ALGEBRAIC THEORY OF LINEAR FEEDBACK SYSTEMS WITH FULL AND DECENTRALIZED COMPENSATORS

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Abstract

This work presents an algebraic theory for linear, time-invariant multiinput-multioutput control systems. Due to the algebraic setting, this theory applies to lumped as well as distributed, continuous-time as well as discrete-time systems. The fundamental problem of stability, the class of all stabilizable plants, the class of all stabilizing compensators and all achievable closed-loop input-output maps are solved for control system configurations with full output-feedback or decentralized output-feedback compensators.

The general algebraic setting and the factorization approach are explained in Chapter Two; the contribution of this chapter is in collecting and simplifying the fundamental results used in algebraic system theory and presenting new results in coprime factorizations.

Using coprime factorizations of the plant and the compensator, the stability of system configurations with full compensators are considered in Chapter Three; the first configuration is the standard unity-feedback system in which the plant and the compensator each have one (vector-)input and one (vector-)output and full feedback is allowed from the plant output to the compensator. The second configuration represents the most general interconnection of two subsystems: the plant and the compensator each have two (vector-)inputs and two (vector-)outputs;

full feedback is allowed from one of the plant outputs to one of the compensator inputs.

In Chapter Four, the unity-feedback system is constrained to have a decentralized compensator, resulting in a block-diagonal structure for the compensator; the two-channel decentralized compensation case is considered in detail and the results are extended to m-channel decentralized control systems. The most important contributions of Chapters Three and Four are the parametrization of all stabilizing full and decentralized compensators and achievable input-output maps for each compensation scheme. All compensator design problems aimed at satisfying performance goals (disturbance rejection, asymptotic tracking, robust performance, sensitivity minimization, H^{∞}–norm minimization, constrained optimization problems) rely on this parametrization of all stabilizing compensators.

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List of Symbols

I/O	input-output
MIMO	multiinput-multioutput
a := b	a is defined as b
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Z}$	integer numbers
$\mathbb{C}_+$	complex numbers with nonnegative real part
I_n	nxn identity matrix
det$A$	the determinant of matrix A
H	principal ring
J	group of units of H
I	multiplicative subset of H
G	ring of fractions of H associated with I
G_s	Jacobson radical of G
$m(H)$	the set of matrices with elements in H .
$\mathcal{U}$	a closed subset of $\mathbb{C}_+$, symmetric with respect to the real axis
$\bar{\mathcal{U}}$	$\mathcal{U} \cup \{\infty\}$
$R_{\mathcal{U}}(s)$	ring of proper scalar rational functions which are analytic in $\mathcal{U}$
$\mathbb{R}_p(s)$	ring of proper scalar rational functions with real coefficients
$\mathbb{R}_{sp}(s)$	ring of strictly proper scalar rational functions with real coefficients

r.c. (l.c.)	right-coprime (left-coprime)
r.f.r. (l.f.r.)	right-fraction (left-fraction) representation
r.c.f.r. (l.c.f.r.)	right-coprime-fraction (left-coprime-fraction) representation
b.c. (b.c.f.r.)	bicoprime (bicoprime-fraction representation)
g.c.d.	greatest-common-divisor
l.c.m.	least-common-multiple
t.z.	transmission zero
$S(P, C)$	unity-feedback system
$\Sigma(\hat{P}, \hat{C})$	two-input two-output plant and compensator configuration
$S(P, C_d)$	two-channel decentralized feedback system
$S(P, C_d)_m$	m-channel decentralized feedback system.

Chapter One

Introduction

This work presents a general algebraic theory for linear, time-invariant (l.t-i), multiinput-multioutput (MIMO) control systems. This theory applies to lumped as well as distributed, continuous-time as well as discrete-time systems. The focus here is on three feedback configurations: The standard unity-feedback system $S(P, C)$, the more general configuration $\Sigma(\hat{P}, \hat{C})$ with two (vector-)input two (vector-)output plant and compensator, and the two-channel decentralized feedback configuration $S(P, C_d)$, with extensions to the m-channel decentralized configuration $S(P, C_d)_m$.

A unified, straightforward algebraic theory is developed in this work starting with the following general idea: Suppose that we are given an n_i-input n_o-output l.t-i plant P; then there is an n_o-input n_i-output compensator C that stabilizes P in the unity-feedback configuration $S(P, C)$. In fact, it is well-known that we can start with a right-coprime factorization $N_p D_p^{-1}$ or a left-coprime factorization $\tilde{D}_p^{-1}\tilde{N}_p$ of P and parametrize the class of *all* stabilizing compensators for P in this configuration. Now we may ask:

(i) What if we started with a bicoprime factorization $N_{pr} D^{-1} N_{pl}$ of P ? Can we convert $N_{pr} D^{-1} N_{pl}$ to either $N_p D_p^{-1}$ or $\tilde{D}_p^{-1}\tilde{N}_p$ and thus use the well-known parametrization of the class of all stabilizing compensators?

(ii) What if the n_oxn_i matrix P were a subblock of an $(\eta_i + n_i)$-input $(\eta_o + n_o)$-output plant $\hat{P}$? Can we stabilize $\hat{P}$ by only allowing feedback from the n_o outputs to the n_i inputs of P? What is the class of all $(\eta_o + n_o)$x$(\eta_i + n_i)$ plants $\hat{P}$ that can be stabilized by such partial feedback in the configuration $\Sigma(\hat{P}, \hat{C})$? What is the class of all stabilizing $(\eta_i' + n_o)$-input $(\eta_o' + n_i)$-output compensators $\hat{C}$ in the configuration $\Sigma(\hat{P}, \hat{C})$? How many free design parameters does $\hat{C}$ have? What are the achievable input-output (I/O) maps of $\Sigma(\hat{P}, \hat{C})$? Can we diagonalize the map from the external-input to some output of $\hat{P}$ while preserving the stability of $\Sigma(\hat{P}, \hat{C})$? After achieving stability and diagonalization, do we still have free parameters to satisfy other design objectives?

(iii) What if the output-vector of P is partitioned into n_{o1} local outputs y_1 and n_{o2} local outputs y_2 and the input-vector of P is partitioned into n_{i1} local inputs u_1 and n_{i2} local inputs u_2, and feedback is allowed only from y_1 to u_1 and from y_2 to u_2, resulting in a block-diagonal compensator structure? What is the class of all P that can be stabilized by such decentralized output-feedback? Can we parametrize the class of all stabilizing decentralized compensators C_d ? How do we generalize decentralized stabilization to an m-channel plant P, with (local) outputs $y_1, \ldots, y_m$ and (local) inputs $u_1, \ldots, u_m$?

The set of all stabilizing compensators and achievable performance in various feedback configurations has attracted much attention; the characterization of all possible designs shows exactly what the limitations are on achievable performance. Stabilizing compensators were characterized in [You.1] for the lumped continuous-time and discrete-time cases. Later, an algebraic formulation was given in [Des.1] to include the lumped and distributed continuous-time and discrete-time cases. Using algebraic tools, [Zam.1] considered stable plants, characterized all stabilizing compensators and established bounds on closed-loop performance. These methods were used for design in [Des.2]. Further results in parametrization were given in [Per.1], [Che.1], [Sae.1], [Ohm.1] and [Vid.2]; a general algebraic design procedure, which enables design with non-square plants and controllers and extends the parametrizations of [You.1] and [Per.1], was obtained in [Des.3]. An excellent review of research in this area and related work until 1985 can be found in [Vid.1].

Various feedback configurations have been used to satisfy stability and other performance specifications. In the classical unity-feedback configuration $S(P, C)$ (shown in Figure 3.1 in Chapter Three), the class of all stabilizing compensators is parametrized by one free parameter matrix Q. All closed-loop I/O maps depend on this parameter, and hence, if one performance requirement is met by choosing Q, then there is no more freedom left in the design. So then, the disturbance-to-output map cannot be decoupled from the external-input-to-output map with this scheme. For a single-input single-output plant, a number of different feedback schemes were briefly discussed in [Hor.1]; among them was the two-degrees-of-freedom design. A two-input one-output compensator was proposed in [Åst.1] and later developed in [Per.1, Des.3, Vid.1]. The class of all stabilizing two-input one-output compensators is parametrized by two parameter matrices; hence, using this two-parameter scheme, the disturbance-to-output map is independent of the exogenous-input-to-output map. A two-parameter compensation scheme was also used in [Des.5], where the plant was more general, with a measured-output used in feedback, and an actual output, which is expected to satisfy certain performance criteria.

A much more advanced scheme, which generalizes the unity-feedback system $S(P, C)$ and the two-parameter scheme, is the two-(vector)input two-(vector)output plant and compensator configuration $\Sigma(\hat{P}, \hat{C})$ (see for example [Net.1]). In this case, the class of all stabilizing compensators has four parameter matrices. Each input-output map of $\Sigma(\hat{P}, \hat{C})$ is an affine (or linear) map in one of these parameter matrices.

Decoupling the map from the external-input to the output of an MIMO plant is extremely desirable from an engineering point-of-view since each output of a diagonal system can be manipulated by a single input, which does not affect any of the other outputs. Diagonalization of the I/O map as a performance specification was studied extensively, mostly using state-space techniques [see, for example, Dio.1, Zam.2]. Using a one-parameter compensator C placed in the *feedback-loop*, [Ham.1] gave conditions for a plant P to be decoupled using output-feedback; in the lumped continuous-time case, using this scheme, there is no "proper" compensator that decouples a plant whose inverse has off-diagonal polynomial terms (because with strictly proper plant and proper compensator, the inverse of the resulting diagonal I/O map is $[P(I+CP)^{-1}]^{-1} = (I+CP)P^{-1}$, which approaches P^{-1} as $s \to \infty$). This configuration introduces the constraint that the polynomial part of P^{-1} must be diagonal; this problem does not arise with a two-parameter compensation scheme. In a more general algebraic setting, decoupling of linear time-invariant MIMO systems over unique factorization domains was considered in [Dat.1] and conditions for the existence of a decoupling dynamic or static *state* feedback were established in the case that the system is internally stable and reachable. Later in [Des.4], a two-parameter compensation scheme was used for diagonalization; the plant was assumed to be more general as in [Des.5]; it was shown that diagonalization can be achieved independently of the disturbance-to-output map.

In large scale systems (for example, power systems, computer communication networks, chemical process control systems, transportation networks, socioeconomic systems) it is often desirable or required due to geographic, economic or other practical considerations, to construct the feedback control or decision strategy of a system based on a constrained measurement or information pattern. An important case of constrained controllers is decentralized control in which only local outputs are utilized by local feedback controllers, resulting in a block-diagonal compensator structure. The design of local decoupled controllers that require no information from the other channels is clearly desirable but not all plants can be stabilized in this fashion. It is important to know the constraints on plants which can be stabilized by decentralized feedback as well as the class of all stabilizing block-diagonal compensators. A comparison of this class to the parametrization of all stabilizing *centralized* compensators shows that, even when the plant satisfies the conditions for decentralized stabilizability, decentralized compensators form only a small subset of all possible designs that would achieve stabilization.

This work is organized as follows: Chapter Two collects all algebraic facts and lemmas which will be used in studying control systems. The standard ring definitions (entire ring, principal ring, ideal of a ring, ring of fractions) can be found in many texts in algebra [Bou.1, Coh.1, Jac.1, Lan.1, Mac.1] or in [Vid.1]. In Chapter Three, the unity-feedback configuration $S(P, C)$ and the general configuration $\Sigma(\hat{P}, \hat{C})$ are studied in detail. The class of all plants that can be stabilized by decentralized feedback and the class of all stabilizing decentralized compensators for two-channel and m-channel systems are obtained in Chapter Four. The results of Chapters Three and Four are combined to exhibit the class of all H–stabilizing compensators for the plant $\hat{P}$, where the second output is partitioned into two channels, i.e., the system $\Sigma(\hat{P}, \hat{C})$ is restricted to a decentralized feedback-loop.

The contribution of this work is in its unified approach to different stabilization schemes by using the same tools of analysis and the same factorization techniques collected under Chapter Two. Consequently, it is possible to compare compensator design with a unity-feedback system, a general two-input two-output system, and a decentralized output-feedback scheme. The parametrizations of stabilizing full and decentralized compensators presented here are extremely important in disturbance rejection, asymptotic tracking, robust performance and costrained optimization problems.

Chapter Two

Algebraic Background

2.1. Introduction

The purpose of this chapter is to clearly separate algebraic facts from system properties, to introduce the algebraic framework and to collect relevant definitions, known facts and important lemmas. These will be used repeatedly in Chapters 3 and 4 to study control systems.

If H is a principal ring (also called principal ideal domain, [Coh.1, Mac.1, Jac.1, Lan.1, Vid.1]), and if I is a multiplicative subset of H, then *any* matrix P whose entries are in the ring of fractions $H / I =: G$ of H associated with the subset I can be factorized as $N_p D_p^{-1}$ and as $\tilde{D}_p^{-1} \tilde{N}_p$, where N_p, D_p, $\tilde{N}_p$, $\tilde{D}_p$ all have entries in H; this would *not* be the case if H were *any* ring.

Some well-known rings such as $\mathbb{R}[s]$ (the ring of polynomials in s with real coefficients), $R_u(s)$ (the ring of stable rational functions in s with real coefficients), $\mathbb{Z}$ (the ring of integers), are principal rings.

Factorizations in principal rings are important tools in the algebraic theory of control systems. If the system is represented by a transfer function P whose entries are in $\mathbb{R}_p(s)$ (the ring of proper rational functions in s with real coefficients), then P can be factorized in $\mathbb{R}[s]$ or in $R_u(s)$.

This chapter is organized as follows: The algebraic notation and some important properties of principal rings are presented in Section 2.2; factorizations in H are defined in Section 2.3. Various generalized Bezout identities are presented in Section 2.4; using these Bezout identities, a bicoprime factorization of the form $N_{pr} D^{-1} N_{pl}$ is reduced into a right-factorization $N_p D_p^{-1}$ or a left-factorization $\tilde{D}_p^{-1} \tilde{N}_p$. Solutions (for $(\tilde{D}_c, \tilde{N}_c)$) of the equation $\tilde{D}_c D_p + \tilde{N}_c N_p = A$, where

$N_p D_p^{-1}$ is a right-coprime factorization of a (given) P , are presented in Section 2.5; this is particularly useful in compensator design with the unity-feedback system $S(P, C)$. Matrices partitioned into four sub-blocks are studied in Section 2.6; this will be especially useful in Section 3.3 for the analysis of a general system $\Sigma(\hat{P}, \hat{C})$. An important lemma, which is very useful in decentralized control, is presented in Section 2.7; slightly different forms of this lemma can be found in [And.1].

Although the results of this chapter are completely algebraic, their system-theoretic importance will be demonstrated by their use in the subsequent chapters.

2.2. Algebraic framework

In this section we introduce the algebraic setting; due to its generality, the results we present in the subsequent chapters apply to distributed or lumped, continuous-time or discrete-time control systems.

2.2.1. Notation [Coh.1, Mac.1, Lan.1, Vid.1]:

H is a principal ring (i.e., an entire commutative ring in which every ideal is principal).

$J \subset H$ is the group of units of H (i.e., $x \in J$ implies $x^{-1} \in H$).

$I \subset H$ is a multiplicative subset, $0 \notin I$, $1 \in I$ (i.e., $x \in I, y \in I$ implies $xy \in I$).

$G = H / I := \{ n / d : n \in H, d \in I \}$ is the ring of fractions of H associated with I .

G_s is the Jacobson radical of G; $G_s := \{ x \in G : (1+xy)^{-1} \in G,$ for all $y \in G \}$.

2.2.2. Example (Rational functions in s) :

Let $\mathcal{U} \supset \mathbb{C}_+$ be a closed subset of $\mathbb{C}$, which is symmetric about the real axis, and let $\mathbb{C} \setminus \mathcal{U}$ be nonempty; let $\bar{\mathcal{U}} := \mathcal{U} \cup \{ \infty \}$. The ring of proper scalar rational functions (with real coefficients) which are analytic in $\mathcal{U}$, denoted by $R_{\mathcal{U}}(s)$, is a principal ring. Now let H be $R_{\mathcal{U}}(s)$; for this principal ring, J , I , G , G_s are interpreted as follows:

By definition of J, $f \in J$ implies that f is a proper rational function, which has neither poles nor zeros in $\bar{\mathcal{U}}$. We choose I to be the multiplicative subset of $R_{\mathcal{U}}(s)$ such that $f \in I$ implies that $f(\infty)$ is a nonzero constant in $\mathbb{R}$; equivalently, $I \subset R_{\mathcal{U}}(s)$ is the set of proper, but not *strictly proper*, real rational functions which are analytic in $\mathcal{U}$. The ring of fractions $R_{\mathcal{U}}(s) / I$ is then the ring of *proper* rational functions $\mathbb{R}_p(s)$. The Jacobson radical of the ring $\mathbb{R}_p(s)$ is the set of strictly proper rational functions $\mathbb{R}_{sp}(s)$.

2.2.3. Facts:

(i) The multiplicative subset I is the set of units of G which are in H .

(ii) Let $A \in \mathcal{M}(H)$, $B \in \mathcal{M}(G)$; then

(a) $A^{-1} \in \mathcal{M}(H)$ iff $\det A \in J$ (A is then called H–unimodular);

(If we choose H as the *specific* principal ring $R_u(s)$, then we say R_u–unimodular.)

(b) $B^{-1} \in \mathcal{M}(G)$ iff $\det B \in I$ (B is then called G–unimodular).

(iii) Let $Y \in \mathcal{M}(G_S)$, $X, Z \in \mathcal{M}(G)$; then $XY, YZ \in \mathcal{M}(G_S)$ and $(I+XY)^{-1}$, $(I+YZ)^{-1} \in \mathcal{M}(G)$.

2.2.4. Lemma:

(i) Let $a, b \in H$; then $ab \in J$ if and only if $a \in J$ and $b \in J$.

(ii) Let $c, d \in H$; then $cd \in I$ if and only if $c \in I$ and $d \in I$.

Proof:

(i) Clearly, $a, b \in J$ implies that $ab \in J$ since J is a (multiplicative) group. To show the converse, let $ab =: u$; by assumption, $u^{-1} \in H$. Therefore, $b \in H$ has inverse $(u^{-1}a) \in H$ since $(u^{-1}a)b = 1$, and hence, $b \in J$. Similarly, $a(bu^{-1}) = 1$ implies that $a \in H$ has inverse $(bu^{-1}) \in H$ and hence, $a \in J$.

(ii) Clearly, $c, d \in I$ implies that $cd \in I$ since I is a multiplicative subset. To show the converse, let $cd =: v \in I$; then $v^{-1} \in G$. Therefore, $c \in H$ has inverse $(dv^{-1}) \in G$ and $d \in H$ has inverse $(v^{-1}c) \in G$; hence, c and $d \in I$.

□

2.3. Coprime-fraction representations

We now define right, left, and bicoprime factorizations in H for matrices with elements in G.

2.3.1. Definitions (Coprime factorizations in H):

(i) The pair (N_p, D_p), where $N_p, D_p \in \mathcal{m}(H)$, is called *right-coprime* (r.c.) iff there exist $U_p, V_p \in \mathcal{m}(H)$ such that

$$V_p D_p + U_p N_p = I ; \tag{2.3.1}$$

(ii) the pair (N_p, D_p) is called a *right-fraction representation* (r.f.r.) of $P \in \mathcal{m}(G)$ iff

$$D_p \text{ is square, } \det D_p \in I \text{ and } P = N_p D_p^{-1} ; \tag{2.3.2}$$

(iii) the pair (N_p, D_p) is called a *right-coprime-fraction representation* (r.c.f.r.) of $P \in \mathcal{m}(G)$ iff (N_p, D_p) is an r.f.r. of P and (N_p, D_p) is r.c.

(iv) The pair $(\tilde{D}_p, \tilde{N}_p)$, where $\tilde{D}_p, \tilde{N}_p \in \mathcal{m}(H)$, is called *left-coprime* (l.c.) iff there exist $\tilde{U}_p, \tilde{V}_p \in \mathcal{m}(H)$ such that

$$\tilde{N}_p \tilde{U}_p + \tilde{D}_p \tilde{V}_p = I ; \tag{2.3.3}$$

(v) the pair $(\tilde{D}_p, \tilde{N}_p)$ is called a *left-fraction representation* (l.f.r.) of $P \in \mathcal{m}(G)$ iff

$$\tilde{D}_p \text{ is square, } \det\tilde{D}_p \in I \text{ and } P = \tilde{D}_p^{-1} \tilde{N}_p ; \tag{2.3.4}$$

(vi) the pair $(\tilde{D}_p, \tilde{N}_p)$ is called a *left-coprime-fraction representation* (l.c.f.r.) of $P \in \mathcal{m}(G)$ iff $(\tilde{D}_p, \tilde{N}_p)$ is an l.f.r. of P and $(\tilde{D}_p, \tilde{N}_p)$ is l.c.

(vii) The triple (N_{pr}, D, N_{pl}), where $N_{pr}, D, N_{pl} \in \mathcal{m}(H)$, is called a *bicoprime-fraction representation* (b.c.f.r.) of $P \in \mathcal{m}(G)$ iff the pair (N_{pr}, D) is *right-coprime*, the pair (D, N_{pl}) is *left-coprime*, $\det D \in I$ and $P = N_{pr} D^{-1} N_{pl}$.

Note that $P \in \mathcal{M}(G)$ is sometimes given as $P = N_{pr} D^{-1} N_{pl} + S_p$, where $S_p \in \mathcal{M}(H)$ and (N_{pr}, D, N_{pl}) is a bicoprime (b.c.) triple. In this case, the b.c.f.r. is given by (N_{pr}, D, N_{pl}, S_p) [Vid.1].

□

Every $P \in \mathcal{M}(G)$ has an r.c.f.r. (N_p, D_p), an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$, and a b.c.f.r. (N_{pr}, D, N_{pl}) in H because H is a principal ring [Vid.1].

2.3.2. Lemma:

Let $\begin{bmatrix} Y_p \\ \cdots \\ X_p \end{bmatrix} = E \begin{bmatrix} D_p \\ \cdots \\ N_p \end{bmatrix}$, and let $\begin{bmatrix} \tilde{X}_p & \vdots & \tilde{Y}_p \end{bmatrix} = \begin{bmatrix} \tilde{N}_p & \vdots & \tilde{D}_p \end{bmatrix} F$, where E, $F \in \mathcal{M}(H)$ are H–unimodular; then

(i) the pair (N_p, D_p) is r.c. if and only if the pair (X_p, Y_p) is r.c.,

(ii) the pair $(\tilde{D}_p, \tilde{N}_p)$ is l.c. if and only if the pair $(\tilde{Y}_p, \tilde{X}_p)$ is l.c.

Proof:

(i) From Definition 2.3.1, (N_p, D_p) is r.c. iff there exist U_p, $V_p \in \mathcal{M}(H)$ such that

$$\begin{bmatrix} V_p & \vdots & U_p \end{bmatrix} \begin{bmatrix} D_p \\ \cdots \\ N_p \end{bmatrix} = I = \begin{bmatrix} V_p & \vdots & U_p \end{bmatrix} E^{-1} \begin{bmatrix} Y_p \\ \cdots \\ X_p \end{bmatrix}$$; equivalently, (X_p, Y_p) is r.c.

(ii) Similar to proof of part (i).

□

2.3.3. Lemma:

Let (N_p, D_p) be an r.c.f.r. and let $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of $P \in \mathcal{M}(G)$; then

(i) (X_p, Y_p) is also an r.f.r. (r.c.f.r.) of P if and only if $(X_p, Y_p) = (N_p R, D_p R)$ for some G–unimodular (H–unimodular, respectively) $R \in \mathcal{M}(H)$,

(ii) $(\tilde{Y}_p, \tilde{X}_p)$ is also an l.f.r. (l.c.f.r.) of P if and only if $(\tilde{Y}_p, \tilde{X}_p) = (L\tilde{N}_p, L\tilde{D}_p)$ for some G–unimodular (H–unimodular, respectively) $L \in \mathcal{M}(H)$.

Proof:

(i) (<=) If $(X_p, Y_p) = (N_pR, D_pR)$ for some G–unimodular (or H–unimodular) $R \in \mathcal{M}(H)$, then $\det Y_p = \det D_p \det R \in I$; hence $N_pD_p^{-1} = X_pY_p^{-1} = P$ and (X_p, Y_p) is an r.f.r. of P. Now if $R \in \mathcal{M}(H)$ is actually H–unimodular, then by the Bezout identity (2.3.1),

$$R^{-1}V_pD_pR + R^{-1}U_pN_pR = R^{-1}V_pY_p + R^{-1}U_pX_p = I ,$$

and hence, (X_p, Y_p) is also r.c.

(=>) Let (X_p, Y_p) be an r.f.r. of P ; then $\det Y \in I$ and $N_pD_p^{-1} = X_pY_p^{-1}$. From the Bezout identity (2.3.1), since $\det D_p \in I$, we obtain

$$V_pY_p + U_pX_p = D_p^{-1}Y_p =: R \in \mathcal{M}(H), \tag{2.3.5}$$

where $R^{-1} = Y_p^{-1}D_p \in G$; hence $R \in \mathcal{M}(H)$ is G–unimodular. Clearly, $Y_p = D_pR$ and $X_p = N_pD_p^{-1}Y_p = N_pR$.

If the pair (X_p, Y_p) is actually r.c., then there are matrices V_y , $U_x \in \mathcal{M}(H)$, such that

$$V_yY_p + U_xX_p = I ;$$

and hence,

$$V_yD_p + U_xN_p = Y_p^{-1}D_p = R^{-1} \in \mathcal{M}(H). \tag{2.3.6}$$

From equations (2.3.5)-(2.3.6), $R \in \mathcal{M}(H)$ is H–unimodular.

(ii) Similar to proof of part (i).

□

2.3.4. Generalized Bezout Identity for (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$:

Let (N_p, D_p) be an r.c. pair and let $(\tilde{D}_p, \tilde{N}_p)$ be an l.c. pair, and let $\tilde{N}_pD_p = \tilde{D}_pN_p$, where $N_p \in H^{n_o x n_i}$, $D_p \in H^{n_i x n_i}$, $\tilde{D}_p \in H^{n_o x n_o}$, $\tilde{N}_p \in H^{n_o x n_i}$; then there are matrices V_p , U_p , $\tilde{U}_p$, $\tilde{V}_p \in \mathcal{M}(H)$ such that

$$\begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} . \tag{2.3.7}$$

Equation (2.3.7) is called a *generalized Bezout identity*; note that $\det D_p$ and $\det\tilde{D}_p$ need not be in I.

2.3.5. Definition (Doubly-coprime fraction representation):

(i) If the generalized Bezout identity (2.3.7) holds, then $((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ is called a *doubly-coprime* pair.

(ii) If $P = N_p D_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$, then $((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ is called a *doubly-coprime-fraction* representation of P.

2.3.6. Generalized Bezout identities for (N_{pr}, D, N_{pl}):

Let (N_{pr}, D, N_{pl}) be a b.c. triple, where $N_{pr} \in H^{n_o \times n}$, $D \in H^{n \times n}$, $N_{pl} \in H^{n \times n_i}$; then we have two generalized Bezout identities:

(i) For the r.c. pair (N_{pr}, D), there are matrices $V_{pr}, U_{pr}, \tilde{X}, \tilde{Y}, \tilde{U}, \tilde{V} \in \mathfrak{m}(H)$ such that

$$\begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N_{pr} & \tilde{V} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix}; \tag{2.3.8}$$

equation (2.3.8) is of the form

$$M_r M_r^{-1} = I_{n+n_o}. \tag{2.3.9}$$

(ii) For the l.c. pair (D, N_{pl}) there are matrices $V_{pl}, U_{pl}, X, Y, U, V \in \mathfrak{m}(H)$ such that

$$\begin{bmatrix} D & -N_{pl} \\ U & V \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix}; \tag{2.3.10}$$

equation (2.3.10) is of the form

$$M_l M_l^{-1} = I_{n+n_i}. \tag{2.3.11}$$

Note that $\det D$ need not be in I in equations (2.3.8) and (2.3.10).

2.3.7. Lemma:

Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of $P \in \mathcal{M}(G)$; then $P \in \mathcal{M}(H)$ if and only if $D^{-1} \in \mathcal{M}(H)$.

Proof:

If $D^{-1} \in \mathcal{M}(H)$ then clearly $P = N_{pr}D^{-1}N_{pl} \in \mathcal{M}(H)$. To show the converse, let $N_{pr}D^{-1}N_{pl} \in \mathcal{M}(H)$. From equation (2.3.10), $N_{pr}D^{-1}N_{pl}U_{pl} = N_{pr}D^{-1}(I_n - DV_{pl}) = N_{pr}D^{-1} - N_{pr}V_{pl} \in \mathcal{M}(H)$; equivalently, $N_{pr}D^{-1} \in \mathcal{M}(H)$. Furthermore, by equation (2.3.8), $U_{pr}N_{pr}D^{-1} = (I_n - V_{pr}D)D^{-1} = D^{-1} - V_{pr} \in \mathcal{M}(H)$; equivalently, $D^{-1} \in \mathcal{M}(H)$.

□

2.3.8. Comments:

(i) Let P be given as $N_{pr}D^{-1}N_{pl} + S_p$, where (N_{pr}, D, N_{pl}) is a b.c. triple and $S_p \in \mathcal{M}(H)$; then $N_{pr}D^{-1}N_{pl} + S_p \in \mathcal{M}(H)$ if and only if $N_{pr}D^{-1}N_{pl} \in \mathcal{M}(H)$ and hence, by Lemma 2.3.7, $P \in \mathcal{M}(H)$ if and only if $D^{-1} \in \mathcal{M}(H)$.

