GRAPHICAL *U*-HURWITZ TESTS FOR A CLASS OF POLYNOMIALS: A GENERALIZATION OF KHARITONOV'S STABILITY THEOREM

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ABSTRACT

In this paper we derive several graphical U-Hurwitz tests for certain classes of polynomials. The classes may be defined in terms of linear equality and inequality constraints on the polynomial coefficients, and the undesirable set U may be any closed subset of the complex plane. The analysis is motivated by the proof of Kharitonov's Stability Theorem.

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1. INTRODUCTION

In 1978 V. L. Kharitonov [1] published a stability theorem for classes of polynomials defined by letting each coefficient vary independently in a specified (but arbitrary) interval. This remarkable result states that the whole class of polynomials is Hurwitz if and only if *four* special, well-defined polynomials are Hurwitz. The original proof has been considerably simplified and extended by many authors; [2] provides a good starting point, with additional references provided there.

In this paper, we use the *analytical methods* used in [2], together with some convexity arguments, to prove a vast generalization of Kharitonov's Theorem. The class of polynomials is generalized to polytopes in coefficient space, so that arbitrary linear dependencies can be considered. The set of acceptable polynomial zeroes is generalized to arbitrary open subsets of $\mathbb{C}$, which allows various stability and performance criteria to be considered, for both continuous time and discrete time systems. The results are non-conservative.

The drawback is that the tests are no longer finite (compare with Kharitonov's four polynomials!). Indeed, we provide two graphical tests. The first is a root locus type test, while the second is a Nyquist type test. Finally, we extend the root locus type result to provide a method for determining precisely the domain of polynomial zeroes for the polytopic class of polynomials mentioned above.

The key idea in the proofs is to consider the image of polynomials under the evaluation map; that is, to consider the codomain of polynomial functions. Thus we explore the relationship among three spaces: the coefficient space of polynomials, the space $\mathbb{C}$ of possible zero locations (the domain of polynomial functions), and the space $\mathbb{C}$ of polynomial evaluations (the codomain). This analytical method was first applied to the Kharitonov problem by Dasgupta [3], and was used extensively in [2].

2. THEORETICAL RESULTS

We denote by P^n the $n+1$ dimensional space of n^{th} order polynomials

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

parameterized by coefficients. Given a finite set of polynomials $K = \{k_1(\cdot), \ldots, k_m(\cdot)\} \subset P^n$, we consider the class N of polynomials which are convex combinations of $k_1(\cdot), \ldots, k_m(\cdot)$; i.e. $N = \text{co}(K)$. We denote by $\text{Ed}(N)$ the edges of the polytope N. We restrict our attention to polytopes N which do not intersect the subspace of P^n defined by $a_n = 0$; that is, we consider only sets K of polynomials whose n^{th} degree coefficients all have the same sign.

Remark: In particular, N might be a parallelepiped, so that each coefficient is allowed to vary independently in a fixed interval. But N may actually be any polytope in $\mathbb{R}^{n+1}$ (the coefficient space); i.e. any bounded set defined by a finite number of linear equality and inequality constraints-- so long as $a_n \neq 0$ on N. Thus affine dependency of coefficients is allowed.

For all $s \in \mathbb{C}$ we denote the evaluation map from P^n to $\mathbb{C}$ by $e_s(\cdot)$; i.e. $e_s(p)=p(s)\ \forall\ p \in P^n$. Since

$$e_s(p) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

we see that e_s is linear (in the coefficients of $p(\cdot)$). Finally, we define $H(s) := e_s(N) = \{p(s) : p \in N\}$. Because e_s is linear, $e_s(\cdot)$ and co$(\cdot)$ commute (as set maps), and we have

$$H(s) = e_s(\text{co}(K)) = \text{co}(e_s(K)) = \text{co}\{k_1(s), \ldots, k_m(s)\}.$$

So $H(s)$ is a convex polygon in $\mathbb{C}$. The following technical lemma may be obvious from geometric intuition, and the proof is left to the Appendix.

Technical Lemma: $\partial H(s) \subset e_s(\text{Ed}(N))$.

The result says that any complex number in an edge of the polygonal image of N is the image of a polynomial in an edge of N (see Figure 1). (It certainly does *not* say that *every* polynomial in Ed(N) maps into $\partial H(s)$, nor that $e_s^{-1}(\partial H(s))$ contains *only* edges.)

Finally, we consider closed subsets U of the complex plane which are "undesirable" or "forbidden" sets of polynomial zero locations. Thus we say that a polynomial $p(\cdot)$ is <u>U-Hurwitz</u> if and only if $p(s) \neq 0\ \forall s \in U$. The main result is the following theorem.

