NORMAL FORMS FOR CONSTRAINED NONLINEAR DIFFERENTIAL EQUATIONS PART II: BIFURCATION

by

L. O. Chua and H. Oka

Memorandum No. UCB/ERL M87/93

24 September 1987

NORMAL FORMS FOR CONSTRAINED NONLINEAR DIFFERENTIAL EQUATIONS PART II: BIFURCATION

by

L. O. Chua and H. Oka

Memorandum No. UCB/ERL M87/93

24 September 1987

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NORMAL FORMS FOR CONSTRAINED NONLINEAR DIFFERENTIAL EQUATIONS PART II: BIFURCATION

by

L. O. Chua and H. Oka

Memorandum No. UCB/ERL M87/93

24 September 1987

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NORMAL FORMS FOR CONSTRAINED NONLINEAR DIFFERENTIAL EQUATIONS
PART II: BIFURCATION†

Leon O. Chua and Hiroe Oka††

Abstract

Applying the theory developed in *Part I* we re-examine the classic *singular perturbaton problem* in terms of *unfoldings* of a generalized nonlinear vector field. Our novel approach is based on a *bifurcation* point of view.

1. UNFOLDINGS OF GENERALIZED VECTOR FIELDS

In *Part I* [1], constrained systems are formulated on a manifold and an equivalence relation is introduced which allows us to develop a method for obtaining normal forms for constrained systems which works in the same way as that for vector fields.

Let us return to the Van der Pol equation considered earlier in *Section 1:* in *Part I* [1]:

$$\left.\begin{aligned} \varepsilon\dot{x} &= x - x^3/3 + y \\ \dot{y} &= -x \end{aligned}\right\} \tag{1.1}$$

where $x, y \in \mathbf{R}$. This is a family of ODE's with a parameter ε. If we assume $\varepsilon = 0$, then equation (1.1) becomes, in our formulation, a constrained system of corank 1 on $\mathbf{R}^2$; namely,

$$(\mathrm{A}, \mathrm{v}) = \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x - \dfrac{x^3}{3} + y \\ -x \end{bmatrix}\right]. \tag{1.2}$$

Many previous works have been published on this equation. The main objective in these works is to obtain some information on the behavior of the solutions for sufficiently small but non-zero ε via a small perturbation from those for $\varepsilon = 0$. Since ε appears in front of $\dot{x}$ in (1.1), this problem is not an ordinary one, but belongs rather to a class of *singular perturbation problems.*

In our present formulation, the Van der Pol equation (1.1) is regarded as a ε-*family of constrained systems.*

†This research is supported in part by the Office of Naval Research, Contract N00014-86-K-0351 and by the National Science Foundation, Grant MIP-8614000.

††L. O. Chua is with the University of California, Berkeley, CA 94720.
H. Oka is with the Department of Mathematics, Kyoto University, Kyoto, 606, Japan.

$$(A_\varepsilon, v_\varepsilon) = \left[\begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x - \frac{x^3}{3} + y \\ -x \end{bmatrix} \right], \tag{1.3}$$

which is of corank 1 only for $\varepsilon = 0$. Otherwise, it has a corank equal to 0. We will consider this family as an *unfolding* of Eq. (1.2).

Definition 1.1: unfolding

A family (A_μ, v_μ) of constrained systems parameterized by μ is called an *unfolding* of the constrained system (A, v) if $(A_0, v_0) = (A, v)$ holds.

■

Hence a singular perturbation problem for ODE's, such as

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = g(x, y, \varepsilon) \end{cases}$$

where $(x, y) \in \mathbf{R}^r \times \mathbf{R}^{n-r}$, is, in our formulation, to be interpreted as a study of an unfolding,

$$\left[\begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} f(x, y, \varepsilon) \\ g(x, y, \varepsilon) \end{bmatrix} \right] \tag{1.4}$$

of the constrained system

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} f(x, y, 0) \\ g(x, y, 0) \end{bmatrix} \right]. \tag{1.5}$$

This approach is based on the same idea from the bifurcation theory for vector fields: the system (1.5) for $\varepsilon = 0$ corresponds to a degenerate singularity, and we are interested in the dynamical aspect of the system (1.4) for ε sufficiently near 0.

Since we have given several types of normal forms of corank 1 on $\mathbf{R}^2$, in *Section 4*, of Part I [1], we will make an attempt here to study their unfoldings in order to illustrate our method. Our approach will help reveal the many rich phenomena associated with singular perturbation problems of ODE's.

Example 1.2: Rapid point

Consider a 1-parameter unfolding,

$$\left[\begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \tag{1.6}$$

of a normal form of a 2-dimensional constrained system of corank 1; namely, $\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$. (See Proposition 4.12 in [1]). Equation (1.6) can also be written as follows in the form of an ODE:

$$\begin{cases} \varepsilon \dot{x} = 1 \\ \dot{y} = 0 \end{cases}.$$

The phase portrait for this equation for small positive ε is shown in Fig. 1 where all trajectories flow rapidly from left to right along parallel horizontal lines.

Example 1.3: regular point on the characteristic surface

Consider next the 1-parameter unfolding

$$\left[\begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \pm x \\ 1 \end{bmatrix} \right],$$

or equivalently,

$$\begin{cases} \varepsilon \dot{x} = \pm x \\ \dot{y} = 1 \end{cases},$$

of the normal form (a) of *Proposition 4.17* from *Part I;* namely,

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \pm x \\ 1 \end{bmatrix} \right].$$

The phase portrait of this system for small positive ε is shown in Fig. 2. Here we can see a *slow* motion of order ε^0 in a neighborhood of the characteristic surface $x = 0$, and a *rapid* motion of order ε^{-1} flowing out of this surface in Fig. 2(a), and into this surface in Fig. 2(b)..

In general, for a point p on a characteristic surface S of a constrained system of corank 1 on $\mathbf{R}^2$, we say that the point p is of *stable type* (resp.; *unstable type)* if the 1st-order normal form around this point is of the form:

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -x \\ 1 \end{bmatrix} \right] \left(resp., \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x \\ 1 \end{bmatrix} \right] \right)$$

The set of all points of *stable type* (resp., *unstable* type) is called the *stable part* (resp.; *unstable part*) of S and is denoted by S_-(resp.; S_+). See Fig. 2(b) (resp.; Fig. 2(a)). The set $S_r = S_+ \cup S_-$ is called the *regular part* of S.

For more details, see Fenichel [2], who treats the singular perturbation problem around a regular point on a characteristic surface in a general manner.

Example 1.4: Impasse point

Consider a 1-parameter unfolding,

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm y + ax^2\\ 1 \pm x\end{bmatrix}\right],$$

or equivalently,

$$\begin{cases}\varepsilon \dot{x} = \pm y + ax^2\\ \dot{y} = 1 \pm x\end{cases} \tag{1.7}$$

of the normal form (a_2') of *Proposition 4.17* from *Part I;* namely,

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm y + ax^2\\ 1 \pm x\end{bmatrix}\right].$$

For brevity, let us choose the "+" sign in the upper right hand side, and assume $a > 0$ in (1.7). This system has a solution which moves along the stable part S_- of the characteristic surface $S = \{y = -ax^2\}$ until the trajectory arrives at the neighborhood of the origin, where it moves rapidly to the right along a horizontal line, as shown in Fig. 3. In other words, for $\varepsilon \neq 0$, an orbit starting near the stable part S_- of S slowly moves along S_- with a velocity of $0(1)$ until it reaches a neighborhood of the origin. Then the trajectory changes into a rapid horizontal motion with the velocity of $0(1/\varepsilon)$. Hence, the *origin* is the point where the trajectory velocity changes from a slow motion to a fast motion. In general, such a point is called an *impasse point [2]*. In the limit when $\varepsilon \to 0$, the trajectory executes an instantaneous *jump* upon reaching an impasse point. Such jump phenomenon has been investigated in depth by Ikegami [4].

Example 1.5: Equilibrium

Consider a 2-parameter unfolding

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\alpha y \pm x\\ ay\end{bmatrix}\right],$$

parametrized by (ε, α) or equivalently,

$$\begin{cases}\varepsilon \dot{x} = \alpha y \pm x\\ \dot{y} = ay\end{cases}.$$

of the normal form (b_1) of *Proposition 4.18* from *Part I;* namely,

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm x\\ ay\end{bmatrix}\right].$$

Let us choose the "−" sign in the upper right-hand side and assume $a > 0$. Figure 4 shows the phase portrait of this system for small $\varepsilon > 0$. This phase portrait shows the local structure near an equilibrium point which lies on a regular part of the characteristic surface.

Example 1.6: Canard

Consider the 2-parameter unfolding

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}y \pm x^2\\ \alpha \pm x\end{bmatrix}\right]$$

or equivalently,

$$\begin{cases}\varepsilon \dot{x} = y \pm x^2\\ \dot{y} = \alpha \pm x\end{cases} \tag{1.8}$$

represents a 2-parameter unfolding of the normal form (b_2') of *Proposition 4.18* from *Part I;* namely,

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}y \pm x^2\\ \pm x\end{bmatrix}\right].$$

This system is known to have a peculiar solution called a "*canard,*" ("duck," in English) which was first introduced by Benoit et al. [5] using *non-standard analysis.* Since we do not have space to discuss the concept of non-standard analysis, we will give only a rough definition of this peculiar solution. Roughly speaking, a "canard" solution is a trajectory, a part of which is included in an ε-neighborhood of the characteristic surface S, which moves from the stable part S_- to the unstable part S_+. For example, if we choose the "$-$" sign in the right-hand sides of (1.8) and assume $\alpha = 0$, then the solution of (1.8) is given by:

$$x(t) = -\frac{t}{2}\,,\; y(t) = \frac{t^2}{4} - \frac{\varepsilon}{2} \tag{1.9}$$

This orbit lies along an ε-neighborhood of the characteristic surface $S = \{y = x^2\}$ from $t = -\infty$ to $t = +\infty$, as shown in Fig. 5. Of course, a "canard" solution does *not* have to be restricted along the *entire* characteristic surface from $t = -\infty$ to $t = +\infty$. In general, it may enter or leave the ε-neighborhood as shown in Fig. 6. Observe, however, that the solution associated with an *impasse point* in Fig. 3, though superficially resembling the trajectory in Fig. 6 near the "fold" point, is *not* a "canard" because it does not contain any point on the unstable part S_+. In terms of *non-standard analysis,* the "canard" solutions associated with the system

$$\begin{cases}\varepsilon \dot{x} = y - f(x)\\ \dot{y} = \alpha - x\end{cases} \tag{1.10}$$

has already been analyzed in detail, and their asymptotic expansions have been derived. See [5-7,9] for details. Moreover, "canard" solutions of 3-dimensional ODE's are discussed in Benoit [6]. Here, we will present only briefly the following theorem due to M. Diener (See [9]):

Theorem 1.7

Suppose the function $f(x)$ has a fold point x_0, i.e., $f'(x_0) = 0$ but $f''(x_0) \neq 0$. Then there exists a parameter value $\alpha = \alpha_0$ such that the above system (1.10) has a canard solution around the point $x = x_0$.

Although we have shown directly that there exists a canard solution in the system (1.8) when $\alpha = 0$, the results due to Benoit et al. [5] (See also Zvonkin et al. [9]) imply that this system (with $\alpha = 0$) has additional "canard" solutions which are different from (1.9). Moreover, the system (1.8) with $\alpha \neq 0$ has canard solutions if α is *sufficiently small* and *negative.*

Example 1.8: Bifurcation of characteristic surfaces

Consider the two-parameter unfolding

$$\left[\begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha \pm x^2 + ay^2 \\ 1 \pm x \end{bmatrix}\right]$$

or equivalently,

$$\begin{cases} \varepsilon \dot{x} = \alpha \pm x^2 + ay^2 \\ \dot{y} = 1 \pm x \end{cases}$$

of the normal form (a_4') of *Proposition 4.17* from *Part I*, namely,

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \pm x^2 + ay^2 \\ 1 \pm x \end{bmatrix}\right].$$

For simplicity, let us choose the "+" sign in the right-hand side. The phase portrait of this system for small $\varepsilon > 0$ is shown in Fig. 7 for the case $a > 0$, and in Fig. 8 for the case $a < 0$.

Figure 8 shows the change in the phase portraits when we decrease the value of α from a positive value to a negative value. The solutions in Figs. 8 (b), (c) and (d), which are also "canards," are described in [7]. It is interesting to point out that the changes in the phase portraits from (b) to (e) occur within a very small range of parameter values.

Example 1.9: saddle-node singularity

Recall the 1st-order normal form (b_1) of *Proposition 4.18* from *Part I;* namely,

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \pm x \\ ay \end{bmatrix}\right].$$

If the 1-jet of a constrained system is equivalent to (b_1) with $a = 0$ (the degenerate case), its non-degenerate 2nd-order normal form is given by,

$$\left[\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}\pm x\\ \pm y^2\end{bmatrix}\right]$$

Consider a 2-parameter unfolding of this normal form

$$\left[\begin{bmatrix}\varepsilon & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}\pm x\\ \alpha \pm y^2\end{bmatrix}\right],$$

or equivalently,

$$\begin{cases}\varepsilon \dot{x} = \pm x\\ \dot{y} = \alpha \pm y^2\end{cases}. \tag{1.11}$$

This system exhibits the *saddle-node bifurcation* of equilibria along the regular part of the characteristic surface. If we choose the "−" sign in the first equation and the "+" sign in the second equation in the right-hand side of (1.11), the associated phase portraits for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$ are shown in Fig. 9.

As illustrated in the above examples, even if we simply restrict ourselves to 2-dimensional constrained systems of corank 1, we can observe various types of dynamical behaviors in their unfoldings. These examples have been studied in details by many authors. Since "a normal form" is originally defined for local systems around a point in the phase space, the phenomena described above describe various *local structures* of constrained systems. For instance, Fig. 10 shows the phase portrait associated with the Van der Pol equation, where the local structure around the points A, B, C, and D correspond to the phase portraits given in *Examples 1.2, 1.3, 1.4, and 1.5;* respectively. These observations suggest that it is useful to study the *unfoldings of normal forms of constrained systems* in order to uncover what types of phase portraits are possible for constrained systems. In particular, it can reveal which types of phase portrait are more robust and hence often observed, as well as those which are less likely to be observed. It will also show how a phase portrait changes when the system is slightly perturbed. Thus, the normal form for constrained systems plays the same role as that for vector fields (ODE's). Therefore our point of view in this paper is to regard a singular perturbation problem for ODE's as a bifurcation problem for constrained systems, by enlarging the space of systems being investigated.

From a practical point of view, the most important unfoldings are those which are *persistent;* namely, a family whose bifurcation behavior does not change in an essential way upon the introduction of small perturbations. We call such an unfolding a *versal unfolding* for constrained system.

Since we are only concerned with the local behaviors of constrained systems near a point $x_0 \in M$ determined by the finite (say, up to k) order terms of the Taylor expansion at this point, it follows that we need to introduce the concept of *unfoldings* in the sense of k-jets; e.g., a family (A_μ^k, v_μ^k) of k-jets of a constrained system parametrized by μ is called an *unfolding* of a k-jet of a constrained system (A^k, v^k) if it satisfies

$(A_0^k, v_0^k) = (A^k, v^k)$.

Similar to the case of vector fields, it is important to obtain versal unfoldings with a minimal number of parameters, called *miniversal unfoldings.* We will not give here a systematic discussion on miniversal unfoldings for constrained systems. The reason is that a "versality <=> transversality" argument similar to that for vector fields does not hold in its analogous form. The difficulty is as follow: in order to discuss versal unfoldings, we need the space $\bigcup_{r=0}^{n} \mathcal{C}\mathcal{X}^{(r)}(M)$, which is, however, neither a vectorspace nor a manifold. In contrast, recall the space of k-jets of vector fields forms a vector space.

Nevertheless, the "versality <=> transversality" argument works for k-jets of constrained systems of *corank* ≤ 1, if we restrict our unfoldings to those having some additional properties, such as those defined below.

Definition 1.10: regular unfolding

An unfolding (A_μ^k, v_μ^k), $\mu \in \mathbf{R}^m$ of a k-jet of constrained system (A^k, v^k) at x_0 is called a *regular unfolding,* if the following conditions hold:

[U] There exist a neighborhood $U = U_1 \times U_2 \subset M \times \mathbf{R}^m$ of $(x_0, 0)$, and a representative (A_μ, v_μ) of (A_μ^k, v_μ^k) defined on U, so that A_μ is of constant corank on U_1 for each fixed μ.

[R] $\left.\frac{d}{d\mu}\right|_{\mu=0} \det A_\mu(x_0) \neq 0 \in \mathbf{R}^m$.

Since the above restriction does not exclude the family of constrained systems

$$\begin{cases} \varepsilon\dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \qquad x \in \mathbf{R}^r, y \in \mathbf{R}^{n-r} , \tag{1.1}$$

the class of "regular unfoldings" is large enough to treat such perturbation problems.

Definition 1.11: k-versal unfolding

(i) Let (A_μ^k, v_μ^k) and $(\tilde{A}_\lambda^k, \tilde{v}_\lambda^k)$ be unfoldings of a constrained system (A^k, v^k). We say (A_μ^k, v_μ^k) is *induced* from $(\tilde{A}_\lambda^k, \tilde{v}_\lambda^k)$ if there exists a family $(P_\mu^k, \phi_\mu^{k+1})$ of (k,k+1)-jets of transformations with (P_0^k, ϕ_0^{k+1}) = identity, and a transformation of parameters $\lambda = \lambda(\mu)$ satisfying $\lambda(0) = 0$ such that

$$(A_\mu^k, v_\mu^k) = (P_\mu^k, \phi_\mu^{k+1})_\# \left[\tilde{A}_{\lambda(\mu)}^k, \tilde{v}_{\lambda(\mu)}^k\right]$$

holds for μ close to zero.

(ii) We call a regular unfolding (A_μ^k, v_μ^k) of (A^k, v^k) *k-versal,* if any regular unfolding $(\tilde{A}_\lambda^k, \tilde{v}_\lambda^k)$ of (A^k, v^k) is induced from (A_μ^k, v_μ^k).

It can be shown that the versal unfoldings in the above sense are characterized by "versality <=>

transversality" arguments, but we will not give here the detail, and only give some results of calculations of versal unfoldings of normal forms obtained in the previous section. (For the detail, see Oka and Kokubu [11].)