(ii) Let (N_p, D_p) be an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of $P \in \mathcal{M}(G)$; then by Lemma 2.3.7, $P \in \mathcal{M}(H)$ if and only if $D_p^{-1} \in \mathcal{M}(H)$ and equivalently, $\tilde{D}_p^{-1} \in \mathcal{M}(H)$. This follows from reducing a b.c.f.r. to an r.c.f.r. if $N_{pl} = I$ and $S_p = 0$ or to an l.c.f.r. if $N_{pr} = I$ and $S_p = 0$.

2.4. Right- or left-coprime factorizations from bicoprime factorizations

Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of $P \in \mathcal{M}(G)$. We obtain an r.c.f.r. (N_p, D_p) and an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ for P from (N_{pr}, D, N_{pl}) in Proposition 2.4.1 below. In Example 2.4.3, we apply Proposition 2.4.1 to the state-space representation of a matrix P with rational function entries, and show that the result in [Net.2] is a special case of our general theory.

2.4.1. Proposition:

Let $P \in \mathcal{M}(G)$. Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P; hence, equations (2.3.8)-(2.3.10) hold. Under these conditions,

$$(N_p, D_p) := (N_{pr}X, Y) \text{ is an r.c.f.r. of } P, \tag{2.4.1}$$

$$(\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \tilde{X} N_{pl}) \text{ is an l.c.f.r. of } P, \tag{2.4.2}$$

where $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{M}(H)$ are defined in equations (2.3.8)-(2.3.10).

2.4.2. Comments:

(i) Using equations (2.3.8)-(2.3.10) we obtain a generalized Bezout identity for the doubly-coprime pair ($(N_{pr}X, Y)$, $(\tilde{Y}, \tilde{X} N_{pl})$):

$$\begin{bmatrix} V + UV_{pr}N_{pl} & UU_{pr} \\ -\tilde{X} N_{pl} & \tilde{Y} \end{bmatrix} \begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X & \tilde{V} + N_{pr}V_{pl}\tilde{U} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.4.3}$$

Note the similarity between equations (2.3.7) and (2.4.3). Equation (2.4.3) is of the form

$$\hat{M}\hat{M}^{-1} = I_{n_i+n_o}. \tag{2.4.4}$$

(ii) If, instead of $N_{pr}D^{-1}N_{pl}$, the plant is given by $P = N_{pr}D^{-1}N_{pl} + S_p$, where $S_p \in \mathcal{M}(H)$, then an r.c.f.r. and an l.c.f.r. are given by:

$$(N_p, D_p) := (N_{pr}X + S_pY, \ Y), \tag{2.4.5}$$

$$(\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \ \tilde{X} N_{pl} + \tilde{Y} S_p), \tag{2.4.6}$$

and the generalized Bezout identity (2.4.3) is replaced by:

$$\begin{bmatrix} V+UV_{pr}N_{pl}-UU_{pr}S_p & UU_{pr} \\ -\tilde{X}N_{pl}-\tilde{Y}S_p & \tilde{Y} \end{bmatrix}\begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X+S_pY & \tilde{V}+N_{pr}V_{pl}\tilde{U}-S_pU_{pl}\tilde{U} \end{bmatrix}=\begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.4.7}$$

□

Proof of Proposition 2.4.1:

By assumption, $P = N_{pr}D^{-1}N_{pl}$, and equations (2.3.8)-(2.3.10) hold. Clearly $N_{pr}X$, Y , $\tilde{Y}$, $\tilde{X}N_{pl} \in \mathcal{m}(H)$. We must show that $(N_{pr}X , Y)$ is an r.c. pair with $\det Y \in \mathcal{I}$ and that $(\tilde{Y} , \tilde{X}N_{pl})$ is an l.c. pair with $\det\tilde{Y} \in \mathcal{I}$:

By equation (2.4.3), $(N_{pr}X , Y)$ is an r.c. pair and $(\tilde{Y} , \tilde{X}N_{pl})$ is an l.c. pair; more specifically, if $(N_{pr}X, Y) =: (N_p, D_p)$ and $(\tilde{Y}, \tilde{X}N_{pl}) =: (\tilde{D}_p, \tilde{N}_p)$, then

$$V_pD_p + U_pN_p = I_{n_i}, \quad \tilde{N}_p\tilde{U}_p + \tilde{D}_p\tilde{V}_p = I_{n_o}, \tag{2.4.8}$$

where $V_p := V + UV_{pr}N_{pl}$, $U_p := UU_{pr}$, $\tilde{U}_p := U_{pl}\tilde{U}$, $\tilde{V}_p := \tilde{V} + N_{pr}V_{pl}\tilde{U}$. (2.4.9)

Now from equations (2.3.8)-(2.3.10),

$$\det D = \det(\begin{bmatrix} D & 0 \\ 0 & I_{n_o} \end{bmatrix} M_rM_r^{-1}) = \det(\begin{bmatrix} I_n - \tilde{U}\tilde{X} & \tilde{U} \\ -\tilde{X} & I_{n_o} \end{bmatrix}\begin{bmatrix} I_n & 0 \\ 0 & \tilde{Y} \end{bmatrix} M_r^{-1}) = \det\tilde{Y}\,\det M_r^{-1}; \tag{2.4.10}$$

$$\det D = \det(M_lM_l^{-1}\begin{bmatrix} D & 0 \\ 0 & I_{n_i} \end{bmatrix}) = \det(M_l\begin{bmatrix} I_n & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} I_n - XU & X \\ -U & I_{n_i} \end{bmatrix}) = \det M_l \det Y . \tag{2.4.11}$$

Since M_r , M_l are H–unimodular by equations (2.3.8)-(2.3.11), and since $\det D \in \mathcal{I}$ by assumption, equations (2.4.10)-(2.4.11) imply that

$$\det Y = \det M_l^{-1}\det D \in \mathcal{I}, \tag{2.4.12}$$

$$\det\tilde{Y} = \det M_r \det D \in \mathcal{I}. \tag{2.4.13}$$

Now by equation (2.3.10), $N_{pl}Y = DX$ and hence,

$$PY = N_pD_p^{-1}Y = N_{pr}X . \tag{2.4.14}$$

Similarly, by equation (2.3.8), $\tilde{Y}N_{pr} = \tilde{X}D$ and hence,

$$\tilde{Y}P = \tilde{Y}N_pD_p^{-1} = \tilde{X}N_{pl} . \qquad (2.4.15)$$

By equations (2.4.12)-(2.4.13), $Y^{-1} \in \mathcal{m}(G)$ and $\tilde{Y}^{-1} \in \mathcal{m}(G)$; therefore, equations (2.4.14)-(2.4.15) imply:

$$P = N_{pr}XY^{-1} = \tilde{Y}^{-1}\tilde{X}N_{pl} , \qquad (2.4.16)$$

where $(N_{pr}X, Y)$ is an r.c. pair and $(\tilde{Y}, \tilde{X}N_{pl})$ is an l.c. pair.

□

2.4.3. Example:

Let H be $R_u(s)$ as in Example 2.2.2. Let $P \in \mathbb{R}_p(s)^{n_o x n_i}$ be represented by its state-space representation

$$\dot{x} = Ax + Bu ,$$
$$y = Cx ,$$

where (C, A, B) is $\bar{\mathcal{U}}$–stabilizable and $\bar{\mathcal{U}}$–detectable. Then $P = (s+a)^{-1}C[(s+a)^{-1}(sI-A)]^{-1}B$, where $-a \in \mathbb{C}\backslash\bar{\mathcal{U}}$. The pair $((s+a)^{-1}C, (s+a)^{-1}(sI-A))$ is r.c. in $R_u(s)$, the pair $((s+a)^{-1}(sI-A), B)$ is l.c. in $R_u(s)$, and $\det[(s+a)^{-1}(sI-A)] \in \mathcal{I}$. Therefore, (N_{pr}, D, N_{pl}) = $((s+a)^{-1}C, (s+a)^{-1}(sI-A), B)$ is a b.c.f.r. of P . Choose $K \in \mathbb{R}^{n_i x n}$ and $F \in \mathbb{R}^{n x n_o}$ such that $(A-BK)$ and $(A-FC)$ have all eigenvalues in $\mathbb{C}\backslash\bar{\mathcal{U}}$. Let $G_K := (sI_n - A + BK)^{-1}$ and let $G_F := (sI_n - A + FC)^{-1}$; then G_K , $G_F \in \mathcal{m}(R_u(s)) \cap \mathcal{m}(\mathbb{R}_{sp}(s))$ and hence, $(s+a)(sI_n - A + BK)^{-1} = (s+a)G_K \in \mathcal{m}(R_u(s))$ and $(s+a)(sI_n - A + FC)^{-1} = (s+a)G_F \in \mathcal{m}(R_u(s))$. For this special b.c.f.r., equations (2.3.8) and (2.3.10) become:

$$\begin{bmatrix} (s+a)G_F & (s+a)G_FF \\ -CG_F & I_{n_o} - CG_FF \end{bmatrix} \begin{bmatrix} (s+a)^{-1}(sI_n - A) & -F \\ (s+a)^{-1}C & I_{n_o} \end{bmatrix} = I_{n+n_o} ; \qquad (2.4.17)$$

$$\begin{bmatrix} (s+a)^{-1}(sI_n - A) & -B \\ (s+a)^{-1}K & I_{n_i} \end{bmatrix} \begin{bmatrix} (s+a)G_K & (s+a)G_KB \\ -KG_K & I_{n_i} - KG_KB \end{bmatrix} = I_{n+n_i} . \qquad (2.4.18)$$

Matching the entries of equations (2.4.17) and (2.4.18) with those of (2.4.8) and (2.4.10), respectively, from equation (2.4.3), we obtain a generalized Bezout identity for this special case:

$$\begin{bmatrix} I_{n_i} + KG_F B & KG_F F \\ -CG_F B & I_{n_o} - CG_F F \end{bmatrix} \begin{bmatrix} I_{n_i} - KG_K B & -KG_K F \\ CG_K B & I_{n_o} + CG_K F \end{bmatrix} = I_{n_i+n_o} . \quad (2.4.19)$$

Comparing equations (2.4.3) and (2.4.19), $(CG_K B, (I_{n_i} - KG_K B))$ is an r.c. pair and $((I_{n_o} - CG_F F), CG_F B)$ is an l.c. pair. Note that equation (2.4.19) gives the same coprime factorizations and the Bezout identity entries obtained in [Net.2, equations (1)-(4)].

□

Let a, $b \in H$; we say that a *is equivalent to* b (denoted by $a \approx b$) iff there exists $u \in J$ such that $a = bu$. Clearly, $a \approx 1$ iff $a \in J$. "$\approx$" is an equivalence relation on H.

In Corollary 2.4.4 below, we use the generalized Bezout identities (2.3.7), (2.3.8), (2.3.10), and Proposition 2.4.1 to show that $\det D_p \approx \det \tilde{D}_p \approx \det D$; (thus, if any one of $\det D_p$, $\det \tilde{D}_p$, $\det D$ is in I, then the other two are also in I). We use the following Bezout identity: If $Q \in \mathcal{M}(H)$ is an arbitrary matrix that has elements in H, the generalized Bezout identity (2.3.7) can be rewritten as:

$$\begin{bmatrix} V_p - Q\tilde{N}_p & U_p + Q\tilde{D}_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde{U}_p - D_p Q \\ N_p & \tilde{V}_p - N_p Q \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}; \quad (2.4.20)$$

equation (2.4.20) is of the form

$$MM^{-1} = I_{n_i+n_o} . \quad (2.4.21)$$

2.4.4. Corollary:

Let (N_p, D_p) be an r.c. pair, $(\tilde{D}_p, \tilde{N}_p)$ be an l.c. pair and (N_{pr}, D, N_{pl}) be a b.c. triple, where N_p, D_p, $\tilde{D}_p$, $\tilde{N}_p$, N_{pr}, D, N_{pl} $\in \mathcal{M}(H)$. Let $\tilde{N}_p D_p = \tilde{D}_p N_p$ and let $(N_{pr}XR, YR) = (N_p, D_p)$, where R is some H–unimodular matrix, with X, Y as in equation (2.3.10); then for all $Q \in \mathcal{M}(H)$,

$$\det D_p \det M = \det \tilde{D}_p , \tag{2.4.22}$$

$$\det(\tilde{V}_p - N_p Q)\det M = \det(V_p - Q\tilde{N}_p), \tag{2.4.23}$$

where M is given by equations (2.4.20)-(2.4.21); hence,

$$\det[\ (V_p - Q\tilde{N}_p)D_p\ ] = \det[\ \tilde{D}_p(\tilde{V}_p - N_p Q)\ ] . \tag{2.4.24}$$

Furthermore,

$$\det D_p \approx \det \tilde{D}_p \approx \det D . \tag{2.4.25}$$

□

Note that $\det D_p$ and $\det \tilde{D}_p$ are *not* assumed to be in I in Corollary 2.4.4. Equation (2.4.25) was proved in [Vid.1] by assuming that $\det D_p \in I$, $\det \tilde{D}_p \in I$, $\det D \in I$.

Proof:

From equations (2.4.20)-(2.4.21) we obtain

$$\begin{bmatrix} D_p & 0 \\ 0 & I_{n_o} \end{bmatrix} M = \begin{bmatrix} I_{n_i} - (\tilde{U}_p + D_p Q)\tilde{N}_p & -(\tilde{U}_p + D_p Q) \\ \tilde{N}_p & I_{n_o} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ 0 & \tilde{D}_p \end{bmatrix}, \tag{2.4.26}$$

and

$$\begin{bmatrix} I_{n_i} & 0 \\ 0 & (\tilde{V}_p - N_p Q) \end{bmatrix} M = \begin{bmatrix} I_{n_i} & (U_p + Q\tilde{D}_p) \\ -N_p & I_{n_o} - N_p(U_p + Q\tilde{D}_p) \end{bmatrix} \begin{bmatrix} V_p - Q\tilde{N}_p & 0 \\ 0 & I_{n_o} \end{bmatrix} . \tag{2.4.27}$$

Equations (2.4.22) and (2.4.23) follow by taking determinants of both sides of equations (2.4.26) and (2.4.27), respectively. Now multiplying both sides of equation (2.4.23) by $\det D_p$, and using equation (2.4.22) we obtain

$$\det(\tilde{V}_p - N_p Q)\det M \det D_p = \det(V_p - Q\tilde{N}_p)\det D_p = \det(\tilde{V}_p - N_p Q)\det\tilde{D}_p \ ; \quad (2.4.28)$$

hence equation (2.4.24) follows since $\det(V_p - Q\tilde{N}_p)\det D_p = \det[\,(V_p - Q\tilde{N}_p)D_p\,]$ and $\det\tilde{D}_p \det(\tilde{V}_p - N_p Q) = \det[\,\tilde{D}_p(\tilde{V}_p - N_p Q)\,]$. (Note that H is a commutative ring.)

By equation (2.4.22), since $\det M \in J$, clearly

$$\det D_p \approx \det\tilde{D}_p \ . \quad (2.4.29)$$

Now by Proposition 2.4.1, the b.c. triple (N_{pr}, D, N_{pl}) reduces to an r.c. pair $(N_{pr}X, Y)$, or an l.c. pair $(\tilde{Y}, \tilde{X} N_{pl})$. By assumption, $(N_p, D_p) = (N_{pr}XR, YR)$ for some H–unimodular $R \in \mathcal{M}(H)$; therefore, by equation (2.4.12),

$$\det D_p = \det Y \det R = \det M_l^{-1} \det D \det R \ ; \quad (2.4.30)$$

since $\det M_l \in J$ and $\det R \in J$, equation (2.4.30) implies that

$$\det D_p \approx \det D \ . \quad (2.4.31)$$

Finally, equation (2.4.25) follows from equations (2.4.31) and (2.4.29).

□

2.4.5. Comment:

Let H be the ring $R_{\mathcal{U}}(s)$ as in Example 2.2.2. Let P be given by $N_p D_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p = N_{pr} D^{-1} N_{pl}$; then the $\bar{\mathcal{U}}$–poles of P are the $\bar{\mathcal{U}}$–zeros of $\det D_p$ (and equivalently, of $\det\tilde{D}_p$ and of $\det D$). We denote the $\bar{\mathcal{U}}$–zeros of $\det D_p$ by

$$\mathcal{Z}[\,\det D_p\,] := \{\, s_o \in \bar{\mathcal{U}} : \det D_p(s_o) = 0 \,\} \ ; \quad (2.4.32)$$

it follows from Corollary 2.4.4, equation (2.4.25), that

$$\mathcal{Z}[\,\det D_p\,] = \mathcal{Z}[\,\det\tilde{D}_p\,] = \mathcal{Z}[\,\det D\,] \ . \quad (2.4.33)$$

□

2.5. Implications of Bezout identities

If $P \in \mathcal{M}(G_S)$, then by Fact 2.2.3.(iii), $N_p = PD_p \in \mathcal{M}(G_S)$; similarly, $\tilde{N}_p = \tilde{D}_p P \in \mathcal{M}(G_S)$.

In Lemma 2.5.1 below, we show that if $N_p \in \mathcal{M}(G_S)$, and $((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ is doubly-coprime pair satisfying the generalized Bezout identity (2.3.7), then D_p^{-1}, $\tilde{D}_p^{-1}$, $(V_p - Q\tilde{N}_p)^{-1}$, $(\tilde{V}_p - N_p Q)^{-1} \in \mathcal{M}(G)$. Consequently, $\tilde{N}_p = \tilde{D}_p N_p D_p^{-1} \in \mathcal{M}(G_S)$ as well.

2.5.1. Lemma:

Let $N_p \in \mathcal{M}(G_S)$; let $((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ be a doubly-coprime pair satisfying the generalized Bezout identity (2.3.7). Under these conditions,

$$\det D_p \in I \quad \text{and} \quad \det \tilde{D}_p \in I ; \tag{2.5.1}$$

and, for all $Q \in H$,

$$\det(V_p - Q\tilde{N}_p) \in I \quad \text{and} \quad \det(\tilde{V}_p - N_p Q) \in I . \tag{2.5.2}$$

Proof:

By Fact 2.2.3.(iii), and by assumption, for any $Q \in \mathcal{M}(H)$, $(U_p + Q\tilde{D}_p)N_p \in \mathcal{M}(G_S)$ and consequently, $(I_{n_i} - (U_p + Q\tilde{D}_p)N_p)^{-1} \in \mathcal{M}(G)$; hence by equation (2.4.20),

$$\det[\,(V_p - Q\tilde{N}_p)D_p\,] = \det[\,I_{n_i} - (U_p + Q\tilde{D}_p)N_p\,] \in I . \tag{2.5.3}$$

By Lemma 2.2.4.(ii), equation (2.5.3) holds if and only if $\det D_p \in I$ and $\det(\tilde{V}_p - Q\tilde{N}_p) \in I$; hence by Corollary 2.4.4, (2.5.1) follows from equation (2.4.22) and (2.5.2) follows from equation (2.4.23), where $\det M \in J$.

□

2.5.2. Lemma:

Let (N_p, D_p) be an r.c. pair and $(\tilde{D}_p, \tilde{N}_p)$ be an l.c. pair satisfying the generalized Bezout identity (2.3.7). Consider the equations

$$\tilde{D}_c D_p + \tilde{N}_c N_p = A \ , \tag{2.5.4}$$

and

$$\tilde{N}_p N_c + \tilde{D}_p D_c = B \ , \tag{2.5.5}$$

where $A \in H^{n_i \times n_i}$, $B \in H^{n_o \times n_o}$. Under these conditions

(i) $(\tilde{D}_c, \tilde{N}_c)$, with $\tilde{D}_c, \tilde{N}_c \in \mathcal{m}(H)$, is a solution of equation (2.5.4) if and only if

$$\left[\tilde{D}_c \ \vdots \ \tilde{N}_c \right] = \left[A \ \vdots \ Q \right] \begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} , \tag{2.5.6}$$

for some $Q \in \mathcal{m}(H)$.

(ii) (N_c, D_c), with $N_c, D_c \in \mathcal{m}(H)$, is a solution of equation (2.5.5) if and only if

$$\begin{bmatrix} -N_c \\ D_c \end{bmatrix} = \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} \begin{bmatrix} -Q \\ B \end{bmatrix} , \tag{2.5.7}$$

for some $Q \in \mathcal{m}(H)$.

Proof:

(i) (<=) Suppose that equation (2.5.6) holds; then by equation (2.3.7), $\tilde{D}_c D_p + \tilde{N}_c N_p =$
$\left[\tilde{D}_c \ \vdots \ \tilde{N}_c \right] \begin{bmatrix} D_p \\ N_p \end{bmatrix} = \left[A \ \vdots \ Q \right] \begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p \\ N_p \end{bmatrix} = \left[A \ \vdots \ Q \right] \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix} = A$ and hence, equation (2.5.4) holds.

(=>) By assumption, equation (2.5.4) holds; hence

$$\left[\tilde{D}_c \ \vdots \ \tilde{N}_c \right] \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} = \left[A \ \vdots \ Q \right] , \tag{2.5.8}$$

where $Q := -\tilde{D}_c \tilde{U}_p + \tilde{N}_c \tilde{V}_p \in \mathcal{m}(H)$. Post-multiplying both sides of equation (2.5.8) by the H-unimodular matrix $\begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix}$ and using equation (2.3.7), we obtain the solution given by equation (2.5.6).

(ii) Similar to part (i), and again follows from the Bezout identity (2.3.7).

□

2.5.3. Comments:

(i) In equations (2.5.4) and (2.5.5), if $A = I_{ni}$ and $B = I_{no}$, then (2.5.4) is a left-Bezout identity for the l.c. pair $(\tilde{D}_c, \tilde{N}_c)$ and (2.5.5) is a right-Bezout identity for the r.c. pair (N_c, D_c). Let (N_p, D_p) be an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of $P \in \mathcal{M}(G)$. If $(\tilde{D}_c, \tilde{N}_c)$ is an l.c.f.r. and (N_c, D_c) is an r.c.f.r. of some $C \in \mathcal{M}(G)$, then equations (2.5.4), (2.5.5), with equation (2.3.7), imply that

$$\begin{bmatrix} \tilde{D}_c & \tilde{N}_c \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -N_c \\ N_p & D_c \end{bmatrix} = \begin{bmatrix} I_{ni} & 0 \\ 0 & I_{no} \end{bmatrix} . \tag{2.5.9}$$

Note the similarity between equations (2.4.20) and (2.5.9). By Lemma 2.5.2, $(\tilde{D}_c, \tilde{N}_c) = ((V_p - Q\tilde{N}_p), (U_p + Q\tilde{D}_p))$, $(N_c, D_c) = ((\tilde{U}_p + D_p Q), (\tilde{V}_p - N_p Q))$, for some $Q \in \mathcal{M}(H)$.

(ii) If $P \in \mathcal{M}(H)$, then we can choose an r.c.f.r. for P as (P, I_{ni}) and an l.c.f.r. for P as (I_{no}, P). With $P = N_p = \tilde{N}_p$, $D_p = I_{ni}$, $\tilde{D}_p = I_{no}$, in equation (2.3.7) we can choose $V_p = I_{ni}$, $\tilde{V}_p = I_{no}$, $U_p = \tilde{U}_p = 0$; hence, for some $Q \in \mathcal{M}(H)$, $(\tilde{D}_c, \tilde{N}_c) = ((I_{ni} - QP), Q)$ is an l.c.f.r. and $(N_c, D_c) = (Q, (I_{no} - PQ))$ is an r.c.f.r. of $C \in \mathcal{M}(G)$.

(iii) If $P \in \mathcal{M}(G_S)$, then by Lemma 2.5.1,

$$\det(V_p - Q\tilde{N}_p) \approx \det(\tilde{V}_p - N_p Q) \in \mathcal{I} , \tag{2.5.10}$$

for all $Q \in \mathcal{M}(H)$.

Let H be $R_{\mathcal{U}}(s)$ as in Example 2.2.2; then $\det(V_p - Q\tilde{N}_p) \not\equiv 0$ for *almost all* $Q \in \mathcal{M}(R_{\mathcal{U}}(s))$ [Vid.1]. Now we find a $Q \in \mathcal{M}(R_{\mathcal{U}}(s))$ such that equation (2.5.10) holds: If $\det D_p \in \mathcal{J}$ (equivalently, $D_p^{-1} \in \mathcal{M}(R_{\mathcal{U}}(s))$ and $\tilde{D}_p^{-1} \in \mathcal{M}(R_{\mathcal{U}}(s))$), then $V_p = D_p^{-1}$ and $\tilde{V}_p = \tilde{D}_p^{-1}$ satisfies the generalized Bezout identity (2.3.7); clearly, V_p and $\tilde{V}_p$ are $R_{\mathcal{U}}$–unimodular and hence, equation (2.5.10) holds with $Q = 0$. Without loss of generality, we assume that $\det D_p \notin \mathcal{J}$ in our search for Q below.

Choose $h \in R_u(s)$ such that

$$1 + h\,(\det D_p) \in \mathbb{R}_{sp}(s)\,, \tag{2.5.11}$$

(note that one choice for h is $-\det D_p^{-1}(\infty)$). Let

$$Q := h\,(\det D_p)(D_p^{-1})\tilde{U}_p \in \mathcal{M}(R_u(s))\,; \tag{2.5.12}$$

note that $(\det D_p)D_p^{-1} \in \mathcal{M}(R_u(s))$. By the generalized Bezout identity (2.3.7), $D_p(V_p - Q\tilde{N}_p) = I_{n_i} - (1 + h\,\det D_p)\tilde{U}_p\tilde{N}_p$. By Fact 2.2.3.(iii) and equation (2.5.11), $(1 + h\,\det D_p)\tilde{U}_p\tilde{N}_p \in \mathcal{M}(\mathbb{R}_{sp}(s))$. Therefore

$$\det D_p \det(V_p - Q\tilde{N}_p) = \det(I_{n_i} - (1 + h\,\det D_p)\tilde{U}_p\tilde{N}_p) \in I\,; \tag{2.5.13}$$

by Lemma 2.2.4.(ii), equation (2.5.13) holds if and only if $\det D_p \in I$ and $\det(V_p - Q\tilde{N}_p) \in I$. Consequently, equation (2.5.10) holds for the choice of $Q \in \mathcal{M}(R_u(s))$ in equation (2.5.12).

(iv) In Lemma 2.3.2, we could also start by assuming that a l.c. pair $(\tilde{D}_c, \tilde{N}_c)$ together with a r.c. pair (N_c, D_c) satisfy the generalized Bezout identity (2.5.14) below:

$$\begin{bmatrix} V_c & U_c \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix} \begin{bmatrix} D_c & -\tilde{U}_c \\ N_c & \tilde{V}_c \end{bmatrix} = \begin{bmatrix} I_{n_o} & 0 \\ 0 & I_{n_i} \end{bmatrix}. \tag{2.5.14}$$

Under these conditions, (N_p, D_p) , with $N_p\,, D_p \in \mathcal{M}(H)$, is a solution of equation (2.5.4) if and only if

$$\begin{bmatrix} -N_p \\ D_p \end{bmatrix} = \begin{bmatrix} D_c & -\tilde{U}_c \\ N_c & \tilde{V}_c \end{bmatrix} \begin{bmatrix} -Q_p \\ A \end{bmatrix},$$

for some $Q_p \in \mathcal{M}(H)$; similarly, $(\tilde{D}_p, \tilde{N}_p)$, with $\tilde{D}_p\,, \tilde{N}_p \in \mathcal{M}(H)$, is a solution of equation (2.5.5) if and only if

$$\begin{bmatrix} \tilde{D}_p & \vdots & \tilde{N}_p \end{bmatrix} = \begin{bmatrix} B & \vdots & Q_p \end{bmatrix} \begin{bmatrix} V_c & U_c \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix},$$

for some $Q_p \in \mathcal{M}(H)$.

□

2.6. Coprime factorizations of $\hat{P}$

In this section we consider an $(\eta_o + n_o)x(\eta_i + n_i)$ matrix $\hat{P} \in \mathcal{m}(G)$, partitioned as

$$\hat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P \end{bmatrix} \in G^{(\eta_o+n_o)x(\eta_i+n_i)}, \quad \text{where } P \in G^{n_o x n_i}. \tag{2.6.1}$$

2.6.1. Lemma:

Let $\hat{P} \in \mathcal{m}(G)$ be as in equation (2.6.1); then there exist N_{11}, N_{12}, N_{21}, N_p, D_{11}, D_{21}, $D_p \in \mathcal{m}(H)$, and $\tilde{D}_{11}, \tilde{D}_{12}, \tilde{D}_p, \tilde{N}_{11}, \tilde{N}_{12}, \tilde{N}_{21}, \tilde{N}_p \in \mathcal{m}(H)$, such that

$$\text{(i)} \quad (N_{\hat{P}}, D_{\hat{P}}) =: \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_p \end{bmatrix}, \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_p \end{bmatrix} \right) \text{ is an r.c.f.r. of } \hat{P}, \tag{2.6.2}$$

$$\text{(ii)} \quad (\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) =: \left(\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_p \end{bmatrix}, \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix} \right) \text{ is an l.c.f.r. of } \hat{P}, \tag{2.6.3}$$

where

$$(N_p, D_p) \text{ is an r.f.r. of } P, \text{ and } (\tilde{D}_p, \tilde{N}_p) \text{ is an l.f.r. of } P.$$

2.6.2. Comments:

(i) In Lemma 2.6.1, it is only claimed that (N_p, D_p) is an r.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ is an l.f.r. of the 2-2 sub-block P of $\hat{P}$; these fraction representations are *not* necessarily *coprime*. However, $\begin{bmatrix} N_{12} \\ N_p \end{bmatrix}$ is right-coprime with D_p by equation (2.6.2), and $\tilde{D}_p$ is left-coprime with $\begin{bmatrix} \tilde{N}_{21} & \vdots & \tilde{N}_p \end{bmatrix}$ by equation (2.6.3).