Theorem: Suppose $U \subset \mathbb{C}$ with $\partial U \subset U$ (i.e. U closed), and with N, Ed(N) and $H(s)$ as defined above. Then the following three conditions are equivalent:

1. N is U-Hurwitz;
2. (a) Ed(N) is U-Hurwitz, and
   (b) in each component of ∂U there is some $\hat{s}$ with $0 \notin H(\hat{s})$;
3. (a) $0 \notin H(s)\ \forall s \in \partial U$, and
   (b) there is some $p \in N$ with $p(\cdot)$ U-Hurwitz.

Proof: <u>1⇒2 and 3:</u> Suppose N is Hurwitz. Clearly 2(a) and 3(b) are satisfied. Now if $s \in \partial U$ and $0 \in H(s)$, then there is a polynomial $p(\cdot) \in N$ with $p(s)=0$. Thus $p(\cdot)$ has a zero in $\partial U \subset U$ and is not U-Hurwitz, which contradicts the assertion that N is U-Hurwitz. Thus $0 \notin H(s)\ \forall s \in \partial U$.

<u>2⇒3:</u> First we note that $H(s)$ moves continuously with s (i.e. $k_i(s)$ is continous, and $H(s) = \text{co}\{k_1(s), \ldots, k_m(s)\}$). Considering any component C of ∂U, from 2(b) there is an $\hat{s}$ in

C with $0 \notin H(\hat{s})$. Since C is connected, if $0 \in H(\tilde{s})$ for some $\tilde{s}$ in C, then there must be some $\bar{s}$ in C with $0 \in \partial H(\bar{s}) \subset e_s(\mathrm{Ed}(N))$, so that Ed($N$) is not U-Hurwitz, contradicting 2(a). Thus no such $\tilde{s}$ exists.

3⇒1: 3(b) provides a u-Hurwitz $p \in N$. N is clearly pathwise connected, and the zeroes of a polynomial vary continuously with respect to its coefficients when the leading coefficient is bounded away from zero. So suppose $\hat{p} \in N$ with $\hat{p}(\cdot)$ not U-Hurwitz. Then on any path in N connecting $\hat{p}(\cdot)$ and $p(\cdot)$, there is another polynomial $\tilde{p}(\cdot) \in N$ with a zero in ∂U, say at $\tilde{s}$. But this implies that $\tilde{p}(\tilde{s})=0$ so that $0 \in H(\tilde{s})$, which contradicts 3(a).

Although the condition 2(b) is easy enough to check for reasonable choices of U, 2(b) can be eliminated if U is pathwise connected and unbounded. The proposition below is proved by constructing a set $U^* \subset U$ which satisfies 2(a) and 2(b). The details are relegated to the Appendix.

Proposition: Suppose $U \subset \mathbb{C}$ is closed, pathwise connected and unbounded. Then condition 2(a) in the Theorem is equivalent to 2(a) and 2(b) (and thus is equivalent to 1 and 3).

3. VERIFICATION OF U-HURWITZ PROPERTIES -- GRAPHICAL TESTS

Condition 2 of the Theorem, or the subsequent Proposition, immediately yields a reduced test for checking that N is U-Hurwitz. Instead of searching the whole (possibly n-dimensional) polytope N for a non-U-Hurwitz polynomial, we search only its one-dimensional edges.

In addition to reducing the computational requirement to perform the test, we obtain a testing procedure which essentially indicates to the engineer when she has checked "enough" polynomials; namely, she can plot the zero-locus of each edge, checking to see if it crosses into U. She can continue filling in points until the locus "looks" continuous, and all the successive increments are much smaller than the distance of the locus to ∂U.

Of course, N may have very many edges (a k-dimensional parallelepiped has $n 2^{n-1}$, for instance), so unless the process is at least partially automated, the test can be quite tedious.

We now turn our attention to Condition 3 of the Theorem to develop a more computationally efficient and much less tedious test, and we consider the example where U is the complement of the open unit disk (the discrete time stability problem). Condition 3 indicates that, after checking any one polynomial for the U-Hurwitz property, we need to verify that $0 \notin \mathrm{co}\{k_1(e^{j\theta}), \ldots, k_m(e^{j\theta})\}\ \forall \theta \in [0,2\pi]$. To perform this test, we use the nearest point function:

$$\mathrm{Nr}(S) = \underset{s \in S}{argmin}\{\| s \|\}$$

defined for any $S \subset \mathbb{C}$. In particular, when S is the convex hull of a finite number of points (as $H(s)$ is), there are very efficient finite algorithms for calculating $\mathrm{Nr}(S)$.[1] In fact, for m reasonably large, the major computational requirement in calculating $\mathrm{Nr}(\mathrm{co}\{k_1(s), \ldots, k_m(s)\})$ should be in evaluating $k_1(s), \ldots, k_m(s)$.

