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NORMAL FORMS FOR NONLINEAR VECTOR FIELDS—PART II: APPLICATIONS[†]

Leon O. Chua and Hiroshi Kokubu[††]

Abstract

This paper applies the normal form theory for nonlinear vector fields from *Part I* to several examples of vector fields whose Jacobian matrix is a typical Jordan form which gives rise to interesting bifurcation behavior. The normal forms derived from these examples are based on Ushiki's method, which is a refinement of Takens' method. A comparison of the normal forms derived by *Poincaré 's* method, *Takens'* method, and *Ushiki's* method is also given.

For vector fields imbued with some form of *symmetry*, we impose additional constraints in the normal form algorithm from *Part I* in order that the resulting normal form will inherit the same form of symmetry.

The normal forms of a given vector field is then used to derive its *versal unfoldings* in the form of an n-parameter family of vector fields. Such unfoldings are powerful tools for analyzing the *bifurcation* phenomena of vector fields when the parameter changes. Moreover, since the *local* bifurcation structure around a highly degenerate singularity can include some *global* bifurcation phenomena observed from a less degenerate family of vector fields, it follows that the concepts of normal form and versal unfolding are useful tools for analyzing such degenerate singularities.

1. INTRODUCTION

In *Part I* of this paper, we presented a detailed algorithm for deriving the normal forms of smooth vector fields on $\mathbf{R}^n$. In this paper, this algorithm will be used in several important applications of the normal form theory. In *Section 1,* we will derive the well-known normal forms associated with several typical Jordan normal forms. We will compare our results, which is based on Ushiki's method, with those derived from Poincaré and Takens' methods. In *Section 2,* we will investigate some important consequences of *symmetry* on the normal forms of vector fields. In the final *Section 3,* we will apply our results to the *versal unfoldings* of vector fields and explore its relationship to *Bifurcation theory.*

2. EXAMPLES OF NON-DEGENERATE NORMAL FORMS

In this section, we will derive the normal forms associated with the following 3 important Jordan normal forms:

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$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.1}$$

$$\left[\begin{array}{cc:c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{array}\right] \tag{2.2}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3}$$

Example 2.1. Hopf type

Consider the class of smooth vector fields on $\mathbf{R}^2$ which vanish at the origin where the linear part is given by (2.1); namely, those vector fields having the 1-jet

$$v_1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \tag{2.4}$$

on $\mathbf{R}^2$. To simplify our computation for this example, it is advantageous to introduce the *complex* coordinates

$$\xi = x + iy \tag{2.5}$$

$$\bar{\xi} = x - iy \tag{2.6}$$

where $i = \sqrt{-1}$. The corresponding differential operators are:

$$\frac{\partial}{\partial \xi} = \frac{1}{2} \left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right] \tag{2.7}$$

$$\frac{\partial}{\partial \bar{\xi}} = \frac{1}{2} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right] \tag{2.8}$$

In terms of the new complex coordinate system $(\xi, \bar{\xi})$, the 1-jet in (2.4) assumes the form:

$$v_1 = i \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{2.9}$$

Similarly, the following Lie bracket formulas in terms of $(\xi, \bar{\xi})$ are useful in the following calculations.

$$\left[\xi^k \bar{\xi}^l \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] = (1-k-l)\, \xi^k \bar{\xi}^l \frac{\partial}{\partial \xi} \tag{2.10}$$

$$\left[\xi^k\bar{\xi}^l \frac{\partial}{\partial\xi}, \xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] = (1-k-l)\,\xi^k\bar{\xi}^l \frac{\partial}{\partial\xi} \tag{2.11}$$

$$\left[\xi^k\bar{\xi}^l \frac{\partial}{\partial\xi}, \xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] = (1-k+l)\,\xi^k\bar{\xi}^l \frac{\partial}{\partial\xi} \tag{2.12}$$

$$\left[\xi^k\bar{\xi}^l \frac{\partial}{\partial\bar{\xi}}, \xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] = (-1-k+l)\,\xi^k\bar{\xi}^l \frac{\partial}{\partial\bar{\xi}} \tag{2.13}$$

Observe that the set $\left\{\xi^k\bar{\xi}^l \frac{\partial}{\partial\xi}, \xi^k\bar{\xi}^l \frac{\partial}{\partial\bar{\xi}}\right\}$ for all k, l where $k+l = p$ forms a basis for the vector space H_p of homogeneous vector fields of degree p. It follows from (2.10)-(2.13) that the linear maps

$$Y \in H_p \mapsto \left[Y, \xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \in H_p \tag{2.14}$$

and

$$Y \in H_p \mapsto \left[Y, \xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \in H_p \tag{2.15}$$

are both represented by *diagonal matrices*. For example, for $p = 1$, the linear map,

$$H_1 \to H_1, Y \mapsto \left[Y, \xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{2.16}$$

is represented by

$$\begin{array}{c} \xi\frac{\partial}{\partial\xi} \\ \bar{\xi}\frac{\partial}{\partial\xi} \\ \xi\frac{\partial}{\partial\bar{\xi}} \\ \bar{\xi}\frac{\partial}{\partial\bar{\xi}} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \xi\frac{\partial}{\partial\xi} \quad \bar{\xi}\frac{\partial}{\partial\xi} \quad \xi\frac{\partial}{\partial\bar{\xi}} \quad \bar{\xi}\frac{\partial}{\partial\bar{\xi}} \end{array} \tag{2.17}$$

Consider next the 2nd order normal form problem. The vector space H_2 in this case is spanned by

$$\left\{\xi^2\frac{\partial}{\partial\xi}, \xi\bar{\xi}\frac{\partial}{\partial\xi}, \bar{\xi}^2\frac{\partial}{\partial\xi}, \xi^2\frac{\partial}{\partial\bar{\xi}}, \xi\bar{\xi}\frac{\partial}{\partial\bar{\xi}}, \bar{\xi}^2\frac{\partial}{\partial\bar{\xi}}\right\} \tag{2.18}$$

In terms of this basis, the linear map (2.15) is represented by the following *non-singular* diagonal matrix:

$$
\begin{bmatrix}
-1 & & & & & \\
 & 1 & & & & \\
 & & 3 & & & \\
 & & & -3 & & \\
 & & & & -1 & \\
 & & & & & 1
\end{bmatrix}
\tag{2.19}
$$

Since the image of (2.19) is the whole space H_2, $\dim B_2 = 6$ where B_2 is defined in (5.29) of [1]. It follows that $\dim G_2 = 0$ and the 2nd order components in the normal form is completely eliminated in view of *Theorem 5.4* from [1].

By repeating the same procedure, we find the linear map (2.15) on H_3 is represented by an 8×8 diagonal matrix, whose diagonal entries are given below with its associated basis component in H_3:

$$
\begin{array}{cccccccc}
-2 & 0 & 2 & 4 & -4 & -2 & 0 & 2 \\
\xi^3 \dfrac{\partial}{\partial \xi} & \xi^2\bar{\xi} \dfrac{\partial}{\partial \xi} & \xi\bar{\xi}^2 \dfrac{\partial}{\partial \xi} & \bar{\xi}^3 \dfrac{\partial}{\partial \xi} & \xi^3 \dfrac{\partial}{\partial \bar{\xi}} & \xi^2\bar{\xi} \dfrac{\partial}{\partial \bar{\xi}} & \xi\bar{\xi}^2 \dfrac{\partial}{\partial \bar{\xi}} & \bar{\xi}^3 \dfrac{\partial}{\partial \bar{\xi}}
\end{array}
\tag{2.20}
$$

It follows from (2.20) that $\dim B_3 = 6$ and hence $\dim G_3 = 2$. Let us choose the complementary space G_3 spanned by

$$
\left\{ \xi\bar{\xi} \left(\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right), \xi\bar{\xi} \left(\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right) \right\}
\tag{2.21}
$$

and consider the following reduced 3rd order normal form problem on G_3:

$$
\frac{d}{dt} g_3(t) = -\pi_3 \left(\left[Y^2, v^2 + g_3(t) \right]_3 \right)
\tag{2.22}
$$

where

$$
v^2 = v_1 = i \left(\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right)
\tag{2.23}
$$

The (reduced) infinitesimal generator Y^2 must satisfy

$$
[Y^2, v^2]^2 = [Y^2, v_1]^2 = 0
\tag{2.24}
$$

that is, $Y^2 = Y_1 + Y_2$ must satisfy

$$
[Y_1, v_1] = 0 \quad \text{and} \quad [Y_2, v_1] = 0 \,.
\tag{2.25}
$$

Since the linear map $Y_2 \to [Y_2, v_1]$ is non-singular in view of (2.19), we must have $Y_2 = 0$ and hence Y^2 is of the form

$$Y^2 = Y_1 = A\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + B\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{2.26}$$

It follows from (2.21) that $g_3(t)$ in (2.22) is of the form

$$g_3(t) = \alpha(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + i\,\beta(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{2.27}$$

and the right hand side of (2.22) becomes, in view of (2.26),

$$\begin{aligned} -\pi_3\left([Y^2, v^2 + g_3(t)]_3\right) &= -\pi_3\left([Y_1, v_1 + g_3(t)]_3\right) \\ &= -\pi_3\left[\left[A\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + B\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right],\right.\right. \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\left.\left.\alpha(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + i\,\beta(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]\right]\right] \\ &= -\pi_3\left[2A\,\alpha(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + 2Ai\,\beta(t)\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]\right] \\ &= -2A\,\alpha(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] - i2A\,\beta(t)\xi\bar{\xi}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \end{aligned} \tag{2.29}$$

Equating the corresponding components from (2.27) and (2.29), we obtain the differential equations

$$\frac{d}{dt}\alpha(t) = -2A\,\alpha(t) \tag{2.30}$$

$$\frac{d}{dt}\beta(t) = -i2A\,\beta(t) \tag{2.31}$$

whose solutions are:

$$\alpha(t) = \alpha(0)e^{-2At} \tag{2.32}$$

$$\beta(t) = \beta(0)e^{-i2At} \tag{2.33}$$

If $\alpha(0) \neq 0$, we can choose $2A = \log|\alpha(0)|$ so that at $t = 1$, we have $\alpha(1) = \pm 1$. If $\alpha(0) = 0$ but $\beta(0) \neq 0$, we can choose $2A = \log i\,|\beta(0)|$ so that at $t = 1$, we have $\beta(1) = \pm i$.

If $\alpha(0) = \beta(0) = 0$, then $\alpha(t) = \beta(t) = 0$.

Therefore, the 3rd order normal forms associated with the 1-jet (2.9) are as follows:

(i) non-degenerate 3rd order normal form

$$\pm \xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] + i\left[1 + \beta\xi\bar{\xi}\right]\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{2.34}$$

(ii) degenerate 3rd order normal form (case 1)

$$i(1 \pm \xi\bar{\xi})\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{2.35}$$

(iii) degenerate 3rd order normal form (case 2)

$$i\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{2.36}$$

In terms of the polar coordinate (r, θ), where

$$x = r \cos\theta \quad , \quad y = r \sin\theta \tag{2.37}$$

the vector fields $\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}$ and $i\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right]$ can be identified with the vector fields $r \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ respectively. Hence, the normal forms (2.34)-(2.36) simplify further to the following expressions in terms of (r, θ):

(i) non-degenerate 3rd order normal form: $\pm r^3 \frac{\partial}{\partial r} + (1+ \beta r^2) \frac{\partial}{\partial \theta}$ (2.38)

(ii) degenerate 3rd order normal form (case 1): $(1 \pm r^2) \frac{\partial}{\partial \theta}$ (2.39)

(iii) degenerate 3rd order normal form (case 2): $\frac{\partial}{\partial \theta}$ (2.40)

Using the same procedure, we can of course obtain higher order normal forms. It turns out that for the non-degenerate case (i), *all terms* in the normal form beyond the 5th order can be eliminated:

Proposition 2.2

The non-degenerate kth order normal form of vector fields on $\mathbf{R}^2$ with the 1-jet $\frac{\partial}{\partial \theta}$ is given by

$$(\pm r^2 + \alpha r^4) r \frac{\partial}{\partial r} + (1 + \beta r^2) \frac{\partial}{\partial \theta} \tag{2.41}$$

for *any* $k \geq 5$

This proposition can be proved by *induction*, similar to that of *Proposition 5.2* from *Part I* [1]. The proof is given in the *Appendix*.

Example 2.3: Hopf and Zero-interaction type

Consider the class of smooth vector fields on $\mathbf{R}^3$ which vanish at the origin where the linear part is given by (2.2); namely, those vector fields having the 1-jet

$$v_1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \tag{2.42}$$

on $\mathbf{R}^3$. The eigenvalues associated with the Jacobian matrix are $\pm i$ and 0.

Just as *Example 2.1*, our calculations are greatly simplified by introducing the complex coordinate $(\xi, \bar{\xi}, z)$ in place of (x, y, z). The simplification is due in part to the following simple formulas

$$\left[\xi^k \bar{\xi}^l \frac{\partial}{\partial z}, \xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] = -(k+l)\xi^k \bar{\xi}^l \frac{\partial}{\partial z} \tag{2.43}$$

$$\left[\xi^k \bar{\xi}^l \frac{\partial}{\partial z}, \xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] = (-k+l)\xi^k \bar{\xi}^l \frac{\partial}{\partial z} \tag{2.44}$$

in addition to (2.10)-(2.13). In terms of the coordinate $(\xi, \bar{\xi}, z)$, the 1-jet v_1 is represented by

$$v_1 = i\left(\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right) \tag{2.45}$$

The linear map

$$Y \in H_p \mapsto [Y, v_1] \in H_p \tag{2.46}$$

of the vector space H_p of homogeneous vector fields of degree p has a *diagonal* matrix representation with respect to the basis

$$\left\{\xi^k \bar{\xi}^l z^m \frac{\partial}{\partial \xi}, \xi^k \bar{\xi}^l z^m \frac{\partial}{\partial \bar{\xi}}, \xi^k \bar{\xi}^l z^m \frac{\partial}{\partial z}\right\}, \ k + l + m = p \tag{2.47}$$

For $p = 2$, the diagonal entries are given by:

$$
\left.
\begin{array}{cccccc}
-1 & 1 & 0 & 3 & 2 & 1 \\
\xi^2 \frac{\partial}{\partial \xi} & \xi\bar{\xi} \frac{\partial}{\partial \xi} & \xi z \frac{\partial}{\partial \xi} & \bar{\xi}^2 \frac{\partial}{\partial \xi} & \bar{\xi} z \frac{\partial}{\partial \xi} & z^2 \frac{\partial}{\partial \xi} \\
\\
-3 & -1 & -2 & 1 & 0 & -1 \\
\xi^2 \frac{\partial}{\partial \bar{\xi}} & \xi\bar{\xi} \frac{\partial}{\partial \bar{\xi}} & \xi z \frac{\partial}{\partial \bar{\xi}} & \bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}} & \bar{\xi} z \frac{\partial}{\partial \bar{\xi}} & z^2 \frac{\partial}{\partial \bar{\xi}} \\
\\
-2 & 0 & -1 & 2 & 1 & 0 \\
\xi^2 \frac{\partial}{\partial z} & \xi\bar{\xi} \frac{\partial}{\partial z} & \xi z \frac{\partial}{\partial z} & \bar{\xi}^2 \frac{\partial}{\partial z} & \bar{\xi} z \frac{\partial}{\partial z} & z^2 \frac{\partial}{\partial z}
\end{array}
\right\} \tag{2.48}
$$