(ii) Let $\hat{P} = N_{\hat{P}} D_{\hat{P}}^{-1} = \tilde{D}_{\hat{P}}^{-1} \tilde{N}_{\hat{P}}$, where $(N_{\hat{P}}, D_{\hat{P}})$ is an r.c. pair as in equation (2.6.2) and $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ is an l.c. pair as in equation (2.6.3). By Lemma 2.3.7, $\hat{P} \in \mathcal{m}(H)$ if and only if $D_{\hat{P}}^{-1} \in \mathcal{m}(H)$, where

$$D_{\hat{P}}^{-1} = \begin{bmatrix} D_{11}^{-1} & 0 \\ -D_p^{-1}D_{21}D_{11}^{-1} & D_p^{-1} \end{bmatrix} \tag{2.6.4}$$

and equivalently, $\tilde{D}_{\hat{P}}^{-1} \in \mathcal{M}(H)$, where

$$\tilde{D}_{\hat{P}}^{-1} = \begin{bmatrix} \tilde{D}_{11}^{-1} & -\tilde{D}_{11}^{-1}\tilde{D}_{12}\tilde{D}_p^{-1} \\ 0 & \tilde{D}_p^{-1} \end{bmatrix} . \tag{2.6.5}$$

□

Proof of Lemma 2.6.1:

$\hat{P} \in \mathcal{M}(G)$ has an r.c.f.r. in H (call it (X, Y)), and an l.c.f.r. in H (call it $(\tilde{Y}, \tilde{X})$).

(i) Recalling the Hermite column-form [Vid.1, Appendix B], there exists an H–unimodular $R \in \mathcal{M}(H)$ such that $D := YR \in \mathcal{M}(H)$ is in the (block-triangular) form given by equation (2.6.2), where we choose to denote the 2-2 entry of $D_{\hat{P}}$ by D_p. Let $N =: XR \in \mathcal{M}(H)$, where we denote the sub-blocks in $N_{\hat{P}}$ as in equation (2.6.3), with $N_p \in \mathcal{M}(H)$ as the 2-2 sub-block. Since $R \in \mathcal{M}(H)$ is H–unimodular, by Lemma 2.3.3.(i), $(N_{\hat{P}}, D_{\hat{P}})$ is also an r.c.f.r. of $\hat{P}$; therefore $\det Y \det R = \det D_{\hat{P}} \in I$.

Now equation (2.6.2) implies that $\det(YR) = \det D_{\hat{P}} = \det D_{11} \det D_p \in I$; hence by Lemma 2.2.4.(ii), $\det D_{11} \in I$ and $\det D_p \in I$. So from equations (2.6.1)-(2.6.2), $P = N_p D_p^{-1}$, where (N_p, D_p) is an r.f.r. of P.

(ii) The proof is similar to that of part (i). Pre-multiplying $\tilde{Y}$ by an H–unimodular $L \in \mathcal{M}(H)$, we obtain the Hermite row-form in equation (2.6.3); by Lemma 2.3.3.(ii), $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ is also an l.c.f.r. of $\hat{P}$. Since $\det L \det \tilde{Y} = \det \tilde{D}_{\hat{P}} = \det \tilde{D}_{11} \det \tilde{D}_p \in I$, we conclude that $(\tilde{D}_p, \tilde{N}_p)$ is an l.f.r. of P.

□

2.7. A useful rank condition

The purpose of this section is to prove an important lemma, which will be useful especially in Chapter Four.

Let H be the ring of proper stable rational functions $R_u(s)$, as in Example 2.2.2. Let

$$\max_{K \in \kappa} rank\ F(K)$$

denote the maximum rank that the matrix $F(K)$ has, as K varies over a specified set κ.

2.7.1. Lemma:

Let $A \in \mathbb{C}^{\eta x \gamma}, B \in \mathbb{C}^{\rho x \gamma}, \tilde{A} \in \mathbb{C}^{\tilde{\rho} x \tilde{\gamma}}, \tilde{B} \in \mathbb{C}^{\tilde{\rho} x \tilde{\eta}}$.

(i) If, for all $K \in \mathbb{R}^{\rho x \eta}$,

$$\max_{K \in \mathcal{M}(\mathbb{R})} rank \left[B + KA \right] < \min \{ \rho, \gamma \}, \tag{2.7.1}$$

then

$$rank \begin{bmatrix} B \\ A \end{bmatrix} = \max_{K \in \mathcal{M}(\mathbb{R})} rank \left[B + KA \right]. \tag{2.7.2}$$

(ii) If, for all $\tilde{K} \in \mathbb{R}^{\tilde{\gamma} x \tilde{\eta}}$,

$$\max_{\tilde{K} \in \mathcal{M}(\mathbb{R})} rank \left[\tilde{B} + \tilde{A}\tilde{K} \right] < \min \{ \tilde{\rho}, \tilde{\eta} \}, \tag{2.7.3}$$

then

$$rank \left[\tilde{B} \quad \tilde{A} \right] = \max_{\tilde{K} \in \mathcal{M}(\mathbb{R})} rank \left[\tilde{B} + \tilde{A}\tilde{K} \right]. \tag{2.7.4}$$

□

An important application of Lemma 2.7.1 is given in Corollary 2.7.2 below: It shows that there is a *real constant output-feedback* which "moves" all $\mathcal{U}$–poles of $P = N_p D_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p = N_{pr} D^{-1} N_{pl}$; in other words, there is a $K \in \mathbb{R}^{n_i x n_o}$ such that det[$D_p + KN_p$] has no zeros at the $\mathcal{U}$–zeros of detD_p (equivalently, there is a $K \in \mathbb{R}^{n_i x n_o}$ such that the $\mathcal{U}$–zeros of det[$\tilde{D}_p + \tilde{N}_p K$] are disjoint from those of det$\tilde{D}_p$ and there is a $K \in \mathbb{R}^{n_i x n_o}$ such that the $\mathcal{U}$

–zeros of det[$D + N_{pl} K N_{pr}$] are disjoint from those of det$D$). Note that the region $\mathcal{U}$ can be chosen to include all open-loop poles of P if we wish to prove that *all* poles can be "moved" by real constant output-feedback.

2.7.2. Corollary:

Let (N_p, D_p) be an r.c.f.r., $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r., (N_{pr}, D, N_{pl}) be a b.c.r.f. of $P \in \mathcal{M}(\mathbb{R}_p(s))$, where N_p , D_p , $\tilde{D}_p$, $\tilde{N}_p$, N_{pr} , D , N_{pl} $\in \mathcal{M}(R_{\mathcal{U}}(s))$. Under these conditions,

(i) there is a $K \in \mathbb{R}^{n_i x n_o}$ such that, for all $s_o \in \mathcal{Z}$[detD_p],

$$rank \left[D_p(s_o) + K N_p(s_o) \right] = n_i ; \tag{2.7.5}$$

(ii) there is a $K \in \mathbb{R}^{n_i x n_o}$ such that, for all $s_o \in \mathcal{Z}$[det$\tilde{D}_p$],

$$rank \left[\tilde{D}_p(s_o) + \tilde{N}_p(s_o) K \right] = n_o ; \tag{2.7.6}$$

(iii) there is a $K \in \mathbb{R}^{n_i x n_o}$ such that, for all $s_o \in \mathcal{Z}$[det$D$],

$$rank \left[D(s_o) + N_{pl}(s_o) K N_{pr}(s_o) \right] = n . \tag{2.7.7}$$

□

Since the state-space representation of P given in Example 2.4.3 is a special bicoprime-fraction representation, Corollary 2.7.2.(iii) implies that, for (A, B, C) minimal, there is a real constant output-feedback such that the closed-loop eigenvalues (i.e., the zeros of det($sI_n - A + BKC$)) are different from the open-loop eigenvalues (i.e., the zeros of det($sI_n - A$)). Intuitively, all eigenvalues associated with controllable-and-observable modes can be "moved" by some real constant output-feedback. Corollary 2.7.2. *does not* imply that the eigenvalues can be moved arbitrarily; in general, we need *dynamic* output-feedback to push the poles into the region of stability $\mathbb{C} \setminus \bar{\mathcal{U}}$.

Proof of Lemma 2.7.1:

(i) Call $\hat{K}$ the maximizer of the left-hand-side of equation (2.7.1), i.e., $\hat{K}$ is the $\rho \times \eta$ real matrix that maximizes $rank\begin{bmatrix} B + KA \end{bmatrix}$; by equation (2.7.1), $r := rank\begin{bmatrix} B + \hat{K}A \end{bmatrix} < \min\{\rho, \gamma\}$. So there are R_u–unimodular matrices L, R (resulting from elementary row operations and elementary column operations, respectively), such that $L\begin{bmatrix} B + \hat{K}A \end{bmatrix} R = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where the 0 in the bottom right is $(\rho - r) \times (\gamma - r)$, with $r < \min\{\rho, \gamma\}$.

Now since $\hat{K}$ is the maximizer, for all $\hat{K}_2 \in \mathbb{R}^{(\rho - r) \times \eta}$, $rank(L[B + (\hat{K} + L^{-1}\begin{bmatrix} 0 \\ \hat{K}_2 \end{bmatrix})A]R) \leq r$. Let $\begin{bmatrix} L(B + \hat{K}A)R \\ AR \end{bmatrix} =: \begin{bmatrix} I_r & 0 \\ 0 & 0 \\ \bar{A} & \hat{A} \end{bmatrix}$; then $rank(L(B + \hat{K}A)R + \begin{bmatrix} 0 \\ \hat{K}_2 \end{bmatrix} AR) = rank\begin{bmatrix} I_r & 0 \\ \hat{K}_2\bar{A} & \hat{K}_2\hat{A} \end{bmatrix} \leq r$ implies that $rank \hat{K}_2\hat{A} = 0$ for all $\hat{K}_2$, and hence $\hat{A}$ is the zero matrix.

The proof concludes by the following equalities: $rank\begin{bmatrix} B \\ A \end{bmatrix} = rank(\begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B + \hat{K}A \\ A \end{bmatrix} R) =$

$$rank\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ \bar{A} & 0 \end{bmatrix} = r = rank\begin{bmatrix} B + \hat{K}A \end{bmatrix} = \max_{K \in \mathcal{M}(\mathbb{R})} rank\begin{bmatrix} B + KA \end{bmatrix}.$$

(ii) Similar to proof of part (i).

□

To prove Corollary 2.7.1, the well-known rank-tests are useful (see, for example, [Cal.1]); we state and prove them to make the discussion complete.

2.7.3. Lemma (Rank-tests):

(i) Let $N_p \in \mathcal{R}_u(s)^{n_o x n_i}$, $D_p \in \mathcal{R}_u(s)^{n_i x n_i}$; then (N_p, D_p) is r.c. if and only if

$$rank \begin{bmatrix} D_p(s) \\ N_p(s) \end{bmatrix} = n_i \ , \ \text{for all } s \in \bar{\mathcal{U}} . \tag{2.7.8}$$

(ii) Let $\tilde{D}_p \in \mathcal{R}_u(s)^{n_o x n_o}$, $\tilde{N}_p \in \mathcal{R}_u(s)^{n_o x n_i}$; then $(\tilde{D}_p, \tilde{N}_p)$ is l.c. if and only if

$$rank \begin{bmatrix} D_p(s) & N_p(s) \end{bmatrix} = n_o \ , \ \text{for all } s \in \bar{\mathcal{U}} . \tag{2.7.9}$$

□

Note that the rank-tests in (2.7.8) or (2.7.9) need to be performed only at the $\bar{\mathcal{U}}$–zeros of $\det D_p$ (equivalently, at the $\bar{\mathcal{U}}$–zeros of $\det\tilde{D}_p$), since they already hold for all other $s \in \bar{\mathcal{U}}$.

Proof:

(i) (N_p, D_p) is an r.c. pair if and only if there is an $\mathcal{R}_u$–unimodular matrix E (labeled as $\begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix}$) such that

$$\begin{bmatrix} V_p(s) & U_p(s) \\ -\tilde{N}_p(s) & \tilde{D}_p(s) \end{bmatrix} \begin{bmatrix} D_p(s) \\ N_p(s) \end{bmatrix} = \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix} ; \tag{2.7.10}$$

since the matrix E has rank $n_i + n_o$ for all $s \in \bar{\mathcal{U}}$, equation (2.7.10) holds if and only if the rank condition (2.7.8) holds.

(ii) Similar to part (i). $(\tilde{D}_p, \tilde{N}_p)$ is l.c. if and only if there is an $\mathcal{R}_u$–unimodular matrix F (labeled as $\begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix}$) such that

$$\begin{bmatrix} \tilde{D}_p(s) & \tilde{N}_p(s) \end{bmatrix} \begin{bmatrix} D_p(s) & -\tilde{U}_p(s) \\ N_p(s) & \tilde{V}_p(s) \end{bmatrix} = \begin{bmatrix} I_{n_o} & 0 \end{bmatrix} ; \tag{2.7.11}$$

since the matrix F has rank $n_i + n_o$ for all $s \in \bar{\mathcal{U}}$, equation (2.7.11) holds if and only if the rank condition (2.7.9) holds.

□

Proof of Corollary 2.7.2:

Suppose, for a contradiction, that there is an $s_o \in \mathcal{Z}[\ \det D_p\ ]$ such that

$$rank\left[\ D_p(s_o)+KN_p(s_o)\ \right] < n_i\ ,\ \text{for all } K \in \mathbb{R}^{n_i x n_o}\ . \qquad (2.7.12)$$

Applying Lemma 2.7.1.(i), with $D_p(s_o) =: B$, $N_p(s_o) =: A$, $\rho := n_i =: \gamma$, $n_o =: \eta$, equation (2.7.12) implies that $rank\begin{bmatrix} D_p(s_o) \\ N_p(s_o) \end{bmatrix} < n_i$; but this contradicts the fact that (N_p, D_p) is right-coprime.

(ii) Similar to proof of part (i) and again follows from Lemma 2.7.1.(i).

(iii) Suppose, for a contradiction, that there is an $s_o \in \mathcal{Z}[\ \det D\ ]$ such that

$$rank\left[\ D(s_o)+N_{pl}(s_o)KN_{pr}(s_o)\ \right] < n_i\ ,\ \text{for all } K \in \mathbb{R}^{n_i x n_o}\ ; \qquad (2.7.13)$$

equivalently, for all $K \in \mathbb{R}^{n_i x n_o}$,

$$rank\begin{bmatrix} D(s_o) & -N_{pl}(s_o) \\ KN_{pr}(s_o) & I_{ni} \end{bmatrix} < n+n_i\ . \qquad (2.7.14)$$

(The equivalence of equations (2.7.13) and (2.7.14) follows by performing elementary row and column operations on the matrix in equation (2.7.14).)

Post-multiplying the matrix in equation (2.7.14) by the R_u–unimodular matrix $M_l(s_o)^{-1} =$ $\begin{bmatrix} V_{pl}(s_o) & X(s_o) \\ -U_{pl}(s_o) & Y(s_o) \end{bmatrix}$, and using the generalized Bezout identity (2.3.10), we conclude that equation (2.7.14) holds if and only if

$$rank\left[\ Y(s_o)+KN_{pr}(s_o)X(s_o)\ \right] < n_i\ . \qquad (2.3.17)$$

Recall that, by Proposition 2.4.1, $(N_{pr}X, Y)$ is a r.c.f.r. of P ; applying Lemma 2.7.1 as in part (i) of the proof, equation (2.7.15) implies that

$$rank\begin{bmatrix} Y(s_o) \\ N_{pr}(s_o)X(s_o) \end{bmatrix} < n_i\ ; \qquad (2.7.16)$$

but equation (2.7.16) contradicts the right-coprimeness of $(N_{pr}X, Y)$.

□

Chapter Three

Control Systems with Full Output-Feedback Compensators

3.1. Introduction

This chapter presents an algebraic theory for two linear, time-invariant (l.t-i), multiinput-multioutput (MIMO) control systems: the classical unity-feedback system $S(P, C)$ and the more general system configuration $\Sigma(\hat{P}, \hat{C})$. Due to the general algebraic setting, the results apply to lumped as well as distributed, continuous-time as well as discrete-time systems in these configurations.

In the unity-feedback system $S(P, C)$, the plant has only one (vector-)input, and one (vector)-output, which is used in feedback to the compensator; the plant model considers only additive inputs or disturbances, which pass through the actuators in the plant. In general, however, there may be inputs (for example, disturbances, initial conditions, noise, manual commands) which are applied directly to the plant without going through the actuators; hence, the map from the directly applied inputs to the plant outputs may be different than the map from the additive inputs to the plant outputs. Furthermore, the regulated output variable of the plant (for example, tracking error, actuator states) may not be accessible or may be different than the measured output (for example, ideal sensor outputs), which is utilized by the compensator. For instance, the temperature of the blades in jet engines cannot be measured; to prevent the blades from burning, controllers are designed based on measurements of other variables like air flow, angular velocity and fuel rate.

The unity-feedback configuration $S(P, C)$ is studied in Section 3.2. The system $S(P, C)$ is called H–stable if and only if all closed-loop input-output (I/O) maps are H–stable. The H–stability condition for $S(P, C)$ is stated in Theorem 3.2.5 in terms of each factorization of P

and C used in the analysis. The class of all compensators that H–stabilize the plant P is parametrized in Theorem 3.2.8; compensator design using the configuration $S(P, C)$ is called one-degree-of-freedom design due to the single free parameter matrix Q of the H–stabilizing compensator [Hor.1]. Although a right-coprime or a left-coprime factorization of the plant are commonly used in obtaining this parametrization, it is also possible to start with a bicoprime factorization and use Proposition 2.4.1 to reduce $N_{pr}D^{-1}N_{pl}$ to $N_pD_p^{-1}$ or to $\tilde{D}_p^{-1}\tilde{N}_p$ (see equations (3.2.29)-(3.2.30)). The class of all achievable maps for $S(P, C)$ is obtained by using the class of all stabilizing compensators; all closed-loop I/O maps in the H–stabilized $S(P, C)$ are affine maps in Q (see equation (3.2.38)).

The system configuration $\Sigma(\hat{P}, \hat{C})$ represents the most general interconnection of two physical systems, a plant $\hat{P}$ and a compensator $\hat{C}$. This general system configuration $\Sigma(\hat{P}, \hat{C})$ is studied in Section 3.3; the plant and the compensator each have two (vector-)inputs and two (vector-)outputs. The measured output y of $\hat{P}$ is used in feedback, but the output z is the actual output of the plant (the output in the performance specifications); the point is that z and y are not the same. The input v is considered as a disturbance, noise or an external command applied directly to the plant. The compensator output y', which is utilized by the plant in feedback, can be considered as the ideal actuator inputs; the output z' of $\hat{C}$ can be used for performance monitoring or fault diagnosis. The input v' of $\hat{C}$ is considered as the independent control input like commands or initial conditions. The signals u and u', which appear at the interconnection of $\hat{P}$ and $\hat{C}$, model possible additive disturbances noise, interference and loading.

The conditions for H–stability of $\Sigma(\hat{P}, \hat{C})$ are stated in Theorem 3.3.5. Intuitively, only those plants which have "instabilities that the feedback-loop can remove" can be considered for H–stabilization; these plants are called Σ–admissible. The restriction on the class of H–stabilizable $\hat{P}$ is due to the feedback being applied only through the second inputs and outputs. The class of Σ–admissible $\hat{P}$ is given in Theorem 3.3.9; the class of all H–stabilizing compensators for Σ–admissible plants is given in Theorem 3.3.11. The 2-2 block of $\hat{C}$ is essentially in a feedback configuration like $S(P, C)$ of Section 3.2; so the set of all C that H–stabilizes the feedback-loop is already known from the previous section.

In the unity-feedback configuration $S(P, C)$, the class of all C that H–stabilize P is parametrized by one parameter matrix Q ; including this parameter matrix Q that comes from C, the set of all $\hat{C}$ that H–stabilize $\hat{P}$ is parametrized by *four* H–stable matrices and hence, we call the system $\Sigma(\hat{P}, \hat{C})$ a four-degrees-of-freedom design (or four-parameter design) [Net.1]. This is clearly much more advantageous and general than two-degrees-of-freedom design with a two-input one-output compensator [see, for example, Vid.1, Des.3,4]; $\Sigma(\hat{P}, \hat{C})$ can obviously be reduced to two parameter design by taking $C_{11} = 0$ and $C_{12} = 0$. The class of all achievable maps for $\Sigma(\hat{P}, \hat{C})$ involves the four compensator parameters; *each* closed-loop I/O map achieved by the H–stabilized $\Sigma(\hat{P}, \hat{C})$ depends on one and *only one* of these four parameter matrices Q_{11}, Q_{12}, Q_{21}, Q. Clearly, several performance specifications can be imposed on the closed-loop performance of $\Sigma(\hat{P}, \hat{C})$.

In Section 3.4, we consider the decoupling problem; namely, find $\hat{C}$ such that, for the given $\hat{P}$, the I/O map $H_{zv'} : v' \mapsto z$ of $\Sigma(\hat{P}, \hat{C})$ is diagonal. Assuming that N_{12} is nonsingular, it is always possible to choose $Q_{21} \in \mathcal{M}(H)$ such that $H_{zv'} = N_{12}Q_{21}$ is diagonal. Diagonalization with this configuration does not involve the feedback-loop and the parameter Q of C ; hence, decoupling the I/O map $H_{zv'}$ is independent of the I/O maps that are affine functions in Q. On the other hand, in the unity-feedback configuration $S(P, C)$, diagonalizing the map $H_{yu'} : u' \mapsto y$ would depend on the choice for Q such that $N_p(U_p + Q\tilde{D}_p)$ is diagonal, and hence, diagonalizing the map $H_{yu'}$ in $S(P, C)$ may not be possible for certain plants.

3.2. The unity-feedback system $S(P, C)$

In this section we consider the system $S(P, C)$ in Figure 3.1.

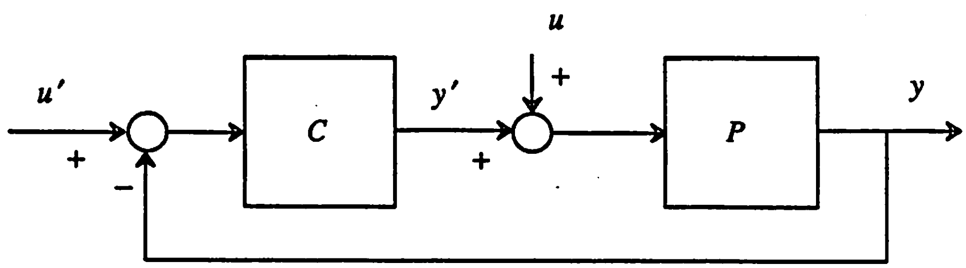

Figure 3.1. The unity-feedback system $S(P, C)$

3.2.1. Assumptions:

(A) The plant $P \in G^{n_o x n_i}$.

Let (N_p, D_p) be an r.c.f.r., $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r., (N_{pr}, D, N_{pl}) be a b.c.f.r. of P, where $N_p \in H^{n_o x n_i}$, $D_p \in H^{n_i x n_i}$, $\tilde{D}_p \in H^{n_o x n_o}$, $\tilde{N}_p \in H^{n_o x n_i}$, $N_{pr} \in H^{n_o x n}$, $D \in H^{n x n}$, $N_{pl} \in H^{n x n_i}$.

(B) The compensator $C \in G^{n_i x n_o}$.

Let $(\tilde{D}_c, \tilde{N}_c)$ be an l.c.f.r. and (N_c, D_c) be an r.c.f.r. of C, where $\tilde{D}_c \in H^{n_i x n_i}$, $\tilde{N}_c \in H^{n_i x n_o}$, $N_c \in H^{n_i x n_o}$, $D_c \in H^{n_o x n_o}$.

If P satisfies Assumption 3.2.1 (A), we have the generalized Bezout identities (2.3.7), (2.3.8), (2.3.10) given in Section 2.3.

3.2.2. Closed-loop I/O maps of $S(P, C)$:

Consider Figure 3.1; let

$$\bar{y} := \begin{bmatrix} y \\ y' \end{bmatrix}, \quad \bar{u} := \begin{bmatrix} u \\ u' \end{bmatrix}. \tag{3.2.1}$$

The map $H_{\overline{yu}} : \overline{u} \mapsto \overline{y}$ is called the I/O map of $S(P,C)$. In terms of P and C, $H_{\overline{yu}}$ is given by

$$H_{\overline{yu}} = \begin{bmatrix} P(I_{n_i}+CP)^{-1} & P(I_{n_i}+CP)^{-1}C \\ -CP(I_{n_i}+CP)^{-1} & (I_{n_i}+CP)^{-1}C \end{bmatrix} . \tag{3.2.2}$$

3.2.3. Analysis of $S(P,C)$:

We analyze the system $S(P,C)$ shown in Figure 3.1 by factorizing P and C as

(i) $P = N_p D_p^{-1}$, $C = \tilde{D}_c^{-1}\tilde{N}_c$,

(ii) $P = \tilde{D}_p^{-1}\tilde{N}_p$, $C = N_c D_c^{-1}$,

(iii) $P = N_{pr} D^{-1} N_{pl}$, $C = \tilde{D}_c^{-1}\tilde{N}_c$,

(iv) $P = N_{pr} D^{-1} N_{pl}$, $C = N_c D_c^{-1}$.

(i) Analysis of $S(P,C)$ with $P = N_p D_p^{-1}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$:

Let $P = N_p D_p^{-1}$, $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_p, D_p) is r.c. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see Figure 3.2); ξ_p denotes the pseudo-state of P.

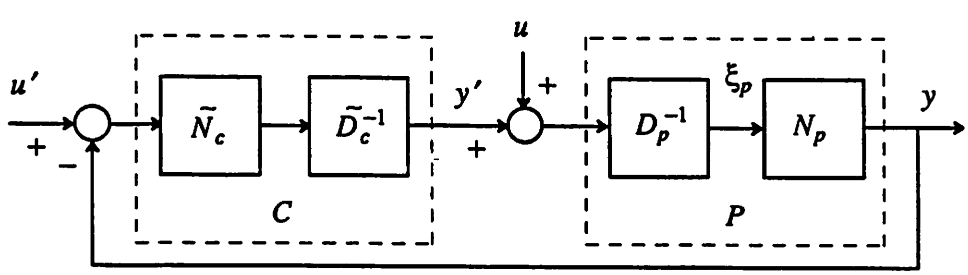

Figure 3.2. $S(P,C)$ with $P = N_p D_p^{-1}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$

$S(P,C)$ is then described by equations (3.2.3)-(3.2.4):

$$\left[\tilde{D}_c D_p + \tilde{N}_c N_p \right] \xi_p = \left[\tilde{D}_c \; \vdots \; \tilde{N}_c \right] \begin{bmatrix} u \\ u' \end{bmatrix} , \tag{3.2.3}$$

$$\begin{bmatrix} N_p \\ D_p \end{bmatrix} \xi_p = \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_{n_i} & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix} . \tag{3.2.4}$$

Equations (3.2.3)-(3.2.4) are of the form

$$D_{H1}\xi_p = N_{L1}\bar{u}$$

$$N_{R1}\xi_p = \bar{y} - S_{H1}\bar{u} \,.$$

If $\det D_{H1} \in \mathcal{I}$ (equivalently, if the system described by equations (3.2.3)-(3.2.4) is well-posed), then the I/O map $H_{\overline{yu}}$ is given by

$$H_{\overline{yu}} = N_{R1}D_{H1}^{-1}N_{L1} + S_{H1} \in \mathcal{M}(G)\,.$$

By elementary row and column operations on the matrices in equations (3.2.3)-(3.2.4), using Lemma 2.3.2, it is easy to see that $(N_{R1}, D_{H1}, N_{L1}, S_{H1})$ is a b.c.f.r. of $H_{\overline{yu}}$.

(ii) Analysis of $S(P, C)$ with $P = \tilde{D}_p^{-1}\tilde{N}_p$ and $C = N_cD_c^{-1}$:

Let $P = \tilde{D}_p^{-1}\tilde{N}_p$, $C = N_cD_c^{-1}$, where $(\tilde{D}_p, \tilde{N}_p)$ is l.c. and (N_c, D_c) is r.c. (see Figure 3.3); ξ_c denotes the pseudo-state of C.

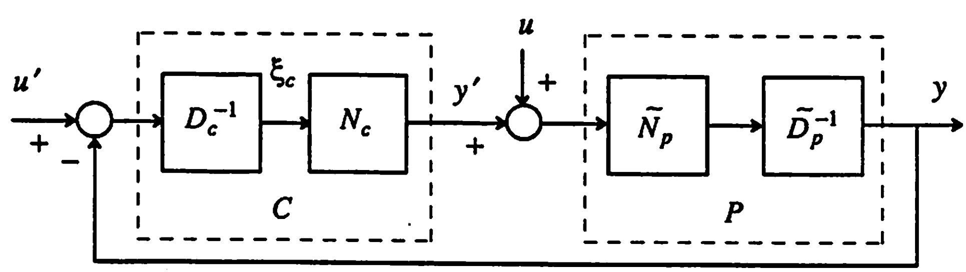

Figure 3.3. $S(P, C)$ with $P = \tilde{D}_p^{-1}\tilde{N}_p$ and $C = N_cD_c^{-1}$

$S(P, C)$ is then described by equations (3.2.5)-(3.2.6):

$$\begin{bmatrix} \tilde{D}_pD_c + \tilde{N}_pN_c \end{bmatrix} \xi_c = \begin{bmatrix} -\tilde{N}_p & \vdots & \tilde{D}_p \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}, \tag{3.2.5}$$

$$\begin{bmatrix} -D_c \\ N_c \end{bmatrix} \xi_c = \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 & -I_{n_o} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}. \tag{3.2.6}$$

Equations (3.2.5)-(3.2.6) are of the form

$$D_{H2}\xi_c = N_{L2}\bar{u}$$

$$N_{R2}\xi_c = \bar{y} - S_{H2}\bar{u} \ .$$

By elementary row and column operations on the matrices in equations (3.2.5)-(3.2.6), using Lemma 2.3.2, we conclude again that $(N_{R2}, D_{H2}, N_{L2}, S_{H2})$ is a b.c.f.r. of $H_{\bar{y}\bar{u}}$.