Theorem 1.12: versal unfoldings

(1) The one-parameter family

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1\\ 0\end{bmatrix}\right] \quad \text{or equivalently,} \quad \begin{cases}\varepsilon\dot{x} = 1\\ \dot{y} = 0\end{cases}$$

is an infinite-versal unfolding of the infinite-order normal form

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1\\ 0\end{bmatrix}\right]. \quad \text{(rapid point: } \textit{Part I, Proposition } 4.12)$$

(2) The family

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\alpha+(1+\beta)x+\gamma xy\\ 1\end{bmatrix}\right] \quad \text{or} \quad \begin{cases}\varepsilon\dot{x} = \alpha+(1+\beta)x+\gamma xy\\ \dot{y} = 1\end{cases}$$

is a 2-versal unfolding of the normal form

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm x\\ 1\end{bmatrix}\right], \quad \left[\begin{matrix}\text{regular slow point on the characteristic}\\ \text{surface: } \textit{Part I, Proposition } 4.17\ (a_1)\end{matrix}\right]$$

(3) The family

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\alpha+\beta x\pm y+(a+\gamma)x+\delta xy\\ 1\pm(1+\zeta)x\end{bmatrix}\right] \quad \text{or} \quad \begin{cases}\varepsilon\dot{x} = \alpha+\beta x\pm y+(a+\gamma)x+\delta xy\\ \dot{y} = 1\pm(1+\zeta)x\end{cases}$$

is a 2-versal unfolding of the normal form

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm y+ax\\ 1\pm x\end{bmatrix}\right]. \quad \left[\begin{matrix}\text{impasse point: } \textit{Part I,}\\ \textit{Proposition } 4.17\ (a_2')\end{matrix}\right]$$

(4) The family

$$\left[\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\alpha+\beta x\pm y\pm x^2+\gamma xy\\ \delta\pm x+\zeta x^2\end{bmatrix}\right] \quad \text{or} \quad \begin{cases}\varepsilon\dot{x} = \alpha+\beta x\pm y\pm x^2+\gamma xy\\ \dot{y} = \delta\pm x+\zeta x^2\end{cases}$$

is a 2-versal unfolding of the normal form

$$\left[\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\pm y\pm x^2\\ \pm x\end{bmatrix}\right]. \quad \left[\text{canard: } \textit{Part I, Proposition } 4.18\ (b_2')\right]$$

Here ε, α, β, γ, δ, ζ are unfolding parameters.

2. APPENDIX

Appendix I: Fiber Bundles

In this Appendix, we will give a quick introduction to the concept of fiber bundles and some related applications which are necessary to understand this paper. Roughly speaking, a *fiber bundle* is a generalization of a direct-product space; it consists of four objects: a *total space* E, a *base space* M, a *fiber* F, and a *projection* π from E to M. Before stating the formal definition, it is instructive to study the following concrete examples which inspire their generalization to fiber bundles.

Example I.1: Cylinder

Let E be a cylinder $S^1 \times I$, where S^1 denotes the unit circle and I denotes the closed interval $[-1, 1]$. We denote S^1 by M and I by F and define the projection

$$\pi : E \triangleq S^1 \times I \rightarrow M \triangleq S^1$$

by neglecting the second component $F = I$ of the direct product $M \times F \triangleq S^1 \times I$. A geometrical interpretation of the image of the projection π is shown in Fig. A.1.

The above 4 objects, $(E, M, F; \pi) = (S^1 \times I, S^1, I; \pi)$, constitute an example of a *trivial fiber bundle*. Here, "trivial" is a technical term used to mean that the *total space* E has a *direct-product* structure. The name *fiber bundle* comes from the observation that E consists of pre-images $\pi^{-1}(x)$ for each $x \in M$, which looks like a collection of fibers forming the surface of a cylinder, as depicted in Fig. A.2. Consequently, we call the pre-image $\pi^{-1}(x)$ the *fiber of* $x \in M$. Observe that $\pi^{-1}(x)$ is homeomorphic to F.

Example I.2: Möbius band

The Möbius band E is formed by first twisting the band $[-\pi, \pi] \times I$ and then joining the two end edges together, as depicted in Fig. A.3. Assuming $M = S^1$ and $F = I$, we can define the natural projection[†]

$$\pi : \textit{Möbius band } E \rightarrow M = S^1$$

by neglecting the second component $F = I$. Therefore, the Möbius band is another example of a fiber bundle over S^1 with fiber I. However, the Möbius band E is *not* a trivial fiber bundle because E does *not* have a *direct-product* structure. (Recall that E is not just $[-\pi, \pi] \times I$; an additional twisting transformation is involved). Moreover the Möbius band E is *not orientable*. In contrast, the cylinder $S^1 \times I$ from *Example I.1* is orientable in the sense that it has a well-defined *inside* and *outside surface*. Nevertheless, the Möbius band is *locally trivial* in the sense that for each $x \in M$, there exists a neighborhood V_x of x such that $\pi^{-1}(V_x)$ is homeomorphic to the direct product $V_x \times I$; namely, $\pi^{-1}(V_x) \simeq V_x \times I$, where the symbol $\simeq$ denotes a

---

[†]Note that unlike the cylinder in Fig. A.1, where its boundary has two components (the top and bottom boundary), the boundary of the Möbius band has only one component which is equal to the *sum* of the top *and* bottom boundaries of the rectangle in Fig. A.3.

In this example, the total space E also consists of a bundle of fibers; namely,

$$E = \bigcup_{x \in M} \pi^{-1}(x)$$

where each fiber $\pi^{-1}(x)$ is homeomorphic to $F = I$, as shown in Fig. A.5.

We will now generalize the above examples to an *abstract* object called a fiber bundle.

Definition I.3. Fiber bundle

A *fiber bundle*† is a collection $(E, M, F; \pi)$ of smooth manifolds E, M, F, and a smooth mapping π satisfying the following two properties:

(i) [projection property] The mapping

$$\pi : E \to M$$

called the *projection*, is *surjective*.

(ii) [local triviality] For each $x \in M$, there exists an *open* neighborhood V in M, and a *diffeomorphism* $\Phi_V : \pi^{-1}(V) \to V \times F$ such that the diagram shown in Fig. A.6(a) *commutes;* i.e.,

$$\pi \Phi_V^{-1}(V \times F) = \pi'(V \times F)$$

where π' denotes the projection of the direct product $V \times F$ into V; i.e., $\pi'(x, f) = x$ for all $(x, f) \in V \times F$. In particular, the collection $(E, M, F; \pi)$ is called the *fiber bundle E over M with fiber F and projection π*; or simply the *fiber bundle E* when the identity of M, F, and π are obvious from the context. The space M is called the *base space*, or base manifold, F is called the *standard fiber*, and $\pi^{-1}(x)$ is called the *fiber of x*, which is denoted by E_x.

When the standard fiber F is a *vector space* V we call $(E, M, V; \pi)$ a *vector bundle.*

Example I.4.

Recall from *Example I.1* that the cylinder $E = S^1 \times I$ is a trivial fiber bundle over S^1. In a similar way, we can show that the *infinite cylinder* in Fig. A.7(a) is also a trivial fiber bundle over S^1 with fiber $F = \mathbf{R}$, and the natural projection π defined by neglecting the second component of $S^1 \times \mathbf{R}$. Moreover, in this case, $E = S^1 \times \mathbf{R}$ can be endowed with a vector bundle structure because the fiber $F = \mathbf{R}$ has a vector space structure. In contrast, $E = S^1 \times I$ from *Example I.1* is *not* a vector bundle because $F = I$ is *not* a vector space.

We can consider the infinite cylinder as another fiber bundle by interchanging the roles of S^1 and $\mathbf{R}$, as shown in Fig. A.7(b). Although the resulting product space $E' = \mathbf{R} \times S^1$ is also a trivial fiber bundle (with base manifold $M' = \mathbf{R}$ and standard fiber $F' = S^1$), it is *not* a vector bundle because $F = S^1$ is *not* a vector

† Rigorously speaking, this definition is actually for a *fiber space*, while the notion of fiber bundle requires a few additional properties. For a precise definition of the fiber bundle and some related topics, see e.g., Schutz [12] or Kobayashi and Nomizu [8].

space.

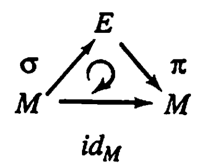
We will consider next the concept of a *section* of a fiber bundle.

Definition I.5. Section

A *section* of a fiber bundle $(E, M, F; \pi)$ is a smooth mapping

$$\sigma : M \to E$$

which satisfies $\pi \circ \sigma = id_M$, where id_M denotes the *identity map* on M. In other words, σ makes the following diagram *commute:*



In the special case where the fiber bundle is *trivial;* i.e., $E = M \times F$, a section σ must have the form

$$\sigma(x) = \left[x, f(x)\right] \ , \ f : M \to F \tag{I.1}$$

Conversely, for any smooth mapping $f : M \to F$, the map σ defined by (I.1) is a section of the trivial bundle $M \times F$. Note that in this special case, the section σ can be interpreted as the *graph* of a mapping f, as shown in Fig. A.8. Since the fiber bundle is a generalization of a direct-product space, the notion of a *section* can be interpreted as a generalization of the *graph of a mapping.*

Example I.6.

Consider the Mӧbius bands as a fiber bundle over S^1 with fiber $I = [-1,1]$. If we define the map σ_0 by assigning† $0 \in I$ in each fiber $\pi^{-1}(x)$, as the image of x, then this point in E projects under π back into x. Hence, σ_0 is a *section* of the Mӧbius band. This section is shown in Fig. A.9 as the lightly drawn closed loop made up of all mid points of the band.

It is easy to verify that any section σ of the Mӧbius band *must* cross 0 in the fiber $I = [-1,1]$, an example of which is depicted in Fig. A.9 by the *bold* closed loop. To see this, try drawing a closed loop along a Mӧbius strip *without* crossing the middle loop. Such a loop must necessarily make 2 "revolutions," compared to only *one* in the loop which crosses the *mid* point, before it returns to the original point. This implies that σ is multi-valued and hence such a closed loop is *not* a section.

Let us consider next an important example of a fiber bundle; namely; the *tangent bundle* of a smooth manifold M. For simplicity, we assume in the sequel that the manifold M is contained in a suitable Euclidean space $\mathbf{R}^m$.

---

†Recall an element of E is a *point* on the Mӧbius strip.

For each $x \in M$, let $T_x M$ denote the set of all tangent vectors of M at x, called the *tangent space* at x, and let TM denote the union of $T_x M$ for all $x \in M$; i.e.,

$$TM \triangleq \bigcup_{x \in M} T_x M$$

To show that TM is also a *manifold*, let us define a local coordinate system on TM as follow:

(1) Define the projection $\pi : TM \to M$ of each $\xi_x \in T_x M$ by $\pi(\xi_x) = x$. It follows that $\pi^{-1}(x) = T_x M$.

(2) Choose a local coordinate ϕ around $x \in M$; namely,

$$\phi : \hat{V} \to V, \hat{y} = (y_1, \cdots, y_n) \to y$$

where $\hat{V}$ denotes an open set in $\mathbf{R}^n$ and V denotes a local coordinate neighborhood of x in M. Such a local coordinate always exists since M is a manifold, and the Jacobian matrix $D\phi(\hat{y})$ is well-defined since M is contained in $\mathbf{R}^m$. Hence, any tangent vector ξ_y at y near x can be expressed as

$$(\xi_y^1, \xi_y^2, \cdots, \xi_y^n) \in \mathbf{R}^n$$

by means of the local coordinate ϕ; i.e.,

$$\xi_y = D\phi(\hat{y}) \begin{bmatrix} \xi_y^1 \\ \vdots \\ \xi_y^n \end{bmatrix}$$

Let us denote the standard basis of the vector space formed by the tangent vectors at y by the notation $\left[\frac{\partial}{\partial x_i}\right]_y$, $i = 1, 2, \cdots, n$; i.e.,

$$\left[\frac{\partial}{\partial x_i}\right]_y = D\phi(\hat{y}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th position}$$

In terms of the standard basis, each tangent vector ξ_y can be represented by

$$\xi_y^1 \left[\frac{\partial}{\partial x_1}\right]_y + \xi_y^2 \left[\frac{\partial}{\partial x_2}\right]_y + \cdots + \xi_y^n \left[\frac{\partial}{\partial x_n}\right]_y$$

Using the projection π and the local coordinate ϕ defined above, we can now define

$$\Phi_V : \hat{V} \times \mathbf{R}^n \to \pi^{-1}(V)$$

via the mapping $(\hat{y}, \xi_y^1, \cdots, \xi_y^n) \to \xi_y$. It is easy to verify that the collection $\left\{ \left[\pi^{-1}(V), \Phi_V \right] \right\}$ associated with each $x \in M$ defines a local coordinate system in TM. Hence TM is a manifold.

Observe that the projection $\pi : TM \to M$ is surjective, and the local coordinate system $\left\{ \left[\pi^{-1}(V), \Phi_V \right] \right\}$ satisfies the condition of *local triviality;* namely, a local direct product $\hat{V} \times \mathbf{R}^n$. Moreover, each fiber $\pi^{-1}(x) = T_x M$ is isomorphic to $\mathbf{R}^n$ and is therefore an n-dimensional. Consequently, the 4 objects $\{TM, M, \mathbf{R}^n; \pi\}$ constitutes a *vector bundle* with the standard fiber $\mathbf{R}^n$, as depicted in Fig. A.10. The manifold TM is called the *tangent bundle of* M.

Example I.7. Tangent bundle of n-sphere S^n

Consider the n-dimensional sphere S^n in the (n+1)-dimensional Euclidean space $\mathbf{R}^{n+1}$; i.e., $S^n \subset \mathbf{R}^{n+1}$. If we denote the usual Euclidean inner product by the notation $\langle , \rangle$, then for $x \in S^n$, the tangent space $T_x S^n$ at x is identified with the set

$$T_x S^n = \left\{ u \in \mathbf{R}^{n+1} \mid \langle x, u \rangle = 0 \right\}$$

as shown in Fig. A.11. It follows that

$$TS^n = \left\{ (x, u) \in S^n \times \mathbf{R}^{n+1} \mid \langle x, u \rangle = 0 \right\} \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$$

is a tangent bundle.

To be more concrete, consider the 1-dimensional sphere S^1 in $\mathbf{R}^2$, as shown in Fig. A.12. In this case, it is convenient to identify $\mathbf{R}^2$ with the complex plane $\mathbb{C}$. Hence, the unit circle S^1 can be represented compactly by

$$S^1 = \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\} = \left\{ e^{i\theta} \,\middle|\, 0 \le \theta \le 2\pi \right\}$$

where $i \triangleq \sqrt{-1}$. At any point $z = e^{i\theta} \in S^1$, the tangent space $T_z S^1$ is represented by

$$T_z S^1 = \left\{ \lambda i z \mid \lambda \in \mathbf{R} \right\}.$$

Hence,

$$TS^1 = \left\{ (z, \lambda iz) \mid |z| = 1, \lambda \in \mathbf{R} \right\}$$

is the tangent bundle of the unit circle S^1.

In this case, we can define the mapping

$$\tau : T S^1 \rightarrow S^1 \times \mathbf{R}$$

by

$$(z, \lambda iz) \mapsto (z, \lambda) .$$

Since τ is a diffeomorphism preserving each fiber of $T S^1$, it follows that the tangent bundle $T S^1$ is globally trivial, i.e.,

$$T S^1 \simeq S^1 \times \mathbf{R}^1$$

Consequently, the tangent bundle of the unit circle is diffeomorphic to the *infinite cylinder* in *Example I.4*.

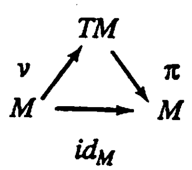
In contrast, it can be shown that the tangent bundle $T S^2$ is *not* trivial.

We are now ready to define a *vector field* on a manifold M.

Definition I.8: Vector Field on Manifold

Any *section* $v : M \rightarrow TM$ of a tangent bundle TM of a smooth manifold M is called a *vector field* on M.

It follows from the definition of a section that v is a vector field on M if the following diagram commutes; namely, $\pi \circ v = id_M$:



Note that for any $x \in M$, $v(x) \in T_x M$. It follows that v assigns a tangent vector $v(x)$ at x to each point $x \in M$. Therefore, the above definition coincides with our intuitive notion of a vector field. In fact, we can even visualize a vector field on S^1 by using the property $T S^1 \simeq S^1 \times \mathbf{R}^1$; namely, let us cut the cylinder $S^1 \times \mathbf{R}^1$ and identify $T S^1$ with $I \times \mathbf{R}^1$ for $I = [0, 2\pi]$, as shown in Fig. A.13. Hence, a point $\xi_0 \in T S^1$ is identified with the point $(\theta, \lambda) \in I \times \mathbf{R}$. It follows that a vector field v on S^1 is any mapping

$$\lambda : I \rightarrow \mathbf{R}^1$$

which satisfies $\lambda(0) = \lambda(2\pi)$.

For the remaining part of *Appendix I*, we will consider the notion of a mapping between two vector bundles. Recalling that vector bundles have vector spaces as their fibers and noting that the *natural homomorphism* between vector spaces is a *linear* mapping, it follows that a mapping between vector bundles must *preserve* fibers and map each fiber linearly. This observation motivates our next definition.

Definition. I.9: Bundle Map

Let (E, M, V, π) and (E', M', V', π') be smooth vector bundles. A smooth mapping

$$R : E \to E'$$

is called a *bundle map* if, for each $x \in M$, R maps each fiber $E_x = \pi^{-1}(x)$ *linearly* into the fiber $E'_{x'}$ for some $x' \in M'$. In other words, there exists a smooth map

$$r : M \to M'$$

such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{R} & E' \\ \pi \downarrow & \curvearrowright & \downarrow \pi' \\ M & \xrightarrow{r} & M' \end{array}$$

i.e., $\pi' \circ R = r \circ \pi$ holds, *and* for any $x \in M$,

$$R \mid_{E_x} : E_x \simeq V \to V' \simeq E'_{r(x)}$$

is a *linear map* between vector spaces. The map $r : M \to M'$ is called a *base map* since M and M' are the bases of the respective vector bundles.

In the special case where $(E, M, V, \pi) = (E', M', V', \pi')$ and where the base map is the *identity* map id_M of M, the bundle map is called a *bundle endomorphism.*

If in addition a bundle endomorphism is *invertible*, we call it a *bundle automorphism.*

It is important to note that a bundle endomorphism $R : E \to E$ assigns, to each point $x \in M$, a *linear map*

$$R_x = R \mid_{E_x} : E_x \to E_x \ .$$

Hence, we can regard the bundle endomorphism as a map

$$R : x \mapsto R_x \ .$$

Let us denote the set of all such linear maps of fibers of E by $End(E)$; i.e.,

$$End(E) = \{R_x : E_x \to E_x \text{ are linear maps for all } x \in M\} .$$

It follows from the above observation that a bundle endomorphism R can be interpreted as a mapping

$$R : M \to End(E)$$

where $x \mapsto R_x$.

