL M87/81

September 1987

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NORMAL FORMS FOR NONLINEAR VECTOR FIELDS — PART I: THEORY AND ALGORITHM[†]

Leon O. Chua and Hiroshi Kokubu[††]

Abstract

Normal forms are powerful analytical tools for studying the qualitative behavior of nonlinear vector fields. This 2-part *tutorial* is aimed for the non-specialist in general, and circuit theorist in particular.

Part I of this paper provides the basic concept and foundation on the modern theory of normal forms for nonlinear vector fields. After stating the *Poincaré* and the *Takens normal form*, this paper focuses on the latest refinements due to *Ushiki.*

For pedagogical reasons, the familiar Jordan form is first derived and shown to be an appropriate normal form for matrices. Rather than using a standard linear algebraic approach, our formulation is based on the "method of infinitesimal deformation" which generalizes naturally to nonlinear vector fields.

1. INTRODUCTION

The concept of *normal forms* of nonlinear vector fields has emerged as an important analytical tool for investigating the qualitative behavior of nonlinear dynamical systems [1-2]. Roughly speaking, the normal form of a vector field is the *simplest* member of an equivalence class of vector fields, all exhibiting the same qualitative behavior. For example, consider the family of all linear systems described by $\dot{x} = Ax$, where A is any $n \times n$ real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since A is diagonalizable, the qualitative behavior of the above family is identical to that of $\dot{x} = \Lambda x$, where Λ is a diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ along its diagonal. In this case, we say Λx is the normal form of the above equivalence class of linear vector fields. Clearly, it is much simpler to investigate the qualitative behavior of this family of vector fields by working with Λx rather than Ax.

In the more general class of linear vector fields where A is not diagonalizable, the normal form of this equivalence class of vector fields is also given by Λx, where Λ denotes the *Jordan form* [3] of A.

It is much harder to generalized the concept of *normal form* to equivalence classes of *nonlinear* vector fields. Major advances have been made over the past decade, however, that though still undergoing development, there is now a systematic procedure for formulating the normal form of nonlinear vector fields. Our main

[†]This research is supported in part by the Office of Naval Research, Contract N00014-86-K-0351 and by the National Science Foundation, Grant MIP-8614000.

[††]L. O. Chua is with the University of California, Berkeley, CA.
H. Kokubu is with the Department of Mathematics, Kyoto University, Kyoto, 606, Japan.

objective in this 2-part *tutorial* is to extract the main results from this rather complicated mathematical subject and to rewrite them in a form that can be more easily understood and applied by the non-specialist.

The notion of *normal form* originated from Poincaré's *formal linearization theorem* for *formal* vector fields[†] of the form

$$\dot{x} = v(x) = \Lambda x + v_2(x) + v_3(x) + \cdots + v_k(x) + \cdots \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ denotes an n-dimensional vector of complex numbers, Λ denotes an $n \times n$ complex matrix, and $v_k(x)$ denotes a *homogenous* polynomial of degree k in x. Note that Λx represents the *linear* part of the nonlinear vector field $v(x)$, where Λ is the Jacobian matrix of $v(x)$ at the origin. For simplicity, let us assume that the $n \times n$ matrix Λ is diagonalizable.

The eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of Λ are said to be *resonant* if a relationship of the form

$$\lambda_k = \sum_{i=1}^{n} m_i \lambda_i \tag{1.2}$$

holds for *non-negative* integers m_i satisfying $\sum_{i=1}^{n} m_i \geq 2$ for some index k satisfying $1 \leq k \leq n$. Otherwise, we say $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are *non-resonant.*[††] For example, $\{\lambda_1, \lambda_2\} = \{i, -i\}$ are resonant because $\lambda_1 = m\lambda_1 + (m-1)\lambda_2$ and $\lambda_2 = (m-1)\lambda_1 + m\lambda_2$ for any integer $m>1$. Here, we can choose either $k = 1, m_1 = m$, $m_2 = m-1$, or $k = 2, m_1 = m-1$ and $m_2 = m$.

For each combination of $\{m_1, m_2, \ldots, m_n\}$ satisfying (1.2), the monomial $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in the right handside of (1.1) is called the *resonant monomial* corresponding to the *resonant condition* (1.2) associated with λ_k.

To transform (1.1) into its normal form, Poincaré introduces a formal coordinate transformation of the form

$$x = \psi(y) = y + \psi_2(y) + \psi_3(y) + \cdots + \psi_k(y) + \cdots \tag{1.3}$$

where $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$ denotes the new coordinate system, and $\psi_k(y)$ denotes a *homogeneous* polynomial of degree k in y. We are now ready to state the Poincaré normal form theorem.

Theorem 1.1 (Poincaré Normal Form)

[†] By *formal*, we mean the question of *convergence* of an infinite series is ignored.

[††] The terminologies here are due to Poincaré and should not be confused with traditional usage in circuit theory.

A formal vector field

$$\dot{x} = v(x) \quad , x \in \mathbb{C}^n \tag{1.1}$$

can be transformed into the *Poincaré normal form*

$$\dot{y} = \Lambda y + w(y) \quad , y \in \mathbb{C}^n \tag{1.4}$$

by an appropriate formal coordinate transformation $x = \psi(y)$, where Λ denotes the Jacobian matrix of $v(x)$ at the origin, and each component $w_k(y)$ of $w = (w_1, w_2, \ldots, w_n)$ consists of *all* resonant monomials corresponding to the resonant condition (1.2) associated with the eigenvalue λ_k of Λ.

It follows from *Theorem 1.1* that if the eigenvalues of Λ are *non-resonant*, then $w(y) = 0$ and the *non-linear* vector field (1.1) can be transformed into a *linear* vector field.

Example 1.1

Consider the case $n = 2$ and $\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. The eigenvalues $\{i, -i\}$ are resonant and the resonant monomials corresponding to $\lambda_1 = i$ are of the form $(y_1 y_2)^m y_1$ for $m = 1, 2, 3, \ldots$. Those corresponding to $\lambda_2 = -i$ are of the form $(y_1 y_2)^m y_2$. It follows from *Theorem 1.1* that the Poincaré normal form is given by:

$$\dot{y}_1 = iy_1 + a_1 y_1^2 y_2 + a_2 y_1^3 y_2^2 + a_3 y_1^4 y_2^3 + \cdots \tag{1.5}$$

$$\dot{y}_2 = -iy_2 + b_1 y_1 y_2^2 + b_2 y_1^2 y_2^3 + b_3 y_1^3 y_2^4 + \cdots \tag{1.6}$$

■

Observe that each component of the normal form equation is an infinite series. We will usually truncate the higher-order terms and focus our attention only to those terms up to the *k*th order, henceforth referred to as the "*k*th order normal form."

By other clever choices of coordinate transformation, it is sometimes possible, especially in the case where Λ contains *multiply degenerate eigenvalues*, to obtain a much simpler normal form than the one prescribed above. We will focus our attention in this paper on two recent normal form results due to Takens [4] and Ushiki [5], respectively.

In 1974, Takens [4] gave a rather geometric set-up of normal forms for vector fields using the *Lie bracket* operation. His result is summarized as follows:†

Theorem 1.2 (Takens normal form)

†Readers not familiar with the Lie bracket for vector fields should consult *Appendix 1*.

Let v be a vector field on $\mathbf{R}^n$ vanishing at the origin O. Expand v into a Taylor series at O, namely,

$$v = v_1 + v_2 + \cdots + v_k + \cdots \tag{1.7}$$

where v_k denotes the kth order term in the expansion. Let b_k denote the Lie bracket $[Y_k, v_1]$ between Y_k and v_1,[†] where Y_k denotes some homogeneous vector field of degree k.

If $\quad v_k = b_k + g_k = [Y_k, v_1] + g_k \quad$ (1.8)

then there exists a coordinate transformation ϕ which fixes the origin O such that

$$\phi_* v = v_1 + v_2 + \cdots + v_{k-1} + g_k + \cdots \tag{1.9}$$

where $\phi_* v$ denotes the transformed vector field of v by ϕ. That is, ϕ does *not* change the terms in v up to the (k-1)th order but eliminates the b_k component of the kth order term v_k.

It follows from *Theorem 1.2* that every component $[Y_k, v_1]$ of v, $k = 1, 2, \ldots, n$, can be eliminated by a coordinate transformation. In particular, if $v_k = [Y_k, v_i]$, i.e., if $g_k = 0$, then the entire kth order term can be eliminated by a coordinate transformation.

A similar normal form approach, couched in Poincaré's style, is due to Arnold [2]. Other more analytical approaches, rather than geometric, are given in [6-7].

Takens normal form essentially makes use only information from the *linear* part v_1 of the nonlinear vector field v in forming the Lie bracket $[Y_k, v_1]$. By exploiting the higher order components, it is sometimes possible to eliminate additional terms from the Takens normal form via further coordinate transformations. Although Takens [4,8,9] was aware of this possibility, it was Ushiki who gave the detailed calculations needed to obtain a normal form equation which is truly the *simplest* possible in the sense that no other terms can be eliminated by any coordinate transformation [5,10,11].

Our main objective of this paper is to present a detailed and precise explanation of Ushiki's normal form for the non-specialist. Consequently, detailed proofs will be given of the most basic results and numerous examples will be given to illustrate the theory.

The remaining parts of this paper (Part I) contain 4 sections. In *Section 2*, we use an unconventional approach, called the *method of infinitesimal deformation*, to derive the *Jordan form* as an appropriate normal form for linear vector fields described by arbitrary matrices. We choose this approach, rather than the standard linear algebraic approach, because it generalizes naturally to *nonlinear* vector fields. This generalization is made in a fairly abstract setting in *Section 3*. The detailed normal form formulation is given in *Sections 4 and 5*, where *Section 4* contains the basic strategy and *Section 5* presents an explicit algorithm due to Ushiki.

[†]For each point $x \in \mathbf{R}^n$, the Lie bracket $[Y_k(x), v_1(x)]:\mathbf{R}^n \to \mathbf{R}^n$ maps x into $DY_k(x)\, v_1(x) - Dv_1(x)\, Y_k(x)$.

2. JORDAN NORMAL FORM FOR MATRICES: ANOTHER PERSPECTIVE

An $n \times n$ matrix A can be transformed into several equivalent forms, the *simplest* of which is called the *normal form.* The choice of a normal form is not *unique*, however, since it depends on the criterion used in comparing the "simplicity" between 2 matrices. For dynamical systems described by a *linear* vector field $\dot{x} = Ax$, the well-known *Jordan form* is generally regarded as the most appropriate normal form.

In this section, we will derive this normal form via an unconventional approach, called the *method of infinitesimal deformation* because the same technique can be generalized to derive the normal forms for *non-linear* vector fields.

Let $M(n, \mathbb{R})$ denote the *vector space* of all real $n \times n$ matrices and let $GL(n, \mathbb{R})$ denote the *group* of all *non-singular* real matrices of the same order.

Definition 2.1: Conjugate operation P_*

Two matrices A and A' in $M(n, \mathbb{R})$ are said to be *conjugate* of each other iff there exists some P in $GL(n, \mathbb{R})$ such that $A' = PAP^{-1}$. In this case, we call A' the transformed matrix of A via the transformation P and denote it by $P_* A$. Here, $GL(n, \mathbb{R})$ is considered as the *group of transformations* of $M(n, \mathbb{R})$.

The conjugacy relation defined above is an *equivalence relation* on $M(n, \mathbb{R})$ and the associated equivalence class is called the *conjugacy class.*

Definition 2.2: Normal form of A

A *normal form* of a given matrix A is a representative of the conjugacy class of A.

The Jordan form of A clearly qualifies as a normal form: its non-zero entries consist of either the eigenvalues or the integer 1.

For any Y in $M(n, \mathbb{R})$, the *exponential matrix* of Y, denoted by e^Y is defined by the limit of the following absolutely convergent series:

$$e^Y \triangleq 1 + Y + \frac{Y^2}{2!} + \frac{Y^3}{3!} + \cdots + \frac{Y^n}{n!} + \cdots \tag{2.1}$$

We define the *exponential map*

$$\exp : \mathbb{R} \times M(n, \mathbb{R}) \to GL(n, \mathbb{R}) \tag{2.2}$$

by

$$(t, Y) \mapsto e^{tY} \tag{2.3}$$

Note that for *fixed* Y, the family of matrices e^{tY} parametrized by t is a *one-parameter group* in $GL(n, \mathbb{R})$; that is,

$$e^{(t+s)Y} = e^{tY} \cdot e^{sY} \tag{2.4}$$

We call Y the *infinitesimal generator* of the one-parameter group e^{tY}.

