$\bar{\beta}$-Continuity

and Its Application to Rational Beta-splines

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University of California at Berkeley Technical Report No. UCB/CSD 88/442
August 1988

Abstract

This paper provides a rigorous mathematical foundation for geometric continuity of *rational* Beta-splines of arbitrary order. A function is said to be n^{th} order $\bar{\beta}$-continuous if and only if it satisfies the Beta-constraints for a fixed value of $\bar{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)$. Sums, differences, products, quotients, and scalar multiples of $\bar{\beta}$-continuous scalar-valued functions are shown to also be $\bar{\beta}$-continuous scalar-valued functions (for the same value of $\bar{\beta}$). Using these results, it is shown that the rational Beta-spline basis functions are $\bar{\beta}$-continuous for the same value of $\bar{\beta}$ as the corresponding integral basis functions. It follows that the rational Beta-spline curve and tensor product surface are geometrically continuous.

KEYWORDS: Geometric Continuity, Reparametrization, Equivalent Parametrization, Beta-constraints, $\bar{\beta}$-continuity, Shape Parameters, Chain Rule, Basis Functions, Beta-splines, Rational Splines, Integral Splines

This work was supported in part by the National Science Foundation under Grant Number CCR-8451997, the Control Data Corporation, AT&T Bell Laboratories, an IBM Faculty Development Award, and the Natural Sciences and Engineering Research Council of Canada.

1. Introduction

Polynomial and piecewise polynomial parametric curves and surfaces are standard in computer graphics and computer aided geometric design (CAGD). Unfortunately, many simple shapes -- including most conic sections and quadric surfaces -- cannot be represented exactly with polynomial or even piecewise polynomial parametrizations. On the other hand, *rational* polynomial parametrizations can exactly represent all planar curves of zero genus†, all conic sections, and all quadric surfaces. For this reason, rational parametric representations have recently been gaining widespread acceptance in CAGD. The growing popularity of the non-uniform rational B-spline (*NURB*) is largely due to this concern for the exact representation of simple shapes.

Beta-splines were introduced into CAGD by Barsky.[1,2] These splines are *geometrically continuous* piecewise polynomials with scalar *shape parameters* which can be adjusted to alter the shape of a curve or surface without moving its control vertices. Like any piecewise polynomial parametric representation, Beta-splines cannot exactly represent many simple standard shapes. To attain exact representations, it is necessary to consider rational Beta-splines.

Barsky[3,4] investigated rational Beta-splines of arbitrary order. His investigation requires a rigorous proof that rational Beta-spline curves and surfaces are geometrically continuous. The purpose of this paper is to lay a firm mathematical foundation for the geometric continuity of rational Beta-splines.

Joe[28] studied rational cubic and quartic Beta-splines. By appealing to the explicit formulas for the *Beta-constraints*[6,7,8,13,14] he showed that these curves and surfaces are indeed geometrically continuous. However, for higher degrees, this procedure would be extremely tedious and difficult because the explicit formulas for the higher order Beta-constraints are quite complicated. Here we study the general case by returning to first principles: *reparametrization* and the *chain rule*. We shall show that when scalar-valued

† Loosely speaking, an implicit polynomial curve $f(x,y)=0$ of degree d has zero genus if it has $\frac{(d-1)(d-2)}{2}$ self intersections. For a more rigorous discussion and further details see.[33]

functions satisfy the Beta-constraints for the same value of $\bar{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)$, then the sum, difference, product, quotient, and scalar multiple of these functions again satisfy the Beta-constraints for this same value of $\bar{\beta}$. From these results, it is shown that the rational Beta-spline basis functions satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the corresponding integral basis functions. We then apply these results to prove that the rational Beta-spline curve and tensor product surface are geometrically continuous.

The next section gives the formal definition of geometric continuity, states the Beta-constraints, and introduces the notion of $\bar{\beta}$-continuity. In Section 3, results about the $\bar{\beta}$-continuity of various combinations of scalar-valued functions are proved. In Section 4, these results are applied to show that the rational Beta-spline basis functions satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the corresponding integral basis functions and therefore can be used to construct geometrically continuous rational Beta-spline curves and tensor product surfaces. Section 5 summarizes our results.

