Three Characterizations of

Geometric Continuity

for Parametric Curves

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Parametric spline curves are typically constructed so that the first n parametric derivatives agree where the curve segments abut. This type of continuity condition has become known as C^n or n^{th} order *parametric continuity*. It has previously been shown that the use of parametric continuity disallows many parametrizations which generate geometrically smooth curves.

A relaxed form of n^{th} order parametric continuity has been developed and dubbed n^{th} order *geometric continuity* and denoted G^n. These notes explore three characterizations of geometric continuity. First, the concept of *equivalent parametrizations* is used to view geometric continuity as a measure of continuity that is *parametrization independent*, that is, a measure that is *invariant under reparametrization*. The second characterization develops necessary and sufficient conditions, called *Beta-constraints*, for geometric continuity of curves. Finally, the third characterization shows that two curves meet with G^n continuity if and only if their arc length parametrizations meet with C^n continuity.

G^n continuity provides for the introduction of n quantities known as *shape parameters* which can be made available to a designer in a computer aided design environment to modify the shape of curves without moving control vertices.

Several applications of geometric continuity are presented. First, composite Bèzier curves are stitched together with G^1 and G^2 continuity using geometric constructions. Then, a subclass of the Catmull-Rom splines based on geometric continuity and possessing shape parameters is discussed. Finally, quadratic G^1 and cubic G^2 Beta-splines are developed using the geometric constructions for the geometrically continuous Bèzier segments.

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1. Parametric Representation

A *parametric* function defines a mapping from a *domain parameter space*, into geometric or *Euclidean* space. The definition of a curve involves functions of a single parameter, whereas for a surface it uses functions of a pair of parameters. Specifically, in the case of curves, the parametric function defines a mapping from u into Euclidean two-space as $\mathbf{Q}(u) = [x(u), y(u)]$ or into Euclidean three-space as $\mathbf{Q}(u) = [x(u), y(u), z(u)]$. This function can be used to define a curve by letting u range over some interval $[u_0, u_f]$ of the u axis. For a surface, the parametric function is a mapping from u,v into three-space as $\mathbf{Q}(u,v) = [x(u,v), y(u,v), z(u,v)]$. A surface is then defined using this function by letting u and v range over some rectangle $[u_0, u_f] \times [v_0, v_f]$ in the u,v plane. Note that the use of **boldface** is to demonstrate that the function is vector-valued.

In the case of curves, if the domain parameter is thought of as time, the parametric function is used to locate the position of the particle in space at a given instant. As time passes, the particle sweeps out a path, thereby tracing the curve. A parametric function therefore defines more than just a path; there is also information about the direction and speed of the particle as it moves along the path.

2. Piecewise Representation and Smoothness

In recent years, computer-aided geometric design and modelling (CAGDM) has relied heavily on mathematical descriptions of objects based on a special kind of parametric function known as a *parametric spline function*. A parametric spline function is a *piecewise* function where each of the pieces is a parametric function. The pieces of a curve are known as *segments* while those of a surface are called *patches*. An important aspect of these functions is the manner in which the segments are joined together. The locations where the pieces of the function abut are called *joints*, in the case of curves, and *borders*, in the case of surfaces. The equations that govern this joining are called *continuity constraints*. In CAGDM, the continuity constraints are typically chosen to impart a given order of smoothness to the spline. The order of smoothness chosen will naturally be application dependent. For some applications, such as architectural drawing, it is sufficient for the curves to be continuous only in position. Other applications, such as the design of mechanical parts, require first or second order smoothness.

We have been intentionally vague about what is meant by ''smoothness''. In fact, there is more than one type of smoothness; the type that is used should be application dependent. For instance, if parametric splines are being used to define the path of an object in an animation system, it is important for the object to *move* smoothly. It is therefore not enough for the *path* of the object to be smooth, the *speed* of the object as it moves along the path must also be continuous. This type of motion can be guaranteed by requiring continuity of position and the *first parametric derivative vector*, also known as the *velocity vector*. If higher order continuity is required, one can demand continuity of the *second parametric* derivative, or *acceleration vector*. However, in many CAGDM applications, only the resulting path is important; the rate at which the points along the curve are swept out is irrelevant. This second notion of smoothness allows discontinuities in speed as long as the resulting path is geometrically smooth. We shall refer to the first kind of smoothness as *parametric continuity* and to the second kind as *geometric continuity*.

Since spline curves and surfaces are defined as piecewise functions, care must be taken to smoothly ''stitch'' the curve segments or surface patches together where they abut. The issue of exactly what is meant by ''smooth'' is a surprisingly subtle one, ultimately leading to the distinction between parametric and geometric continuity. We present here an abbreviated development of geometric continuity; more complete treatments can be found in[4, 5, 13, 15] Geometric continuity has become an important topic of research, and recent work has been reported in[10, 17, 18, 24, 25] On a historical note, n^{th} order geometric continuity has its roots in first and second order geometric continuity, the ideas of which appeared in various forms in.[1, 19, 20, 22, 28, 29, 30]

In the following, it will be important to maintain the distinction between *parametrizations* and *curves*. A parametrization is a function which describes a curve. A curve is the image of a parametrization. There can be many different parametrizations that describe the same curve.

3. Parametric Continuity

Let us now examine how continuity has classically been specified for parametric functions in CAGDM. We refer to this measure of continuity as parametric continuity. As mentioned in the previous section, it is typical to stitch pieces of parametric functions together to obtain a parametric spline. Borrowing concepts from fields such as numerical analysis and approximation theory, it seems reasonable to require that the derivatives of the pieces agree at the joint.

Consider the situation shown in Figure 1 where two C^∞ parametrizations (a parametrization is C^∞ if it is infinitely differentiable) $\mathbf{q}(u)$, $u \in [0,1]$, and $\mathbf{r}(t)$, $t \in [0,1]$ meet at a common point such that

$$\mathbf{r}(0) = \mathbf{q}(1).$$

These parametrizations are said to meet with n^{th} *order parametric continuity*, denoted C^n, if the first n parametric derivatives match at the common point; that is, if

$$\mathbf{r}^{(i)}(0) = \mathbf{q}^{(i)}(1), \quad i=1,\cdots,n, \tag{1}$$

where superscript (i) denotes the i^{th} derivative. Unfortunately, parametric continuity does not capture our intuitive notion of smoothness, as demonstrated by the following example. This shows that it is possible for the first derivative vector to be discontinuous even though the curve possesses a physically continuous unit tangent vector throughout its length.

Example 1: Figure 2 shows the two parametrizations $\mathbf{q}(u)$ and $\mathbf{r}(u)$ defined by

$$\mathbf{q}(u) = (2u, u), \quad u \in [0,1]$$
$$\mathbf{r}(t) = (4t+2, 2t+1) \quad t \in [0, \tfrac{1}{2}].$$

These parametrizations meet with positional continuity at the point (2,1). Note, how-

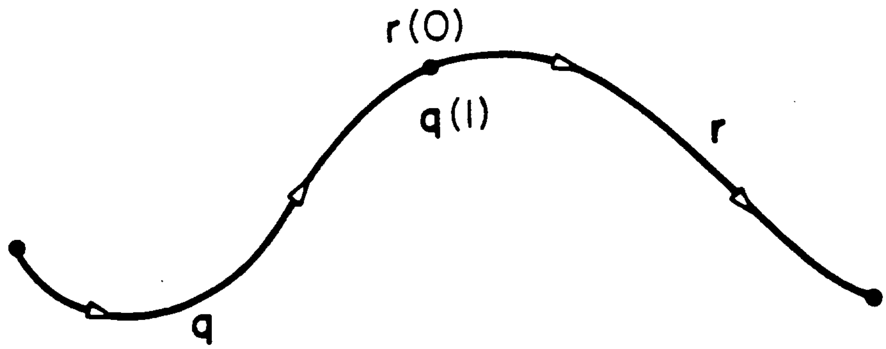

Figure 1: Two parametrized curves q and r meeting at a common point.