The crucial step in the method of infinitesimal deformation is to calculate the *derivative*

$$\frac{d}{dt}\bigg|_{t=0} (e^{tY})_* A \tag{2.5}$$

henceforth called the *infinitesimal deformation* of A by Y, where

$$(e^{tY})_* A \triangleq e^{tY} \cdot A \cdot (e^{tY})^{-1} = e^{tY} \cdot A \cdot e^{-tY} \tag{2.6}$$

in view of *Definition 2.1*. Differentiating (2.6) with respect to t and evaluating the result at t=0, we found the infinitesimal deformation is given by the matrix

$$YA - AY \triangleq [Y, A] \tag{2.7}$$

where the Lie bracket notation is adopted here for simplicity. We can summarize the above result as follow:

Proposition 2.3: Infinitesimal Deformation of a Matrix

The infinitesimal deformation of the matrix A is given by

$$\frac{d}{dt}\bigg|_{t=0} (e^{tY})_* A = [Y, A] \tag{2.8}$$

■

Since e^{tY} is a group,

$$\frac{d}{d\tau}\bigg|_{\tau=t} (e^{\tau Y})_* A = \frac{d}{d\tau}\bigg|_{\tau=0} (e^{(\tau+t)Y})_* A$$

$$= \frac{d}{d\tau}\bigg|_{\tau=0} e^{\tau Y}(e^{tY}Ae^{-tY})e^{-\tau Y}$$

$$= \frac{d}{d\tau}\bigg|_{\tau=0} (e^{\tau Y})_* \{(e^{tY})_* A\} = [Y, (e^{tY})_* A] \tag{2.9}$$

Introducing the notation

$$A_t \triangleq (e^{tY})_* A \tag{2.10}$$

in (2.8) and (2.9), we obtain the following *linear* differential equation

$$\boxed{\frac{d}{dt} A_t = [Y, A_t]} \tag{2.11}$$

on $M(n, \mathbb{R})$. Note that (2.11) consists of a system of n^2 linear differential equations corresponding to the n^2 elements of A_t. Solving (2.11) with the *initial* condition $A_0 = A$, we obtain the one-parameter family $(e^{tY})_* A$

of transformed matrices. If we visualize A_0 as an initial point in the n^2-dimensional Euclidean space, then the solution of (2.11) is a trajectory parametrized by the time t. To reconstruct the matrix solution of (2.11) at any time $t = t_k$, we simply identify the n^2 coordinates of this trajectory at $t = t_k$. This trajectory is *uniquely* specified once Y and the initial matrix A_0 are given. Hence, the linear differential equation (2.11) specifies the evolution of A_t for any given Y and A_0.

Example 2.4

Choose $A_0 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $A_t = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix}$. In this care, $n = 2$ and (2.11) becomes

$$\frac{d}{dt}\begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix} = \left[\begin{bmatrix} p & q \\ r & s \end{bmatrix}, \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix}\right]$$

$$= \begin{bmatrix} qc_t - rb_t & -q(a_t - d_t) + (p - s)b_t \\ r(a_t - d_t) - (p - s)c_t & -(qc_t - rb_t) \end{bmatrix} \tag{2.12}$$

To obtain as simple a solution as possible, let us choose $p = s$, $q = 0$, and $r = -1$ as the elements of Y. The resulting solution can then be solved trivially to obtain

$$\left.\begin{aligned} b_t &= b_0 = 1 \\ a_t &= a_0 - rb_t t = -1 + t \\ d_t &= d_0 + rb_t t = 1 - t \\ c_t &= c_0 + \int_0^t r(a_t - d_t)dt \\ &= -1 + 2\int_0^t (1 - t)dt = -1 + 2t - t^2 = -(1 - t)^2 \end{aligned}\right\} \tag{2.13}$$

Observe that the "simplest" solution occurs when $t = 1$ in

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.14}$$

In fact, we have obtained

$$\exp\begin{bmatrix} p & 0 \\ -1 & p \end{bmatrix}_* \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.15}$$

where the resulting matrix is precisely the Jordan form of $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$!

Since all matrix solutions of (2.11) with the initial matrix $A_0 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ are equivalent to each other, it suffices to choose the simplest solution $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ corresponding to the point on the above trajectory in the 4-dimensional space at $t = 1$. ■

The preceding approach for deriving the Jordan form is the fundamental idea used to develop the normal form theory in the following sections; namely, choose an appropriate infinitesimal generator Y and integrate the associated linear differential equation (2.11). Needless to say, for *linear* vector fields, our above approach is less efficient than the usual linear algebraic method. However, our approach becomes extremely useful for nonlinear vector fields.

Remark

The preceding method of infinitesimal deformation assumes a transformation of the form e^{tY}. This transformation is not completely general because its determinant

$$\det e^{tY} = e^{\text{trace}(tY)} > 0$$

is always positive; whereas in the general transformation group $GL(n, \mathbf{R})$, matrices having a negative determinant are also present. Because of this restriction, some matrices may not be reducible to the Jordan form via the preceding method. For example, if we replace $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ from Example (2.4) with another matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, we would obtain the normal form $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ instead of the Jordan form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This lack of generality, however, is only superficial because we can always reduce the transformed matrix into the Jordan form by another transformation having a negative determinant. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.16}$$

The above remark applies also to the general framework for normal forms in the following sections.

3. GENERAL FRAMEWORK FOR NORMAL FORMS: A UNIFIED APPROACH

The same approach used in the preceding section for deriving the normal forms of *linear* vector fields can be generalized to a much larger class of vector fields. For complete generality, we will present a *unified* approach in this section in an *abstract* setting so that the normal forms derived in the following sections, as well as elsewhere [12], will clearly be seen as special cases. In our unified approach, the general framework for normal forms requires a *vector space* M of abstract objects and a *group* G of transformations. In fact, for even greater generality, we can generalize the vector space M to a *manifold*.

For linear vector fields considered in *Section 2*, we have $M = M(n, \mathbf{R})$ and $G = GL(n, \mathbf{R})$. For the class of vector fields to be considered in the following sections, M is, roughly speaking, the set of all vector fields on $\mathbf{R}^n$ and G is the group of all coordinate transformations.

For each x in M and g in G, we denote the *transformed object* by $g_* x$. We say x and x' in M are *equivalent* iff $x' = g_* x$ holds for some g in G. A *normal form* of x is a *representative* of the equivalence class of x. This representative is chosen to be the "simplest" member according to some criterion of comparison.

Suppose there exists a local one-parameter group $g(t)$ ($t \in \mathbf{R}$) in G satisfying

$$g(t+s)=g(t)\cdot g(s) \tag{3.1}$$

for sufficiently small t and s. Here, "local" implies that "t" is sufficiently small. If our group G of transformations is so tame that it admits a differentiable structure and $g(t)$ is smooth in t, then the following expression

$$\left.\frac{d}{dt}\right|_{t=0} g(t)=Y$$

is well defined and the resulting object Y is called the *infinitesimal generator* of $g(t)$. In order to specify the infinitesimal generator Y, we frequently use the notation $\exp(tY)$ instead of $g(t)$, even though $\exp(t\,Y)$ is not necessarily the exponential map defined in (2.1).

In this paper, the set of infinitesimal generators is denoted by $\mathcal{G}$. Sometimes, $\mathcal{G} = M$, as in the case of all $n \times n$ matrices or vector fields on $\mathbf{R}^n$. However, in more general cases, such as those considered in [12], $\mathcal{G} \neq M$.

Since $\exp(tY)$ is an element of $\mathcal{G}$, for each x in M, the transformed object $(\exp(tY))_* x$ is a well-defined member of M parametrized by t. We define the *infinitesimal deformation* of x by Y by the derivative

$$\left.\frac{d}{dt}\right|_{t=0} (\exp(tY))_* x \tag{3.2}$$

Since M is a vector space, this derivative can be considered as an element of M, which we denote by $\{Y, x\}$. Since $\exp(tY)$ is a one-parameter group, all the steps used in the preceding section for deriving the linear differential equation (2.11) is also applicable, *mutatis-mutandis*, to the above abstract version of (2.5). Hence, by a mere change of symbols, we obtain the following differential equation

$$\boxed{\frac{d}{dt} x_t = \{Y, x_t\}} \tag{3.3}$$

for the abstract vector space, where

$$x_t \triangleq (\exp(tY))_* x \tag{3.4}$$

Following the procedure from *Section 2*, we can derive the normal form of any x in M by choosing first an appropriate Y and then solving the differential equation (3.3). In the next section, we will illustrate this general approach for deriving normal forms for the class of nonlinear vector fields on $\mathbf{R}^n$ which vanishes at the origin.

4. NORMAL FORMS FOR VECTOR FIELDS: BASIC STRATEGY

In this section, the vector space M of the general framework for normal forms will be specialized to the class of all *smooth* (i.e., C^∞) *vector fields*† defined on a neighborhood of the origin O of $\mathbf{R}^n$ and which vanish at O. We will denote the set of all such vector fields by $\mathcal{X}_0(n)$, or simply $\mathcal{X}_0$.

When one considers the dynamical behavior of vector fields, it is often the case that only the terms of *finite* order, say k, are essential in the analysis. Hence, we may neglect all terms in $\mathcal{X}_0$ beyond order k. Since the formulation of the normal form requires various coordinate transformations and derivative operations, and the evaluation of Lie brackets, it is not clear whether the higher-order terms can be truncated at each intermediate calculation steps, or whether one has to work with the complete expansion and then truncate only at the final stage. Since the latter would have been extremely messy, it is highly desirable to truncate at all intermediate steps. In order to do this rigorously and to avoid ambiguity, we will apply the concept and notation of *k-jets*, which is reviewed in *Appendix 2*.

Let $\mathcal{X}_0^k$ denote the vector space of all *k-jets* of the vector fields in $\mathcal{X}_0$ at O (obtained by truncating all terms of degree greater than k). Let j^k and $j^{k,l}$ denote the natural projections

$$j^k : \mathcal{X}_0 \to \mathcal{X}_0^k \tag{4.1}$$

$$j^{k,l} : \mathcal{X}_0^k \to \mathcal{X}_0^l \quad (k \geq l) \tag{4.2}$$

respectively. Hence, if $v \in \mathcal{X}_0$ is a smooth vector field, then $v' = j^k v$ can be identified with a vector field v' obtained by truncating all terms of the Taylor expansion of v at O beyond the degree k. Similarly, $v'' = j^{k,l}\, v'$ truncates further all terms of v' (whose highest order term has degree k) beyond degree $l \leq k$. It follows from Proposition A2.4 that j^k and $j^{k,l}$ are *surjective vector space homomorphisms* [13]. Let $H_k \triangleq Ker(j^{k,k-1})$ denote the *kernel* of $j^{k,k-1}$, i.e., the set of all elements of $\mathcal{X}_0^k$ which maps to O. Clearly, H_k consists of all vector fields described by a *homogeneous* polynomial of degree k. It follows that

$$\mathcal{X}_0^k = H_1 \oplus H_2 \oplus + \cdots \oplus H_k \tag{4.3}$$

where $\oplus$ denotes the *direct sum* operation. Hence, every v^k in $\mathcal{X}_0^k$ can be decomposed uniquely into the form

$$v^k = v_1 + v_2 + \cdots + v_k \tag{4.4}$$

where $v_i \in H_i$. Here and in the sequel, the suffix i represents the ith order part, while the superfix k represents the k-jet.

From here on, we assume the vector space M of the general framework for normal forms to be $\mathcal{X}_0^k$, i.e.,

---

†A function $f : \mathbf{R}^n \to \mathbf{R}^n$ is said to be C^∞ iff all *partial derivatives* of $f(\cdot)$ exist for all orders $1, 2, \cdots$.

all nonlinear vector fields of degree up to k. It remains for us to pick an appropriate group G of transformations.