Consequently, the complementary space G_2 can be chosen with the basis

$$
\left\{ z \left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right], z \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right], \xi\bar{\xi} \frac{\partial}{\partial z}, z^2 \frac{\partial}{\partial z} \right\} \tag{2.49}
$$

The reduced 2nd order normal form problem is given by

$$
\frac{d}{dt} g_2(t) = -\pi_2 \left(\left[Y_1, v_1 + g_2(t) \right]_2 \right) \tag{2.50}
$$

where Y_1 satisfies

$$
[Y_1, v_1] = 0 \tag{2.51}
$$

That is, Y_1 belongs to the kernel of the linear map $Y \mapsto [Y, v_1]$ on H_1. Since this linear map is represented by a diagonal matrix whose diagonal entries are

$$
\begin{array}{ccccccccc}
0 & 2 & 1 & -2 & 0 & -1 & -1 & 1 & 0 \\
\xi \frac{\partial}{\partial \xi} & \bar{\xi} \frac{\partial}{\partial \xi} & z \frac{\partial}{\partial \xi} & \xi \frac{\partial}{\partial \bar{\xi}} & \bar{\xi} \frac{\partial}{\partial \bar{\xi}} & z \frac{\partial}{\partial \bar{\xi}} & \xi \frac{\partial}{\partial z} & \bar{\xi} \frac{\partial}{\partial z} & z \frac{\partial}{\partial z}
\end{array} \tag{2.52}
$$

it follows that Y_1 is of the form

$$
Y_1 = A \left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] + B \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] + Cz \frac{\partial}{\partial z} \tag{2.53}
$$

On the other hand, we can write

$$
\begin{aligned}
g_2(t) = {} & \alpha(t) z \left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] + i\,\beta(t) z \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] \\
& + \gamma(t) \xi\bar{\xi} \frac{\partial}{\partial z} + \delta(t) z^2 \frac{\partial}{\partial z}
\end{aligned} \tag{2.54}
$$

Substituting (2.53) and (2.54) into the right hand side of (2.50), we obtain

$$
\begin{aligned}
-\pi_2\left(\left[Y, v_1 + g_2(t)\right]_2\right) &= -C\,\alpha(t)z\left[\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] - iC\,\beta(t)z\left[\xi\frac{\partial}{\partial\xi}-\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \\
&\quad -(2A-C)\,\gamma(t)\xi\bar{\xi}\frac{\partial}{\partial z} - C\,\delta(t)z^2\frac{\partial}{\partial z}
\end{aligned}
\tag{2.55}
$$

where we have made use of the following calculations of the Lie bracket $[Y, v]$:

Y \ v	$z\left[\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$z\left[\xi\frac{\partial}{\partial\xi}-\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$\xi\bar{\xi}\frac{\partial}{\partial z}$	$z^2\frac{\partial}{\partial z}$
$\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}$	0	0	$2\xi\bar{\xi}\frac{\partial}{\partial z}$	0
$\xi\frac{\partial}{\partial\xi}-\bar{\xi}\frac{\partial}{\partial\bar{\xi}}$	0	0	0	0
$z\frac{\partial}{\partial z}$	$z\left[\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$z\left[\xi\frac{\partial}{\partial\xi}-\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$-\xi\bar{\xi}\frac{\partial}{\partial z}$	$z^2\frac{\partial}{\partial z}$

Equating the corresponding coefficients in (2.54) and (2.55), we obtain the following 4 linear differential equations:

$$\dot{\alpha} = -C\,\alpha \tag{2.56}$$

$$\dot{\beta} = -C\,\beta \tag{2.57}$$

$$\dot{\gamma} = -(2A-C)\gamma \tag{2.58}$$

$$\dot{\delta} = -C\,\delta \tag{2.59}$$

This implies that, generically, the coefficient γ and one of the coefficients α, β, and δ can be normalized to ± 1. Consequently, in terms of the *cylindrical coordinate* (r, θ, z), the non-degenerate 2nd order normal form can be chosen to be any one of the following 3 forms:

$$\text{(a)}\quad \alpha rz\frac{\partial}{\partial r} + (1+\beta z)\frac{\partial}{\partial\theta} + (s_1 r^2 + s_2 z^2)\frac{\partial}{\partial z} \tag{2.60}$$

$$\text{(b)}\quad s_2 rz\frac{\partial}{\partial r} + (1+\beta z)\frac{\partial}{\partial\theta} + (s_1 r^2 + \delta z^2)\frac{\partial}{\partial z} \tag{2.61}$$

$$\text{(c)}\quad \alpha rz\frac{\partial}{\partial r} + (1+s_2 z)\frac{\partial}{\partial\theta} + (s_1 r^2 + \delta z^2)\frac{\partial}{\partial z} \tag{2.62}$$

where $s_1, s_2 = \pm 1$.

We can repeat the same algorithm to derive higher order normal forms. Here, we will present only the result of the *non-degenerate 3rd order normal form* for case (a) of the preceding 2-jets:

$$(\alpha z + az^2)\, r \frac{\partial}{\partial r} + (1+\beta z + bz^2) \frac{\partial}{\partial \theta} + (s_1 r^2 + s_2 z^2 + cz^3) \frac{\partial}{\partial z} \tag{2.63}$$

where $s_1, s_2 = \pm 1$.

Example 2.4: Triple-zero type

Consider the class of smooth vector fields on $\mathbb{R}^3$ which vanish at the origin where the linear part is given by (2.3); namely, those vector fields having the 1-jet

$$v_1 = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \tag{2.64}$$

on $\mathbb{R}^3$. The eigenvalues associated with the Jacobian matrix consist of 3 zeros. Unlike the preceding two examples, the linear map

$$L_p : Y \in H_p \mapsto [Y, v_1] \in H_p \tag{2.65}$$

for this example does *not* have a diagonal matrix representation. We should expect therefore much more tedious calculations in this example.

Let us examine first the kernel of the linear map

$$L_1 : H_1 \to H_1 \quad , \quad Y \mapsto [Y, v_1] \tag{2.66}$$

In terms of the basis

$$\left\{ x \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, z \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, y \frac{\partial}{\partial z}, z \frac{\partial}{\partial z}, \right\} \tag{2.67}$$

the map L_1 is represented by the matrix

$$\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array} \right]$$

Hence, the kernel of L_1 is spanned by

$$\left\{z \frac{\partial}{\partial x}, y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right\}. \tag{2.68}$$

Next, let us choose a basis for H_2 as follows:

$$\left\{\begin{matrix} x^2 \frac{\partial}{\partial x}, xy \frac{\partial}{\partial x}, xz \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, yz \frac{\partial}{\partial x} z^2 \frac{\partial}{\partial x} \\ x^2 \frac{\partial}{\partial y}, xy \frac{\partial}{\partial y}, xz \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, yz \frac{\partial}{\partial y}, z^2 \frac{\partial}{\partial y} \\ x^2 \frac{\partial}{\partial z}, xy \frac{\partial}{\partial z}, xz \frac{\partial}{\partial z}, y^2 \frac{\partial}{\partial z}, yz \frac{\partial}{\partial z}, z^2 \frac{\partial}{\partial z} \end{matrix}\right\} \tag{2.69}$$

The linear map L_2 with respect to the above basis has the following matrix representation:

$$\left[\begin{array}{c|c|c}
\begin{matrix} 0&0&0&0&0&0 \\ -2&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&-2&0&0 \\ 0&0&0&0&-1&0 \end{matrix} & \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{matrix} & \bigcirc \\ \hline
\bigcirc & \begin{matrix} 0&0&0&0&0&0 \\ -2&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&-2&0&0 \\ 0&0&0&0&-1&0 \end{matrix} & \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{matrix} \\ \hline
\bigcirc & \bigcirc & \begin{matrix} 0&0&0&0&0&0 \\ -2&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&-2&0&0 \\ 0&0&0&0&-1&0 \end{matrix}
\end{array}\right] \tag{2.70}$$

Analyzing the linear map L_2, we obtain the following basis for

B_2 = images of L_2:

$$\left\{\begin{matrix} xy \frac{\partial}{\partial x}, xz \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, yz \frac{\partial}{\partial x}, z^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y}, \\ xz \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, yz \frac{\partial}{\partial y}, z^2 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z}, \\ xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z} - y^2 \frac{\partial}{\partial z}, yz \frac{\partial}{\partial z}, z^2 \frac{\partial}{\partial z} \end{matrix}\right\} \tag{2.71}$$

Hence; we can choose the complementary space G_2 to B_2 in H_2 to be the subspace spanned by

$$\left\{x^2 \frac{\partial}{\partial z}, xy \frac{\partial}{\partial z}, xz \frac{\partial}{\partial z}, y^2 \frac{\partial}{\partial z}\right\}. \tag{2.72}$$

Here we must remember that the projection $\pi_2 : H_2 \to G_2$ gives the following:

$$\begin{aligned} \pi_2\left[xy \frac{\partial}{\partial y}\right] &= \pi_2\left[xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z} - y^2 \frac{\partial}{\partial z} + xz \frac{\partial}{\partial z} + y^2 \frac{\partial}{\partial z}\right] \\ &= xz \frac{\partial}{\partial z} + y^2 \frac{\partial}{\partial z}, \end{aligned} \tag{2.73}$$

$$\pi_2\left[x^2 \frac{\partial}{\partial y}\right] = \pi_2\left[x^2 \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z} + 2xy \frac{\partial}{\partial z}\right] = 2xy \frac{\partial}{\partial z}, \tag{2.74}$$

and

$$\begin{aligned} \pi_2\left[x^2 \frac{\partial}{\partial x}\right] &= \pi_2\left[x^2 \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} + 2xy \frac{\partial}{\partial y}\right] = 2\pi_2\left[xy \frac{\partial}{\partial y}\right] \\ &= 2xz \frac{\partial}{\partial z} + 2y^2 \frac{\partial}{\partial z}. \end{aligned} \tag{2.75}$$

It follows from (2.67) that a vector field $Y_1 \in H_1$ satisfying,

$$[Y_1, v_1] = 0, \tag{2.76}$$

is of the form

$$Y_1 = Az \frac{\partial}{\partial x} + B\left[y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}\right] + C\left[x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right]. \tag{2.77}$$

Our 2nd order normal form problem is therefore given by

$$\begin{aligned} \frac{d}{dt} g_2(t) &= -\pi_2\left(\left[Y_1, v_1 + g_2(t)\right]_2\right) \\ &= -\pi_2\left(\left[Y_1, g_2(t)\right]\right) \end{aligned} \tag{2.78}$$

where

$$g_2(t) = \alpha(t)\, x^2 \frac{\partial}{\partial z} + \beta(t)\, xy \frac{\partial}{\partial z} + \gamma(t)\, xz \frac{\partial}{\partial z} + \delta(t)\, y^2 \frac{\partial}{\partial z}, \tag{2.79}$$

To compute the right hand side of (2.78), it is convenient to derive the following table:

$Y \backslash v$	$x^2 \frac{\partial}{\partial z}$	$xy \frac{\partial}{\partial z}$	$xz \frac{\partial}{\partial z}$	$y^2 \frac{\partial}{\partial z}$
$z \frac{\partial}{\partial x}$	$2xz \frac{\partial}{\partial z} - x^2 \frac{\partial}{\partial x}$	$yz \frac{\partial}{\partial z} - xy \frac{\partial}{\partial x}$	$z^2 \frac{\partial}{\partial z} - xz \frac{\partial}{\partial x}$	$-y^2 \frac{\partial}{\partial x}$
$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$	$x^2 \frac{\partial}{\partial z}$	$xy \frac{\partial}{\partial z}$	$xz \frac{\partial}{\partial z}$	$y^2 \frac{\partial}{\partial z}$

Remark 2.5

Since $y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} = v_1$, we observe that $\left[y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, v_1 \right] \in B_2$ does not affect (2.78). Consequently, there is no need to evaluate any Lie bracket involving $y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}$.

Using the above table, we evaluate

$$
\begin{aligned}
[Y_1, g_2] = {} & A\,\alpha \left[2xz \frac{\partial}{\partial z} - x^2 \frac{\partial}{\partial x} \right] + A\,\beta \left[yz \frac{\partial}{\partial z} - xy \frac{\partial}{\partial x} \right] \\
& + A\,\gamma \left[z^2 \frac{\partial}{\partial z} - xz \frac{\partial}{\partial x} \right] - A\,\delta y^2 \frac{\partial}{\partial x} + C\,\alpha x^2 \frac{\partial}{\partial z} \\
& + C\,\beta xy \frac{\partial}{\partial z} + C\,\gamma xz \frac{\partial}{\partial z} + C\,\delta y^2 \frac{\partial}{\partial z} .
\end{aligned}
\tag{2.80}
$$

Therefore, the right hand side of (2.78) becomes

$$
\begin{aligned}
-\pi_2 \left[[Y_1, g_2] \right] = {} & - C\,\alpha x^2 \frac{\partial}{\partial z} - C\,\beta xy \frac{\partial}{\partial z} - (2A\,\alpha + C\gamma) xz \frac{\partial}{\partial z} \\
& - C\,\delta y^2 \frac{\partial}{\partial z} + A\,\alpha \pi_2 \left[x^2 \frac{\partial}{\partial x} \right] .
\end{aligned}
\tag{2.81}
$$

From (2.75), we obtain

$$
A\,\alpha \pi_2 \left[x^2 \frac{\partial}{\partial x} \right] = 2A\,\alpha xz \frac{\partial}{\partial z} + 2A\,\alpha y^2 \frac{\partial}{\partial z}
\tag{2.82}
$$

Hence,

$$
\begin{aligned}
-\pi_2 \left[[Y_1, g_2] \right] = {} & - C\,\alpha x^2 \frac{\partial}{\partial z} - C\,\beta xy \frac{\partial}{\partial z} - C\,\gamma xz \frac{\partial}{\partial z} \\
& + (2A\,\alpha - C\,\delta) y^2 \frac{\partial}{\partial z} ,
\end{aligned}
\tag{2.83}
$$

Equating the corresponding coefficients of (2.79) and (2.83), we obtain the following 4 linear differential equations:

$$\begin{cases} \dot{\alpha} = -C\,\alpha \\ \dot{\beta} = -C\,\beta \\ \dot{\gamma} = -C\,\gamma \\ \dot{\delta} = 2A\,\alpha - C\,\delta\,. \end{cases} \tag{2.84}$$

This equation generically gives,

$$\alpha = \pm 1 \quad \text{and} \quad \delta = 0\,,$$

Hence, the non-degenerate 2nd order normal form is given by:

$$y\,\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + (sx^2 + \beta xy + \gamma xz)\,\frac{\partial}{\partial z}\,, \tag{2.85}$$

where $s = \pm 1$.