ever, that their first derivative vectors don't match:

$$\mathbf{q}^{(1)}(1) = (2,1)$$
$$\mathbf{r}^{(1)}(0) = (4,2)$$

implying that

$$\mathbf{r}^{(1)}(0) \neq \mathbf{q}^{(1)}(1).$$

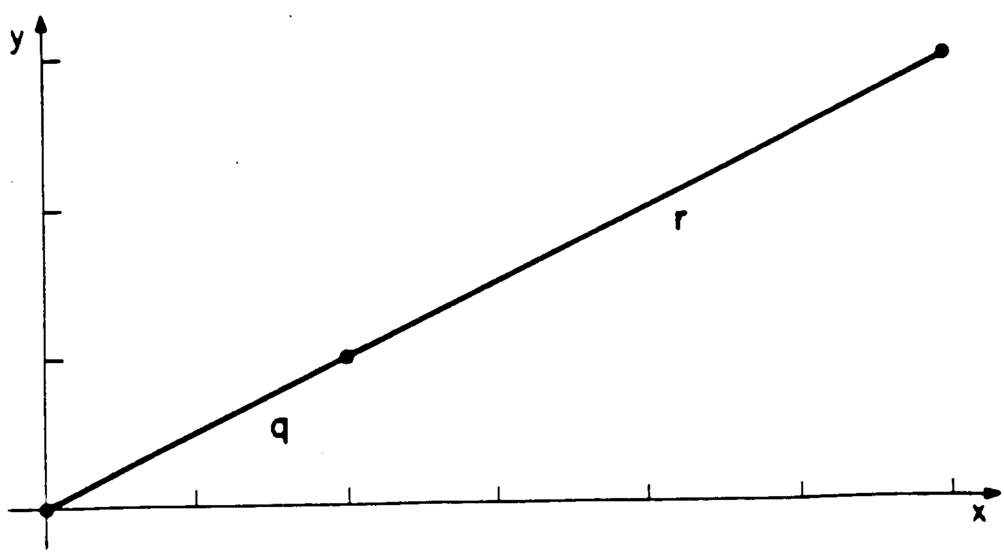

Figure 2: A discontinuous first derivative with a continuous unit tangent vector.

These line segments are collinear and have a continuous unit tangent vector, namely $(\frac{4}{5},\frac{3}{5})$, even though there is a jump in the first derivative vector at the joint. In other words, these parametrizations do not meet with first order parametric continuity even though the curve segments appear to meet very smoothly. ∎

This example provides insight into the shortcomings of parametric continuity. Parametric continuity places too much emphasis on the particulars of the parametrizations. It does not necessarily reflect the smoothness of the resulting composite curve; rather, it is a measure of smoothness for parametrizations. Also, parametric continuity disallows many parametrizations that would generate visually smooth curves. To really understand how to avoid situations like the one in Example 1, we turn now to the ideas of *reparametrization* and *equivalent parametrizations*.

4. Reparametrization and Equivalent Parametrizations

Recall that a parametrization $Q(u)$ is said to be C^n if it is n times continuously differentiable. Let $Q(u)$, $u \in [u_0, u_f]$, and $\tilde{Q}(\tilde{u})$, $\tilde{u} \in [\tilde{u}_0, \tilde{u}_f]$, be two *regular* C^n parametrizations. (A parametrization is regular if its first derivative vector never vanishes.) These parametrizations are said to be *equivalent*, that is, they describe the same *oriented curve*, if there exists a regular C^n function $f : [\tilde{u}_0, \tilde{u}_f] \Rightarrow [u_0, u_f]$ such that

(i) $\tilde{Q}(\tilde{u}) = Q(f(\tilde{u}))$. That is, $\tilde{Q} = Q \circ f$.

(ii) $f([\tilde{u}_0, \tilde{u}_f]) = [u_0, u_f]$

(iii) $f^{(1)} > 0$

Intuitively, Q and $\tilde{Q}$ trace out the same set of points in the same order. We also say that Q has been reparametrized to obtain $\tilde{Q}$, and we call f an *orientation preserving change of variables* (see Figure 3). The following example illustrates a concrete example of equivalent parametrizations.

Example 2: Let q be as in Figure 2, and let $\tilde{q}$ be defined by

$$\tilde{q}(\tilde{u}) = (4\tilde{u}, 2\tilde{u}), \quad \tilde{u} \in [0, \frac{1}{2}].$$

To show that $q(u) = (2u, u)$ and $\tilde{q}(\tilde{u}) = (4\tilde{u}, 2\tilde{u})$ are equivalent parametrizations, we observe that

$$\tilde{q}(\tilde{u}) = q(2\tilde{u}), \quad \text{for all } \tilde{u} \in [0, \frac{1}{2}].$$

Thus, we have found a mapping $f:[0,\frac{1}{2}] \Rightarrow [0,1]$ defined by $f(\tilde{u}) = 2\tilde{u}$ that satisfies property (i) of equivalent parametrizations. It is easily verified that f satisfies the other two properties as well. We therefore conclude that $\mathbf{q}$ and $\tilde{\mathbf{q}}$ describe the same oriented curve, which in this case is the oriented line segment from (0,0) to (2,1). ∎

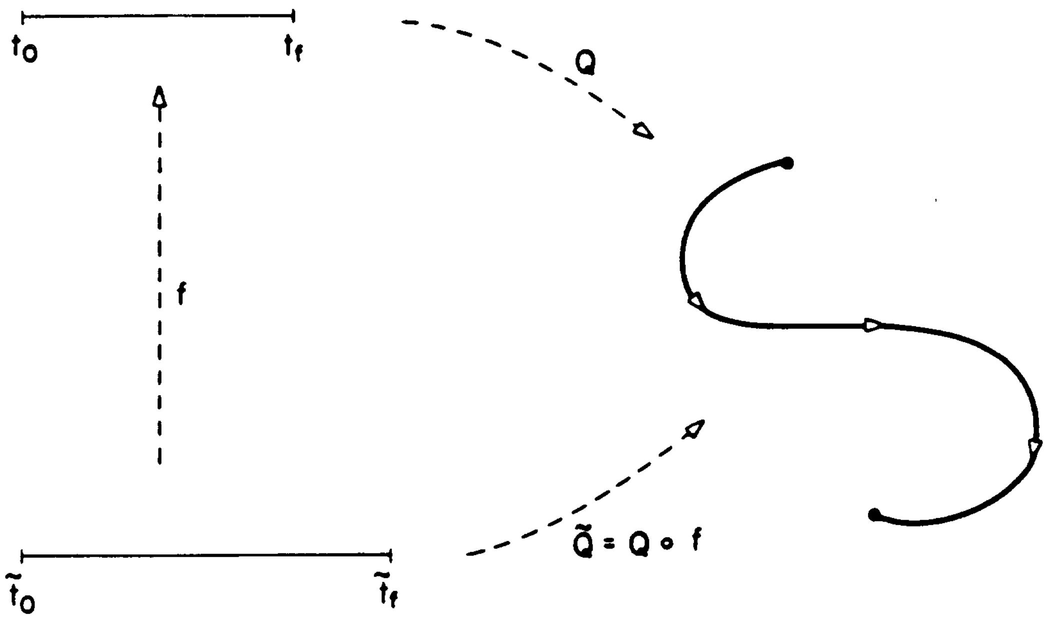

Figure 3: Q is reparametrized by f to obtain $\tilde{Q}$.