Let v be a vector field on $\mathbf{R}^n$ which vanishes at O and let ϕ be a coordinate transformation of $\mathbf{R}^n$ satisfying $\phi(0) = 0$. Then the transformed vector field $\phi_* v$ by ϕ is given by

$$\tilde{v}(y) \triangleq (\phi_* v)(y) = D\phi(\phi^{-1}(y)) \cdot v(\phi^{-1}(y)) \tag{4.5}$$

This transformation is best depicted by the following commutative diagram:

$$\begin{array}{ccc} \mathbf{R}^n_x & \xrightarrow{v} & \mathbf{R}^n_{v(x)} \\ y = \phi(x) \downarrow & & \downarrow D\phi(x) \\ \mathbf{R}^n_y & \xrightarrow{\tilde{v}} & \mathbf{R}^n_{\tilde{v}(y)} \end{array}$$

The above transformation was chosen because we have identified the vector field v with the ordinary differential equation (ODE)

$$\dot{x} = v(x) \tag{4.6}$$

and under the change in coordinates $x \to \phi(x) \triangleq y$, we have

$$\dot{y} = D\phi(x)\,\dot{x} = D\phi(x)v(x) = D\phi(\phi^{-1}(y)) \cdot v(\phi^{-1}(y)) \tag{4.7}$$

which we would like to identify with (4.5).

It follows from *Appendix 2* that if v and v' are *k-jet equivalent* at O, i.e., if they have identical terms up to order k, then $\phi_* v$ and $\phi_* v'$ are also k-jet equivalent. Moreover, the higher-order part of ϕ (beyond k) does *not* affect the k-jet of $\phi_* v$. Consequently, if ϕ and ϕ' are k-jet equivalent at O, then $\phi_* v$ and $\phi_*' v$ are also k-jet equivalent. Thus, we can define the following transformation group:

Let Diff_0 denote the group of coordinate transformations of $\mathbf{R}^n$ having origin O as a fixed point and let Diff_0^k denote the k-jets of Diff_0 at O. The following proposition summarizes the properties of Diff_0^k.

Proposition 4.1

(1) Diff_0^k forms a *group*.

(2) For ϕ^k in Diff_0^k and v^k in $\mathcal{X}_0^k$, the transformed vector field $\phi^k{}_* \, v^k$ is well defined and is given by

$$\phi^k{}_* \, v^k = j_0^k(\phi_* \, v) \tag{4.8}$$

where ϕ and v are representatives† of ϕ^k and v^k, respectively. Moreover,

$$\phi^k{}_* \, (\psi^k{}_* \, v^k) = (\phi^k \circ \psi^k)_* \, v^k \tag{4.9}$$

holds for all $\phi^k, \psi^k \in \mathrm{Diff}_0^k$ and for all $v^k \in \mathcal{X}_0^k$, where "$\circ$" denotes the "composition" operation.

Proof

(1) We define the group *multiplication* operation between ϕ^k and ψ^k in Diff_0^k by

$$\phi^k \circ \psi^k = j_0^k \, (\phi \circ \psi) \tag{4.10}$$

where ϕ and ψ are representatives of ϕ^k and ψ^k, respectively. This binary operation is well defined because, by the chain rule, the composition $\phi \circ \psi$ up to order k is determined by the *derivatives* of ϕ and ψ of order only up to k. The above multiplication operation clearly satisfies the axioms defining a group.

(2) By the same reasoning as above, (4.8) is well defined. To prove (4.9), let ϕ, ψ, and v be representatives of ϕ^k, ψ^k, and v^k, respectively. Then

$$\phi^k{}_*(\psi^k{}_* \, v^k) = \phi^k{}_* \, j_0^k(\psi_* \, v) = j_0^k(\phi_*(\psi_* \, v))$$

$$= j_0^k((\phi \circ \psi)_* \, v) = j_0^k(\phi \circ \psi)_* \, j_0^k \, v = (\phi^k \circ \psi^k)_* \, v^k$$

■

Let us choose Diff_0^k as the group G of the general framework for normal forms. Our next task will be to calculate the *infinitesimal deformation* (3.2). It is instructive to consider first a simple example.

Example 4.2

Consider a vector field v defined by the following ODE on $\mathbb{R}^2$

$$\dot{x} = y + xy \quad , \quad \dot{y} = -y^2 \tag{4.11}$$

Let us identify (4.11) with the *differential operator*

$$v = (y + xy) \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} \tag{4.12}$$

The *quadratic* terms $xy \dfrac{\partial}{\partial x} - y^2 \dfrac{\partial}{\partial y}$ of v can be expressed compactly by the following *Lie bracket* (see

† ϕ is said to be a *representative* of ϕ^k iff the part of ϕ up to order k is identical to ϕ^k.

Appendix 1)†

$$xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} = \left[xy\frac{\partial}{\partial y}\ ,\ y\frac{\partial}{\partial x}\right] \tag{4.13}$$

where $Y \triangleq y\frac{\partial}{\partial x}$ coincides with the *linear* part of v in (4.11). The vector field $X \triangleq xy\frac{\partial}{\partial y}$ in (4.13) can be identified with the ODE

$$\dot{x} = 0\ ,\quad \dot{y} = xy \tag{4.14}$$

whose solution can be trivially obtained as follows:

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 e^{x_0 t} \end{aligned} \tag{4.15}$$

where $x(0) = x_0$ and $y(0) = y_0$ is the initial condition.

Corresponding to the above solution, consider the following family of transformations, parametrized by t:

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y\, e^{xt} \end{aligned} \tag{4.16}$$

In terms of the new coordinates $(\tilde{x}, \tilde{y})$, the ODE (4.11) becomes, for each fixed parameter value t,††

$$\dot{\tilde{x}} = \dot{x} = y + xy = (1+\tilde{x})\tilde{y}\ e^{-\tilde{x}t} \tag{4.17a}$$

$$\begin{aligned} \dot{\tilde{y}} &= \dot{y}\ e^{xt} + tye^{xt}\ \dot{x} \\ &= -y^2 e^{xt} + ty(y+xy)e^{xt} \\ &= [(t-1) + t\tilde{x}]\tilde{y}^2\ e^{-\tilde{x}t} \end{aligned} \tag{4.17b}$$

Expanding $e^{-\tilde{x}t}$ in a Taylor series about the origin, the *quadratic* terms of the transformed vector field (4.17) is given by (parametrized by t):

---

†Using the Lie bracket operation $[X, Y] = DY \cdot X - DX \cdot Y$ between two vector fields X and Y on $\mathbf{R}^2$ with $X = \begin{bmatrix} 0 \\ xy \end{bmatrix}$ and $Y = \begin{bmatrix} y \\ 0 \end{bmatrix}$, we obtain $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ xy \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ y & x \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} xy \\ -y^2 \end{bmatrix}$. We remark here that in applying this definition of Lie bracket operation to the case of matrices, the sign is opposite to that of Lie bracket for matrices. Compare formulae (2.8) and (4.23).

††To avoid confusing with the independent time variable t associated with $\dot{x}$ and $\dot{y}$, one could replace t in (4.16) with τ.

$$[\tilde{y}(-\tilde{x}t) + \tilde{x}\tilde{y}] \frac{\partial}{\partial \tilde{x}} + (t-1)\tilde{y}^2 \frac{\partial}{\partial \tilde{y}} = (1-t)\left[\tilde{x}\tilde{y} \frac{\partial}{\partial \tilde{x}} - \tilde{y}^2 \frac{\partial}{\partial \tilde{y}}\right] \tag{4.18}$$

Observe that if we choose $t = 1$ in (4.18), i.e., if we choose the coordinate transformation

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= ye^x \end{aligned} \tag{4.19}$$

then the quadratic term of the transformed vector field will be eliminated. This is precisely the goal of Takens normal form (*Theorem 1.2*).

The vector field resulting from the transformation (4.19) would still contains higher order terms beyond order 2. Let us now consider a *different* transformation

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y + xy \end{aligned} \tag{4.20}$$

In this case, the transformed vector field becomes

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x} = y + xy = \tilde{y} \\ \dot{\tilde{y}} &= (1+x)\dot{y} + \dot{x}y = (1+x)(-y^2) + (y + xy)y = 0 \end{aligned} \tag{4.21}$$

Note that the transformed vector field $\tilde{y} \dfrac{\partial}{\partial \tilde{x}}$ is *linear* and the vector field v in (4.11) is therefore linearizable by the transformation (4.20). Observe that the two transformations (4.19) and (4.20) are *2-jet equivalent* at 0. ■

Recall next that a vector field Y on $\mathbf{R}^n$ generates a flow ϕ^t, that is, a local 1-parameter group of transformations of $\mathbf{R}^n$. This transformation is obtained by solving the associated ODE $\dot{x} = Y(x)$ with the initial condition $x(0) = x_0$; namely,

$$x(t) = \phi^t(x_0) \tag{4.22}$$

If Y vanishes at 0, then $\phi^t(0) = 0$. To derive the normal form, we need the following fundamental formula from differential geometry:

Theorem 4.3: Infinitesimal deformation of vector fields

Let ϕ^t be a flow generated by $Y \in \mathcal{X}_0$. Then the following formula holds:

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t{}_* v = \lim_{t \to 0} \frac{1}{t} (\phi^t{}_* v - v) = -[Y, v] \tag{4.23}$$

For a proof of this standard result, see p. 15 of [14]. Here, we will demonstrate the validity of (4.23) with an example.

Example 4.4

Let v in (4.23) be the vector field considered earlier in Example 4.2, namely,

$$v = (y + xy)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} \tag{4.24}$$

Let Y in (4.23) be the vector field defined by

$$Y = xy\frac{\partial}{\partial y} \tag{4.25}$$

The flow generated by Y is obtained from the solution

$$x(t) = x_0 \quad , y(t) = y_0 e^{x_0 t} \tag{4.26}$$

of the associated ODE

$$\dot{x} = 0 \quad , \quad \dot{y} = xy \quad , \quad x(0) = x_0 \quad , \quad y(0) = y_0. \tag{4.27}$$

The resulting 1-parameter group $\{\phi^t\}$ of transformations is therefore given by

$$\phi^t(x,y) = (x, ye^{xt}) \tag{4.28}$$

For each value of the parameter t, the transformed vector field $\phi^t{}_* \, v$ is given by

$$(1+x)ye^{-xt}\frac{\partial}{\partial x} + [(1+x)t - 1]y^2e^{-xt}\frac{\partial}{\partial y}. \tag{4.29}$$

Diferentiating $\phi^t{}_* v$ with respect to t at $t = 0$, we obtain:

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t{}_* v = -xy(1+x)\frac{\partial}{\partial x} + y^2(1+2x)\frac{\partial}{\partial y} \tag{4.30}$$

One can easily check that (4.30) coincides with the Lie bracket

$$-[Y,v] \triangleq -\left[xy\frac{\partial}{\partial y},\ (y+xy)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}\right] \tag{4.31}$$

as predicted by *Theorem 4.3*. ■

Since we have set up $M = \mathcal{X}_0^k$ and $G = \mathrm{Diff}_0^k$ as a framework for our normal forms, we must translate *Theorem 4.3* to corresponding k-jet spaces.

Corollary 4.5: Infinitesimal deformation for k-jets of vector fields.

Let $(\phi^k)^t$ be the local 1-parameter group of k-jets of transformations in Diff_0^k which is generated by $Y^k \in \mathcal{X}_0^k$. Then

$$\left.\frac{d}{dt}\right|_{t=0} (\phi^k)^t{}_* \, v^k = -[Y^k, v^k]^k \quad , \quad v^k \in \mathcal{X}_0^k \tag{4.32}$$

holds, where $[Y^k, v^k]^k$ denotes the k-jet of $[Y^k, v^k]$ at 0.

Proof. Let us first verify that $(\phi^k)^t$ is well defined: Indeed, if Y and Y' are representatives of Y^k, i.e., if Y and Y' are k-jet equivalent at 0, then the flow ϕ^t generated by Y and the flow ϕ'^t generated by Y' are also k-jet equivalent at 0 for every t. This property is proved in *Appendix 3*.