Now we state as fact that $\mathrm{Nr}(\mathrm{co}\{k_1(s), \ldots, k_m(s)\})$ is a continuous function of s. We propose to plot the locus of $\hat{\mathrm{Nr}}(\theta) := \mathrm{Nr}(\mathrm{co}\{k_1(e^{j\theta}), \ldots, k_m(e^{j\theta})\})$ for $0 \le \theta \le 2\pi$. There are two possibilities: either $\hat{\mathrm{Nr}}(\theta) = 0$ for some $\theta \in [0, 2\pi]$ or $\hat{\mathrm{Nr}}(\theta)$ circles the origin n times without intersecting the origin. The engineer can continue to fill in the locus until she's confident that one of the two conditions has been met, in much the same spirit as the Nyquist stability test, where we also count encirclements and look for zero intersections. If the locus touches the origin, N is not U-Hurwitz and a non-U-Hurwitz polynomial in N has been demonstrated. If the locus circles the origin n times without intersecting the origin, then N must be U-Hurwitz.

Remark: Instead of plotting the complex number $\hat{\mathrm{Nr}}(\theta)$, we could just as easily plot the real number $|\hat{\mathrm{Nr}}(\theta)|$ as a continuous function of θ. We would simply check to see if the graph ever hits zero, filling in the plot as necessary to get a confident answer. We prefer the complex plane test, however, as counting encirclements adds one more degree of confidence that the user has tested enough points, and we simply aren't aware of any disadvantage of this test. Another benefit of the complex plane test has to do with its (somewhat esoteric) resemblance to the Nyquist criterion--the engineering community may except it more readily as a stability analysis tool since it "feels" more familiar. For whatever reasons, our limited experience in presenting these results to practicing engineers does indicate a definite preference for the complex plane test.

It should be clear that the test generalizes to arbitrary closed $U \subset \mathbb{C}$. After checking any one polynomial, one notes which components of U^c contains a zero. There are at most n such components. Then the loci of $\mathrm{Nr}(\mathrm{co}\{k_1(s), \ldots, k_m(s)\})$ must be plotted for each component of ∂U which is contained in the boundary of one of the components of U^c that contained a zero of the test polynomial. In general, this reduced set of components of ∂U may still be infinite, although such strange choices of U are of little engineering interest. If U^c, the set of acceptable polynomial zeroes, is a union of *simply connected* subsets of $\mathbb{C}$ (connected sets without holes), then there will be at most n loci to plot. Of course, each of these

[1] We refer, for example, to the two point method (see [4]) and the method of Wolfe [5].

n components of ∂U will have to be appropriately parameterized.

Finally, we note that one application of this test which has generated some interest involves putting a small disk around some or all of the zeroes of a given nominal polynomial to generate a variety of sensitivity analyses. The components of ∂U are easily parameterized for this case.

4. DOMAINS OF POLYNOMIAL ZEROES

We now return to condition 2 of the Theorem to address a more fundamental question than the one addressed in the previous section. Given the polytope N of polynomials as previously defined, without having an *a priori* "desired" subset of zero locations to consider, can we determine precisely the domain D of zeroes of polynomials in N? The solution we propose does not precisely determine D, but comes sufficiently close in the following sense: we find a set $D^* \supset D$ such that the boundary of D^* is contained in the boundary of D. In this sense, the set obtained is not conservative. However, D may have "holes" in it that D^* does not share. For most engineering applications, this is of little consequence. As with the Proposition of Section 2, we restrict our attention to polytopes N which do not contain polynomials with vanishing leading coefficient; i.e. $a_n \neq 0 \ \forall p \in N$.

The concept is quite simple. As in Section 3, we propose to plot the zero loci for every edge of N. Let E denote the total locus of edge zeroes; i.e. $E := \{s : p(s) = 0, p \in \mathrm{Ed}(N)\}$. E is bounded since N is bounded and a_n is bounded away from zero. Consider the set $E^c = \mathbb{C} \setminus E$. E^c can have only one unbounded component which we denote C. We define $D^* := C^c$. So D^* is the union of E with all of the bounded components of E^c. An engineer with the zero locus (i.e. E) in front of him would simply shade in every enclosed region to display D^*.