The differential operator (2.85) can be identified with the differential equation:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= sx^2 + \beta xy + \gamma xz\,, \end{aligned}\right\} \tag{2.86}$$

which is equivalent to the following 3rd order differential equation:

$$\dddot{x} = sx^2 + \beta x\dot{x} + \gamma x\ddot{x}\,. \tag{2.87}$$

Consider next the 3rd order normal form problem. The basis for the vector space H_3 can be chosen as follows:

$$\left\{x^k y^l z^m\,\frac{\partial}{\partial x}\,,\, x^k y^l z^m\,\frac{\partial}{\partial y}\,,\, x^k y^l z^m\,\frac{\partial}{\partial z}\right\},\ k + l + m = 3 \tag{2.88}$$

In terms of this basis, the linear map (lexicographic order)

$$L_3 : H_3 \to H_3\,;\ Y_3 \to [Y_3, v_1] \tag{2.89}$$

is represented by the 30×30 matrix,

$$\begin{bmatrix} A & \vdots & I & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & A & \vdots & I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & \vdots & A \end{bmatrix} \tag{2.90}$$

where I denotes a 10×10 unit matrix, and

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix} . \tag{2.91}
$$

It is not difficult to obtain the image B_3 of the linear map L_3 : B_3 is spanned by

$$
\left\{
\begin{array}{l}
x^2y \frac{\partial}{\partial x}, xyz \frac{\partial}{\partial x}, xz^2 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, y^2z \frac{\partial}{\partial x}, yz^2 \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial x}, \\
xz^2 \frac{\partial}{\partial y}, y^2z \frac{\partial}{\partial y}, yz^2 \frac{\partial}{\partial y}, z^3 \frac{\partial}{\partial y}, yz^2 \frac{\partial}{\partial z}, z^3 \frac{\partial}{\partial z}, \\
x^3 \frac{\partial}{\partial y} - 3x^2 y \frac{\partial}{\partial z}, x^2z \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial y} + xyz \frac{\partial}{\partial z}, \\
3xyz \frac{\partial}{\partial z} + y^3 \frac{\partial}{\partial z}, x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z}, \\
x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z}, x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial x} - y^2z \frac{\partial}{\partial z}, \\
xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z}, y^3 \frac{\partial}{\partial y} - 3y^2z \frac{\partial}{\partial z}, xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z}
\end{array}
\right\} \tag{2.92}
$$

Hence, we can choose the following basis for the complementary space G_3:

$$
\left\{ x^3 \frac{\partial}{\partial z}, x^2y \frac{\partial}{\partial z}, x^2z \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial z}, xyz \frac{\partial}{\partial z}, y^2z \frac{\partial}{\partial z} \right\} \tag{2.93}
$$

In order to solve the 3rd order normal form problem

$$
\frac{d}{dt} g_3(t) = -\pi_3 \left(\left[Y^2, v^2 + g_3(t) \right]_3 \right) \tag{2.94}
$$

we must choose the infinitesimal generator Y^2 such that

$$
[Y^2, v^2]^2 = 0 \tag{2.95}
$$

that is, we must satisfy

$$[Y_1, v_1] = 0 \tag{2.96}$$

and

$$[Y_1, v_2] + [Y_2, v_1] = 0 \tag{2.97}$$

As is discussed on the 2nd order normal form problem, Y_1 is of the form,

$$Y_1 = Az\,\frac{\partial}{\partial x} + Bv_1 + C\left[x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z}\right] \tag{2.98}$$

Hence, If $A \neq 0$ or $C \neq 0$, then $[Y_1, v_2]$ does not belong to B_2 = Image of L_2; in view of (2.81), whereas $[Y_2, v_1] \in B_2$. Therefore (2.96) and (2.97) implies

$$Y_1 = \text{constant} \cdot v_1 = K \cdot v_1 \tag{2.99}$$

Hence, (2.97) becomes

$$\begin{aligned} [Y_1, v_2] + [Y_2, v_1] &= [Kv_1, v_2] + [Y_2, v_1] \\ &= [Y_2 - Kv_2, v_1] = 0 \end{aligned} \tag{2.100}$$

It follows from (2.100) that

$$Y_2 = Kv_2 + (\text{kernel of } L_2)\,. \tag{2.101}$$

Here, the kernel of L_2 is spanned by

$$\left\{\begin{matrix} z^2\,\dfrac{\partial}{\partial x}, yz\,\dfrac{\partial}{\partial x} + z^2\,\dfrac{\partial}{\partial y}, 2xz\,\dfrac{\partial}{\partial x} - y^2\,\dfrac{\partial}{\partial x}, \\ xz\,\dfrac{\partial}{\partial x} + yz\,\dfrac{\partial}{\partial y} + z^2\,\dfrac{\partial}{\partial z} \end{matrix}\right\} \tag{2.102}$$

Thus we can choose

$$\begin{aligned} Y^2 = K(v_1{+}v_2) + Az^2\,\frac{\partial}{\partial z} + B\left[yz\,\frac{\partial}{\partial x} + z^2\,\frac{\partial}{\partial y}\right] \\ + C\left[2xz\,\frac{\partial}{\partial x} - y^2\,\frac{\partial}{\partial x}\right] + D\left[xz\,\frac{\partial}{\partial x} + yz\,\frac{\partial}{\partial y} + z^2\,\frac{\partial}{\partial z}\right] \end{aligned} \tag{2.103}$$

On the other hand, from (2.93), $g_3(t)$ is of the form

$$g_3(t) = a(t)x^3 \frac{\partial}{\partial z} + b(t)x^2y \frac{\partial}{\partial z} + c(t)x^2z \frac{\partial}{\partial z}$$
$$+d(t)xy^2 \frac{\partial}{\partial z} + e(t)xyz \frac{\partial}{\partial z} + f(t)y^2z \frac{\partial}{\partial z} . \tag{2.104}$$

Observe that

$$\begin{aligned} \left[Y_2, v^2 + g_3(t)\right]_3 &= \left[Y_1, g_3(t)\right] + [Y_2, v_2] \\ &= \left[Kv_1, g_3(t)\right] + \left[Kv_2 + (\text{kernel}), v_2\right] \\ &= \left[Kv_1, g_3(t)\right] + [Kv_2, v_2] + \left[(\text{kernel}), v_2\right] . \end{aligned} \tag{2.105}$$

Since the first term $\left[Kv_1, g_3(t)\right]$ belongs to B_3, it vanishes under the projection $\pi_3 : H_3 \rightarrow G_3$. The second term vanishes by itself. Therefore only the third term [(kernel), v_2] is essential: The Lie bracket [(kernel), v_2] for various kernels and v_2 are summarized as follow:

(kernel) \ v_2	$x^2 \frac{\partial}{\partial z}$	$xy \frac{\partial}{\partial z}$	$xz \frac{\partial}{\partial z}$
$z^2 \frac{\partial}{\partial x}$	$2xz^2 \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial x}$	$yz^2 \frac{\partial}{\partial z} - 2xyz \frac{\partial}{\partial x}$	$z^3 \frac{\partial}{\partial z} - 2xz^2 \frac{\partial}{\partial x}$
$yz \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y}$	$2xyz \frac{\partial}{\partial z} - x^2y \frac{\partial}{\partial x} - 2x^2z \frac{\partial}{\partial y}$	$y^2z \frac{\partial}{\partial z} + xz^2 \frac{\partial}{\partial z} - xy^2 \frac{\partial}{\partial x} - 2xyz \frac{\partial}{\partial y}$	$yz^2 \frac{\partial}{\partial z} - xyz \frac{\partial}{\partial x} - 2xz^2 \frac{\partial}{\partial y}$
$2xz \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial x}$	$4x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} - 2x^3 \frac{\partial}{\partial x}$	$2xyz \frac{\partial}{\partial z} - y^3 \frac{\partial}{\partial z} - 2x^2y \frac{\partial}{\partial x}$	$2xz^2 \frac{\partial}{\partial z} - y^2z \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial x}$
$x^2 \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z}$	$-x^3 \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y}$	$-x^2y \frac{\partial}{\partial x} - xy^2 \frac{\partial}{\partial y}$	$-x^2z \frac{\partial}{\partial x} - xyz \frac{\partial}{\partial y}$

In this table, the following terms belong to B_3:

$$\left\{ yz^2 \frac{\partial}{\partial z}, xyz \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial z}, xz^2 \frac{\partial}{\partial x}, x^2y \frac{\partial}{\partial x}, xz^2 \frac{\partial}{\partial y} \right\} \tag{2.106}$$

Moreover,

$$2xz^2 \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial x} = 2\left[xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z}\right] - 2\left[x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z}\right],$$

$$\begin{aligned} 2xyz \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial y} &= 2xyz \frac{\partial}{\partial z} - 4xyz \frac{\partial}{\partial z} - 2\left[x^2z \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial z}\right] \\ &= -2xyz \frac{\partial}{\partial z} - 2\left[x^2z \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial z}\right], \end{aligned}$$

$$y^2z\,\frac{\partial}{\partial z} + xz^2\,\frac{\partial}{\partial z} - xy^2\frac{\partial}{\partial x} - 2xyz\,\frac{\partial}{\partial y}$$

$$= y^2z\,\frac{\partial}{\partial z} - 2y^2z\,\frac{\partial}{\partial z} + \left[xz^2\,\frac{\partial}{\partial z} + 2y^2z\,\frac{\partial}{\partial z}\right]$$

$$-y^2z\,\frac{\partial}{\partial z} - \left[xy^2\,\frac{\partial}{\partial x} - y^2z\,\frac{\partial}{\partial z}\right] + 2y^2z\,\frac{\partial}{\partial z} - 2\left[xyz\,\frac{\partial}{\partial y} + y^2z\,\frac{\partial}{\partial z}\right]$$

$$= \left[xz^2\,\frac{\partial}{\partial z} + 2y^2z\,\frac{\partial}{\partial z}\right] - \left[xy^2\,\frac{\partial}{\partial x} - y^2z\,\frac{\partial}{\partial z}\right] - 2\left[xyz\,\frac{\partial}{\partial y} + y^2z\,\frac{\partial}{\partial z}\right],$$

$$4x^2z\,\frac{\partial}{\partial z} - 2xy^2\,\frac{\partial}{\partial z} - 2x^3\,\frac{\partial}{\partial x}$$

$$= 4x^2z\,\frac{\partial}{\partial z} - 2xy^2\,\frac{\partial}{\partial z} - 6x^2z\,\frac{\partial}{\partial z} - 12xy^2\,\frac{\partial}{\partial z} - 2\left[x^3\,\frac{\partial}{\partial x} - 3x^2z\,\frac{\partial}{\partial z} - 6xy^2\,\frac{\partial}{\partial z}\right]$$

$$= -\,2x^2z\,\frac{\partial}{\partial z} - 14xy^2\,\frac{\partial}{\partial z} - 2\left[x^3\,\frac{\partial}{\partial x} - 3x^2z\,\frac{\partial}{\partial z} - 6xy^2\,\frac{\partial}{\partial z}\right]$$

$$2xyz\,\frac{\partial}{\partial z} - y^3\,\frac{\partial}{\partial z} = 2xyz\,\frac{\partial}{\partial z} + 3xyz\,\frac{\partial}{\partial z} - \left[y^3\,\frac{\partial}{\partial z} + 3xyz\,\frac{\partial}{\partial z}\right]$$

$$= 5xyz\,\frac{\partial}{\partial z} - \left[y^3\,\frac{\partial}{\partial z} + 3xyz\,\frac{\partial}{\partial z}\right]$$

$$2xz^2\,\frac{\partial}{\partial z} - y^2z\,\frac{\partial}{\partial z} - 2x^2z\,\frac{\partial}{\partial x}$$

$$= -\,4y^2z\,\frac{\partial}{\partial z} + 2\left[xz^2\,\frac{\partial}{\partial z} + 2y^2z\,\frac{\partial}{\partial z}\right] - y^2z\,\frac{\partial}{\partial z} + 4y^2z\,\frac{\partial}{\partial x} - 2\left[x^2z\,\frac{\partial}{\partial x} + 2y^2z\,\frac{\partial}{\partial z}\right]$$

$$= -\,y^2z\,\frac{\partial}{\partial z} + 2\left[xz^2\,\frac{\partial}{\partial z} + 2y^2z\,\frac{\partial}{\partial z}\right] - 2\left[x^2z\,\frac{\partial}{\partial x} + 2y^2z\,\frac{\partial}{\partial z}\right]$$

$$-x^3 \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y}$$

$$= -3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} - \left[x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z}\right]$$

$$- x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} - \left[x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z}\right]$$

$$= -4x^2z \frac{\partial}{\partial z} - 8xy^2 \frac{\partial}{\partial z} - \left[x^3 \frac{\partial}{\partial z} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z}\right] - \left[x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z}\right]$$

$$- xy^2 \frac{\partial}{\partial y} = xyz \frac{\partial}{\partial z} - \left[xy^2 \frac{\partial}{\partial y} + xyz \frac{\partial}{\partial z}\right],$$

$$- x^2z \frac{\partial}{\partial x} - xyz \frac{\partial}{\partial y} = 2y^2z \frac{\partial}{\partial z} - \left[x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z}\right] + y^2z \frac{\partial}{\partial z} - \left[xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z}\right]$$

$$= 3y^2z \frac{\partial}{\partial z} - \left[x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z}\right] - \left[xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z}\right].$$

In all of the above expressions, the terms enclosed by parentheses belong to B_3. Hence

$$\begin{aligned}
-\pi_3\left(\left[Y^2, v^2 + g_3(t)\right]_3\right) &= -Bs\left[-2xyz \frac{\partial}{\partial z}\right] - Cs\left[-2x^2z \frac{\partial}{\partial z} - 14xy^2 \frac{\partial}{\partial z}\right] \\
&\quad - C\beta\left[5xyz \frac{\partial}{\partial z}\right] - C\gamma\left[-y^2z \frac{\partial}{\partial z}\right] \\
&\quad - Ds\left[-4x^2z \frac{\partial}{\partial z} - 8xy^2 \frac{\partial}{\partial z}\right] - D\beta\left[xyz \frac{\partial}{\partial z}\right] - D\gamma\left[3y^2z \frac{\partial}{\partial z}\right] \qquad (2.107) \\
&= (2Cs + 4Ds)x^2z \frac{\partial}{\partial z} + (14Cs + 8Ds)xy^2 \frac{\partial}{\partial z} \\
&\quad + (2Bs - 5C\beta - D\beta)xyz \frac{\partial}{\partial z} + \left[C\gamma - 3D\gamma\right]y^2z \frac{\partial}{\partial z}.
\end{aligned}$$

Finally, equating the corresponding coefficients of (2.104) and (2.107), we obtain the following system of differential equations,

$$\left.\begin{aligned}
\dot{a} &= 0\,,\ \dot{b} = 0 \\
\dot{c} &= 2Cs + 4Ds \\
\dot{d} &= 14Cs + 8Ds \\
\dot{e} &= 2Bs - 5C\,\beta - D\,\beta \\
\dot{f} &= C\,\gamma - 3D\,\gamma
\end{aligned}\right\} \tag{2.108}$$

for solving the 3rd order normal form problem. By an appropriate choice of B, C, and D, we can choose

$$d = e = f = 0$$

at $t = 1$ and obtain the following *non-degenerate 3rd order normal form:*

$$v_2 + (ax^3 + bx^2y + cx^2z)\,\frac{\partial}{\partial z}\,. \tag{2.109}$$

Equation (2.109) can be identified with the differential equation

$$\left.\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= sx^2 + \beta xy + \gamma xz + ax^3 + bx^2y + cx^2z
\end{aligned}\right\} \tag{2.110}$$

or, equivalently,

$$\begin{aligned}
\dddot{x} &= sx^2 + \beta x\dot{x} + \gamma x\ddot{x} + ax^3 + bx^2\dot{x} + cx^2\ddot{x} \\
&= (sx^2+ax^3) + (\beta x+bx^2)\,\dot{x} + (\gamma x+cx^2)\,\ddot{x}\,.
\end{aligned} \tag{2.111}$$

We will conclude this section by giving a comparison of the normal forms derived via *Poincaré 's method, Takens' method and Ushiki's method.* Since the *goal* for obtaining normal forms of vector fields is to eliminate as many *monomials* from each order as possible, we will list in the following tables the number of monomials of each degree that is still present in the normal form. For example, a number "2" under the column for degree 3 for Takens method means that there are 2 monomials of degree 3 in the Takens normal form. Similarly, the number "0" under the same column would mean that the resulting normal form has no 3rd degree terms. Hence "0" corresponds to the ideal situation where all terms of a given degree are eliminated.