The k-jet $[Y^k, v^k]^k$ is also well defined because both Y^k and v^k vanish at 0. Consequently, by taking the k-jet of both sides of (4.23), we obtain (4.32). ■

Let us summarize what we have established so far: we have chosen the framework for normal forms with $M = \mathcal{X}_0^k$, $G = \text{Diff}_0^k$, and the infinitesimal deformation given by (4.32). It follows from (3.3) that (4.32) can be recast into an ODE

$$\boxed{\frac{d}{dt} v_t^k = -[Y^k, \; v_t^k]^k} \tag{4.33}$$

upon defining $v_t^k \triangleq (\phi^k)^t{}_*$. Following the strategy described in the preceding section, we can obtain the *kth order normal form* of v^k by solving (4.33) for some appropriate choices of Y^k's. Consequently, our next task is to solve (4.33).

Our basic strategy is to solve (4.33) recursively. First we let $k = 1$ and derive the 1*st* order normal form. Then we proceed to $k = 2$ by an appropriate transformation which allows us to find the simplest but equivalent 2-jets without affecting the previously derived 1-jet. This procedure can in principle be repeated to derive the normal form of any order.

To derive the 1*st* order normal form of $v^1 \in \mathcal{X}_0^1$, let $v = Ax + \ldots$ be its representative. Let $\phi = Px + \ldots$ be a representative of $\phi^1 \in \text{Diff}_0^1$. Since ϕ is a diffeomorphism at 0, P is non-singular and we have

$$\phi_* \, v = (PAP^{-1})x + \ldots \tag{4.34}$$

Note that the 1-jet equivalence relation on $\mathcal{X}_0^1$ coincides with the conjugate relation on $M(n, \mathbb{R})$ (see *Definition 2.1*). Hence, the *1st order normal forms for vector fields* coincides with the *Jordan normal forms* for matrices.

Our strategy for solving (4.33) can be outlined as follow: First, we choose a Jordan normal form to be the 1-jet v^1 and consider the 2-jet $v^2 = v_1 + v_2$ with $v_1 = v^1$. Next we seek an appropriate transformation to change v^2 into a simpler form. The 2*nd* order normal form is *not* unique and depends on the degree of degeneracy of the original 2-jet v^2. The most usual case involving the least degenerate 2-jet is called the *non-degenerate 2nd order normal form.* Although the 2nd order normal forms corresponding to different degenerate

2-jets are different in form, they all contain the *the same* 1-jet; namely, the corresponding Jordan normal form.

The next step is to choose the *2nd* order normal form, non-degenerate or otherwise, as the 2-jet v^2 and consider the *3rd* order normal form problem for 3-jets of vector-fields having the above 2-jet. We then proceed to reduce the *3rd* order terms to the simplest possible form via another transformation which does *not* affect the lower order terms. Such a transformation can always be found in view of the following important property:

Lemma 4.6 (key Lemma)

Let v^k be a k-jet of vector field in $\mathcal{X}_0^k$ and let v^{k-1} be its $(k-1)th$ order part, i.e.

$$j_0^{k,k-1}\ v^k = v^{k-1} \tag{4.35}$$

If a vector field $Y^k \in \mathcal{X}_0^k$ satisfies

$$[Y^k,\ v^k]^{k-1} = 0 \tag{4.36}$$

then the flow $(\phi^k)^t$ generated by Y^k does *not* affect the previous $(k-1)$-jet v^{k-1}; i.e.,

$$j_0^{k,k-1}\ (\phi^k)^t{}_*\ v^k = v^{k-1} \tag{4.37}$$

Proof. Let ϕ, Y, and v be representatives of ϕ^k, Y^k, and v^k, respectively. It suffices to prove

$$j_0^{k-1}\ \phi^t{}_*\ v = j_0^{k-1}\ v = v^{k-1} \tag{4.38}$$

Recall that

$$\frac{d}{dt}\ \phi^t{}_*v = -[Y,\phi^t{}_*v] \tag{4.39}$$

where $\phi^t{}_*v$ can be expanded into a Taylor series about 0 with respect to t; namely,

$$\phi^t{}_*\ v = v - t[Y,v] + \frac{t^2}{2}\ [Y,\ [Y,v]]$$

$$-\frac{t^3}{3!}\ [Y,[Y,[Y,v]]] + \frac{t^4}{4!}\ [Y,[Y,[Y,[Y,v]]]] + \cdots \tag{4.40}$$

The above "nested" Lie brackets resulted from the following iterative substitutions:

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t{}_*\ v = -[Y,\ \phi^t{}_*\ v\ |_{t=0}] = -[Y,v] \tag{4.41}$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \phi^t{}_*\ v = -\left.\frac{d}{dt}\right|_{t=0} [Y,\ \phi^t{}_*v]$$

$$= -\left[Y, \left.\frac{d}{dt}\right|_{t=0} \phi^t{}_*\ v\right] = [Y,[Y,v]] \tag{4.42}$$

. . . . . .

Now, (4.36) implies that

$$j_0^{k-1} \; [Y, v] = 0 \tag{4.43}$$

It follows from (4.43) and property (4) of *Proposition A1-3* that

$$j_0^{k-1} \; [Y, [Y, \cdots, [Y, v] \cdots]] = 0 \tag{4.44}$$

Applying j_0^{k-1} to both sides of (4.40) and using (4.44), we obtain (4.38). ■

5. NORMAL FORMS FOR VECTOR FIELDS: EXPLICIT ALGORITHM

Our recursive algorithm for deriving normal forms of vector field consists of fixing a $(k-1)$-jet v^{k-1} of vector field v and deriving the kth order normal form by simplifying the kth order term h_k of v by an appropriate one-parameter group of k-jet transformations $(\phi^k)^t \in \text{Diff}_0^k$ which leaves the $(k-1)$-jet v^{k-1} unchanged. By the key Lemma 4.6, the generator Y^k of $(\phi^k)^t$ needs only satisfy the constraint

$$[Y^k, v^k]^{k-1} = 0 \tag{5.1}$$

Under this condition, we study the differential equation

$$\frac{d}{dt} v^k(t) = -[Y^k, v^k(t)]^k \tag{5.2}$$

where $v^k(t) = (\phi^k)^t{}_* \, v^k$. Since the $(k-1)$-jet v^{k-1} of $v^k(t)$ is invariant, it follows that $\frac{d}{dt} v^{k-1} = 0$ and (5.2) can be considered as an ODE

$$\boxed{\frac{d}{dt} h_k(t) = -[Y^k, v^{k-1} + h_k(t)]_k} \tag{5.3}$$

on the subspace H_k, namely, the set of all homogeneous vector fields of order k, where $h_k(t)$ is the kth order part of $v^k(t)$, i.e., $v^k(t) = v^{k-1} + h_k(t)$, and $[\cdot, \cdot]_k$ denotes the kth order part of $[\cdot, \cdot]$. We will henceforth call (5.3) under the condition (5.1) the kth *order normal form problem on* H_k. In the following illustrative examples, we will show how (5.3) can be interpreted as a *linear* ODE which can be easily solved.

Example 5.1: Simple-zero type

Consider all vector fields on the real line, $\mathbf{R}^1$, having a zero *linear* part, i.e.,

$$v^1 = 0 \tag{5.4}$$

It follows that (5.1) is satisfied for *any* 2-jet Y^2. Since the vector space H_k of vector fields on $\mathbf{R}^1$ is 1-dimensional whose basis is $x^k \frac{\partial}{\partial x}$, we can express v^2 and Y^2 as follows:

$$v^2 = a\ x^2 \frac{\partial}{\partial x} \tag{5.5}$$

$$Y^2 = (Ax + Bx^2) \frac{\partial}{\partial x} \tag{5.6}$$

Since

$$[Y^2, v^2]^2 = \left[Ax\frac{\partial}{\partial x}\ ,\ ax^2 \frac{\partial}{\partial x} \right] = Aax^2 \frac{\partial}{\partial x} \tag{5.7}$$

it follows from (5.2), (5.5) and (5.7) that the solution on H_2 along the basis $x^2 \frac{\partial}{\partial x}$ must satisfy the *linear* ODE

$$\frac{d}{dt}\, a(t) = -Aa(t) \tag{5.8}$$

whose solution is given by:

$$a(t) = a(0)e^{-At} \tag{5.9}$$

If $a(0) = a \neq 0$ (non-degenerate case), then we can choose $A = \log |a(0)|$ so that at $t = 1$, we have $a(1) = \pm 1$, depending on the sign of $a(0)$. In other words, in the non-degenerate case, the simplest coefficient that we can choose for a in (5.5) is ± 1.

If $a(0) = 0$ (degenerate case), then $a(t) = 0$ and a is simply chosen to be 0.

Hence, we have obtained the following two 2*nd* order normal forms for the vector field (5.5):

(i) $v^2 = \pm x^2 \frac{\partial}{\partial x}$ (non-degenerate 2nd order normal form)

(ii) $v^2 = 0$ (degenerate 2nd order normal form)

Let us proceed to solve the 3*rd* order normal form problem assuming a non-degenerate 2nd order normal form; namely, define

$$v^3 = (\pm x^2 + \alpha x^3) \frac{\partial}{\partial x} \tag{5.10}$$

$$Y^3 = (Ax + Bx^2 + Cx^3) \frac{\partial}{\partial x} \tag{5.11}$$

To satisfy condition (5.1) for $k = 3$, we must have

$$[Y^3, v^3]^2 = \left[Ax\frac{\partial}{\partial x}\ ,\ \pm x^2 \frac{\partial}{\partial x} \right] = \pm Ax^2 \frac{\partial}{\partial x} = 0 \tag{5.12}$$

which is possible iff $A = 0$. It follows that

$$[Y^3, v^2 + h_3(t)]_3 = \left[(Bx^2 + Cx^3)\frac{\partial}{\partial x}, (\pm x^2 + \alpha x^3)\frac{\partial}{\partial x}\right]_3 = 0 \tag{5.13}$$

and (5.3) becomes

$$\frac{d}{dt}\alpha(t) = 0 \tag{5.14}$$

whose solution is $\alpha(t) = \alpha(0) = \alpha$. Therefore, the non-degenerate *3rd* order normal form is given by:

$$v^3 = (\pm x^2 + \alpha x^3)\frac{\partial}{\partial x} \tag{5.15}$$

To obtain the *4th* order normal form, define

$$v^4 = (\pm x^2 + \alpha x^3 + px^4)\frac{\partial}{\partial x} \tag{5.16}$$

$$Y^4 = (Ax + Bx^2 + Cx^3 + Dx^4)\frac{\partial}{\partial x} \tag{5.17}$$

It follows from (5.1) that

$$[Y^4, v^4]^3 = \left[Ax\frac{\partial}{\partial x}, \pm x^2\frac{\partial}{\partial x}\right] + \left[Ax\frac{\partial}{\partial x}, \alpha x^3\frac{\partial}{\partial x}\right] + \left[Bx^2\frac{\partial}{\partial x}, \pm x^2\frac{\partial}{\partial x}\right]$$

$$= (\pm Ax^2 + 2A\alpha x^3)\frac{\partial}{\partial x} = 0 \tag{5.18}$$

Hence, $A = 0$. Since

$$[Y^4, v^4]^4 = (B\alpha \mp C)x^4\frac{\partial}{\partial x} \tag{5.19}$$

(5.3) becomes

$$\frac{d}{dt}p(t) = -B\alpha \pm C \tag{5.20}$$

By choosing $B = 0$, $C = 1$ and $t = \mp p(0)$, the solution of (5.20) is given by

$$p(t) = P(0) \pm t = 0 \tag{5.21}$$

Consequently, the non-degenerate *4th order normal* form is given by

$$v^4 = (\pm x^2 + \alpha x^3)\frac{\partial}{\partial x} \tag{5.22}$$

In fact, the following proposition proves that all terms of degree greater than 3 can be set to zero in the higher order normal forms.

Proposition 5.2

The non-degenerate *kth order normal form* of vector fields on $\mathbf{R}^1$ with vanishing 1-jet (i.e., no linear terms) is given by $(\pm x^2 + \alpha x^3)\frac{\partial}{\partial x}$ for $k \geq 3$.