In what follows we will equip a vector bundle structure with the set $End(E)$ so that the above map $R : M \to End(E)$ is a *section.*

Let $\tilde{\pi} : End(E) \to M$ be a map defined by $R_x \to x$ and, for the vector space V, let $End(V)$ denote the set of all linear maps from V into itself. Since E is the total space of a vector bundle (E, M, V, π) with the standard fiber V, the inverse image $\tilde{\pi}^{-1}(x)$, for any $x \in M$, is

$$\tilde{\pi}^{-1}(x) = \{R_x : E_x \to E_x \text{ are linear maps }\}$$

which is isomorphic to $End(V)$ because E_x is isomorphic to V. Hence, we have a vector bundle $End(E)$ over M with the standard fiber $End(V)$ and the projection $\tilde{\pi}$, which we call the *endomorphism bundle* of the vector bundle (E, M, V, π). A bundle endomorphism $R : E \to E$ can now be identified with a *section* of the endomorphism bundle. Such an identification plays an important role in *Section 3* of this paper, and in *Appendix II.*

Appendix II: Tensor Bundle and Tensor Field

Let U and V be vector spaces over $\mathbf{R}$. Let $M(U,V)$ denote the vector space generated by the pairs (u,v) where $u \in U$ and $v \in V$. In other words, $M(V,V)$ consists of linear combinations of finite number of pairs (u,v). Let N denote the vector subspace of $M(U,V)$ which is spanned by elements of $M(U,V)$ of the form

$$(u_1+u_2, v) - (u_1, v) - (u_2, v)$$

$$(u, v_1+v_2) - (u, v_1) - (u, v_2)$$

$$(ru, v) - r(u, v)$$

$$(u, rv) - r(u, v)$$

where u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$, and $r \in \mathbf{R}$.

Definition. II.1. Tensor Product

The *tensor product* of U and V is defined to be the *quotient vector space* $M(U,V)/N$, and is denoted by $U \otimes V$.

There is a natural bilinear map

$$U \times V \to M(U,V) \to M(U,V)/N \triangleq U \otimes V$$

We denote the image of $(u,v) \in U \times V$ by $u \otimes v$.

Proposition II.2. Properties of Tensor Product

(i) Let $e_1, e_2, \cdots, e_m$ be a basis of U and let $f_1, f_2, \cdots, f_n$ be a basis of V. Then

$$\{e_i \otimes f_j : i = 1,2, \cdots, m, j = 1,2, \cdots, n\}$$

is a basis of $U \otimes V$.

2) $dim\ U \otimes V = (dim\ U)(dim\ V)$

3) Let $Hom(U,V)$ denote the set of all *linear* maps from U into V. Then $Hom(U,V)$ is *isomorphic* to $V \otimes U^*$, where U^* is the *dual* vector space of U.

Proof.

(1) Since any element of $M(U,V)$ can be written in the form

$$\sum_p r_p(u_p, v_p) \quad , \quad r_p \in \mathbf{R}$$

where $(u_p, v_p) \in M(U,V)$, $p = 1,2, \cdots, p_0$, for some p_0 it follows that the elements of $U \otimes V$ has the form

$$\sum_p r_p(u_p \otimes v_p)$$

Since u_p and v_p can be expressed in terms of their respective basis; namely,

$$u_p = \sum_{i=1}^{m} \alpha_{p_i} e_i \quad \text{and} \quad v_p = \sum_{j=1}^{n} \beta_{p_j} f_j$$

where $\alpha_{p_i}, \beta_{p_j} \in \mathbf{R}$, it follows that

$$\sum_p (u_p \otimes v_p) = \sum_{p,i,j} r_p\ \alpha_{p_i}\ \beta_{p_j}\ (e_i \otimes f_j)$$

Hence, every element of $U \otimes V$ is spanned by $e_i \otimes f_j$. The *linear independence* of $\{e_i \otimes f_j\}$ follows from the linear independence of $\{e_i\}$ and $\{f_j\}$. For a more precise proof, see [8].

(2) This property follows directly from *property (1).*

(3) From *property (1),* the tensor product $V \otimes U^*$ is spanned by $\{f_j \otimes e_i^*\}$, where $\{e_i^*\}$ is the *dual basis* of $\{e_i\}$, i.e., e_i^* is the *linear functional*

$$e_i^* : U \to \mathbf{R}, u \to \alpha_i .$$

for any $u = \sum_{i=1}^{m} \alpha_i e_i \in U$.

To each $f_j \otimes e_i^*$, let us assign a linear map from U to V via the matrix representation

$$A_{ij} = \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} \quad \text{row } j$$

column i

in terms of the bases $\{e_i\}$ of U and $\{f_j\}$ of V. We can define a map h by this assignment and extend it linearly to the entire vector space $V \otimes U^*$. Since

$$h : f_j \otimes e_i^* \mapsto A_{ij}$$

gives a one-to-one correspondence between the bases of $V \otimes U^*$ and those of $Hom(U, V)$, it follows that h is an isomorphism.

■

Observe that in the special case where $U = V$, the above property (3) reduces to the following isomorphism:

$$V \otimes V^* \leftrightarrows End(V)$$

where $\leftrightarrows$ denotes an isomorphism. We call $V \otimes V^*$ the *tensor space* of type (1,1).

Recall now the endomorphism bundle $\left[End(E), M, End(V)\right]$ of a vector bundle (E, M, V) defined earlier in *Appendix I*. If we identify the vector space $End(V)$ with $V^* \otimes V$, whose isomorphism has just been established, then we can also identify the vector bundle $End(E)$ with a vector bundle over M whose standard fiber is the tensor space $V \otimes V^*$. We denote this vector bundle by $(E \otimes E^*, M, V \otimes V^*)$ and call it the *tensor bundle* of (E, M, V) of type (1,1).

In the remaining part of this appendix, we will consider the special case of the tensor bundle of a tangent bundle TM; i.e., $TM \otimes T^*M$, where T^*M denotes the *cotangent bundle* of M.

Recall from *Appendix I* that a bundle endomorphism R of TM can be considered as a *section* of the endomorphism bundle $End(TM)$. Since $End(TM)$ can be identified with the tensor bundle $TM \otimes T^*M$, it follows that the bundle endomorphism R of TM is a section of the tensor bundle $TM \otimes T^*M$; i.e., a mapping

$$R : x \mapsto R(x) \in T_xM \otimes T_x^*M \ .$$

Since the vector space $T_xM \otimes T_x^*M$ is spanned by

$$\left[\frac{\partial}{\partial x_j}\right]_x \otimes (dx_i)_x \ , \ x \in M$$

where $\left\{(dx_i)_x\right\}$ and $\left\{\left[\frac{\partial}{\partial x_j}\right]_x\right\}$ denote a basis of T_x^*M and T_xM, respectively, in terms of a local coordinate system around $x \in \dot{M}$, it follows that the map R can be expressed as

$$R = \sum_{i,j} R_{ij}(x) \frac{\partial}{\partial x_j} \otimes dx_i$$

We call the map R a *tensor field* on M of type (1,1).

Appendix III. Transformation Group of Constrained Systems (or Generalized Vector Fields)

Our purpose of this appendix is to discuss the structure of the set $G = AUT(TM) \rtimes Diff(M)$ of all transformations of constrained systems or generalized vector fields. Here, a transformation consists of a pair (P, ϕ), where P denotes a bundle automorphism f TM and ϕ is a diffeomorphism of M. Our first task is to prove that the set G forms a group.

Proposition III.1.

The set G forms a *group* under the multiplication operation

$$(P, \phi) \cdot (Q, \psi) = (P \circ T\phi \circ Q \circ T\phi^{-1}, \phi \circ \psi) \tag{III.1}$$

where (P, ϕ) and (Q, ψ) are *transformations* of a constrained system, or generalized vector field, and $T\phi$ denotes the tangent map of ϕ.

Proof.

We will show the group axioms are satisfied.

(i) *Associativity*

For any (P, ϕ), (Q, ψ), and $(R, \zeta) \in G$, we will prove that

$$\left[(P, \phi) \cdot (Q, \psi)\right] \cdot (R, \zeta) = (P, \phi) \cdot \left[(Q, \psi) \cdot (R, \zeta)\right]. \tag{III.2}$$

First observe that *composition* among *diffeomorphisms* is associative; namely, $(\phi \circ \psi) \circ \zeta = \phi \circ (\psi \circ \zeta)$. Next, observe that the *chain rule* operation is *functorial;* namely, $T(\phi \circ \psi) = T\phi \circ T\psi$. Applying these two properties repeatedly to the left-hand side (l.h.s.) and the right-hand side (r.h.s.) of (III.2), respectively, we obtain:

$$\begin{aligned} l.h.s. &= \left[(P, \phi) \cdot (Q, \psi)\right] \cdot (R, \zeta) = (P \circ T\phi \circ Q \circ T\phi^{-1}, \phi \circ \psi) \cdot (R, \zeta) \\ &= \left[(P \circ T\phi \circ Q \circ T\phi^{-1}) \circ T(\phi \circ \psi) \circ R \circ T(\phi \circ \psi)^{-1}, (\phi \circ \psi) \circ \zeta\right] \\ &= \left[P \circ T\phi \circ Q \circ T\psi \circ R \circ T\psi^{-1} \circ T\phi^{-1}, (\phi \circ \psi) \circ \zeta\right] \end{aligned} \tag{III.3}$$

$$\begin{aligned} r.h.s. &= (P, \phi) \cdot \left[(Q, \psi) \cdot (R, \zeta)\right] = (P, \phi) \cdot (Q \circ T\psi \circ R \circ T\psi^{-1}, \psi \circ \zeta) \\ &= \left[P \circ T\phi \circ (Q \circ T\psi \circ R \circ T\psi^{-1}) \circ T\phi^{-1}, \phi \circ (\psi \circ \zeta)\right] \end{aligned} \tag{III.4}$$

Since (III.3) and (III.4) are identical, (III.2) holds.

(ii) *Existence of unit element*

We claim that (Id_{TM}, id_M) is the *unit element* of G, where Id_{TM} denotes the identity map (bundle automorphism) of TM and id_M denotes the identity map of M. Indeed, for any $(P, \phi) \in G$, we have

$$(Id_{TM}, id_M) \cdot (P, \phi) = \left[Id_{TM} \circ T(id_M) \circ P \circ T(id_M)^{-1}, id_M \circ \phi \right] = (P, \phi)$$

and

$$(P, \phi) \cdot (Id_{TM}, id_M) = (P \circ T\phi \circ Id_{TM} \circ T\phi^{-1}, \phi \circ id_M) = (P, \phi).$$

(iii) *Existence of inverse element*

We claim that the *inverse* of any element $(P, \phi) \in G$ is given by $(T\phi^{-1} \circ P^{-1} \circ T\phi, \phi^{-1})$, where P^{-1} (resp., ϕ^{-1}) is the inverse of P (resp., ϕ). Indeed,

$$\begin{aligned}(P, \phi) \cdot (T\phi^{-1} \circ P^{-1} \circ T\phi, \phi^{-1}) &= \left[P \circ T\phi \circ (T\phi^{-1} \circ P^{-1} \circ T\phi) \circ T\phi^{-1}, \phi \circ \phi^{-1} \right] \\ &= (Id_{TM}, id_M)\end{aligned}$$

$$\begin{aligned}(T\phi^{-1} \circ P^{-1} \circ T\phi, \phi^{-1}) \cdot (P, \phi) &= \left[(T\phi^{-1} \circ P^{-1} \circ T\phi) \circ T\phi^{-1} \circ P \circ (T\phi^{-1})^{-1}, \phi^{-1} \circ \phi \right] \\ &= (Id_{TM}, id_M)\end{aligned}$$

■

Remark

Since the set $\left\{ (P, id_M) \in G \mid P \text{ is any bundle automorphism} \right\}$ forms a normal subgroup of G, the group G is said to be a *semi-direct product group* of the "group of bundle automorphisms" and the "group of diffeomorphisms," in view of the structure induced by the above *multiplication* operation. In order to indicate it, we use the symbol   instead of $\times$.

Proposition III.2

The group G acts on the set of all constrained systems, (or generalized vector fields); that is, the following formulae hold:

(i) $(Id_{TM}, id_M)_{\#}(\mathrm{A}, \mathrm{v}) = (\mathrm{A}, \mathrm{v})$

(ii) $(Q, \psi)_{\#} \left[(P, \phi)_{\#}(\mathrm{A}, \mathrm{v}) \right] = \left[(Q, \psi) \cdot (P, \phi) \right]_{\#}(\mathrm{A}, \mathrm{v})$ for any (P, ϕ), $(Q, \psi) \in G$, and for any $(\mathrm{A}, \mathrm{v}) \in \mathcal{CX}(M)$ $\left[\text{or } \mathcal{GX}(M) \right]$.

Proof. Recall from definition (2.10) of *Section 2* that

$$(P,\phi)_\#(\mathbf{A},\mathbf{v}) \triangleq (P\circ T\phi\circ\mathbf{A}\circ T\phi^{-1}, P\circ T\phi\circ\mathbf{v}\circ\phi^{-1}).$$

Applying this definition, we obtain

(i) $(Id_{TM}, id_M)_\#(\mathbf{A},\mathbf{v}) = (Id_{TM}\circ T(id_M)\circ\mathbf{A}\circ T(id_M)^{-1}, Id_{TM}\circ T(id_M)\circ\mathbf{v}\circ id_M^{-1}) = (\mathbf{A},\mathbf{v})$.

(ii) $(Q,\psi)_\#\left[(P,\phi)_\#(\mathbf{A},\mathbf{v})\right]$

$$\begin{aligned}
&= (Q,\psi)_\#(P\circ T\phi\circ\mathbf{A}\circ T\phi^{-1}, P\circ T\phi\circ\mathbf{v}\circ\phi^{-1})\\
&= \left[Q\circ T\psi\circ(P\circ T\phi\circ\mathbf{A}\circ T\phi^{-1})\circ T\psi^{-1}, Q\circ T\psi\circ(P\circ T\phi\circ\mathbf{v}\circ\phi^{-1})\circ\psi^{-1}\right]\\
&= (Q\circ T\psi\circ P\circ T\phi\circ\mathbf{A}\circ T\phi^{-1}\circ T\psi^{-1}, Q\circ T\psi\circ P\circ T\phi\circ\mathbf{v}\circ\phi^{-1}\circ\psi^{-1})\\
&= (Q\circ T\psi\circ P\circ T\psi^{-1}\circ T\psi\circ T\phi\circ\mathbf{A}\circ T\phi^{-1}\circ T\psi^{-1}, Q\circ T\psi\circ P\circ T\psi^{-1}\circ T\psi\circ T\phi\circ\mathbf{v}\circ\phi^{-1}\circ\psi^{-1})\\
&= \left[(Q\circ T\psi\circ P\circ T\psi^{-1})\circ T(\psi\circ\phi)\circ\mathbf{A}\circ T(\psi\circ\phi)^{-1}, (Q\circ T\psi\circ P\circ T\psi^{-1})\circ T(\psi\circ\phi)\circ\mathbf{v}\circ(\psi\circ\phi^{-1})\right]\\
&= (Q\circ T\psi\circ P\circ T\psi^{-1}, \psi\circ\phi)_\#(\mathbf{A},\mathbf{v})\\
&= \left[(Q,\psi)\cdot(P,\phi)\right]_\#(\mathbf{A},\mathbf{v}).
\end{aligned}$$

■

Appendix IV. Proof of: "*Y_* and R_** are well-defined"

In order to prove that the mappings Y_* and R_* defined in *Section 3* from *TM* to $T(TM)$ are well-defined, let us first review the actions of the coordinate transformations of *TM* and $T(TM)$. From the definition of the *tangent bundle* in *Appendix I*, we can induce a coordinate transformation of *TM* from that of M. In particular, let (x,ξ) and (y,ζ) represent the local coordinates of a point on *TM*, where x and y are related by a coordinate transformation $y = \phi(x)$ on M. Denoting the corresponding transformation of (x,ξ) on *TM* by $T\phi(x,\xi)$, we obtain

$$(y,\zeta) = T\phi(x,\xi) = \left[\phi(x), D\,\phi(x)\cdot\xi\right]$$

In other words, a coordinate change ϕ of x in M "induces" a coordinate change $T\phi$ of (x,ξ) in *TM*. We can iterate this transformation rule recursively to induce a similar transformation on $T(TM)$ since *TM* itself can be considered as a manifold. Denoting this transformation by $T^2\phi$, we obtain

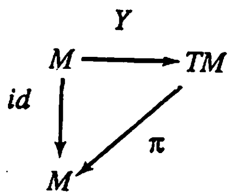
$$\begin{aligned}
(y,\zeta,w,v) = T\phi^2(x,\xi,v,\eta) &= \left[T\phi(x,\xi), D\left[T\phi(x,\xi)\right]\cdot(v,\eta)\right]\\
&= \left[\phi(x), D\,\phi(x)\cdot\xi, D\,\phi(x)\cdot v, D\,\phi(x)\cdot\eta + D^2\phi(x)\cdot v\cdot\xi\right]
\end{aligned}$$

Since both Y_* and R_* are mappings from *TM* into $T(TM)$, for them to be well defined, we must prove the following two relationships hold for *any* coordinate transformation ϕ of M:

$$T^2\phi \circ Y_* = Y_* \circ T\phi \tag{IV.1}$$

$$T^2\phi \circ R_* = R_* \circ T\phi \tag{IV.2}$$

The mapping Y is a well-defined vector field on M; i.e.,



or $Y : M \rightarrow TM$, with $\pi \circ Y = id_M$

Observe that in terms of the local coordinate system (x, ξ) on TM, the image $Y(x)$ of x on M must be written with 2 components; namely,

$$Y(x) = \left[x, \tilde{Y}(x)\right]$$

We will call the *second* component $\tilde{Y}(x)$ of $Y(x)$ as the *principal part* of the vector field Y. To avoid clutter, however, we will often abuse our notation and simply denote a vector field by its principal part and ignore writing the first component. Since Y is well defined, it must satisfy

$$T\phi \circ Y = Y \circ \phi$$

or

$$\left[\phi(x), D\,\phi(x) \cdot \tilde{Y}(x)\right] = \left[\phi(x), \tilde{Y}\left[\phi(x)\right]\right]$$

It follows that

$$D\,\phi(x) \cdot Y(x) = Y\left[\phi(x)\right] \tag{IV.3}$$

Similarly, since a *bundle endomorphism*

$$R : TM \rightarrow TM$$

is also well defined, it must satisfy

$$T\phi \circ R = R \circ T\phi$$

or

$$\left[\phi(x), D\,\phi(x) \cdot R(x) \cdot \xi\right] = \left[\phi(x), R\left[\phi(x)\right] \cdot D\,\phi(x) \cdot \xi\right]$$

It follows that

$$D\,\phi(x) \cdot R(x) = R\left[\phi(x)\right] \cdot D\,\phi(x) \tag{IV.4}$$

We are now ready to prove (IV.1) by writing

$$T^2\phi \circ Y_*(x,\xi) = T^2\phi \left[x,\xi,Y(x),DY(x)\cdot\xi\right]$$

$$= \left[\phi(x),D\,\phi(x)\cdot\xi,D\,\phi(x)\cdot Y(x),D\,\phi(x)\cdot DY(x)\cdot\xi + D^2\phi(x)\cdot Y(x)\cdot\xi\right] \tag{IV.5}$$

and

$$Y_* \circ T\phi(x,\xi) = Y_*\left[\phi(x),D\,\phi(x)\cdot\xi\right]$$

$$= \left[\phi(x),D\,\phi(x)\cdot\xi,Y\left[\phi(x)\right],DY\left[\phi(x)\right]\cdot D\,\phi(x)\cdot\xi\right] \tag{IV.6}$$

Differentiating (IV.3) with respect to x, we obtain

$$D^2\phi(x)\cdot Y(x) + D\,\phi(x)\cdot DY(x) = DY\left[\phi(x)\right]\cdot D\,\phi(x) \tag{IV.7}$$

Substituting (IV.3) and (IV.7) into (IV.5), we obtain (IV.6). Hence (IV.1) holds for any coordinate system.

