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Abstract

Stability of the unity-feedback interconnection of a *linear* and a *nonlinear* subsytem is considered. If the linear subsystem has a left-coprime factorization, the nonlinear subsystem is shown to have a specific normalized right-coprime factorization. If the linear subsystem also has a *normalized* right-coprime factorization, we obtain a parametrization of the set of all stabilizing nonlinear subsystems; this parametrization can be interpreted as: i) that of all stabilizing nonlinear compensators for a given *linear* plant or ii) that of all nonlinear fractional perturbations of a possibly nonlinear plant stabilized by a given *linear* compensator.

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I. Introduction

The problem of characterizing all linear time-invariant compensators which stabilize a linear time-invariant plant in the unity-feedback configuration has been solved using tools of algebraic control theory; the characterization is obtained by finding solutions of certain Bezout identities [You.1, Des.1, Vid.1,2]. A generalization of this approach to linear input-output maps can be found in [Fei.1]; see [Man.1] for the time-varying continuous-time case. In [Kha.1], the set of all stabilizing discrete-time possibly nonlinear time-varying compensators for a discrete-time linear time-invariant plant is obtained using periodic compensators and two-step compensation schemes. In [Des.2,3], the set of all stabilizing compensators for an *incrementally stable nonlinear* plant (e.g. *stable* linear plant) is obtained. In [Ham.1], using left and right factorizations of a class of causal nonlinear discrete-time plants, a complete parametrization of the set of all stable solutions U , V of the equation $UN + VD = M$ is given.

In this paper, we consider the nonlinear unity-feedback configuration where one of the two subsystems (either the plant or the compensator) is specified by a *linear* (not necessarily time-invariant) map. Since the plant and the compensator appear symmetrically in the stability analysis of the unity-feedback system, we choose to derive the results for a fixed linear plant. Assuming that the linear plant has a "generalized" left-coprime factorization, we show that all nonlinear stabilizing compensators have normalized right-coprime factorizations which satisfy a Bezout-like identity. In the case where the linear plant also has a normalized right-coprime factorization, we obtain the set of all solutions satisfying the identity; in fact, we obtain a parametrization of the set of all nonlinear stabilizing compensators. Interchanging the roles of the plant and the compensator, this result gives the set of all nonlinear plant perturbations which maintain feedback system stability for a given linear compensator.

II. Notation

(e.g. [Wil.1, Saf.1, Des.4]) Let $\mathcal{T} \subset \mathbb{R}$ and let $\mathcal{V}$ be a normed vector space. Let $\zeta := \{ F \mid F : \mathcal{T} \to \mathcal{V} \}$ be the vector space of $\mathcal{V}$–valued functions on $\mathcal{T}$. For any $T \in \mathcal{T}$, the projection map $\Pi_T : \zeta \to \zeta$ is defined by $\Pi_T F(t) := \begin{cases} F(t) & t \le T \,,\ t \in \mathcal{T} \\ \theta_\zeta & t > T \,,\ t \in \mathcal{T} \,, \end{cases}$ where θ_ζ is the zero element in ζ. Let $\Lambda \subset \zeta$ be a normed vector space which is closed under the family of projection maps $\{ \Pi_T \}_{T \in \mathcal{T}}$. For any $F \in \Lambda$, let the norm $\| \Pi_{(.)} F \| : \mathcal{T} \to \mathbb{R}_+$ be a nondecreasing function. The *extended space* Λ_e is defined by

$$\Lambda_e := \{ F \in \zeta \mid \forall T \in \mathcal{T},\ \Pi_T F \in \Lambda \} .$$

A map $F : \Lambda_e \to \Lambda_e$ is said to be *causal* iff, for all $T \in \mathcal{T}$, Π_T commutes with $\Pi_T F$; equivalently, $\Pi_T F = \Pi_T F \Pi_T$.

In the following we will be considering two function spaces closely related to Λ_e. The superscripts i and o refer to "input" and "output", respectively. Let Λ_e^i and Λ_e^o be extended function spaces analogous to Λ_e except that their functions take values in the normed spaces $\mathcal{V}^i$ and $\mathcal{V}^o$, respectively; the associated projections Π_T are redefined accordingly.