Proof. We have already proved the above assertion for $k = 4$. Let us prove the case for $k \geq 5$ *by induction.*

Suppose the above assertion holds for some $k \geq 4$, namely,

$$v^k = (\pm x^2 + \alpha x^3)\frac{\partial}{\partial x} \tag{5.23}$$

If we choose

$$v^{k+1} = (\pm x^2 + \alpha x^3 + px^{k+1})\frac{\partial}{\partial x} \tag{5.24}$$

and

$$Y^{k+1} = Ax^k\frac{\partial}{\partial x} \tag{5.25}$$

then

$$[Y^{k+1}, v^{k+1}]^k = 0 \tag{5.26}$$

and (5.1) holds. Since

$$[Y^{k+1}, v^k + h_{k+1}(t)]_{k+1} = \pm 2Ax^{k+1} \tag{5.27}$$

the differential equation (5.3) becomes

$$\frac{d}{dt}p(t) = \mp 2A \tag{5.28}$$

Therefore, if we choose $A = \frac{1}{2}$ and $t = \pm p(0)$, then $p(t) = 0$, and the $(k+1)$th order normal form is also given by $(\pm x^2 + \alpha x^3)\frac{\partial}{\partial x}$ ∎

The preceding calculation is relatively easy since our vector field is defined on $\mathbf{R}^1$. The computation becomes much more complicated for vector fields on $\mathbf{R}^n$, where $n \geq 2$. In order to overcome this complexity, we can reduce the linear differential equation (5.3) which is defined on the subspace H_k, to another equation which is defined on a subspace G_k of somewhat lower dimension. This subspace is defined as follow.

Definition 5.3

Let B_k be the subspace of H_k consisting of the images of the linear map

$$L : H_k \to H_k \ , Y_k \mapsto [Y_k, v_1] \tag{5.29}$$

where $Y_k \in H_k$ and $v_1 \in H_1$.

The subspace G_k is defined as a complementary subspace to B_k of H_k. We denote the *natural projection* of H_k along B_k by:

$$\pi_k : H_k \rightarrow G_k \tag{5.30}$$

The geometrical interpretation of B_k, H_k and π_k is depicted in Fig. 1. The following fundamental theorem will greatly reduce our normal form calculations on $\mathbf{R}^n$.

Theorem 5.4: Reduction Theorem

The *k*th order normal form problem

$$\frac{d}{dt} h_k(t) = -[Y^k, v^{k-1} + h_k(t)]_k \tag{5.31}$$

on H_k with

$$[Y^{k-1}, v^{k-1}]^{k-1} = 0 \tag{5.32}$$

can be reduced to the problem

$$\frac{d}{dt} g_k(t) = -\pi_k([Y^{k-1}, v^{k-1} + g_k(t)]_k) \tag{5.33}$$

on G_k under the same condition (5.32), where $g_k(t) \in G_k$.

More precisely, if we arrive at some $\tilde{g}_k$ by integrating (5.33) with (5.32) from the initial point $g_k(0)$, then we can also deform $h_k(0)$, satisfying $\pi_k(h_k(0)) = g_k(0)$, to $\tilde{g}_k$ itself by integrating (5.31) with (5.32) for some appropriate choice of Y^k. In particular, the terms belonging to B_k can be eliminated.

A geometrical interpretation of *Theorem 5.4* is shown in Fig. 2.

Remarks

(1) Note that, in (5.31), the superfix of Y is k, whereas that in (5.33) is $k-1$.

(2) The choice of G_k (and, as a result, π_k) is not unique. If we replace G_k with another complementary space G_k', the resulting normal forms are transformed to each other up to order k, but have different forms.

(3) The last statement of Theorem 5.4 corresponds to *Takens normal form theorem* (see Theorem 1.2).

(4) It is often possible to do better than just eliminating B_k, by solving (5.33). This further simplification of Takens normal form is due to Ushiki [5] who gave a systematic procedure for achieving the simplest possible normal form. Our approach in this paper is based on Ushiki's algorithm.

(5) Two systematic and general methods for computing Takens normal forms up to any desired higher order terms have recently been developed in Cushman and Sanders [16], and in Elphick et al. [17]. Moreover, normal forms have been computed using *symbolic* manipulations by Rand and Armbruster [18].

Proof of the Reduction Theorem

Let us prove first that if $\pi_k(h_k(0)) = g_k(0)$, then we can deform $h_k(0)$ to $g_k(0)$. Since we can decompose $h_k(0)$ into

$$h_k(0) = b_k(0) + g_k(0) \tag{5.34}$$

where $b_k(0) \in B_k$ is of the form

$$b_k(0) = [Y_k, v_1] \tag{5.35}$$

for some $Y_k \in H_k$ in view of (5.29). Consider the differential equation

$$\frac{d}{dt} h_k(t) = -[Y_k, v^{k-1} + h_k(t)]_k \tag{5.36}$$

where Y_k satisfies the condition

$$[Y_k, v^{k-1}]^{k-1} = 0 \tag{5.37}$$

Since

$$-[Y_k, v^{k-1} + h_k(t)]_k = -[Y_k, v_1] = -b_k(0) \tag{5.38}$$

the differential equation (5.36) becomes

$$\frac{d}{dt} h_k(t) = -b_k(0) \tag{5.39}$$

whose solution is

$$h_k(t) = h_k(0) - t\ b_k(0) \tag{5.40}$$

Hence, choosing t=1 in (5.40), we obtain

$$h_k(1) = h_k(0) - b_k(0) = g_k(0) \tag{5.41}$$

It follows from the above argument that the b_k component can be deformed into any desired form. Next, let us project (5.31) on G_k by means of π_k.

Since $h_k(t) = b_k(t) + g_k(t)$ and $b_k(t) = [Z_k, v_1]$ for some $Z_k \in H_k$, we have

$$\begin{aligned}[Y^k, v^{k-1} + h_k(t)]_k &= [Y^k, v^{k-1} + [Z_k, v_1] + g_k(t)]_k \\ &= [Y^k, v^{k-1} + g_k(t)]_k + [Y^k, [Z_k, v_1]]_k \end{aligned} \tag{5.42}$$

Applying Jacobi's identity (*Proposition A1-3(3)* in *Appendix 1*), we have

$$[Y^k, [Z_k, v_1]]_k = -[Z_k, [v_1, Y^k]]_k - [v_1, [Y^k, Z_k]]_k$$

$$= -[Z_k, [v_1, Y_1]] - [v_1, [Y_1, Z_k]]. \tag{5.43}$$

The first term in (5.43) vanishes since $[v_1, Y_1] = 0$ from (5.1) and the second term belongs to B_k.

On the other hand,

$$[Y^k, v^{k-1} + g_k(t)]_k = [Y^{k-1}, v^{k-1} + g_k(t)]_k + [Y_k, v^{k-1} + g_k(t)]_k$$

$$= [Y^{k-1}, v^{k-1} + g_k(t)]_k + [Y_k, v_1] \tag{5.44}$$

where the last term in (5.44) belongs to B_k. It follows from (5.42)–(5.44) that

$$\frac{d}{dt} g_k(t) = -\pi_k([Y^{k-1}, v^{k-1} + g_k(t)]_k)$$

Since this equation depends only on g_k, and not on b_k, it can be solved within G_k. The proof is, thus, completed. ■

To demonstrate the usefulness of this theorem, and to illustrate the preceding algorithm for deriving the normal forms of vector fields, let us consider a non-trivial example in complete details.

Example 5.5: Double-zero type

Consider all vector fields on the real plane $\mathbf{R}^2$ which vanish at the origin 0, and whose *linear* part v_1 at 0 is equivalent to $y\frac{\partial}{\partial x}$. Hence, its Jacobian matrix at 0 is equivalent to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with a double zero eigenvalue. Of course, by an appropriate linear transformation, we may assume, without loss of generality, that v_1 itself is defined by $y\frac{\partial}{\partial x}$.

Consider first the *2nd order normal form problem* on H_2, the real vector space whose elements are homogeneous polynomials of degree 2; i.e., linear combinations of x^2, xy, and y^2 along $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Hence, this vector space is spanned by

$$\left\{ \begin{matrix} x^2\frac{\partial}{\partial x}, & xy\frac{\partial}{\partial x}, & y^2\frac{\partial}{\partial x} \\ x^2\frac{\partial}{\partial y}, & xy\frac{\partial}{\partial y}, & y^2\frac{\partial}{\partial y} \end{matrix} \right\} \tag{5.45}$$

in the sense that any element of H_2 is a linear sum of the above basis vectors. Hence,

$$\dim H_2 = 6 \tag{5.46}$$

Since $v_1 = y\frac{\partial}{\partial x}$, the subspace B_2 consists of all elements of the form $\left[Y_2, y\frac{\partial}{\partial x} \right] \in H_2$, where Y_2 is any element in H_2. To find a basis for B_2, it suffices to find the image of the 6 basis vectors in (5.45). For example, since $\left[x^2\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}\right] = -2xy\frac{\partial}{\partial x}$, $x^2\frac{\partial}{\partial x} \to -2xy\frac{\partial}{\partial x}$. Similarly, $\left[xy\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}\right] = -y^2\frac{\partial}{\partial x}$ gives $xy\frac{\partial}{\partial x} \mapsto -y^2\frac{\partial}{\partial x}$, $\left[y^2\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}\right] = 0$ gives $y^2\frac{\partial}{\partial x} \mapsto 0$, $\left[x^2\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] = x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}$ gives $x^2\frac{\partial}{\partial y} \mapsto x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}$, $\left[xy\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$ gives $xy\frac{\partial}{\partial y} \mapsto xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$, and $\left[y^2\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] = y^2\frac{\partial}{\partial x}$ gives $y^2\frac{\partial}{\partial y} \mapsto y^2\frac{\partial}{\partial x}$. The above result can be summarized by the following matrix representation of the linear map

$$H_2 \to H_2, Y_2 \mapsto [Y_2, v_1] \tag{5.47}$$

$$\begin{array}{c} x^2\frac{\partial}{\partial x} \\ xy\frac{\partial}{\partial x} \\ y^2\frac{\partial}{\partial x} \\ x^2\frac{\partial}{\partial y} \\ xy\frac{\partial}{\partial y} \\ y^2\frac{\partial}{\partial y} \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{5.48}$$
$$\quad x^2\frac{\partial}{\partial x} \quad xy\frac{\partial}{\partial x} \quad y^2\frac{\partial}{\partial x} \quad x^2\frac{\partial}{\partial y} \quad xy\frac{\partial}{\partial y} \quad y^2\frac{\partial}{\partial y}$$

Note that B_2 is spanned by

$$\left\{xy\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y}\right\} \tag{5.49}$$

in the sense that the image of the above 6 basis vectors can be expressed as a linear sum of the 4 vectors in (5.49). Hence B_2 is a 4-dimensional subspace of H_2. The complementary space G_2 must therefore be only of dimension 2. Indeed, we can choose, among others,

$$\left\{x^2\frac{\partial}{\partial y}, xy\frac{\partial}{\partial y}\right\} \tag{5.50}$$

as a set of basis vectors for G_2. Clearly, any element of H_2 can be decomposed into a component in B_2, spanned by (5.49), and a component in G_2, spanned by (5.50). For example,

$$\underbrace{x^2\frac{\partial}{\partial x}}_{\in\, H_2} = \underbrace{\left[x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}\right]}_{\in\, B_2} + \underbrace{2xy\frac{\partial}{\partial y}}_{\in\, G_2} \tag{5.51}$$

The projection map $\pi_2 : H_2 \to G_2$ therefore gives

$$\pi_2\left[x^2\frac{\partial}{\partial x}\right] = 2xy\frac{\partial}{\partial y} \tag{5.52}$$

Every element of $g_2 \in G_2$ is a linear sum of the basis vectors in (5.50); namely,

$$g_2 = \alpha x^2\frac{\partial}{\partial y} + \beta xy\frac{\partial}{\partial y} \tag{5.53}$$

for some real numbers α and β.

