THE EFFECT OF DISCRETIZED FEEDBACK IN A CLOSED LOOP SYSTEM

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Memorandum No. UCB/ERL M87/87

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ABSTRACT

When a continuous time control law is implemented using a digital computer, the closed loop system may not have the same stability properties as the system with a true continuous controller due to delay and digitization errors. Using a Lyapunov analysis, this paper shows that, for linear systems and a class of nonlinear systems with discretized feedback, some stability properties can be preserved if the sampling frequency is properly chosen. In particular, we propose a *variable sampling interval scheme* for linear systems. This scheme is desirable when (1) computer resources are tightly shared by many tasks or (2) power consumption is critical. The effect of truncation error is also studied.

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1. Introduction

The discretization of linear systems is well understood (for example, see [1] [2]). z-domain analysis is certainly the best way of designing a controller if this controller is to be implemented using a digital computer. However, in some cases z-domain analysis may result in an impractical design. An example of such a design is the *dead beat* controller which, theoretically, can bring the state from any location in the state space to the origin in n steps where n is the dimension of the system. It is clear that, unless the sampling frequency is very low or the state variable is always close to the origin, the required control effort can be unreasonably high. On the other hand, a computer simulated continuous time controller can perform very close to a system with a true continuous time controller if the sampling frequency is properly chosen. For a linear system, we proposed a *variable sampling interval scheme* which cannot be obtained by z-domain analysis. This scheme is desirable when computer resources are tightly shared by many tasks or when power consumption is critical (such as in space applications), since the power consumption of some integrated circuits is proportional to the system clock rate. With this variable sampling interval scheme, the system clock (or sampling frequency) can be slowed down or sped up as needed.

Continuous time control of nonlinear systems has been studied very extensively recently [3,4,5]. It is not surprising that this design methodology usually leads to a control law which is a nonlinear function of the plant output. Except for some extremely simple cases, this nonlinear control law must be implemented using a digital computer which, of course, does not provide the instantaneous response that is required to realize a continuous time function. Unlike a linear system, most nonlinear systems can not be easily discretized. A great deal of work has been done in the analysis of sampled data non-linear systems. The emphasis has been on obtaining a state space model for the non-linear system after sampling as a power series in the sampling interval (see e.g.[6,7,8]). However, the behavior between samples is somewhat more difficult to understand. Thus, we take the position that we will construct a discrete controller as a sampled version of a continuous controller with a appropriate sampling frequency. It is intuitively clear that, if the sampling frequency is high enough, the performance of the closed loop system with a digital computer in the loop will be close to the performance of the system with a true continuous time controller. Using Lyapunov analysis, this paper shows that, this intuition is correct for a class of nonlinear systems. The effect of truncation error is also studied.

2. Definition of Discretized Feedback

Consider the system represented by the state equation

$$\dot{x} = f(x) + g(x)u(x) \tag{2.1}$$

* Research supported in part by NSF under PYI grant DMC84-51129.

where $u(x)$ is a stablizing state feedback. In practice, this term is usually generated by a digital computer at discrete times and is held constant between samples. We represent this *discretized feedback system* by the following equation.

$$\dot{x} = f(x) + g(x)u(x_k) \tag{2.2}$$

Before we define the function x_k, we must first examine the timing sequence in a typical digital computer based controller. In such a system, the central processor unit (CPU) is interrupted by various service requests. A service request is granted if the CPU is idle or is running a lower priority service routine. Usually, a real time control service request has a relatively high priority and is initiated by a timer which generates a request at a fixed interval.

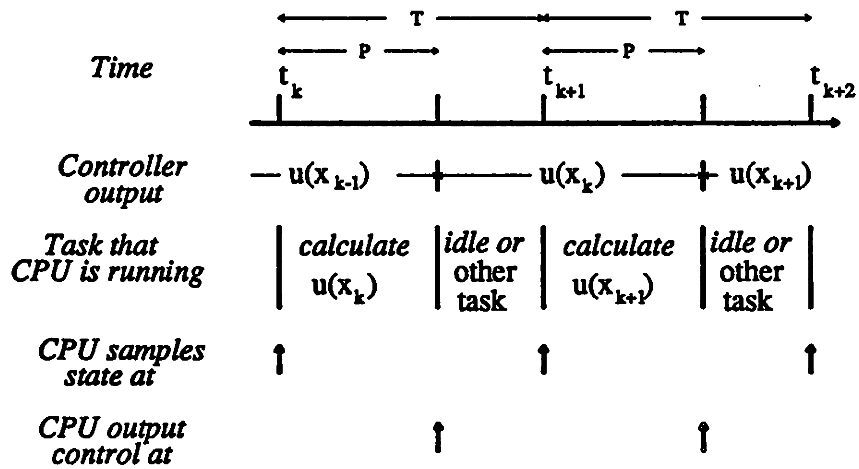

Figure 1

Figure (1) shows the sequence of events in a control cycle. At time t_k, the CPU starts to execute the control routine in response to the request generated by the timer. In the control routine, the CPU first reads the current state and, after some time ρ required to execute the control algorithm, the CPU outputs the control $u(x_k)$ to the plant and this $u(x_k)$ is held constant between time $t_k+\rho$ and $t_{k+1}+\rho$ while the CPU is either idle or running other tasks.

x_k can now be defined as follows:

$$x_k(t) \triangleq \begin{cases} 0 & \text{for } t_0 \le t < t_0 + \rho \\ x(t_k) & \text{for } t_k+\rho < t < t_{k+1}+\rho \text{ and } k \neq 0 \end{cases} \tag{2.3}$$

Due to the computer delay we just described, we need the following assumption on the *uncontrolled* system:

$$\dot{y} = f(y). \tag{2.4}$$

Let A denote the region of attraction of the equilibrium 0 of system (2.4).

(A1) Given a closed set B_1 and a bounded open set B_2 and $B_1 \subset B_2 \subset A$, there exists a $s(B_1, B_2) > 0$ such that, if the initial condition $y(t_o) \in B_1$, then the trajectory of (2.4) $y(t_o+t) \in B_2$, for all $t \le s(B_1,B_2)$.