It turns out that even for the remaining monomials, some coefficients can be set to ± 1, rather than remaining arbitrary in the general case. This further refinement, due to Ushiki [2,3], is indicated by a *parentheses*. For example, the number 2 + (1) in the table under degree 2 means that there are 3 monomials of degree 2 present in the normal form, one of which has a coefficient equal to ± 1.

The following tables are based on the 5 examples derived in *Part I* [1] as well as in this section.† They

†A similar comparison is given in Ushiki [3]. Our tables include, however, corrections to some errors in [3].

correspond to non-degenerate normal forms from each example.

Case 1: Simple-zero type in $\mathbf{R}^1$ *(Example 5.1, Part I)*

The Jacobian matrix in this case is a scalar: (0)

degree	2nd	3rd	4th	5th	6th	7th
Poincaré	1	1	1	1	1	1
Takens	1	1	1	1	1	1
Ushiki	(1)	1	0	0	0	0

Case 2: Hopf type in $\mathbf{R}^2$ *(Example 2.1)*

The Jacobian matrix in this case is: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

degree	2nd	3rd	4th	5th	6th	7th
Poincaré	0	2	0	2	0	2
Takens	0	2	0	2	0	2
Ushiki	0	1+(1)	0	1	0	0

Case 3: Hopf and Zero-interaction type in $\mathbf{R}^3$ *(Example 2.3)* in cylindrical coordinate

The Jacobian matrix in this case is: $\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$

degree	2nd	3rd	4th	5th
Poincaré	4	6	7	9
Takens	4	6	7	9
Ushiki	2 + (2)	3	9	2

Case 4: Double-zero type in $\mathbf{R}^2$ *(Example 5.5, Part I)*

The Jacobian matrix in this case is: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Since the *linear* part in this example is *not* diagonalizable, Poincaré's method is not applicable.

degree	2nd	3rd	4th
Takens	2	2	2
Ushiki	1 + (1)	1	1

Case 5: Triple-zero type in $\mathbf{R}^3$ *(Example 2.4)*

The Jacobian matrix in this case is : $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Again, Poincaré's method is not applicable because the linear part is not diagonalizable.

degree	2nd	3rd
Takens	4	6
Ushiki	2 + (1)	3

Remarks

1. Both Poincaré and Takens' methods give the same result for vector fields having a diagonalizable linear part.
2. Poincaré's method does *not* apply, in its original form, to *non-diagonalizable* vector fields.
3. Takens' method is applicable to *all* cases.
4. Ushiki's method may be considered as a refinement of Takens' method.
5. Takens' is aware that further refinements are possible and have in fact derived them for cases 1 and 2 [5,6].
6. This paper follows Ushiki's approach since his method gives an explicit algorithm for general vector fields.

■

3. NORMAL FORM WITH SYMMETRIES

Many vector fields of practical interest are imbued with some form of *symmetry;* e.g., reflection symmetry, point symmetry, rotation symmetry, etc. In such cases, it is natural that their normal forms should exhibit the same symmetries. Our goal in this section is to show how such additional constraints can be imposed upon the algorithm and in the preceding section.

Definition 3.1: Vector fields with symmetry

A *symmetry* for a vector field v on $\mathbf{R}^n$ is a *diffeomorphism* $\gamma : \mathbf{R}^n \to \mathbf{R}^n$ satisfying

$$\gamma_* v = v \tag{3.1}$$

where

$$\gamma_* v(x) \triangleq D\gamma\left[\gamma^{-1}(x)\right] \cdot v\left[\gamma^{-1}(x)\right] \tag{3.2}$$

Example 3.2:

Let v be a vector field on $\mathbf{R}^2$ defined by

$$v(x,y) = (1+y^2, x^2y) . \tag{3.3}$$

Then v exhibits a symmetry via the diffeomorphism

$$\gamma(x,y) = (x,-y) , \tag{3.4}$$

To see this, we calculate

$$D\gamma(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \gamma^{-1}(x,y) = (x,-y) \tag{3.5}$$

and

$$\gamma_* v(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1+y^2 \\ -x^2y \end{bmatrix} = v(x,y) \tag{3.6}$$

It is easy to see that the set of all symmetries for a vector field v forms a *group* under composition. This group is called the *symmetry group* of v. For the above example, the symmetry group contains at least four elements $\{id, \gamma, \delta, \gamma \circ \delta\}$, where $\delta : (x,y) \to (-x,y)$, since $\gamma \circ \gamma = \delta \circ \delta = id$ and $\gamma \circ \delta = \delta \circ \gamma$.

A group made of two elements is frequently denoted by $\mathbf{Z}_2$. In this section, we shall restrict our consideration to vector fields whose symmetry group contains $Z_2 = \{id, \gamma\}$ as a subgroup. To derive a normal form with symmetry γ, we must find appropriate transformations which preserve the same symmetry γ, after subjecting the vector field to these transformations, as in the preceding section. In other words, the resulting normal form must also exhibit the symmetry γ. Using our notation from [1], our abstract objects in this section consists of a subspace ${}_\gamma\mathcal{X}_0$ of smooth vector fields $\mathcal{X}_0$ which vanish at 0 and exhibit the symmetry γ.

Our next proposition characterizes the class of transformations that preserves symmetry.

Proposition 3.3

Suppose a vector field v exhibits a symmetry γ. Then a transformed vector field $\phi_* v$ also exhibits the same symmetry γ if ϕ *commutes* with γ, i.e., $\gamma \circ \phi = \phi \circ \gamma$, or equivalently,

$$\gamma \circ \phi \circ \gamma^{-1} = \phi \tag{3.7}$$

Proof.

Since $\gamma_* v = v$ and $\phi \circ \gamma = \gamma \circ \phi$,

$$\gamma_*(\phi_* v) = (\gamma \circ \phi)_* v = (\phi \circ \gamma)_* v = \phi_*(\gamma_* v) = \phi_* v \tag{3.8}$$

■

Proposition 3.4

If a local one-parameter group $\{\phi^t\}$ of transformations is generated by a vector field Y, i.e.,

$$\phi^t = \exp tY \ , \tag{3.9}$$

then the local one-parameter group

$$\tilde{\phi}^t = \gamma \circ \phi^t \circ \gamma^{-1} \tag{3.10}$$

is generated by $\gamma_* Y$.

Proof.

Since ϕ^t is the flow of the vector field Y,

$$\frac{d}{dt}\, \phi^t(x) = Y\left[\phi^t(x)\right] \tag{3.11}$$

holds. Thus,

$$\begin{aligned}
\frac{d}{dt}\, \tilde{\phi}^t(x) &= \frac{d}{dt}\, (\gamma \circ \phi^t \circ \gamma^{-1})(x) = D\gamma\left[\phi^t \circ \gamma^{-1}(x)\right] \cdot \frac{d}{dt}\, \phi^t \left[\gamma^{-1}(x)\right] \\
&= D\,\gamma\left[\phi^t \circ \gamma^{-1}(x)\right] \cdot Y\left[\phi^t(\gamma^{-1}(x))\right] \\
&= D\,\gamma\left[\gamma^{-1}(\tilde{\phi}^t(x))\right] \cdot Y\left[\gamma^{-1}(\tilde{\phi}^t(x))\right] \\
&= (\gamma_* Y)\left[\tilde{\phi}^t(x)\right] .
\end{aligned} \tag{3.12}$$

This completes the proof.

■

It follows from the above propositions that the set

$${}_{\gamma}\mathrm{Diff}_0 = \{\phi : \mathbf{R}^n \to \mathbf{R}^n \text{ , diffeomorphism} ,\ \phi(0) = 0 ,\ \gamma \circ \phi = \phi \circ \gamma\} \tag{3.13}$$

can be chosen as our transformation group and ${}_{\gamma}\mathcal{X}_0$ can be chosen as its infinitesimal generators. Once we fix this transformation group and space, the remaining steps for obtaining the normal form are identical to the preceding algorithm, as illustrated by the following example.

Example 3.4

Consider vector fields on $\mathbf{R}^1$ with a vanishing 1-jet; i.e.,

$$v_1 = 0 \ , \quad v(0) = v'(0) = 0 \ . \tag{3.14}$$

Let us impose the symmetry

$$\gamma(x) = -x \ . \tag{3.15}$$

The space ${}_{\gamma}\mathcal{X}_0$ can be decomposed into the homogeneous parts

$$ {}_{\gamma}\mathcal{X}_0 = {}_{\gamma}H_1 \oplus {}_{\gamma}H_2 \oplus {}_{\gamma}H_3 \oplus {}_{\gamma}H_4 \oplus \cdots , \tag{3.16} $$

where each part ${}_{\gamma}H_k$ consists of homogeneous vector fields v_k of order k satisfying the symmetry constraint

$$ \gamma_* v_k = v_k , \tag{3.17} $$

Hence, for even k, we have

$$ {}_{\gamma}H_k = 0 \tag{3.18} $$

and for odd k, we have

$$ {}_{\gamma}H_k = H_k . \tag{3.19} $$

In other words,

$$ {}_{\gamma}\quad{}_{0} = H_1 \oplus H_3 \oplus H_5 \oplus \cdots . \tag{3.20} $$

Since we have $v^2 = 0$, our next step is to obtain the simplest 3rd order part; i.e., consider the 3rd order normal form problem with $v^2 = 0$ under the symmetry $\gamma(x) = -x$.

The 1-jet $v_1 = 0$ implies that the linear map

$$ L_k : {}_{\gamma}H_k \to {}_{\gamma}H_k , \; Y_k \to [Y_k , v_1] \tag{3.21} $$

is the zero map for every k. Hence, the complementary space ${}_{\gamma}G_k$ to the image of L_k is simply ${}_{\gamma}H_k$, and the normal form problem is already reduced:

$$ \frac{d}{dt} g_k(t) = - \left[Y^{k-1} , v^{k-1} + g_k(t) \right]_k \tag{3.22} $$

with

$$ \left[Y^{k-1} , v^{k-1} \right]^{k-1} = 0 , \; \gamma_* Y^{k-1} = Y^{k-1} . \tag{3.23} $$

For $k = 3$,

$$ \frac{d}{dt} g_3(t) = - \left[Y^2 , g_3(t) \right]_3 = - \left[Y_1 , g_3(t) \right] \tag{3.24} $$

where

$$ [Y^2 , v^2]^2 = 0 \tag{3.25} $$

automatically holds for $v^2 = 0$. Thus, Y^2 need only satisfy $\gamma_* Y^2 = Y^2$; hence $Y_2 = 0$. Defining

$$ Y_1 = Ax \frac{\partial}{\partial x} \tag{3.26} $$

and

$$g_3(t) = \alpha(t)x^3 \frac{\partial}{\partial x} , \qquad (3.27)$$

we obtain

$$- \left[Y_1 , g_3(t) \right] = - 2A \, \alpha(t)x^3 \qquad (3.28)$$

Equating the coefficients of (3.27) and (3.28), we obtain the differential equation

$$\dot{\alpha} = - 2A \, \alpha . \qquad (3.29)$$

It follows that we can always choose the value of A so that the non-degenerate 3rd order normal form with symmetry γ is given by

$$v^3 = \pm x^3 \frac{\partial}{\partial x} . \qquad (3.30)$$

Similar to *Prop. 5.2* from *Part I* [1], we can prove that the γ-symmetric kth order normal form of vector fields on $\mathbf{R}^1$ with a vanishing 1-jet is given by

$$(\pm x^3 + ax^5) \frac{\partial}{\partial x} \qquad a \in \mathbf{R} . \qquad (3.31)$$

for $k \geq 5$.

Following the same procedure, we have derived the following *non-degenerate normal forms with* $\mathbf{Z}_2$*-symmetries* for several typical examples:

2-dimensional case

(i) non-degenerate 3rd order normal form with the symmetry $(x, y) \rightarrow (-x, -y)$, whose linear part has *double*- zero eigenvalues:

$$v^3 = y \frac{\partial}{\partial x} + (sx^3 + ax^2y) \frac{\partial}{\partial y} , \ s = \pm 1 \qquad (3.32)$$

(ii) non-degenerate 3rd order normal form with the symmetry $(x, y) \rightarrow (-x, y)$, whose linear part has *double*- zero eigenvalues:

$$v^3 = s_1 xy \frac{\partial}{\partial x} + (s_2 x^2 + ay^2 + by^3) \frac{\partial}{\partial y} , \ s_1 = \pm 1 , \ s_2 = \pm 1 \qquad (3.33)$$

3-dimensional case

(iii) non-degenerate 3rd order normal form with the symmetry $(x, y, z) \rightarrow (-x, -y, -z)$, whose linear part has *triple*-zero eigenvalues:

$$v^3 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (sx^3+ax^2y+bx^2z+cxyz+dxz^2)\frac{\partial}{\partial z}\ ,\ s = \pm 1 \tag{3.34}$$

(iv) non-degenerate 3rd order normal form with the symmetry $(x,y,z) \to (-x,-y,z)$, whose linear part has *triple*-zero eigenvalues:

$$v^3 = y\frac{\partial}{\partial x} + (s_1xz+ayz+bx^3+cx^2y+dyz^2)\frac{\partial}{\partial y} + (s_2x^2+ez^2+fz^3)\frac{\partial}{\partial z}\ ,\ s_1 = \pm 1\ ,\ s_2 = \pm 1 \tag{3.35}$$

(v) non-degenerate 3rd order normal form with the symmetry $(x,y,z) \to (x,y,-z)$, whose linear part has *triple*-zero eigenvalues:

$$v^3 = y\frac{\partial}{\partial x} + (s_1x^2+axy+bx^3+cxz^2+dyz^2)\frac{\partial}{\partial y} + (exz+s_2z^3)\frac{\partial}{\partial z}\ ,\ s_1 = \pm 1\ ,\ s_2 = \pm 1 \tag{3.36}$$

4. VERSAL UNFOLDINGS

Our preceding normal form theory consists basically of methods for simplifying ordinary differential equations (ODE) in a neighborhood of their singular points. We have presented various examples of normal forms having several eigenvalues with a *zero real part;* namely, multiple zero eigenvalues, or pure imaginary eigenvalues, etc. All such eigenvalues which lie on the *imaginary* axis in the complex plane are called *central* eigenvalues. A singular point of a vector field is said to be *hyperbolic* if it does not contain any central eigenvalues. From the *Hartman-Grobman theorem,* we know that the local phase portraits of vector fields around *hyperbolic* singular points are determined by their associated 1-jets, and the *center manifold theorem* reduces a vector field with both central and non-central eigenvalues to that with only central eigenvalues. Hence, we will only consider vector fields around a singular point where all of its eigenvalues are central.