It follows from (5.33) of the Reduction theorem that the *reduced 2nd order normal form problem* becomes

$$\frac{d}{dt}g_2(t) = -\pi_2([Y_1, g_2(t)]) \tag{5.54}$$

where Y_1 must satisfy (5.37), i.e.,

$$[Y_1, v_1] = 0 \tag{5.55}$$

In general, Y_1 is defined by $(ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y}$. Hence, $[Y_1, v_1] = (cx + (d - a)y)\frac{\partial}{\partial x} - cy\frac{\partial}{\partial y} = 0$ iff $c = 0$ and $a = d$. Consequently, to satisfy (5.55), Y_1 must assume the special form

$$Y_1 = A\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] + By\frac{\partial}{\partial x} \tag{5.56}$$

Substituting (5.53) and (5.56) into (5.54), we obtain

$$\frac{d}{dt}\left[\alpha(t)x^2\frac{\partial}{\partial y} + \beta(t)xy\frac{\partial}{\partial y}\right]$$

$$= -\pi_2\left[\left[A\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] + By\frac{\partial}{\partial x},\ \alpha(t)x^2\frac{\partial}{\partial y} + \beta(t)xy\frac{\partial}{\partial y}\right]\right]$$

$$= -\pi_2\left\{A\,\alpha(t)x^2\frac{\partial}{\partial y} + A\,\beta(t)xy\frac{\partial}{\partial y} + 2B\,\alpha(t)xy\frac{\partial}{\partial y} - B\,\alpha(t)x^2\frac{\partial}{\partial x}\right.$$

$$+ B\beta(t)y^2\frac{\partial}{\partial y} - B\beta(t)xy\frac{\partial}{\partial x}\Big]$$

$$= -A\alpha(t)x^2\frac{\partial}{\partial y} - \Big[A\beta(t) + 2B\alpha(t)\Big]xy\frac{\partial}{\partial y} + B\alpha(t)\pi_2\Big[x^2\frac{\partial}{\partial x}\Big]$$

$$= -A\alpha(t)x^2\frac{\partial}{\partial y} - A\beta(t)xy\frac{\partial}{\partial y} \tag{5.57}$$

where the last expression results from cancellation of terms due to (5.52). Extracting the differential equation along each basis vector in G_2, we obtain

$$\frac{d}{dt}\alpha(t) = -A\alpha(t) \tag{5.58}$$

$$\frac{d}{dt}\beta(t) = -A\beta(t) \tag{5.59}$$

The solutions of these two uncoupled linear differential equations are:

$$\alpha(t) = \alpha(0)e^{-At} \tag{5.60}$$

$$\beta(t) = \beta(0)e^{-At} \tag{5.61}$$

If $\alpha(0) \neq 0$, we can choose $A = \log|\alpha(0)|$ so that at $t = 1$, we have

$$\alpha(1) = \alpha(0)e^{-A} = \pm 1 \tag{5.62}$$

In this case, $\beta = \beta(0)e^{-A}$

If $\alpha(0) = 0$ (degenerate case 1) and $\beta(0) \neq 0$, we can similarly choose $\beta(1) = \pm 1$ and $\alpha = 0$.

If $\alpha(0) = \beta(0) = 0$ (degenerate case 2), the 2-jet vanishes and the normal form degenerates into a 1-jet.

Hence, depending on the degree of degeneracy, the 2nd order normal form problem for the vector field $v^1 = y\frac{\partial}{\partial x}$ has the following solutions:

(i) non-degenerate 2nd order normal form:

$$v^2 = y\frac{\partial}{\partial x} + (\pm x^2 + \beta xy)\frac{\partial}{\partial y} \tag{5.63}$$

(ii) degenerate 2nd order normal form (case 1)

$$v^2 = y\frac{\partial}{\partial x} \pm xy\frac{\partial}{\partial y} \tag{5.64}$$

(iii) degenerate 2nd order normal form (case 2)

$$v^2 = y\frac{\partial}{\partial x} \tag{5.65}$$

Remark.

If we choose another complementary space G_2' to be the one spanned by the basis vectors

$$\left\{x^2\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial y}\right\} \tag{5.66}$$

instead of (5.50), the corresponding 2nd order normal forms are:

(i) non-degenerate 2nd order normal form:

$$v^2 = (y \pm x^2)\frac{\partial}{\partial x} + \beta'\, x^2\frac{\partial}{\partial y} \tag{5.67}$$

(ii) degenerate 2nd order normal form (case 1)

$$v^2 = y\frac{\partial}{\partial x} \pm x^2\frac{\partial}{\partial y} \tag{5.68}$$

(iii) degenerate 2nd order normal form (case 2)

$$v^2 = y\frac{\partial}{\partial x} \tag{5.69}$$

Let us proceed next to solve the 3*rd* order normal form problem in H_3 using the non-degenerate 2nd order normal form (5.63). Recall H_3 consists of all degree-3 homogeneous polynomials; namely, linear sums of x^3, x^2y, xy^2, y^3 along $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Hence, H_3 is an 8-dimensional space spanned by the following 8 basis vectors:

$$\left\{\begin{matrix} x^3\frac{\partial}{\partial x}, & x^2y\frac{\partial}{\partial x}, & xy^2\frac{\partial}{\partial x}, & y^3\frac{\partial}{\partial x} \\ x^3\frac{\partial}{\partial y}, & x^2y\frac{\partial}{\partial y}, & xy^2\frac{\partial}{\partial y}, & y^3\frac{\partial}{\partial y} \end{matrix}\right\} \tag{5.70}$$

Following the same procedure in the construction of the matrix representation in (5.47), we obtain the following matrix representation for the linear map

$$H_3 \to H_3,\ Y_3 \mapsto [Y_3,\ v_1] \tag{5.71}$$

with respect to the basis (5.70):

$$
\begin{array}{c}
x^3\frac{\partial}{\partial x} \\
x^2y\frac{\partial}{\partial x} \\
xy^2\frac{\partial}{\partial x} \\
y^3\frac{\partial}{\partial x} \\
x^3\frac{\partial}{\partial y} \\
x^2y\frac{\partial}{\partial y} \\
xy^2\frac{\partial}{\partial y} \\
y^3\frac{\partial}{\partial y}
\end{array}
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\tag{5.72}
$$

$$
x^3\frac{\partial}{\partial x} \quad x^2y\frac{\partial}{\partial x} \quad xy^2\frac{\partial}{\partial x} \quad y^3\frac{\partial}{\partial x} \quad x^3\frac{\partial}{\partial y} \quad x^2y\frac{\partial}{\partial y} \quad xy^2\frac{\partial}{\partial y} \quad y^3\frac{\partial}{\partial y}
$$

It follows from (5.72) that the image subspace B_3 is spanned by 6 basis vectors; namely

$$
\left\{x^2y\frac{\partial}{\partial x},\ xy^2\frac{\partial}{\partial x},\ y^3\frac{\partial}{\partial x},\ x^3\frac{\partial}{\partial x} - 3x^2y\frac{\partial}{\partial y},\ xy^2\frac{\partial}{\partial y},\ y^3\frac{\partial}{\partial y}\right\} \tag{5.73}
$$

Hence $\dim B_3 = 6$ and the complementary space G_3 has dimension 2. Let us choose G_3 to be the subspace of H_3 spanned by the following 2 basis vectors:

$$
\left\{x^3\frac{\partial}{\partial y},\ x^2y\frac{\partial}{\partial y}\right\} \tag{5.74}
$$

It follows from (5.73) and (5.74) that the projection $\pi_3 : H_3 \to G_3$ maps $x^3\frac{\partial}{\partial x}$ as follow:

$$
\pi_3\left[x^3\frac{\partial}{\partial x}\right] = \pi_3\left[x^3\frac{\partial}{\partial x} - 3x^2y\frac{\partial}{\partial y} + 3x^2y\frac{\partial}{\partial y}\right] = 3x^2y\frac{\partial}{\partial y} \tag{5.75}
$$

In order to satisfy the constraint (5.32), i.e.,

$$
[Y^2, v^2]^2 = 0 \tag{5.76}
$$

it suffices to verify that

$$
[Y_1, v_1] = 0 \tag{5.77}
$$

$$
[Y_2, v_1] + [Y_1, v_2] = 0 \tag{5.78}
$$

To check this, let

$$Y^2 = Y_1 + Y_2 = A\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] + By\frac{\partial}{\partial x}$$

$$+ (C_1x^2 + C_2xy + C_3y^2)\frac{\partial}{\partial x}$$

$$+ (D_1x^2 + D_2xy + D_3y^2)\frac{\partial}{\partial y} \tag{5.79}$$

Using the non-degenerate 2nd order normal form (5.63); i.e.,

$$v^2 = y\frac{\partial}{\partial x} + (\pm x^2 + \beta\, xy)\frac{\partial}{\partial y} \tag{5.80}$$

we calculate

$$[Y_1, v_2] = \left[A\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] + By\frac{\partial}{\partial x},\ (\pm x^2 + \beta xy)\frac{\partial}{\partial y}\right]$$

$$= \mp Bx^2\frac{\partial}{\partial x} - B\beta xy\frac{\partial}{\partial x} \pm Ax^2\frac{\partial}{\partial y} + (A\beta \pm 2B)xy\frac{\partial}{\partial y} + B\beta y^2\frac{\partial}{\partial y} \tag{5.81}$$

and

$$[Y_2, v_1] = \{D_1x^2 + (-2C_1 + D_2)xy + (-C_2 + D_3)y^2\}\frac{\partial}{\partial x} + (-2D_1xy - D_2y^2)\frac{\partial}{\partial y} \tag{5.82}$$

Substituting (5.81) and (5.82) into (5.78) and equating the corresponding coefficients to zero, we obtain

$$\left.\begin{aligned} &\mp B + D_1 = 0, && -B\beta + (-2C_1 + D_2) = 0 \\ &-C_2 + D_3 = 0, && \pm A = 0 \\ &A\beta \pm 2B - 2D_1 = 0, && B\beta - D_2 = 0 \end{aligned}\right\} \tag{5.83}$$

Hence, we must set

$$A = 0,\ C_1 = 0,\ C_2 = D_3,\ D_1 = \pm B,\ D_2 = B\beta \tag{5.84}$$

in (5.79); i.e.,

$$Y^2 = By\frac{\partial}{\partial x} + C_2xy\frac{\partial}{\partial x} + C_3y^2\frac{\partial}{\partial x}$$

$$\pm Bx^2\frac{\partial}{\partial y} + B\beta xy\frac{\partial}{\partial y} + C_2y^2\frac{\partial}{\partial y}$$

$$= Bv^2 + C_2\left[xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right] + C_3y^2\frac{\partial}{\partial x} \tag{5.85}$$

The reduced 3rd order normal form problem is therefore given by

$$\frac{d}{dt}g_3 = -\pi_3([Y^2,\ v^2 + g_3]_3) \tag{5.86}$$

where

$$g_3 = (ax^3 + bx^2y)\frac{\partial}{\partial y} \tag{5.87}$$

in view of (5.74), and Y^2 is given by (5.85). Since

$$[Bv^2,\ v^2 + g_3]_3 = [Bv^1,\ g_3] \in B_3 \tag{5.88}$$

we need only consider

$$\left[C_2\left[xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right] + C_3y^2\frac{\partial}{\partial x}\ ,\ v^2\right]$$

$$= \mp C_2x^3\frac{\partial}{\partial x} - C_2\beta x^2y\frac{\partial}{\partial x}$$

$$\pm C_3\left[2xy^2\frac{\partial}{\partial y} - 2x^2y\frac{\partial}{\partial x}\right] + C_3\beta\left[y^3\frac{\partial}{\partial y} - 2xy^2\frac{\partial}{\partial x}\right] \tag{5.89}$$

Substituting (5.87) and (5.89) into corresponding terms in (5.86), we obtain

$$\frac{d}{dt}\left[a(t)x^3 + b(t)x^2y\right]\frac{\partial}{\partial y}$$

$$= -\pi_3\left[\mp C_2x^3\frac{\partial}{\partial x} + (-C_2\beta \mp 2C_3)x^2y\frac{\partial}{\partial x} - 2C_3\beta xy^2\frac{\partial}{\partial x} \pm 2C_3xy^2\frac{\partial}{\partial y} + C_3\beta y^3\frac{\partial}{\partial y}\right]$$

$$= \pm 3C_2x^2y\frac{\partial}{\partial y} \tag{5.90}$$

where the last expression results from (5.74) and (5.75). Equating the coefficients of corresponding terms, we obtain the following 2 uncoupled linear differential equations:

$$\frac{d}{dt}\ a(t) = 0 \tag{5.91}$$

$$\frac{d}{dt}\ b(t) = \pm 3C_2 \tag{5.92}$$

The solutions are given trivially by

$$a(t) = a(0) \tag{5.93}$$

$$b(t) = b(0) \pm 3C_2 t \tag{5.94}$$

Hence, if we choose $C_2 = \mp \frac{1}{3} b(0)$, then at t=1, we have $b(1) = 0$; i.e., we can set b=0 in (5.87). It follows that the 3rd order normal form for the non-degenerate vector field $v^2 = y\frac{\partial}{\partial x} + (\pm x^2 + \beta xy)\frac{\partial}{\partial y}$ is as follow:

non–degenerate 3rd order normal form:

$$V^3 = y\frac{\partial}{\partial x} + (\pm x^2 + \beta xy + ax^3)\frac{\partial}{\partial y} \tag{5.95}$$

Following the same procedure, we can derive also various degenerate 3rd order normal forms, as well as higher order normal forms. In Part II of this paper, we will apply this procedure to derive the normal forms of several typical examples.