The existence of equivalent parametrizations means that there are many distinct parametrizations that describe the same oriented curve. Differential geometers are therefore careful to distinguish between properties of a particular parametrization and properties of the oriented curve it describes. This distinction can be made more precise by separating out the properties of parametrizations that remain invariant under reparametrization. Mathematically, let $\mathbf{q}$ and $\tilde{\mathbf{q}}$ be equivalent parametrizations, and let $Property(\mathbf{q})$ represent some property of $\mathbf{q}$ (that is, some statement about $\mathbf{q}$). $Property(\mathbf{q})$ is *intrinsic* if and only if

$$Property(\mathbf{q}) = Property(\tilde{\mathbf{q}}).$$

Intrinsic properties are shared by all equivalent parametrizations, and can therefore be interpreted as fundamental properties of the curve being described. As an example, we ask: Is the first derivative vector an intrinsic property? By definition, the first derivative vector is intrinsic if and only if

$$\mathbf{q}^{(1)} = \bar{\mathbf{q}}^{(1)}, \tag{2}$$

where $\mathbf{q}$ and $\bar{\mathbf{q}}$ are arbitrary equivalent parametrizations. To determine if Equation (2) does in fact hold, we use the *chain rule* from calculus:

$$\begin{aligned}
\mathbf{q}^{(1)} &= (\bar{\mathbf{q}} \circ f)^{(1)} \quad \textit{(by definition of equivalence)} \\
&= \bar{\mathbf{q}}^{(1)} f^{(1)} \quad \textit{(by the chain rule)} \\
&\neq \bar{\mathbf{q}}^{(1)} \quad \textit{(since } f^{(1)} \textit{ is not necessarily } 1)
\end{aligned}$$

Thus, the first derivative vector is *not* an intrinsic property, and is therefore not a fundamental property of an oriented curve. There is, however, is a closely related property that is intrinsic; namely, the *unit tangent vector*:

$$\begin{aligned}
\text{Unit Tangent Vector } (\mathbf{q}) &\equiv \frac{\mathbf{q}^{(1)}}{|\mathbf{q}^{(1)}|} \quad \textit{(by definition of unit tangent vector)} \\
&= \frac{(\bar{\mathbf{q}} \circ f)^{(1)}}{|(\bar{\mathbf{q}} \circ f)^{(1)}|} \quad \textit{(by definition of equivalence)} \\
&= \frac{\bar{\mathbf{q}}^{(1)} f^{(1)}}{|\bar{\mathbf{q}}^{(1)} f^{(1)}|} \quad \textit{(by the chain rule)} \\
&= \frac{\bar{\mathbf{q}}^{(1)} f^{(1)}}{|\bar{\mathbf{q}}^{(1)}| f^{(1)}} \quad \textit{(since } f^{(1)} > 0) \\
&= \frac{\bar{\mathbf{q}}^{(1)}}{|\bar{\mathbf{q}}^{(1)}|} \\
&= \text{Unit Tangent Vector } (\bar{\mathbf{q}})
\end{aligned}$$

From the point of view of differential geometry, Example 1 can now be explained as follows: C^n continuity requires equality of derivative vectors, and since derivative vectors are not intrinsic, C^n continuity can be destroyed simply by reparametrizing one of the curves. What is needed then is a measure of continuity that is *parametrization independent*. In other words, we would like a measure of continuity that is *invariant under reparametrization*, that is, one which remains valid after arbitrary reparametrization. The following definition of geometric continuity provides such a measure.

Definition 1: Let $\mathbf{q}(u)$ and $\mathbf{r}(t)$ be two regular C^n parametrizations meeting at a point $\mathbf{J}$. They meet with n^{th} order *geometric continuity*, denoted G^n, if there exists a parametrization $\tilde{\mathbf{q}}$ equivalent to $\mathbf{q}$ such that $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^n continuity at the joint $\mathbf{J}$.

It can be readily verified that geometric continuity is intrinsic in the sense that if $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity, and if $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ are any pair of equivalent parametrizations for $\mathbf{q}$ and $\mathbf{r}$, respectively, then $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ also meet with G^n continuity.

To develop some familiarity with the definition, let us apply it to the parametrizations of Example 1. In particular, if we choose $\tilde{\mathbf{q}}$ to be the equivalent parametrization constructed in Example 2, then we see that

$$\tilde{\mathbf{q}}^{(1)}(\frac{1}{2}) = (4,2)$$

$$\mathbf{r}^{(1)}(0) = (4,2)$$

implying that

$$\mathbf{r}^{(1)}(0) = \tilde{\mathbf{q}}^{(1)}(\frac{1}{2}).$$

Thus, $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^1 continuity at $\mathbf{J} = (2,1)$; hence, $\mathbf{q}$ and $\mathbf{r}$ meet with G^1 continuity.

The characterization of geometric continuity based on the existence of equivalent parametrizations can be summarized as: *Don't base continuity on the parametrizations at hand; reparametrize if necessary to find ones that meet with C^n continuity.* Although this is a useful theoretical tool for probing the intricacies of geometric continuity, there are other characterizations that are also of practical significance. We now briefly present two such equivalent characterizations.

5. Beta-Constraints

Let $\mathbf{q}(u), u \in [0,1]$ and $\mathbf{r}(t), t \in [0,1]$ be two regular C^n parametrizations meeting with G^n continuity at $\mathbf{r}(0) = \mathbf{q}(1)$, as was shown in Figure 1. According to Definition 1, there must exist an orientation preserving change of variables $u:[\tilde{u}_0,1] \Rightarrow [0,1]$ such that

$$\mathbf{r}^{(i)}(0) = \tilde{\mathbf{q}}^{(i)}(1), \quad i=1, \cdots, n \tag{3}$$

where

$$\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u})), \quad \tilde{u} \in [\tilde{u}_0, 1].$$

For simplicity (and without loss of generality) we have chosen $\tilde{u}_f = 1$. Using the chain rule, derivatives of $\tilde{\mathbf{q}}$ can be expanded in terms of derivatives of $\mathbf{q}$ and u. If the chain rule is applied i times, $\tilde{\mathbf{q}}^{(i)}$ can be expressed as a function, call it CR_i, of the first i derivatives of $\mathbf{q}$ and the first i derivatives of u:

$$\tilde{\mathbf{q}}^{(i)}(\tilde{u}) = CR_i(\mathbf{q}^{(1)}(u(\tilde{u})), \cdots, \mathbf{q}^{(i)}(u(\tilde{u})), u^{(1)}(\tilde{u}), \cdots, u^{(i)}(\tilde{u})), \quad i=1, \cdots, n. \tag{4}$$

Evaluating this expression at $\tilde{u} = 1$ and using the fact that $u(1)=1$, we find that

$$\tilde{\mathbf{q}}^{(i)}(1) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), u^{(1)}(1), \cdots, u^{(i)}(1)), \quad i=1, \cdots, n$$

This can be rewritten as

$$\tilde{\mathbf{q}}^{(i)}(1) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), \beta_1, \cdots, \beta_i), \quad i=1, \cdots, n \tag{5}$$

by performing the substitutions

$$\beta_j = u^{(j)}(1), \quad j=1, \cdots, i.$$

The quantities $\beta_1, \cdots, \beta_i$ are real numbers, and since $u^{(1)}(1)>0$ (property (iii) of an orientation preserving change of variables), we can conclude that $\beta_1>0$. Substituting Equation (5) into Equation (3) yields the so-called *Beta-constraints*:

$$\mathbf{r}^{(i)}(0) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), \beta_1, \cdots, \beta_i), \quad i=1, \cdots, n. \tag{6}$$

This argument shows that if $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity, then there exist real parameters $\beta_1, \cdots, \beta_n$, with $\beta_1>0$, commonly called *shape parameters*, satisfying the Beta-constraints. More important for applications, the converse is also true. To be precise, the parametrizations $\mathbf{q}(u), u \in [0,1]$ and $\mathbf{r}(t), t \in [0,1]$ meet with G^n continuity at $\mathbf{r}(0) = \mathbf{q}(1)$ if and only if there exist real numbers $\beta_1, \cdots, \beta_n$ with $\beta_1>0$ such that Equations (6) are satisfied. For a formal proof the reader is referred to.[4, 5, 13]

As an example of the form of the Beta-constraints, the constraints for G^4 continuity are

$$\mathbf{r}^{(1)}(0) = \beta_1 \mathbf{q}^{(1)}(1) \tag{7.1}$$

$$\mathbf{r}^{(2)}(0) = \beta_1^2\mathbf{q}^{(2)}(1) + \beta_2\mathbf{q}^{(1)}(1) \tag{7.2}$$

$$\mathbf{r}^{(3)}(0) = \beta_1^3\mathbf{q}^{(3)}(1) + 3\beta_1\beta_2\mathbf{q}^{(2)}(1) + \beta_3\mathbf{q}^{(1)}(1) \tag{7.3}$$

$$\mathbf{r}^{(4)}(0) = \beta_1^4\mathbf{q}^{(4)}(1) + 6\beta_1^2\beta_2\mathbf{q}^{(3)}(1) + (4\beta_1\beta_3 + 3\beta_2^2)\mathbf{q}^{(2)}(1) + \beta_4\mathbf{q}^{(1)}(1). \tag{7.4}$$

where β_2 and β_3 are arbitrary, but β_1 is constrained to be positive.