The above conclusion appears, at first sight, to be rather academic especially from an engineering context because almost all Jacobian matrices associated with physical systems are *hyperbolic;* i.e., all eigenvalues have a *non-zero* real part. This observation is true only for *fixed* vector fields. For *families* of vector fields, however, central eigenvalues will always be encountered. In particular, we will show that *simple* central eigenvalues (i.e., multiplicity one) are inevitable in one-parameter families of vector fields, while *multiple* central eigenvalues are inevitable in multi-parameter families of vector fields.

Example 4.1

Consider a vector field $x^2 \frac{\partial}{\partial x}$ on $\mathbf{R}^1$. The origin is a singular point whose linear part vanishes. Hence, we have a zero *central* eigenvalue at the origin.

If we perturb this vector field into

$$(\mu+x^2) \frac{\partial}{\partial x} , \quad \mu \neq 0 \tag{4.1}$$

then we have a simple singular point at $x = \pm \sqrt{-\mu}$, $\mu < 0$ whose eigenvalues are given by $\pm \sqrt{-\mu}$. In this example, we have succeeded in avoiding central eigenvalues by perturbing the vector field slightly. This example shows that it is possible to avoid singular points with a *simple* zero eigenvalue in the space of vector fields having a *non-degenerate* 2nd order term; namely, all vector fields whose 2-jet is given by (4.1).

Unfortunately, it can be proved that in the space of one-parameter *families* of vector fields, it is impossible to avoid central eigenvalues. In particular, if a one-parameter family $f(x,\mu) \frac{\partial}{\partial x}$ is sufficiently close to the one-parameter family $(\mu+x^2) \frac{\partial}{\partial x}$, as well as to its derivatives, then there exists a parameter value μ_0 close to zero such that $f(x,\mu_0) \frac{\partial}{\partial x}$ has a singular point x_0 whose 1-jet vanishes but whose 2-jet does *not* vanish.† In other words, there exist a nearby one-parameter vector field which has a *central* eigenvalues at some parameter value $\mu_0 \neq 0$.

In general, if a singular point with a *central* eigenvalue is avoidable in a (k-1)-parameter families of vector fields, but is inevitable in k-parameter families, then we say the singularity has a *codimension k*. Hence, it follows from the above example that a singularity with a vanishing 1-jet but non-vanishing 2-jet has a *codimension 1*.

A singularity of codimension k should therefore be studied in the space of k-parameter families of vector fields. We are therefore concerned with the following two problems:

(1) Find the codimension of singularities.

(2) Study the process by which singularities are formed in a *typical family* of vector fields, called a *versal family* of the singularity, whose precise definition will be given later.

Let us begin with the versal families for matrices which was first studied by Arnold [8-9]. Let $A(\mu)$ denote a family of $n \times n$ matrices which depend smoothly on $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$.

---

†It is important to note that the one-parameter family of vector fields $(\mu+x^2) \frac{\partial}{\partial x}$ is only one point in the space of one-parameter *families* (plural!) of vector fields. There are infinitely many other families of vector fields, parametrized by μ, which are different but "close" to $(\mu+x^2) \frac{\partial}{\partial x}$.

Definition 4.2

(i) A family $A(\mu)$ is an *unfolding* of a matrix A_0 if $A(0) = A_0$.

(ii) Let $A(\mu)$ and $\tilde{A}(\tilde{\mu})$ be unfoldings of a matrix A_0, where μ and $\tilde{\mu}$ need not have the same dimension. We say $A(\mu)$ is *induced from* $\tilde{A}(\tilde{\mu})$ if there exist an unfolding $C(\mu)$ of the identity matrix and a transformation $\tilde{\mu} = \phi(\mu)$ satisfying $\phi(0) = 0$ such that

$$A(\mu) = C(\mu) \cdot \tilde{A}\left[\phi(\mu)\right] \cdot C(\mu)^{-1} \tag{4.2}$$

holds for any μ close to zero.

(iii) An unfolding $A(\mu)$ of A_0 is *versal* if every unfoldings of A_0 is induced from $A(\mu)$.

Example 4.3

(1) For any 2×2 matrix A_0, the family

$$A(\mu) = A_0 + \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix} \tag{4.3}$$

is a *versal* family because any unfolding of A_0 can be written as

$$A_0 + \begin{bmatrix} \phi_1(\lambda) & \phi_2(\lambda) \\ \phi_3(\lambda) & \phi_4(\lambda) \end{bmatrix} \tag{4.4}$$

where $\phi_i(0) = 0$.

(2) Let A_0 denote the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then the family

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{bmatrix} \tag{4.5}$$

is *versal* because any unfolding

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \phi_1(\lambda) & \phi_2(\lambda) \\ \phi_3(\lambda) & \phi_4(\lambda) \end{bmatrix} \tag{4.6}$$

is *conjugate* to the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \phi_3(1+\phi_2) - \phi_1\phi_4 & \phi_1 + \phi_4 \end{bmatrix} \tag{4.7}$$

under the transformation

$$C = \begin{bmatrix} 1+\phi_2(\lambda) & 0 \\ -\phi_1(\lambda) & 1 \end{bmatrix} \tag{4.8}$$

Remark 4.4

The above example shows that the matrix $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has at least 2 kinds of versal unfoldings; namely,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{bmatrix}. \tag{4.9}$$

Therefore, it is important to obtain versal unfoldings with the *minimum* number of parameters. Such versal unfoldings are said to be *miniversal.*

We will show later that the versal unfolding $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{bmatrix}$ is miniversal.

In order to characterize versal and miniversal unfoldings, we need the following concepts from differential topology:

Definition 4.5

(1) Let L be a finite dimensional vector space over $\mathbf{R}$ and let X, Y be its subspaces: $X, Y \subset L$. We say X and Y are *transversal* (in L) and denote this property by the symbol

$$X \pitchfork Y \quad (\text{in } L) \tag{4.10}$$

if the sum $X+Y$ is the whole space L (see Fig. 1). This implies that

$$\dim L \leq \dim X + \dim Y . \tag{4.11}$$

If the equality holds, then X and Y are said to be *minitransversal.*

(2) Let M and N be smooth manifolds and let P be a submanifold of N. We say a smooth mapping $f : M \to N$ is *transversal to P at a point $x \in M$* if one of the following conditions holds:

(i) $f(x) \notin P$

(ii) $f(x) = p \in P$ and

$$Df(x) \cdot T_x M \pitchfork T_p P \text{ in } T_p N \tag{4.12}$$

where $Df(x)$ is the tangent map (see Fig. 2)

$$Df(x) : T_x M \to T_{f(x)} N . \tag{4.13}$$

If $Df(x) \cdot T_x M$ and $T_p P$ are minitransversal, then we say f is *minitransversal* to P at $x \in M$.

In considering an unfolding $A(\mu)$ of A_0, we may regard it as a smooth mapping

$$A : \mu \rightarrow A(\mu) \tag{4.14}$$

from the parameter space Π to $M(n, \mathbf{R})$.

Our next theorem is fundamental in versal unfoldings.

Theorem 4.6

An unfolding $A(\mu)$ of $A_0 \in M(n, \mathbf{R})$ is (mini-)versal if and only if $A(\mu)$ is (mini-)transversal to $\mathcal{O}(A_0)$ at $\mu = 0$, where $\mathcal{O}(A_0)$ is a submanifold of $M(n, \mathbf{R})$ defined by

$$\mathcal{O}(A_0) = \left\{ CA_0C^{-1} \mid C \in GL(n, \mathbf{R}) \right\}. \tag{4.15}$$

Remark 4.7

(1) This theorem is an example of the "*versality* <=> *transversality*" principle which originated from the singularity theory of *smooth* functions [10]. This principle works for various objects.

(2) If an unfolding $A(\mu)$ of A_0 is transversal to $\mathcal{O}(A_0)$ at A_0, then this type of singularity is inevitable in $A(\mu)$; that is, if we slightly perturb $A(\mu)$ to $A'(\mu')$, as depicted in Fig. 3, then the family $A'(\mu')$ still has a matrix A_0' which is conjugate to A_0. If $A(\mu)$ is mini-versal, then $A(\mu)$ is an unfolding with the *least* number of parameters which contains A_0 in an inevitable manner. This shows that the number of parameters of a miniversal unfolding is equal to the codimension of A_0 in the space $M(n, \mathbf{R})$ of matrices.

(3) This theorem can be easily proved using the inverse function theorem. See [8,9] for the proof.

Corollary 4.7

(1) A *miniversal unfolding* of the $n \times n$ matrix

$$A_0 = \left.\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}\right\} n \tag{4.16}$$

is given by

$$A_0 + \begin{bmatrix} & \bigcirc & \\ \cdots & & \cdots \\ \mu_1 & \cdots & \mu_n \end{bmatrix} \tag{4.17}$$

(2) A *miniversal unfolding* of the matrix

$$A_0 = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{4.18}$$

is given by

$$\begin{bmatrix} \mu_1 & -\omega \\ \omega & \mu_1 \end{bmatrix} \tag{4.19}$$

where ω is close to ω_0.

(3) A *miniversal unfolding* of $A_0 = 0$ is given by

$$\begin{bmatrix} \mu_{11} & \cdots & \mu_{1n} \\ \vdots & & \vdots \\ \mu_{n1} & \cdots & \mu_{nn} \end{bmatrix}. \tag{4.20}$$

Proof

Each statement can be proved by checking the minitransversality of $A(\mu)$ to $\mathcal{O}(A_0)$. See [8,9] for the proof.

■

We are now ready to consider vector fields.

Definition 4.8

(1) Let v_0 be a vector field on $\mathbb{R}^n$ vanishing at the origin 0. We say a family $v(\mu)$ of vector fields on $\mathbb{R}^n$ is an *unfolding* of v_0 if $v(0) = v_0$ holds. We do *not* assume $v(\mu)$ vanishes at 0 for $\mu \neq 0$.

(2) Let $v(\mu)$ and $w(\lambda)$ be *unfoldings* of v_0. We say $w(\lambda)$ is *induced from* $v(\mu)$ if there exist a family $\phi(\lambda)$ of diffeomorphisms with

$$\phi(0) = \text{identity} \tag{4.21}$$

and a transformation of parameters $\mu = \mu(\lambda)$ with

$$\mu(0) = 0 \tag{4.22}$$

such that

$$w(\lambda) = \phi(\lambda)_* \, v\left[\mu(\lambda)\right] \tag{4.23}$$

for λ close to zero.

(3) We say an unfolding $v(\mu)$ of v_0 is *k-versal* if any unfolding of v_0 coincides with an induced unfolding up to order k.

Remark 4.9

Instead of coordinate transformations, we may consider the case where $v\left[\mu(\lambda)\right]$ is *locally topologically equivalent* to $w(\lambda)$. (See [8,9] for the definition.) Under this equivalence, $v(\mu)$ is said to be *topologically versal.*

Example 4.10

An unfolding $(\mu+x^2)\dfrac{\partial}{\partial x}$ of the vector field $x^2\dfrac{\partial}{\partial x}$ is 2-versal. To prove this, let $\phi(x,\lambda)\dfrac{\partial}{\partial x}$ be any unfolding of $x^2\dfrac{\partial}{\partial x}$. Since $\phi(x,0)=x^2$, $\phi_{xx}(0,0)=2\neq 0$. Therefore, by the implicit function theorem, the equation $\phi_x(x,\lambda)=0$ can be solved for x as a function of λ; namely,

$$x=\eta(\lambda)\ ,\ \text{with}\ \ \eta(0)=0\ . \tag{4.24}$$

By a family of coordinate transformations,

$$X=\frac{\phi_{xx}(\eta(\lambda),\lambda)}{2}\cdot\left\{x-\eta(\lambda)\right\}, \tag{4.25}$$

the vector fields $\phi(x,\lambda)\dfrac{\partial}{\partial x}$ can be transformed into the form

$$\left\{\psi(\lambda)+X^2+O(X^3)\right\}\frac{\partial}{\partial X}\ . \tag{4.26}$$

Hence, upon choosing $\mu=\psi(\lambda)$, (4.26) coincides with $(\mu+x^2)\dfrac{\partial}{\partial x}$ up to order two.

The general "*versality* <=> *transversality*" principle also works for this case. In particular, by changing $M(n,\mathbf{R})$ to

$$\mathcal{X}_0^k=H_1\oplus H_2\oplus\ \cdots\ \oplus H_k$$

and $\mathcal{O}(A_0)$ to

$$\mathcal{O}(v_0)=\{\phi^k{}_*v_0\mid\phi^k\in\mathrm{Diff}_0^k\}\ ,\ v_0\in\mathcal{X}_0^k\ ,$$

we can prove the following theorem by using the same reasoning as that of Theorem 4.6:

Theorem 4.11

An unfolding $v(\mu)$ of $v_0\in\mathcal{X}_0^k$ is *(mini-)k-versal* if and only if v is *(mini-)transversal* to $\mathcal{O}(v_0)\subset\mathcal{X}_0^k$ at $\mu=0$.

Remark 4.12

Note that we restrict the whole space to $\mathcal{X}_0^k$, *not* $\mathcal{X}^k$. This means that any unfolding $v(\mu)$ of $v_0\in\mathcal{X}_0^k$ should vanish at the origin; i.e., $v(\mu)(0)=0$. Therefore *Theorem 4.11* does *not* imply Example 4.10 because in that case, the unfolding $(\mu+x^2)\dfrac{\partial}{\partial x}$ does not vanish at O. If we apply this theorem to

$v_0 = x^2 \frac{\partial}{\partial x}$, we would obtain a 2-versal unfolding

$$(\mu x + x^2) \frac{\partial}{\partial x} \tag{4.27}$$

under the condition that the origin is alway the singular point.

Nevertheless, we presented the above theorem because we can easily obtain an unfolding without the trivial singularity 0 from the k-versal unfolding in $\mathcal{X}_0^k$, and because the following general corollary holds.

Corollary 4.13

Suppose $v_0 \in \mathcal{X}_0^k$ is a *non-degenerate* kth order normal form with a specified 1-jet $Ax \frac{\partial}{\partial x}$. Then a k-versal unfolding of v_0 in $\mathcal{X}_0^k$ is obtained upon adding a linear unfolding

$$\Phi(\mu) x \frac{\partial}{\partial x} \tag{4.28}$$

where $A + \Phi(\mu)$ is a versal unfolding of the matrix A. In other words, a versal unfolding of a non-degenerate normal form can be obtained by adding a linear versal unfolding.