Appendix 1. Lie bracket for vector fields

Let X and Y be smooth vector fields on $\mathbf{R}^n$. We can identify them with smooth mappings from $\mathbf{R}^n$ to itself:

$$X, Y : \mathbf{R}^n \rightarrow \mathbf{R}^n \tag{A.1}$$

In terms of the standard coordinates $x = (x_1, x_2, \cdots, x_n)$ of $\mathbf{R}^n$, we obtain the following coordinate representation of X and Y:

$$\left.\begin{aligned} X(x) &= (X_1(x), X_2(x), \cdots, X_n(x)) \\ Y(x) &= (Y_1(x), Y_2(x), \cdots, Y_n(x)) \end{aligned}\right\} \tag{A.2}$$

It is often convenient to identify vector fields with *first order differential operators*; namely,

$$\left.\begin{aligned} X &= X_1(x)\frac{\partial}{\partial x_1} + X_2(x)\frac{\partial}{\partial x_2} + \cdots + X_n(x)\frac{\partial}{\partial x_n} \\ Y &= Y_1(x)\frac{\partial}{\partial x_1} + Y_2(x)\frac{\partial}{\partial x_2} + \cdots + Y_n(x)\frac{\partial}{\partial x_n} \end{aligned}\right\} \tag{A.3}$$

Depending on the context, we will use one of these 3 vector field representations throughout this paper. For the subject of this Appendix; namely, Lie brackets, we will adopt the differential operator representation almost exclusively.

Proposition A1.1

Let X and Y be vector fields denoted by the first order differential operators (A.3). Then the differential operator $XY - YX$ is also of first order, and hence is itself a vector field.

Proof.

Since $X = \sum_{i=1}^{n} X_i \dfrac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{n} Y_j \dfrac{\partial}{\partial x_j}$,

$$\begin{aligned} XY &\triangleq \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{n} Y_j \frac{\partial}{\partial x_j} \right] \\ &= \sum_{i,j=1}^{n} \left[X_i \frac{\partial Y_i}{\partial x_i} \frac{\partial}{\partial x_j} + X_i Y_j \frac{\partial^2}{\partial x_i \, \partial x_j} \right] \end{aligned} \tag{A.4}$$

Similarly,

$$YX = \sum_{i,j=1}^{n} \left[Y_j \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial x_i} + Y_j X_i \frac{\partial^2}{\partial x_i \, \partial x_j} \right] \tag{A.5}$$

Therefore

$$XY - YX = \sum_{i,j=1}^{n} \left[x_i \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial x_i} \right]$$

$$= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \left[\frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j \right] \right\} \frac{\partial}{\partial x_i} \tag{A.6}$$

$$\triangleq \sum_{i=1}^{n} Z_i \frac{\partial}{\partial x_i}$$

Since $XY - YX$ is also a first order differential operator, it is a vector field on $\mathbf{R}^n$ ■

Definition A1.2. Lie bracket $[X, Y]$

The vector field $XY - YX$ is called the *Lie bracket* of vector fields X and Y and will henceforth be denoted by $[X, Y]$.

The following properties of Lie brackets are needed in this paper:

Proposition A1.3

Let X, Y, Z denote vector fields and let a, b denote real numbers. Then:

(1) If we identify the vector fields with mappings of $\mathbf{R}^n$ as in (A.1), then the Lie bracket $[X, Y]$ is itself a mapping defined by

$$[X, Y] : \mathbf{R}^n \to \mathbf{R}^n$$

$$x \mapsto DY(x) \cdot X(x) - DX(x) \cdot Y(x) \tag{A.7}$$

where $DX(x)$ and $DY(x)$ denote the Jacobian matrix of X and Y at x, respectively.

(2) The Lie bracket operation is *bilinear*; i.e.,

$$\left. \begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z] \\ [X, aY + bZ] &= a[X, Y] + b[X, Z] \end{aligned} \right\} \tag{A.8}$$

and is skew-symmetric; i.e.,

$$[X, Y] = -[Y, X] . \tag{A.9}$$

(3) Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{A.10}$$

(4) If X and Y are homogeneous polynomial vector fields of degree k and l, respectively, i.e., each component X_i of X, and Y_j of Y is a homogeneous polynomial of degree k and l, respectively, then $[X, Y]$

is a homogeneous vector field of order $k+l-1$.

Proof. (1) is obvious from the coordinate representation (A.6). Also, (2) and (3) are easily checked by using this representation. To avoid redundancy, we will prove only (3). Let

$$X = \sum_i X_i \frac{\partial}{\partial x_i} , Y = \sum_j Y_j \frac{\partial}{\partial x_j} , Z = \sum_k Z_k \frac{\partial}{\partial x_k} \tag{A.11}$$

$$[X, Y] = \sum_i \left\{ \sum_j \left[\frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j \right] \right\} \frac{\partial}{\partial x_i} \tag{A.12}$$

If we define

$$[X, Y]_i = \sum_j \left[\frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j \right] \tag{A.13}$$

then

$$\begin{aligned}
[[X, Y], Z] &= \sum_i \left\{ \sum_k \left[\frac{\partial Z_i}{\partial x_k} \cdot [X, Y]_k - \frac{\partial [X,Y]_i}{\partial x_k} Z_k \right] \right\} \frac{\partial}{\partial x_i} \\
&= \sum_{i,j,k} \left\{ \frac{\partial Z_i}{\partial x_k} \left[\frac{\partial Y_k}{\partial x_j} X_j - \frac{\partial X_k}{\partial x_j} Y_j \right] \right. \\
&\quad \left. - \frac{\partial}{\partial x_k} \left[\frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j \right] Z_k \right\} \frac{\partial}{\partial x_i} \\
&= \sum_{i,j,k} \left\{ \frac{\partial Z_i}{\partial x_k} \left[\frac{\partial Y_k}{\partial x_j} X_j - \frac{\partial X_k}{\partial x_j} Y_j \right] - \left[\frac{\partial Y_i}{\partial x_j} \frac{\partial X_j}{\partial x_k} - \frac{\partial X_i}{\partial x_j} \frac{\partial Y_j}{\partial x_k} \right] Z_k \right. \\
&\quad \left. - \left[\frac{\partial^2 Y_i}{\partial x_k \, \partial x_j} X_j - \frac{\partial^2 X_i}{\partial x_k \, \partial x_j} Y_j \right] Z_k \right\} \frac{\partial}{\partial x_i}
\end{aligned} \tag{A.14}$$

By a cyclic permutation of X, Y, and Z, we obtain similarly:

$$\begin{aligned}
[[Y, Z], X] &= \sum_{i,j,k} \left\{ \frac{\partial X_i}{\partial x_k} \left[\frac{\partial Z_k}{\partial x_j} Y_j - \frac{\partial Y_k}{\partial x_j} Z_j \right] - \left[\frac{\partial Z_i}{\partial x_j} \frac{\partial Y_j}{\partial x_k} - \frac{\partial Y_i}{\partial x_j} \frac{\partial Z_j}{\partial x_k} \right] X_k \right. \\
&\quad \left. - \left[\frac{\partial^2 Z_i}{\partial x_k \, \partial x_j} Y_j - \frac{\partial^2 Y_i}{\partial x_k \, \partial x_j} Z_j \right] X_k \right\} \frac{\partial}{\partial x_i}
\end{aligned} \tag{A.15}$$

$$[[Z, X], Y] = \sum_{i,j,k} \left\{ \frac{\partial Y_i}{\partial x_k} \left[\frac{\partial X_k}{\partial x_j} Z_j - \frac{\partial Z_k}{\partial x_j} X_j \right] - \left[\frac{\partial X_i}{\partial x_j} \frac{\partial Z_j}{\partial x_k} - \frac{\partial Z_i}{\partial x_j} \frac{\partial X_j}{\partial x_k} \right] Y_k \right.$$

$$\left. - \left[\frac{\partial^2 X_i}{\partial x_k \, \partial x_j} Z_j - \frac{\partial^2 Z_i}{\partial x_k \, \partial x_j} X_j \right] Y_k \right\} \frac{\partial}{\partial x_i} \tag{A.16}$$

Adding (A.14), (A.15), and (A.16), we obtain the Jacobi identity (A.10). ■

(4) Since

$$[X, Y]_i = \sum_j \left[\frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j \right] \tag{A.17}$$

the degree of the ith component of $[X, Y]_i$ is equal to $k+l-1$. ■

Appendix 2. Jet

In this Appendix, the basic notion of *jets* and its properties will be presented. Only a restrictive treatment will be given for simplicity. Readers are referred to [15] for a more general treatment.

Let f and g denote smooth maps from a neighborhood of the origin O in $\mathbf{R}^m$ to $\mathbf{R}^n$.

Definition A2.1: k-jet equivalence

We say f and g are *k-jet equivalent at O* iff every partial derivatives up to order k of f and g at 0 coincide, that is,

$$\left[\frac{\partial}{\partial x_1} \right]^{k_1} \cdots \left[\frac{\partial}{\partial x_n} \right]^{k_n} (f - g)(0) = 0 \tag{A.18}$$

for all $k_1, k_2, \ldots, k_n$ with $0 \le k_1 + k_2 + \ldots + k_n \le k$.

Example A2.2

Let f and g be functions on $\mathbf{R}^1$ defined by

$$f(x) = 2x - x^2 \quad \text{and} \quad g(x) = 2x + 3x^2$$

Then $f'(0) = g'(0) = 2$ but $f''(0) \neq g''(0)$. Hence, f and g are 1-jet equivalent, but *not* 2-jet equivalent.

Let $C^\infty(U, \mathbf{R}^n)$ denote the set of all smooth maps from U to $\mathbf{R}^n$. Then the k-jet equivalence at 0 defines an equivalence relation on $C^\infty(U, \mathbf{R}^n)$.

Definition A2.3. k-jet

The k-jet equivalence class of $f \in C^\infty(U, \mathbf{R}^n)$ is called the *k-jet* of f at 0 and is denoted by $j_0^k f$. We denote the set of all k-jets at 0 by $J_0^k C^\infty(U, \mathbf{R}^n)$.

Proposition A2.4

(1) $J_0^k\, C^\infty(U, \mathbf{R}^n)$ forms a vector space over $\mathbf{R}$.

(2) Let j^k denote the map defined by

$$j^k : C^\infty(U, \mathbf{R}) \to J_0^k\, C^\infty(U, \mathbf{R}) \qquad \text{(A.19)}$$

$$f \to j_0^k f$$

Then j^k is a *surjective vector space homomorphism.*

(3) If $k \geq l$, the map

$$j^{k,l} : J_0^k\, C^\infty(U, \mathbf{R}) \to J_0^l\, C^\infty(U, \mathbf{R}) \qquad \text{(A.20)}$$

$$j_0^k f \to j_0^l f$$

is well defined and is also a surjective vector space homomorphism.

(4) Every element in $Ker(j^{k,k-1})$ is represented by a homogeneous polynomial of degree k.

Proof.

(1) Let us define the addition and scalar multiplication by

$$j_0^k f + j_0^k g = j_0^k(f + g) \qquad \text{(A.21)}$$

and

$$r \cdot j_0^k f = j_0^k(r \cdot f) \quad , \quad r \in \mathbf{R} \qquad \text{(A.22)}$$

These operations are well defined and hence $J_0^k\, C^\infty(U, \mathbf{R}^n)$ has a vector space structure.

(2) It is a direct consequence of the definition of the above vector space operations that j^k is a vector space homomorphism. Surjectivity of j^k is obvious.

(3) If f and f' are k-jet equivalent at 0, then f and f' are l-jet equivalent at 0 for $l \leq k$. Thus, the map $j^{k,l}$ is well defined. Hence, property (3) follows the same arguments as (2).