6. Arc Length Parametrization

The next characterization of geometric continuity is based on arc length parametrizations. It is possible to show that two parametrizations meet with G^n continuity if and only if the corresponding arc length parametrizations meet with C^n continuity.[13]

To gain a better understanding of this characterization, consider the cases of $n=1$ and $n=2$ in more detail. The case $n=1$ requires that the first derivatives with respect to arc length agree. But the first derivative with respect to arc length is the unit tangent vector. Thus, the case $n=1$ is equivalent to requiring that the unit tangent vectors agree at the joint **J**. Similarly, for $n=2$, the second derivative with respect to arc length is required to be continuous. The second derivative with respect to arc length is the curvature vector. Hence, for $n=2$, the unit tangent and curvature vectors must match at the joint. This can be stated more formally as a theorem for G^1 and G^2 continuity:

Theorem 1: Two parametrizations meet with G^1 continuity if and only if they have a common unit tangent vector; they meet with G^2 continuity if and only if they have common unit tangent and curvature vectors.

These are exactly the requirements of G^1 and G^2 continuity as originally developed for the Beta-spline representation in[1,3] and which appeared in various forms in.[19,20,22,28,29,30] This characterization therefore has the appeal that it represents a generalization of previous definitions.

7. Applications

Having derived the Beta-constraints, the general idea is to construct splines that satisfy them instead of requiring that parametric derivatives match. Since these constraints are stated in terms of $\beta_1, \cdots, \beta_n$, the resulting spline will have these quantities as shape parameters; they should not, however, be confused with the *domain parameter*. Changing one of the β's will, in general, change the shape of the composite curve, but always in such a way that geometric smoothness is maintained; it is for this reason that we call the β's shape parameters.

Referring back to the Beta-constraints, note that the shape parameter β_i is introduced in the constraint relating the i^{th} derivatives of the parametrizations in question. For example, β_1 is introduced in (7.1), and therefore controls the discrepancy between the first parametric derivatives, but always in such a way that the resulting composite curve is geometrically smooth. Suppose that $\beta_1 = 1$, implying that the first parametric derivatives agree. In this case, the shape parameter β_2 controls the discrepancy between the second parametric derivatives. If $\beta_1 = 1$ and $\beta_2 = 0$, the first two Beta-constraints reduce to the constraints for C^2 continuity. In general, if $\beta_1 = 1$ and $\beta_2 = \cdots = \beta_n = 0$, then G^n continuity reduces to C^n, showing that for regular parametrizations geometric continuity is a strict generalization of parametric continuity.

It is also important to realize that the shape parameters are *local* to a joint. If the composite curve being constructed comprises many curve segments, each of the joints possesses its own set of shape parameters. Thus, for a composite curve having m joints generated by G^n parametrizations, a total of mn shape parameters are introduced. In some applications, it is convenient to associate the same values of the n shape parameters with each of the joints, thereby making the assignment of shape parameters *global* to the composite curve.

8. Geometric Continuity for Composite Bězier Curves

As an application of geometric continuity, consider the problem of stitching Bězier curves together with G^1 and G^2 continuity. We first recall several important facts concerning Bězier curves.[2,6,7,8,9,11,21]

A Bèzier curve uses a simple and efficient formulation where the curve is defined solely in terms of a set of control vertices connected in a sequence to form a *control polygon* (Figure 4). The curve mimics the overall shape of the control polygon, but interpolates only the first and last vertices of the control polygon. The curve is defined by a polynomial whose degree is equal to the number of edges in the control polygon (that is, the number of control vertices minus one). It follows immediately from this definition that the formulation has *global*, not *local*, control; that is, the motion of a control vertex affects the shape of the entire curve. Likewise the curve is infinitely differentiable by virtue of being a polynomial.

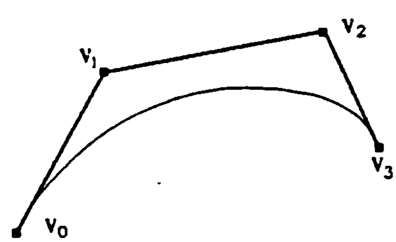

Figure 4: Bèzier curve and its control polygon.

A Bèzier curve $\mathbf{q}(u), u \in [0,1]$ of degree d defined by a control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_i, \cdots, \mathbf{V}_d>$, takes the form

$$\mathbf{q}(u) = \sum_{i=0}^{d} \mathbf{V}_i B_i^d(u), \quad u \in [0,1]$$

where $B_i^d(u)$ is the i^{th} *Bernstein* polynomial of degree d

$$B_i^d(u) = \binom{d}{i} u^i (1-u)^{d-i}, \quad i=0, \cdots, d.$$

It is frequently desirable to decouple the number of control vertices from the degree of the curve, and to have local control as well. In the Bèzier formulation, it is easy to raise the degree of the curve, by creating a new control polygon that generates the exact same curve with a (degenerate) polynomial of higher order (Figure 5). If we have the control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_i, \cdots, \mathbf{V}_d>$, the following formula gives the control polygon $<\mathbf{W}_0, \cdots, \mathbf{W}_i, \cdots, \mathbf{W}_{d+1}>$:

$$\mathbf{W}_i = (\frac{i}{d+1})\mathbf{V}_{i-1} + (1 - \frac{i}{d+1})\mathbf{V}_i \qquad i = 0, \cdots, d+1$$

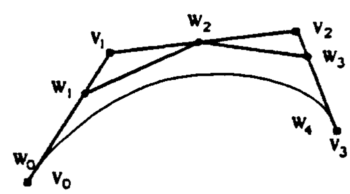

Figure 5: Raising the degree of a Bèzier curve from cubic to quartic.

Describing the curve $\mathbf{q}(u)$ in Bèzier form has many advantages: the shape is intuitively related to the control vertices, there is an easy geometric construction for the curve, splitting a curve into two spans is also geometrically easy, and the relationships between the parametric derivatives at u=0 and u=1 and the control vertices are simply expressed. Specifically, we will use the following properties.

(i) **Position:** $\mathbf{q}(u)$ interpolates $\mathbf{V}_0$ at u=0, and $\mathbf{V}_d$ at u=1:

$$\mathbf{q}(0) = \mathbf{V}_0 \tag{8.1a}$$

$$\mathbf{q}(1) = \mathbf{V}_d . \tag{8.1b}$$

(ii) **First Derivatives:** The initial first derivative vector is in the direction of the vector from $\mathbf{V}_0$ to $\mathbf{V}_1$, and the final first derivative vector is in the direction of the vector from $\mathbf{V}_{d-1}$ to $\mathbf{V}_d$. More precisely, the initial and final first derivative vectors are:

$$\mathbf{q}^{(1)}(0) = d(\mathbf{V}_1 - \mathbf{V}_0) \tag{8.2a}$$

$$\mathbf{q}^{(1)}(1) = d(\mathbf{V}_d - \mathbf{V}_{d-1}). \tag{8.2b}$$

(iii) **Second Derivatives:** The initial second derivative vector depends only on $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$, and the final second derivative vector depends only on $\mathbf{V}_{d-2}$, $\mathbf{V}_{d-1}$, and $\mathbf{V}_d$; specifically,

$$\mathbf{q}^{(2)}(0) = d(d-1)(\mathbf{V}_0 - 2\mathbf{V}_1 + \mathbf{V}_2) \tag{8.3a}$$

$$\mathbf{q}^{(2)}(1) = d(d-1)(\mathbf{V}_{d-2} - 2\mathbf{V}_{d-1} + \mathbf{V}_d). \tag{8.3b}$$

To obtain local control, we use a piecewise representation of the curve. The entire curve is composed of curve segments, each of which is a Bèzier polynomial. The problem we now wish to address is to maintain some amount of continuity at the joints; specifically,

Given: The shape parameters β_1 and β_2, and the control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_i, \cdots, \mathbf{V}_d>$ defining the parametrization

$$\mathbf{q}(u) = \sum_{i=0}^{d} \mathbf{V}_i B_i^d(u), \quad u \in [0,1],$$

find: constraints on the control polygon $\mathbf{W}_0, \cdots, \mathbf{W}_d$ defining the parametrization

$$\mathbf{r}(t) = \sum_{j=0}^{d} \mathbf{W}_j B_j^d(t), \quad t \in [0,1]$$

such that: $\mathbf{r}$ and $\mathbf{q}$ meet with G^1 (or G^2) continuity at $\mathbf{q}(1)$ with respect to β_1 (and β_2) (see Figure 6).