Similarly, we can prove (IV.2) by writing

$$T^2\phi \circ R_*(x,\xi) = T^2\phi(x,\xi,0,R(x)\cdot\xi) = \left[\phi(x),D\,\phi(x)\cdot\xi,0,D\,\phi(x)\cdot R(x)\cdot\xi\right] \tag{IV.8}$$

and

$$R_* \circ T\phi(x,\xi) = R_*\left[\phi(x),D\,\phi(x)\cdot\xi\right] = \left[\phi(x),D\,\phi(x)\cdot\xi,0,R\left[\phi(x)\right]\cdot D\,\phi(x)\cdot\xi\right] \tag{IV.9}$$

Substituting (IV.4) into (IV.8), we obtain (IV.9). Hence (IV.2) also holds for any coordinate system. This completes our proof.

■

Appendix V. Proof of: "$\exp t(R^k,Y^{k+1})$ *are well-defined*"

Let (R,Y) and (R',Y') be representatives of the $(k,k+1)$-jet (R^k,Y^{k+1}) at $x_0 \in M$ of an infinitesimal generator.

Proposition V.1

The local one-parameter groups $\underline{\exp}\,t(R,Y)$ and $\underline{\exp}\,t(R',Y')$ in G are $(k,k+1)$-jet equivalent at x_0 for any t. Furthermore, the $(k,k+1)$-jet

$$\underline{\exp}\,t(R^k,Y^{k+1}) \triangleq j_{x_0}^{k,k+1}\,\underline{\exp}\,t(R,Y)$$

forms a local one-parameter group in $J_{x_0}^{k,k+1}G$.

Proof. Recall that $\underline{\exp}\,t(R,Y)$ is defined by

$$\underline{\exp}\,t(R,Y) = \sigma^{-1}\circ\exp t(R_*+Y_*)$$

where σ is the group isomorphism on its image

$$\sigma : AUT(TM) \rtimes Diff(M) \rightarrow Diff(TM)$$

$$(P,\phi) \mapsto P \circ T\phi$$

and $R_* + Y_*$ is a vector field on TM defined by $(R_*+Y_*)(x,\xi) = \left[x,\xi,Y(x),[R(x)+DY(x)]\xi\right]$ for a local coordinate (x,ξ) of TM (see Chapter 3).

Since (R,Y) and (R',Y') are $(k,k+1)$-jet equivalent at x_0, the vector fields (R_*+Y_*) and $(R_*'+Y_*')$ on TM are $(k+1)$-jet equivalent at $(x_0,0)$. Moreover, these vector fields vanish at $(x_0,0)$, for $Y(x_0)=0$. Hence, the flows $\exp t(R_*+Y_*)$ and $\exp t(R_*'+Y_*')$ are also $(k+1)$-jet equivalent at $(x_0,0)$. (See Appendix 3 of [13]). These flows can be written in the form

$$\exp t(R_*+Y_*)(x,\xi) = \left[\phi^t(x), F^t(x)\xi\right]$$

$$\exp t(R_*'+Y_*')(x,\xi) = \left[\phi'^t(x), F'^t(x)\xi\right]$$

where ϕ^t and ϕ'^t are diffeomorphisms of M, and F^t and F'^t are bundle isomorphisms of TM covering ϕ^t and ϕ'^t, respectively. The $(k+1)$-jet equivalence of $\exp t(R_*+Y_*)$ and $\exp t(R_*'+Y_*')$ at $(x_0,0)$ implies the $(k,k+1)$-jet equivalence of (F^t,ϕ^t) and (F'^t,ϕ'^t) at x_0. It follows that $\sigma^{-1}(F^t,\phi^t)$ and $\sigma^{-1}(F'^t,\phi'^t)$ are also $(k,k+1)$-jet equivalent. This completes the proof of the first part of Proposition V.1.

It remains to prove

$$\underline{\exp}(t+s)(R^k,Y^{k+1}) = \underline{\exp}\, t(R^k,Y^{k+1}) \cdot \underline{\exp}\, s(R^k,Y^{k+1}) .$$

This follows upon taking the $(k,k+1)$-jet of both sides of

$$\underline{\exp}(t+s)(R,Y) = \underline{\exp}\, t(R,Y) \cdot \underline{\exp}\, s(R,Y)$$

where (R,Y) is a representative of (R^k,Y^{k+1}), and observing that

$$j_{x_0}^{k,k+1}\left\{\underline{\exp}\, t(R,Y) \cdot \underline{\exp}\, s(R,Y)\right\} = \left[j_{x_0}^{k,k+1}\underline{\exp}\, t(R,Y)\right] \cdot \left[j_{x_0}^{k,k+1}\underline{\exp}\, s(R,Y)\right]$$

■

Appendix VI. Proof of the Reduction Theorem for Constrained System Normal Forms

The basic outline of the proof of this theorem is similar to that in [13] for the *Reduction Theorem 5.4*. There is a significant difference, however, between the bracket product { , } for generalized vector fields, which does *not* satisfy the *Jacobi identity*, and the Lie bracket for vector fields, which does. It is necessary therefore for us to devise another approach in place of the Jacobi identity.

Consider the kth-order normal form problem

$$\frac{d}{dt} h_k(t) = -\left\{\xi^k, a^{k-1} + h_k(t)\right\}_k \tag{VI.1}$$

with

$$\left\{\xi^{k-1}, a^{k-1}\right\}^{k-1} = 0 \qquad \text{(VI.2)}$$

where

$$\xi^k = (R^k, Y^{k+1}) \in J^{k,k+1}_{x_0} \mathcal{G}\mathcal{X},$$

$$a^{k-1} = (\mathrm{A}^{k-1}, \mathrm{v}^{k-1}) \in J^{k-1}_{x_0} \mathcal{C}\mathcal{X}^r,$$

and

$$h_k = (\mathrm{A}_k, \mathrm{v}_k) \in H_k \mathcal{C}\mathcal{X}^r .$$

Choose the subspace

$$B_k = \left\{J^{k,k+1}_{x_0} \mathcal{G}\mathcal{X}, a_0\right\}_k$$

and let $\hat{B}_k$ denote the complementary space of B_k in $H_k \mathcal{C}\mathcal{X}^r$.

We will prove first that for any two elements h_k and h_k' satisfying $\pi_k(h_k) = \pi_k(h_k')$, we can deform h_k to h_k' by integrating (VI.1) with (VI.2). Indeed, since $h_k - h_k' \in B_k$, h_k is of the form

$$h_k = \{\xi_k, a_0\} + h_k'$$

for some ξ_k . Consider the differential equation

$$\frac{d}{dt} h_k(t) = - \{\xi_k, a^{k-1} + h_k(t)\}_k$$

under the initial condition $h_k(0) = h_k$. Note that the condition (VI.2) is satisfied for γ_k. Since $\{\xi_k, a^{k-1} + h_k\}_k = \{\xi_k, a_0\}$, it follows that the solution of this differential equation is given by

$$h_k(t) = h_k(0) - t\{\xi_k, a_0\}$$

Hence, by choosing t = 1, we obtain

$$h_k(1) = h_k(0) - \{\xi_k, a_0\} = h_k - \{\xi_k, a_0\} = h_k'$$

which proves our preceding assertion.

It suffices therefore to prove that the normal form problem (VI.1)-(VI.2) reduces to the one on $\hat{B}_k$; namely,

$$\frac{d}{dt} \hat{b}_k(t) = - \pi_k \left[\left\{\xi^{k-1}, a^{k-1} + \hat{b}_k(t)\right\}_k\right] \qquad \text{(VI.3)}$$

with (VI.2), as in the reduction theorem for vector field normal forms. Putting $h_k(t) = b_k(t) + \hat{b}_k(t)$, and

$b_k(t) = \left\{\zeta_k(t), a_0\right\}$ for some $\zeta_k(t) \in H_{k,k+1}\mathcal{G}\mathcal{X}$, we obtain

$$\left\{\xi^k, a^{k-1} + h_k(t)\right\}_k = \left\{\xi^{k-1}, a^{k-1} + \hat{b}_k(t)\right\}_k + \left\{\xi_k, a^{k-1} + \hat{b}_k(t)\right\}_k + \left\{\xi^k, b_k(t)\right\}_k$$

$$= \left\{\xi^{k-1}, a^{k-1} + \hat{b}_k(t)\right\}_k + \left\{\xi_k, a_0\right\} + \left\{\xi_0, \left\{\zeta_k(t), a_0\right\}\right\}.$$

Since the second term belongs to B_k, we must prove that the third term also belongs to B_k. If it can be proved, then the projection of the equation (VI.1) by π_k becomes (VI.3), which completes the proof, since (VI.3) depends only on $\hat{b}_k$, not on b_k, and therefore, is solved within $\hat{B}_k$.

For $\xi_0 = (R_0, Y_1)$, $a_0 = (A_0, v_0)$, $\zeta_k = (Z_k, t_{k+1})$,

$$\left\{\xi_0, \{\zeta_k, a_0\}\right\} = \left\{(R_0, Y_1), \left\{(Z_k, t_{k+1}), (A_0, v_0)\right\}\right\}$$

$$= \left\{(R_0, Y_1), \left[Z_k A_0 - \mathcal{L}_{t_{k+1}} A_0, Z_k v_0 - [t_{k+1}, v_0]\right]\right\}$$

$$= \left[R_0(Z_k A_0 - \mathcal{L}_{t_{k+1}} A_0) - \mathcal{L}_{Y_1}(Z_k A_0 - \mathcal{L}_{t_{k+1}} A_0),\ R_0\left[Z_k v_0 - [t_{k+1}, v_0]\right] - \left[Y_1, Z_k v_0 - [t_{k+1}, v_0]\right]\right]$$

Note that condition (VI.2) implies $\{\xi_0, a_0\} = 0$; i.e.,

$$\left\{(R_0, Y_1), (A_0, v_0)\right\} = \left[R_0 A_0 - \mathcal{L}_{Y_1} A_0, R_0 v_0 - [Y_1, v_0]\right] = 0 .$$

Hence, we have

$$R_0 A_0 = \mathcal{L}_{Y_1} A_0 \tag{VI.4}$$

and

$$R_0 v = [Y_1, v_0] \tag{VI.5}$$

Before proceeding further, we pause here to give a Lemma, whose proof is given in Chapter 1, Section 3 of [8]:

Lemma VI.1

$$\mathcal{L}_u(TS) = (\mathcal{L}_u T)S + T(\mathcal{L}_u S) \tag{VI.6}$$

$$[u, Tw] = (\mathcal{L}_u T)w + T[u, w] \tag{VI.7}$$

$$\mathcal{L}_{[u,w]}T = \mathcal{L}_u\mathcal{L}_w T - \mathcal{L}_w \mathcal{L}_u T \qquad \text{(VI.8)}$$

holds for (jets of) bundle endomorphism T, S, and (jets of) vector fields u, v.

Using the above Lemma, we can deform the first component of $\left\{\xi_0, \{\zeta_k, a_0\}\right\}$ as follows:

$$R_0(Z_k A_0 - \mathcal{L}_{t_{k+1}}A_0) - \mathcal{L}_{Y_1}(Z_k A_0 - \mathcal{L}_{t_{k+1}}A_0)$$

$$= R_0 Z_k A_0 - R_0(\mathcal{L}_{t_{k+1}}A_0) - \mathcal{L}_{Y_1}(Z_k A_0) + \mathcal{L}_{Y_1}(\mathcal{L}_{t_{k+1}}A_0)$$

$$= R_0 Z_k A_0 - R_0 (\mathcal{L}_{t_{k+1}}A_0) - (\mathcal{L}_{Y_1}Z_k) A_0 - Z_k(\mathcal{L}_Y A_0) + \mathcal{L}_{Y_1}(\mathcal{L}_{t_{k+1}}A_0)$$

in view of (VI.6)

$$= R_0 Z_k A_0 - \mathcal{L}_{t_{k+1}}(R_0 A_0) + (\mathcal{L}_{t_{k+1}}R_0) A_0 - (\mathcal{L}_{Y_1}Z_k) A_0 - Z_k(\mathcal{L}_{Y_1}A_0)$$

$$+ \mathcal{L}_{[Y, t_{k+1}]}A_0 + \mathcal{L}_{t_{k+1}}(\mathcal{L}_{Y_1}A_0)$$

in view of (VI.6) and (VI.8)

$$= R_0 Z_k A_0 - \mathcal{L}_{t_{k+1}}(\mathcal{L}_{Y_1}A_0) + (\mathcal{L}_{t_{k+1}}R_0) A_0 - (\mathcal{L}_{Y_1}Z_k) A_0$$

$$- Z_k R_0 A_0 + \mathcal{L}_{[Y, t_{k+1}]} A_0 + \mathcal{L}_{t_{k+1}}(\mathcal{L}_{Y_1}A_0)$$

in view of (VI.4)

$$= (R_0 Z_k - Z_k R_0 + \mathcal{L}_{t_{k+1}}R_0 - \mathcal{L}_{Y_1}Z_k) A_0 - \mathcal{L}_{-[Y, t_{k+1}]}A_0 .$$

Similarly, for the second component, we have

$$R_0\left[Z_k v_0 - [t_{k+1}, v_0]\right] - \left[Y_1, Z_k v_0 - [t_{k+1}, v_0]\right]$$

$$= R_0 Z_k v_0 - R_0[t_{k+1}, v_0] - [Y_1, Z_k v_0] + \left[Y_1, [t_{k+1}, v_0]\right]$$

$$= R_0 Z_k v_0 - [t_{k+1}, R_0 v_0] + (\mathcal{L}_{t_{k+1}}R_0) v_0 - (\mathcal{L}_{Y_1}Z_k) v_0 - Z_k[Y_1, v_0] + \left[Y_1, [t_{k+1}, v_0]\right]$$

$$= R_0 Z_k v_0 - \left[t_{k+1}, [Y_1, v_0]\right] + (\mathcal{L}_{t_{k+1}}R_0) v_0$$

$$- (\mathcal{L}_{Y_1}Z_k) v_0 - Z_k R_0 v_0 - \left[t_{k+1}, [v_0, Y_1]\right] - \left[v_0, [Y_1, t_{k+1}]\right]$$

in view of the Jacobi identity and (VI.5)

$$= (R_0 Z_k - Z_k R_0 + \mathcal{L}_{t_{k+1}}R_0 - \mathcal{L}_{Y_1}Z_k) v_0 - \left[- [Y_1, t_{k+1}], v_0\right] .$$

It follows that $\left\{\xi_0, \{\zeta_k, a_0\}\right\}$ is of the form $\left\{(X_k, u_{k+1}), (A_0, v_0)\right\}$ for

$X_k = R_0 Z_k - Z_k R_0 + \mathcal{L}_{t_{k+1}} R_0 - \mathcal{L}_{Y_1} Z_k$, $u_{k+1} = -[Y_1, t_{k+1}]$. Hence $\left\{\xi_0, \{\zeta_k, a_0\}\right\} \in B_k$. This completes our proof of the reduction theorem for constrained system normal forms.

■

Appendix VII. Normal Forms for Regular Slow Point

Let $(\mathbf{A}, \mathbf{v})$ denote an m-dimensional constrained system of corank 1 whose leading part $(\mathbf{A}_0, \mathbf{v}_0)$ is equivalent to

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix}, \begin{bmatrix} 0 \\ e_{m-1} \end{bmatrix}\right]$$

where I_{m-1} denotes the unit matrix of order (m-1) and

$$e_{m-1} \triangleq [\underbrace{1\,0\,\cdots\,0}_{m-1 \text{ components}}]^T$$

Proposition VII.1

> The non-degenerate infinite order normal form of $(\mathbf{A}, \mathbf{v})$ is given by
>
> $$\left[\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix}, \begin{bmatrix} \pm x \\ e_{m-1} \end{bmatrix}\right]$$

Proof. To avoid clutter, we will use the following notations:

$$(x, y, \mathbf{z}) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{m-2},$$

$$\mathbf{z} = (z_1, z_2, \cdots, z_{m-2}) \triangleq (z_i),\ 1 \le i \le m-2$$

$$\mathbf{k} = (k_1, k_2, \cdots, k_{m-2}) \triangleq (k_i),\ 1 \le i \le m-2$$

$$\mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2} \cdots z_{m-1}^{k_{m-1}}$$

$$|\mathbf{k}| = \sum_{i=1}^{m-2} k_i,\ 1_\alpha = (0, 0, \cdots \underset{\substack{\uparrow \\ \alpha th \text{ position}}}{1}, 0 \cdots 0)$$

$$\frac{\partial}{\partial \mathbf{z}} \otimes d\,\mathbf{z} = \sum_{i=1}^{m-2} \frac{\partial}{\partial z_i} \otimes dz_i$$

To consider the nth-order normal form problem, let us define

$$a_0 = \left[\frac{\partial}{\partial y} \otimes dy + \frac{\partial}{\partial z} \otimes dz \,,\, \frac{\partial}{\partial y}\right]$$

and compute $\{\xi_n, a_0\}$, where $\xi_n \in H_{n,n+1}\mathcal{G}\mathcal{X}$. For any homogeneous polynomial $f(x,y,z)$ of order n of x, y, and z, we have

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial x} \otimes dx, 0\right], a_0\right\} = 0$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial x} \otimes dy, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial x} \otimes dy, f\frac{\partial}{\partial x}\right]$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial x} \otimes dz_i, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial x} \otimes dz_i, 0\right]$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial y} \otimes dx, 0\right], a_0\right\} = 0$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial y} \otimes dy, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial y} \otimes dy, f\frac{\partial}{\partial y}\right]$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial y} \otimes dz_i, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial y} \otimes dz_i, 0\right]$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial z_i} \otimes dx, 0\right], a_0\right\} = 0$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial z_i} \otimes dy, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial z_i} \otimes dy, f\frac{\partial}{\partial z_i}\right]$$

$$\left\{\left[f(x,y,z)\frac{\partial}{\partial z_i} \otimes dz_j, 0\right], a_0\right\} = \left[f\frac{\partial}{\partial z_i} \otimes dz_j, 0\right]$$

For all $i, j, \mathbf{k}$ with $i + j + |\mathbf{k}| = n+1$, we have

$$\left\{\left[0,\, x^i y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x}\right],\, a_0\right\}$$

$$= \left[j x^i y^{j-1} \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x} \otimes dy + \sum_{\alpha} k_\alpha x^i y^j \mathbf{z}^{\mathbf{k}-1_\alpha} \frac{\partial}{\partial x} \otimes dz_{k\alpha},\, j x^i y^{j-1} \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x}\right]$$

$$\left\{\left[0,\, x^i y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial y}\right],\, a_0\right\}$$

$$= \left[-i x^{i-1} y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial y} \otimes dx,\, j x^i y^{j-1} \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial y}\right]$$

$$\left\{\left[0,\, x^i y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial z_l}\right],\, a_0\right\}$$

$$= \left[-i x^{i-1} y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial z_l} \otimes dx,\, j x^i y^{j-1} \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial z_l}\right]$$

The preceding computation shows that the complementary space $\hat{B}_n$ to the image B_n of the linear map

$$\xi_n \in H_{n,n+1}\mathcal{G}\mathcal{X} \rightarrow \{\xi_n, a_0\} \in H_n\,\mathcal{C}\mathcal{X}^1$$

can be identified as the subspace spanned by

$$\left\{\left[0,\, x^i y^i \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x}\right],\, \left[0,\, x^{i'} y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial y}\right],\, \left[0,\, x^{i'} y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial z_l}\right]\right\}$$

for $i + j + |\mathbf{k}| = i' + j + |\mathbf{k}| = n + 1,\ i' \neq 0,\ 1 \leq l \leq m-2$. It suffices to consider the reduced normal form problem on this complementary space. The preceding computation also shows that the space

$$\mathcal{G}_{n-1} \triangleq \left\{\xi_{n-1} \in H_{n-1,n}\mathcal{G}\mathcal{X} \mid \{\xi_{n-1}, a_0\}_{n-1} = 0\right\}$$

is spanned by

$$\left[f(x,y,\mathbf{z}) \frac{\partial}{\partial x} \otimes dx, 0\right],\ \left[f(x,y,\mathbf{z}) \frac{\partial}{\partial y} \otimes dx, 0\right],$$

$$\left[f(x,y,\mathbf{z}) \frac{\partial}{\partial z_l} \otimes dx, 0\right],\ \left[0, \mathbf{z}^{\mathbf{n}} \frac{\partial}{\partial y}\right],\ \left[0, \mathbf{z}^{\mathbf{n}} \frac{\partial}{\partial z_l}\right],$$

$$\left[-jx^i y^{j-1} \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x} \otimes dy - \sum_{\alpha=1}^{m-2} k_\alpha x^i y^j \mathbf{z}^{\mathbf{k}-1_\alpha} \frac{\partial}{\partial x} \otimes dz_{k_\alpha}, x^i y^j \mathbf{z}^{\mathbf{k}} \frac{\partial}{\partial x} \right]$$

for $1 \le l \le m-2$, $i+j+|\mathbf{k}| = n$, $|\mathbf{n}| = n$, and f is any homogeneous polynomial of order $n-1$.