(iii) Analysis of $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$:

Let $P = N_{pr}D^{-1}N_{pl}$, $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_{pr}, D, N_{pl}) is b.c. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see Figure 3.4); ξ_x denotes the pseudo-state of P.

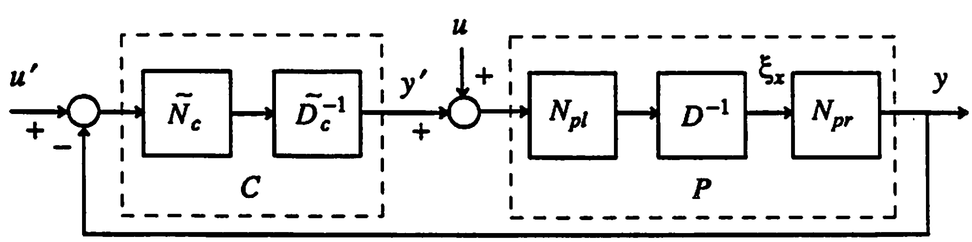

Figure 3.4. $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$

$S(P, C)$ is then described by equations (3.2.7)-(3.2.8):

$$\begin{bmatrix} D & \vdots & -N_{pl} \\ \cdots & & \cdots \\ \tilde{N}_c N_{pr} & \vdots & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y' \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \tilde{N}_c \end{bmatrix} \begin{bmatrix} u \\ \cdots \\ u' \end{bmatrix} , \tag{3.2.7}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_i} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y' \end{bmatrix} = \begin{bmatrix} y \\ \cdots \\ y' \end{bmatrix} . \tag{3.2.8}$$

Equations (3.2.7)-(3.2.8) are of the form

$$D_{H3}\xi_3 = N_{L3}\bar{u}$$

$$N_{R3}\xi_3 = \bar{y} \ .$$

If det$D_{H3} \in I$, then the system is well-posed; again by elementary row and column operations on the matrices in equations (3.2.7)-(3.2.8), using Lemma 2.3.2, (N_{R3}, D_{H3}, N_{L3}) is a b.c.f.r. of $H_{\overline{yu}}$.

(iv) Analysis of $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = N_cD_c^{-1}$:

Let $P = N_{pr}D^{-1}N_{pl}$, $C = N_cD_c^{-1}$, where (N_{pr}, D, N_{pl}) is b.c. and (N_c, D_c) is r.c. (see Figure 3.5).

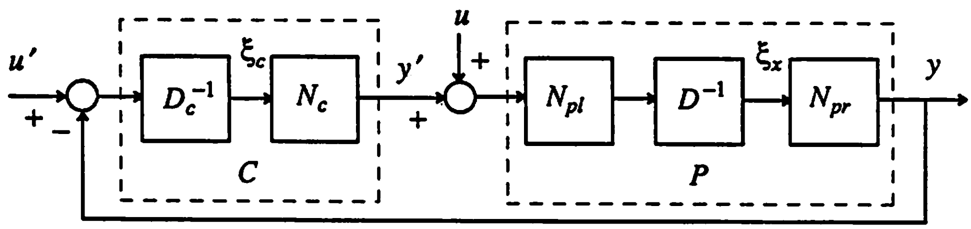

Figure 3.5. $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = N_cD_c^{-1}$

$S(P, C)$ is then described by equations (3.2.9)-(3.2.10):

$$\begin{bmatrix} D & \vdots & -N_{pl}N_c \\ \cdots & & \cdots \\ N_{pr} & \vdots & D_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_c \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_o} \end{bmatrix} \begin{bmatrix} u \\ \cdots \\ u' \end{bmatrix}, \tag{3.2.9}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & N_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_c \end{bmatrix} = \begin{bmatrix} y \\ \cdots \\ y' \end{bmatrix}. \tag{3.2.10}$$

Equations (3.2.9)-(3.2.10) are of the form

$$D_{H4}\xi_4 = N_{L4}\overline{u}$$

$$N_{R4}\xi_4 = \overline{y} .$$

If det$D_{H4} \in I$, then the system is well-posed; by elementary row and column operations on the matrices in equations (3.2.9)-(3.2.10), using Lemma 2.3.2, (N_{R4}, D_{H4}, N_{L4}) is a b.c.f.r. of $H_{\overline{yu}}$.

□

3.2.4. Definition (H–stability):

The system $S(P, C)$ is said to be *H–stable* iff $H_{\overline{yu}} \in \mathcal{m}(H)$.

If $H_{\overline{yu}} \in \mathcal{m}(H)$, then we also say that $H_{\overline{yu}}$ is H–stable.

If we choose a *specific* principal ring like $R_u(s)$, then we say R_u–stable.

3.2.5. Theorem (H–stability of $S(P, C)$):

Consider the system $S(P, C)$ shown in Figure 3.1. Let Assumptions 3.2.1 (A) and (B) hold. Under these conditions, the following five conditions are equivalent:

(i) $S(P, C)$ is H–stable;

(ii) $D_{H1} := \left[\tilde{D}_c D_p + \tilde{N}_c N_p \right]$ is H–unimodular; (3.2.11)

(iii) $D_{H2} := \left[\tilde{D}_p D_c + \tilde{N}_p N_c \right]$ is H–unimodular; (3.2.12)

(iv) $D_{H3} := \begin{bmatrix} D & -N_{pl} \\ \tilde{N}_c N_{pr} & \tilde{D}_c \end{bmatrix}$ is H–unimodular; (3.2.13)

(v) $D_{H4} := \begin{bmatrix} D & -N_{pl} N_c \\ N_{pr} & D_c \end{bmatrix}$ is H–unimodular. (3.2.14)

3.2.6. Comments:

(i) Post-multiplying the matrix D_{H3} in equation (3.2.13) by the H–unimodular matrix M_l^{-1} defined in the generalized Bezout identity (2.3.10), we obtain

$$D_{H3} M_l^{-1} = \begin{bmatrix} I_n & 0 \\ \tilde{N}_c N_{pr} V_{pl} - \tilde{D}_c U_{pl} & \tilde{N}_c N_{pr} X + \tilde{D}_c Y \end{bmatrix}.$$

But D_{H3} is H–unimodular if and only if $D_{H3} M_l^{-1}$ is H–unimodular; hence, condition (3.2.13) holds if and only if

$$\tilde{D}_c Y + \tilde{N}_c N_{pr} X \quad \text{is } H\text{–unimodular}. \tag{3.2.15}$$

Note that the H–unimodularity condition (3.2.15) is the same as condition (3.2.11) since $(N_{pr} X, Y)$ is an r.c.f.r. of P by Proposition 2.4.1.

Similarly, pre-multiplying the matrix D_{H4} in equation (3.2.14) by the H–unimodular matrix M_r defined in the generalized Bezout identity (2.3.8), we obtain

$$M_r D_{H4} = \begin{bmatrix} I_n & -V_{pr} N_{pl} N_c + U_{pr} D_c \\ 0 & \tilde{X} N_{pl} N_c + \tilde{Y} D_c \end{bmatrix} .$$

But D_{H4} is H–unimodular if and only if $M_r D_{H4}$ is H–unimodular; hence, condition (3.2.14) holds if and only if

$$\tilde{X} N_{pl} N_c + \tilde{Y} D_c \quad \text{is } H\text{–unimodular} . \tag{3.2.16}$$

Note that condition (3.2.16) is the same as condition (3.2.12) are the same since $(\tilde{Y} , \tilde{X} N_{pl})$ is an l.c.f.r. of P by Proposition 2.4.1.

(ii) If condition (3.2.11) (equivalently, (3.2.12)) holds, then by normalization we obtain

$$\tilde{D}_c D_p + \tilde{N}_c N_p = I_{n_i} , \tag{3.2.17}$$

and

$$\tilde{N}_p N_c + \tilde{D}_p D_c = I_{n_o} . \tag{3.2.18}$$

With $P = N_p D_p^{-1} = \tilde{D}_p^{-1} \tilde{N}_p$, $C = \tilde{D}_c^{-1} \tilde{N}_c = N_c D_c^{-1}$, equations (3.2.17)-(3.2.18) are equivalent to

$$\begin{bmatrix} \tilde{D}_c & \tilde{N}_c \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -N_c \\ N_p & D_c \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} . \tag{3.2.19}$$

Equation (3.2.19) is the same as equation (2.5.9) and is a useful form of the generalized Bezout identity.

□

Proof of Theorem 3.2.5:

(i) <=> (ii):

Suppose that we analyze $S(P, C)$ with P factorized as $N_p D_p^{-1}$ and C factorized as $\tilde{D}_c^{-1} \tilde{N}_c$; then equations (3.2.3)-(3.2.4) describe the system. Let $S(P, C)$ be H–stable; then by Definition 3.2.4, $H_{\overline{yu}} \in \mathcal{M}(H)$, and in particular, $H_{y'u} : u \mapsto y'$ is given by

$$H_{y'u} = -CP(I_{n_i} + CP)^{-1} = (I_{n_i} + CP)^{-1} - I_{n_i} \in \mathcal{M}(H) . \tag{3.2.20}$$

Since $D_p^{-1} \in \mathcal{M}(G)$ and $\tilde{D}_c^{-1} \in \mathcal{M}(G)$, by equations (3.2.3) and (3.2.20),

$$D_{H1}^{-1} = D_p^{-1}(I_{n_i} + CP)^{-1}\tilde{D}_c^{-1} \in \mathcal{M}(G), \tag{3.2.21}$$

(equivalently, $S(P, C)$ is well-posed). Thus $(N_{R1}, D_{H1}, N_{L1}, S_{H1})$ is a b.c.f.r. of $H_{\overline{yu}}$. By Lemma 2.3.7, $H_{\overline{yu}} \in \mathcal{M}(H)$ implies that $D_{H1}^{-1} \in \mathcal{M}(H)$; equivalently, condition (3.2.11) holds.

Conversely, if D_{H1}^{-1} is H–unimodular, then clearly equations (3.2.3)-(3.2.4) describe a well-posed system and hence, $H_{\overline{yu}}$ is given by $N_{R1}D_{H1}^{-1}N_{L1} + S_{H1}$. By Lemma 2.3.7, $H_{\overline{yu}} \in \mathcal{M}(H)$ and hence, $S(P, C)$ is H–stable.

(ii) <=> (iii) <=> (iv) <=> (v):

Condition (3.2.11) holds if and only if

$$\det(\tilde{D}_c D_p + \tilde{N}_c N_p) \approx 1, \tag{3.2.22}$$

if and only if

$$\det\tilde{D}_c \det(I_{n_i} + CP)\det D_p \approx 1. \tag{3.2.23}$$

Now $\det(I_{n_i} + CP) = \det(I_{n_o} + PC)$. By Corollary 2.4.4 (equation (2.4.29)) $\det D_p \approx \det\tilde{D}_p$; similarly, $\det\tilde{D}_c \approx \det D_c$. Therefore, equation (3.2.23) is equivalent to

$$\det\tilde{D}_p \det(I_{n_o} + PC)\det D_c \approx 1; \tag{3.2.24}$$

equivalently,

$$\det(\tilde{D}_p D_c + \tilde{N}_p N_c) \approx 1, \tag{3.2.25}$$

and hence, condition (3.2.12) holds.

Now by Proposition 2.4.1, $(N_{pr}X, Y)$ is an r.c.f.r. and $(\tilde{Y}, \tilde{X} N_{pl})$ is a l.c.f.r. of P, where the Bezout identities (2.3.8)-(2.3.10) hold. Therefore, from Comment 3.2.6.(i), condition (3.2.13) is equivalent to (3.2.11) and condition (3.2.14) is equivalent to (3.2.12).

□

3.2.7. Definition (H–stabilizing compensator C):

(i) C is called an *H–stabilizing compensator for P* (later abbreviated as: C *H–stabilizes P*) iff $C \in G^{n_i \times n_o}$ satisfies Assumption 3.2.1 (B) *and* the system $S(P, C)$ is H–stable.

(ii) The set

$$\mathbf{S}(P) := \{ \ C : C \ \textit{H–stabilizes} \ P \ \} \tag{3.2.26}$$

is called the *set of all* H–stabilizing compensators for P .

3.2.8. Theorem (Set of all H–stabilizing compensators for P):

Let $P \in \mathcal{m}(G_S)$ and let P satisfy Assumptions 3.2.1 (A); then the set $\mathbf{S}(P)$ of all H–stabilizing compensators C for P is given by equation (3.2.27) and equivalently, by equation (3.2.28) below:

$$\mathbf{S}(P) = \{ \ C = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p) \ : \ Q \in \mathcal{m}(H) \ \} \, ; \tag{3.2.27}$$

$$\mathbf{S}(P) = \{ \ C = (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1} \ : \ Q \in \mathcal{m}(H) \ \} \, ; \tag{3.2.28}$$

where the matrices V_p , U_p , $\tilde{V}_p$, $\tilde{U}_p$ in equations (3.2.27)-(3.2.28) satisfy the generalized Bezout identity (2.3.7).

Equations (3.2.27) and (3.2.28) give a *parametrization* of all H–stabilizing compensators for P ; in each case, the map $Q \mapsto C$ is bijective and, for the same $Q \in \mathcal{m}(H)$, (3.2.27) and (3.2.28) give the same C .

3.2.9. Comments:

(i) (All H–stabilizing compensators based on a b.c.f.r. of P):

By Proposition 2.4.1, $(N_{pr}X, Y)$ is an r.c.f.r. and $(\tilde{Y}, \tilde{X} N_{pl})$ is an l.c.f.r. of P ; with this doubly-coprime-fraction representation of P , the set $\mathbf{S}(P)$ of all H–stabilizing compensators is given by:

$$\mathbf{S}(P) = \{ \; (V + UV_{pr}N_{pl} - Q\tilde{X}\,N_{pl})^{-1}(UU_{pr} + Q\tilde{Y}\,) \; : \; Q \; \in \; \mathcal{M}(H) \; \} \; ; \qquad (3.2.29)$$

equivalently,

$$\mathbf{S}(P) = \{ \; (U_{pl}\tilde{U} \; + YQ)(\tilde{V} \; + N_{pr}V_{pl}\tilde{U} \; - N_{pr}XQ)^{-1} \; : \; Q \; \in \; \mathcal{M}(H) \; \} \; ; \qquad (3.2.30)$$

where the matrices in equations (3.2.29)-(3.2.30) satisfy the generalized Bezout identities (2.3.8)-(2.3.10).

Following Comment 2.4.2, a generalized Bezout identity for the doubly-coprime pair $(\,(N_{pr}X\,,Y)\,,(\tilde{Y}\,,\tilde{X}\,N_{pl})\,)$ is given by equation (2.4.3); comparing the two generalized Bezout identities (2.4.3) and (2.3.7), it is easy to see that equation (3.2.29) is equivalent to equation (3.2.27) and equation (3.2.30) is equivalent to equation (3.2.28).

(ii) (All H–stabilizing compensators for H–stable P):

If $P \in \mathcal{M}(H)$, then following Comment 2.5.3.(ii), the set $\mathbf{S}(P)$ of all H–stabilizing compensators is given by:

$$\mathbf{S}(P) = \{ \; C = (I_{n_i} - QP)^{-1}Q \; : \; Q \; \in \; \mathcal{M}(H) \; \} \; ;$$

equivalently,

$$\mathbf{S}(P) = \{ \; C = Q(I_{n_o} - PQ)^{-1} \; : \; Q \; \in \; \mathcal{M}(H) \; \} \; .$$

(iii) (All H–stabilizing compensators when $P \in \mathcal{M}(G)$)

In Theorem 3.2.8, if we assume that $P \in \mathcal{M}(G)$ but not $\mathcal{M}(G_S)$, then in equations (3.2.27)-(3.2.28) (and equivalently, (3.2.29)-(3.2.30)) we choose $Q \in \mathcal{M}(H)$ such that

$$\det(V_p - Q\tilde{N}_p) \in I \quad (\text{ equivalently, } \det(\tilde{V}_p - N_pQ) \in I \;) \qquad (3.2.31)$$

because equation (3.2.31) is *not* automatically satisfied for *all* $Q \in \mathcal{M}(H)$.

If H is $R_u(s)$ as in Example 2.2.2, then following Comment 2.5.3.(iii), $Q \in \mathcal{M}(R_u(s))$ can be chosen as in equation (2.5.12) to satisfy equation (3.2.31); in other words, with Q as in (2.5.12), we have a proper compensator where the denominator is given by $\tilde{D}_c = (V_p - QN_p)$ (or $D_c = (\tilde{V}_p - N_pQ)$).

(iv) (All P such that $S(P, C)$ is H-stable):

It is trivial to observe that P and C are symmetric in $S(P,C)$: Let $C \in \mathcal{m}(G_S)$, $C = \tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1}$, be given, let $(\tilde{D}_c, \tilde{N}_c)$ and (N_c, D_c) satisfy the generalized Bezout identity (2.5.14). Under these conditions, the set of all $P \in \mathcal{m}(G)$ for which $S(P, C)$ is H-stable is given by equation (3.2.27P) and equivalently, equation (3.2.28P) below:

$$\{ P = (\tilde{U}_c + D_c Q_p)(\tilde{V}_c - N_c Q_p)^{-1} : Q_p \in \mathcal{m}(H) \} ; \tag{3.2.27P}$$

$$\{ P = (V_c - Q_p \tilde{N}_c)^{-1}(U_c + Q_p \tilde{D}_c) : Q_p \in \mathcal{m}(H) \} . \tag{3.2.28P}$$

If $C \in \mathcal{m}(G)$, then $Q_p \in \mathcal{m}(H)$ should be chosen so that $\det(\tilde{V}_c - N_c Q_p) \in I$ (equivalently, $\det(V_c - Q_p \tilde{N}_c) \in I$).

□

Proof of Theorem 3.2.8:

By normalizing equations (3.2.11) and (3.2.12), $S(P, C)$ is H-stable if and only if equations (3.2.17)-(3.2.18) (and hence, the generalized Bezout identity (3.2.19)) hold. By Lemma 2.5.2, $(\tilde{D}_c, \tilde{N}_c)$, where $\tilde{D}_c$, $\tilde{N}_c \in \mathcal{m}(H)$, is a solution of equation (3.2.17) if and only if

$$\begin{bmatrix} \tilde{D}_c & \vdots & \tilde{N}_c \end{bmatrix} = \begin{bmatrix} I_{n_i} & \vdots & Q \end{bmatrix} \begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} =: \begin{bmatrix} I_{n_i} & \vdots & Q \end{bmatrix} \bar{M} \tag{3.2.32}$$

for some $Q \in \mathcal{m}(H)$; similarly, (N_c, D_c) , where N_c , $D_c \in \mathcal{m}(H)$, is a solution of equation (3.2.18) if and only if

$$\begin{bmatrix} -N_c \\ D_c \end{bmatrix} = \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} \begin{bmatrix} -Q \\ I_{n_o} \end{bmatrix} =: \bar{M}^{-1} \begin{bmatrix} -Q \\ I_{n_o} \end{bmatrix} \tag{3.2.33}$$

for some $Q \in \mathcal{m}(H)$. Now by Lemma 2.5.1, since $P \in \mathcal{m}(G_S)$ implies that $N_p = P D_p \in \mathcal{m}(G_S)$, for all $Q \in \mathcal{m}(H)$, $\det(V_p - Q\tilde{N}_p) \in I$ and $\det(\tilde{V}_p - N_p Q) \in I$.

Finally, if C is given by the expression in equation (3.2.27) or (3.2.28), then C satisfies Assumption 3.2.1 (B) and $S(P, C)$ is H-stable. Conversely, if $(\tilde{D}_c, \tilde{N}_c)$ is an l.c.f.r. and (N_c, D_c) is an r.c.f.r. of an H-stabilizing C for the plant P , then $(\tilde{D}_c, \tilde{N}_c)$ is given by equation

(3.2.32) and (N_c, D_c) is given by equation (3.2.33), for some $Q \in \mathcal{M}(H)$; hence, C satisfies the expressions in (3.2.27)-(3.2.28).

Now let $C_1 = \tilde{D}_{c1}^{-1}\tilde{N}_{c1}$, $C_2 = \tilde{D}_{c2}^{-1}\tilde{N}_{c2}$ be two H–stabilizing compensators; hence, C_1 and C_2 are given by the expression in (3.2.27). By equation (3.2.32)

$$\left[\tilde{D}_{c1} \;\vdots\; \tilde{N}_{c1}\right]\tilde{M}^{-1} = \left[I_{n_i} \;\vdots\; Q_1\right] = \tilde{D}_{c1}\left[I_{n_i} \;\vdots\; C_1\right]\tilde{M}^{-1}, \tag{3.2.34}$$

for some $Q_1 \in \mathcal{M}(H)$, and

$$\left[\tilde{D}_{c2} \;\vdots\; \tilde{N}_{c2}\right]\tilde{M}^{-1} = \left[I_{n_i} \;\vdots\; Q_2\right] = \tilde{D}_{c2}\left[I_{n_i} \;\vdots\; C_2\right]\tilde{M}^{-1}, \tag{3.2.35}$$

for some $Q_2 \in \mathcal{M}(H)$. But $C_1 = C_2$ in equations (3.2.34)-(3.2.35) if and only if $\left[I_{n_i} \;\vdots\; C_1\right]\tilde{M}^{-1} = \tilde{D}_{c1}^{-1}\left[I_{n_i} \;\vdots\; Q_1\right] = \tilde{D}_{c2}^{-1}\left[I_{n_i} \;\vdots\; Q_2\right]$; equivalently, $\tilde{D}_{c1} = \tilde{D}_{c2}$, and $\tilde{D}_{c1}^{-1}Q_1 = \tilde{D}_{c2}^{-1}Q_2$; hence, $Q_1 = Q_2$. This shows that, for each $C \in \mathbf{S}(P)$, there is a unique $Q \in \mathcal{M}(H)$ such that $C = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p)$.

Now suppose that C_1 has an l.c.f.r. $(\tilde{D}_{c1}, \tilde{N}_{c1})$ but C_2 is given by an r.c.f.r. (N_{c2}, D_{c2}) as in equation (3.2.28). So by equations (3.2.33) and (3.2.34),

$$\left[\tilde{D}_{c1} \;\vdots\; \tilde{N}_{c1}\right]\tilde{M}^{-1}\tilde{M}\begin{bmatrix} -N_{c2} \\ \cdots \\ D_{c2} \end{bmatrix} = \left[I_{n_i} \;\vdots\; Q_1\right]\begin{bmatrix} -Q_2 \\ \cdots \\ I_{n_o} \end{bmatrix}. \tag{3.2.36}$$

But $C_1 = C_2$ if and only if $\tilde{N}_{c1}D_{c2} = \tilde{D}_{c1}N_{c2}$; from equation (3.2.36), $C_1 = C_2$ if and only if

$$\left[-\tilde{D}_{c1}N_{c2} + \tilde{N}_{c1}D_{c2}\right] = Q_1 - Q_2 = 0 .$$

We conclude that $C_1 = (V_p - Q_1\tilde{N}_p)^{-1}(U_p + Q_1\tilde{D}_p)$ equals $C_2 = (V_p - Q_2\tilde{N}_p)^{-1}(U_p + Q_2\tilde{D}_p)$ if and only if $Q_1 = Q_2$.

□

3.2.10. Achievable I/O maps of $S(P, C)$:

The set

$$\mathrm{A}(P) := \{\ H_{\overline{yu}}\ :\ C\ \ H\text{–stabilizes}\ P\ \} \tag{3.2.37}$$

is called the set of *all achievable I/O maps* of the unity-feedback system $S(P, C)$.

By Theorem 3.2.8, the compensator C H–stabilizes P if and only if $C \in \mathrm{S}(P)$ given in equations (3.2.27)-(3.2.28). substituting $\tilde{D}_c^{-1}\tilde{N}_c = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p)$ or $N_c D_c^{-1} = (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1}$ for C into equation (3.2.2), we obtain the set of all achievable I/O maps in equation (3.2.38) below:

$$\mathrm{A}(P) = \left\{\ H_{\overline{yu}} = \begin{bmatrix} N_p(V_p - Q\tilde{N}_p) & N_p(U_p + Q\tilde{D}_p) \\ -(\tilde{U}_p + D_p Q)\tilde{N}_p & D_p(U_p + Q\tilde{D}_p) \end{bmatrix} : Q \in \mathcal{m}(H)\ \right\} . \tag{3.2.38}$$

Note that each closed-loop map of $S(P, C)$ is an affine map in the parameter matrix $Q \in \mathcal{m}(H)$.

Compensator design using the configuration $S(P, C)$ is called one-degree-of-freedom design [Hor.1] or one-parameter design [Vid.1] since all achievable maps are parametrized by the single parameter matrix Q.

3.3. The general feedback system $\Sigma(\hat{P}, \hat{C})$

In this section we consider the general feedback system $\Sigma(\hat{P}, \hat{C})$ shown in Figure 3.6; the $(\eta_o + n_o)\mathrm{x}(\eta_i + n_i)$ plant $\hat{P} \in \mathcal{M}(G)$ is partitioned as

$$\hat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P \end{bmatrix} \in G^{(\eta_o+n_o)\mathrm{x}(\eta_i+n_i)} , \quad \text{where } P \in G^{n_o \mathrm{x} n_i} ; \tag{3.3.1}$$

similarly, $\hat{C} \in G^{(\eta_o'+n_i)\mathrm{x}(\eta_i'+n_o)}$ is partitioned as

$$\hat{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C \end{bmatrix} \in G^{(\eta_o'+n_i)\mathrm{x}(\eta_i'+n_o)} , \quad \text{where } C \in G^{n_i \mathrm{x} n_o} . \tag{3.3.2}$$

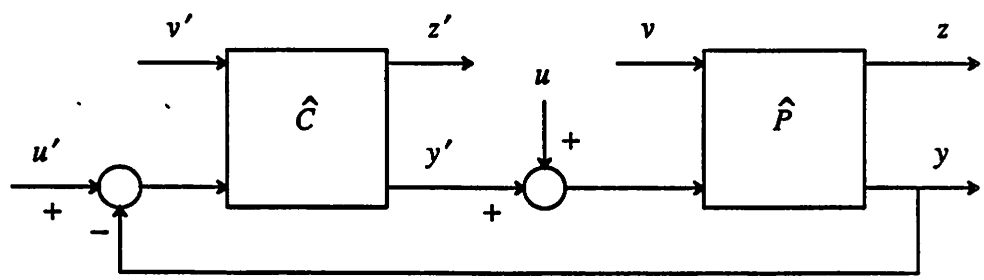

Figure 3.6. The feedback system $\Sigma(\hat{P}, \hat{C})$

3.3.1. Assumptions:

(A) The plant $\hat{P} \in G^{(\eta_o+n_o)\mathrm{x}(\eta_i+n_i)}$ is partitioned as in equation (3.3.1).

By Lemma 2.6.1, $\hat{P}$ has an r.c.f.r. $(N_{\hat{P}}, D_{\hat{P}})$ and an l.c.f.r. $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ which satisfy equations (3.3.3)-(3.3.4) below:

$$\text{(i)} \qquad (N_{\hat{P}}, D_{\hat{P}}) =: (\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_p \end{bmatrix} , \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_p \end{bmatrix}) , \tag{3.3.3}$$

$$\text{(ii)} \qquad (\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) =: (\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_p \end{bmatrix} , \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix}) , \tag{3.3.4}$$

where

(N_p, D_p) is an r.f.r. of P , and $(\tilde{D}_p, \tilde{N}_p)$ is an l.f.r. of P .

(B) The compensator $\hat{C} \in G^{(\eta_o'+n_i)\times(\eta_i'+n_o)}$ is partitioned as in equation (3.3.2).

By Lemma 2.6.1 applied to $\hat{C}$, the compensator $\hat{C}$ has an l.c.f.r. $(\tilde{D}_{\hat{C}}, \tilde{N}_{\hat{C}})$ and an r.c.f.r. $(N_{\hat{C}}, D_{\hat{C}})$ which satisfy equations (3.3.5)-(3.3.6) below:

$$(\tilde{D}_{\hat{C}}, \tilde{N}_{\hat{C}}) =: \left(\begin{bmatrix} \tilde{D}'_{11} & \tilde{D}'_{12} \\ 0 & \tilde{D}_c \end{bmatrix}, \begin{bmatrix} \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{21} & \tilde{N}_c \end{bmatrix} \right), \tag{3.3.5}$$

$$(N_{\hat{C}}, D_{\hat{C}}) =: \left(\begin{bmatrix} N'_{11} & N'_{12} \\ N'_{21} & N_c \end{bmatrix}, \begin{bmatrix} D'_{11} & 0 \\ D'_{21} & D_c \end{bmatrix} \right), \tag{3.3.6}$$

where

$(\tilde{D}_c, \tilde{N}_c)$ is an l.f.r. of C, and (N_c, D_c) is an r.f.r. of C.

□

By Lemma 2.3.3, any *other* r.c.f.r. of $\hat{P}$ is given by $(N_{\hat{P}}R, D_{\hat{P}}R)$, where $(N_{\hat{P}}, D_{\hat{P}})$ is the r.c.f.r in equation (3.3.3) and $R \in \mathcal{m}(H)$ is H–unimodular. Similarly, any other l.c.f.r. of $\hat{P}$ is given by $(L\tilde{D}_{\hat{P}}, L\tilde{N}_{\hat{P}})$, where $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ is the l.c.f.r. in equation (3.3.4) and $L \in \mathcal{m}(H)$ is H–unimodular. Note that the pair (N_p, D_p) in equation (3.3.3) is *not* necessarily r.c. and the pair $(\tilde{D}_p, \tilde{N}_p)$ in equation (3.3.4) is *not* necessarily l.c. Similar comments apply to coprime-fraction representations of $\hat{C}$.