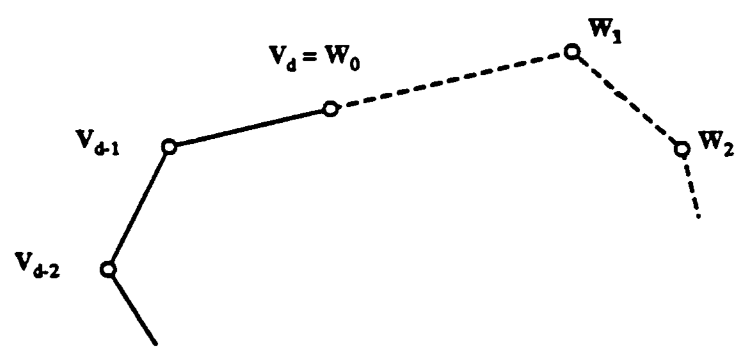

Figure 6: Situation for stitching two Bèzier curves $\mathbf{r}$ and $\mathbf{q}$ together with G^n continuity.

Since a Bèzier curve interpolates its first and last control vertices, we can guarantee C^0 (and hence G^0) continuity by setting $\mathbf{W}_0 = \mathbf{V}_d$, as shown in Figure 6. To achieve G^1 continuity for a given $\beta_1>0$, we can find $\mathbf{W}_1$ by recalling Equation (7.1) and using Equation (8.2) to yield

$$d(\mathbf{W}_1 - \mathbf{W}_0) = d\beta_1(\mathbf{V}_d - \mathbf{V}_{d-1}), \quad \beta_1>0.$$

Simplification and rearrangement yields

$$\mathbf{W}_1 = \mathbf{W}_0 + \beta_1(\mathbf{V}_d - \mathbf{V}_{d-1}), \quad \beta_1>0,$$

and since $\mathbf{W}_0 = \mathbf{V}_d$,

$$\mathbf{W}_1 = \mathbf{V}_d + \beta 1(\mathbf{V}_d - \mathbf{V}_{d-1}), \quad \beta 1 > 0. \tag{9}$$

Geometrically, Equation (9) states that $\mathbf{W}_1$ must lie on the ray starting at $\mathbf{V}_d$ ($= \mathbf{W}_0$), extending in the direction of the vector from $\mathbf{V}_{d-1}$ to $\mathbf{V}_d$. The length of the segment $\mathbf{W}_0\mathbf{W}_1$ relative to the length of $\mathbf{V}_{d-1}\mathbf{V}_d$ is given by the parameter $\beta 1$. Thus, given $\mathbf{V}_{d-1}$, $\mathbf{V}_d$ and $\beta 1 > 0$, the control vertices $\mathbf{W}_0$ and $\mathbf{W}_1$ can be determined geometrically as shown in Figure 7, or algorithmically using the following construction:

$$(1)\ \mathbf{W}_0 \Leftarrow \mathbf{V}_d \tag{10.1}$$

$$(2)\ \mathbf{W}_1 \Leftarrow \mathbf{W}_0 + \beta 1(\mathbf{V}_d - \mathbf{V}_{d-1}) \tag{10.2}$$

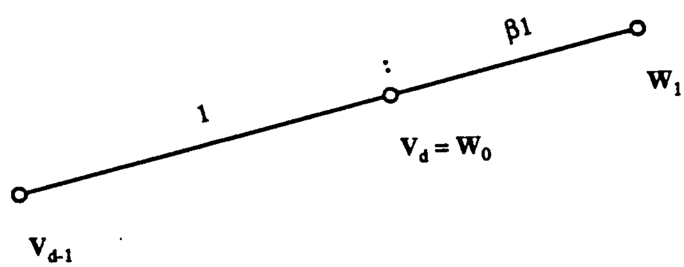

Figure 7: The construction of $\mathbf{W}_0$ and $\mathbf{W}_1$ to achieve G^1 continuity for a given $\beta 1$.

Once $\mathbf{W}_0$ and $\mathbf{W}_1$ have been constrained subject to G^1 continuity, the control vertex $\mathbf{W}_2$ can be constrained to guarantee G^2 continuity for a given $\beta 2$ by recalling Equation (7.2) and using Equation (8.3) to yield

$$d(d-1)(\mathbf{W}_0 - 2\mathbf{W}_1 + \mathbf{W}_2) = \beta 1^2\, d(d-1)(\mathbf{V}_{d-2} - 2\mathbf{V}_{d-1} + \mathbf{V}_d) + \beta 2\, d(\mathbf{V}_d - \mathbf{V}_{d-1}).$$

Solving for $\mathbf{W}_2$ yields

$$\mathbf{W}_2 = 2\mathbf{W}_1 - \mathbf{W}_0 + \beta 1^2(\mathbf{V}_{d-2} - 2\mathbf{V}_{d-1} + \mathbf{V}_d) + \frac{\beta 2(\mathbf{V}_d - \mathbf{V}_{d-1})}{d-1} \tag{11a}$$

Substituting $\mathbf{V}_d$ for $\mathbf{W}_0$ and Equation (9) for $\mathbf{W}_1$, and rearranging yields

$$\mathbf{W}_2 = \beta 1^2\mathbf{V}_{d-2} - (2\beta 1^2 + 2\beta 1 + \frac{\beta 2}{d-1})\mathbf{V}_{d-1} + (\beta 1^2 + 2\beta 1 + \frac{\beta 2}{d-1} + 1)\mathbf{V}_d \tag{11b}$$

Rather than the algebraic approach given above for the determination of $\mathbf{W}_2$, a more geometric approach was developed by Farin[19] and later improved upon by Boehm.[10] For our purposes, it is most convenient to think of the approach of Farin and Boehm as a convenient factorization of Equation (11), each term of which has a well-defined geometric interpretation.

The Farin-Boehm construction takes as input the control polygon $<\mathbf{V}_{d-2},\mathbf{V}_{d-1},\mathbf{V}_d>$ and the shape parameters $\beta_1>0$ and β_2, and produces as output the control vertices $\mathbf{W}_0$, $\mathbf{W}_1$, and $\mathbf{W}_2$ such that the curves meet with G^2 continuity with respect to β_1 and β_2. The construction may be stated as:

$$(1)\ \gamma \Leftarrow \frac{(d-1)(1+\beta_1)}{\beta_2+\beta_1(d-1)(1+\beta_1)} \tag{12.1}$$

$$(2)\ \mathbf{W}_0 \Leftarrow \mathbf{V}_d \tag{12.2}$$

$$(3)\ \mathbf{W}_1 \Leftarrow \mathbf{W}_0+\beta_1(\mathbf{V}_d-\mathbf{V}_{d-1}) \tag{12.3}$$

$$(4)\ \mathbf{T} \Leftarrow \mathbf{V}_{d-1}+\beta_1^2\gamma(\mathbf{V}_{d-1}-\mathbf{V}_{d-2}) \tag{12.4}$$

$$(5)\ \mathbf{W}_2 \Leftarrow \mathbf{W}_1+\frac{1}{\gamma}(\mathbf{W}_1-\mathbf{T})$$

The geometric interpretation of this construction is shown in Figure 8.