From this corollary, we can obtain the following *versal unfoldings* automatically:

$$\text{(i)} \quad A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \ (\omega \neq 0) \tag{4.29}$$

In polar coordinate:

$$(\mu \pm r^2 + \alpha r^4)\, r \frac{\partial}{\partial r} + (\omega + \beta r^2) \frac{\partial}{\partial \theta} \tag{4.30}$$

(k-versal for any $k \geq 5$)

$$\text{(ii)} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.31}$$

$$y \frac{\partial}{\partial x} + (\mu_1 x + \mu_2 y \pm x^2 + \beta xy + ax^3) \frac{\partial}{\partial y} \tag{4.32}$$

(3-versal)

$$\text{(iii)} \quad A = \begin{bmatrix} 0 & -\omega & \vdots & 0 \\ \omega & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 \end{bmatrix} \quad (w \neq 0)$$

$$(\mu_1+\alpha z+az^2)\, r\, \frac{\partial}{\partial r} + (\omega+\beta z+bz^2)\, \frac{\partial}{\partial \theta} + (\mu_2 z+s_1 r^2+s_2 z^2+cz^3)\, \frac{\partial}{\partial z} \quad (s_i = \pm 1) \tag{4.33}$$

(3-versal)

(iv) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$y\, \frac{\partial}{\partial x} + z\, \frac{\partial}{\partial y} + (\mu_1 x+\mu_2 y+\mu_3 z+sx^2+\beta xy+\gamma xz+ax^3+bx^2y+cx^2z)\, \frac{\partial}{\partial z} \quad (s = \pm 1) \tag{4.34}$$

(3-versal)

(v) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with the symmetry $(x, y) \to (-x, -y)$

$$y\, \frac{\partial}{\partial x} + (\mu_1 x+\mu_2 y+sx^3+ax^2y)\, \frac{\partial}{\partial y} \quad (s = \pm 1) \tag{4.35}$$

(3-versal)

(vi) $A = \begin{bmatrix} 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 \end{bmatrix}$ with the symmetry $(x, y, z) \to (-x, -y, z)$

$$\begin{aligned} &y\, \frac{\partial}{\partial x} + (\mu_1 x+\mu_2 y+s_1 xz+ayz+bx^3+cx^2y+dyz^2)\, \frac{\partial}{\partial y} \\ &\quad + (\mu_3 z+s_2 x^2+ez^2+fz^3)\, \frac{\partial}{\partial z} \quad (s_i = \pm 1) \end{aligned} \tag{4.36}$$

(3-versal).

The proof of *Corollary 4.13* consists of showing the addition of a linear versal unfolding gives a transversal family to $\mathcal{O}(v_0)$. This is assured by the non-degeneracy assumption since k-jets equivalent to non-degenerate normal form of order k forms an open set in

$$H_2 \oplus H_3 \oplus \cdots \oplus H_k .$$

Due to versal unfoldings, we can study the change in the phase portraits of vector fields near a vector field with central eigenvalues in a systematic manner. This study constitutes a part of *bifurcation theory*. Unfortunately, since we have only k-versal unfoldings for *finite* k, our considerations must be restricted to truncated vector fields of finite order. Such truncation sometimes gives rise to very complicated and subtle problems which are beyond the scope of this paper. Therefore our descriptions in what follows are incomplete. The reader is referred to [7,9,11] and the references therein for more details.

Let us begin with the simplest case

$$(\mu+x^2)\frac{\partial}{\partial x} \text{ on } \mathbf{R}^1 \tag{4.37}$$

which is proved to be 2-versal in *Example 4.10.* Equation (4.37) is equivalent to the following differential equation:

$$\dot{x} = \mu + x^2 . \tag{4.38}$$

When the parameter μ changes from a positive to a negative value, the bifurcation behavior of this ODE is given in the x-space in Fig. 4, or in the (μ, x)-space in Fig. 5. Such a bifurcation gives birth to a pair of stable and unstable singular points, and is called a *saddle-node bifurcation.*

Our next example is given by (4.1). In polar coordinate, (4.1) can be identified with the ODE

$$\begin{cases} \dot{r} = (\mu+r^2+\alpha r^4)r \\ \dot{\theta} = \omega+\beta r^2 \qquad , \ \omega \neq 0 \end{cases} \tag{4.39}$$

Since $\omega \neq 0$, say $\omega > 0$, we have $\dot{\theta} > 0$ for sufficiently small r. Hence, the local phase portrait near the origin changes as shown in the (x, y)-space in Fig. 6, and in the (μ, x, y)-space in Fig. 7, when we vary the parameter μ from a positive to a negative value. Observe that a limit cycle is born from a singular point. This oscillation mechanism is called the *Hopf Bifurcation.*

Remark 4.14

The above arguments are all based on truncated equations. But fortunately, the result is true for any ODE with a non-degenerate lower order jet. More precisely,

(i) For a one-parameter family $v(\mu)$ of vector fields on $\mathbf{R}^1$, if $v(\mu_0)$ can be transformed into the *non-degenerate 2nd order* normal form $\pm x^2 \frac{\partial}{\partial x}$ for some μ_0, and if $\left.\frac{\partial}{\partial \mu}\right|_{\mu=\mu_0} v(\mu) \neq 0$, then $v(\mu)$ exhibits the *saddle-node bifurcation* in a neighborhood of μ_0.

(ii) Similarly, for a one-parameter family $v(\mu)$ on $\mathbf{R}^2$, if $v(\mu_0)$ can be transformed into the *non-degenerate 3rd order normal form*

$$\pm r^3 \frac{\partial}{\partial r} + (\omega+\beta r^2)\frac{\partial}{\partial \theta} , \ \omega \neq 0 \tag{4.40}$$

in polar coordinate and if $\left.\frac{\partial}{\partial \mu}\right|_{\mu=\mu_0} v(\mu) \neq 0$, then $v(\mu)$ exhibits the *Hopf bifurcation* in a neighborhood of μ_0.

More complicated bifurcation phenomena can be observed from the family (4.2). Although this family assumes the origin is a singular point, we can eliminate this assumption by making the transformation $x \to x - x_0$. The resulting family, truncated up to 2nd order, is of the form

$$y \frac{\partial}{\partial x} + (\mu_1+\mu_2 x \pm x^2+\beta xy) \frac{\partial}{\partial y} \tag{4.41}$$

Hence, this is 2-versal. It follows from the result of Bogdanov [12-13] (see also Arnold [9]) that this family exhibits the 2-parameter bifurcation phenomena shown in Fig. 8. All of the phase portraits in Fig. 8 are shown near the origin and the parameters μ_1, μ_2 are also chosen close to zero. This is an example of a *local* 2-parameter bifurcation. It is remarkable to observe that, in this bifurcation, both the saddle-node bifurcation and the Hopf bifurcation are present, in addition to a third bifurcation phenomenon which yields a homoclinic orbit.

Now we can appreciate a *fundamental observation* from the above *local* bifurcation theory; namely, a *global* bifurcation can be observed from a local bifurcation of a *more degenerate* singularity. For example, although a limit cycle can be observed from a vector field with a hyperbolic singular point, which requires a *global* analysis, the Hopf bifurcation predicts a limit cycle in an arbitrarily small neighborhood of a central singular point, which requires only a *local* analysis. Similarly, although the bifurcation phenomena in Fig. 9 can be observed from a 1-parameter family of vector fields, which requires a global analysis, we can observe them in an arbitrarily small neighborhood of the origin in the above 2-parameter family (4.41), which requires only a local analysis.

Bifurcations of the other *2-parameter families* (4.33) and (4.35) have also been studied in [5], [7], [9], [14], etc. It remains to consider the bifurcation phenomena in *3-parameter families*, such as (4.34), (4.36), etc. For such families, even *strange attractors* can be expected to bifurcate locally. Some analytical results, as well as results based on numerical simulations [4,15], have been reported recently. However, we are very far from a complete understanding of these extremely complex phenomena. *Part 3* of this paper will therefore be devoted to this subject.

APPENDIX. PROOF OF PROPOSITION 2.2

Let us begin with the 3-jet

$$v^3 = sr^3 \frac{\partial}{\partial r} + (1+\beta r^2) \frac{\partial}{\partial \theta} \ , \ s = \pm 1 \tag{A.1}$$

of a vector field v on $\mathbb{R}^2$, expressed in polar coordinate (r, θ), corresponding to the non-degenerate 3rd order normal form (2.38). Recall the following useful formulas from (2.12) and (2.13):

$$\left[\xi^k \bar{\xi}^l \frac{\partial}{\partial \xi} \ , \ \xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] = (1-k+l) \xi^k \bar{\xi}^l \frac{\partial}{\partial \xi} \tag{A.2}$$

$$\left[\xi^k \bar{\xi}^l \frac{\partial}{\partial \bar{\xi}} \ , \ \xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] = (-1-k+l) \xi^k \bar{\xi}^l \frac{\partial}{\partial \bar{\xi}} \tag{A.3}$$

Observation (a)

If $n = k+l$ is *even*, then $-k+l$ is also even and hence $\pm 1-k+l \neq 0$ in (A.2) and (A.3). This implies that the mapping

$$L_k : H_k \rightarrow H_k \ ; \ Y \rightarrow [Y, v_1]$$

is *surjective* for all *even* k; i.e.,

$$B_k = \text{Image } L_k = H_k \ , \text{ for all even } k \tag{A.5}$$

Observation (b)

If $n = k+l$ is *odd*, then the map L_k has a non-empty kernel only for $k-l = \pm 1$.

It follows from Observation (a) that all even order terms in the higher order normal forms are eliminated. It suffices therefore for us to examine only the *odd* order terms $n = 2m+1$, $m \geq 1$, *inductively*. For simplicity, we denote the non-degenerate (2m+1)th order normal form by $v_{(m)}$. Hence, $v_{(1)}$ denotes the 3-jet v^3; i.e., $v_{(1)} \triangleq v^3$.

For $m \geq 2$, we can choose the complementary space G_{2m+1} to be the subspace spanned by

$$\left\{ (\xi \bar{\xi})^m \left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] , (\xi \bar{\xi})^m \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] \right\} \tag{A.6}$$

Hence, to determine the non-degenerate 5th order normal form $v_{(2)}$, the complementary space G_5 is spanned by

$$\left\{ (\xi \bar{\xi})^2 \left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] , (\xi \bar{\xi})^2 \left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] \right\} \tag{A.7}$$

The corresponding 5th order normal form problem is given by:

$$\frac{d}{dt}\, g_5(t) = -\,\pi_5 \left[\left[Y^4, v^3 + g_5(t) \right]_5 \right] \tag{A.8}$$

where Y^4 satisfies the constraint

$$[Y^4, v^3]^4 = 0 \tag{A.9}$$

This implies:

$$[Y_1, v_1] = 0 \tag{A.10}$$

$$[Y_2, v_1] = 0 \tag{A.11}$$

$$[Y_3, v_1] + [Y_1, v_3] = 0 \tag{A.12}$$

and

$$[Y_4, v_1] + [Y_2, v_3] = 0 \tag{A.13}$$

It follows from (2.19), (A.10) and (A.11) that

$$Y_1 = A\left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] + B\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{A.14}$$

and

$$Y_2 = 0 \tag{A.15}$$

Now, (2.34) and (A.14) give

$$\begin{aligned}[Y_1, v_3] &= \left[Y_1, s\, \xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] + i\,\beta\xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right]\right] \\ &= 2A\, s\, \xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] + 2i\, A\, \beta\xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right]\end{aligned} \tag{A.16}$$

It follows from (A.12) and (A.16) that $A = 0$ and $Y_3 \in$ kernel of L_3. Hence, we can choose

$$Y_3 = C\, \xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] + D\, \xi\bar{\xi}\left[\xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}\right] \tag{A.17}$$

By a similar analysis, (A.13) implies $Y_4 = 0$. Consequently, the normal form problem (A.8) reduces to:

$$\frac{d}{dt}\, g_5(t) = -\,\pi_5 \left[[Y_1, g_5(t)] + [Y_3, v_3] \right] \tag{A.18}$$

where

$$g_5(t) = a(t)(\xi\bar{\xi})^2 \left[\xi \frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + b(t)(\xi\bar{\xi})^2 \left[\xi \frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{A.19}$$

Since

$$Y_1 = B\left[\xi \frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] = -\,i\,B\,v_1 \tag{A.20}$$

we have

$$[Y_1, g_5] = i\,B\,[g_5, v_1] = 0 \tag{A.21}$$

On the other hand,

$$[Y_3, v_3] = 2(C\,\beta - Ds)(\xi\bar{\xi})^2 \left[\xi \frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{A.22}$$

Equating the corresponding coefficients of (A.19) and (A.22), we obtain the differential equations:

$$\dot{a} = 0 \tag{A.23}$$

$$\dot{b} = 2(Ds - C\,\beta)\ ,\quad s = \pm 1 \tag{A.24}$$

It follows from (A.23) and (A.24) that $a(t) = a(0)$, and $b(t) = b(0) + 2(Ds - C\,\beta)\,t$. By choosing $C = 0$, and an appropriate D, we can always make $b(t) = 0$ at $t = 1$. Hence, the resulting 5th order normal form becomes

$$v_{(2)} = v_{(1)} + a\,r^5 \frac{\partial}{\partial r} \tag{A.25}$$

$$= (sr^2 + ar^4)\,r\frac{\partial}{\partial r} + (1+\beta r^2)\frac{\partial}{\partial\theta}\ ,\quad s = \pm 1 \tag{A.26}$$

Let us consider now the general *(2m+1)th order* normal form problem on G_{2m+1}:

$$\frac{d}{dt}\,g_{2m+1}(t) = -\,\pi_{2m+1}\left(\left[Y^{2m}, v_{(2)} + g_{2m+1}(t)\right]_{2m+1}\right) \tag{A.27}$$

We claim that

$$g_{2m+1} = 0\ ,\quad m \geq 3 \tag{A.28}$$

To prove (A.28), consider the following observations:

(i)

$$Y_{2m-1} = P(\xi\bar{\xi})^{m-1}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + Q(\xi\bar{\xi})^{m-1}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{A.29}$$

satisfies

$$[Y_{2m-1}, v_{(2)}]^{2m} = 0 \tag{A.30}$$

Since $Y_{2m-1} \in$ kernel of L_{2m-1},

$$[Y_{2m-1}, v_{(2)}]^{2m} = [Y_{2m-1}, v_1] = 0 \tag{A.31}$$

(ii) The calculation for the Lie bracket $[Y_{2m-1}, v_3]$ for various combinations of Y_{2m-1} and v_3 is summarized below:

Y_{2m-1} \ v_3	$(\xi\bar{\xi})\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$(\xi\bar{\xi})\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$
$(\xi\bar{\xi})^{m-1}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$(4-2m)(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$2(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$
$(\xi\bar{\xi})^{m-1}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	$(2-2m)(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right]$	0

It follows from observation (i) that we can always choose Y_{2m-1} as the infinitesimal generator; namely,

$$Y^{2m} = Y_{2m-1} \tag{A.32}$$

Hence, the normal form problem (A.27) becomes

$$\begin{aligned} -\pi_{2m+1}&\left(\left[Y_{2m-1}, v_{(2)} + g_{2m+1}\right]_{2m+1}\right) \\ &= -\pi_{2m+1}\left([Y_{2m-1}, v_3]\right) \\ &= (2m-4)Ps(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \\ &\quad + \{(2m-2)Qs + 2P\beta\}\, i\,(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \end{aligned} \tag{A.33}$$

Equating the corresponding coefficients of (A.33) and

$$g_{2m+1}(t) = \phi(t)(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] + i\,\psi(t)(\xi\bar{\xi})^{m}\left[\xi\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right] \tag{A.34}$$

we obtain:

$$\dot{\phi} = (2m-4)Ps \tag{A.35}$$

$$\dot{\psi} = 2P\,\beta + (2m-2)Qs \tag{A.36}$$

Since $s \neq 0$ and $m \geq 3$, we can always choose an appropriate P and Q at t = 1 so that

$$\phi(1) = \psi(1) = 0 \tag{A.37}$$

It follows that the $(2m+1)$th order contribution to the normal form is eliminated for all $m \geq 3$. This completes our proof of Proposition 2.2

■

FIGURE CAPTIONS

Fig. 1. Subspaces X and Y in this picture are transversal because any point in L can be decomposed into a point in X and a point in Y. Note that X and Y would not be transversal if they are collinear in this picture.