Now consider the reduced 1st order normal form problem

$$\frac{d}{dt} \hat{b}_1(t) = -\pi_1 \left[\{\xi_0, a_0 + \hat{b}_1(t)\}_1 \right], \; \hat{b}_1 \in \hat{B}_1, \; \xi_0 \in \mathcal{G}_0 .$$

Since

$$\hat{B}_1 = \left[0, x \frac{\partial}{\partial x}\right], \left[0, y \frac{\partial}{\partial x}\right], \left[0, z_l \frac{\partial}{\partial x}\right], \left[0, x \frac{\partial}{\partial y}\right], \left[0, x \frac{\partial}{\partial z_l}\right]$$

$$\mathcal{G}_0 = \left\langle \begin{array}{c} \left[\frac{\partial}{\partial x} \otimes dx, 0\right], \left[\frac{\partial}{\partial y} \otimes dx, 0\right], \left[\frac{\partial}{\partial z_p} \otimes dx, 0\right], \\ \left[0, z_p \frac{\partial}{\partial y}\right], \left[0, z_p \frac{\partial}{\partial z_q}\right], \left[0, x \frac{\partial}{\partial x}\right], \\ \left[-\frac{\partial}{\partial x} \otimes dy, y \frac{\partial}{\partial x}\right], \left[-\frac{\partial}{\partial x} \otimes dz_p, z_p \frac{\partial}{\partial x}\right] \end{array} \right\rangle ,$$

we can write

$$\hat{b}_1(t) = \alpha(t) \left[0, x \frac{\partial}{\partial x}\right] + \beta(t) \left[0, y \frac{\partial}{\partial x}\right]$$
$$+ \sum_{l=1}^{m-2} \gamma_l(t) \left[0, z_l \frac{\partial}{\partial x}\right] + \delta(t) \left[0, x \frac{\partial}{\partial y}\right] + \sum_{l=1}^{m-2} \varepsilon_l(t) \left[0, x \frac{\partial}{\partial z_l}\right]$$

and

$$\xi_0 = A \left[\frac{\partial}{\partial x} \otimes dx, 0\right] + B \left[\frac{\partial}{\partial y} \otimes dx, 0\right]$$
$$+ \sum_{p=1}^{m-2} C_p \left[\frac{\partial}{\partial z_p} \otimes dx, 0\right] + \sum_{p=1}^{m-2} D_p \left[0, z_p \frac{\partial}{\partial y}\right]$$
$$+ \sum_{1 \le p,q \le m-2} E_{pq} \left[0, z_p \frac{\partial}{\partial z_q}\right] + F \left[0, x \frac{\partial}{\partial x}\right]$$
$$+ G \left[-\frac{\partial}{\partial x} \otimes dy, y \frac{\partial}{\partial x}\right] + \sum_{p=1}^{m-2} H_p \left[-\frac{\partial}{\partial x} \otimes dz_p, z_p \frac{\partial}{\partial x}\right] .$$

Using these bases, we calculated the bracket expressions summarized in *Table VII.1*. From these expressions, we obtain the following system of differential equations:

$$\dot{\alpha} = A\,\alpha$$

$$\dot{\beta} = A\,\beta + F\,\beta - G\,\alpha$$

$$\dot{\gamma}_l = A\,\gamma_l - D_l\beta - \sum_p E_{lp}\gamma_p + F\,\gamma_l - H_l\alpha$$

$$\dot{\delta} = B\,\alpha + \sum_{l=1}^{m-2} D_l\,\varepsilon_l - F\,\delta$$

$$\dot{\varepsilon}_l = C_l\alpha + \sum_p E_{pl}\varepsilon_p - F\,\varepsilon_l$$

If $\alpha(0) \neq 0$, we can choose suitable A, B, C_p, D_p, E_{pq}, F, G, and H_p so that

$$\alpha(1) = \pm 1, \beta(1) = \gamma_l(1) = \delta(1) = \varepsilon_l(1) = 0\,.$$

Hence, the nondegenerate 1st-order normal form is

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix}, \begin{bmatrix} \pm x \\ e_{m-1} \end{bmatrix}\right].$$

To obtain the higher order normal forms, consider

$$\frac{d}{dt}\,\hat{b}_n(t) = -\,\pi_n\left[\{\xi_{n-1}, a^1 + \hat{b}_n(t)\}_n\right]$$

where

$$\{\xi_{n-1}, a^1 + \hat{b}_n(t)\}_n = \{\xi_{n-1}, a_1\} = \left\{\xi_{n-1}, \left[0, \pm x\,\frac{\partial}{\partial x}\right]\right\}.$$

We can eliminate all n-jets for $n \geq 2$ with the help of *Table VII.2*, which lists the bracket multiplications of $\xi_{n-1} \in \;_{n-1}$ and $x\,\dfrac{\partial}{\partial x}$.

REFERENCES

[1] L. O. Chua and H. Oka, "Normal forms for constrained nonlinear differential equations--Part I: Theory," *IEEE Trans. on Circuits and Systems*, Special Issue on Chaos and Bifurcation, to appear.

[2] L. O. Chua and A. C. Deng, "Impasse Points," in preparation.

[3] N. Fenichel, "Geometric singular perturbation theory for ordinary differential equations," *J. Diff. Equations*, vol. 31, pp. 53-98, 1979.

[4] G. Ikegami, "Singular perturbations in foliation," (preprint).

[5] E. Benoit, S. L. Callot, F. Diener, and M. Diener, "Chasse au canard," *Collectanea Mathematica*, vol. 32, pp. 37-119, 1980.

[6] E. Benoit, "Canards de $\mathbb{R}^3$," Thesis, Nice, 1983.

[7] M. Diener, "The canard unchained or how fast/slow dynamical systems bifurcate," *Math. Intelligencer*, vol. 6, pp. 38-49, 1984.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I, Interscience Publisher, New York, 1963.

[9] A. K. Zvonkin and M. A. Shubin, "Non-standard analysis and singular perturbations of ordinary differential equations," *Russ. Math. Surveys*, vol. 39, pp. 77-127, 1984.

[10] H. Oka, "Constrained systems, characteristic surfaces, and normal forms," *Japan J. Appl. Math.*, vol. 4, pp. 393-431, 1987.

[11] H. Oka and H. Kokubu, "Versal unfolding of constrained systems," in *Dynamical Systems and Applications*, Edited by N. Aoki, World Scientific Publication, Singapore, 1988.

[12] B. Schutz, *Geometrical Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1980.

[13] L. O. Chua and H. Kokubu, "Normal forms of nonlinear vector fields--Part I: Theory and Algorithm," Electrical Research Laboratory, Memorandum UCB/ERL M87/81, University of California, Berkeley, July 1, 1987.

FIGURE CAPTIONS

Fig. 1. Phase portrait of constrained system defined by $\varepsilon\dot{x} = 1$ and $\dot{y} = 0$. The *double arrowheads* denote *rapid* motions.

Fig. 2. (a) Phase portraits associated with $\varepsilon\dot{x} = x, \dot{y} = 1$. (b) Phase portraits associated with $\varepsilon\dot{x} = -x$, $\dot{y} = 1$. As usual, the *double arrowhead* denote a *rapid* motion of the trajectories.

Fig. 3. Phase portrait associated with $\varepsilon\dot{x} = y + ax^2, \dot{y} = 1 + x$, where $a > 0$.

Fig. 4. Phase portrait associated with $\varepsilon\dot{x} = \alpha y - x, \dot{y} = ay$, where $a > 0$.

Fig. 5. Phase portrait associated with $\varepsilon\dot{x} = y - x^2, \dot{y} = \alpha - x$. The trajectory stradling along the parabola is called a "canard."

Fig. 6. A typical "canard" trajectory associated with the phase portrait of Fig. 11 may approach the ε-neighborhood of S_- at some time to t_1 and leaves ε-neighborhood of S_+ at sometime t_2.

Fig. 7. Phase portraits associated with $\varepsilon\dot{x} = \alpha + x^2 + ay^2, \dot{y} = 1 + x$, where $a > 0$.

Fig. 8. Phase portraits associated with $\varepsilon\dot{x} = \alpha + x^2 + ay^2, \dot{y} = 1 + x$, where $a < 0$.

Fig. 9. Phase portraits associated with $\varepsilon\dot{x} = x, \dot{y} = \alpha + y^2$.

Fig. 10. Phase portrait associated with the Van der Pol equation.

FIGURE CAPTIONS FOR APPENDIX

Fig. A.1. In this example of a trivial fiber bundle, the total space E is a cylinder, the base space M is the unit circle, the fiber F is the unit interval, and the projection π is the obvious map of the cylinder into the unit circle.

Fig. A.2. The surface of the cylinder can be thought of as a sheet made of vertical fibers, one of which is shown in bold. Observe that each fiber projects naturally into a *point* x on the circumference of the unit circle.

Fig. A.3. A Möbius band is made by first twisting one end of a ribbon and then pasting the two end edges together.

Fig. A.4. Geometrical interpretation of a *locally-trivial* fiber bundle: in any neighborhood V_x of x, $\pi^{-1}(V_x)$ has the same structure as that of Fig. A.2; namely, a narrow band made of parallel fibers.

Fig. A.5. Each fiber $\pi^{-1}(x)$ of the Möbius band is homeomorphic to unit interval I. Moreover, because of the twisting operation, the top and bottom boundaries of the ribbon (prior to the twist) now form a contiguous loop; namely, starting from any point on either boundary and traversing consistently on the boundary along any direction, one eventually returns to the original point after having traversed all points on both the top and the bottom boundaries exactly once. In other words, the boundary is homeomorphic to a circle S^1.

Fig. A.6. (a) This diagram *commutes* namely $\pi' \circ \Phi_V = \pi$. (b) Geometrical interpretation of the commutative diagram of a fiber bundle. Note that shaded region $\pi^{-1}(V)$ is diffeomorphic to that of the local direct-product $V \times F$.

Fig. A.7. (a) By choosing $E = S^1 \times \mathbf{R}$, this infinite cylinder is not only a trivial fiber bundle but also a *vector* bundle because in this case, $F \triangleq \mathbf{R}$ is a vector space. (b) By choosing $E' = \mathbf{R} \times S^1$, this infinite cylinder is a trivial fiber bundle but not a vector bundle because in this case, $F \triangleq S^1$ is not a vector space.

Fig. A.8. For a trivial fiber bundle, a *section* σ can be interpreted as the *graph* of a *single-valued* function f.

Fig. A.9. Two examples of a *section* of a Möbius band: The *first* section σ_0 is formed by the union of the *middle* points of all fibers. The *second* section, shown by the *bold* closed curve, is any closed loop drawn on the surface of the Möbius band which *crosses* the first section.

Fig. A.10. For each point x on a manifold M, the plane tangent to M at x is denoted by T_xM. The picture on the left shows 2 tangent planes T_xM and T_yM. The collection of all such tangent planes over all points of M is the tangent bundle TM. The picture on the right shows each fiber T_xM at x is diffeomorphic to $\mathbf{R}^n$.

Fig. A.11. An n-dimensional tangent plane on an n-dimensional sphere S^n in $\mathbf{R}^{n+1}$. The collection of all such tangent planes overall points of S^n is the tangent bundle of S^n.

Fig. A.12. Special case of Fig. A.11 drawn for $n = 1$. Here, the collection of all tangent lines to the unit circle S^1 is the tangent bundle of S^1.

Fig. A.13. A vector field on S^1 can be identified with a mapping $\lambda: I \rightarrow \mathbf{R}^1$, with $\lambda(0) = \lambda(2\pi)$.