Without this assumption, the *uncontrolled* system may escape out of the attraction region before the first control output $u(x_0)$ can be generated by the computer.

In the following sections, we will investigate the effect of such discretization and delay on the feedback loop.

3. Linear system with discretized feedback

This section gives an upper bound on the sampling interval. If the sampling interval is lower then this bound, the system with discretized feedback remains exponentially stable provided that the system with a continuous time controller is exponentially stable.

Consider the following linear system

$$\dot{x} = Ax + Bv \tag{3.1}$$

where $v=Fx$ is the stabilizing state feedback. The following theorem shows that if $v=Fx_k$ (as defined in (2.3)) and if ρ is so small that it can be neglected, the system remains stable for some properly chosen sampling interval $t_{k+1}-t_k$.

We propose two methods of estimating the sampling interval. The first method takes advantage of the fact that a linear differential equation has a closed form solution. The second method uses an approach which is similar to that described in section 4.

Theorem: (1)

Consider system (3.1) with $v=Fx$ is exponentially stable and computation time $\rho=0$. Then, there exists a $T>0$ such that, if $t_{k+1}-t_k<T$ for all k, the system with discretized feedback represented by

$$\dot{x} = Ax + BFx_k \tag{3.2}$$

is exponentially stable.

Proof of theorem 1 : (Method 1)

Since $A+BF$ is exponentially stable, given $Q>0$, there exists a $P>0$ satisfying the following Lyapunov equation

$$(A+BF)^T P + P(A+BF) = -Q \tag{3.3}$$

Let

$$V(t) = x^T(t)Px(t) \tag{3.4}$$

be a Lyapunov function for system (3.1). Then,

$$\dot{V}(t) = -x^T(t)Qx(t) + x(t)^T(P+P^T)BF(x_k - x(t)) \tag{3.5}$$

Between time t_k and t_{k+1}, $x(t)$ can be expressed as

$$x(t) = e^{A(t-t_k)}x_k + \int_0^{t-t_k} e^{A(t-t_k-s)}BFx_k\,ds \triangleq G(\tau)\,x_k \tag{3.6}$$

where $\tau \triangleq t-t_k$ and $G(\tau)$ is defined as

$$G(\tau) \triangleq e^{A\tau} + \int_0^{\tau} e^{A(\tau-s)}BF\,ds \tag{3.7}$$

Substituting equation (3.6) in (3.5), we get

$$\dot{V}(t) = -x_k^T G(\tau)^T Q G(\tau) x_k + x_k^T G^T(\tau)[P+P^T]BF[I-G(\tau)]x_k \tag{3.8}$$

$$\triangleq -x_k^T \hat{Q}(\tau) x_k$$

where $\hat{Q}(\tau)$ is defined as

$$\hat{Q}(\tau) \triangleq G(\tau)^T Q G(\tau) + G^T(\tau)[P+P^T]BF[I-G(\tau)]. \tag{3.9}$$

It is clear that $\hat{Q}(0)=Q$. In fact, by the continuity of a solution of a differential equation, there exists a T_a such that, for all $\tau<T_a$, $\hat{Q}(\tau)$ is positive definite. Therefore, if the sampling interval is kept less than T_a and $x_k \neq 0$, then $\dot{V}<0$, and using the fact that V is bounded below by 0, this implies that $\lim_{t\to\infty} V(t)=c$ where c is a non-negative constant. Suppose $c\neq 0$, then from (3.4), we have $\lim_{t\to\infty} \| x(t) \|^2 > c/\| P \| > 0$ and it follows from (3.6) that, for any fixed $\tau_a<T_a$, $\lim_{t\to\infty} \| x_k \| > c/(\| G(\tau_a) \|^2 \| P \|) \triangleq e > 0$. From (3.8), $\lim_{t\to\infty} \dot{V} < -\hat{q}e < 0$ where $\hat{q}$ is defined as the minimum singular value of $\hat{Q}(\tau_a)$. This contradicts the hypothesis that $\lim_{t\to\infty} V(t)=c>0$. Therefore, $\lim_{t\to\infty} V(t)=0$ and, hence, system (3.2) is asymptotically stable.

To show that the system is exponentially stable, we need $G(\tau)$ to be invertible. It is clear that $G(0)=I$. Thus, there exists a T_s such that, for all $\tau<T_s$, $G^{-1}(\tau)$ exists. From equation (3.6) and (3.8), for a given $\tau_o<\min\{T_s,T_a\}$, we have

$$\dot{V}(t) = -x^T G^{-1}(\tau_o)^T \hat{Q} G^{-1}(\tau_o) x < -\alpha \| x \|^2$$

where α is defined as the minimum singular value of $G^{-1}(\tau_o)^T \hat{Q} G^{-1}(\tau_o)$. Now, $\dot{V}$ is bounded above by a negative quadratic function of $\| x \|$. By Lyapunov theorem [11], system (3.2) is exponentially stable.

Q.E.D

In the above proof, for asymptotic stability, we limit the sampling interval to stay below an upper bound T_a to insure the positive definiteness of $\hat{Q}$. In fact, our *variable sampling interval scheme* is based on whether $x_k^T \hat{Q}(\tau) x_k > 0$ (i.e. $\dot{V}<0$) for each x_k rather than the positive definiteness of $\hat{Q}(\tau)$. The unit ball i.e. $\{ x \mid \| x \| = 1 \}$ is divided into segments. Then, a table is generated that records the minimum τ^*, over each segment, of the maximum τ satisfying $x^T \hat{Q}(\tau) x \geq \varepsilon > 0$. At time t_k, the sampling interval between t_k and t_{k+1} is determined depending on the segment which $x_k/\| x \|$ belongs to. Here we use the fact that if $(x/\| x \|)^T \hat{Q}(\tau)(x/\| x \|) > 0$, $x^T \hat{Q}(\tau) x > 0$. The following is an example of this approach.