A causal map $H : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e$ is said to be *S-stable* [Des.3] iff there exists a continuous nondecreasing function $\phi_H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall (u_1, u_2) \in \Lambda^o \times \Lambda^i ,\ \ \| H(u_1, u_2) \| \le \phi_H(\| u_1 \| + \| u_2 \|).$$

An S-stable map need not be continuous. Note that the composition and the sum of S-stable maps are S-stable.

A *feedback system* is said to be *well-posed* iff, for all allowed inputs, all of the signals in the system are (uniquely) determined by causal functions of the inputs.

A well-posed (nonlinear) *feedback system* is called *S-stable* iff, for all allowed inputs, all of the signals in the feedback system are determined by causal S-stable maps.

A causal (nonlinear) map $P : \Lambda_e^i \to \Lambda_e^o$ is said to have a *right factorization* $(N_p , D_p ; X_p)$ iff there exist causal S-stable maps N_p , D_p , such that

(i) $D_p : X_p \subset \Lambda_e^i \rightarrow \Lambda_e^i$ is bijective and has a causal inverse,

and (ii) $N_p : X_p \rightarrow \Lambda_e^o$, with $N_p [X_p] = P [\Lambda_e^i]$,

and (iii) $P = N_p D_p^{-1}$ [Vid.3, Ham.2].

X_p is called the *factorization space* of the right factorization $(N_p , D_p ; X_p)$ [Ham.2].

$(N_p , D_p ; X_p)$ is said to be a *normalized right-coprime factorization* of $P : \Lambda_e^i \rightarrow \Lambda_e^o$ iff

(i) $(N_p , D_p ; X_p)$ is a right factorization of P ,

and (ii) there exist causal S-stable maps $U_p : \Lambda_e^o \rightarrow X_p$ and $V_p : \Lambda_e^i \rightarrow X_p$ such that $U_p N_p + V_p D_p = I_{X_p}$, where I_{X_p} denotes the identity map on X_p .

III. Main Results

We consider the unity-feedback configuration $S(P, C)$ shown in Figure 1: the *linear plant* is given by a causal *linear* (not necessarily time-invariant) map $P : \Lambda_e^i \to \Lambda_e^o$ and the compensator is given by a (possibly nonlinear) causal map $C : \Lambda_e^o \to \Lambda_e^i$. We assume that the the linear map P has a "left-coprime factorization".

Assumption 3.1: The causal linear map $P : \Lambda_e^i \to \Lambda_e^o$ has the following properties:

1) There exist causal *linear* S-stable maps $\tilde{N} : \Lambda_e^i \to \Lambda_e^o$ and $\tilde{D} : \Lambda_e^o \to \Lambda_e^o$, where $\tilde{D}$ is bijective and has a causal inverse, such that

$$P = \tilde{D}^{-1}\tilde{N} \ . \tag{1a}$$

2) There exist causal S-stable (not necessarily linear) maps $\tilde{U} : \Lambda_e^o \to \Lambda_e^i$ and $\tilde{V} : \Lambda_e^o \to \Lambda_e^o$ such that

$$\tilde{N}\tilde{U} + \tilde{D}\tilde{V} = I_{\Lambda_e^o} \ . \tag{1b}$$

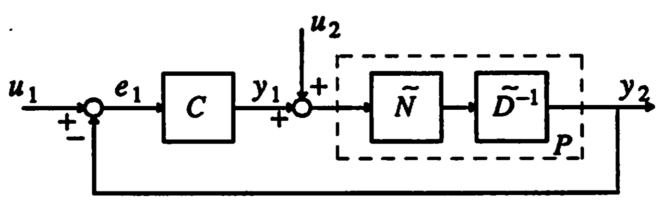

Figure 1 The unity-feedback system $S(P, C)$

Theorem 3.2: (S-stability of $S(P,C)$) Consider the unity-feedback system $S(P,C)$ in Figure 1, where the causal *linear* map $P : \Lambda_e^i \to \Lambda_e^o$ satisfies Assumption 3.1; then $S(P,C)$ is well-posed and S-stable if and only if the causal (not necessarily linear) map $C : \Lambda_e^o \to \Lambda_e^i$ has a normalized right-coprime factorization $(N_{cr}, D_{cr}; X \subset \Lambda_e^o)$ such that

$$\tilde{N} N_{cr} + \tilde{D} D_{cr} = I_X \ . \tag{2}$$

Proof:

" if "

Let $(N_{cr}, D_{cr}; X \subset \Lambda_e^o)$ be a normalized right-coprime factorization of $C : \Lambda_e^o \to \Lambda_e^i$. From the summing node equations in Figure 1, we obtain

$$e_1 = D_{cr}\xi_c = u_1 - y_2 \tag{3a}$$

$$\tilde{D}\, y_2 = \tilde{N}\,(u_2 + y_1) = \tilde{N}\,(u_2 + N_{cr}\xi_c)\ , \tag{3b}$$

where $\xi_c \in X$. From equations (3a-b), using the *linearity* of $\tilde{N}$ and $\tilde{D}$ and equation (2), we obtain

$$\xi_c = \begin{bmatrix} \tilde{D} & -\tilde{N} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{4}$$

Equation (4) determines a causal S-stable map $(u_1, u_2) \mapsto \xi_c$; since N_{cr} and D_{cr} are S-stable, the map $(u_1, u_2) \mapsto (e_1, e_2)$ is S-stable, hence $S(P, C)$ is well-posed and S-stable.

" only if "

By well-posedness and S-stability of $S(P, C)$, C has a right factorization $(N_c, D_c; \Lambda_e^o)$; namely $N_c = C(I + PC)^{-1}$, $D_c = (I + PC)^{-1}$. Using the right factorization $(N_c, D_c; \Lambda_e^o)$ of C in the summing node equations of $S(P, C)$ and using *linearity* of $\tilde{N}$ and $\tilde{D}$, we obtain

$$(\tilde{N} N_c + \tilde{D} D_c)\xi_c = \begin{bmatrix} \tilde{D} & -\tilde{N} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{5}$$

where $\xi_c \in \Lambda_e^o$. By well-posedness of $S(P, C)$ and the existence of the causal inverse D_c^{-1} , there exists a *causal* map $(u_1, u_2) \mapsto \xi_c$ (which need not be S-stable even if $S(P, C)$ is). Choose inputs by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \tilde{V} \\ -\tilde{U} \end{bmatrix} v\ , \tag{6}$$

where $v \in \Lambda_e^o$. Substituting equation (6) in (5) and using equation (1b), we obtain

$$(\tilde{N} N_c + \tilde{D} D_c)\xi_c = v\ , \tag{7}$$

which determines a causal map $v \mapsto \xi_c$. Hence the inverse map

$$(\tilde{N} N_c + \tilde{D} D_c)^{-1} : \Lambda_e^o \to \Lambda_e^o$$

is defined on Λ_e^o and is causal. Solving equation (7) for ξ_c and recalling that $e_1 = D_c \xi_c$ and $y_1 = N_c \xi_c$, we obtain

$$e_1 = D_c(\tilde{N} N_c + \tilde{D} D_c)^{-1} v\ , \tag{8a}$$

$$y_1 = N_c(\tilde{N} N_c + \tilde{D} D_c)^{-1} v \ . \tag{8b}$$

By S-stability of $S(P, C)$, equations (8a) and (8b) determine causal S-stable maps $v \mapsto e_1$ and $v \mapsto y_1$, respectively. Hence

$$(N_c(\tilde{N} N_c + \tilde{D} D_c)^{-1} \ , \ D_c(\tilde{N} N_c + \tilde{D} D_c)^{-1} \ ; \ \Lambda_e^o \ )$$

is a right factorization, which is in fact a normalized right-coprime factorization of C, since

$$\tilde{N} \ [N_c(\tilde{N} N_c + \tilde{D} D_c)^{-1}] + \tilde{D} \ [D_c(\tilde{N} N_c + \tilde{D} D_c)^{-1}] = I_{\Lambda_e^o} \ .$$

□

In the following lemma, we give the set of all solutions of equation (2), where the causal S-stable linear maps $\tilde{N}$ and $\tilde{D}$ are given.