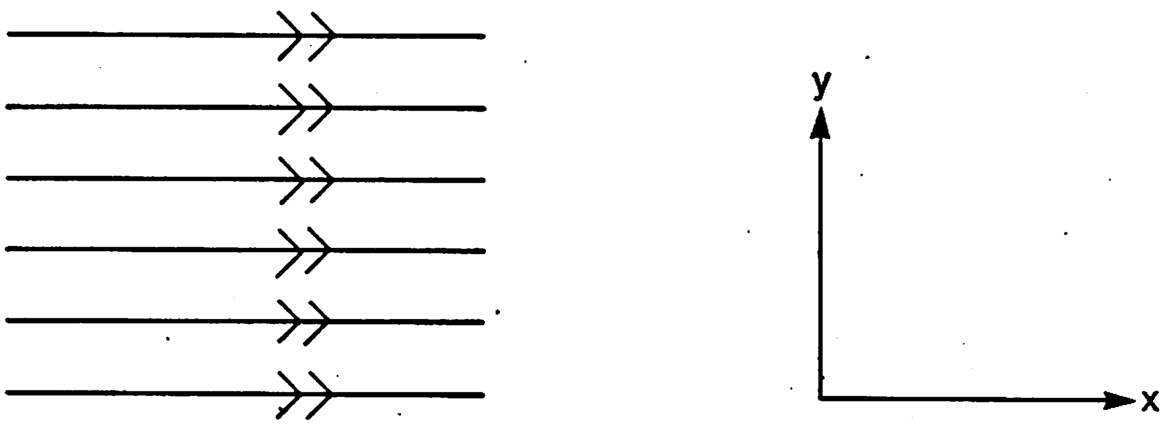


Fig. 1

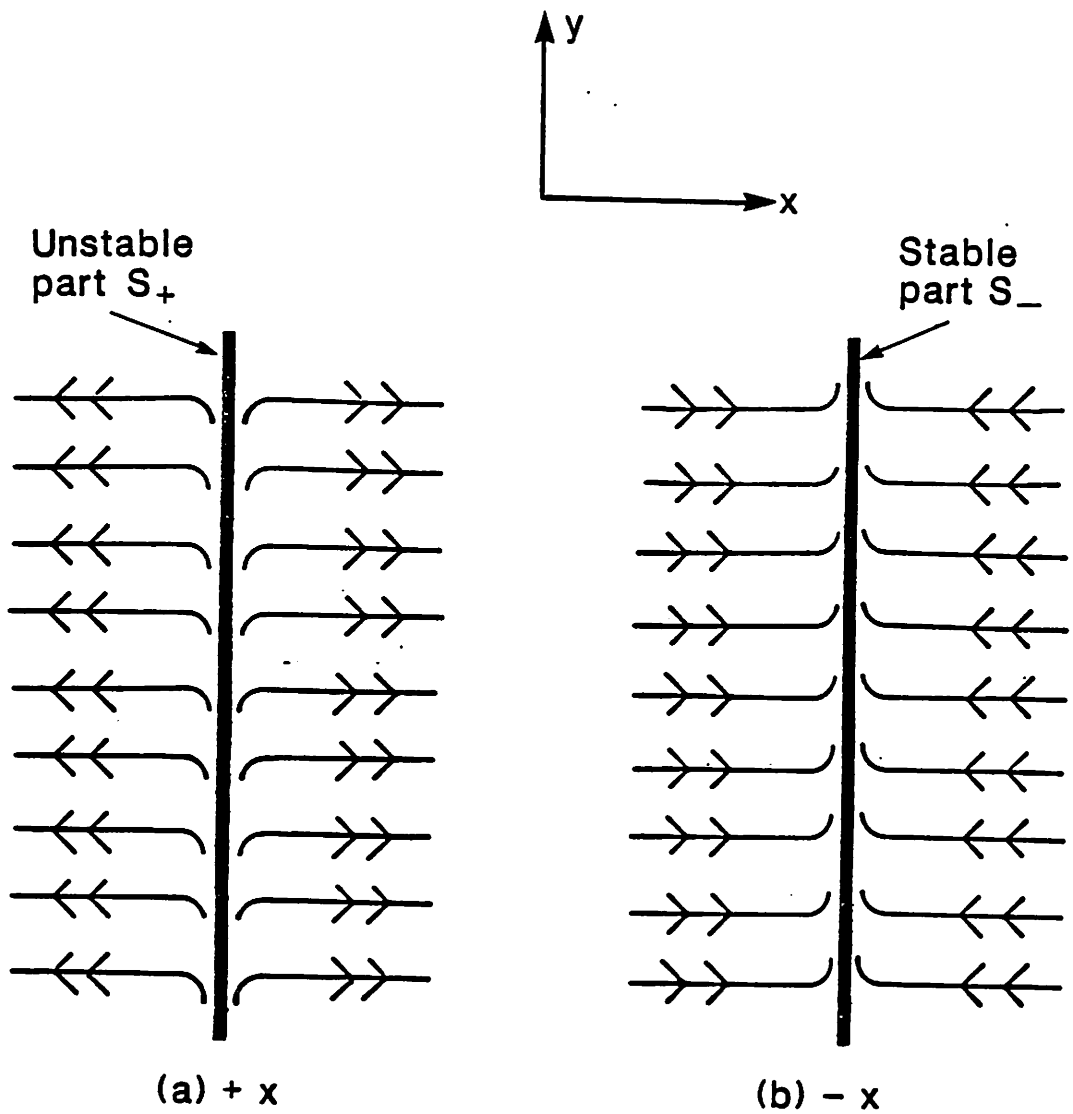


Fig. 2

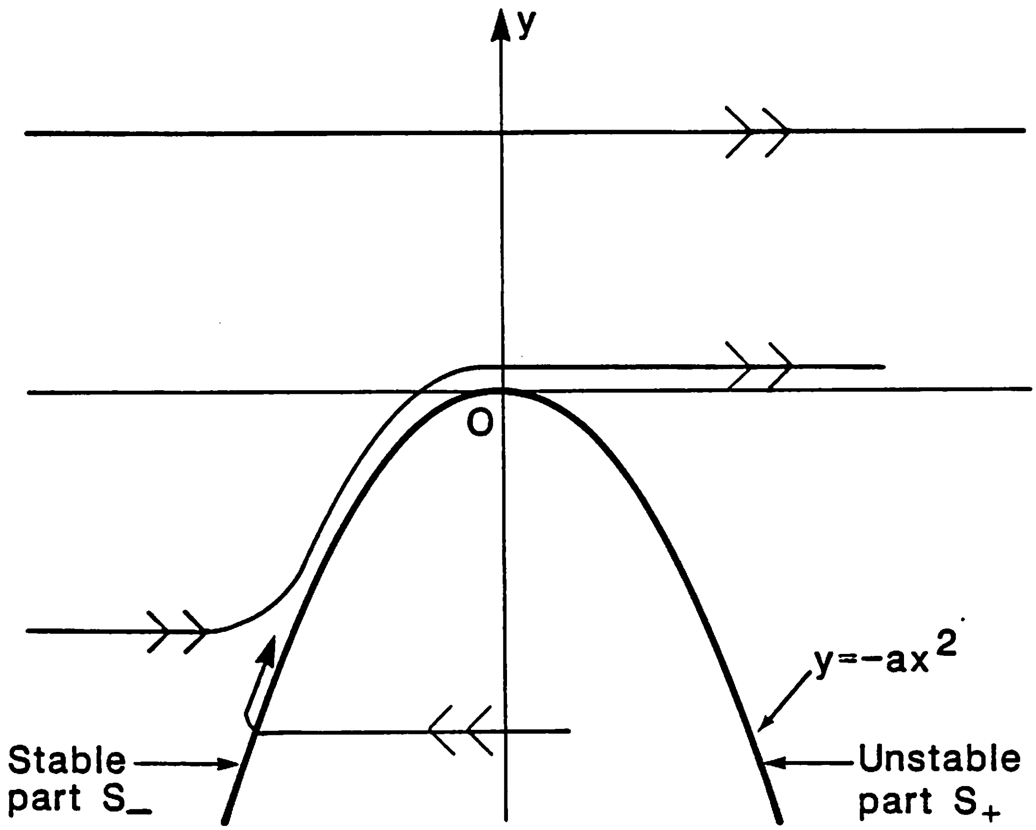


Fig.3

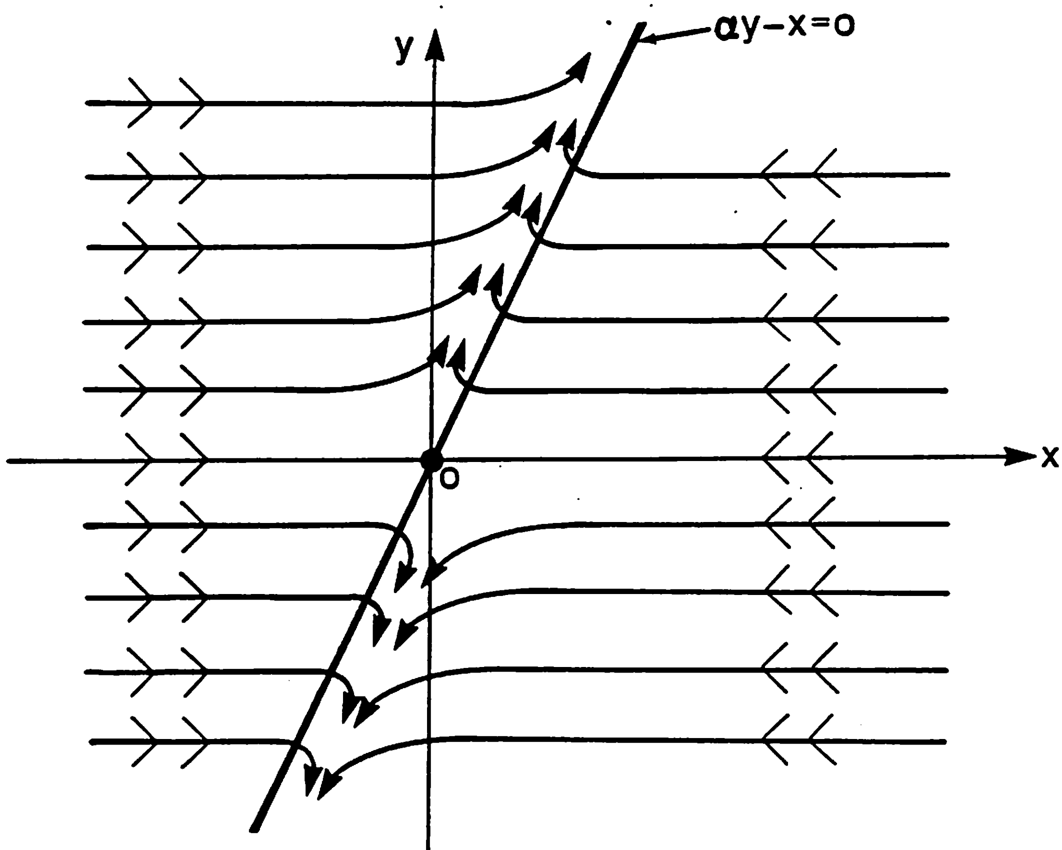


Fig.4

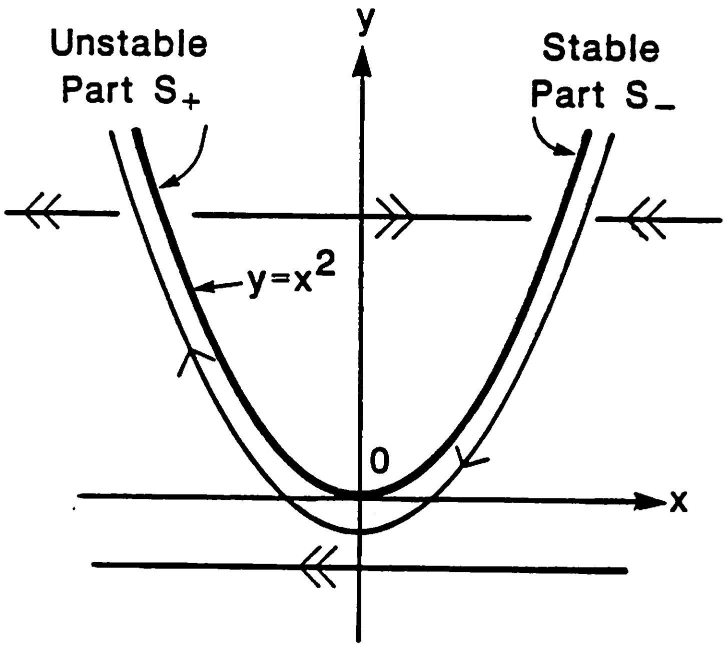


Fig. 5

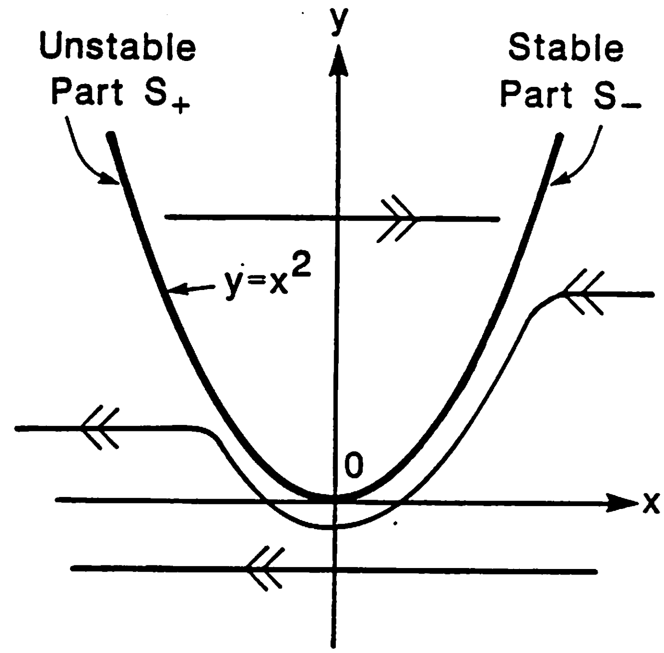


Fig. 6

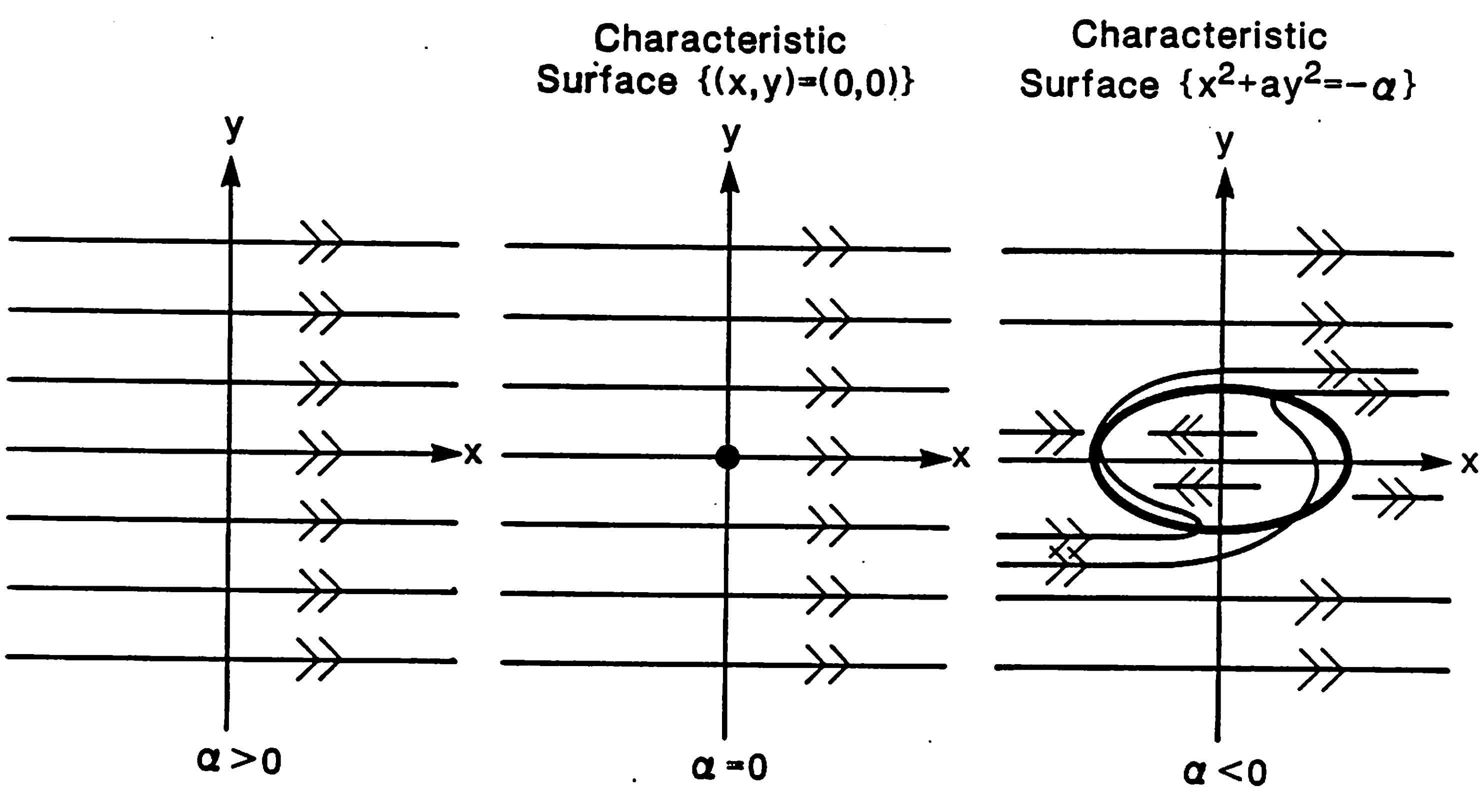


Fig. 7

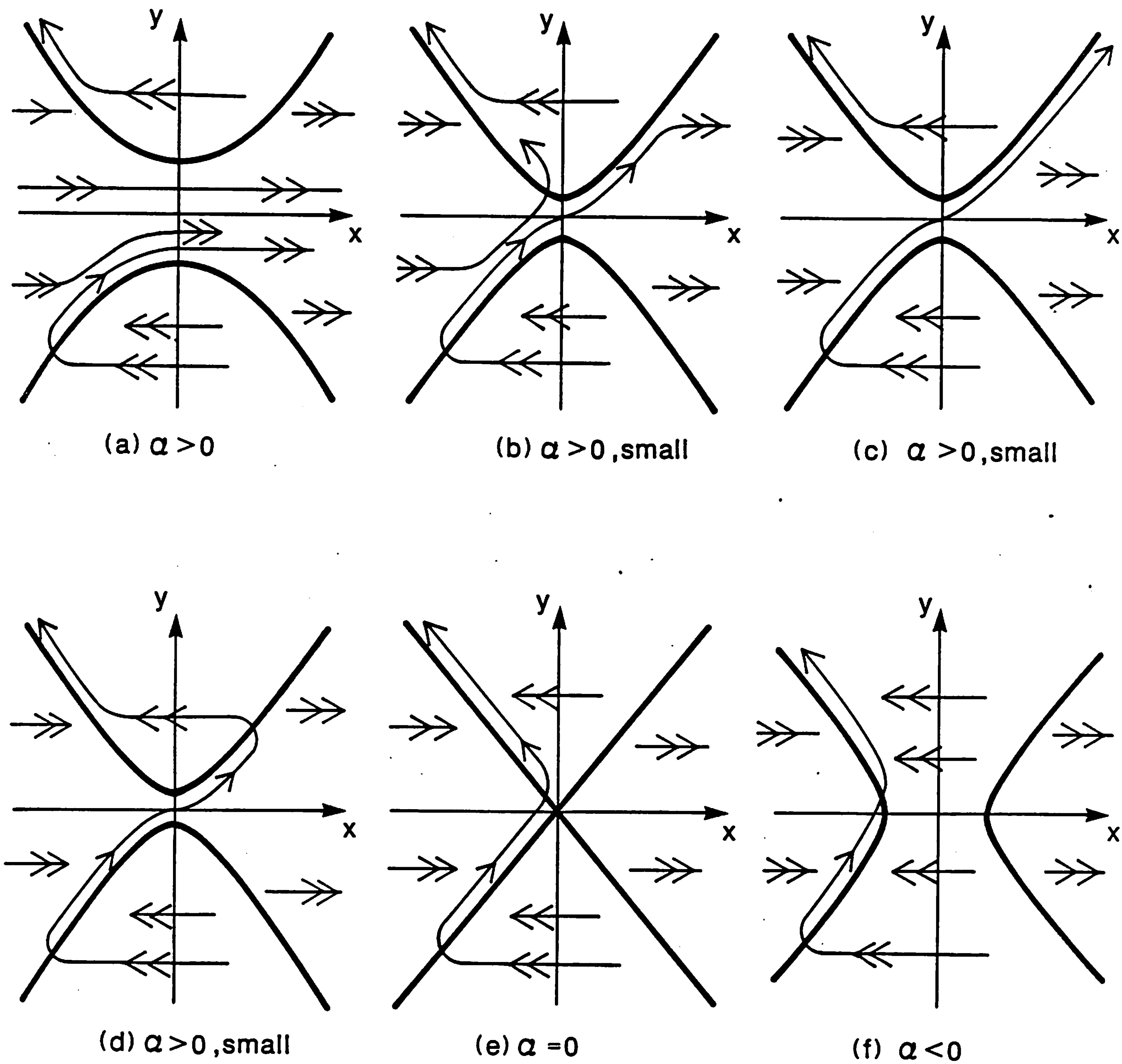


Fig.8

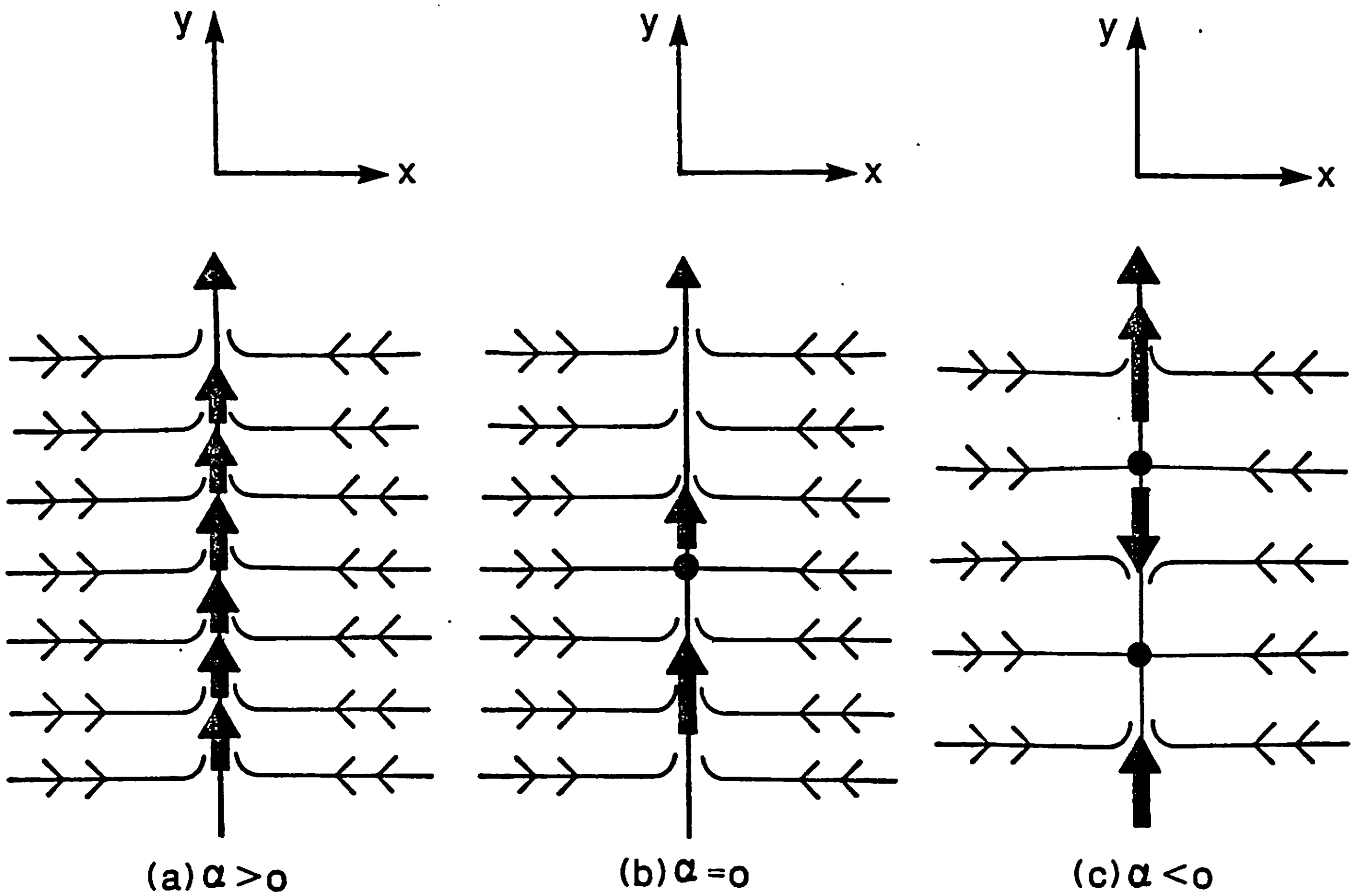


Fig. 9

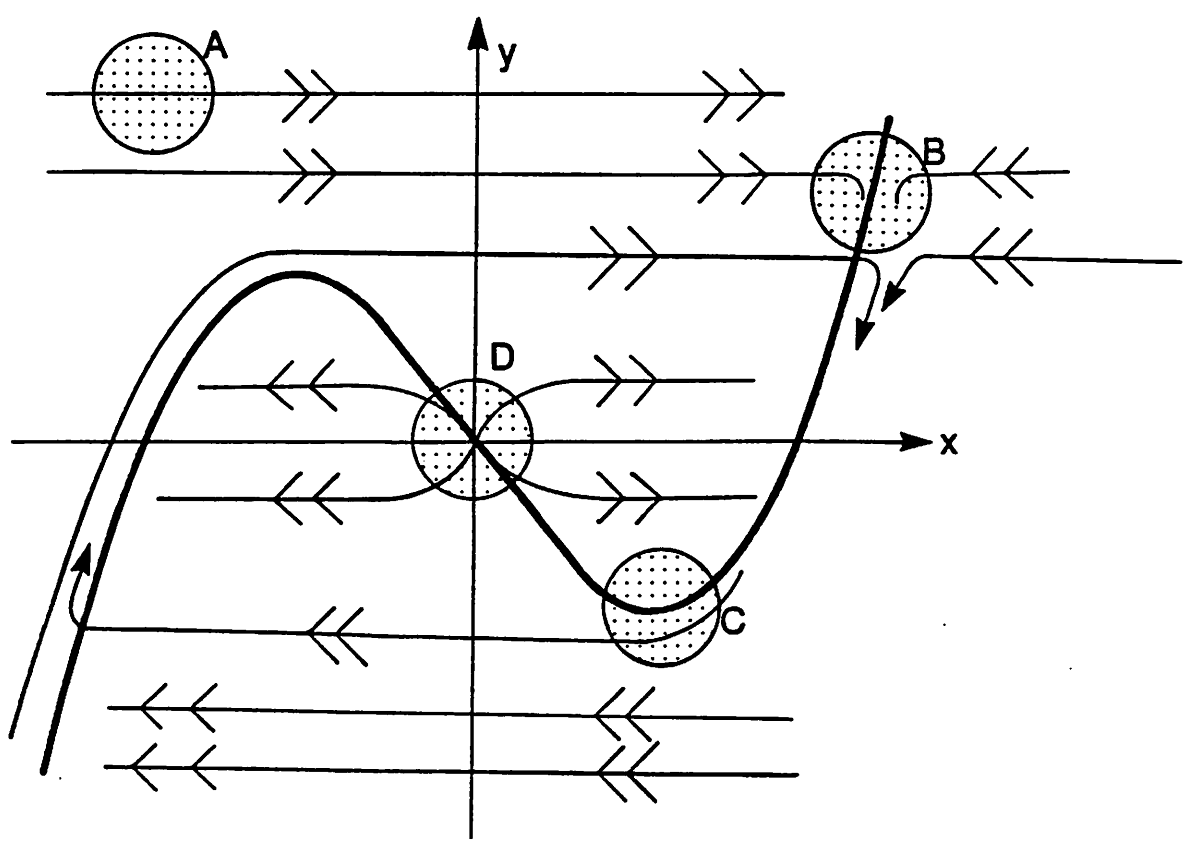


Fig. 10

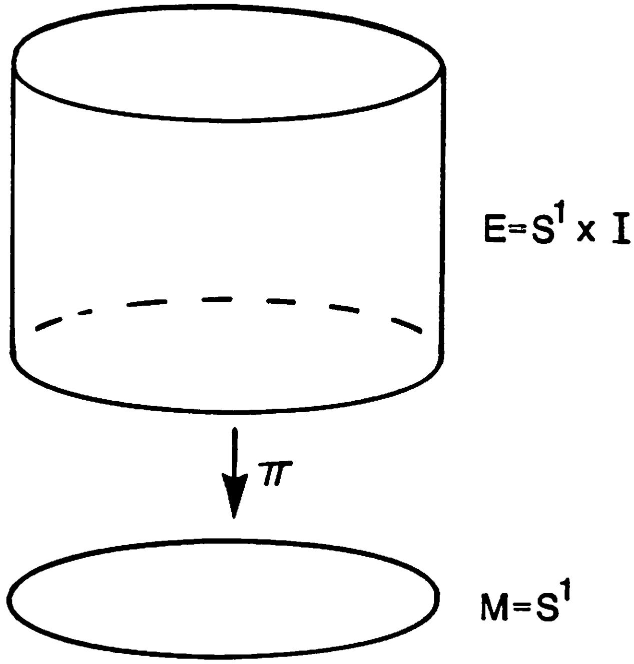


Fig.A1

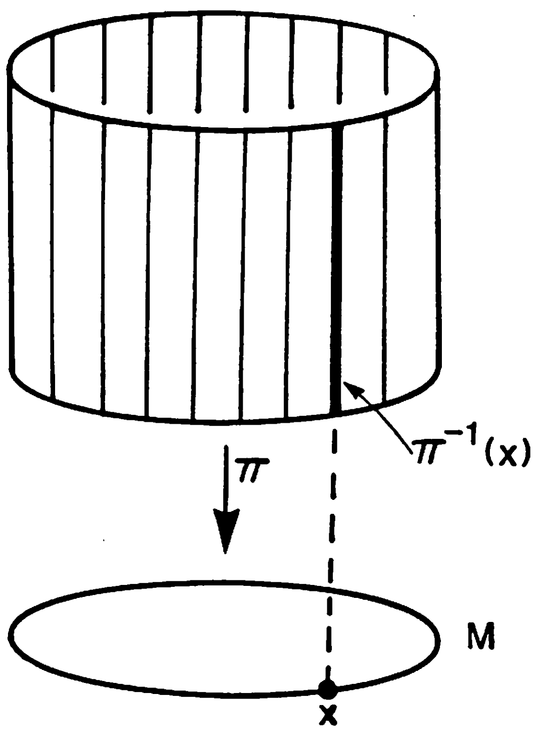


Fig.A2

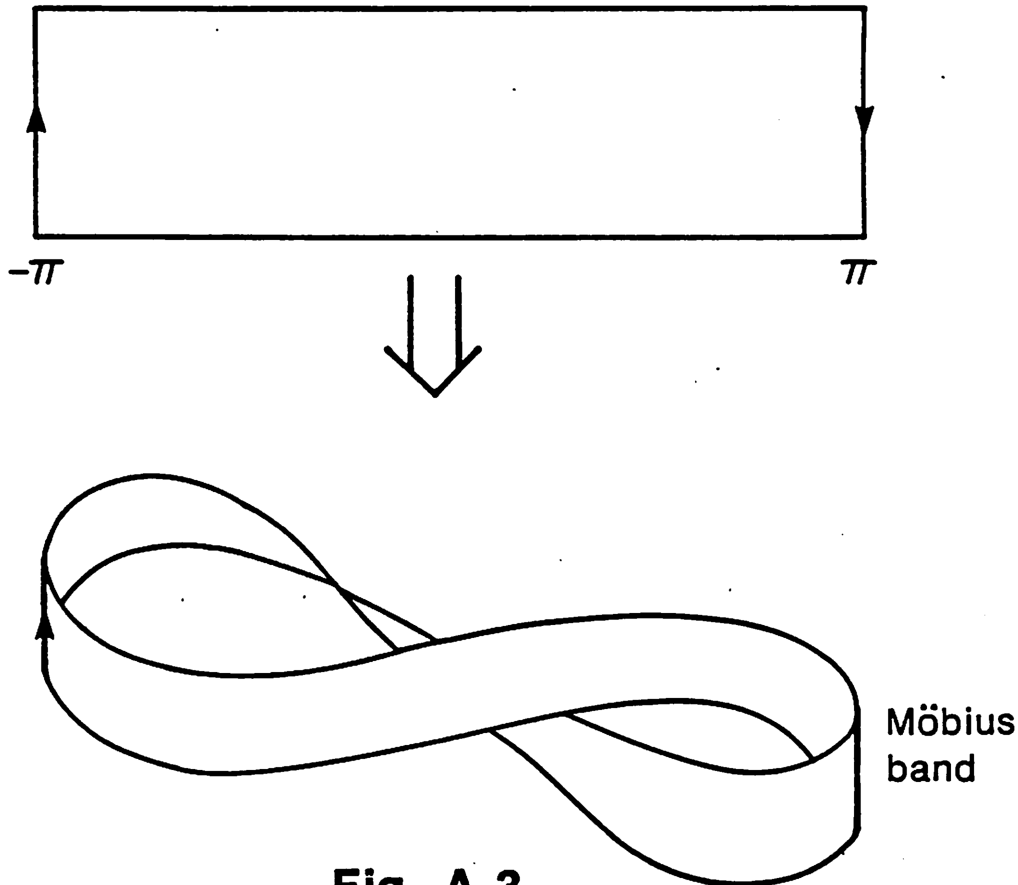


Fig. A.3

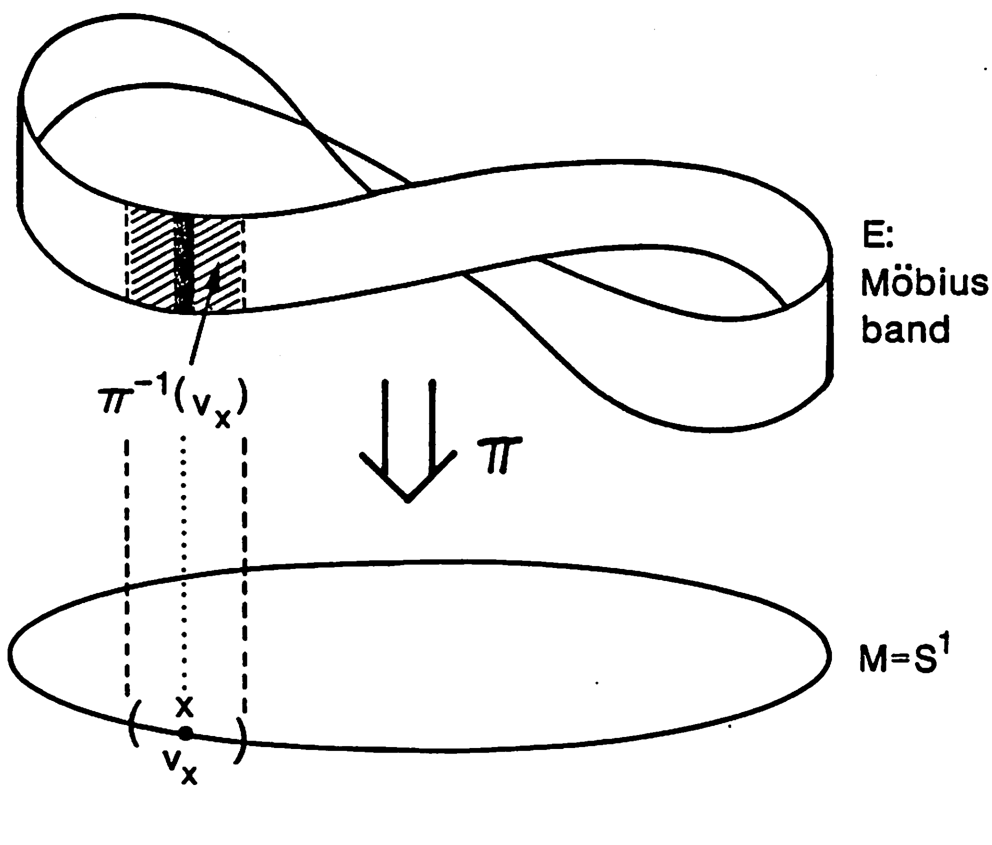


Fig. A.4

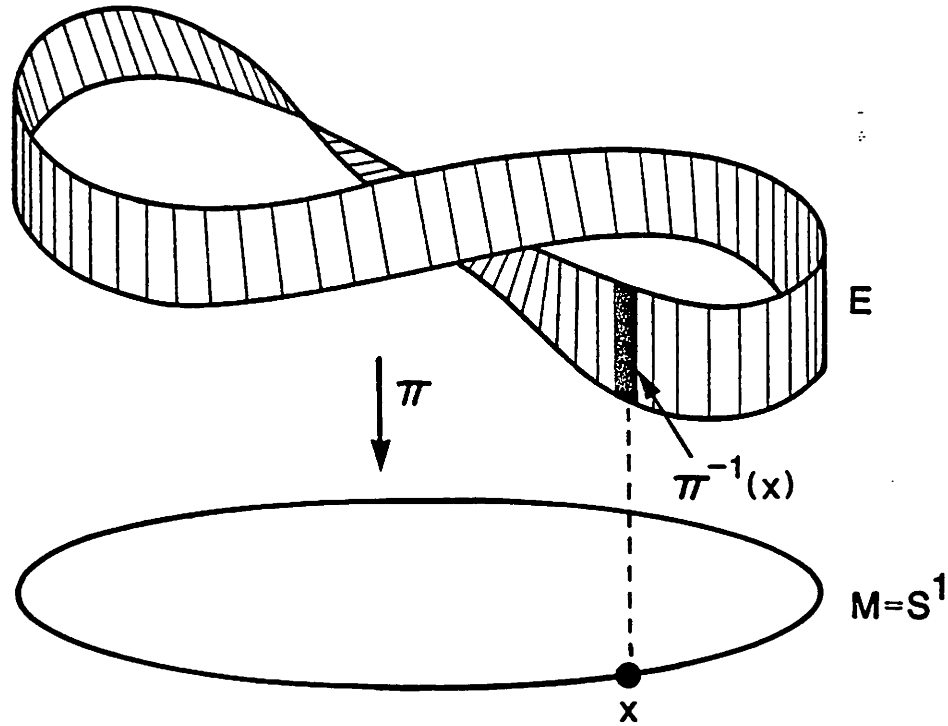


Fig.A.5

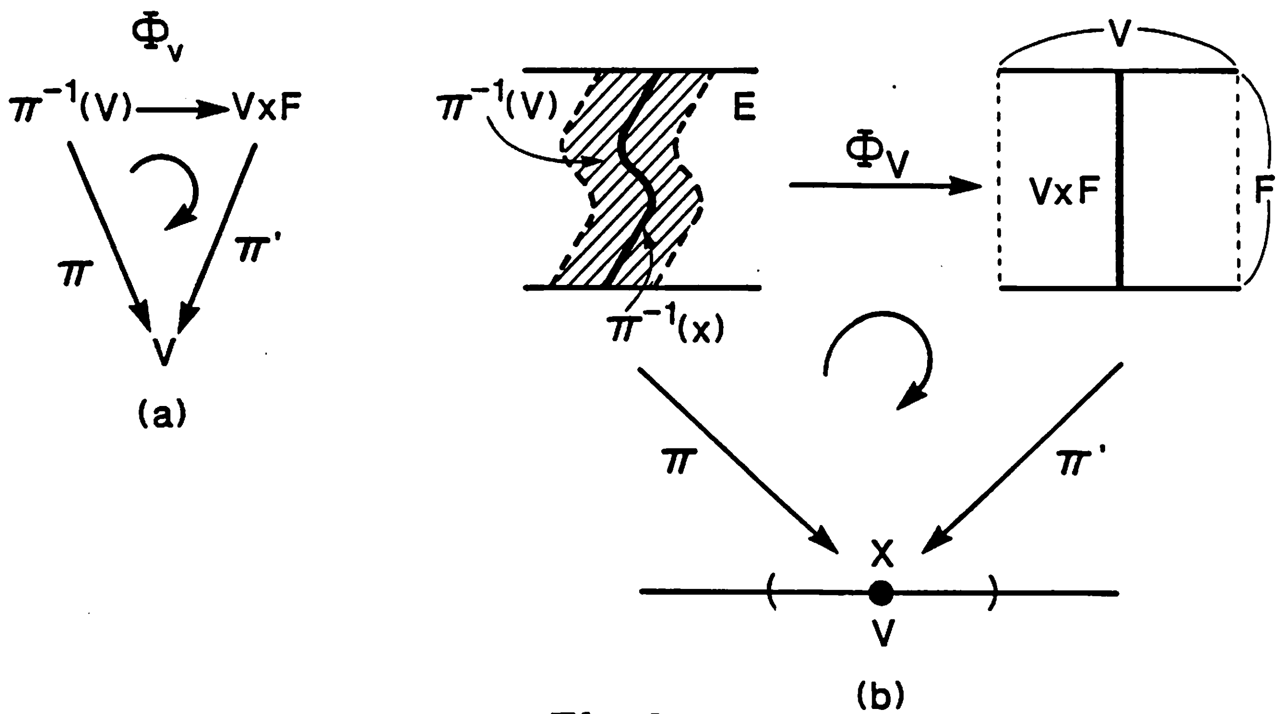


Fig.A.6

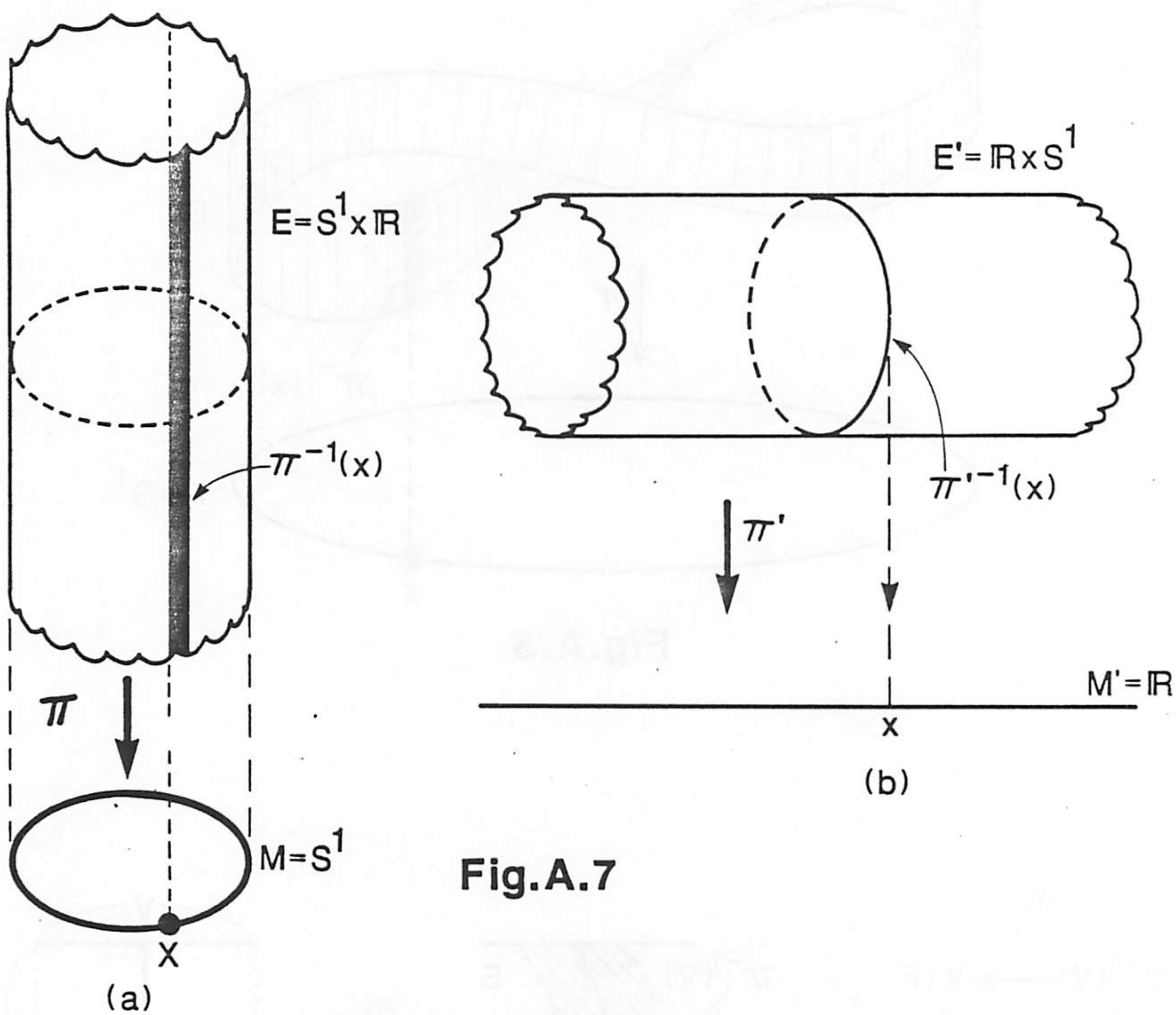


Fig.A.7

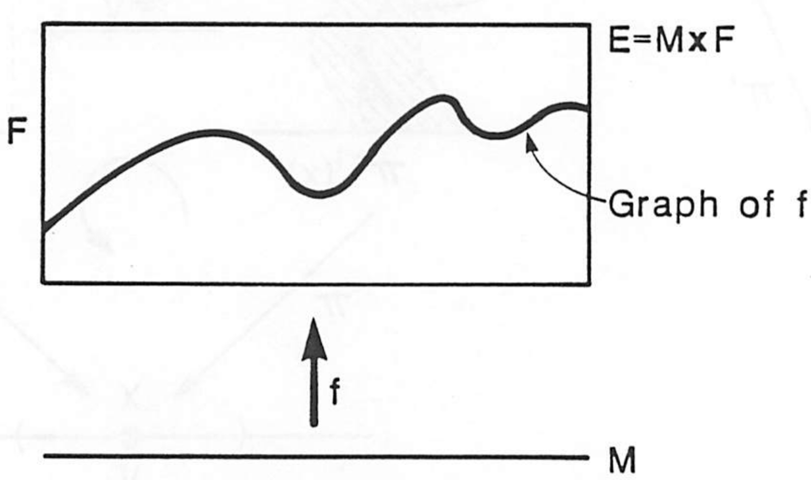


Fig.A.8

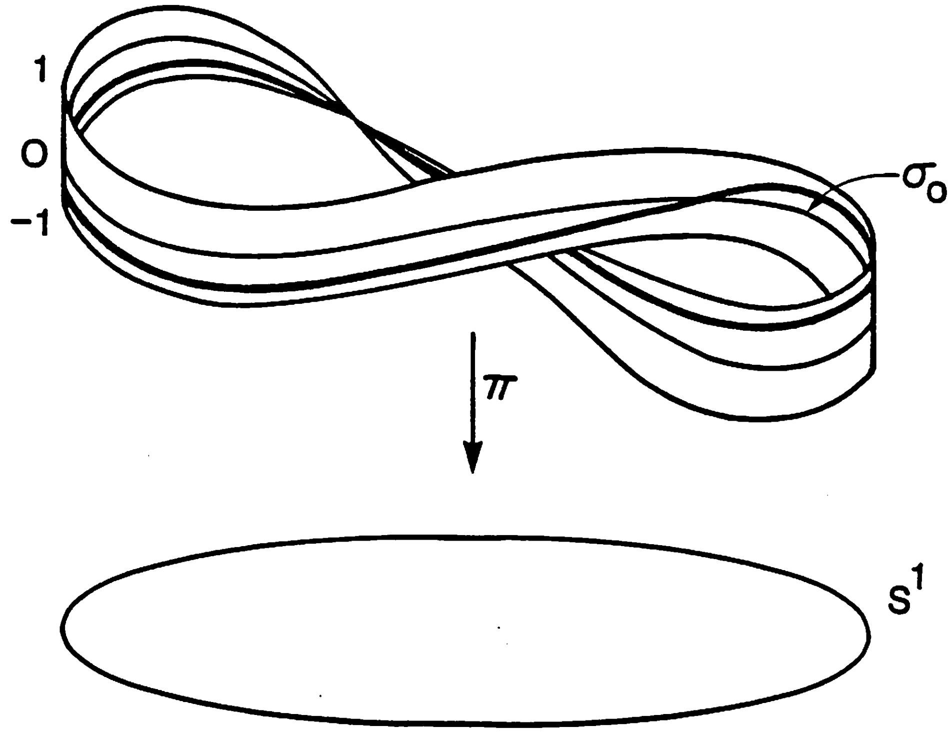


Fig.A.9

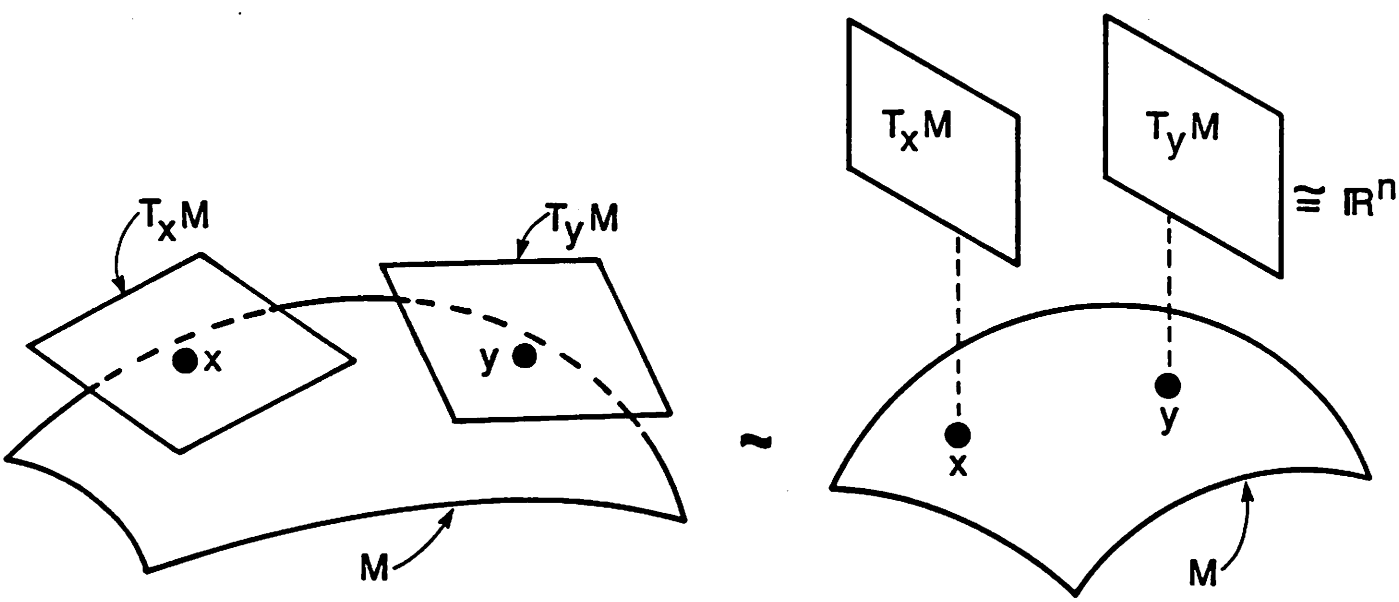


Fig.A.10

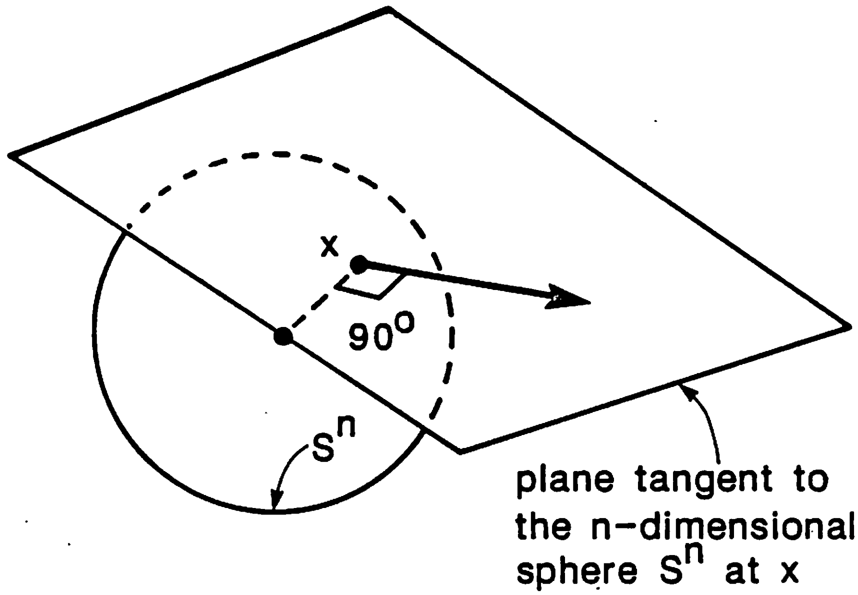


Fig.A.11

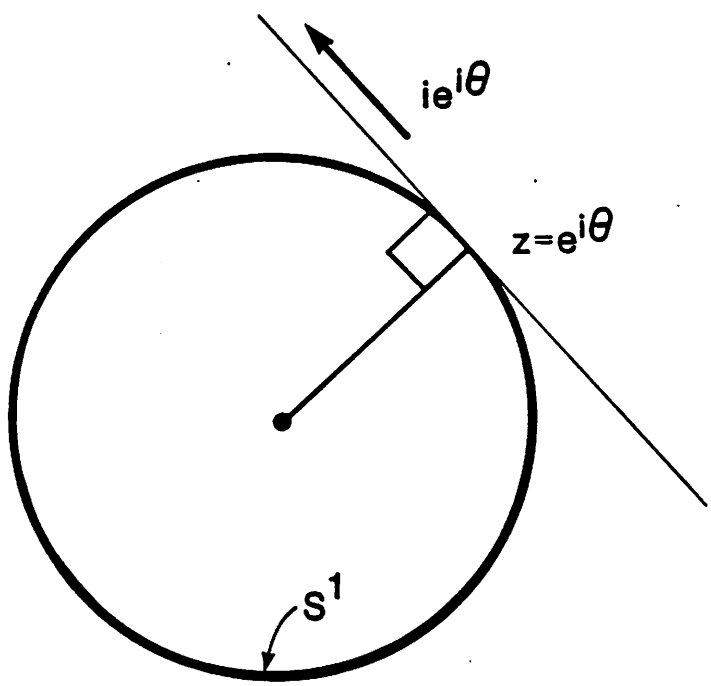


Fig.A.12

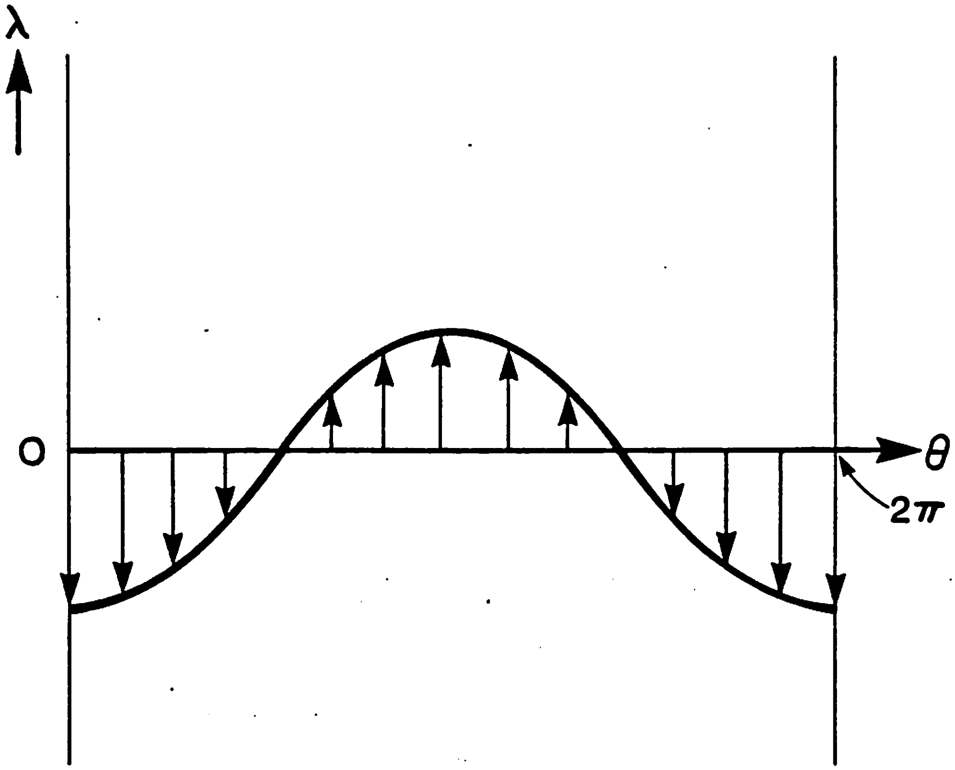


Fig. A.13

Table VII.1. Bracket expressions for $\{\xi_0, \hat{b}_1\}$, where $\xi_0 \in \mathfrak{g}_0$ and $\hat{b}_1 \in \hat{B}_1$.