3.3.2. Closed-loop I/O maps of $\Sigma(\hat{P}, \hat{C})$:

Consider Figure 3.6; let

$$\hat{y} := \begin{bmatrix} z \\ y \\ z' \\ y' \end{bmatrix}, \quad \hat{u} := \begin{bmatrix} v \\ u \\ v' \\ u' \end{bmatrix}.$$

The map $H_{\hat{y}\hat{u}} : \hat{u} \mapsto \hat{y}$ is called the I/O map of $\Sigma(\hat{P}, \hat{C})$. In terms of $\hat{P}$ and $\hat{C}$, $H_{\hat{y}\hat{u}}$ is given by

$$H_{\hat{y}\hat{u}} = \begin{bmatrix} P_{11} - P_{12}T^{-1}CP_{21} & P_{12}T^{-1} & P_{12}T^{-1}C_{21} & P_{12}T^{-1}C \\ (I_{n_o} - PT^{-1}C)P_{21} & PT^{-1} & PT^{-1}C_{21} & PT^{-1}C \\ -C_{12}(I_{n_o} - PT^{-1}C)P_{21} & -C_{12}PT^{-1} & C_{11} - C_{12}PT^{-1}C_{21} & C_{12}(I_{n_o} - PT^{-1}C) \\ -T^{-1}CP_{21} & T^{-1} - I_{n_i} & T^{-1}C_{21} & T^{-1}C \end{bmatrix}, \quad (3.3.7)$$

where $T := (I_{n_i} + CP)$.

3.3.3. Analysis of $\Sigma(\hat{P}, \hat{C})$:

We analyze the system $\Sigma(\hat{P}, \hat{C})$ shown in Figure 3.6 by factorizing $\hat{P}$ as $N_{\hat{p}}D_{\hat{p}}^{-1}$ and $\hat{C}$ as $\tilde{D}_{\hat{c}}^{-1}\tilde{N}_{\hat{c}}$ (see Figure 3.7); $\hat{\xi}_p$ denotes the pseudo-state of $\hat{P}$.

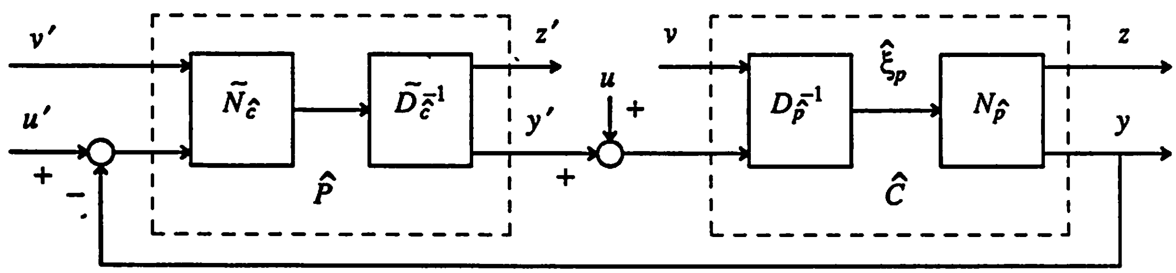

Figure 3.7. The system $\Sigma(\hat{P}, \hat{C})$ with $\hat{P} = N_{\hat{p}}D_{\hat{p}}^{-1}$ and $\hat{C} = \tilde{D}_{\hat{c}}^{-1}\tilde{N}_{\hat{c}}$

$\Sigma(\hat{P}, \hat{C})$ is then described by equations (3.3.8)-(3.3.9):

$$\begin{bmatrix} D_{11} & 0 & \vdots & 0 & 0 \\ D_{21} & D_p & \vdots & 0 & -I_{n_i} \\ \cdots & \cdots & & \cdots & \cdots \\ \tilde{N}'_{12}N_{21} & \tilde{N}'_{12}N_p & \vdots & \tilde{D}'_{11} & \tilde{D}'_{12} \\ \tilde{N}_c N_{21} & \tilde{N}_c N_p & \vdots & 0 & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \hat{\xi}_p \\ \cdots \\ z' \\ y' \end{bmatrix} = \begin{bmatrix} I_{\eta_i + n_i} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \tilde{N}_{\hat{c}} \end{bmatrix} \begin{bmatrix} v \\ u \\ \cdots \\ v' \\ u' \end{bmatrix}, \quad (3.3.8)$$

$$\begin{bmatrix} N_{\hat{p}} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{\eta_o' + n_i} \end{bmatrix} \begin{bmatrix} \hat{\xi}_p \\ \cdots \\ z' \\ y' \end{bmatrix} = \begin{bmatrix} z \\ y \\ \cdots \\ z' \\ y' \end{bmatrix}. \quad (3.3.9)$$

Equations (3.3.8)-(3.3.9) are of the form

$$\hat{D}_H \hat{\xi} = \hat{N}_L \hat{u}$$
$$\hat{N}_R \hat{\xi} = \hat{y} \ ;$$

by elementary row and column operations on the matrices in equations (3.3.8)-(3.3.9), using Lemma 2.3.2, it is easy to see that $(\hat{N}_R, \hat{D}_H, \hat{N}_L)$ is a b.c. triple, with $\hat{N}_R, \hat{D}_H, \hat{N}_L \in \mathcal{M}(H)$. If $\det\hat{D}_H \in I$, then the I/O map $H_{\hat{y}\hat{u}}$ is given by

$$H_{\hat{y}\hat{u}} = \hat{N}_R \hat{D}_H^{-1} \hat{N}_L \in \mathcal{M}(G).$$

The definition of H–stability for $\Sigma(\hat{P}, \hat{C})$ is analogous to the H–stability definition for $S(P, C)$:

3.3.4. Definition (H–stability):

The system $\Sigma(\hat{P}, \hat{C})$ is said to be *H–stable* iff $H_{\hat{y}\hat{u}} \in \mathcal{M}(H)$.

3.3.5. Theorem (H–stability of $\Sigma(\hat{P}, \hat{C})$):

Consider the system $\Sigma(\hat{P}, \hat{C})$ shown in Figure 3.6. Let Assumptions 3.3.1 (A) and (B) hold. Under these conditions, the following three conditions are equivalent:

(i) $\Sigma(\hat{P}, \hat{C})$ is *H–stable* ;

(ii) $\hat{D}_H$ is H–unimodular ; (3.3.10)

(iii) D_{11} is H–unimodular , and (3.3.11)

$\tilde{D}'_{11}$ is H–unimodular , and (3.3.12)

$\left[\tilde{D}_c D_p + \tilde{N}_c N_p \right]$ is H–unimodular. (3.3.13)

D_{11} and $\tilde{D}_{11}$ are defined in equations (3.3.3) and (3.3.4), respectively.

3.3.6. Comments:

(i) Condition (3.3.10) of Theorem 3.3.5 is equivalent to $\det\hat{D}_H \in J$; by equation (3.3.8),

$$\det\hat{D}_H = \det D_{11} \det\tilde{D}'_{11} \det(\tilde{D}_c D_p + \tilde{N}_c N_p). \quad (3.3.14)$$

By Lemma 2.3.4.(i), $\det\hat{D}_H \in J$ if and only if each of the three factors in equation (3.3.14) is in J ; hence, by equations (3.3.3) and (3.3.5), $\det\hat{D}_H \in J$ if and only if

$$\det D_{11} = \det D_{\hat{p}}(\det D_p)^{-1} \in J \quad (\text{equivalently}, \det D_{\hat{p}} \simeq \det D_p), \tag{3.3.15}$$

and

$$\det\tilde{D}'_{11} = \det\tilde{D}_{\hat{c}}(\det\tilde{D}_c)^{-1} \in J \quad (\text{equivalently}, \det\tilde{D}_{\hat{c}} \simeq \det\tilde{D}_c), \tag{3.3.16}$$

and

$$\det(\tilde{D}_c D_p + \tilde{N}_c N_p) \in J \quad (\text{equivalently}, \det(\tilde{D}_c D_p + \tilde{N}_c N_p) \simeq 1). \tag{3.3.17}$$

Due to equation (3.3.14), condition (3.3.10) of Theorem 3.3.5 is equivalent to conditions (3.3.11)-(3.3.12)-(3.3.13).

(ii) By normalization, conditions (3.3.11)-(3.3.12)-(3.3.13) of Theorem 3.3.5 can be written as:

$$D_{11} = I_{\eta_i} \text{ and} \tag{3.3.18}$$

$$\tilde{D}'_{11} = I_{\eta_o'} \text{ and} \tag{3.3.19}$$

$$\tilde{D}_c D_p + \tilde{N}_c N_p = I_{n_i}. \tag{3.3.20}$$

Equation (3.3.20) is in fact a right-Bezout identity for the r.c.f.r. (N_p, D_p) of P and a left-Bezout identity for the l.c.f.r. $(\tilde{D}_c, \tilde{N}_c)$ of C ; by equations (3.3.18)-(3.3.20), if $\Sigma(\hat{P}, \hat{C})$ is H-stable, then the r.c.f.r. $(N_{\hat{p}}, D_{\hat{p}})$ of $\hat{P}$ in equation (3.3.3) and the l.c.f.r. $(\tilde{D}_{\hat{c}}, \tilde{N}_{\hat{c}})$ of $\hat{C}$ in equation (3.3.6) can be written as:

$$(N_{\hat{p}}, D_{\hat{p}}) = \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_p \end{bmatrix}, \begin{bmatrix} I_{\eta_i} & 0 \\ D_{21} & D_p \end{bmatrix} \right), \tag{3.3.21}$$

where (N_p, D_p) is a right-*coprime*-fraction representation of P ,

$$(\tilde{D}_{\hat{c}}, \tilde{N}_{\hat{c}}) = \left(\begin{bmatrix} I_{\eta_o'} & \tilde{D}'_{12} \\ 0 & \tilde{D}_c \end{bmatrix}, \begin{bmatrix} \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{21} & \tilde{N}_c \end{bmatrix} \right), \tag{3.3.22}$$

where $(\tilde{D}_c, \tilde{N}_c)$ is a left-*coprime*-fraction representation of C .

(iii) From equation (3.3.14), using $\det(I_{n_i} + CP) = \det(I_{n_o} + PC)$, we can express $\det\hat{D}_H$ also as:

$$\det\hat{D}_H = \det D_{11} \det D_p \det\tilde{D}'_{11} \det\tilde{D}_c \det(I_{n_o} + PC). \tag{3.3.23}$$

Now by Corollary 2.4.4 applied to $\hat{P}$ and $\hat{C}$, using equations (3.3.3)-(3.3.6), we obtain

$$\det D_{\hat{P}} \simeq \det\tilde{D}_{\hat{P}} \quad (\text{equivalently}, \det D_{11}\det D_p \simeq \det\tilde{D}_{11}\det\tilde{D}_p) \tag{3.3.24}$$

and

$$\det\tilde{D}_{\hat{C}} \simeq \det D_{\hat{C}} \quad (\text{equivalently}, \det\tilde{D}'_{11}\det\tilde{D}_c \simeq \det D'_{11}\det D_c); \tag{3.3.25}$$

hence, substituting equations (3.3.24)-(3.3.24) into equation (3.3.23), we obtain

$$\det\hat{D}_H = \det\tilde{D}_{11}\det D'_{11}\det(\tilde{D}_p D_c + \tilde{N}_p N_c). \tag{3.3.26}$$

Therefore, if we analyze the system $\Sigma(\hat{P}, \hat{C})$ with $\hat{P}$ factorized as $\tilde{D}_{\hat{P}}^{-1}\tilde{N}_{\hat{P}}$ and $\hat{C}$ factorized as $N_{\hat{C}}D_{\hat{C}}^{-1}$, by normalization, condition (iii) of Theorem 3.3.5 is equivalent to

$$\tilde{D}_{11} = I_{\eta_o} \quad \text{and} \tag{3.3.27}$$

$$D'_{11} = I_{\eta_i'} \quad \text{and} \tag{3.3.28}$$

$$\tilde{D}_p D_c + \tilde{N}_p N_c = I_{n_o} . \tag{3.3.29}$$

As in equations (3.3.18)-(3.3.20) above, we conclude that if $\Sigma(\hat{P}, \hat{C})$ is H–stable, then the l.c.f.r. of $\hat{P}$ in equation (3.3.4) and the r.c.f.r. of $\hat{C}$ in equation (3.3.6) can be written as:

$$(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) = (\begin{bmatrix} I_{\eta_o} & \tilde{D}_{12} \\ 0 & \tilde{D}_p \end{bmatrix} , \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix}), \tag{3.3.30}$$

where $(\tilde{D}_p, \tilde{N}_p)$ is a left-*coprime*-fraction representation of P ,

$$(N_{\hat{C}}, D_{\hat{C}}) = (\begin{bmatrix} N'_{11} & N'_{12} \\ N'_{21} & N_c \end{bmatrix} , \begin{bmatrix} I_{\eta_i'} & 0 \\ D'_{21} & D_c \end{bmatrix}), \tag{3.3.31}$$

where (N_c, D_c) is a right-*coprime*-fraction representation of C .

(iv) If $\Sigma(\hat{P}, \hat{C})$ is H–stable, then $\hat{P}$ has an r.c.f.r. as shown in equation (3.3.21), and an l.c.f.r. as shown in equation (3.3.30); also $\hat{C}$ has an l.c.f.r. and an r.c.f.r. as shown in equations (3.3.22) and (3.3.31), respectively. Under these conditions,

$$D_{\hat{P}}^{-1} = \begin{bmatrix} I_{\eta_i} & 0 \\ -D_p^{-1}D_{21} & D_p^{-1} \end{bmatrix} , \quad (\tilde{D}_{\hat{P}}^{-1} = \begin{bmatrix} I_{\eta_o} & -\tilde{D}_{12}\tilde{D}_p^{-1} \\ 0 & \tilde{D}_p^{-1} \end{bmatrix}), \tag{3.3.32}$$

$$\tilde{D}_{\hat{c}}^{-1} = \begin{bmatrix} I_{\eta_o'} & -\tilde{D}'_{12}\tilde{D}_c^{-1} \\ 0 & \tilde{D}_c^{-1} \end{bmatrix}, \qquad (D_{\hat{c}}^{-1} = \begin{bmatrix} I_{\eta_i'} & 0 \\ -D_c^{-1}D'_{21} & D_c^{-1} \end{bmatrix}). \tag{3.3.33}$$

Conditions (3.3.11)-(3.3.12)-(3.3.13) can be interpreted as follows: $\Sigma(\hat{P}, \hat{C})$ is H–stabilized if and only if 1) the only source of "instability" in the plant $\hat{P}$ is D_p (equivalently, $\tilde{D}_p$) 2) *and* the only source of "instability" in the compensator $\hat{C}$ is $\tilde{D}_c$ (equivalently, D_c) 3) *and* the *feedback-loop* (with P and C) is H–stable. Note that the H–stability of the "feedback-loop" is equivalent to the H–stability of the unity-feedback system $S(P, C)$; indeed, equation (3.3.20) is identical to (3.2.12).

□

Proof of Theorem 3.3.5:

(i) <=> (ii)

The proof is similar to Theorem 3.2.5. Let $\Sigma(\hat{P}, \hat{C})$ be H–stable; then by Definition 3.3.4, $H_{\hat{y}\hat{u}} \in \mathcal{M}(H)$, and in particular, $H_{y'u} \in \mathcal{M}(H)$, where $H_{y'u}$ is given by equation (3.2.20). Since $\hat{P} = N_{\hat{p}}D_{\hat{p}}^{-1}$ and $\hat{C} = \tilde{D}_{\hat{c}}^{-1}\tilde{N}_{\hat{c}}$ implies that $D_{\hat{p}}^{-1} \in \mathcal{M}(G)$ and $\tilde{D}_{\hat{c}}^{-1} \in \mathcal{M}(G)$, by equations (3.2.20) and (3.3.23), $\hat{D}_H^{-1} \in \mathcal{M}(G)$ (equivalently, $\Sigma(\hat{P}, \hat{C})$ is well-posed); hence, $(\hat{N}_R, \hat{D}_H, \hat{N}_L)$ is a b.c.f.r. of $H_{\hat{y}\hat{u}}$. By Lemma 2.3.7, $H_{\hat{y}\hat{u}} \in \mathcal{M}(H)$ if and only if $\hat{D}_H^{-1} \in \mathcal{M}(H)$ and hence, the equivalence of conditions (i) and (ii) follows.

(ii) <=> (iii)

Following Comment 3.3.6.(i), equation (3.3.14) implies that $\det\hat{D}_H \in J$ if and only if $\det D_{11} \in J$, $\det\tilde{D}'_{11} \in J$ and $\det(\tilde{D}_c D_p + \tilde{N}_c N_p) \in J$.

□

Theorem 3.3.5 shows that not all plants $\hat{P}$ can be H–stabilized by some compensator $\hat{C}$ in the configuration $\Sigma(\hat{P}, \hat{C})$; the restriction on the class of plants is a consequence of the feedback being applied only from the output to the input of P . Plants which can be H–stabilized in the configuration $\Sigma(\hat{P}, \hat{C})$ are called Σ–admissible. Clearly, $\hat{C}$ must also be Σ–admissible to H–stabilize $\hat{P}$.

3.3.7. Definition (H–stabilizing compensator $\hat{C}$):

(i) $\hat{C}$ is called an *H–stabilizing compensator for $\hat{P}$* (later abbreviated as: $\hat{C}$ *H–stabilizes* $\hat{P}$) iff $\hat{C} \in G^{(\eta_o'+n_i)\times(\eta_i'+n_o)}$ satisfies Assumption 3.3.1 (B) *and* the system $\Sigma(\hat{P}, \hat{C})$ is H–stable.

(ii) The set

$$\hat{S}(\hat{P}) := \{ \ \hat{C} : \hat{C} \ H\text{–stabilizes} \ \hat{P} \ \} \tag{3.3.34}$$

is called the *set of all H–stabilizing compensators for $\hat{P}$* .

3.3.8. Definition (Σ–admissibility):

$\hat{P} \in \mathcal{M}(G)$ is called *Σ–admissible* iff $\hat{P}$ can be H–stabilized by some $\hat{C} \in \mathcal{M}(G)$.

Let $(N_{\hat{p}}, D_{\hat{p}})$ be an r.c.f.r. of $\hat{P}$; by Theorem 3.3.5, $\hat{P}$ is Σ–admissible if and only if two conditions are satisfied: 1) $\det D_{\hat{p}} \approx \det D_p$ and 2) (N_p, D_p) is a right-*coprime*-fraction representation of P . In terms of the l.c.f.r. $(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}})$ of $\hat{P}$, again by Theorem 3.3.5, $\hat{P}$ is Σ–admissible if and only if 1) $\tilde{D}_{\hat{p}} \approx \det\tilde{D}_p$ and 2) $(\tilde{D}_p, \tilde{N}_p)$ is a left-*coprime*-fraction representation of P . The necessity of these conditions follows from Definition 3.3.8 and Theorem 3.3.5; the sufficiency follows by observing that if $(N_{\hat{p}}, D_{\hat{p}})$ and $(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}})$ satisfy these conditions, then the system $\Sigma(\hat{P}, \hat{C})$ is made H–stable by choosing a $\hat{C}$ that satisfies equations (3.3.12) and (3.3.13).

We now parametrize the class of all Σ–admissible plants $\hat{P}$ and then we parametrize the class of all H–stabilizing compensators $\hat{C}$ for Σ–admissible $\hat{P}$.

3.3.9. Theorem (Class of Σ–admissible $\hat{P}$):

Let $\hat{P} \in \mathcal{M}(G)$ be partitioned as in equation (3.3.1); then $\hat{P}$ is Σ–admissible if and only if $\hat{P}$ has an r.c.f.r. in the form given by equation (3.3.35) and an l.c.f.r. given by equation (3.3.36) below:

$$(N_{\hat{p}}, D_{\hat{p}}) = \left(\begin{bmatrix} \hat{N}_{11} & N_{12} \\ \tilde{V}_p \tilde{N}_{21} & N_p \end{bmatrix}, \begin{bmatrix} I_{\eta_i} & 0 \\ -\tilde{U}_p \tilde{N}_{21} & D_p \end{bmatrix} \right), \tag{3.3.35}$$

$$(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}}) = \left(\begin{bmatrix} I_{\eta_o} & -N_{12} U_p \\ 0 & \tilde{D}_p \end{bmatrix}, \begin{bmatrix} \hat{N}_{11} & N_{12} V_p \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix} \right), \tag{3.3.36}$$

where (N_p, D_p) is an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ is an l.c.f.r. of P ; the pairs (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$, together with U_p , V_p , $\tilde{U}_p$, $\tilde{V}_p$, satisfy the generalized Bezout identity (2.3.7), and $\hat{N}_{11}$, N_{12} , $\tilde{N}_{21} \in \mathcal{M}(H)$ are free parameter matrices.

3.3.10. Comments:

(i) By Theorem 3.3.5, $\hat{P}$ is Σ–admissible if and only if an r.c.f.r. $(N_{\hat{p}}, D_{\hat{p}})$ of $\hat{P}$ satisfies equation (3.3.21) and an l.c.f.r. $(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}})$ of $\hat{P}$ satisfies equation (3.3.30). Another point of view is the following: suppose that (N_p, D_p) is an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ is an l.c.f.r. of P , and that the generalized Bezout identity (2.3.7) holds. From this information, we generate the class of all Σ–admissible plants by choosing three completely free matrices $\hat{N}_{11}$, N_{12} , $\tilde{N}_{21} \in \mathcal{M}(H)$ and forming the r.c. pair $(N_{\hat{p}}, D_{\hat{p}})$ in equation (3.3.35) or the l.c. pair $(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}})$ in equation (3.3.36); with this assignment of $(\tilde{D}_{\hat{p}}, \tilde{N}_{\hat{p}})$ and $(N_{\hat{p}}, D_{\hat{p}})$, $\hat{P} := N_{\hat{p}} D_{\hat{p}}^{-1} = \tilde{D}_{\hat{p}}^{-1} \tilde{N}_{\hat{p}}$ is a Σ–admissible plant. Note that $\det D_{\hat{p}} \in I$ (equivalently, $\det \tilde{D}_{\hat{p}} \in I$) follows from $\det D_p \in I$ (equivalently, $\det \tilde{D}_p \in I$).

(ii) Theorem 3.3.9 states that the class of all Σ–admissible plants is parametrized by only *three* free matrices $\hat{N}_{11}$, N_{12} , $\tilde{N}_{21} \in \mathcal{M}(H)$.

(iii) We can consider the following "easy check for Σ–admissibility of $\hat{P}$" as a corollary to Theorem 3.3.9: Suppose that we are given a $\hat{P} \in \mathcal{M}(G)$, partitioned as in equation (3.3.1), and that the coprime-fraction representations $N_p D_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$ of P satisfy the generalized Bezout identity (2.3.7); then $\hat{P}$ is Σ–admissible if and only if the three conditions in equation (3.3.37) below hold:

$$P_{11} - P_{12}D_p U_p P_{21} \in \mathcal{M}(H) \quad \text{and} \quad P_{12}D_p \in \mathcal{M}(H) \quad \text{and} \quad \tilde{D}_p P_{21} \in \mathcal{M}(H). \qquad (3:3.37)$$

We justify that Σ–admissibility of $\hat{P}$ is equivalent to equation (3.3.37) as follows: If $\hat{P}$ is Σ–admissible, then by Theorem 3.3.9, $P_{11} = \hat{N}_{11} + N_{12}D_p^{-1}\tilde{U}_p\tilde{N}_{21}$, $P_{12} = N_{12}D_p^{-1}$, $P_{21} = \tilde{D}_p^{-1}\tilde{N}_{21}$ and $P = N_p D_p^{-1}$; hence, equation (3.3.37) holds. To show the converse, choose any $\hat{C} \in \mathcal{M}(G)$ such that equations (3.3.12)-(3.3.13) hold; then the closed-loop I/O map in equation (3.3.7) is in H because equation (3.3.37) holds and hence, by Definition 3.3.8, $\hat{P}$ is Σ–admissible.

□

Proof of Theorem 3.3.9:

By Theorem 3.3.5, $\hat{P}$ is Σ–admissible if and only if the r.c.f.r. $(N_{\hat{P}}, D_{\hat{P}})$ and the l.c.f.r. $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ of $\hat{P}$, given in Assumption 3.3.1, satisfy equations (3.3.21) and (3.3.30), respectively. Now since $\hat{P} = N_{\hat{P}}D_{\hat{P}}^{-1} = \tilde{D}_{\hat{P}}^{-1}\tilde{N}_{\hat{P}}$, we obtain

$$\tilde{N}_{12}D_p + (-\tilde{D}_{12})N_p = N_{12}, \qquad (3.3.38)$$

$$-\tilde{N}_p D_{21} + \tilde{D}_p N_{21} = \tilde{N}_{21}, \qquad (3.3.39)$$

$$\tilde{N}_{12}D_{21} - \tilde{D}_{12}N_{21} = N_{11} - \tilde{N}_{11}. \qquad (3.3.40)$$

By Lemma 2.5.2.(i), $(\tilde{N}_{12}, \tilde{D}_{12})$ is a solution of equation (3.3.38) (with $\tilde{N}_{12}, \tilde{D}_{12} \in \mathcal{M}(H)$) if and only if

$$\left[\tilde{N}_{12} \ \vdots \ -\tilde{D}_{12} \right] = \left[N_{12} \ \vdots \ \hat{Q} \right] \begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix}, \qquad (3.3.41)$$

for some $\hat{Q} \in \mathcal{M}(H)$; similarly, by Lemma 2.5.2.(ii), (D_{21}, N_{21}) is a solution of equation

(3.3.39) (with $D_{21}, N_{21} \in \mathcal{M}(H)$) if and only if

$$\begin{bmatrix} D_{21} \\ N_{21} \end{bmatrix} = \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} \begin{bmatrix} -\tilde{Q} \\ \tilde{N}_{21} \end{bmatrix}, \tag{3.3.42}$$

for some $\tilde{Q} \in \mathcal{M}(H)$. Substituting equations (3.3.41)-(3.3.42) into (3.3.40) and using the generalized Bezout identity (2.3.7), we obtain

$$\left[\tilde{N}_{12} \ \vdots \ -\tilde{D}_{12} \right] \begin{bmatrix} D_{21} \\ N_{21} \end{bmatrix} = \left[N_{12} \ \vdots \ \hat{Q} \right] \begin{bmatrix} -\tilde{Q} \\ \tilde{N}_{21} \end{bmatrix} = N_{11} - \tilde{N}_{11}. \tag{3.3.43}$$

Using equations (3.3.41)-(3.3.42), the r.c.f.r. $(N_{\hat{P}}, D_{\hat{P}})$ and the l.c.f.r. $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ become:

$$(N_{\hat{P}}, D_{\hat{P}}) = \left(\begin{bmatrix} N_{11} & N_{12} \\ \tilde{V}_p \tilde{N}_{21} - N_p \tilde{Q} & N_p \end{bmatrix}, \begin{bmatrix} I_{n_i} & 0 \\ -\tilde{U}_p \tilde{N}_{21} - D_p \tilde{Q} & D_p \end{bmatrix} \right), \tag{3.3.44}$$

$$(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) = \left(\begin{bmatrix} I_{n_o} & -N_{12} U_p - \hat{Q} \tilde{D}_p \\ 0 & \tilde{D}_p \end{bmatrix}, \begin{bmatrix} \tilde{N}_{11} & N_{12} V_p - \hat{Q} \tilde{N}_p \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix} \right). \tag{3.3.45}$$

Let $R := \begin{bmatrix} I_{n_i} & 0 \\ \tilde{Q} & I_{n_i} \end{bmatrix} \in \mathcal{M}(H)$, let $L := \begin{bmatrix} I_{n_o} & \hat{Q} \\ 0 & I_{n_o} \end{bmatrix} \in \mathcal{M}(H)$. Using equation (3.3.43), let

$$\hat{N}_{11} := N_{11} + N_{12} \tilde{Q} = \tilde{N}_{11} + \hat{Q} \tilde{N}_{21}. \tag{3.3.46}$$

Since R and L are H–unimodular, by Lemma 2.3.3, $(N_{\hat{P}} R, D_{\hat{P}} R)$, which is the same as the r.c.f.r. given in (3.3.35), is also an r.c.f.r. and $(L \tilde{D}_{\hat{P}}, L \tilde{N}_{\hat{P}})$, which is the same as the l.c.f.r. given in (3.3.36), is also an l.c.f.r. of $\hat{P}$; we conclude that $\hat{P}$ is Σ–admissible if and only if an r.c.f.r. and an l.c.f.r. of $\hat{P}$ are given by equations (3.3.35) and (3.3.36), respectively.

□

3.3.11. Theorem (Set of all H–stabilizing compensators $\hat{C}$ for $\hat{P}$):

Let $\hat{P} \in \mathcal{M}(G)$ be Σ–admissible with $P \in \mathcal{M}(G_S)$; let (N_p, D_p) be an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of P , and let the generalized Bezout identity (2.3.7) hold. Under these conditions, the set $\hat{\mathrm{S}}(\hat{P})$ of all H–stabilizing compensators $\hat{C}$ for $\hat{P}$ is given by equation (3.3.47) and equivalently, by equation (3.3.48) below:

$$\hat{\mathrm{S}}(\hat{P}) = \{ \ \hat{C} = \begin{bmatrix} I_{\eta_o'} & -Q_{12}\tilde{N}_p \\ 0 & V_p - Q\tilde{N}_p \end{bmatrix}^{-1} \begin{bmatrix} Q_{11} & Q_{12}\tilde{D}_p \\ Q_{21} & U_p + Q\tilde{D}_p \end{bmatrix} :$$

$$Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H) \ \} , \tag{3.3.47}$$

$$\hat{\mathrm{S}}(\hat{P}) = \{ \ \hat{C} = \begin{bmatrix} Q_{11} & Q_{12} \\ D_pQ_{21} & \tilde{U}_p + D_pQ \end{bmatrix} \begin{bmatrix} I_{\eta_i'} & 0 \\ -N_pQ_{21} & \tilde{V}_p - N_pQ \end{bmatrix}^{-1} :$$

$$Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H) \ \} , \tag{3.3.48}$$

Equations (3.3.47) and (3.3.48) give a parametrization of all H–stabilizing compensators for $\hat{P}$; each of these equations defines a bijection from $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$ to $\hat{C} \in \hat{\mathrm{S}}(\hat{P})$. For the same $(Q_{11}, Q_{12}, Q_{21}, Q)$, equations (3.3.47)-(3.3.48) give the same $\hat{C} \in \hat{\mathrm{S}}(\hat{P})$.