Fig. 2. Geometrical interpretation of the *transversality* of a smooth mapping f between two manifolds M and N. Here, the manifold M on the left maps into the bold manifold $f(M)$. In particular, $x \in M$ maps to $p \in f(M)$. Here, f is transversal because the tangent to $f(M)$ at p intersects the tangent to the submanifold P at p with a finite angle.

Fig. 3. Geometrical interpretation showing the perturbation of an unfolding $A(\mu)$ into $A'(\mu')$ preserves the transversality property.

Fig. 4. Phase portraits illustrating the *saddle-node bifurcation* in the x-space, parametrized by μ.

Fig. 5. Phase portraits illustrating the *saddle-node bifurcation* in the (μ, x)-space.

Fig. 6. Phase portraits illustrating the Hopf Bifurcation on the radial coordinate (on the left) and on the (x,y)-plane, parametrized by μ. For $\mu>0$ and $\mu=0$, the trajectory is an expanding spiral ($\dot{r}>0$ and $\dot{\theta}>0$). For $\mu<0$, a circular limit cycle with radius r_0 is born. All nearby trajectories inside or outside are repelled from the limit cycle.

Fig. 7. Phase portrait illustrating the Hopf Bifurcation in the (μ,x,y)-space. Note that each cross section parallel to the (x,y)-plane on the right ($\mu>0$) or on the (x,y)-plane ($\mu=0$) itself consists of an expanding spiral as in Fig. 6. On the other hand, each cross section with the parabola on the left ($\mu<0$) is a circle, which corresponds to a limit cycle.

Fig. 8. Phase portraits illustrating the 2-parameter bifurcation phenomena associated with the vector field (4.41).

Fig. 9. Phase portraits exhibiting a global bifurcation producing a homoclinic orbit associated with a 1-parameter family of vector fields.

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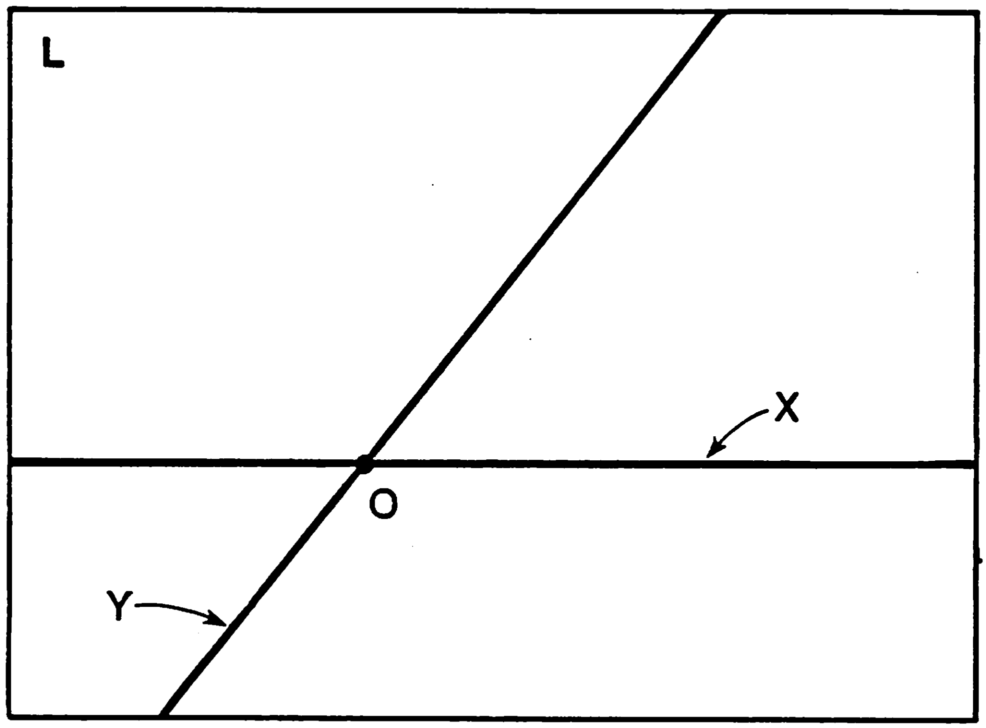


Fig.1

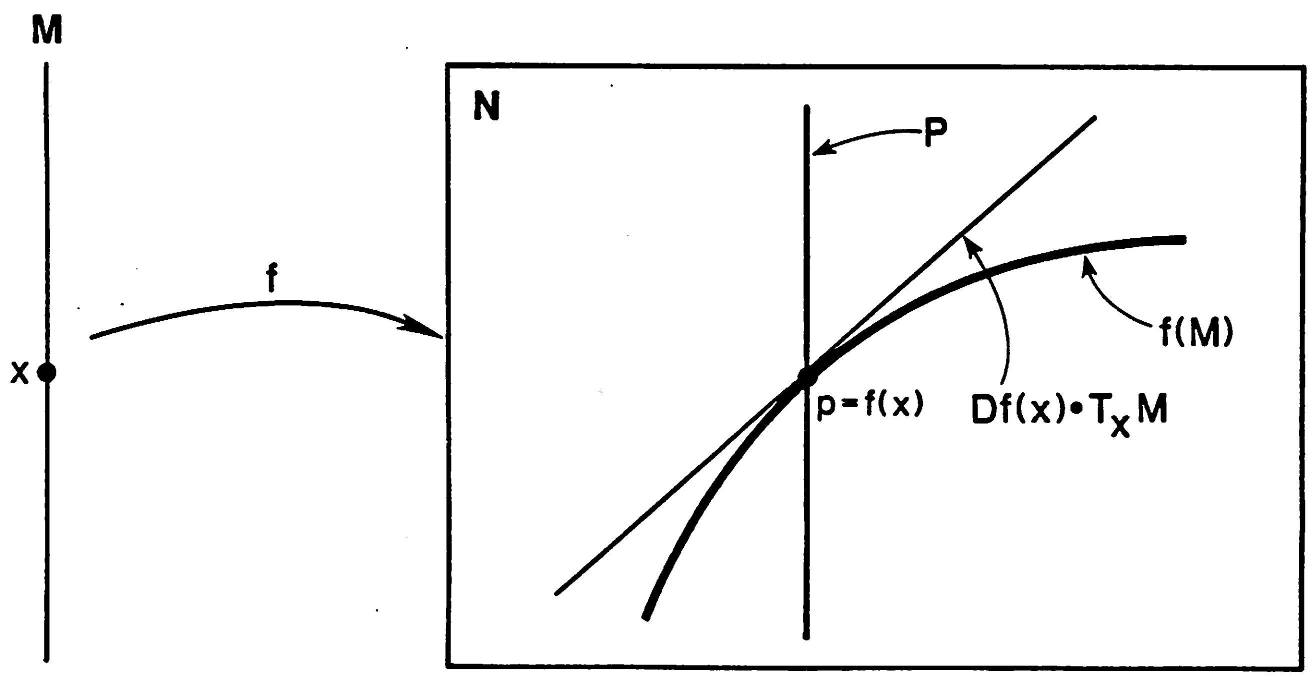


Fig.2

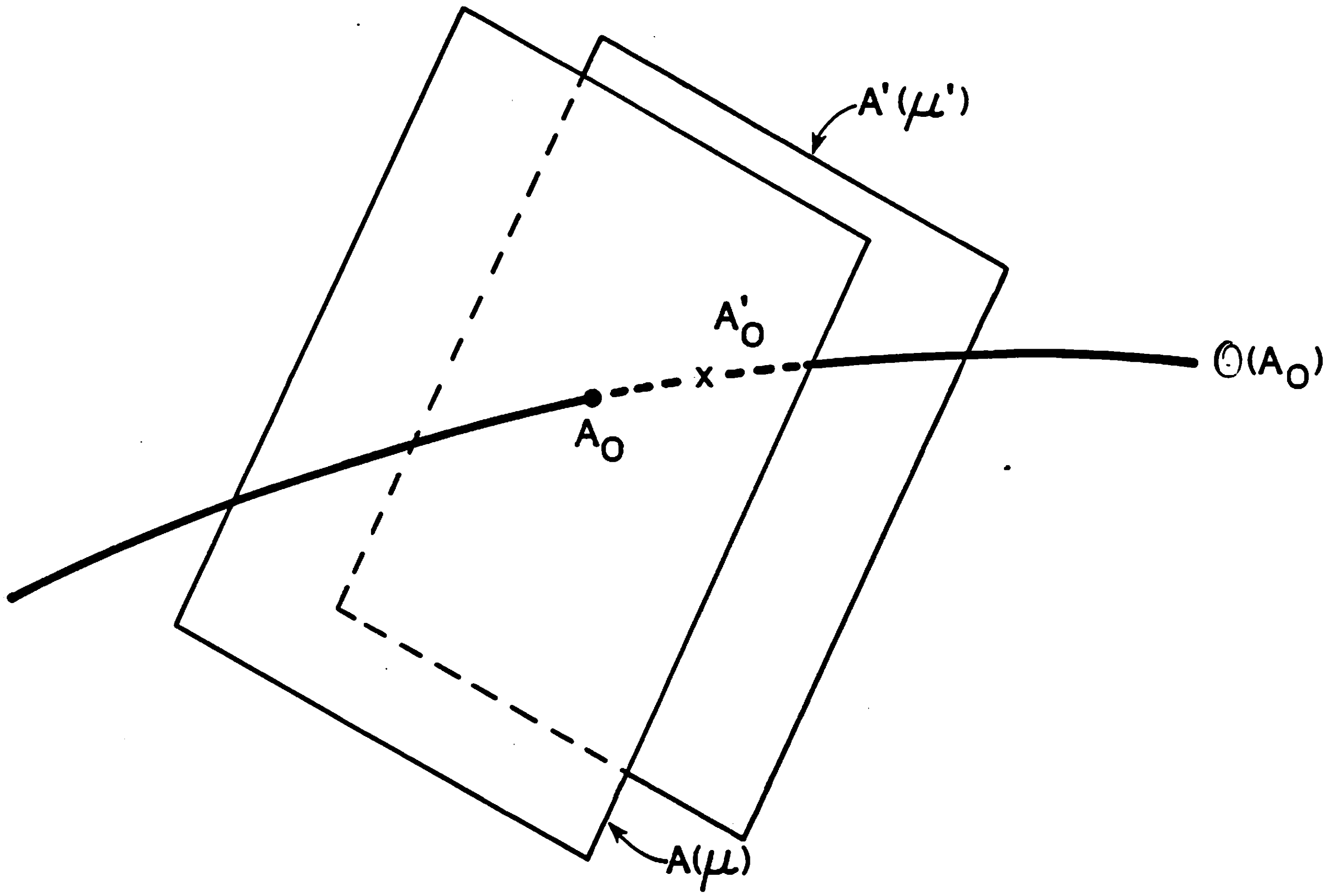

A'(μ')
A'O
𝒪(AO)
x
AO
A(μ)