It is clear that $\partial D^* \subset E$, so that every $\alpha \in \partial D^*$ is a zero of some polynomial in N, which justifies the claim of non-conservatism made above. Of course, we still must show that $D \subset D^*$; i.e. that $p(s) \neq 0$ $\forall s \in C, \forall p(\cdot) \in N$. So suppose $\hat{s} \in C$ and $\hat{p} \in N$ with $\hat{p}(\hat{s}) = 0$. Since C is connected and unbounded, we can find a closed set U which contains $\hat{s}$; indeed, a path from $\hat{s}$ to infinity will suffice.[2] Now U and N satisfy the hypotheses of the Proposition of Section 2, so N is U-Hurwitz. Thus $\hat{p}(\hat{s}) \neq 0$, which contradicts the assertion.

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[2] Since C is unbounded, there is a sequence of points $s_n \in C$ with $|s_n| \to \infty$. Since C is connected, we can connect $\hat{s}$ to s_1 and s_k to s_{k+1}, $k=1,2,3,\ldots$. The union of all these paths is the desired path.

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6. REFERENCES

[1] Kharitonov, V. L., "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations," *Differential Equations*, vol. 14, 1979, pp. 1483-1485.

[2] Minnichelli, R. J., Anagnost, J. J. and Desoer, C. A., "An Elementary Proof of Kharitonov's Theorem with Extensions," Memorandum No. UCB/ERL M87/78, Electronic Research Laboratory, University of California, Berkeley, 1987.

[3] Dasgupta, S., *Perspectives on Kharitonov's Theorem: A View from the Imaginary Axis*, Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA, 52242, rec'd 10/87.

[4] Gilbert, E. G., "An Iterative Procedure for Computing the Minimum of a Quadratic Form on a Convex Set," *SIAM Journal of Control*, Vol. 4, No. 1, 1966, pp. 61-79.

[5] Wolfe, P., "Finding the Nearest Point in a Polytope," *Mathematical Programming 11*, 1976, pp. 128-149.

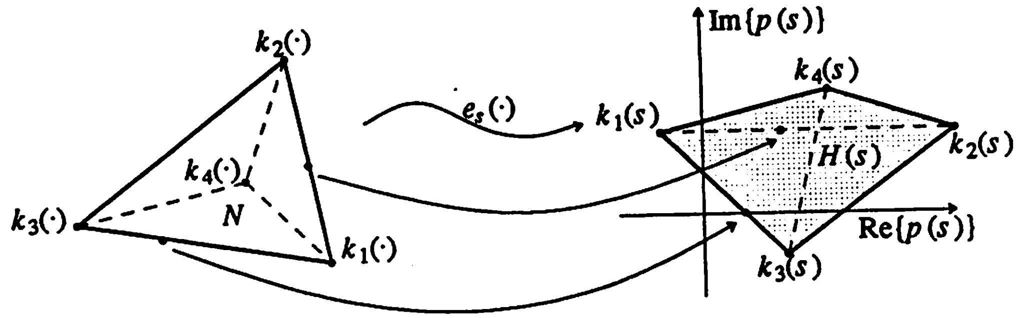

Figure 1(a):
$H(s)$ is the image of N.
In this case the inverse image of each point in $\partial H(s)$ consists of one polynomial in N, an edge polynomial, but not every edge polynomial in N maps into ∂U.

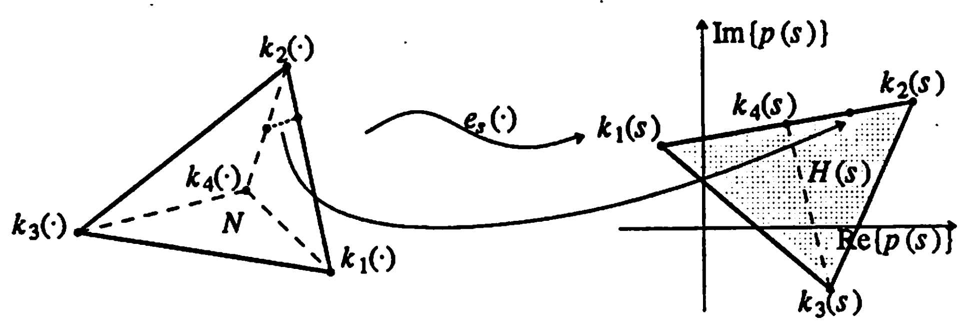

Figure 1(b):
In this case the inverse image of one edge of $H(\hat{s})$ contains a whole face of N, and the inverse image of each point in that edge (except the endpoints) contains two edge polynomials.