$\mathfrak{g}_0$ \ $\hat{B}_1$	$\left(0, x \frac{\partial}{\partial z_\ell}\right)$ ϵ_ℓ	$\left(0, x \frac{\partial}{\partial y}\right)$ δ	$\left(0, z_\ell \frac{\partial}{\partial x}\right)$ γ_ℓ	$\left(0, y \frac{\partial}{\partial x}\right)$ β	$\left(0, x \frac{\partial}{\partial x}\right)$ α
$A\left(\frac{\partial}{\partial x} \otimes dx, 0\right)$	0	0	$\left(0, z_\ell \frac{\partial}{\partial x}\right)$	$\left(0, y \frac{\partial}{\partial x}\right)$	$\left(0, x \frac{\partial}{\partial x}\right)$
$B\left(\frac{\partial}{\partial y} \otimes dx, 0\right)$	0	0	$\left(0, z_\ell \frac{\partial}{\partial y}\right)$	$\left(0, y \frac{\partial}{\partial y}\right)$	$\left(0, x \frac{\partial}{\partial y}\right)$
$C_p\left(\frac{\partial}{\partial z_p} \otimes dx, 0\right)$	0	0	$\left(0, z_\ell \frac{\partial}{\partial z_p}\right)$	$\left(0, y \frac{\partial}{\partial z_p}\right)$	$\left(0, x \frac{\partial}{\partial z_p}\right)$
$D_p\left(0, z_p \frac{\partial}{\partial y}\right)$	$\left(0, \delta_{p\ell}\, x \frac{\partial}{\partial y}\right)$	0	0	$\left(0, -z_p \frac{\partial}{\partial x}\right)$	0
$E_{pq}\left(0, z_p \frac{\partial}{\partial z_q}\right)$	$\left(0, \delta_{p\ell}\, x \frac{\partial}{\partial z_q}\right)$	0	$\left(0, -\delta_{q1}\, z_p \frac{\partial}{\partial x}\right)$	0	0