Fig.3

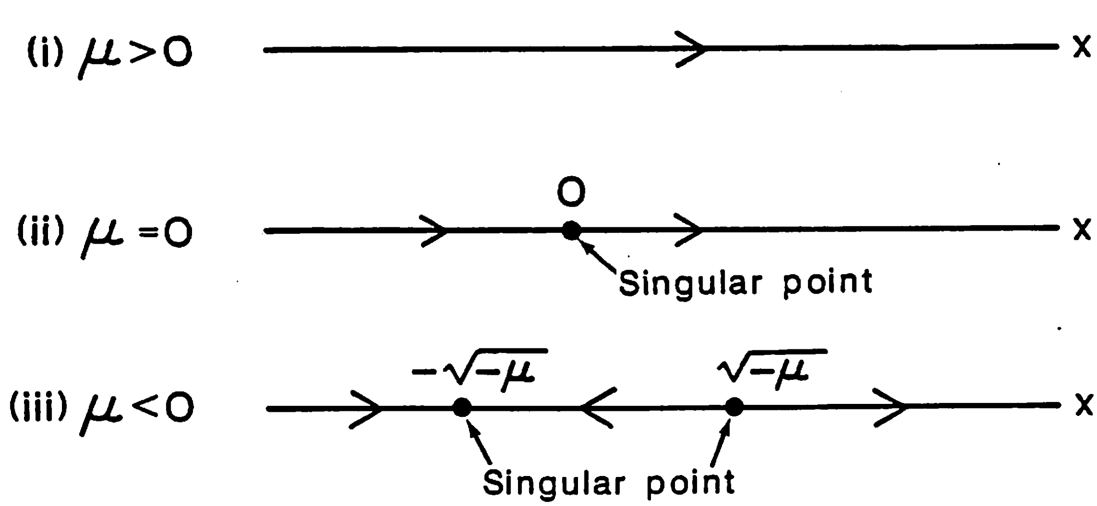

(i) μ>O
x
(ii) μ =O
O
x
Singular point
(iii) μ<O
−√−μ
√−μ
x
Singular point

Fig.4

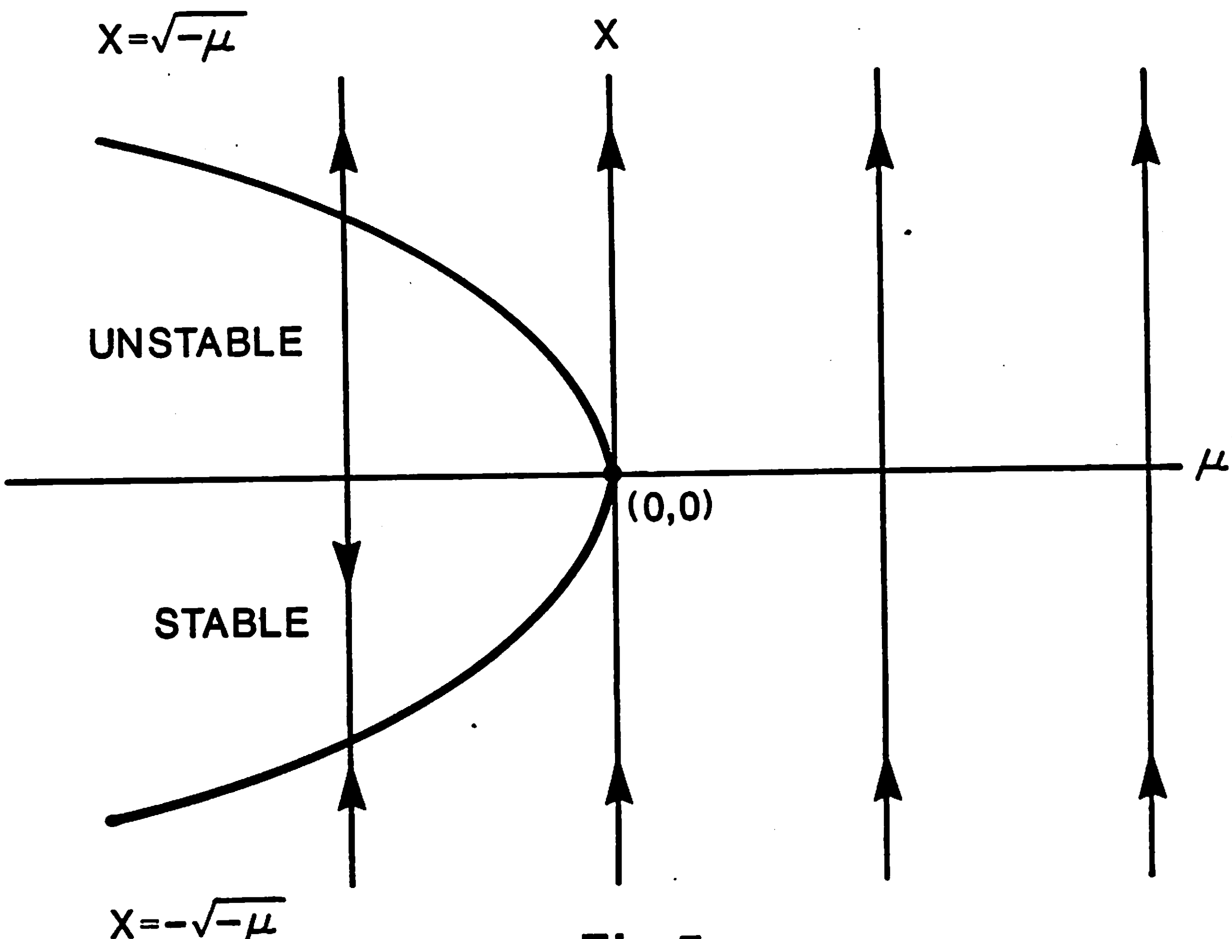


Fig.5

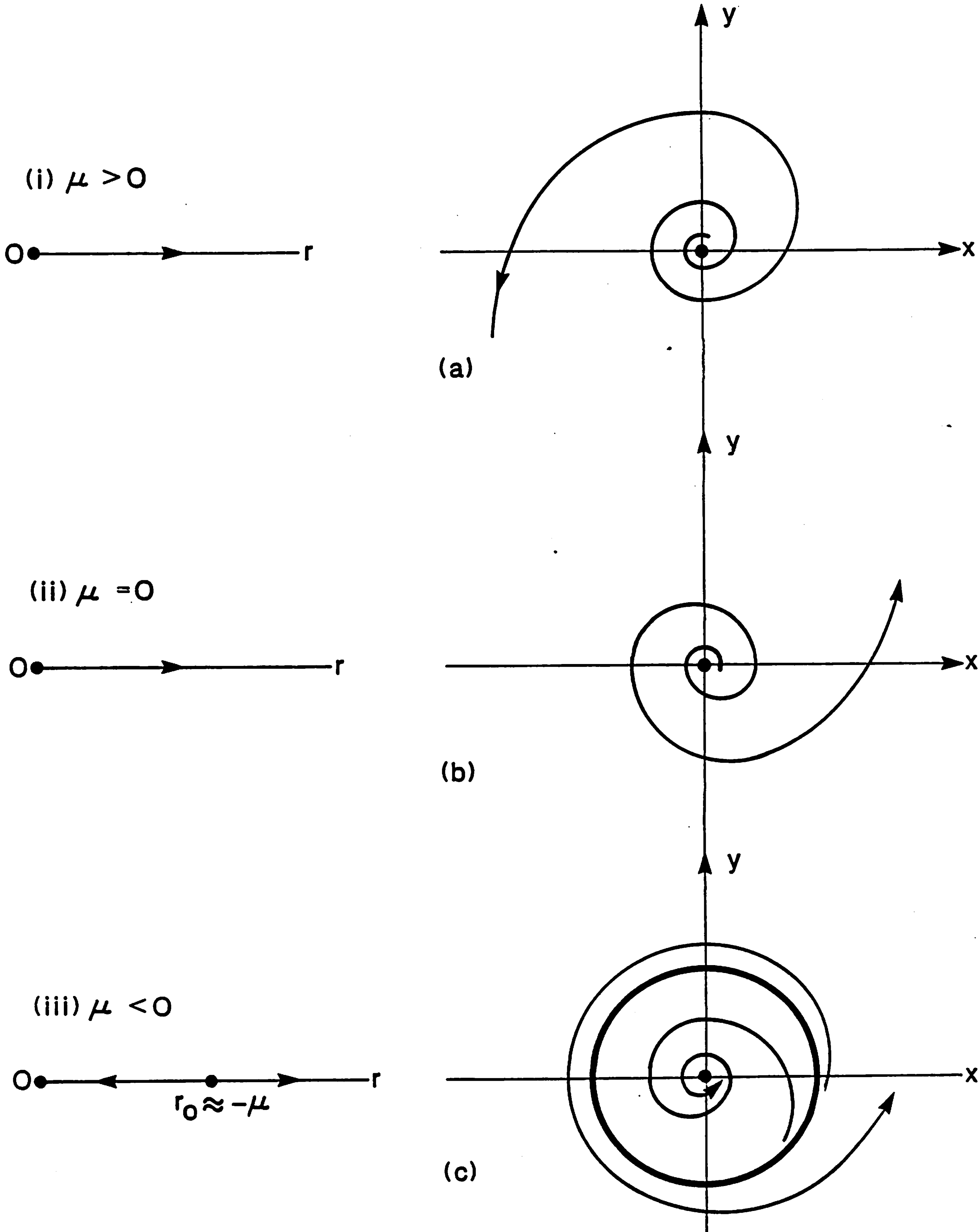


Fig.6

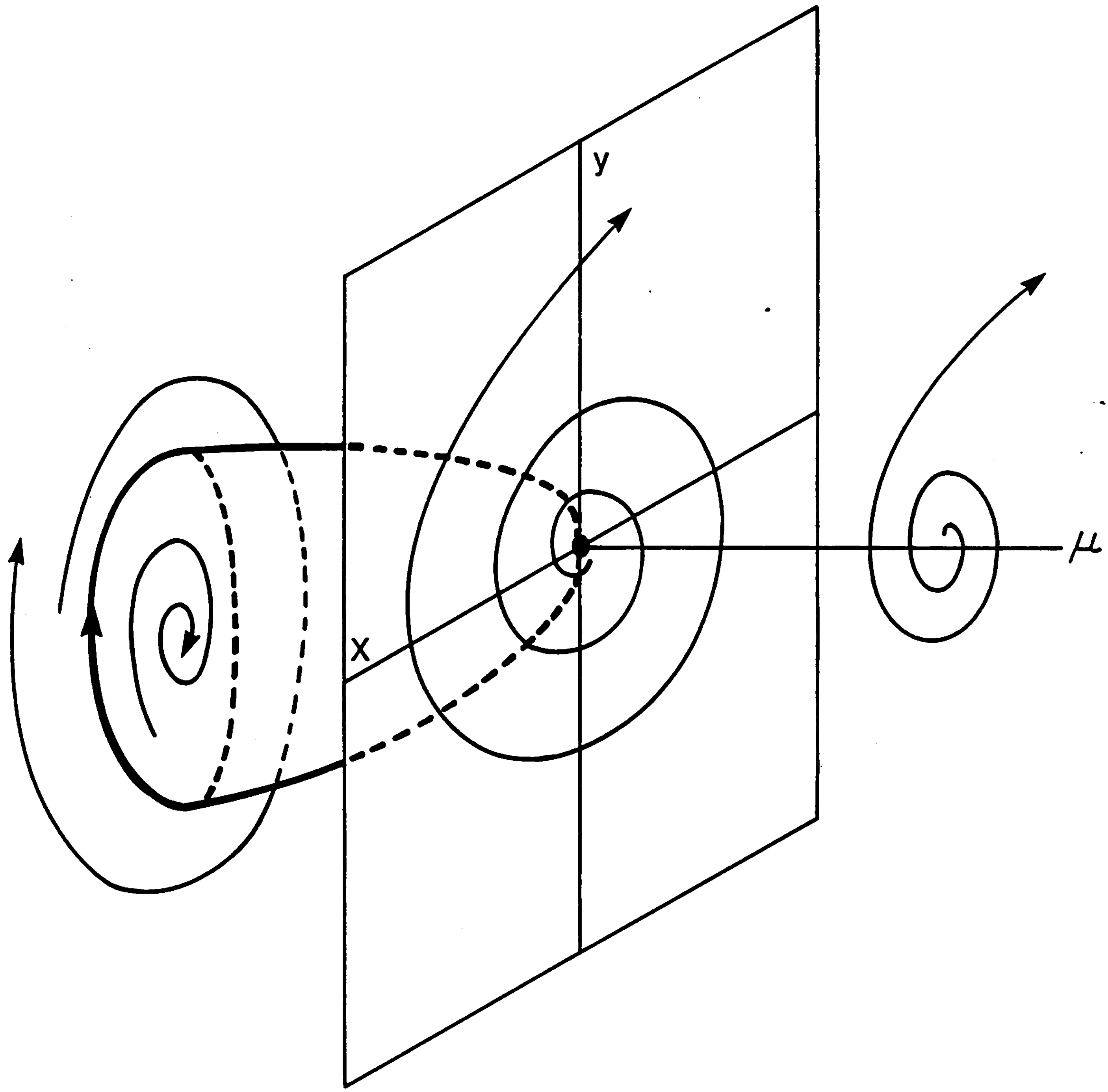
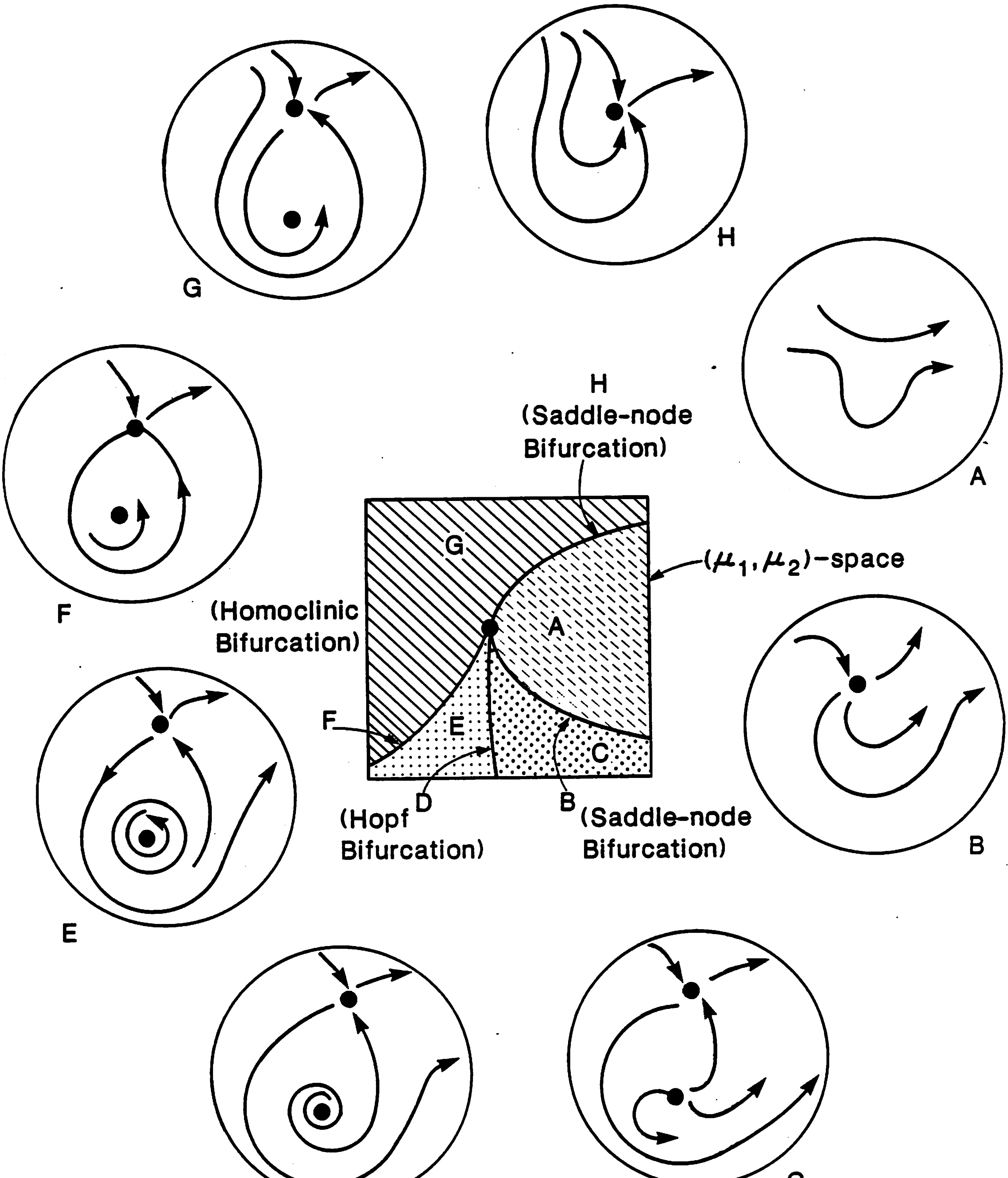


Fig.7



G
H
A
H
(Saddle-node
Bifurcation)
G
(μ_1, μ_2)-space
F
(Homoclinic
Bifurcation)
A
B
E
F
E
C
(Hopf
Bifurcation)
D
B
(Saddle-node
Bifurcation)
D
C

Fig.8

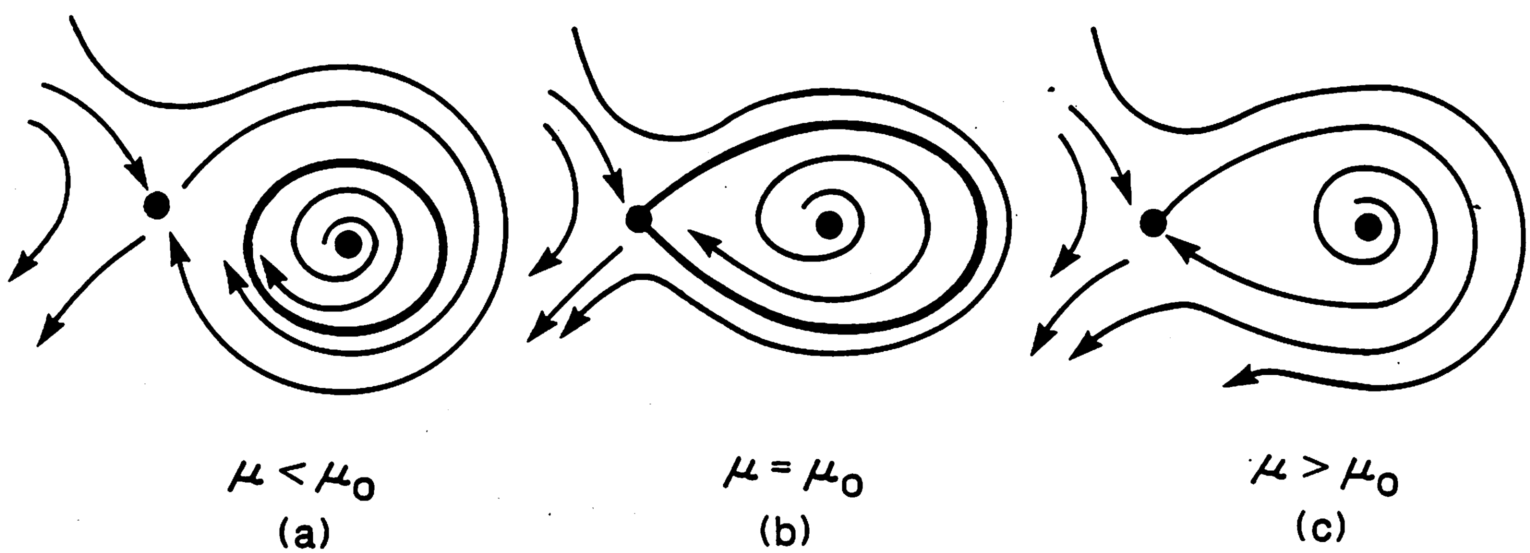


Fig.9