Example:

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \tag{3.10}$$

with the state feedback u

$$u = \begin{bmatrix} -35 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The pair

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1.5881 & 0.0143 \\ 0.0143 & 0.0405 \end{bmatrix}$$

satisfies the Lyapunov function (3.3). In figure 2, the radius at angle $\tan^{-1}(x_2/x_1)$ represents the maximum τ such that $[x_1,x_2]\hat{Q}(\tau)[x_1,x_2]^T \geq \varepsilon > 0$. For example, suppose $x_k=[-3,5]$ (as shown in figure 2), the sampling interval $[t_k, t_{k+1}]$ can be extended to 0.29 second.

From figure 2, we see that, if the sampling interval is kept below 0.07 second, $\dot{V}$ is guaranteed to be negative. With the table look up scheme, the sampling interval may vary from 0.07 second to 0.7 second and still guarantee asymptotic stability. Figure (3) shows the response of the system with a square wave function input. Figure (4) shows the sampling clock. Note that the sampling interval varies from 0.07 to 0.25 second. In this *set point control* simulation, the state variable is $[x_1-v(t), x_2]^T$ where $v(t)$ is a square wave function and $x_1-v(t)$ is the set point error.

Proof of theorem 1 (Method 2)

The same Lyapunov function (3.3) is used in this approach. Let T_s be such a number that, for all $\tau < T_s$, $G(\tau)$ (defined in (3.7)) is invertible. If we choose $T < T_s$, we have

$$G(\tau)^{-1}x(t) = x_k \tag{3.11}$$

and

$$\| G(\tau)^{-1} - I \| \, \| x(t) \| \geq \| x_k - x(t) \| \tag{3.12}$$

Since $\| G(\tau)^{-1} - I \| \to 0$ as $\tau \to 0$, there exists a $\delta > 0$ such that

$$\| G(\tau)^{-1} - I \| < \frac{q}{\| (P+P^T)BF \|} \qquad \text{for all } \tau < \delta \tag{3.13}$$

where $q = \| Q^{-1} \|^{-1}$ i.e. the minimum singular value of Q. Now, take $T = \min(\delta, T_s)$. By equation (3.5), (3.12) and (3.13), for $t_k \leq t < t_{k+1}$, we have

$$\begin{aligned} \dot{V}(t) &\leq -q \| x(t) \|^2 + \| x(t) \| \, \| (P+P^T)BF \| \, \| x_k - x(t) \| \\ &\leq -(q - \| (P+P^T)BF \| \, \| G(\tau)^{-1} - I \| \,) \| x(t) \|^2 \\ &\triangleq -\lambda \| x(t) \|^2 \end{aligned} \tag{3.14}$$

where λ is defined as

$$\lambda \triangleq q - \| (P+P^T)BF \| \, \| G(\tau)^{-1} - I \| > 0 \tag{3.15}$$

Hence, $\dot{V}$ is bounded above by a negative quadratic function of $\| x \|$. By Lyapunov theorem [11], system (3.2) is exponentially stable.

Q.E.D.

The second method gives a more conservative bound on the sampling interval. For the example given earlier (3.10), the maximum sampling interval given by method 2 is 0.008 seconds compared with 0.07 second given by method 1.

Consider the case that $\rho \neq 0$ i.e. the time required to execute the control algorithm is not zero. At $t=t_k$, we can exactly predict the state at $t_k+\rho$ by the following equation.

$$\begin{aligned} x(t_k+\rho) &= e^{A\rho}x_k + \int_0^{\rho} e^{A(t-t_k-\tau)}BFx_{k-1}\, d\tau \\ &\triangleq \Phi(x_{k-1}, x_k) \end{aligned} \tag{3.16}$$

Then, after sampling x_k at t_k, the control u can be generated as $F\Phi(x_{k-1},x_k)=Fx(t_k+\rho)$ and this control is output to the plant at exactly $t=t_k+\rho$. The following theorem can be proved by either method suggested earlier and, therefore, is omitted.

Theorem: (2)

Assume system (3.1) with $v=Fx$ is exponentially stable. There exists a $T>0$ such that if $t_{k+1}-t_k<T$ for all k, the system with discretized feedback represented by

$$\dot{x}=Ax+BF\,\Phi(x_{k-1},x_k)$$

is exponentially stable.

4. Non-linear systems with discretized feedback

As mentioned in the section (1), an exact discretization of a general nonlinear system does not exist. Consequently, one way of designing a nonlinear controller is to, first, ignore the fact that the control law will be implemented in a discretized fashion and, then choose a sampling frequency so that this system will behave similar to the system with a true continuous time controller. In this section, for a class of nonlinear systems, we give a lower bound on the sampling frequency to maintain stability.

In the proof of the following theorems, we required a Lyapunov function for the system with true continuous time controller i.e. system (2.1) which is assumed to be stable in some sense. For this system, a Lyapunov converse theorem [11] guarantees the existence of a Lyapunov function but does not provide it in an explicit form. If the system (2.1) is proved to be stable using a Lyapunov function in the first place, this same function can be used in our analysis. Furthermore, one of the popular design approaches [3] [9] is to exactly linearize system (2.1) by state feedback. Then a controller is designed based on this linearized system. In this case, a Lyapunov function is ready obtained from the linearized system. An example of such a design approach is the *computed torque* controller [10] for a mechanical manipulator. Nevertheless, the converse theorem is quoted in the following proofs.

A Lyapunov converse theorem states that, for every asymptotically stable equilibrium point, there exists a $V(.)$ such that $V(x)$ and $-\dot{V}(x)$ are positive definite over some region containing the equilibrium point. In the case of an exponentially stable equilibrium point, $V(.)$ has the following properties.

$$\dot{V}(x)\le -a\|x\|^2 \qquad \text{for some } a>0 \tag{4.1}$$

$$\left\|\frac{\partial V}{\partial x}\right\|\le b\,\|x\| \qquad \text{for some } b>0 \tag{4.2}$$

We need the following lemma in the proof of theorem (3)

Lemma: (1)

For a given $x_o\in\mathbb{R}^n$, we define the set B as

$$B\triangleq\{x \mid \|x_o-x\|\le c\,\|x_o\|\}$$

where c is a fixed constant and $0<c<1$. Then, there exists a d such that

$$d\,\|x\|\ge\|x_o-x\|$$

for all $x\in B$ and $d(c)\to 0$ as $c\to 0$.