Proof:

By assumption, $\hat{P}$ is Σ–admissible; hence, by Theorem 3.3.5, $\Sigma(\hat{P}, \hat{C})$ is H–stable if and only if equations (3.3.12) and (3.3.13) hold. From Comments 3.3.6.(i) and (ii), $\hat{C}$ H–stabilizes $\hat{P}$ if and only if an l.c.f.r. $(\tilde{D}_{\hat{c}}, \tilde{N}_{\hat{c}})$ of $\hat{C}$ satisfies equation (3.3.22), an r.c.f.r. $(N_{\hat{c}}, D_{\hat{c}})$ of $\hat{C}$ satisfies (3.3.31), where C is such that $(\tilde{D}_c, \tilde{N}_c)$ and (N_c, D_c) satisfy the generalized Bezout identity (3.2.19).

By Theorem 3.2.5, the set of all C which satisfies equation (3.3.20) (and equivalently, (3.3.29)) is given by the set $\mathbf{S}(P)$ in equation (3.2.27) (and equivalently, (3.2.28)). Substituting equations (3.3.22), (3.3.31) into $\tilde{N}_{\hat{C}}D_{\hat{C}} = \tilde{D}_{\hat{C}}N_{\hat{C}}$ and using equations (3.3.27)-(3.3.28), we obtain

$$\tilde{N}'_{12}D_c + (-\tilde{D}'_{12})N_c = N'_{12}, \tag{3.3.49}$$

$$-\tilde{N}_c D'_{21} + \tilde{D}_c N'_{21} = \tilde{N}'_{21}, \tag{3.3.50}$$

$$\tilde{N}'_{12}D'_{21} - \tilde{D}'_{12}N'_{21} = N'_{11} - \tilde{N}'_{11}. \tag{3.3.51}$$

By Lemma 2.5.2, using the generalized Bezout identity (3.2.19), $(\tilde{N}'_{12}, \tilde{D}'_{12})$ is a solution of equation (3.3.49)

$$\left[\tilde{N}'_{12} \;\vdots\; -\tilde{D}'_{12} \right] = \left[N'_{12} \;\vdots\; \hat{Q}' \right] \begin{bmatrix} \tilde{D}_p & \tilde{N}_p \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix}, \tag{3.3.52}$$

and (D'_{21}, N'_{21}) is a solution of equation (3.3.50) if and only if

$$\begin{bmatrix} D'_{21} \\ N'_{21} \end{bmatrix} = \begin{bmatrix} D_c & -N_p \\ N_c & D_p \end{bmatrix} \begin{bmatrix} -\tilde{Q}' \\ \tilde{N}'_{21} \end{bmatrix}, \tag{3.3.53}$$

for some $\hat{Q}' \in \mathcal{M}(H)$ and $\tilde{Q}' \in \mathcal{M}(H)$. Substituting equations (3.3.52)-(3.3.53) into (3.3.22) and (3.3.31), we obtain

$$(\tilde{D}_{\hat{C}}, \tilde{N}_{\hat{C}}) = \left(\begin{bmatrix} I_{n_o'} & -N'_{12}\tilde{N}_p - \hat{Q}'\tilde{D}_c \\ 0 & \tilde{D}_c \end{bmatrix}, \begin{bmatrix} \tilde{N}'_{11} & N'_{12}\tilde{D}_p - \hat{Q}'\tilde{N}_c \\ \tilde{N}'_{21} & \tilde{N}_c \end{bmatrix} \right), \tag{3.3.54}$$

$$(N_{\hat{C}}, D_{\hat{C}}) = \left(\begin{bmatrix} N'_{11} & N'_{12} \\ -N_c\tilde{Q}' + D_p\tilde{N}'_{21} & N_c \end{bmatrix}, \begin{bmatrix} I_{n_i'} & 0 \\ D_c\tilde{Q}' - N_p\tilde{N}'_{21} & D_c \end{bmatrix} \right), \tag{3.3.55}$$

where $(\tilde{D}_c, \tilde{N}_c)$ and (N_c, D_c) are as in equations (3.3.27)-(3.3.28).

Let $L' := \begin{bmatrix} I_{n_o'} & \hat{Q}' \\ 0 & I_{n_i} \end{bmatrix}$, $R' := \begin{bmatrix} I_{n_i'} & 0 \\ \tilde{Q}' & I_{n_o} \end{bmatrix}$; then $(L'\tilde{D}_{\hat{C}}, L'\tilde{N}_{\hat{C}})$ is also an l.c.f.r. and $(N_{\hat{C}}R', D_{\hat{C}}R')$ is also an r.c.f.r. of $\hat{C}$.

Now let $Q_{11} := \tilde{N}'_{11} + \hat{Q}'\tilde{N}'_{21}$; by equation (3.3.51), $\tilde{N}'_{11} + \hat{Q}'\tilde{N}'_{21} = N'_{11} + N'_{12}\tilde{Q}' = Q_{11}$. Let $Q_{12} := N'_{12}$, $Q_{21} := \tilde{N}'_{21}$. Finally, $\hat{C}$ H–stabilizes $\hat{P}$ if and only if $\hat{C}$ has an l.c.f.r. and an r.c.f.r. as in equations (3.3.47) and (3.3.48).

Now let $\hat{C}_1$ and $\hat{C}_2$ be two compensators in $\hat{\mathrm{S}}(\hat{P})$ given by equation (3.3.47); then

$$\hat{C}_1 = \begin{bmatrix} Q_{11} + Q_{12}\tilde{N}_p\tilde{D}_{c1}^{-1} & Q_{12}(\tilde{D}_p + \tilde{N}_p C_1) \\ \tilde{D}_{c1}^{-1}Q_{21} & C_1 \end{bmatrix} \text{ and } \hat{C}_2 = \begin{bmatrix} \hat{Q}_{11} + \hat{Q}_{12}\tilde{N}_p\tilde{D}_{c2}^{-1} & \hat{Q}_{12}(\tilde{D}_p + \tilde{N}_p C_2) \\ \tilde{D}_{c2}^{-1}\hat{Q}_{21} & C_2 \end{bmatrix},$$

where $C_1 = (V_p - Q_1\tilde{N}_p)^{-1}(U_p + Q_1\tilde{D}_p)$, $C_2 = (V_p - Q_2\tilde{N}_p)^{-1}(U_p + Q_2\tilde{D}_p)$. It was shown in the proof of Theorem 3.2.8 that $C_1 = C_2$ if and only if $Q_1 = Q_2$ and $\tilde{D}_{c1} = \tilde{D}_{c2}$; hence, $\hat{C}_1 = \hat{C}_2$ if and only if $Q_{21} = \hat{Q}_{21}$, and $Q_{12}(\tilde{D}_p + \tilde{N}_p N_c D_c^{-1}) = \hat{Q}_{12}(\tilde{D}_p + \tilde{N}_p N_c D_c^{-1})$; thus, using equation (3.3.29), $Q_{12} = \hat{Q}_{12}$ and hence, $Q_{11} = \hat{Q}_{11}$. Therefore, there is a unique set of parameters Q_{11} , Q_{12} , Q_{21} , Q for each H–stabilizing $\hat{C}$. Using $\tilde{D}_{\hat{c}}N_{\hat{c}} = \tilde{N}_{\hat{c}}D_{\hat{c}}$, a similar argument as in the proof of Theorem 3.2.8 shows that, for the same $\hat{C}$, the parameter matrices in (3.3.47) are the same as the ones in (3.3.48).

□

3.3.12. Comments:

(i) Consider Figure 3.8, which shows $\Sigma(\hat{P}, \hat{C})$ where $\hat{P}$ is Σ–admissible; this figure is obtained by taking $\hat{P} = N_{\hat{p}}D_{\hat{p}}^{-1}$, where $(N_{\hat{p}}, D_{\hat{p}})$ is given by equation (3.3.35), and by taking

$$\hat{C} = \begin{bmatrix} I_{\eta_o'} & -Q_{12}\tilde{N}_p \\ 0 & \tilde{D}_c \end{bmatrix}^{-1} \begin{bmatrix} Q_{11} & Q_{12}\tilde{D}_p \\ Q_{21} & \tilde{N}_c \end{bmatrix};$$

note that the only instabilities in $\hat{P}$ and $\hat{C}$ are due to D_p and $\tilde{D}_c$, respectively. In Figure 3.8, $\hat{C}$ and $\hat{P}$ already satisfy equations (3.3.11) and (3.3.12); hence, $\Sigma(\hat{P}, \hat{C})$ is H–stable if and only if the "feedback-loop" is H–stable.

If the ring H is the ring of proper stable rational functions $R_{\mathcal{U}}(s)$ as in Example 2.2.2, then the Σ–admissibility of $\hat{P}$ implies that every $\mathcal{U}$–pole of P_{11} , P_{12} , P_{21} is a $\mathcal{U}$–pole of P =

$N_p D_p^{-1}$, with *at most* the same McMillan degree [Vid.1, Net.1]. Similarly, for $\hat{C}$ to be an H–stabilizing compensator for $\hat{P}$, the $\mathcal{U}$ –poles of C_{11} , C_{12} , C_{21} must be "contained" in the $\mathcal{U}$ –poles of $C = \tilde{D}_c^{-1}\tilde{N}_c$, and C must be chosen so that the feedback-loop is H–stable.

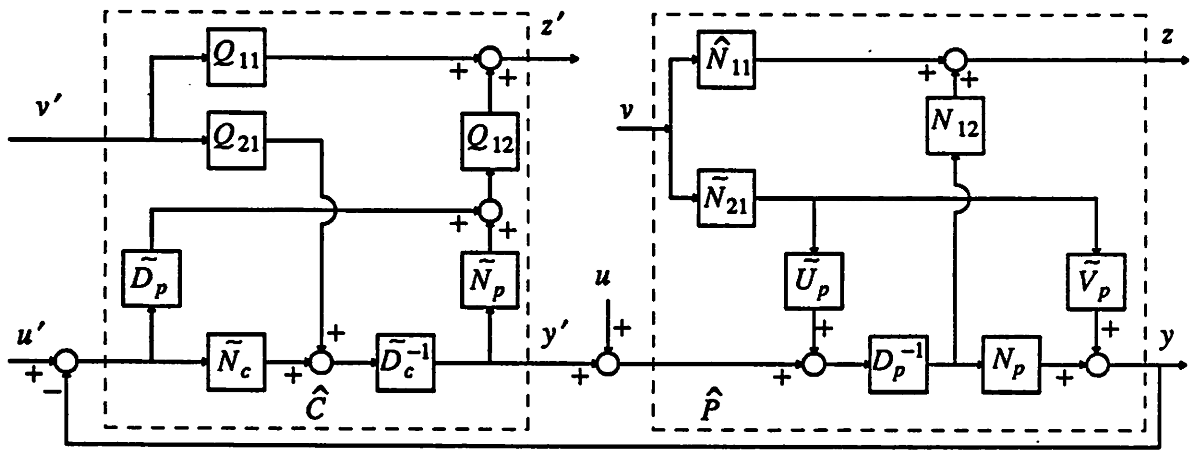

Figure 3.8. The system $\Sigma(\hat{P}, \hat{C})$ with a Σ–admissible plant $\hat{P} = N_{\hat{p}} D_{\hat{p}}^{-1}$;

note the duality between $\hat{C} = \tilde{D}_{\hat{c}}^{-1}\tilde{N}_{\hat{c}}$ and $\hat{P}$.

(ii) The class of all H–stabilizing compensators is *parametrized* by four matrices, Q_{11} , Q_{12} , Q_{21}, $Q \in \mathfrak{m}(H)$; the matrix Q parametrizes the class of all C that H–stabilizes the loop $S(P, C)$. We refer to design with the unity-feedback system $S(P, C)$ as *one-degree-of-freedom* design [Hor.1] because only one parameter matrix is available for design (see Section 3.2). In contrast, for the more general system $\Sigma(\hat{P}, \hat{C})$, there are *four-degrees-of-freedom* because $\hat{C}$ has four completely free matrices in H , which can be chosen to meet performance specifications. For example, in Section 3.4, we use the parameter Q_{21} to diagonalize the input-output map $H_{zv'} : v' \mapsto z$.

3.3.13. Achievable I/O maps of $\Sigma(\hat{P}, \hat{C})$:

The set

$$\hat{A}(\hat{P}) := \{ H_{\hat{y}\hat{u}} : \hat{C} \ H\text{-stabilizes } \hat{P} \} \tag{3.3.56}$$

is called the set of *all achievable I/O maps* of the system $\Sigma(\hat{P}, \hat{C})$.

Substituting for $\hat{C}$ from the expression in equations (3.3.47) and (3.3.48) into the closed-loop I/O map $H_{\hat{y}\hat{u}}$ in equation (3.3.7), we obtain the set of all achievable I/O maps for $\Sigma(\hat{P}, \hat{C})$:

$$\hat{A}(\hat{P}) = \{ H_{\hat{y}\hat{u}} = \begin{bmatrix} \hat{N}_{11} - N_{12}Q\tilde{N}_{21} & \vdots & N_{12}(V_p - Q\tilde{N}_p) & \vdots & N_{12}Q_{21} & \vdots & N_{12}(U_p + Q\tilde{D}_p) \\ (\tilde{V}_p - N_p Q)\tilde{N}_{21} & \vdots & N_p(V_p - Q\tilde{N}_p) & \vdots & N_p Q_{21} & \vdots & N_p(U_p + Q\tilde{D}_p) \\ -Q_{12}\tilde{N}_{21} & \vdots & -Q_{12}\tilde{N}_p & \vdots & Q_{11} & \vdots & Q_{12}\tilde{D}_p \\ -(\tilde{U}_p + D_p Q)\tilde{N}_{21} & \vdots & -(\tilde{U}_p + D_p Q)\tilde{N}_p & \vdots & D_p Q_{21} & \vdots & D_p(U_p + Q\tilde{D}_p) \end{bmatrix}$$

$$: Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H) \} . \tag{3.3.58}$$

Inspection of equation (3.3.58) shows that each closed-loop map achieved by $\Sigma(\hat{P}, \hat{C})$ depends on only one of *four* free *parameters* $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$; in fact, each of these maps is an affine function of *one* parameter only.

If $P_{11} = 0$ and $P_{21} = I_{n_o}$, then v can be viewed as an additive disturbance at the output y ; the disturbance-to-output map $H_{yv} : v \mapsto y$ is given by $(\tilde{V}_p - N_p Q)\tilde{N}_{21} = (\tilde{V}_p - N_p Q)\tilde{D}_p$, which depends on the parameter $Q \in \mathcal{M}(H)$. On the other hand, the external-input to output maps $H_{zv'} = N_{12}Q_{21}$ and $H_{yv'} = N_p Q_{21}$ depend on a *different* parameter Q_{21} . Consequently, output shaping and disturbance rejection can be achieved simultaneously, since $H_{zv'}$ and $H_{yv'}$ are *decoupled* from H_{yv} .

3.4. Achievable diagonal maps

We now consider the problem of achieving a diagonal I/O map for a Σ–admissible plant $\hat{P}$; more precisely, we require the closed-loop map $H_{zv'} : v' \mapsto z$ from the external-input v' to the output z of the $\mathcal{H}$–stabilized $\Sigma(\hat{P}, \hat{C})$ to be diagonal. We obtain the class of all achievable diagonal maps $H_{zv'}$.

Suppose that $\hat{P} \in \mathcal{m}(\mathcal{G})$., partitioned as in equation (3.3.1), is a Σ–admissible plant. We assume that and $\eta_i' = n_i = \eta_o$; consequently, $P_{12} \in \mathcal{G}^{n_i \times n_i}$ is square since there are n_i inputs v' and n_i outputs z. Furthermore, we assume that $N_{12} \in \mathcal{H}^{n_i \times n_i}$ is nonsingular (i.e., $\det N_{12} \not\equiv 0$).

We define two diagonal (nonsingular) matrices Δ_L and Δ_R as follows:

(i) Let $\Delta_{Lk} \in \mathcal{H}$ be a greatest-common-divisor (g.c.d.) of the elements of the k-th row of N_{12}. For $k = 1, \cdots, n_i$, $\Delta_{Lk} \in \mathcal{H}$ exists since $\mathcal{H}$ is a principal ring [Lan.1]. We define Δ_L and $\hat{N}_{12}$ by the following equations:

$$\Delta_L := diag\ [\Delta_{L1}, \cdots, \Delta_{Ln_i}], \tag{3.4.1}$$

$$N_{12} =: \Delta_L \hat{N}_{12}. \tag{3.4.2}$$

By construction, $\det\Delta_L \not\equiv 0$. Note that the diagonal elements Δ_{Lk} of Δ_L are unique except for factors in $\mathcal{J}$.

(ii) By assumption, $\det N_{12} = \det\Delta_L \det\hat{N}_{12} \not\equiv 0$; hence, $\det\hat{N}_{12} \not\equiv 0$. Write the ij-th entry of $\hat{N}_{12}^{-1}$ as $\dfrac{m_{ij}}{d_{ij}}$, where (m_{ij}, d_{ij}) is a coprime pair in $\mathcal{H}$; note that $d_{ij} \not\equiv 0$ since the denominator of each entry is a factor of $\det\hat{N}_{12}$ (i.e., $\det\hat{N}_{12} = d_{ij} a_{ij}$ for some $a_{ij} \in \mathcal{H}$).

Let $\Delta_{Rj} \in \mathcal{H}$ be a least-common-multiple (l.c.m.) of $\{ d_{1j}, \cdots, d_{n_i j} \}$ (i.e., a l.c.m. of the denominators of the elements in the j-th column of $\hat{N}_{12}^{-1}$). For $j = 1, \cdots, n_i$, Δ_{Rj} exists since H is a principal ring. Let

$$\Delta_R := diag\ [\ \Delta_{R1}, \cdots, \Delta_{Rn_i}\ ]\ ; \tag{3.4.3}$$

$\det\Delta_R \not\equiv 0$ since $d_{ij} \not\equiv 0$. The entries Δ_{Rj} of Δ_R are unique except for factors in J. Note that if $\hat{N}_{12}^{-1} \in \mathcal{m}(H)$, then $\Delta_R = I_{n_i}$.

Now for some $b_{ij} \in H$, $\Delta_{Rj} = d_{ij}b_{ij}$; therefore the ij-th element of $\hat{N}_{12}^{-1}\Delta_R$ is $\frac{m_{ij}}{d_{ij}}\Delta_{Rj}$ $= m_{ij}b_{ij} \in H$, and hence,

$$\hat{N}_{12}^{-1}\Delta_R \in \mathcal{m}(H). \tag{3.4.4}$$

Intuitively, if H is $R_u(s)$ as in Example 2.2.2, then we can interpret the diagonal matrices Δ_L and Δ_R as follows: Δ_{Lk} extracts the $\bar{\mathcal{U}}$–zeros that are common to all elements in the k-th row of N_{12} ; Δ_L "book-keeps" the $\bar{\mathcal{U}}$–zeros of $P_{12} = N_{12}D_p^{-1}$ that appear in each entry of some row of N_{12}. Clearly, P_{12} may have other $\bar{\mathcal{U}}$–zeros that cannot be extracted by Δ_L ; these $\bar{\mathcal{U}}$ –zeros are the $\bar{\mathcal{U}}$–zeros of $\det\hat{N}_{12}$ (equivalently, the $\bar{\mathcal{U}}$–poles of $\hat{N}_{12}^{-1}$). Now the diagonal matrix Δ_R makes $\hat{N}_{12}^{-1}\Delta_R$ H–stable, i.e., cancels these $\bar{\mathcal{U}}$–poles. Let $s \in \bar{\mathcal{U}}$ be a zero of Δ_R (hence a $\bar{\mathcal{U}}$–zero of $\det\hat{N}_{12}$); the multiplicity of $s \in \bar{\mathcal{U}}$ in $\det\Delta_R$ may exceed its multiplicity in $\det\hat{N}_{12}$. If $\det\hat{N}_{12} \in H^{n_i \times n_i}$ has n zeros at $s \in \bar{\mathcal{U}}$, then $\det\Delta_R$ has *at most* n^{n_i} zeros at $s \in \bar{\mathcal{U}}$; so Δ_R has at most as many $\bar{\mathcal{U}}$–zeros as $(\det\hat{N}_{12})I_{n_i}$.

3.4.1. Definition (Achievable diagonal $H_{zv'}$):

The set

$$\hat{A}_{zv'}(\hat{P}) := \{\ H_{zv'} : \hat{C}\ \textit{H–stabilizes}\ \hat{P}\ \text{and the map}\ H_{zv'}\ \text{is}\ \textit{diagonal}\ \text{and}\ \textit{nonsingular}\ \} \tag{3.4.5}$$

is called the set of *all achievable diagonal nonsingular maps* $H_{zv'} : v' \mapsto z$.

Clearly, $\hat{A}_{zv'}(\hat{P})$ is a subset of the achievable $v' \mapsto z$ map in $\hat{A}(\hat{P})$, because $\hat{C}$ must be a H–stabilizing compensator; in other words, $\hat{A}_{zv'}(\hat{P})$ is the set of all $N_{12}Q_{21} \in \mathcal{m}(H)$ that are *diagonal* and *nonsingular*. Thus we must choose the parameter $Q_{21} \in \mathcal{M}(H)$ so that $N_{12}Q_{21}$ is diagonal and nonsingular (see equation (3.3.58)). The "minimal" restriction on Q_{21} to achieve diagonal $H_{zv'}$ is given in Theorem 3.4.2 below:

3.4.2. Theorem (Class of all achievable diagonal $H_{zv'}$):

Let $\hat{P} \in \mathcal{M}(G)$ be Σ–admissible, and let $P \in \mathcal{M}(G_S)$; let $N_{12} \in H^{n_i x n_i}$ be nonsingular. Under these conditions,

$$\hat{A}_{zv'}(\hat{P}) = \{ \Delta_L \Delta_R \hat{Q}_{21} : \hat{Q}_{21} \in \mathcal{M}(H) \text{ is diagonal and nonsingular} \} , \qquad (3.4.6)$$

where Δ_L and Δ_R are the diagonal, nonsingular matrices defined by equations (3.4.1) and (3.4.3).

3.4.3. Comments:

(i) The map $H_{zv'} = \Delta_L \Delta_R \hat{Q}_{21}$ (where $\hat{Q}_{21} \in \mathcal{M}(H)$) is an achievable map of $\Sigma(\hat{P}, \hat{C})$ if and only if the compensator parameter Q_{21} is chosen as

$$Q_{21} = \hat{N}_{21}^{-1} \Delta_R \hat{Q}_{21} ; \qquad (3.4.7)$$

where $\hat{Q}_{21} \in H^{n_i x n_i}$ is diagonal and nonsingular. By equation (3.4.4), $Q_{21} \in \mathcal{M}(H)$. Therefore, to achieve diagonalization, from the set $\hat{S}(\hat{P})$ of all H–stabilizing compensators $\hat{C}$, we must choose $C_{21} = \tilde{D}_c^{-1} Q_{21} = (V_p - Q\tilde{N}_p)^{-1} Q_{21}$ as

$$C_{21} = (V_p - Q\tilde{N}_p)^{-1} \hat{N}_{12}^{-1} \Delta_R \hat{Q}_{21} , \qquad (3.4.8)$$

where the matrix $\hat{Q}_{21} \in H^{n_i x n_i}$ is diagonal and nonsingular. Note that in equation (3.4.8), $Q \in H^{n_i x n_o}$ is a free parameter and is *not* used in diagonalizing the I/O map $H_{zv'}$. The other compensator parameters Q_{11} and Q_{12} shown in equations (3.3.47)-(3.3.48) are not used in diagonalizing the map $H_{zv'}$ either.

(ii) If H is $R_u(s)$ as in Example 2.2.2, then the "cost" of diagonalizing the map $H_{zv'}$ is that the number of $\bar{\mathcal{U}}$–zeros are increased. Since Δ_L is a factor of N_{12} , $H_{zv'}$ must have zeros at the $\bar{\mathcal{U}}$ –zeros of Δ_L ; the multiplicity of a $\bar{\mathcal{U}}$–zero of $H_{zv'}$ may be larger than its multiplicity in $\det N_{12}$ due to Δ_R . If Δ_L represents all $\bar{\mathcal{U}}$–zeros of P_{12} (equivalently, if $\hat{N}_{12}^{-1} \in \mathcal{m}(H)$) and if $\hat{Q}_{21}$ is chosen so that it has no $\bar{\mathcal{U}}$–zeros, then the $\bar{\mathcal{U}}$–zeros of the diagonal $H_{zv'}$ have the same multiplicity as in $\det N_{12}$ since $\Delta_R = I_{n_i}$.

Note that the parameter Q_{21} is now restricted to be $\hat{N}_{12}^{-1}\Delta_R \hat{Q}_{21}$ and hence, can no longer be assigned arbitrarily in order to meet other design specifications; the only freedom left is the diagonal nonsingular matrix $\hat{Q}_{21} \in \mathcal{m}(H)$.

(iii) Although we chose to diagonalize the map $H_{zv'}$, we could also choose to diagonalize $H_{yv'} : v' \mapsto y$, the map from the same external-input v' to the output y of $\hat{P}$ (y is the output used in the feedback-loop). In that case, assuming that $n_o = n_i$ and that $N_p \in H^{n_i \times n_i}$ is nonsingular, we would define Δ_{Rp} , Δ_{Lp} , $\hat{N}_p$ from N_p as we did above to obtain Δ_L , Δ_R and $\hat{N}_{12}$ from N_{12}; the set of all achievable nonsingular maps $H_{yv'}$ would then be $\hat{A}_{yv'}(\hat{P})$, where

$$\hat{A}_{yv'}(\hat{P}) = \{ \Delta_{Lp}\Delta_{Rp}\hat{Q}_{21} : \hat{Q}_{21} \in \mathcal{m}(H) \text{ is diagonal and nonsingular } \} .$$

The compensator parameter Q_{21} would then be chosen as

$$\hat{N}_p^{-1}\Delta_{Rp}\hat{Q}_{21} .$$

□

Proof of Theorem 3.4.2:

The map $H_{zv'}$ is an achievable map of $\Sigma(\hat{P},\hat{C})$ if and only if $H_{zv'} = N_{12}Q_{21}$ for some $Q_{21} \in \mathcal{m}(H)$. By equation (3.4.1),

$$H_{zv'} = N_{12}Q_{21} = \Delta_L \hat{N}_{21} Q_{21} \qquad (3.4.9)$$

for some $Q_{21} \in \mathcal{m}(H)$. Now $H_{zv'} \in \mathcal{m}(H)$ is diagonal and nonsingular if and only if

$Q_{21} \in \mathcal{M}(H)$ is such that $\Delta_L \hat{N}_{12} Q_{21}$ is diagonal and nonsingular. Choose Q_{21} as in equation (3.4.7); then by equation (3.4.4), $Q_{21} \in \mathcal{M}(H)$. Clearly, $H_{zv'} = \Delta_L \Delta_R \hat{Q}_{21}$ is an achievable diagonal nonsingular map.

Now if $H_{zv'}$ is a given diagonal map achieved by $\Sigma(\hat{P}, \hat{C})$, then by equation (3.4.7), Δ_L is clearly a factor of $H_{zv'}$. Now suppose, for a contradiction, that

$$H_{zv'} = \Delta_L \hat{\Delta}_R \hat{Q}_{21} , \qquad (3.4.10)$$

where all (diagonal) entries of $\hat{\Delta}_R$ are the same as those of Δ_R except the j-th entry, which is a proper factor of Δ_{Rj} ; i.e., for some $\delta_j \notin J$,

$$\Delta_{Rj} = \hat{\Delta}_{Rj} \delta_j . \qquad (3.4.11)$$

Since Δ_{Rj} is a l.c.m. of d_{ij} , $i = 1, \cdots, n_i$, the k-th row j-th column denominator d_{kj} has that factor δ_j , i.e., $d_{kj} = \delta_j \hat{d}_{kj}$. The kj-th entry of Q_{21} is then $\frac{m_{kj}}{d_{kj}} \hat{\Delta}_{Rj} q_j$, where q_j is the j-th (diagonal) entry of $\hat{Q}_{21}$. Since δ_j is not a factor of $\hat{\Delta}_{Rj}$, the only way that the kj-th entry will be in H is if $q_j = \delta_j q_j'$, for some $q_j' \in H$; $\hat{Q}_{21}$ then becomes $diag[\,1 \cdots \delta_j \cdots 1\,]\hat{Q}'_{21}$. Therefore, $H_{zv'} = \Delta_L \hat{\Delta}_R \, diag[\,1 \cdots \delta_j \cdots 1\,]\hat{Q}'_{21} = \Delta_L \Delta_R \hat{Q}'_{21}$, for some $\hat{Q}'_{21} \in \mathcal{M}(H)$.

□

Chapter Four

Decentralized Control Systems

4.1. Introduction

In Chapter Three we studied two system configurations, $S(P, C)$ and $\Sigma(\hat{P}, \hat{C})$; these systems put no constraints on the structure of the feedback-compensator. We now investigate the consequences of restricting the compensator to be block-diagonal.