2. Geometric Continuity: Equivalent Parametrizations, Reparametrization, Beta-Constraints, and $\bar{\beta}$-continuity

Geometric continuity is a measure of smoothness which has recently been proposed as an alternative to *parametric continuity*, the conventional measure of smoothness for parametric curves and surfaces. In,[6, 7, 8, 13, 14] the notion of n^{th} order geometric continuity for an arbitrary non-negative integer n, called G^n continuity, was introduced. Geometric continuity has become an important topic of research, and recent work has been reported in.[11, 15, 16, 17, 25, 26] On a historical note, n^{th} order geometric continuity has its roots in first and second order geometric continuity, the ideas of which first appeared in various forms in.[1, 18, 19, 20, 29, 31, 32]

The concept of geometric continuity is based upon the notion of *equivalent parametrizations*. This section defines equivalent parametrizations and geometric continuity and then states, without proof, a theorem on Beta-constraints; the notion of $\bar{\beta}$-continuity is then introduced. Much of this material is derived from that presented in.[6, 7, 8, 13, 14]

Intuitively, two parametrizations are *equivalent* if they trace out the same set of points in the same order. We formalize this notion in the following definition:

Definition 1: "Equivalent Parametrization." Let $\mathbf{q}(u)$ and $\tilde{\mathbf{q}}(\tilde{u})$, where $\mathbf{q}: [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^d$, and $\tilde{\mathbf{q}}: [\tilde{u}_{\min}, \tilde{u}_{\max}] \Rightarrow \mathbf{R}^d$, $d>1$, be two *regular* C^n parametrizations (a parametrization is regular if its first derivative vector never vanishes).† These parametrizations are said to be *equivalent* if there exists a C^n function $u:[\tilde{u}_{\min}, \tilde{u}_{\max}] \Rightarrow [u_{\min}, u_{\max}]$ such that

(i) *Composition*: $\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u}))$. That is, $\tilde{\mathbf{q}} = \mathbf{q} \circ u$.

(ii) *Onto*: $u([\tilde{u}_{\min}, \tilde{u}_{\max}]) = [u_{\min}, u_{\max}]$

(iii) *Orientation preserving*: $\dfrac{du}{d\tilde{u}} > 0$

We say that $\mathbf{q}$ has been reparametrized to obtain $\tilde{\mathbf{q}}$, and we call u an *orientation preserving change of variables.*

Definition 2: "Geometric Continuity." Let $\mathbf{q}(u)$ and $\mathbf{r}(t)$, where $\mathbf{q}: [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^d$ and $\mathbf{r}: [t_{\min}, t_{\max}] \Rightarrow \mathbf{R}^d$, $d>1$, be two regular C^n parametrizations that meet at a common point $\mathbf{J}$:

$$\mathbf{r}(t_{\min}) = \mathbf{q}(u_{\max}) = \mathbf{J}.$$

These parametrizations meet with n^{th} order *geometric continuity*, denoted G^n, if and only if there exists a parametrization $\tilde{\mathbf{q}}$ equivalent to $\mathbf{q}$ such that $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^n continuity at the joint $\mathbf{J}$; that is, if and only if

$$\mathbf{r}^{(i)}(t_{\min}) = \tilde{\mathbf{q}}^{(i)}(\tilde{u}_{\max}), \quad i=0,1, \cdots, n.$$

† Boldface is used to demonstrate that the function is vector-valued.

The characterization of geometric continuity based on the existence of equivalent parametrizations can be summarized as: *Do not base smoothness on the parametrizations at hand; reparametrize if necessary to find ones that meet with C^n continuity.*

Using this idea of when two parametrizations meet with n^{th} order geometric continuity, we can also define the notion of a geometrically continuous curve. Intuitively, a curve is geometrically continuous at a point if and only if after segmenting it at that point the two segments meet with geometric continuity. We formalize this notion as follows:

Definition 3: **"Geometrically Continuous Curve."** Let $\mathbf{p}(u)$, where $\mathbf{p}: [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^d, d > 1$, be a curve and let $\mathbf{q} = \mathbf{p}_{|[u_{\min}, u^*]}$ and $\mathbf{r} = \mathbf{p}_{|[u^*, u_{\max}]}$. Then $\mathbf{p}$ is said to be G^n at $\mathbf{p}(u^*)$ if and only if $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity at $\mathbf{p}(u^*)$. But $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity if and only if there exists a parametrization $\bar{\mathbf{q}}$ equivalent to $\mathbf{q}$ such that $\bar{\mathbf{q}}$ and $\mathbf{r}$ meet with C^n continuity. Thus, $\mathbf{p}(u)$ is G^n at $\mathbf{p}(u^*)$ if and only if there is a reparametrization $u: [\bar{u}_{\min}, \bar{u}_{\max}] \Rightarrow [u_{\min}, u_{\max}]$ such that $\mathbf{p}(\bar{u}) = \mathbf{p}(u(\bar{u}))$ is C^n at $\mathbf{p}(\bar{u}^*)$ where $u(\bar{u}^*) = u^*$.

Notice that like parametric continuity, geometric continuity is a local property.

The following theorem precisely characterizes when two parametrizations $\mathbf{q}(u)$ and $\mathbf{r}(t)$ meet with G^n continuity at a point $\mathbf{J}$. Intuitively, since there is a parametrization $\bar{\mathbf{q}}(\bar{u})$ equivalent to $\mathbf{q}(u)$ which meets $\mathbf{r}(t)$ at $\mathbf{J}$ with C^n continuity, we can calculate the i^{th} derivative of $\mathbf{r}(t)$ at $\mathbf{J}$ in terms of the first i derivatives of $\mathbf{q}(u)$ at $\mathbf{J}$ by the chain rule.

Notation: Let $\sum_{j=1}^{i} CR_{ij}(\beta 1, \beta 2, \cdots, \beta i)\mathbf{q}^{(j)}(u), i = 1,2,\cdots,n$ denote the expression for the chain rule applied to $[\mathbf{q}(u(\bar{u}))]^{(i)}$ with βj substituted for $\frac{d^j u}{d\bar{u}^j}, j = 1,2,\cdots,i$.

Theorem 1: **"Beta-constraints."** Let $\mathbf{q}(u)$ and $\mathbf{r}(t)$, where $\mathbf{q}: [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^d$ and

$$\mathbf{r}: [t_{\min}, t_{\max}] \Rightarrow \mathbf{R}^d, d > 1,$$

be two regular C^n parametrizations that meet at a common point **J**; that is,

$$\mathbf{r}(t_{\min}) = \mathbf{q}(u_{\max}) = \mathbf{J}.$$

Then they meet with G^n continuity at **J** if and only if there exist real numbers $\beta_1, \beta_2, \cdots, \beta_n$ with $\beta_1 > 0$ such that

$$\mathbf{r}^{(i)}(t_{\min}) = \sum_{j=1}^{i} CR_{ij}(\beta_1, \beta_2, \cdots, \beta_i)\mathbf{q}^{(j)}(u_{\max}), \quad i = 1,2, \cdots, n. \tag{1}$$

Moreover, if $\mathbf{q}(u)$ and $\mathbf{r}(t)$ satisfy equation (1), then there is a reparametrization $\bar{\mathbf{q}}(\bar{u}) = \mathbf{q}(u(\bar{u}))$ of $\mathbf{q}(u)$ such that $\bar{\mathbf{q}}(\bar{u})$ and $\mathbf{r}(t)$ meet with C^n continuity at **J** and

$$\beta_j = \frac{d^j u}{d\bar{u}^j} \mid_J \quad j = 1,2, \cdots, n.$$

Proof: See.[6, 8]

Equations (1) are called the *Beta-constraints* and $\beta_1, \beta_2, \cdots, \beta_n$ are the *shape parameters* which are found in the Beta-spline.