Lemma 3.3: Let the causal *linear* map $P : \Lambda_e^i \to \Lambda_e^o$ satisfy Assumption 3.1. Suppose also that $(N, D; X_r \subset \Lambda_e^i)$ is a normalized right-coprime factorization of P (hence $P = ND^{-1} = \tilde{D}^{-1} \tilde{N}$, N and D need not be linear maps); under these conditions the set of all causal S-stable solutions

$$N_{cr} : X \subset \Lambda_e^o \to \Lambda_e^i \ , \tag{9a}$$

$$D_{cr} : X \to \Lambda_e^o \tag{9b}$$

of the equation

$$\tilde{N} N_{cr} + \tilde{D} D_{cr} = I_X \tag{2}$$

is given by

$$N_{cr} = \tilde{U} + DQ \tag{10a}$$

$$D_{cr} = \tilde{V} - NQ \tag{10b}$$

where $Q : X \to X_r$ is a causal S-stable not necessarily linear map.

□

Comment: Note that the S-stable maps $\tilde{U}$, $\tilde{V}$, N, D and Q are *not* required to be linear.

Proof of Lemma 3.3: Any pair N_{cr}, D_{cr} specified by equations (10a-b) is a solution of equation (2). Substituting (10a-b) in equation (2) and using *linearity* of $\tilde{N}$ and $\tilde{D}$, we obtain

$$\tilde{N} N_{cr} + \tilde{D} D_{cr} = \tilde{N} \tilde{U} + \tilde{D} \tilde{V} + (\tilde{D} N - \tilde{N} D) Q \ . \tag{11}$$

Using equation (1b) and

$$\tilde{D} N = \tilde{N} D \tag{12}$$

in equation (11), equation (2) is satisfied.

Now suppose that there exist causal S-stable maps N_{cr} , D_{cr} as in (9a-b), satisfying equation (2), then by equation (1b),

$$\tilde{N} N_{cr} + \tilde{D} D_{cr} = I_X = \tilde{N} \tilde{U} + \tilde{D} \tilde{V} \ . \tag{13}$$

Using the *linearity* of $\tilde{N}$ and $\tilde{D}$ and equation (12) in equation (13), we obtain

$$ND^{-1}(N_{cr} - \tilde{U}) = \tilde{V} - D_{cr} \ . \tag{14}$$

Let

$$Q := D^{-1}(N_{cr} - \tilde{U}) : X \to X_r \ . \tag{15}$$

Clearly, the map Q defined in (15) is causal. Since $(N, D; X_r)$ is a normalized right-coprime factorization of P , there exist causal S-stable maps $U : \Lambda_e^o \to X_r$ and $V : \Lambda_e^i \to X_r$ such that

$$UN + VD = I_{X_r} \ . \tag{16}$$

Then by equations (14), (15) and (16), we obtain

$$\begin{aligned} Q &= (UN + VD)Q \\ &= UNQ + VDQ \\ &= U(\tilde{V} - D_{cr}) + V(N_{cr} - \tilde{U}) \ . \end{aligned} \tag{17}$$

Hence the map Q defined in (15) is S-stable; moreover from equations (14) and (15) we obtain equations (10a-b).

□

Theorem 3.4: **(All stabilizing compensators)** Let the causal *linear* map $P : \Lambda_e^i \to \Lambda_e^o$ satisfy Assumption 3.1. Suppose also that $(N, D; X_r \subset \Lambda_e^i)$ is a normalized right-coprime factorization of P (N and D need not be linear); under these conditions the set of all causal stabilizing compensators C in the feedback system $S(P, C)$ is given by

$$\{ C = (\tilde{U} + DQ)(\tilde{V} - NQ)^{-1} \mid Q : X \to X_r \text{ causal S-stable },\ (\tilde{V} - NQ)^{-1} \text{ causal } \} \tag{18}$$

moreover, the map $Q \mapsto C$ in (18) is bijective.