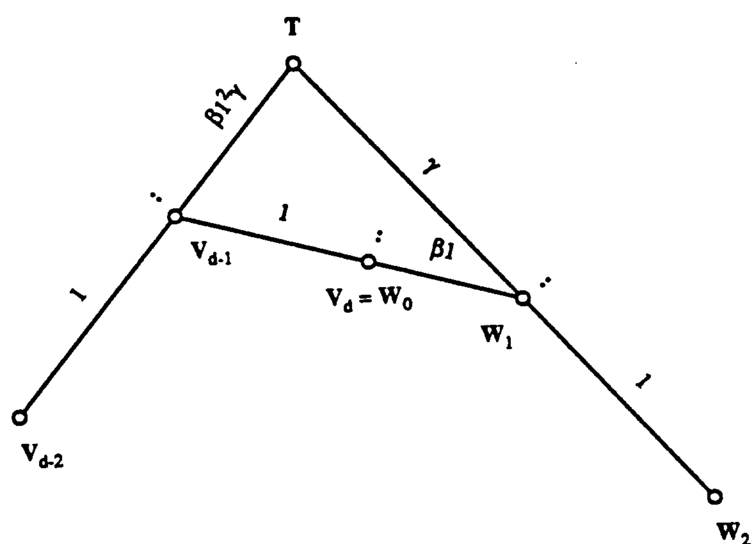

Figure 8: The Farin-Boehm construction.

In other words, only $\mathbf{W}_3$ can be freely chosen if we insist on G^2 geometric continuity *once* β_1 and β_2 are chosen. The crucial point is that the relaxation of the continuity constraints gives us two more degrees of freedom. In particular, we can adjust β_1 and β_2 to be able to ensure G^2 on both sides of the span.

9. Geometric Continuity for Catmull-Rom Splines

Catmull-Rom splines[12] have local control, can be either approximating or interpolating, and are efficiently computable. In[14,16] the authors construct a subclass of the Catmull-Rom splines which has shape parameters by requiring geometric rather than parametric continuity. Some members of this class are interpolating and others are approximating. The set of shape parameter values are associated with the joints of the curve. The shape parameters may be applied globally, affecting the entire curve, or they may be modified locally, affecting only a portion of the curve near the corresponding joint. It is shown that this class results from combining geometrically continuous (Beta-spline) blending functions with a new set of geometrically continuous interpolating functions related to the classical *Lagrange curves*.

The well-known cubic Catmull-Rom spline is a C^1 interpolating spline where the i^{th} segment of the curve can be written in the form

$$\mathbf{q}_i(u) = \sum_{j=-1}^{2} \phi_j(u)\mathbf{V}_{i+j}, \quad u \in [0,1]. \tag{13}$$

Using a process similar to Section 8, the weighting functions $\phi_j(u), j = -1,0,1,2$, are constructed so that $\mathbf{q}_i$ and $\mathbf{q}_{i+1}$ meet with G^1 continuity at their common joint. The resulting functions are

$$\phi_{-1}(u) = -\beta_1^2 \frac{u^3 - 2u^2 + u}{\beta_1 + 1}, \tag{14.1}$$

$$\phi_0(u) = \frac{(\beta_1^2 + \beta_1 + 1)u^3 - (2\beta_1^2 + 2\beta_1 + 1)u^2 + (\beta_1^2 - 1)u + \beta_1 + 1}{\beta_1 + 1}, \tag{14.2}$$

$$\phi_1(u) = -\frac{(\beta_1^2 + \beta_1 + 1)u^3 - (2\beta_1^2 + \beta_1 + 1)u^2 - \beta_1 u}{\beta_1(\beta_1 + 1)}, \tag{14.3}$$

$$\phi_2(u) = \frac{u^3 - u^2}{\beta_1(\beta_1 + 1)}. \tag{14.4}$$

The effect of β_1 on the shape of the spline is shown in Figure 9.

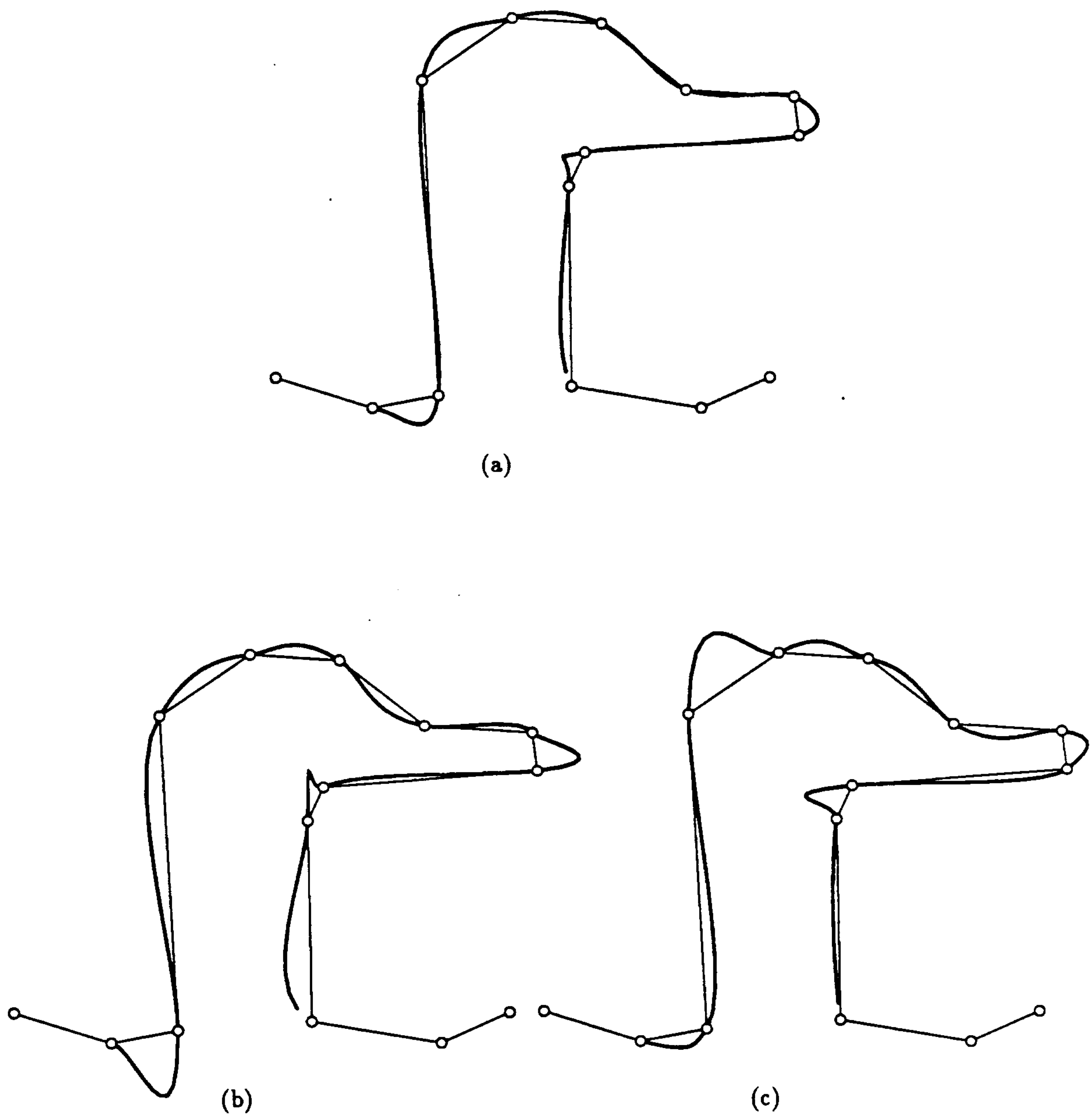

Figure 9: The above curves all share the same control polygon. Curve (a) has $\beta_1=1$, and is therefore equivalent to the C^1 Catmull-Rom spline Curves (b) and (c) have values of β_1 of 1/2 and 2, respectively.

10. Beta-splines

As our final example, we present a development of Beta-splines that is based on the results of Section 8 where Bèzier segments were stitched together with G^1 and G^2 continuity.