Proof:

It is obvious that $(1-c)x_o=\text{argmin}\,\{\|x\| \mid x\in B\}$ i.e. $(1-c)x_o$ is the minimum norm element in B. Therefore,

$$\|x\|\ge(1-c)\|x_o\| \qquad \text{for all } x\in B.$$

Take $d=c/(1-c)$, then, for all $x\in B$,

$$d \| x \| = \frac{c}{(1-c)} \| x \| \geq c \| x_o \| \geq \| x_o - x \|$$

Q.E.D.

The following theorem can be thought of as a generalization of theorem (1).

Theorem: (3)

Let 0 be a exponentially stable equilibrium point of system (2.1) with attraction region A and let $V(.)$ be a Lyapunov function for 0. Assume that assumption (A1) and the following conditions are satisfied.

A2) Equations (4.1) and (4.2) are satisfied in a region B.

A3) $f(.)$ has Lipschitz constant F in B.

A4) $\| g(x) \| \leq G$ for all $x \in B$.

A5) $u(.)$ has Lipschitz constant U in B.

Then, there exists a T, P such that, if $t_{k+1}-t_k<T$ and $\rho<P$, Then, system (2.2) has one of the following properties.

I) If $A=B=\mathrm{R}^n$, there exists T, P such that if, $t_{k+1}-t_k<T$ and $\rho<P$; 0 is the globally exponentially stable equilibrium point.

II) If $B \neq \mathrm{R}^n$, for any set in $B \cap A$ that can be expressed as $\{x \mid V(x)<\lambda\}$ for some $\lambda>0$, there exists T, P such that if, $t_{k+1}-t_k<T$ and $\rho<P$, $\{x \mid V(x)<\lambda\}$ is contained in the attraction region of 0 of system (2.2).

Proof:

let V be a Lyapunov function such that equations (4.1) and (4.2) are satisfied. Consider the $\dot{V}(t)$ along the trajectory of system (2.2),

$$\dot{V}(t) = \frac{\partial V}{\partial x}(f(x)+g(x)u(x_k)) \tag{4.3}$$

$$= \frac{\partial V}{\partial x}[f(x)+g(x)u(x)] + \frac{\partial V}{\partial x} g(x)[u(x_k)-u(x)]$$

$$\leq -a \| x \|^2 + bGU \| x \| \| x_k - x \|$$

where a and b are as defined in (4.1) (4.2). Between $t_k+\rho$ and $t_{k+1}+\rho$, $x(t)$ can be expressed as

$$x(t) = x_k + \int_{t_k}^{t_k+\rho} [f(x)+g(x)u(x_{k-1})]d\tau + \int_{t_k+\rho}^{t} [f(x)+g(x)u(x_k)]d\tau \tag{4.4}$$

hence,

$$\| x(t)-x_k \| = \int_{t_k}^{t} \| g(x)u(x_k) \| d\tau + \int_{t_k}^{t_k+\rho} \| g(x)[u(x_{k-1})-u(x_k)] \| d\tau$$

$$+ \int_{t_k}^{t} \| f(x_k) \| d\tau + \int_{t_k}^{t} \| f(x)-f(x_k) \| d\tau$$

$$\leq (T+P)GU \| x_k \| + PGU \| x_k - x_{k-1} \|$$

$$+ (T+P)F \| x_k \| + \int_{t_k}^{t} F \| x(t)-x_k \| d\tau.$$

Suppose

$$\| x_k - x_{k-1} \| \leq \eta \| x_k \| . \tag{4.5}$$

This η will be determined later in the proof. Then we have

$$\| x(t) - x_k \| \leq [(T+P)(GU+F)+PGU\eta] \| x_k \| + \int_{t_k}^{t} F \| x(\tau) - x_k \| \, d\tau .$$

By the Bellman-Gronwall inequality, we have

$$\begin{aligned} \| x(t) - x_k \| &\leq [(T+P)(GU+F)+PGU\eta] e^{FT} \| x_k \| \\ &\triangleq c(T,P,\eta) \| x_k \| \end{aligned} \tag{4.6}$$

Suppose T, P are so chosen that $c(T,P,\eta)<1$. By lemma (1), there exists a $d(T,P,\eta)$ such that

$$\| x(t) - x_k \| \leq d(T,P,\eta) \| x(t) \| . \tag{4.7}$$

Note that the above equation is equivalent to equation (3.12) and to ask $c<1$ (so lemma 1 is satisfied) is similar to ask $G(\tau)$ to be invertible in (3.12). From the above equation and (4.3), we have

$$\dot{V} \leq [-a + bGUd] \| x \|^2 \tag{4.8}$$

Since, for any constant η, $d(T,P,\eta) = \frac{c}{1-c} \to 0$ as $P, T \to 0$, we shrink T, P further, if necessary, such that

$$\frac{a}{bGU} > d. \tag{4.9}$$

Then, with this T, P, $\dot{V}$ along the trajectory of system (2.2) is bounded above by a negative quadratic function of $\| x \|$. This implies that system (2.2) is global exponentially stable. Now, the question is whether such a η exists. Consider the state $x(t)$ in the first sample interval i.e. $t \in [t_0, t_1]$.

$$x(t) = x(t_0) + \int_{t_0}^{t_0+p} f(x) d\tau + \int_{t_0+p}^{t} f(x) + g(x) u(x_0) d\tau$$

Similar to the argument given earlier (between (4.4) and (4.6)), by choosing sufficiently small P, T, we have the following relation.

$$\| x_0 - x_1 \| \leq \eta_o(P, T) \| x_1 \| \tag{4.10}$$

With this η_o, we can shrink T, P further, if necessary, such that

$$0 < d(T,P,\eta_o) < \min \left\{ \eta_o , \frac{a}{bGU} \right\}$$

With this T, P, equation (4.5), (4,7), and (4.9) are satisfied simultaneously.