(4) Let η be an arbitrary element of $Ker(j^{k,k-1})$. In other words, $\eta \in J_0^k\, C^\infty(U, \mathbf{R}^n)$ and $j^{k,k-1}(\eta) = 0$. Therefore, if we take a representative f of η, then $j_0^{k-1} f = 0$, that is, all derivatives vanish at 0 up to order $k-1$. Thus, f is k-jet equivalent at 0 to its kth-order part. This completes the proof.

■

Appendix 3 Flow in Diff_0^k *generated by a k-jet vector field*

Let $Y^k \in \mathcal{X}_0^k$ and let Y and Y' be representatives of Y^k; i.e., $j_0^k\ Y = j_0^k\ Y' = Y^k$. The purpose of this appendix is to prove the following basic result.

Proposition A3.1

> Let ϕ^t and ϕ'^t denote flows generated by Y and Y', respectively. Then ϕ^t and ϕ'^t are k-jet equivalent at 0 for every t.

Proof. Recall the flow ϕ^t generated by Y is characterized by the following conditions:

$$\frac{d}{dt}\,\phi^t(x) = Y(\phi^t(x)) \tag{A.23}$$

$$\phi^0(x) = x \tag{A.24}$$

Since Y vanishes at 0, the flow ϕ^t fixes 0, that is,

$$\phi^t(0) = 0 \tag{A.25}$$

ϕ'^t also satisfies the same conditions as Y':

$$\frac{d}{dt}\,\phi'^t(x) = Y'(\phi'^t(x)) \tag{A.26}$$

$$\phi'^0(x) = x \tag{A.27}$$

$$\phi'^t(0) = 0 \tag{A.28}$$

It follows from (A.25) and (A.28) that the 0-jet of ϕ^t and ϕ'^t coincide.

Suppose next that the first order derivative of Y is equal to that of Y', that is,

$$D_x\ Y(0) = D_x\ Y'(0) \tag{A.29}$$

Differentiating (A.23) and (A.26) with respect to x, we obtain

$$\frac{d}{dt}\,D_x\ \phi^t(x) = D_x\,\frac{d}{dt}\,\phi^t(x) = D_x\ Y(\phi^t(x))\cdot D_x\ \phi^t(x) \tag{A.30}$$

and

$$\frac{d}{dt}\,D_x\ \phi'^t(x) = D_x\,\frac{d}{dt}\,\phi'^t(x) = D_x\ Y'(\phi'^t(x))\cdot D_x\ \phi'(x) \tag{A.31}$$

Substituting $x = 0$ and subtracting (A.30) from (A.31), we obtain

$$\frac{d}{dt}(D_x\, \phi''(0) - D_x\, \phi'(0)) = D_x\, Y(0) \cdot (D_x\, \phi''(0) - D_x\, \phi'(0)) \tag{A.32}$$

where we have made use of (A.25), (A.28), and (A.29). This differential equation is linear with respect to $D_x\, \phi''(0) - D_x\, \phi'(0)$. By differentiating (A.24) and (A.27) with respect to x at $x = 0$, we obtain the *initial condition*

$$D_x\, \phi'^0(0) - D_x\, \phi^0(0) = 0 \tag{A.33}$$

It follows from (A.32) and (A.33) that

$$D_x\, \phi''(0) - D_x\, \phi'(0) = 0 \tag{A.34}$$

for every t and ϕ^t is therefore 1-jet equivalent to ϕ'^t at 0.

Let us differentiate next (A.30) and (A.31) with respect to x to obtain

$$\frac{d}{dt}\, D_x^2\, \phi^t(x) = D_x^2\, Y(\phi^t(x)) \cdot (D_x\, \phi^t(x))^2 + D_x\, Y(\phi^t(x)) \cdot D_x^2\, \phi^t(x) \tag{A.35}$$

$$\frac{d}{dt}\, D_x^2\, \phi'^t(x) = D_x^2\, Y'(\phi'^t(x)) \cdot (D_x\, \phi'^t(x))^2 + D_x\, Y(\phi'^t(x)) \cdot D_x^2\, \phi'(x) \tag{A.36}$$

Substituting $x = 0$ in (A.35) and (A.36), we obtain

$$\frac{d}{dt}\, D_x^2\, \phi^t(0) = D_x^2\, Y(0) \cdot \left(D_x\, \phi^t(0)\right)^2 + D_x\, Y(0) \cdot D_x\, \phi^t(0) \tag{A.37}$$

$$\frac{d}{dt}\, D_x^2\, \phi'^t(0) = D_x^2\, Y'(0)\left(D_x\, \phi'^t(0)\right)^2 + D_x\, Y'(0) \cdot D_x^2\, \phi'(0) \tag{A.38}$$

If Y and Y' are 2-jet equivalent at 0, then

$$D_x^2\, Y(0) = D_x^2\, Y'(0) \text{ and } D_x Y(0) = D_x Y'(0) \tag{A.39}$$

Moreover, (A.34) implies

$$D_x\, \phi^t(0) = D_x\, \phi'^t(0) \tag{A.40}$$

Hence, we obtain

$$\frac{d}{dt}(D_x^2\, \phi^t(0) - D_x^2\, \phi'^t(0)) = DY(0)(D_x^2\, \phi^t(0) - D_x^2\, \phi'^t(0)) \tag{A.41}$$

Using a similar argument as above, we obtain

$$D_x^2\, \phi^t(0) - D_x^2\, \phi'^t(0) = 0 \tag{A.42}$$

Repeating the above arguments, we obtain

$$j_0^k\ \phi^t = j_0^k\ \phi'^t \tag{A.43}$$

if Y is k-jet equivalent to Y' at 0. ■

FIGURE CAPTIONS

Fig. 1. Geometrical interpretation of B_k, G_k, and π_k.

Fig. 2. Deformation of $h_k(0)$ to a point $\tilde{g}_k$ lying on G_k.

REFERENCES

[1] J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, 1983, 2nd Printing, 1986.

[2] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, 1983.

[3] K. Hoffman and R. Kunze, *Linear Algebra*, 2nd Edition, Prentice Hall, Englewood Cliffs, New Jersey, 1983, 2nd Printing, 1986.

[4] F. Takens, "Singularities of vector fields," *Publ. Math. I.H.E.S.*, vol. 43, 1974, pp. 47-100.

[5] S. Ushiki, "Normal forms for singularities of vector fields," *Japan J. Appl. Math.*, vol. 1, 1984, pp. 1-37.

[6] G. Iooss and D. D. Joseph, *Elementary Stability and Bifurcation Theory*, Springer-Verlag, 1981.

[7] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, 1982.

[8] F. Takens, "Normal forms for certain singularities of vector fields," *Ann. Inst. Fourier*, vol. 23, 1973, pp. 163-195.

[9] F. Takens, "Forced oscillations and bifurcations," *Comm. Math. Inst.*, Rijkuniversitet, Utrecht, vol. 3, 1974, pp. 1-59.

[10] S. Ushiki, "Normal forms for singularities of nonlinear ordinary differential equations, in *Computing Methods in Applied Science and Engineering*, VI, Edited by R. Glowinski and J. L. Lions, North Holland, 1984.

[11] S. Ushiki, H. Oka, and H. Kokubu, "Existence d'attracteurs étranges dans le déploiement d'une singularité dégénérée d'un champ de vecteurs invariant par translation," C.R. Acad. Sc. Paris, t. 298, Série I, n° 3, 1984, pp. 39-42.

[12] L. O. Chua and H. Oka, "Normal forms for constrained nonlinear differential equations—Part I: Theory and Examples," Memorandum No. ERL M87/83, Electronics Research Laboratory, University of California, Berkeley, CA, 94720 September 15, 1987.

[13] S. Lang, *Linear Algebra*, 2nd Edition, Addison-Wesley, Reading, Mass. 1971.

[14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I, Interscience Publishers, 1963.

[15] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer-Verlage, 1973.

[16] R. Cushman and J. Sanders, "Nilpotent normal forms and representation theory of $sl(2,\mathbb{R})$" in *Multiparameter Bifurcation Theory*, Edited by M. Golubitsky and J. Guckenheimer, Contemporary Mathematics, vol. 56, AMS, 1986, pp. 31-51.

[17] C. Elphick, E. Tirapegui, M. E. Brachet, P. Coullet, and G. Iooss, "A simple global characterization for normal forms of singular vector fields," *Physica* 29D, 1987, pp. 95-127.

[18] R. H. Rand and D. Armbruster, *Perturbation Methods, Bifurcation Theory and Computer Algebra,* Appl. Math. Sci., vol. 65, Springer-Verlag, 1987.

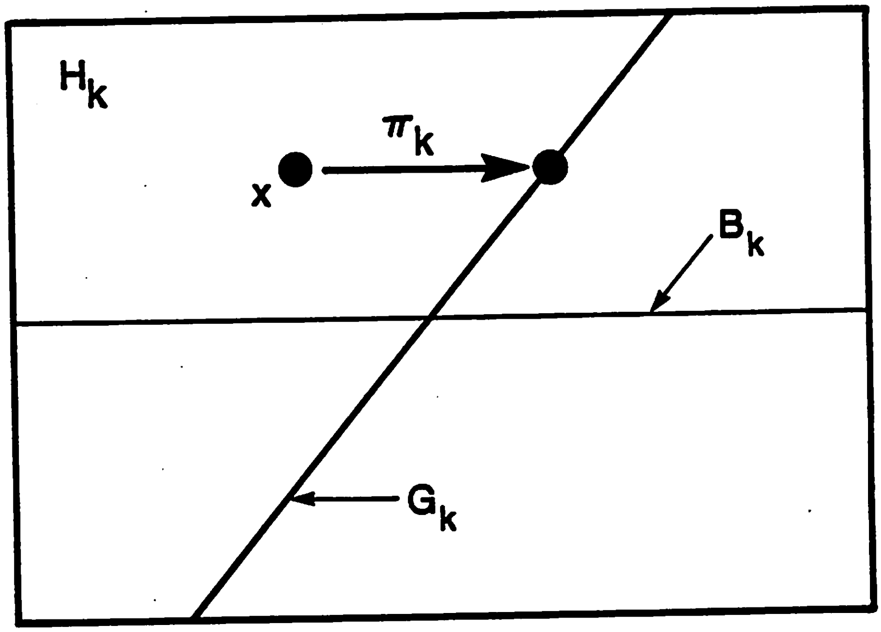

Fig.1

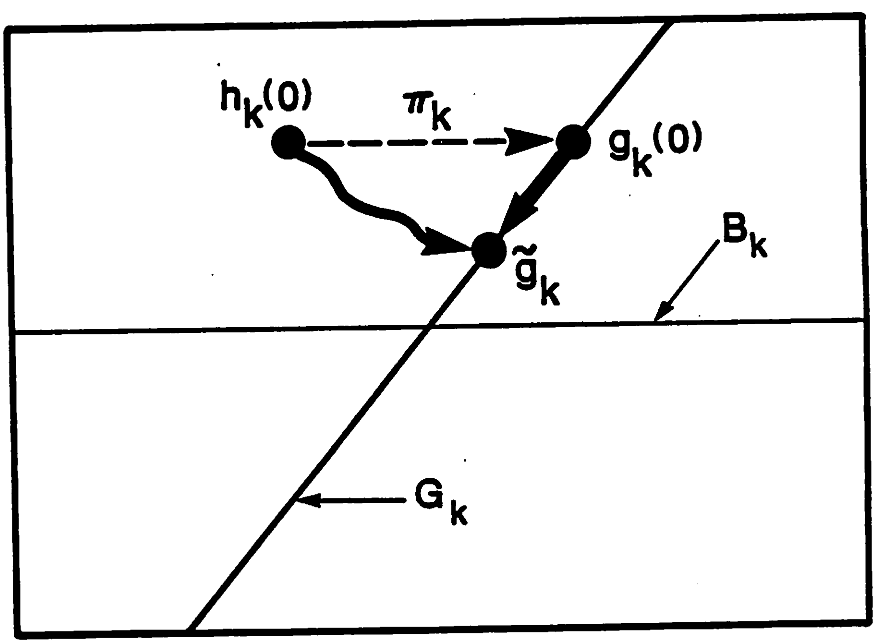

Fig.2