10.1. Quadratic G^1 Beta-splines

Given a control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_i, \cdots, \mathbf{V}_m>$ and a set of shape parameter values $\overline{\beta 1} = (\beta 1_0, \cdots, \beta 1_{m-1})$, the i^{th} segment $\mathbf{q}_i(u)$ of a G^1 quadratic Beta-spline takes the form

$$\mathbf{q}_i(u) = \sum_{r=-1}^{1} \mathbf{V}_{i+r} b_{i+r,r}(\overline{\beta 1};u), \quad u \in [0,1], \quad i = 1, \cdots, m-1, \tag{15}$$

where the functions $b_{i+r,r}(\overline{\beta 1};u)$, called the G^1 Beta-spline blending functions, are quadratic polynomials constructed so that

$$\begin{aligned} \mathbf{q}_{i+1}(0) &= \mathbf{q}_i(1), \quad i=1, \cdots, m-1 \\ \mathbf{q}_{i+1}^{(1)}(0) &= \beta 1_i \mathbf{q}_i^{(1)}(1), \quad i=1, \cdots, m-1. \end{aligned} \tag{16}$$

The Beta-spline blending functions can be determined by symbolically solving a system of linear equations.[1,3,4] A more elegant method, due to Farin[19] and Boehm,[10] proceeds by describing each segment $\mathbf{q}_i(u)$ in Bèzier form. In their approach, the i^{th} segment is written as

$$\mathbf{q}_i(u) = \sum_{r=0}^{2} \mathbf{W}_{i,r} B_r^2(u), \quad i=1, \cdots, m-1 \tag{17}$$

where the Bèzier control polygon $<\mathbf{W}_{i,0}, \mathbf{W}_{i,1}, \mathbf{W}_{i,2}>$ is constructed from the Beta-spline control polygon $<\mathbf{V}_{i-1}, \mathbf{V}_i, \mathbf{V}_{i+1}>$, and the shape parameters $\beta 1_{i-1}$ and $\beta 1_i$ as follows:

(1) The interior Bèzier vertex $\mathbf{W}_{i,1}$ is defined simply by

$$\mathbf{W}_{i,1} \Leftarrow \mathbf{V}_i. \tag{18}$$

(2) The junction vertex $\mathbf{W}_{i,0}$ divides the line segment $\mathbf{V}_{i-1}\mathbf{V}_i$ into relative distances $1:\beta 1_{i-1}$, and the junction vertex $\mathbf{W}_{i,2}$ is set equal to $\mathbf{W}_{i+1,0}$. Algorithmically,

$$\mathbf{W}_{i,0} \Leftarrow \frac{\beta 1_{i-1} \mathbf{V}_{i-1} + \mathbf{V}_i}{1 + \beta 1_{i-1}}. \tag{19}$$

$$\mathbf{W}_{i,2} \Leftarrow \mathbf{W}_{i+1,0} = \frac{\beta 1_i \mathbf{V}_i + \mathbf{V}_{i+1}}{1 + \beta 1_i}.$$

Comparison of Figure 10 and Figure 7 shows that the segments thus constructed do indeed satisfy Equations (16).

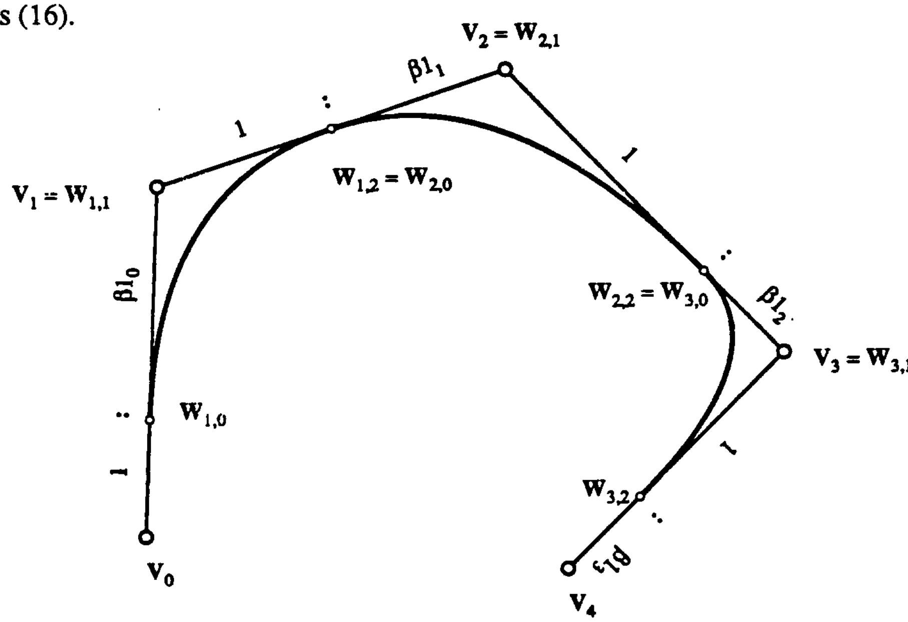

Figure 10: The construction of the G^1 Bèzier control polygons.

Once the Bèzier control polygons have been constructed, each segment can be drawn using standard Bèzier curve algorithms.[6,11,26,27]

10.2. Cubic G^2 Beta-splines

The previous constructions and definitions for quadratic G^1 Beta-splines can be extended to define cubic G^2 Beta-splines. A G^2 Beta-spline is defined by a control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_m>$, and two sets of shape parameter values, $\overline{\beta 1}=(\beta 1_0, \cdots, \beta 1_m)$ and $\overline{\beta 2}=(\beta 2_0, \cdots, \beta 2_m)$, associated with the joints of the curve. The i^{th} segment of the curve, $i=1, \cdots, m-2$, is given by

$$\mathbf{q}_i(u) = \sum_{r=-1}^{2} \mathbf{V}_{i+r} b_{i+r,r}(\overline{\beta 1},\overline{\beta 2};u), \quad u \in [0,1],$$

where the functions $b_{i+r,r}(\overline{\beta 1},\overline{\beta 2};u)$ are cubic polynomial functions constructed so that

$$\begin{aligned} \mathbf{q}_{i+1}(0) &= \mathbf{q}_i(1) \\ \mathbf{q}_{i+1}^{(1)}(0) &= \beta 1_i \mathbf{q}_i^{(1)}(1) \end{aligned} \tag{20}$$

$$\mathbf{q}_{i+1}^{(2)}(0) = \beta 1_i^2 \mathbf{q}_i^{(2)}(1) + \beta 2_i \mathbf{q}_i^{(1)}(1).$$

Rather than constructing the basis functions directly, we follow the approach of Farin and Boehm to construct the Bèzier control polygons of each of the segments. Let $<\mathbf{W}_{i,0}, \cdots, \mathbf{W}_{i,3}>$ denote the Bèzier control polygon of the i^{th} segment, $i=1, \cdots, m-2$. The first step of the construction proceeds by positioning, for each $i=0, \cdots, m-1$, the two interior Bèzier vertices $\mathbf{W}_{i,1}$ and $\mathbf{W}_{i,2}$ on the polygon edge $\mathbf{V}_i\mathbf{V}_{i+1}$ so that the three segments $\mathbf{V}_i\mathbf{W}_{i,1}$, $\mathbf{W}_{i,1}\mathbf{W}_{i,2}$, and $\mathbf{W}_{i,2}\mathbf{V}_{i+1}$ are of relative lengths $\gamma_i{:}1{:}\beta 1_{i+1}^2\gamma_{i+1}$ (see Figure 11(a)), where γ_i is defined as in the construction described by Equations (12) with $d=3$:

$$\gamma_i = \frac{2(1+\beta 1_i)}{\beta 2_i + 2\beta 1_i(1+\beta 1_i)}, \quad i=0, \cdots, m.$$

The second step of the construction positions the junction Bèzier vertices $\mathbf{W}_{i-1,3}$ and equivalently $\mathbf{W}_{i,0}$ at the point on the segment $\mathbf{W}_{i-1,2}\mathbf{W}_{i,1}$ that divides it into relative distances $1{:}\beta 1_i$, as shown in Figure 11(b). This construction guarantees that the Bèzier vertices for adjacent segments are positioned as required by Figure 8. Hence, adjacent segments meet with G^2 (unit tangent and curvature vector) continuity. More specifically, adjacent segments are guaranteed to satisfy Equations (20). The construction for G^2 Beta-splines may be stated algorithmically as follows:

(1a) for $i=0, \cdots, m$, compute γ_i from $\beta 1_i$ and $\beta 2_i$:

$$\gamma_i \Leftarrow \frac{2(1+\beta 1_i)}{\beta 2_i + 2\beta 1_i(1+\beta 1_i)}.$$