The proof for property (II) is similar.

Q.E.D.

As mentioned earlier, if the system is asymptotically stable (instead of exponentially stable) the Lyapunov converse theorem does not guarantee equation (4.1) and (4.2). The following theorem gives a local convergence property without assuming the exponential stability of the original system.

Theorem: (4)

Consider a nonlinear system represented by equation (2.1). Let 0 be the globally asymptotically stable equilibrium point of the system. Assume that assumptions (A1), A(5) are satisfied. Given B_1, B_2, $0 \in B_1 \subset B_2$ and B_2 is bounded. There exists a $T > 0$ such that, if $t_{k+1} - t_k < T$, all the trajectories of system (2.2) starting from B_2 converge to B_1.

Proof:

Let V be a Lyapunov function of system (2.1). For given B_1, B_2, we construct sets L_1 and L_2 as follow. $0 \in L_1 = \{ x \mid V(x) < c \} \subset B_1$ for some $c > 0$. $L_2 = \{ x \mid V(x) < d \} \supset B_2$ where $d = \sup_{x \in B_2} V(x)$. and let $W = \{ x \mid V(x) < e \}$ where $e > d$. For system (2.2), $\dot{V}(x)$ can be represented by the following equation.

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} \tag{4.11}$$

$$= \frac{\partial V}{\partial x} (f(x) + g(x)u(x)) + \frac{\partial V}{\partial x} g(x)(u(x_k) - u(x))$$

The first term of the above equation is a negative definite function. Let

$$-\alpha \triangleq \sup_{x \in (W - L_1)} \frac{\partial V}{\partial x} (f(x) + g(x)u(x)) < 0 \tag{4.12}$$

$$\beta \triangleq \sup_{x \in W} \| \frac{\partial V}{\partial x} g(x) \| \tag{4.13}$$

and

$$\xi \triangleq \sup_{x \in W} \| f(x) + g(x)u(x) \| . \tag{4.14}$$

Let u be the Lipschitz constant of $u(.)$ over W. Now choose T such that

$$T < \min \left\{ \frac{\alpha - \delta}{2\xi\beta U}, s \right\} \tag{4.15}$$

where $\delta > 0$, $\alpha - \delta > 0$ and $s(L_2, W)$ is as defined by (A1). We now show that, with this T, the system indeed has the property claimed in the theorem.

Between time $t_k + \rho$ and $t_{k+1} + \rho$, $x(t)$ can be expressed as

$$x(t) = x_k + \int_{t_k}^{t} f(x) + g(x)u(\tau) \, d\tau \tag{4.16}$$

where

$$u(\tau) = \begin{cases} u(x_{k-1}) & \text{if } t_k \le \tau < t_k + \rho \\ u(x_k) & \text{if } t_k + \rho \le \tau < t_{k+1} + \rho^{\cdot} \end{cases}$$

Then, from equation (4.11), (4.15), (4.16) and the relation $T + P \le 2T$, we have

$$\dot{V}(x(t)) \le \frac{\partial V}{\partial x} (f(x) + g(x)u(x)) + \beta U \| x(t) - x_k \| \tag{4.17}$$

$$\le \frac{\partial V}{\partial x} (f(x) + g(x)u(x)) + \beta U \int_{t_0}^{t} \| f(x) + g(x)u(\tau) \| \, d\tau$$

$$\le -\alpha + \beta U \xi (T + P) \qquad \text{if } x(t) \in W - L_1$$

$$< -\delta \qquad \text{if } x(t) \in W - L_1$$

The last inequality implies that, for any trajectory starting from $B_2 - L_1$, eventually reaches L_1 and trajectories starting from L_1 stay in L_1.

Q.E.D.

5. Non-linear System with discretized dynamical feedback

Consider a nonlinear plant with nonlinear dynamical feedback represented by

$$\dot{x} = f(x) + g(x)z \tag{5.1}$$
$$\dot{y} = h(y) + j(y)v(x)$$
$$z = p(x,y)$$

where the last two equations are the controller. There are many numerical integration methods that can be used to implement such a dynamical control law. But, by far, the easiest and most popular numerical integration method is the forward Euler method. The following theorem states that, by using the forward Euler method, the closed loop system represented by

$$\dot{x} = f(x) + g(x)z_k \tag{5.2}$$
$$y_k = \Delta t\Big[h(y_{k-1}) + j(y_{k-1})v(x_{k-1})\Big] + y_{k-1}$$
$$z_k = p(x_{k-1}, y_k)$$

has properties similar to theorem (4). For simplicity, we assume $\rho = T$ i.e. $x(t_k)$ is read at t_k and the control z_{k+1} is the output at exactly t_{k+1}. This is the case when a CPU is dedicated to the real time control task.

We use the following notation in the following theorem where $V(x,y)$ is a Lyapunov function for the asymptotically stable system (5.1). Note that the state variable of (5.1) is (x,y).

$$V_x(x_1,x_2,y) = \frac{\partial V}{\partial x}\Big|_{(x_1,y)}\Big[f(x_1) + g(x_1)p(x_2,y)\Big]$$
$$V_y(x_1,x_2,y) = \frac{\partial V}{\partial y}\Big|_{(x_1,y)}\Big[h(y) + j(y)v(x_2)\Big]$$
$$H(x,y) = V_x(x,x,y) + V_y(x,x,y)$$

Theorem: (5)

Let 0 denote the globally asymptotically stable equilibrium point of system (5.1) and let $V(x,y)$ be a Lyapunov function for 0 and, hence, $V, -\dot{V}$ are positive definite. Assuming that the following conditions are satisfied.