As an example of the form of the Beta-constraints, the constraints for G^4 continuity are

$$\mathbf{r}^{(1)}(t_{\min}) = \beta_1 \mathbf{q}^{(1)}(u_{\max}) \tag{2.1}$$

$$\mathbf{r}^{(2)}(t_{\min}) = \beta_1^2 \mathbf{q}^{(2)}(u_{\max}) + \beta_2 \mathbf{q}^{(1)}(u_{\max}) \tag{2.2}$$

$$\mathbf{r}^{(3)}(t_{\min}) = \beta_1^3 \mathbf{q}^{(3)}(u_{\max}) + 3\beta_1\beta_2 \mathbf{q}^{(2)}(u_{\max}) + \beta_3 \mathbf{q}^{(1)}(u_{\max}) \tag{2.3}$$

$$\mathbf{r}^{(4)}(t_{\min}) = \beta_1^4 \mathbf{q}^{(4)}(u_{\max}) + 6\beta_1^2\beta_2 \mathbf{q}^{(3)}(u_{\max}) + (4\beta_1\beta_3 + 3\beta_2^2)\mathbf{q}^{(2)}(u_{\max}) + \beta_4 \mathbf{q}^{(1)}(u_{\max}). \tag{2.4}$$

where β_2, β_3, and β_4 are arbitrary, but β_1 is constrained to be positive.

The following definition formalizes the concept of $\bar{\beta}$-continuity.

Definition 4: "**$\bar{\beta}$-continuity.**" Let $\mathbf{p}(u)$, where $\mathbf{p}: [u_{min}, u_{max}] \Rightarrow \mathbf{R}^d, d \geq 1$, be a function. Then $\mathbf{p}$ is said to be n^{th} order $\bar{\beta}$-continuous at u^* if and only if $\mathbf{p}(u)$ satisfies the Beta-constraints for a fixed value of $\bar{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)$ at $\mathbf{p}(u^*)$.

By Theorem 1, a curve in $\mathbf{R}^d, d > 1$, is $\bar{\beta}$-continuous if and only if it is G^n continuous. However, unlike curves in $\mathbf{R}^d, d > 1$, regular scalar-valued functions in $\mathbf{R}^1$ always satisfy the Beta-constraints for some unique value of $\bar{\beta}$. Therefore, all regular scalar-valued functions are $\bar{\beta}$-continuous for some value of $\bar{\beta}$. In the following section, we will study various combinations of scalar-valued $\bar{\beta}$-continuous functions for fixed values of $\bar{\beta}$.

3. Sums, Differences, Products, Quotients, and Scalar Multiples of $\bar{\beta}$-continuous Scalar-valued Functions Are $\bar{\beta}$-continuous

Lemma 1: Let $f(u)$ and $g(u)$, where $f : [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^1$ and $g : [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^1$, be real-valued functions that are $\bar{\beta}$-continuous (for the same value of $\bar{\beta}$) at $u^* \in [u_{\min}, u_{\max}]$.

Then there exists a reparametrization $u(\bar{u})$, where $u : [\bar{u}_{\min}, \bar{u}_{\max}] \Rightarrow [u_{\min}, u_{\max}]$, for which $f(u(\bar{u}))$ and $g(u(\bar{u}))$ are both C^n continuous at u^*.

Proof: Let $\mathbf{p}(u) = (f(u), g(u))$. Since, by assumption, $f(u)$ and $g(u)$ satisfy the Beta-constraints for the same value of $\bar{\beta}$ at u^*, the curve $\mathbf{p}(u)$ satisfies the Beta-constraints for this value as well. Therefore, by applying Theorem 1 to the curve $\mathbf{p}(u)$, there is a reparametrization $u(\bar{u})$, where $u : [\bar{u}_{\min}, \bar{u}_{\max}] \Rightarrow [u_{\min}, u_{\max}]$, such that $\mathbf{p}(u(\bar{u})) = (f(u(\bar{u})), g(u(\bar{u})))$ is C^n continuous at $\mathbf{p}(u^*)$. Hence, $f(u(\bar{u}))$ and $g(u(\bar{u}))$ are C^n continuous at u^*. ∎

We now use Lemma 1 to show that sums, differences, products, quotients, and scalar multiples of $\bar{\beta}$-continuous functions are themselves $\bar{\beta}$-continuous functions (for the same value of $\bar{\beta}$).

Lemma 2: Let $f(u)$ and $g(u)$, where $f : [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^1$ and $g : [u_{\min}, u_{\max}] \Rightarrow \mathbf{R}^1$, be real-valued functions that are $\bar{\beta}$-continuous (for the same value of $\bar{\beta}$) at u^*.