In large scale systems, we often encounter restrictions on the feedback controller structure. These systems have several local control stations; each local controller observes only the corresponding (local) outputs. Such decentralized control of systems results in a block-diagonal controller-matrix structure.

A multi-channel plant P , which has rational function entries, can be stabilized by a decentralized dynamic output-feedback compensator if and only if P has no unstable *decentralized fixed-eigenvalues* (misleadingly called fixed-modes in the literature) with respect to block-diagonal real constant output-feedback [Wan.1]. Decentralized fixed-eigenvalues can be characterized various ways and interpreted in terms of plant transmission-zeros [see for example, And.1, Cor.1, Dav.1, 2,]. An algebraic characterization of fixed-eigenvalues using left-factorizations of the plant is given in [And.1]; the rank-test developed there can also be used to obtain other characterizations in a state-space setting.

Decentralized compensator synthesis methods for linear time-invariant systems are available in the literature; these procedures do not result in an explicit expression for the class of all stabilizing compensators. The original method in [Wan.1] uses state-space techniques to move all unstable controllable and observable modes to the left-half complex plane by applying feedback to each channel sequentially; an algorithm that includes improper plants is given in [Dav.2]. In [Cor.1], if the plant is *strongly-connected*, the system is made stabilizable and detectable through one channel by applying appropriate feedback to all other channels (see also [Vid.3]). In [Güç.1], an N x N plant, which has no unstable fixed-eigenvalues with respect to *diagonal* constant feedback, is considered; using polynomial algebra, an N-step algorithm is given to determine a compensator which moves the poles of this square plant to a symmetric region of stability. This procedure gives one diagonal compensator explicitly.

In this chapter, we obtain the necessary and sufficient conditions on P for stabilizability by a decentralized dynamic compensator in the completely general algebraic framework of Chapter Two; hence, the results are applicable to distributed and lumped, continuous-time and discrete-time systems. Decentralized stabilizability conditions turn out to be certain Smith-form-like structures that must be satisfied by coprime factorizations of the plant P. When the compensator structure is required to be block-diagonal as in decentralized output-feedback, finding the class of all stabilizing decentralized compensators is complicated; the task is to find a *structured* generalized Bezout identity where the coprime factorizations of P satisfy decentralized stabilizability conditions. For plants that satisfy these conditions, we parametrize the class of all stabilizing decentralized compensators; this class has as many parameter matrices as the number of channels (here the parameter matrices satisfy a unimodularity condition).

The chapter is organized as follows: Section 4.2 gives the system description; to simplify derivations, we consider a two-channel MIMO system in detail (see Figure 4.1). Conditions on coprime factorizations of P for decentralized stabilizability and the set of all stabilizing decentralized compensators C_d are given in Section 4.3. In Section 4.4, the main results of Section 4.3 are interpreted when the plant can be represented by a transfer matrix with rational function entries; it is shown that the decentralized stabilizability conditions of Section 4.3 in fact generalize the requirement that the system has no fixed-eigenvalues [And.1, Wan.1]. An algorithm is given for designing stabilizing decentralized compensators for a given strictly proper P based on any of its right-coprime factorizations. In Section 4.5, the main theorems of Section 4.3 are extended to m-channels and interpreted in the special algebraic setting of proper stable rational functions. Finally in Section 4.6, decentralized compensator design is extended to the general system $\Sigma(\hat{P}, \hat{C})$ of Section 3.3.

4.2. System description

Consider the decentralized control system $S(P, C_d)$ shown in Figure 4.1.

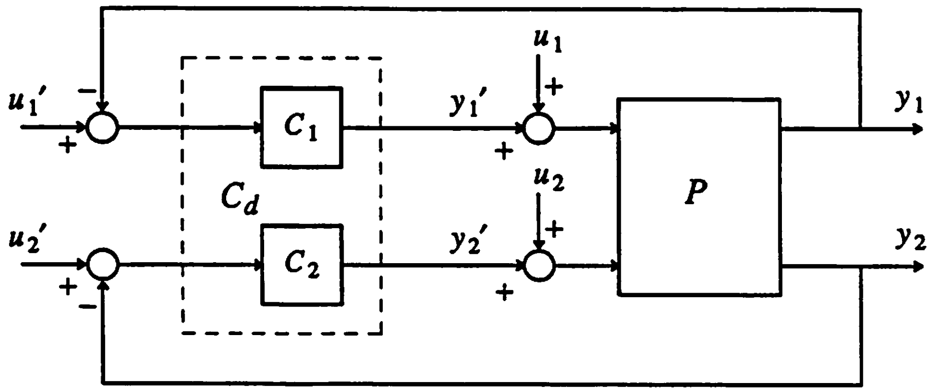

Figure 4.1: The two-channel decentralized control system $S(P, C_d)$.

4.2.1. Assumptions:

(A) Let $P \in G^{n_o \times n_i}$ be a two-channel plant, where $n_o =: n_{o1} + n_{o2}$, $n_i =: n_{i1} + n_{i2}$.

Let (N_p, D_p) be an r.c.f.r. of P, where

$$N_p =: \begin{bmatrix} N_{p1} \\ N_{p2} \end{bmatrix} \in H^{n_o \times n_i}, \quad D_p =: \begin{bmatrix} D_{p1} \\ D_{p2} \end{bmatrix} \in H^{n_i \times n_i}$$

$N_{p1} \in H^{n_{o1} \times n_i}$, $N_{p2} \in H^{n_{o2} \times n_i}$, $D_{p1} \in H^{n_{i1} \times n_i}$, $D_{p2} \in H^{n_{i2} \times n_i}$.

Let $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of P , where

$$\tilde{D}_p =: \left[\tilde{D}_{p1} \vdots \tilde{D}_{p2} \right] \in H^{n_o \times n_o}$$

$$\tilde{N}_p =: \left[\tilde{N}_{p1} \vdots \tilde{N}_{p2} \right] \in H^{n_o \times n_i}$$

$\tilde{D}_{p1} \in H^{n_o \times n_{o1}}$, $\tilde{D}_{p2} \in H^{n_o \times n_{o2}}$, $\tilde{N}_{p1} \in H^{n_o \times n_{i1}}$, $\tilde{N}_{p2} \in H^{n_o \times n_{i2}}$.

Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P where $N_{pr} =: \begin{bmatrix} N_{pr1} \\ N_{pr2} \end{bmatrix} \in H^{n_o \times n}$, $D \in H^{n \times n}$, $N_{pl} =: \begin{bmatrix} N_{pl1} \vdots N_{pl2} \end{bmatrix} \in H^{n \times n_i}$, $N_{pr1} \in H^{n_{o1} \times n}$, $N_{pr2} \in H^{n_{o2} \times n}$, $N_{pl1} \in H^{n \times n_{i1}}$, $N_{pl2} \in H^{n \times n_{i2}}$.

(B) Let $C_d \in G^{n_i \times n_o}$ be a *decentralized* compensator, where $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$, $C_1 \in G^{n_{i1} \times n_{o1}}, C_2 \in G^{n_{i2} \times n_{o2}}$.

Let $(\tilde{D}_{c1}, \tilde{N}_{c1})$ be an l.c.f.r. of C_1 and let $(\tilde{D}_{c2}, \tilde{N}_{c2})$ be an l.c.f.r. of C_2 , where $\tilde{D}_{c1} \in H^{n_{i1} \times n_{i1}}, \tilde{D}_{c2} \in H^{n_{i2} \times n_{i2}}, \tilde{N}_{c1} \in H^{n_{i1} \times n_{o1}}, \tilde{N}_{c2} \in H^{n_{i2} \times n_{o2}}$.

Let $\tilde{D}_c := \begin{bmatrix} \tilde{D}_{c1} & 0 \\ 0 & \tilde{D}_{c2} \end{bmatrix}$, $\tilde{N}_c := \begin{bmatrix} \tilde{N}_{c1} & 0 \\ 0 & \tilde{N}_{c2} \end{bmatrix}$; note that $(\tilde{D}_c, \tilde{N}_c)$ is an l.c.f.r. of C_d if and only if $(\tilde{D}_{c1}, \tilde{N}_{c1})$ is an l.c.f.r. of C_1 and $(\tilde{D}_{c2}, \tilde{N}_{c2})$ is an l.c.f.r. of C_2.

Let (N_{c1}, D_{c1}) be an r.c.f.r. of C_1 and let (N_{c2}, D_{c2}) be an r.c.f.r. of C_2 , where $N_{c1} \in H^{n_{i1} \times n_{o1}}$, $N_{c2} \in H^{n_{i2} \times n_{o2}}$, $D_{c1} \in H^{n_{o1} \times n_{o1}}$, $D_{c2} \in H^{n_{o2} \times n_{o2}}$. Let $D_c := \begin{bmatrix} D_{c1} & 0 \\ 0 & D_{c2} \end{bmatrix}$, $N_c := \begin{bmatrix} N_{c1} & 0 \\ 0 & N_{c2} \end{bmatrix}$; note that (N_c, D_c) is an r.c.f.r. of C_d if and only if (N_{c1}, D_{c1}) is an r.c.f.r. of C_1 and (N_{c2}, D_{c2}) is an r.c.f.r. of C_2.

□

If P satisfies Assumption 4.2.1 (A) we have the generalized Bezout identity (2.3.7) for the doubly-coprime pair $((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$. For the b.c.f.r. (N_{pr}, D, N_{pl}) of P we have the two generalized Bezout identities (2.3.8) and (2.3.10), partitioned as follows: for the r.c. pair (N_{pr}, D), there are matrices V_{pr} , U_{pr} , $\tilde{X}$, $\tilde{Y}$, $\tilde{U}$, $\tilde{V}$ $\in \mathcal{m}(H)$ such that

$$\begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N_{pr} & \tilde{V} \end{bmatrix} =: \begin{bmatrix} V_{pr} & U_{pr1} & U_{pr2} \\ -\tilde{X} & \tilde{Y}_1 & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N_{pr1} & \tilde{V}_1 \\ N_{pr2} & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix} ; \quad (4.2.1)$$

for the l.c. pair (D, N_{pl}), there are matrices $V_{pl}, U_{pl}, X, Y, U, V \in \mathcal{m}(H)$ such that

$$\begin{bmatrix} D & -N_{pl} \\ U & V \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} =: \begin{bmatrix} D & -N_{pl1} & -N_{pl2} \\ U & V_1 & V_2 \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl1} & Y_1 \\ -U_{pl2} & Y_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix} . \tag{4.2.2}$$

Let

$$\bar{y} := \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}, \quad \bar{u} := \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix};$$

the map $H_{\bar{y}\bar{u}} : \bar{u} \mapsto \bar{y}$ is called the I/O map. In terms of P and C_d, $H_{\bar{y}\bar{u}}$ is given by

$$H_{\bar{y}\bar{u}} = \begin{bmatrix} P(I_{n_i} + C_d P)^{-1} & P(I_{n_i} + C_d P)^{-1} C_d \\ -C_d P(I_{n_i} + C_d P)^{-1} & (I_{n_i} + C_d P)^{-1} C_d \end{bmatrix} . \tag{4.2.3}$$

Note that equation (4.2.3) is the same as equation (3.2.2), where C is replaced by C_d.

4.2.2. Definition (H–stability):

The system $S(P, C_d)$ is said to be *H–stable* iff $H_{\bar{y}\bar{u}} \in \mathcal{m}(H)$.

4.2.3. Analysis:

We now analyze the system $S(P, C_d)$ by factorizing P and C_d; as expected, we consider four cases:

(i) Let $P = N_p D_p^{-1}$, let $C = \tilde{D}_c^{-1} \tilde{N}_c$, where (N_p, D_p) is an r.c. pair as in Assumption 4.2.1 (A), and $(\tilde{D}_c, \tilde{N}_c)$ is an l.c. pair as in Assumption 4.2.1 (B) (see Figure 4.2); again, ξ_p denotes the pseudo-state of P.

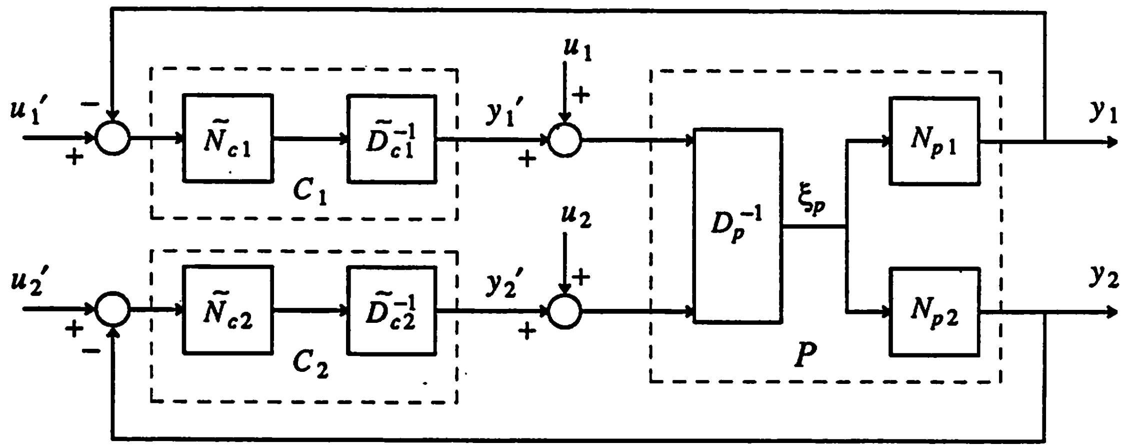

Figure 4.2: $S(P, C_d)$ with $P = N_p D_p^{-1}$ and $C_d = \tilde{D}_c^{-1}\tilde{N}_c$.

$S(P, C_d)$ is then described by equations (4.2.4)-(4.2.5) below:

$$\begin{bmatrix} \tilde{D}_{c1}D_{p1}+\tilde{N}_{c1}N_{p1} \\ \tilde{D}_{c2}D_{p2}+\tilde{N}_{c2}N_{p2} \end{bmatrix} \xi_p = \begin{bmatrix} \tilde{D}_{c1} & 0 & \tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2} & 0 & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}, \tag{4.2.4}$$

$$\begin{bmatrix} N_{p1} \\ N_{p2} \\ D_{p1} \\ D_{p2} \end{bmatrix} \xi_p = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_{ni1} & 0 & 0 & 0 \\ 0 & I_{ni2} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}. \tag{4.2.5}$$

Equations (4.2.4)-(4.2.5) are in the form

$$D_{H1}\xi_p = N_{L1}u$$
$$N_{R1}\xi_p = y - S_{H1}u \ , \tag{4.2.6}$$

where (N_{R1}, D_{H1}) is an r.c. pair and (D_{H1}, N_{L1}) is an l.c. pair. If $\det D_{H1} \in I$, then

$$H_{\bar{y}\bar{u}} = N_{R1}D_{H1}^{-1}N_{L1} + S_{H1} \in \mathcal{m}(G) .$$

$S(P, C_d)$ is H–stable if and only if $D_{H1}^{-1} \in \mathcal{m}(H)$ (equivalently, $\det D_{H1} \in J$ and hence, D_{H1} is H–unimodular). D_{H1} can be expressed several ways:

$$D_{H1} = \tilde{D}_c D_p + \tilde{N}_c N_p = \begin{bmatrix} \tilde{D}_{c1} D_{p1} + \tilde{N}_{c1} N_{p1} \\ \\ \tilde{D}_{c2} D_{p2} + \tilde{N}_{c2} N_{p2} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_{c1} & 0 & \vdots & \tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2} & \vdots & 0 & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} D_p \\ \cdots \\ N_p \end{bmatrix} = \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix} ; \quad (4.2.7)$$

and $\det D_{H1}$ can also be written as $\det D_{H1} = \det \tilde{D}_c \det(I + C_d P) \det D_p$. By normalization and due to the block-diagonal compensator structure, $D_{H1} \in \mathcal{M}(H)$ is H–unimodular if and only if there are *block-diagonal* matrices $V_p := \tilde{D}_c$, $U_p := \tilde{N}_c \in \mathcal{M}(H)$ such that

$$V_p D_p + U_p N_p = I_{n_i} . \quad (4.2.8)$$

Equation (4.2.8) is a Bezout identity where V_p , $U_p \in \mathcal{M}(H)$ are restricted to be *block-diagonal* as shown in equation (4.2.7).

(ii) Let $P = \tilde{D}_p^{-1} \tilde{N}_p$, let $C = N_c D_c^{-1}$, where $(\tilde{D}_p, \tilde{N}_p)$ is an r.c. pair as in Assumption 4.2.1 (A), and (N_c, D_c) is an l.c. pair as in Assumption 4.2.1 (B) (see Figure 4.3); for $i = 1, 2$, ξ_{ci} denotes the pseudo-state of C_i.

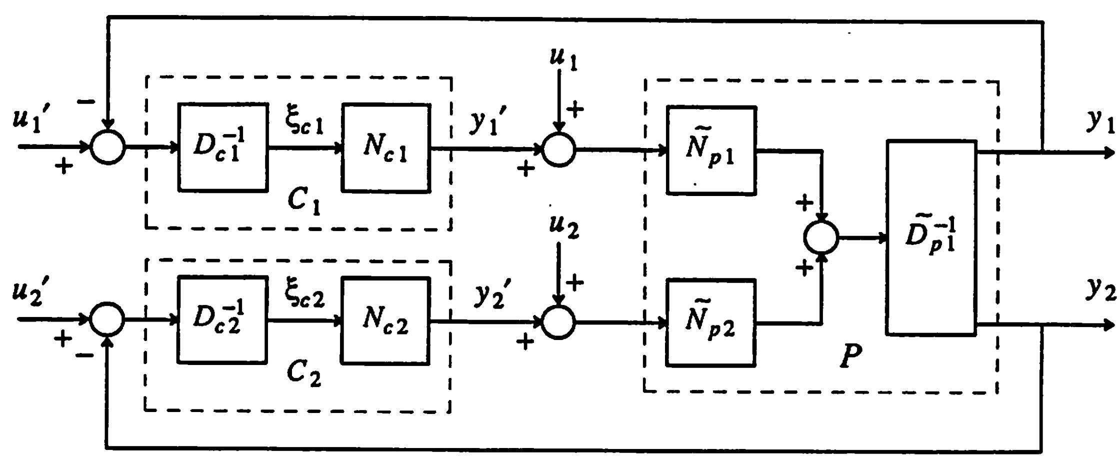

Figure 4.3: $S(P, C_d)$ with $P = \tilde{D}_p^{-1} \tilde{N}_p$ and $C_d = N_c D_c^{-1}$.

$S(P, C_d)$ is then described by equations (4.2.9)-(4.2.10) below:

$$\left[\tilde{D}_{p1}D_{c1}+\tilde{N}_{p1}N_{c1} \;\vdots\; \tilde{D}_{p2}D_{c2}+\tilde{N}_{p2}N_{c2}\right]\begin{bmatrix}\xi_{c1}\\ \cdots \\ \xi_{c2}\end{bmatrix} = \left[-\tilde{N}_{p1} \;\; -\tilde{N}_{p2} \;\; \tilde{D}_{p1} \;\; \tilde{D}_{p2}\right]\begin{bmatrix}u_1\\ u_2\\ u_1'\\ u_2'\end{bmatrix}, \quad (4.2.9)$$

$$\begin{bmatrix}-D_{c1} & \vdots & 0\\ 0 & \vdots & -D_{c2}\\ N_{c1} & \vdots & 0\\ 0 & \vdots & N_{c2}\end{bmatrix}\begin{bmatrix}\xi_{c1}\\ \cdots \\ \xi_{c2}\end{bmatrix} = \begin{bmatrix}y_1\\ y_2\\ y_1'\\ y_2'\end{bmatrix} + \begin{bmatrix}0 & 0 & -I_{n_{o1}} & 0\\ 0 & 0 & 0 & -I_{n_{o2}}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_1'\\ u_2'\end{bmatrix}. \quad (4.2.10)$$

Following similar steps as in case (i) of the analysis, $S(P, C_d)$ is H–stable if and only if

$$D_{H2} := \left[\tilde{D}_{p1}D_{c1}+\tilde{N}_{p1}N_{c1} \;\vdots\; \tilde{D}_{p2}D_{c2}+\tilde{N}_{p2}N_{c2}\right] = \left[\tilde{D}_{p}D_{c}+\tilde{N}_{p}N_{c}\right] \in \mathcal{M}(H)$$

is H–unimodular. D_{H2} can be written also as

$$D_{H2} = \left[-\tilde{N}_{p1} \;\; \tilde{D}_{p1} \;\vdots\; -\tilde{N}_{p2} \;\; \tilde{D}_{p2}\right]\begin{bmatrix}-N_{c1} & 0\\ D_{c1} & 0\\ \cdots & \cdots\\ 0 & -N_{c2}\\ 0 & D_{c2}\end{bmatrix}. \quad (4.2.11)$$

(iii) Let $P = N_{pr}D^{-1}N_{pl}$, let $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. of P as in Assumption 4.2.1 (A), and $(\tilde{D}_c, \tilde{N}_c)$ is an l.c.f.r. of C as in Assumption 4.2.1 (B) (see Figure 4.4).

$S(P, C_d)$ is then described by equations (4.2.12)-(4.2.13) below:

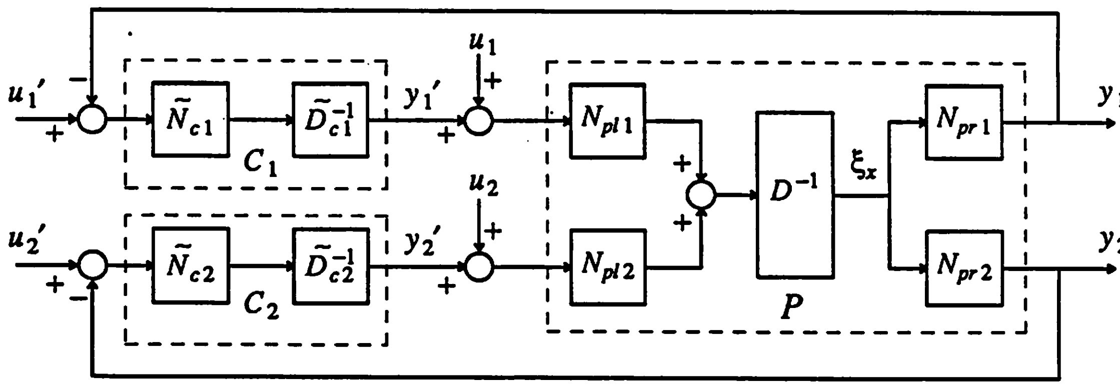

Figure 4.4: $S(P, C_d)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C_d = \tilde{D}_c^{-1}\tilde{N}_c$.

$$\begin{bmatrix} D & \vdots & -N_{pl1} & -N_{pl2} \\ \tilde{N}_{c1}N_{pr1} & \vdots & \tilde{D}_{c1} & 0 \\ \tilde{N}_{c2}N_{pr2} & \vdots & 0 & \tilde{D}_{c2} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} N_{pl1} & N_{pl2} & 0 & 0 \\ 0 & 0 & \tilde{N}_{c1} & 0 \\ 0 & 0 & 0 & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}, \qquad (4.2.12)$$

$$\begin{bmatrix} N_{pr1} & \vdots & 0 & 0 \\ N_{pr2} & \vdots & 0 & 0 \\ 0 & \vdots & I_{n_{i1}} & 0 \\ 0 & \vdots & 0 & I_{n_{i2}} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}. \qquad (4.2.13)$$

The system $S(P, C_d)$ is H–stable if and only if

$$D_{H3} := \begin{bmatrix} D & -N_{pl} \\ \tilde{N}_c N_{pr} & \tilde{D}_c \end{bmatrix} = \begin{bmatrix} D & \vdots & -N_{pl1} & -N_{pl2} \\ \tilde{N}_{c1}N_{pr1} & \vdots & \tilde{D}_{c1} & 0 \\ \tilde{N}_{c2}N_{pr2} & \vdots & 0 & \tilde{D}_{c2} \end{bmatrix} \qquad (4.2.14)$$

is H–unimodular. Post-multiply D_{H3} by $R := \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} \in \mathcal{M}(H)$ defined in equation (4.2.2); then $S(P, C_d)$ is H–stable if and only if

$$\tilde{D}_c Y + \tilde{N}_c N_{pr} X = \begin{bmatrix} \tilde{D}_{c1} & 0 \\ 0 & \tilde{D}_{c2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} \tilde{N}_{c1} & 0 \\ 0 & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} N_{pr1}X \\ N_{pr2}X \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} Y_1 \\ N_{pr1}X \\ \cdots \\ Y_2 \\ N_{pr2}X \end{bmatrix} \text{ is } H\text{–unimodular}. \qquad (4.2.15)$$

By Proposition 2.4.1, the pair $(N_{pr}X, Y)$ is an r.c.f.r. of P. Note that, with $D_p := Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, $N_p := N_{pr}X = \begin{bmatrix} N_{pr1}X \\ N_{pr2}X \end{bmatrix}$, equation (4.2.15) is the same as equation (4.2.7).

(iv) Let $P = N_{pr} D^{-1} N_{pl}$, let $C = N_c D_c^{-1}$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. of P as in Assumption 4.2.1 (A), and (N_c, D_c) is an r.c.f.r. of C as in Assumption 4.2.1 (B) (see Figure 4.5).

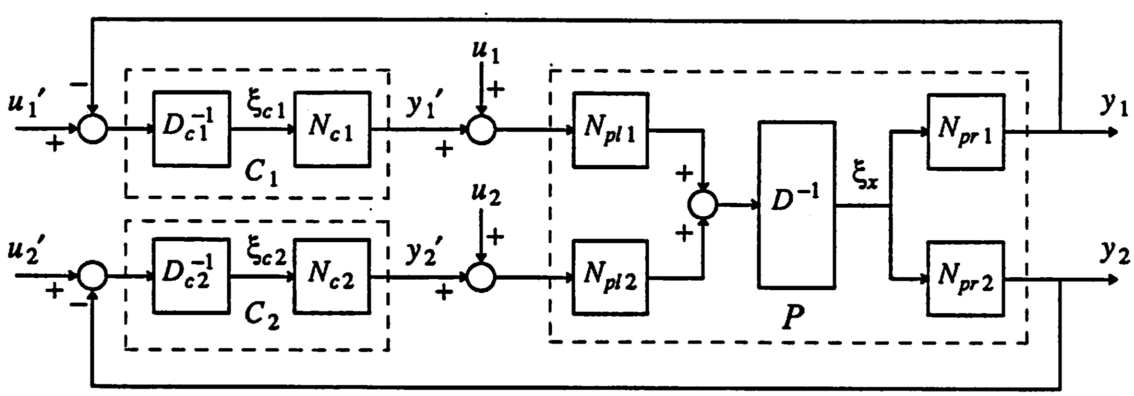

Figure 4.5: $S(P, C_d)$ with $P = N_{pr} D^{-1} N_{pl}$ and $C_d = N_c D_c^{-1}$.

$S(P, C_d)$ is then described by equations (4.2.16)-(4.2.17):

$$\begin{bmatrix} D & \vdots & -N_{pl1}N_{c1} & -N_{pl2}N_{c2} \\ N_{pr1} & \vdots & D_{c1} & 0 \\ N_{pr2} & \vdots & 0 & D_{c2} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_{c1} \\ \xi_{c2} \end{bmatrix} = \begin{bmatrix} N_{pl1} & N_{pl2} & 0 & 0 \\ 0 & 0 & I_{no1} & 0 \\ 0 & 0 & 0 & I_{no2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}, \quad (4.2.16)$$

$$\begin{bmatrix} N_{pr1} & \vdots & 0 & 0 \\ N_{pr2} & \vdots & 0 & 0 \\ 0 & \vdots & N_{c1} & 0 \\ 0 & \vdots & 0 & N_{c2} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ \xi_{c1} \\ \xi_{c2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}. \quad (4.2.17)$$

The system $S(P, C_d)$ is H–stable if and only if

$$D_{H4} := \begin{bmatrix} D & -N_{pl}N_c \\ N_{pr} & D_c \end{bmatrix} = \begin{bmatrix} D & \vdots & -N_{pl1}N_{c1} & -N_{pl2}N_{c2} \\ N_{pr1} & \vdots & D_{c1} & 0 \\ N_{pr2} & \vdots & 0 & D_{c2} \end{bmatrix}$$

is H–unimodular. Pre-multiply D_{H4} by $L := \begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \in \mathcal{M}(H)$ defined in equation (4.2.1); then $S(P, C_d)$ is H–stable if and only if

$$\tilde{X} N_{pl} N_c + \tilde{Y} D_c = \begin{bmatrix} \tilde{X} N_{pl1} & \vdots & \tilde{X} N_{pl2} \end{bmatrix} \begin{bmatrix} N_{c1} & 0 \\ 0 & N_{c2} \end{bmatrix} + \begin{bmatrix} \tilde{Y}_1 & \vdots & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} D_{c1} & 0 \\ 0 & D_{c2} \end{bmatrix}$$

$$= \begin{bmatrix} -\tilde{X} N_{pl1} & \tilde{Y}_1 & \vdots & -\tilde{X} N_{pl2} & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} -N_{c1} & 0 \\ D_{c1} & 0 \\ \cdots & \cdots \\ 0 & -N_{c2} \\ 0 & D_{c2} \end{bmatrix} \text{ is } H\text{–unimodular}. \quad (4.2.18)$$

By Proposition 2.4.1, the pair $(\tilde{Y}, \tilde{X} N_{pl})$ is an l.c.f.r. of P. Note that, with $\tilde{D}_p := \tilde{Y} = \begin{bmatrix} \tilde{Y}_1 & \vdots & \tilde{Y}_2 \end{bmatrix}$, $\tilde{N}_p := \tilde{X} N_{pl} = \begin{bmatrix} \tilde{X} N_{pl1} & \vdots & \tilde{X} N_{pl2} \end{bmatrix}$, equation (4.2.18) is the same as equation (4.2.11).

4.3. Main results

In this section the plant P satisfies Assumption 4.2.1 (A).