Comment: When the linear plant P has a transfer matrix representation with entries in a principal ring [see e.g. Vid.2], Assumption 3.1 holds, moreover P has a normalized right-coprime factorization.

Proof of Theorem 3.4: From Theorem 3.2 and Lemma 3.3, we conclude that (18) holds; moreover the map $Q \mapsto C$ is surjective. We now show that the map $Q \mapsto C$ is also injective:

It suffices to show that

$$(\tilde{U} + DQ_1)(\tilde{V} - NQ_1)^{-1} = (\tilde{U} + DQ_2)(\tilde{V} - NQ_2)^{-1} \tag{19}$$

implies $Q_1 = Q_2$. Using the linearity of $\tilde{N}$ and $\tilde{D}$ and equations (1b) and (12), we obtain

$$\tilde{N}(\tilde{U} + DQ_1) + \tilde{D}(\tilde{V} - NQ_1) = I_X \ , \tag{20a}$$
$$\tilde{N}(\tilde{U} + DQ_2) + \tilde{D}(\tilde{V} - NQ_2) = I_X \ . \tag{20b}$$

Composing equation (20a) on the right with the nonlinear map $(\tilde{V} - NQ_1)^{-1}$ and using equations (19) and (20b), we obtain

$$(\tilde{V} - NQ_1)^{-1}(\tilde{V} - NQ_2) = I_X \ . \tag{21}$$

Using equation (21) in (19), we obtain

$$DQ_1 = DQ_2 \ , \tag{22}$$

hence $Q_1 = Q_2$ since D is bijective.

□

When we interchange the roles of P and C, that is when a *linear* compensator C satisfies Assumption 3.1 and has a normalized right-coprime factorization ND^{-1}, then the set of all causal *stabilizable* plants P in the feedback system $S(P, C)$ are given by

$$\{ P = (\tilde{U} + DQ)(\tilde{V} - NQ)^{-1} \mid Q : X \to X_r \text{ causal S-stable }, (\tilde{V} - NQ)^{-1} \text{ causal } \} \ . \tag{23}$$

If $\tilde{V}$ is assumed to have a causal inverse, for $Q = U(0) + V(0)$, equation (16) shows that $P = \tilde{U}\tilde{V}^{-1}$, hence by (23) $S(\tilde{U}\tilde{V}^{-1}, C)$ is S-stable; then (23) gives a parametrization of the class of all S-stable fractional perturbations of the nominal (not necessarily linear) plant $\tilde{U}\tilde{V}^{-1}$, which result in an S-stable unity-feedback system for the given linear compensator C.

IV. Conclusion

The factorization approach has been extremely useful in the analysis and synthesis of linear time-invariant feedback systems since individual subsystems have right- and left-coprime factorizations. Although a direct generalization of such properties to nonlinear maps seems intractable, a right factorization approach to nonlinear unity-feedback system stability analysis seems to be natural since the existence of right factorizations is a necessary condition for stable feedback systems. The motivation of this paper was to investigate the effect of right- and left-coprime factorization properties of the *linear* subsystem on the *nonlinear* stable unity-feedback system.

In Theorem 3.2, assuming only a left-coprime factorization of the *linear* subsystem, a special normalized right-coprime factorization of the *nonlinear* subsystem becomes a necessary and sufficient condition for closed-loop stability. In Theorem 3.4, assuming that the linear subsystem has also a normalized right-coprime factorization, the set of all special normalized right-coprime factorizations is derived.

With respect to Theorem 3.4, it is interesting to note that the set of all stabilizing nonlinear compensators (for a given linear plant) (or interpreted differently, the set of all stable nonlinear fractional perturbations of a nonlinear plant, stabilized by a given linear compensator) is precisely of the form well-known for the linear case except that certain maps including the free stable parameter Q are nonlinear.

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