Then the sum $f(u) + g(u)$, difference $f(u) - g(u)$, product $f(u)g(u)$, quotient $\frac{f(u)}{g(u)}$, for $g(u) \neq 0$, and scalar multiple $c\, f(u)$ are each functions that are $\bar{\beta}$-continuous (for the same value of $\bar{\beta}$ as for $f(u)$ and $g(u)$) at u^*.

Proof: By Lemma 1, since $f(u)$ and $g(u)$ satisfy the Beta-constraints for the same value of $\bar{\beta}$ at u^*, there exists a reparametrization $u(\bar{u})$ for which $f(u(\bar{u}))$ and $g(u(\bar{u}))$ will both be C^n continuous at u^*. Thus, the sum $f(u(\bar{u})) + g(u(\bar{u}))$, difference $f(u(\bar{u})) - g(u(\bar{u}))$, product $f(u(\bar{u}))g(u(\bar{u}))$, quotient $\frac{f(u(\bar{u}))}{g(u(\bar{u}))}$, and scalar multiple $c\, f(u(\bar{u}))$ are all C^n continuous since they are the sum, difference, product, quotient, and scalar multiple, respectively, of functions that are C^n continuous. Moreover, each function $f(u) + g(u)$, $f(u) - g(u)$, $f(u)g(u)$, $\frac{f(u)}{g(u)}$, for $g(u) \neq 0$, and $c\, f(u)$ satisfies the Beta-constraints for the same value of $\bar{\beta}$ as do $f(u)$ and $g(u)$ since it is the same reparametrization that converts them to C^n functions. Thus, we conclude that these various combinations of functions are each $\bar{\beta}$-continuous (for the same value of $\bar{\beta}$ as for $f(u)$ and $g(u)$) at u^*. ∎

4. Application of $\bar{\beta}$-continuous Functions: The Rational Beta-spline

4.1. Background on the Rational Beta-spline

In ,[3,4] Barsky developed the *rational Beta-spline*. Similar work, restricted to the cubic and quartic cases was reported by Joe in[28] and by Boehm in,[12] respectively. The rational Beta-spline combines the features of the rational form, in general, with those of the Beta-spline representation, specifically. It is important to be able to show that the rational Beta-spline basis functions satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the corresponding integral basis functions and consequently that the rational

Beta-spline curve and tensor product surface are geometrically continuous and have the same shape parameters as their integral counterparts. This we now proceed to do.

Given an ordered sequence of $m+1$ control vertices $\mathbf{V}_i$ and weights w_i, for $i = 0, \cdots, m$, a rational Beta-spline curve, denoted by $\mathbf{Q}(u)$, is defined by

$$\mathbf{Q}(u) = \frac{\sum_{i=0}^{m} w_i \, \mathbf{V}_i \, B_i(u)}{\sum_{r=0}^{m} w_r \, B_r(u)} \tag{3}$$

where we assume that the denominator does not vanish over the parametric domain and where $B_i(u)$ are Beta-spline basis functions. In this approach, each basis function is defined over the entire curve, although it has *local support*. Each basis function is *piecewise*, comprising a sequence of *basis segments*. Such a basis segment is given by a single polynomial whose order is at most the order of the spline curve. For example, the standard cubic Beta-spline curve is of order four, and is a linear combination of cubic basis functions of order four. Details of the various types of Beta-spline basis functions can be found in.[1, 2, 5, 9, 10, 22, 23, 24, 27]

The rational Beta-spline curve defined in equation (3) can be rewritten in a more familiar form as an affine combination of basis functions which are now *rational* basis functions. Rearranging equation (3) yields

$$\mathbf{Q}(u) = \sum_{i=0}^{m} \mathbf{V}_i \left[\frac{w_i \, B_i(u)}{\sum_{r=0}^{m} w_r \, B_r(u)} \right]. \tag{4}$$

Denoting the term in brackets by $R_i(u)$,

$$R_i(u) = \frac{w_i \, B_i(u)}{\sum_{r=0}^{m} w_r \, B_r(u)} \tag{5}$$

and replacing this in equation (4) results in

$$Q(u) = \sum_{i=0}^{m} V_i \, R_i(u). \tag{6}$$

Rewriting the rational Beta-spline curve in the form given by equation (6) reveals a curve formulation that is indistinguishable from the non-rational or *integral* form except that the basis functions are themselves rational.