6. APPENDIX

Technical Lemma: $\partial H(s) \subset e_s(\mathrm{Ed}(N))$.

Proof: Choose $\alpha \in \partial H(s)$. The proof proceeds by induction.

Inductive Hypothesis: Suppose $k>1$ and there is some $p(\cdot) \in e_s^{-1}(\alpha)$ in a k-dimensional outer surface of N. (For $k=n$, this should be interpreted as $p(\cdot) \in N$.) Then there is some $\hat{p}(\cdot) \in e_s^{-1}(\alpha)$ in a $(k-1)$-dimensional outer surface of N.

Proof: If $p(\cdot)$ is in the *boundary* of the k-dimensional outer surface, S, then choose $\hat{p}(\cdot)=p(\cdot)$ and we're done. So suppose $p(\cdot)$ is in the interior of S. We consider the boundary of ∂S, and note that ∂S is a connected (since $k>1$) subset of S "surrounding" $p(\cdot)$. Thus, given $p_1(\cdot) \in \partial S$, we define $p_2(\cdot) \in \partial S$ as the opposite edge of a line segment $\bar{l}$ through $p(\cdot)$ and $p_1(\cdot)$. So $e_s(\bar{l})$ is a line segment (possibly degenerate) in $H(s)$ containing α, with endpoints $p_1(s)$ and $p_2(s)$. Since $e_s(\cdot)$ is linear, either $e_s(\bar{l})$ is degenerate (a single point, namely α) or α lies in the interior of the segment $e_s(\bar{l})$. Since α is on an edge of the *convex* polygon $H(s)$, the line segment $e_s(\bar{l})$ must be contained in the same edge. Since $p_1(\cdot)$ was an arbitrary polynomial in ∂S, we see that the whole boundary ∂S (and, in fact, all of S) maps into the edge containing α. Thus $e_s(\partial S)$ is a line segment contained in that edge, and since $p_1(s)$ and $p_2(s)$ lie on opposite sides of α (or are both equal to α), $e_s(\partial S)$ contains α; i.e. $e_s(\hat{p})=\alpha$ for some $\hat{p}(\cdot) \in \partial S$. So $\hat{p}(\cdot)$ is the desired polynomial.

Now we complete the proof of the Lemma. Since $\alpha \in H(s)=e_s(N)$, there is some $p_n(\cdot) \in N$ with $p_n(s)=\alpha$. We deduce inductively for $k=n, \ldots, 1$ that there is some $p_k(\cdot)$ in some k-dimensional outer surface of N with $p_k(s)=\alpha$. We end up with $p_1(\cdot)$ in a 1-dimensional outer surface of N; that is, $p_1(\cdot) \in \mathrm{Ed}(N)$. So $\alpha \in e_s(\mathrm{Ed}(N))$. ⊠

Proposition: Suppose $U \subset \mathbb{C}$ is closed, pathwise connected and unbounded. Then condition 2(a) in the Theorem is equivalent to 2(a) and 2(b) (and thus is equivalent to 1 and 3).

Proof: Since $a_n \neq 0 \ \forall p(\cdot) \in N$ and N is compact, a_n is bounded away from zero; $|a_n|$ is also bounded above. For $|s|$ large and $p(\cdot) \in N$, $p(s)=a_n s^n(1+O(1/s))$. So $H(s)$ goes to infinity with uniform angle as $|s| \to \infty$ (i.e. $\arg(p_1(s)) - \arg(p_2(s)) \to 0$ as $|s| \to \infty \ \forall p_1(\cdot), p_2(\cdot) \in N$, uniformly). In particular, $0 \notin H(S)$ for $|s|$ sufficiently large. So, since U was assumed to be unbounded, there is some s^* in U with $0 \notin H(s^*)$.

Now suppose condition 2(a) is satisfied, but that $p(\hat{s})=0$ for some $p \in N$, $\hat{s} \in U$. We will show that this leads to a contradiction.

Since U is pathwise connected, we can find a bounded, *simply connected* $U^* \subset U$ (so ∂U^* is connected) with $\hat{s} \in U^*$ and $s^* \in \partial U^*$ (any path from $\hat{s}$ to s^* will suffice). Now U^* satisfies 2(a) and 2(b), implying that N is U^*-Hurwitz. In particular, $p(\cdot)$ is U^*-Hurwitz, which contradicts the assertion that $p(\hat{s})=0$. ⊠

Remark: We can extend the Proposition to sets U which are not necessarily connected, but every component of U is pathwise connected and unbounded; i.e. U is "pathwise connected to infinity." However, we do not forsee any useful applications for such an extension.