$F\left(0, x \frac{\partial}{\partial x}\right)$	$\left(0, -x \frac{\partial}{\partial z_\ell}\right)$	$\left(0, -x \frac{\partial}{\partial y}\right)$	$\left(0, z_\ell \frac{\partial}{\partial x}\right)$	$\left(0, y \frac{\partial}{\partial x}\right)$	0
$G\left(-\frac{\partial}{\partial x} \otimes dy, y \frac{\partial}{\partial x}\right)$	$\left(0, -y \frac{\partial}{\partial z_\ell}\right)$	$\left(0, -y \frac{\partial}{\partial y}\right)$	0	0	$\left(0, -y \frac{\partial}{\partial x}\right)$
$H_p\left(-\frac{\partial}{\partial x} \otimes dz_p, z_p \frac{\partial}{\partial x}\right)$	$\left(0, -z_p \frac{\partial}{\partial z_\ell}\right)$	$\left(0, -z_p \frac{\partial}{\partial y}\right)$	0	0	$\left(0, -z_p \frac{\partial}{\partial x}\right)$

Table VII.2. Bracket expressions for $\{\xi_{n-1}, a_1\}$

$\{\xi_{n-1}, a_1\}$	$\left(0,\ x^{i+1} y^j z^k \frac{\partial}{\partial x}\right)$	$\left(0,\ x^{i+1} y^j z^k \frac{\partial}{\partial}\right)$	$\left(0,\ x^{i+1} y^j z^k \frac{\partial}{\partial z_\ell}\right)$	0	0	$\left(0, (1-i)\ x^i y^j z^k \frac{\partial}{\partial x}\right)$
$\mathcal{G}_{n-1}$	$\left(f \frac{\partial}{\partial x} \otimes dx,\ 0\right)$	$\left(f \frac{\partial}{\partial y} \otimes dx,\ 0\right)$	$\left(f \frac{\partial}{\partial z_\ell} \otimes dx,\ 0\right)$	$\left(0,\ z^n \frac{\partial}{\partial y}\right)$	$\left(0,\ z^n \frac{\partial}{\partial z_\ell}\right)$	$\left(*,\ x^i y^j z^k \frac{\partial}{\partial x}\right)$