(A6) $V_x(x_1,x_2,y)$ has Lipschitz constant $l_1(W)$ in (x_1,x_2) over any given bounded set W.

(A7) $V_y(x_1,x_2,y)$ has Lipschitz constant $l_2(W)$ in x_1 over any given bounded set W.

Given B_1, B_2, $0 \in B_1 \subset B_2$ where B_2 is bounded. Then, there exists a T such that, if $t_{k+1} - t_k < T$, all the trajectories of system (5.2) starting from B_2 converge to B_1.

Proof:

We construct W, L_1, L_2 exactly the same way as in in the proof of theorem (4) i.e.

$$L_1 = \{ (x,y) | V(x,y) < c \}$$

$$L_2 = \{ (x,y) | V(x,y) < d \}$$

$$W = \{ (x,y) | V(x,y) < e \}$$

where $e > d > c$ and $L_1 \subset B_1 \subset B_2 \subset L_2$. Define

$$\xi \triangleq \sup_{(x,y)\in W} (f(x)+g(x)u(y))$$

$$-\alpha \triangleq \sup_{(x,y)\in W-L_1} H(x,y)$$

Since V is a Lyapunov function of 0 of system (5.1), $\dot{V}=H$ is a negative definite function. Hence, $\alpha>0$. Consider the function $V(.)$ along the trajectory of system (5.2).

$$\frac{V^{k+1}-V^k}{\Delta t} = \frac{1}{\Delta t}\left\{ \int_{t_k}^{t_{k+1}} V_x(x,x_{k-1},y_k)dt + V(x_{k+1},y_{k+1}) - V(x_{k+1},y_k) \right\}$$

$$= \frac{1}{\Delta t}\left\{ \int_{t_k}^{t_{k+1}} V_x(x,x_{k-1},y_k)dt + \frac{\partial V}{\partial y}|_{(x_{k+1},y_k)}[h(y_k)+j(y_k)v(x_k)]\Delta t + O(\Delta t^2) \right\}$$

$$= \frac{1}{\Delta t}\left\{ \int_{t_k}^{t_{k+1}} \left[V_x(x,x_{k-1},y_k) + \frac{\partial V}{\partial y}|_{(x_{k+1},y_k)}(h(y_k)+j(y_k)v(x_k)) \right] dt + O(\Delta t^2) \right\}$$

$$= \frac{1}{\Delta t}\left\{ \int_{t_k}^{t_{k+1}} H(x_k,y_k)dt + \int_{t_k}^{t_{k+1}} V_x(x,x_{k-1},y_k) - V_x(x_k,x_k,y_k) + V_y(x_{k+1},x_k,y_k) \right.$$

$$\left. - V_y(x_k,x_k,y_k)dt + O(\Delta t^2) \right\} \tag{5.3}$$

From the above equation, if $(x, y_k) \in W - L_1$, then

$$\frac{V^{k+1}-V^k}{\Delta t} \le -\alpha + l_1 \| (x,x_{k-1})-(x_k,x_k) \| + l_2 \| x_{k+1}-x_k \| + O(\Delta t)$$

$$\le -\alpha + l_1 \| x-x_k \| + l_1 \| x_{k-1}-x_k \| + l_2 \| x_{k+1}-x_k \| + O(\Delta t) \tag{5.4}$$

where $V^k \triangleq V(x(t_k),y_k)$, $l_1=l_1(W)$, and $l_2=l_2(W)$. Furthermore, if $(x, y_k) \in W - L_1$, then

$$\| x-x_k \| = \int_{t_k}^{t_{k+1}} \| f(x)+g(x)u(y_k) \| \, dt \le \xi \Delta t = O(\Delta t) \tag{5.5}$$

and, hence,

$$\| x_{k-1}-x_k \| = O(\Delta t) \tag{5.6}$$

$$\| x_{k+1}-x_k \| = O(\Delta t). \tag{5.7}$$

From (5.6), (5.7), and (5.4), we conclude that, if $(x, y_k) \in W - L_1$, then

$$\frac{V^{k+1} - V^k}{\Delta t} \le -\alpha + O(\Delta t)$$

The above inequality implies that if $(x(t_k), y_k) \in W - L_1$, for sufficiently small Δt, V^k is a strictly decreasing sequence and, hence, $(x(t_i), y_i) \in L_1$ for some i. Now it remains to show that

(1) If $(x(t_0)$, then $y_0) \in B_2$, $(x(t), y_k) \in W$ for all t, k.

(2) Once $(x(t), y_k)$ reaches L_1, then $(x(t), y_k)$ stays in B_1 there after.

Proof(1): Assumption (A1) guarantees the existence of a T such that, if $(x(t_0), y_0) \in B_2$, $(x(t_1), y_1) \in W$. However, between t_1 and t_2, V can increase. From (5.5) and (4.2), we see that, by choosing a sufficiently small T, we can keep $\| x(t) - x(t_1) \|$ so small that $V(x(t), y_1) - V(x(t_1), y_1) < e - V(x(t_1), y_1)$. This implies that $(x(t), y_1) \in W$ for all $t \in [t_1, t_2]$. Since $V(x(t_2), y_2) < V(x(t_1), y_1)$, the same choice of T will guarantee that $(x(t), y_2) \in W$ for all $t \in [t_2, t_3]$. Similarly, we can show that $(x(t), y_k) \in W$ for all t, k.

Proof(2): As shown in the proof(1), $V(x(t), y_k) - V(x(t_k), y_k)$ $(t \in [t_k, t_{k+1}])$ can be kept as small as necessary by keeping T small if $(x, y_k) \in W$. Therefore, if necessary, we can shrink T and/or L_1 to make sure that, once $(x(t_i), y_i) \in L_1$, $(x(t), y_k)$ will stay in B_1. Note that it is possible, for some $j > i$, $(x(t_j), y_j)$ does not belong to L_1 (but belong to B_1). In this case, the sequence V^k $k = j, j+1, \cdots$ will keep decreasing until $(x(t), y_k) \in L_1$ again.