In the following sections, it will be shown that the rational Beta-spline basis functions satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the corresponding integral basis functions and that the rational Beta-spline curve and tensor product surface are geometrically continuous. These facts will be established as applications of the results for $\bar{\beta}$-continuous functions which were derived in Section 3.

4.2. Rational Beta-spline Basis Functions and Corresponding Integral Basis Functions Are $\bar{\beta}$-continuous for the Same Value of $\bar{\beta}$

As an application of the results on various combinations of functions that are $\bar{\beta}$-continuous (for the same value of $\bar{\beta}$), it will now be demonstrated that the rational Beta-spline basis functions are $\bar{\beta}$-continuous for the same value of $\bar{\beta}$ as the corresponding integral basis functions.

Consider the rational Beta-spline basis functions, $R_i(u)$, $i = 0, 1, \cdots, m$, given in Equation (5). First, consider the numerator. The integral basis functions, $B_i(u)$, $i = 0, 1, \cdots, m$, all satisfy the Beta-constraints for the same value of $\bar{\beta}$, by definition. The numerator must therefore satisfy the Beta-constraints for this same value of $\bar{\beta}$ by Lemma 2 since it is simply a constant times an integral basis function. Second, consider the denominator. This is just a sum of terms of the form in the numerator. By application of Lemma 2, the sum of functions that satisfy the Beta-constraints for the same value of $\bar{\beta}$, also satisfies the Beta-constraints for the same value of $\bar{\beta}$. Hence, both the numerator and denominator satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the original integral basis functions. Finally, since the rational Beta-spline basis functions are simply the quotients of functions that satisfy the Beta-constraints for the same value of $\bar{\beta}$, then by Lemma 2, they too must satisfy the Beta-constraints for the same value of $\bar{\beta}$. Thus, the rational Beta-spline basis functions are $\bar{\beta}$-continuous for the same value of $\bar{\beta}$ as the integral basis functions.

4.3. The Rational Beta-spline Curve and Tensor Product Surface Are Geometrically Continuous

Regarding the rational Beta-spline curve as an affine combination of rational basis functions as given by equation (6) facilitates establishing that the curve is geometrically continuous. Consider each component of the curve. First, all the terms in the sum are constant multiples of functions that satisfy the Beta-constraints for the same value of $\bar{\beta}$ as do the original integral basis functions and thus they satisfy the Beta-constraints for the same value of $\bar{\beta}$. Consequently, by Lemma 2, the sum of these terms satisfies the Beta-constraints for this same value of $\bar{\beta}$. Thus, each component of the rational Beta-spline curve satisfies the Beta-constraints for this same value of $\bar{\beta}$, and hence, the curve itself must be geometrically continuous. A similar argument establishes that the tensor product rational Beta-spline surface is geometrically continuous.

5. Conclusion

A function is said to be n^{th} order $\bar{\beta}$-continuous if and only if it satisfies the Beta-constraints for a fixed value of $\bar{\beta}$. Sums, differences, products, quotients, and scalar multiples of $\bar{\beta}$-continuous scalar-valued functions are shown to also be $\bar{\beta}$-continuous scalar-valued functions for this same value of $\bar{\beta}$. In other words, it is demonstrated that $\bar{\beta}$-continuity is preserved by addition, subtraction, multiplication, division, and scalar multiplication. The proofs appeal only to the chain rule and the definition of geometric continuity in terms of reparametrization. In the case of multiplication or division, recent work[21, 30] has shown that $\bar{\beta}$-continuity is the most general definition of continuity for which this holds.

These results on various combinations of $\overline{\beta}$-continuous scalar-valued functions were applied to show that the rational Beta-spline basis functions are $\overline{\beta}$-continuous for the same value of $\overline{\beta}$ as the corresponding integral basis functions, thus providing a rigorous mathematical foundation for the geometric continuity of the rational Beta-spline curve and tensor product surface of arbitrary order.

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