(1b) for $i=0, \cdots, m-1$, compute the interior Bèzier vertices:

$$\mathbf{W}_{i,1} \Leftarrow \frac{(1+\beta 1_{i+1}^2\gamma_{i+1})\mathbf{V}_i + \gamma_i \mathbf{V}_{i+1}}{1 + \gamma_i + \beta 1_{i+1}^2\gamma_{i+1}}$$

$$\mathbf{W}_{i,2} \Leftarrow \frac{\beta 1_{i+1}^2\gamma_{i+1}\mathbf{V}_i + (1+\gamma_i)\mathbf{V}_{i+1}}{1 + \gamma_i + \beta 1_{i+1}^2\gamma_{i+1}}$$

(2) for $i=1, \cdots, m-1$, compute the junction Bèzier vertices:

$$\mathbf{W}_{i-1,3} \Leftarrow \mathbf{W}_{i,0} \Leftarrow \frac{\beta 1_i \mathbf{W}_{i-1,2} + \mathbf{W}_{i,1}}{1 + \beta 1_i}$$

(a)

(b)

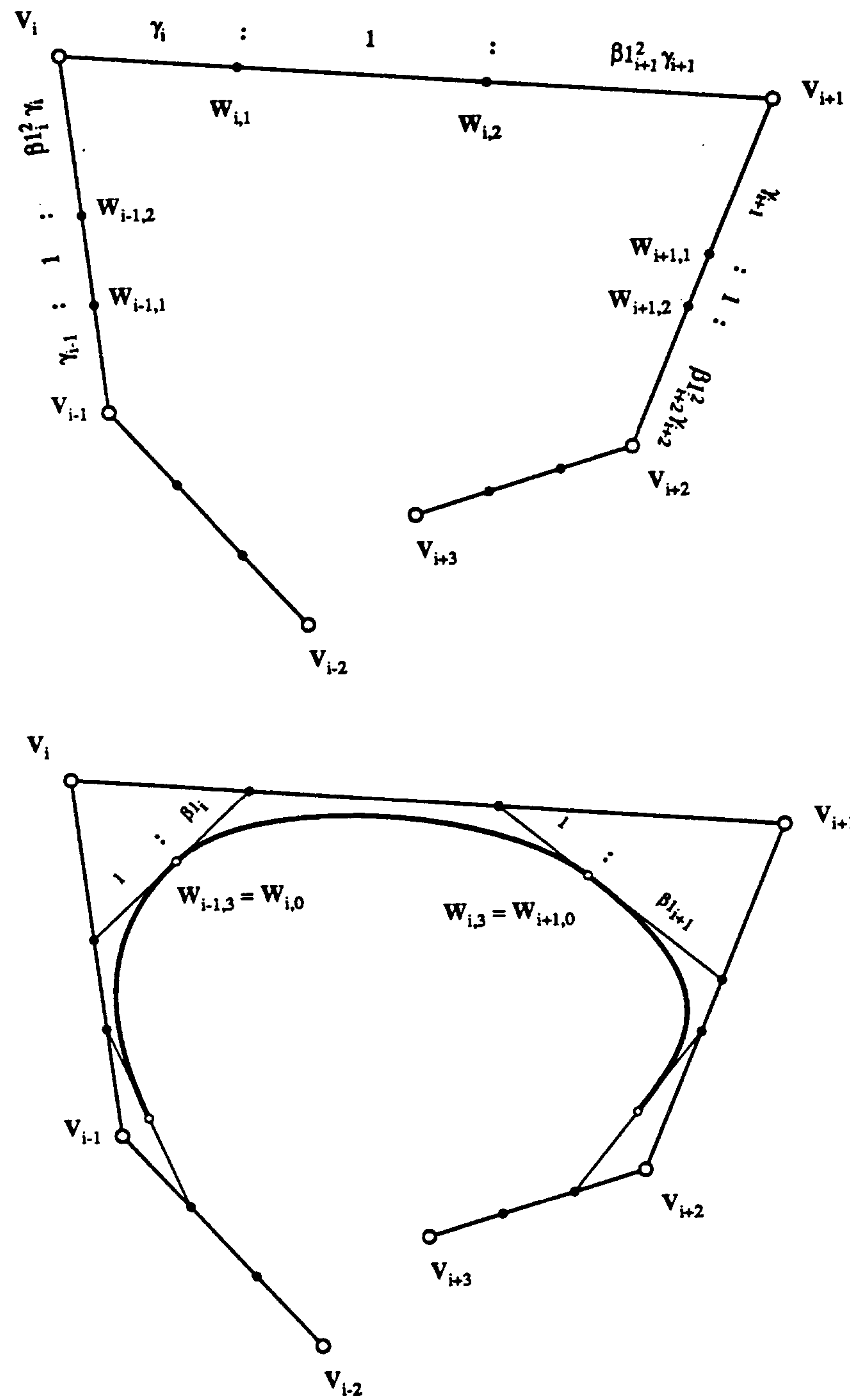

Figure 11: The construction for the G^2 Beta-spline curve.
(a) Construction of the interior Bèzier vertices.
(b) Positioning the junction vertices.

10.3. Designing with Beta-splines and Generalizing to G^n Beta-splines

As pointed out by Fournier & Barsky[21] and Boehm,[10] when designing with Beta-splines it is often more convenient for the designer to directly specify the interior Bèzier vertices, then have the system compute the shape parameters and place the junction vertices. In particular, the method of Farin and Boehm proceeds by having the designer specify a control polygon $<\mathbf{V}_0, \cdots, \mathbf{V}_m>$, and a pair of vertices $\mathbf{W}_{i,1}$ and $\mathbf{W}_{i,2}$ on each leg of the polygon. From this input, the shape parameters are uniquely determined (as long as $\mathbf{W}_{i,1} \neq \mathbf{W}_{i,2}$), and can be computed automatically by the system. The shape parameters are then used to compute the junction vertices, thereby completing the determination of each segment in Bèzier form. The curve can then be drawn using standard techniques for Bèzier curves.[6,11,26,27]

Although we have demonstrated the construction of G^1 and G^2 Beta-splines, we have not established that Beta-splines of all orders exist. It is, in fact, possible to construct G^n Beta-splines for arbitrary $n \geq 1$, as shown by Goodman[23] and Dyn & Micchelli.[17] Thus, given a set of shape parameter values $\overline{\beta_1}, \cdots, \overline{\beta_n}$, it is possible to find piecewise polynomial blending functions $B_i(\overline{\beta_1}, \cdots, \overline{\beta_n}; \overline{u})$ that satisfy the n^{th} order Beta-constraints with respect to the given shape parameters. Goodman and Dyn & Micchelli also show that these functions have local support and their segments have degree $n+1$. However, an algorithm for geometrically constructing the Bèzier control polygons of a G^n Beta-spline for arbitrary n and for arbitrary shape parameters is currently unknown.

11. Conclusion

Geometric continuity is an intrinsic measure of continuity which is appropriate for spline development. Geometric continuity has been shown to be a relaxed form of parametric continuity that is independent of the parametrizations of the curve segments under consideration, but is still sufficient for geometric smoothness of the resulting curve. However, geometric continuity is only appropriate for applications where the particular parametrization used is unimportant since parametric discontinuities are allowed.

Three characterizations of geometric continuity were developed. First, the concept of equivalent parametrizations was used to view geometric continuity as a measure of continuity that is parametrization independent, that is, a measure that is invariant under reparametrization. The second characterization developed the Beta-constraints, which are necessary and sufficient conditions for geometric continuity of curves. Finally, the third characterization showed that two curves meet with G^n continuity if and only if their arc length parametrizations meet with C^n continuity.

By using the Beta-constraints instead of requiring continuous parametric derivatives, n degrees of freedom called shape parameters are introduced. The shape parameters may be made available to a designer in a CAGD environment as a convenient method of changing the shape of the curve without altering the control polygon.

Several examples of splines using geometric continuity were provided: the construction of geometrically continuous Bèzier curves, the development of a subclass of the Catmull-Rom splines based on geometric continuity and possessing shape parameters, and the geometric construction of quadratic G^1 and cubic G^2 Beta-spline curves.

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On page 7, line -3, "$(\frac{4}{5}, \frac{3}{5})$" should be "$(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$".

On page 13, line 4, under equation (7.4), "where β_2 and β_3 are arbitrary, but β_1 is constrained to be positive." should be "where β_2, β_3, and β_4 are arbitrary, but β_1 is constrained to be positive."