Q.E.D.

6. The Effect of Truncation Error

Truncation error is unavoidable in a digital computer due to finite word length. We represent system (2.2) with the consideration of truncation error as

$$\dot{x} = f(x) + g(x)\hat{u}(\hat{x}_k) \tag{6.1}$$

where $\hat{x}_k$ is the digitized version of x_k and $\hat{u}$ is the function with truncation error due to computation. To simplify the analysis, we lump all these digitization errors into $\bar{x}_k$ i.e.

$$\dot{x} = f(x) + g(x)u(\bar{x}_k) \tag{6.2}$$

where

$$\| \bar{x}_k - x_k \| < \varepsilon. \tag{6.3}$$

The following theorems shows that all the theorems in the earlier sections suffer from the truncation error.

Theorem: (6)

Assume that system (3.1) with $v = Fx$ is exponentially stable. If $\rho = 0$, there exists a $T > 0$ such that, if $t_{k+1} - t_k < T$ for all k, all the trajectories of system (6.2) converge to a ball $B = \{ x \mid \| x \| \le c \}$ for some $c > 0$.

Proof:

Similar to equation (3.6), the following equation represents $x(t)$ for $t_k \le t < t_{k+1}$.

$$x(t) = e^{A(t-t_k)}x_k + \int_0^{t-t_k} e^{A(t-t_k-\tau)}BF\bar{x}_k \, d\tau \tag{6.4}$$

$$= \left[e^{A(t-t_k)} + \int_0^{t-t_k} e^{A(t-t_k-\tau)}BF \, d\tau \right] \bar{x}_k + e^{A(t-t_k)}(x_k - \bar{x}_k)$$

$$= G(t)\bar{x}_k + e^{A(t-t_k)}(x_k - \bar{x}_k)$$

where $G(t)$ is defined by equation (3.7). For the same reason stated earlier, $G(t)$ can be kept from becoming singular. So we have

$$G^{-1}x(t) = \bar{x}_k + G^{-1}e^{A(t-t_k)}(x_k - \bar{x}_k) \tag{6.5}$$

and

$$(G^{-1} - I)x(t) - G^{-1}e^{A(t-t_k)}(x_k - \bar{x}_k) = \bar{x}_k - x(t) \tag{6.6}$$

We now chose T exactly the same way as described in the proof (method 2) of theorem (1). From the above equation and similar to equation (3.14), we have

$$\dot{V}(t) \le -\lambda \| x \|^2 + \zeta \| x \| \tag{6.7}$$

where λ is defined as in (3.15) and ζ is defined as

$$\zeta = \frac{1}{2} \| (P+P^T)BF \| \; \| G^{-1}e^{A(t-t_k)} \| \; \| x_k - \bar{x}_k \| \tag{6.8}$$

Equation (6.8) implies that $\dot{V}(x) < 0$ for $x \in \{x \mid \| x \| < \frac{\zeta}{\lambda}\}$ and this implies that all the trajectories converge to $B = \{x \mid \| x \| \le \| P \| \frac{\zeta}{\lambda}\}$. More precisely, all the trajectories converge to the set $\{x \mid x^T P x < \frac{\zeta}{\lambda}\}$.

Q.E.D.

The above theorem can also be proven using method 1 as in section 3. The following theorem shows that, with truncation error, the discretized feedback system we studied in theorem (3) is no longer exponentially stable.

Theorem:(7)

With truncation error, all the trajectories of system (2.2) in Theorem (3) converge to a region which contains the equilibrium point of the original system (2.1).

Proof:

Similar to equation (4.7), we have

$$\dot{V} \le [-a + bGUd\,] \| x \|^2 + bGU \| x \| \varepsilon.$$

Since the first term eventually dominates the second term as $\| x \| \to \infty$, this inequality implies that all trajectories converge to a neighborhood of the original equilibrium point.

Q.E.D.

The following theorem shows that, with the truncation error, theorem (4) is no longer valid for arbitrary B_1, B_2 unless the ε (as defined in (6.3)) can be assumed to be sufficiently small. But, ε is not a design variable. Hence, there may or may not exist such a T that guarantees the property claimed in the theorem. In the proof of the following theorem, we show that, by sacrificing the 'size' of the *attraction region*

$B_2 - B_1$, we increase the 'chance' of existing such a T.

Theorem: (8)

The theorem (4) is true for sufficiently small ε.

Proof:

Similar to equation (4.15), we take

$$0 < T < \min\left\{ \frac{1}{2\xi}\left(\frac{\alpha-\delta}{\beta U} - \varepsilon\right), s \right\}. \tag{6.9}$$

Then, similar to equation (4.17),

$$\begin{aligned}
\dot{V}(x(t)) &\le \frac{\partial V}{\partial x}(f(x) + g(x)u(x)) + \beta U \, \| x(t) - \tilde{x}_k \| \\
&\le \frac{\partial V}{\partial x}(f(x) + g(x)u(x)) + \beta U \, \| x(t) - x_k \| \; + \beta U \varepsilon \\
&\le -\alpha + \beta U(\xi(T+P) + \varepsilon) \qquad \text{if } x(t) \in W - L_1 \\
&< -\delta \qquad \text{if } x(t) \in W - L_1
\end{aligned}$$

It is clear that inequality (6.9) can be satisfied by some T only if

$$\varepsilon < \frac{\alpha - \delta}{\beta U}$$

which is always possible if ε can be assumed sufficiently small. In this case, this theorem can be proved by using the same argument as in the proof of theorem (4). Furthermore, the above inequality is 'easier' to satisfy if the ratio $\frac{\alpha}{\beta}$ is bigger which means that B_2 become smaller and/or B_1 become larger i.e. the attraction region $B_2 - B_1$ is smaller.

Q.E.D.

7. Summary

We have shown that, a system with a discretized feedback and with a properly chosen sampling frequency

(C1) is exponentially stable if the system is linear (theorem (1,2)).

(C2) is asymptotically stable if the system is nonlinear and satisfies some assumptions (theorem (3)).

(C3) is locally stable in the sense that, all the trajectories starting from a bounded area converge to a small neighborhood of the equilibrium point if the system satisfies some assumptions (theorem (4,5)).

We also show that, with the consideration of truncation error, a linear closed loop system, regardless of the choice of the sampling frequency, is no longer exponentially stable. Instead, a weaker form of stability is proved (theorem 6). In the nonlinear case, theorem (7) showed that, if we can assume the truncation effect to be sufficiently small, the closed loop system will have property (C3). Otherwise (this is usually the case), no stability is guaranteed.

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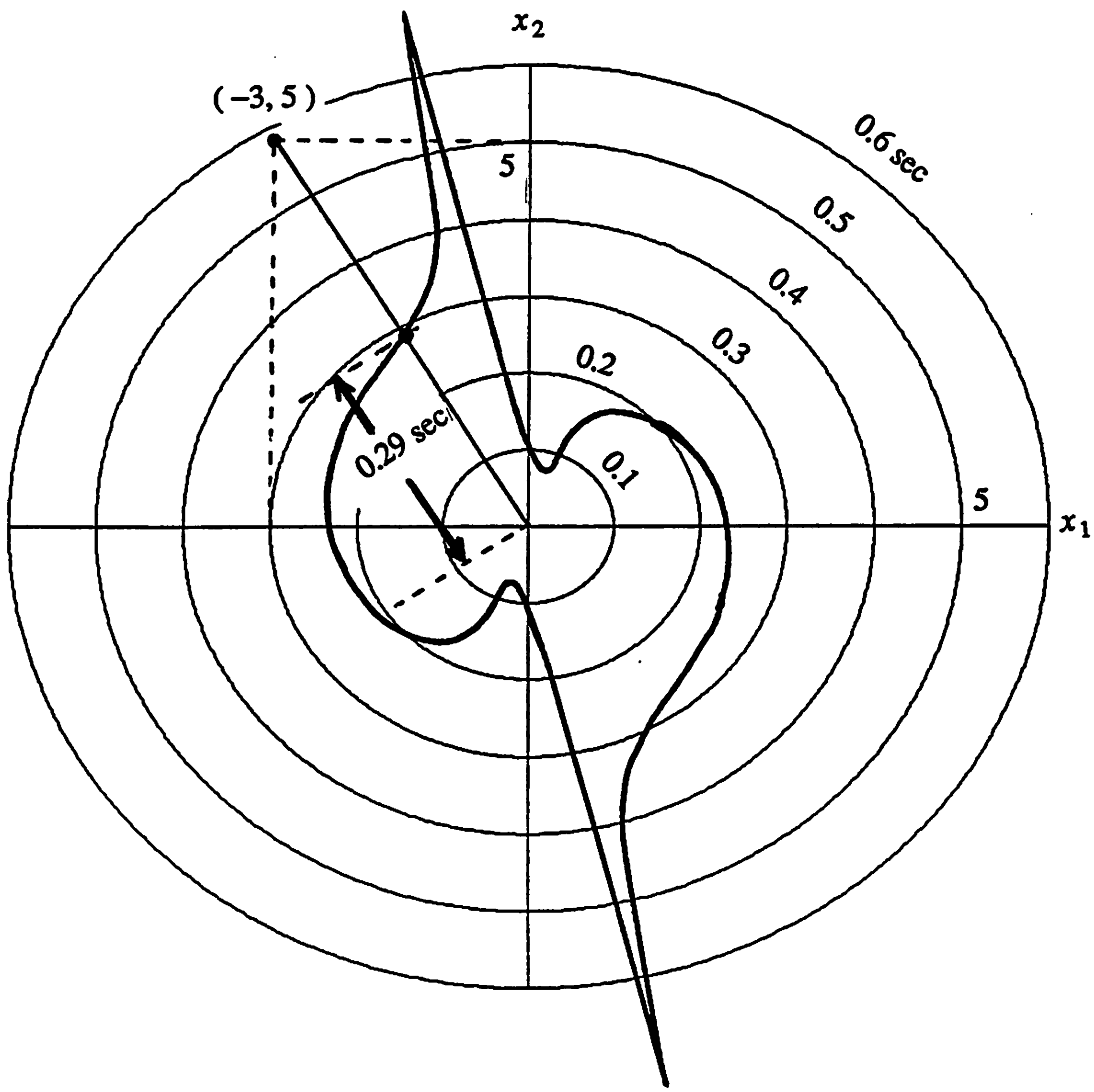

Figure 2

Figure 3.a (X1)

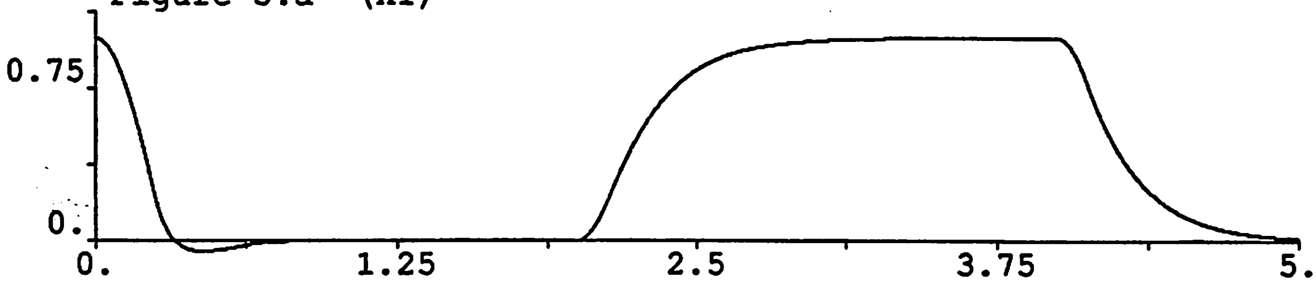

Figure 3.b (X2)

Figure 4 (Sample clock)