4.3.1. Definition (H–stabilizing decentralized compensator):

C_d is called an *H–stabilizing decentralized compensator for P* (later abbreviated as C_d H–stabilizes P) iff $C_d \in G^{n_i x n_o}$ satisfies Assumption 4.2.1 (B) *and* the system $S(P, C_d)$ is H–stable.

4.3.2. Definition (Class of all H–stabilizing decentralized compensators):

The set

$$\mathbf{S}_d(P) := \{ \ C_d : C_d \ H\text{–}stabilizes \ P \ \} \tag{4.3.1}$$

is called the set of *all H–stabilizing decentralized compensators for P* .

4.3.3. Comment:

In Chapter 3 (Theorem 3.2.8) we showed that the set $\mathbf{S}(P)$ of all *centralized* (full-feedback) compensators that H–stabilize P is given by

$$\mathbf{S}(P) = \{ \ C = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p) : Q \in \mathcal{M}(H) \ \}, \tag{4.3.3}$$

where V_p , U_p are as in the generalized Bezout identity (2.3.7). $\mathbf{S}(P)$ can also be expressed in terms of an r.c.f.r. (N_p, D_p) of P :

$$\mathbf{S}(P) = \{ \ C = (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1} : Q \in \mathcal{M}(H) \ \}.$$

Following Comment 3.2.9.(iii), if $P \in \mathcal{M}(G)$ instead of $\mathcal{M}(G_s)$, then $Q \in \mathcal{M}(H)$ should be such that $\det(\tilde{V}_p - N_p Q) \in I$ (equivalently, $\det(V_p - Q\tilde{N}_p) \in I$).

The class of all H–stabilizing *decentralized* compensators $\mathbf{S}_d(P)$ will turn out to be more complicated. (Note that $\mathbf{S}_d(P)$ is a subset of $\mathbf{S}(P)$). For the existence of such *decentralized* compensators, the plant P has to satisfy *additional* conditions which are not required for the existence of *full-feedback* compensators that would achieve H–stabilization; these conditions are due to the block-diagonal structure of the compensator.

□

Theorems 4.3.4R and 4.3.4L below establish the necessary and sufficient conditions on (an r.c.f.r. (N_p, D_p) or an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ of) P for the existence of H–stabilizing decentralized dynamic output-feedback compensators:

4.3.4R. Theorem (Conditions on $P = N_p D_p^{-1}$ for decentralized H–stabilizability):

Let $P \in \mathcal{M}(G_S)$ satisfy Assumption 4.2.1 (A); then there exists an H–stabilizing decentralized compensator C_d for P if and only if P has an r.c.f.r. (N_p, D_p) such that

$$\begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix} =: \begin{bmatrix} D_{11} & D_{12} \\ N_{11} & N_{12} \end{bmatrix} = E_1 \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix}, \tag{4.3.4}$$

where $E_1 \in H^{(n_{i1}+n_{o1})\mathrm{x}(n_{i1}+n_{o1})}$ is H–unimodular and $W_1 \in H^{n_{o1}\mathrm{x}n_{i2}}$, " roman " and"

$$\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix} =: \begin{bmatrix} D_{21} & D_{22} \\ N_{21} & N_{22} \end{bmatrix} = E_2 \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix}, \tag{4.3.5}$$

where $E_2 \in H^{(n_{i2}+n_{o2})\mathrm{x}(n_{i2}+n_{o2})}$ is H–unimodular and $W_2 \in H^{n_{o2}\mathrm{x}n_{i1}}$.

□

Equation (4.3.4) implies that the pair (N_{11}, D_{11}) is r.c. and similarly, equation (4.3.5) implies that the pair (N_{22}, D_{22}) is r.c.

4.3.5L. Theorem (Conditions on $P = \tilde{D}_p^{-1}\tilde{N}_p$ for decentralized H–stabilizability):

Let $P \in \mathcal{m}(G_S)$ satisfy Assumption 4.2.1 (A); then P has an r.c.f.r. (N_p, D_p) which satisfies conditions (4.3.4)-(4.3.5) of Theorem 4.3.4R if and only if P has an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ such that

$$\begin{bmatrix} -\tilde{N}_{p1} & \tilde{D}_{p1} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_{o1}} \\ -W_2 & 0 \end{bmatrix} E_1^{-1} \text{ , and} \tag{4.3.6}$$

$$\begin{bmatrix} -\tilde{N}_{p2} & \tilde{D}_{p2} \end{bmatrix} = \begin{bmatrix} -W_1 & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} E_2^{-1} \text{ ,} \tag{4.3.7}$$

where the H–unimodular matrices $E_1 \in H^{(n_{i1}+n_{o1})x(n_{i1}+n_{o1})}$, $E_2 \in H^{(n_{i2}+n_{o2})x(n_{i2}+n_{o2})}$ and the matrices $W_1 \in H^{n_{o1}xn_{i2}}$, $W_2 \in H^{n_{o2}xn_{i1}}$ are defined in equations (4.3.4)-(4.3.5). Equivalently, there exists an H–stabilizing decentralized compensator C_d for $P \in \mathcal{m}(G_S)$ if and only if P has an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ such that conditions (4.3.6)-(4.3.7) hold for some H–unimodular $E_1, E_2 \in \mathcal{m}(H)$ and some $W_1, W_2 \in \mathcal{m}(H)$.

4.3.6. Comments:

(i) Let (N_p, D_p) be an r.c.f.r. of P; then by Lemma 2.3.3.(i), (X_p, Y_p) is another r.c.f.r. of P if and only if $(X_p, Y_p) = (N_p R, D_p R)$ for some H–unimodular matrix $R \in H^{n_i x n_i}$. By Theorem 4.3.4R, P can be H–stabilized by a decentralized compensator if and only if any r.c.f.r. (X_p, Y_p), $X_p := \begin{bmatrix} X_{p1} \\ X_{p2} \end{bmatrix}$, $Y_p := \begin{bmatrix} Y_{p1} \\ Y_{p2} \end{bmatrix}$, of P is of the form

$$\begin{bmatrix} Y_{p1} \\ X_{p1} \\ \cdots \\ Y_{p2} \\ X_{p2} \end{bmatrix} = \begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix} R = \begin{bmatrix} E_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2 \end{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \\ \cdots & \cdots \\ 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} R \text{ ,} \tag{4.3.8}$$

for some H–unimodular matrix $R \in H^{n_i x n_i}$, where E_1, $E_2 \in \mathcal{m}(H)$ are H–unimodular and $W_1, W_2 \in \mathcal{m}(H)$.

Similarly, let $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of P ; then $(\tilde{Y}_p, \tilde{X}_p)$ is another l.c.f.r. of P if and only if $(\tilde{Y}_p, \tilde{X}_p) = (L\tilde{D}_p, L\tilde{N}_p)$ for some H–unimodular matrix $L \in H^{n_o \times n_o}$. By Theorem 4.3.5L, P can be H–stabilized by a decentralized compensator C_d if and only if any l.c.f.r. $(\tilde{Y}_p, \tilde{X}_p)$, $\tilde{Y}_p := \begin{bmatrix} \tilde{Y}_{p1} & \tilde{Y}_{p2} \end{bmatrix}$, $\tilde{X}_p := \begin{bmatrix} \tilde{X}_{p1} & \tilde{X}_{p2} \end{bmatrix}$ of P is of the form

$$\begin{bmatrix} -\tilde{X}_{p1} & \tilde{Y}_{p1} & \vdots & -\tilde{X}_{p2} & \tilde{Y}_{p2} \end{bmatrix} = L \begin{bmatrix} 0 & I_{n_{o1}} & \vdots & -W_1 & 0 \\ -W_2 & 0 & \vdots & 0 & I_{n_{o2}} \end{bmatrix} \begin{bmatrix} E_1^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2^{-1} \end{bmatrix}, \quad (4.3.9)$$

for some H–unimodular matrix $L \in H^{n_o \times n_o}$.

(ii) In Section 4.4 below, we show that, if H is $R_u(s)$ as in Example 2.2.2, conditions (4.3.4)-(4.3.5) (equivalently, conditions (4.3.6)-(4.3.7)) on $P \in \mathbb{R}_{sp}(s)^{n_o \times n_i}$ are equivalent to the condition that the system $S(P, C_d)$ has *no fixed-eigenvalues* in $\bar{\mathcal{U}}$.

(iii) Suppose that P is given by a b.c.f.r. (N_{pr}, D, N_{pl}) and C_d is given by an l.c.f.r. $(\tilde{D}_c, \tilde{N}_c)$ as in case (iii) of Analysis 4.2.3. Consider equation (4.2.15) and apply Theorem 4.3.4R to the r.c.f.r. $(N_p, D_p) := (N_{pr}X, Y)$ of P; then equation (4.3.8) implies that $P = N_{pr}D^{-1}N_{pl} \in \mathcal{M}(G_s)$ can be H–stabilized by a decentralized compensator C_d if and only if there exists an H–unimodular matrix $R \in H^{n_i \times n_i}$ such that

$$\begin{bmatrix} Y_1 \\ N_{pr1}X \end{bmatrix} = E_1 \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix} R, \text{ and} \quad (4.3.4B)$$

$$\begin{bmatrix} Y_2 \\ N_{pr2}X \end{bmatrix} = E_2 \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} R, \quad (4.3.5B)$$

where $E_1 \in H^{(n_{i1}+n_{o1}) \times (n_{i1}+n_{o1})}$ and $E_2 \in H^{(n_{i2}+n_{o2}) \times (n_{i2}+n_{o2})}$ are H–unimodular, and $W_1 \in H^{n_{o1} \times n_{i2}}$, $W_2 \in H^{n_{o2} \times n_{i1}}$.

Similarly if C_d is given by an r.c.f.r. (N_c, D_c) as in case (iv) of Analysis 4.2.3, then considering equation (4.2.18), we apply Theorem 4.3.5L to the l.c.f.r. $(\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \tilde{X} N_{pl})$ of P. Following equation (4.3.9), P can be H–stabilized by a decentralized compensator C_d if and only if there exists an H–unimodular matrix $L \in H^{n_o \times n_o}$ such that

$$\left[-\tilde{X} N_{pl1} \; \vdots \; \tilde{Y}_1 \right] = L \begin{bmatrix} 0 & I_{n_{o1}} \\ -W_2 & 0 \end{bmatrix} E_1^{-1} , \text{ and} \tag{4.3.6B}$$

$$\left[-\tilde{X} N_{pl2} \; \vdots \; \tilde{Y}_2 \right] = L \begin{bmatrix} -W_1 & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} E_2^{-1} , \tag{4.3.7B}$$

where $E_1^{-1} \in H^{(n_{i1}+n_{o1})\text{x}(n_{i1}+n_{o1})}$ and $E_2^{-1} \in H^{(n_{i2}+n_{o2})\text{x}(n_{i2}+n_{o2})}$ are H–unimodular and $W_1 \in H^{n_{o1}\text{x}n_{i2}}, W_2 \in H^{n_{o2}\text{x}n_{i1}}$.

Equations (4.3.4B)-(4.3.5B) (equivalently, (4.3.6B)-(4.3.7B)) will be useful in Section 4.4 when we explain *"rank-tests"* for decentralized H–stabilizability in terms of the state-space description of P.

□

Proof of Theorem 4.3.4R:

(<=) By assumption, an r.c.f.r. (N_p, D_p) of P satisfies conditions (4.3.4)-(4.3.5); then

$$\begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ N_{11} & N_{12} \\ \cdots & \cdots \\ D_{21} & D_{22} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} E_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2 \end{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \\ \cdots & \cdots \\ 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} . \tag{4.3.10}$$

Refer to equation (4.2.7) and consider the compensators $C_1 = \tilde{D}_{c1}^{-1}\tilde{N}_{c1}$, $C_2 = \tilde{D}_{c2}^{-1}\tilde{N}_{c1}$, where $\tilde{D}_{c1}, \tilde{D}_{c2}, \tilde{N}_{c1}, \tilde{N}_{c2}$ are given by

$$\left[\tilde{D}_{c1} \; \vdots \; \tilde{N}_{c1} \right] = \left[I_{n_{i1}} \; \vdots \; 0 \right] E_1^{-1} , \tag{4.3.11}$$

$$\left[\tilde{D}_{c2} \; \vdots \; \tilde{N}_{c2} \right] = \left[I_{n_{i2}} \; \vdots \; 0 \right] E_2^{-1} . \tag{4.3.12}$$

Since E_1^{-1} , $E_2^{-1} \in \mathcal{M}(H)$, clearly $\tilde{D}_{c1}, \tilde{D}_{c2}, \tilde{N}_{c1}, \tilde{N}_{c2} \in \mathcal{M}(H)$; for $k = 1, 2$, $(\tilde{D}_{ck}, \tilde{N}_{ck})$ is an l.c. pair since E_k is H–unimodular.

With $(\tilde{D}_{c1}, \tilde{N}_{c1})$ as in equation (4.3.11) and $(\tilde{D}_{c2}, \tilde{N}_{c2})$ as in equation (4.3.12), and (N_p, D_p) as in equation (4.3.10), equation (4.2.7) becomes

$$D_{H1} = \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ N_{11} & N_{12} \\ \cdots & \cdots \\ D_{21} & D_{22} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_{i2}} \end{bmatrix} . \tag{4.3.13}$$

Now $P \in \mathcal{M}(G_S)$ implies that $N_p \in \mathcal{M}(G_S)$, hence N_{11} , $N_{22} \in \mathcal{M}(G_S)$ and $\tilde{N}_{c1}N_{11}$, $\tilde{N}_{c2}N_{22} \in \mathcal{M}(G_S)$. We use these facts to establish that $\det\tilde{D}_{c1} \in \mathcal{I}$ and $\det\tilde{D}_{c2} \in \mathcal{I}$: from equation (4.3.13),

$$\tilde{D}_{c1}D_{11} + \tilde{N}_{c1}N_{11} = I_{n_{i1}} , \tag{4.3.14}$$

$$\tilde{D}_{c2}D_{22} + \tilde{N}_{c2}N_{22} = I_{n_{i2}} . \tag{4.3.15}$$

By equation (4.3.14), $\det\tilde{D}_{c1}\det D_{11} = \det(I_{n_{i1}} - \tilde{N}_{c1}N_{11}) \in \mathcal{I}$; therefore, by Lemma 2.2.4.(ii), $\det\tilde{D}_{c1} \in \mathcal{I}$ and $\det D_{11} \in \mathcal{I}$. By equation (4.3.15), $\det\tilde{D}_{c2}\det D_{22} = \det(I_{n_{i2}} - \tilde{N}_{c2}N_{22}) \in \mathcal{I}$; therefore, by Lemma 2.2.4.(ii), $\det\tilde{D}_{c2} \in \mathcal{I}$ and $\det D_{22} \in \mathcal{I}$.

This proves that $(\tilde{D}_{c1}, \tilde{N}_{c1})$ given by equation (4.3.11) is an l.c.f.r. of $C_1 \in \mathcal{M}(G)$, and $(\tilde{D}_{c2}, \tilde{N}_{c2})$ given by equation (4.3.12) is a l.c.f.r. of $C_2 \in \mathcal{M}(G)$. Now since equation (4.3.13) implies that D_{H1} is H–unimodular, with this choice of $(\tilde{D}_{c1}, \tilde{N}_{c1})$ and $(\tilde{D}_{c2}, \tilde{N}_{c2})$, where C_d = $\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{c1}^{-1}\tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2}^{-1}\tilde{N}_{c1} \end{bmatrix}$, the system $S(P, C_d)$ is H–stable. Therefore the decentralized compensator C_d , specified by equations (4.3.11)-(4.3.12), H–stabilizes P .

(=>) By assumption, P satisfies Assumption 4.2.1 (A), and P can be H–stabilized by a decentralized compensator C_d . So (by Definition 4.3.1) C_d satisfies Assumption 4.2.1 (B) and the system $S(P, C_d)$ is H–stable; therefore D_{H1} given by equation (4.2.7) is an H–unimodular matrix, and hence, by normalization, equation (4.3.13) holds for some r.c.f.r. (N_p, D_p) of P. Using the fact that $P \in \mathcal{M}(G_S)$ and the same reasoning as in the sufficiency proof above, equation (4.3.14) implies that (N_{11}, D_{11}) is an r.c. pair and $\det D_{11} \in \mathcal{I}$; similarly, equation (4.3.15) implies that (N_{22}, D_{22}) is an r.c. pair and $\det D_{22} \in \mathcal{I}$.

Now let $(\tilde{D}_{11}, \tilde{N}_{11})$ be an l.c.f.r. of $N_{11}D_{11}^{-1}$ and let $(\tilde{D}_{22}, \tilde{N}_{22})$ be an l.c.f.r. of $N_{22}D_{22}^{-1}$ (note that $\tilde{D}_{11}, \tilde{D}_{22} \in I$); then with $\tilde{U}_{11}, \tilde{V}_{11}, \tilde{U}_{22}, \tilde{V}_{22} \in \mathcal{m}(H)$, we write the following generalized Bezout identities using equation (4.3.13):

$$\begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \\ -\tilde{N}_{11} & \tilde{D}_{11} \end{bmatrix} \begin{bmatrix} D_{11} & -\tilde{U}_{11} \\ N_{11} & \tilde{V}_{11} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & I_{n_{o1}} \end{bmatrix}, \tag{4.3.16}$$

$$\begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \\ -\tilde{N}_{22} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} D_{22} & -\tilde{U}_{22} \\ N_{22} & \tilde{V}_{22} \end{bmatrix} = \begin{bmatrix} I_{n_{i2}} & 0 \\ 0 & I_{n_{o2}} \end{bmatrix}. \tag{4.3.17}$$

The matrices in equations (4.3.16) and (4.3.17) are H–unimodular. Let $E_1^{-1} := \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \\ -\tilde{N}_{11} & \tilde{D}_{11} \end{bmatrix}$ and let $E_2^{-1} := \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \\ -\tilde{N}_{22} & \tilde{D}_{22} \end{bmatrix}$; clearly E_1 and E_2 are H–unimodular matrices with elements in H .

Now let $W_1 := -\tilde{N}_{11}D_{12}+\tilde{D}_{11}N_{12} \in H^{n_{o1} \mathrm{x} n_{i2}}$ and let $W_2 := -\tilde{N}_{22}D_{21}+\tilde{D}_{22}N_{21} \in H^{n_{o2} \mathrm{x} n_{i1}}$; from equations (4.3.13), (4.3.16), (4.3.17) we get

$$\begin{bmatrix} E_1^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2^{-1} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ N_{11} & N_{12} \\ \cdots & \cdots \\ D_{21} & D_{22} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \\ \cdots & \cdots \\ 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix}. \tag{4.3.18}$$

Pre-multiplying both sides of equation (4.3.18) by $\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$, we see that this r.c.f.r. (N_p, D_p) of P satisfies conditions (4.3.4)-(4.3.5) for some H–unimodular $E_1 \in \mathcal{m}(H)$ and some H–unimodular $E_2 \in \mathcal{m}(H)$.

□

Proof of Theorem 4.3.5L:

(=>) By assumption, an r.c.f.r. (N_p, D_p) of P satisfies conditions (4.3.4)-(4.3.5). Consider the following generalized Bezout identity:

$$\begin{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & 0 \end{bmatrix} E_1^{-1} & \begin{bmatrix} 0 & 0 \\ I_{n_{i2}} & 0 \end{bmatrix} E_2^{-1} \\ \cdots & \cdots \\ \begin{bmatrix} 0 & I_{n_{o1}} \\ -W_2 & 0 \end{bmatrix} E_1^{-1} & \begin{bmatrix} -W_1 & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} E_2^{-1} \end{bmatrix} \begin{bmatrix} E_1 \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix} & E_1 \begin{bmatrix} 0 & 0 \\ I_{n_{o1}} & 0 \end{bmatrix} \\ \cdots & \cdots \\ E_2 \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} & E_2 \begin{bmatrix} 0 & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & I_{n_{i2}} \end{bmatrix} & 0 \\ \cdots & \cdots \\ 0 & \begin{bmatrix} I_{n_{o1}} & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} \end{bmatrix} .$$

(4.3.19)

Let $(\tilde{D}_p, \tilde{N}_p)$ be such that $\begin{bmatrix} -\tilde{N}_{p1} & \tilde{D}_{p1} \end{bmatrix}$ is as in equation (4.3.6) and $\begin{bmatrix} -\tilde{N}_{p2} & \tilde{D}_{p2} \end{bmatrix}$ is as in equation (4.3.7); then equation (4.3.19) is of the form

$$\begin{bmatrix} V_{p1} & U_{p1} & V_{p2} & U_{p2} \\ -\tilde{N}_{p1} & \tilde{D}_{p1} & -\tilde{N}_{p2} & \tilde{D}_{p2} \end{bmatrix} \begin{bmatrix} D_{p1} & -\tilde{U}_{p1} \\ N_{p1} & \tilde{V}_{p1} \\ D_{p2} & -\tilde{U}_{p2} \\ N_{p2} & \tilde{V}_{p2} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} , \tag{4.3.20}$$

where $V_{p1} \in H^{n_i \times n_{i1}}$, $U_{p1} \in H^{n_i \times n_{o1}}$, $V_{p2} \in H^{n_i \times n_{i2}}$, $U_{p2} \in H^{n_i \times n_{o2}}$, $\tilde{U}_{p1} \in H^{n_{i1} \times n_o}$, $\tilde{V}_{p1} \in H^{n_{o1} \times n_o}$, $\tilde{U}_{p2} \in H^{n_{i2} \times n_o}$, $\tilde{V}_{p2} \in H^{n_{o2} \times n_o}$ are defined in an obvious manner by comparing equations (4.3.19) and (4.3.20).

We must show that the pair $(\tilde{D}_p, \tilde{N}_p)$, satisfying conditions (4.3.6)-(4.3.7), is in fact an l.c.f.r. of P : If $(\tilde{D}_p, \tilde{N}_p)$ is so that conditions (4.3.6)-(4.3.7) hold, then clearly $\tilde{D}_p \in H^{n_o \times n_o}$ and $\tilde{N}_p \in H^{n_o \times n_i}$. By equation (4.3.20), the pair $(\tilde{D}_p, \tilde{N}_p)$ is l.c.; furthermore,

$$\begin{bmatrix} -\tilde{N}_{p1} & -\tilde{N}_{p2} \end{bmatrix} \begin{bmatrix} D_{p1} \\ D_{p2} \end{bmatrix} + \begin{bmatrix} \tilde{D}_{p1} & \tilde{D}_{p2} \end{bmatrix} \begin{bmatrix} N_{p1} \\ N_{p2} \end{bmatrix} = -\tilde{N}_p D_p + \tilde{D}_p N_p = 0 , \tag{4.3.21}$$

hence, $\tilde{N}_p D_p = \tilde{D}_p N_p$. By Corollary 2.4.4, equations (4.3.20)-(4.3.21) imply that

$$\det \tilde{D}_p \approx \det D_p , \tag{4.3.22}$$

where, $\det D_p \in I$ since (N_p, D_p) is an r.c.f.r. of P ; hence, $\det\tilde{D}_p \in I$ by equation (4.3.22).

Now we have established that $\tilde{D}_p^{-1} \in \mathcal{m}(G)$; therefore equation (4.3.21) implies that $P = \tilde{D}_p^{-1}\tilde{N}_p$. Finally, $(\tilde{D}_p, \tilde{N}_p)$ defined by equations (4.3.6)-(4.3.7) is an l.c.f.r. of P.

By Theorem 4.3.4R, P can be H–stabilized by a decentralized compensator C_d if and only if (N_p, D_p) satisfies conditions (4.3.4)-(4.3.5); we have shown above that conditions (4.3.4)-(4.3.5) imply that an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ satisfies conditions (4.3.6)-(4.3.7). It is entirely similar to show the converse, and thus we omit the proof of sufficiency.

□

Theorem 4.3.4R states that P can be H–stabilized by a decentralized compensator C_d if and only if conditions (4.3.4)-(4.3.5) are satisfied. So in Theorem 4.3.7 below, we assume that some r.c.f.r. (N_p, D_p) of P satisfies these conditions in addition to Assumption 4.2.1 (A) in order to find the class of all H–stabilizing compensators. Equation (4.2.7) is once again the key tool.

4.3.7. Theorem (Set of all H–stabilizing decentralized compensators):

Let $P \in \mathcal{m}(G_S)$ satisfy Assumption 4.2.1 (A); let in addition an r.c.f.r. (N_p, D_p) of P satisfy conditions (4.3.4) and (4.3.5) of Theorem 4.3.4R; equivalently, let an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ of P satisfy conditions (4.3.6) and (4.3.7) of Theorem 4.3.5L. Under these conditions the set $\mathrm{S}_d(P)$ of *all* H–stabilizing decentralized compensators for P is given by

$$\mathrm{S}_d(P) := \left\{ \; C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{c1}^{-1}\tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2}^{-1}\tilde{N}_{c1} \end{bmatrix} : \right.$$

$$\begin{bmatrix} \left[\tilde{D}_{c1} \vdots \tilde{N}_{c1}\right] & \vdots & 0 \\ 0 & \vdots & \left[\tilde{D}_{c2} \vdots \tilde{N}_{c2}\right] \end{bmatrix} = \begin{bmatrix} \left[I_{n_{i1}} \vdots Q_1\right]E_1^{-1} & \vdots & 0 \\ 0 & \vdots & \left[I_{n_{i2}} \vdots Q_2\right]E_2^{-1} \end{bmatrix},$$

for some $Q_1 \in H^{n_{i1} \times n_{o1}}, Q_2 \in H^{n_{i2} \times n_{o2}}$ such that $\left.\det(I_{n_{i2}} - Q_2W_2Q_1W_1) \in J \right\}$; (4.3.23)

equivalently,

$$\mathbf{S}_d(P) := \Big\{ \; C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} N_{c1}D_{c1}^{-1} & 0 \\ 0 & N_{c2}D_{c2}^{-1} \end{bmatrix} :$$

$$\begin{bmatrix} -N_{c1} & 0 \\ D_{c1} & 0 \\ \cdots & \cdots \\ 0 & -N_{c2} \\ 0 & D_{c2} \end{bmatrix} = \begin{bmatrix} E_1\begin{bmatrix} -Q_1 \\ I_{n_{o1}} \end{bmatrix} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & E_2\begin{bmatrix} -Q_2 \\ I_{n_{o2}} \end{bmatrix} \end{bmatrix},$$

for some $Q_1 \in H^{n_{i1} \times n_{o1}}$, $Q_2 \in H^{n_{i2} \times n_{o2}}$ such that $\det(I_{n_{i2}} - Q_2W_2Q_1W_1) \in J \Big\}$ (4.3.24)

the map $(Q_1, Q_2) \mapsto C_d$, Q_1 , $Q_2 \in \mathcal{M}(H)$, such that $\det(I_{n_{i2}} - Q_2W_2Q_1W_1) \in J$, $C_d \in \mathbf{S}_d(P)$, is a bijection; for the same pair (Q_1, Q_2) , equations (4.3.23) and (4.3.24) give the same H–stabilizing C_d .

4.3.8. Comments:

(i) In conditions (4.3.4)-(4.3.5) (equivalently, (4.3.6)-(4.3.7)) if either one of W_1 or W_2 is the zero matrix (i.e., if both of $D_{12} = 0$ and $N_{12} = 0$ in equation (4.3.4) or both of $D_{21} = 0$ and $N_{21} = 0$ in equation (4.3.5)), then for *all* $Q_1, Q_2 \in \mathcal{M}(H)$, $\det(I_{n_i} + QW) := \det\begin{bmatrix} I_{n_{i1}} & Q_1W_1 \\ Q_2W_2 & I_{n_{i2}} \end{bmatrix} =$ $\det(I_{n_{i2}} - Q_2W_2Q_1W_1) = \det(I_{n_{i1}} - Q_1W_1Q_2W_2) = 1$ and hence, the set $\mathbf{S}_d(P)$ in equation (4.3.23) (or (4.3.24)) is parametrized by two *free* parameters Q_1 and $Q_2 \in \mathcal{M}(H)$.

(ii) In Theorem 4.3.7, if $P \in \mathcal{M}(G)$ instead of $\mathcal{M}(G_s)$, then the matrices $Q_1 \in \mathcal{M}(H)$ and $Q_2 \in \mathcal{M}(H)$ should be chosen so that $\det\tilde{D}_{c1} := \det(\begin{bmatrix} I_{n_{i1}} \vdots Q_1 \end{bmatrix} E_1^{-1} \begin{bmatrix} I_{n_{i1}} \\ 0 \end{bmatrix}) \in I$ and $\det\tilde{D}_{c2} := \det(\begin{bmatrix} I_{n_{i2}} \vdots Q_2 \end{bmatrix} E_2^{-1} \begin{bmatrix} I_{n_{i2}} \\ 0 \end{bmatrix}) \in I$ *in addition to* $\det(I_{n_i} + QW) \in J$.

□

Proof of Theorem 4.3.7:

We only prove equation (4.3.23); the proof of (4.3.24) is entirely similar.

If C_d is given by the expression in equation (4.3.23), then C_d H–stabilizes P:

With (N_p, D_p) as in conditions (4.3.4)-(4.3.5) and $(\tilde{D}_c, \tilde{N}_c)$ given as in the expression (4.3.23), we obtain

$$\begin{bmatrix} \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} & \vdots & 0 \\ 0 & \vdots & \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \end{bmatrix} \begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} I_{n_{i1}} \vdots Q_1 \end{bmatrix} E_1^{-1} & \vdots & 0 \\ 0 & \vdots & \begin{bmatrix} I_{n_{i2}} \vdots Q_2 \end{bmatrix} E_2^{-1} \end{bmatrix} \begin{bmatrix} E_1 \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix} \\ \cdots \\ E_2 \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & Q_1 W_1 \\ Q_2 W_2 & I_{n_{i2}} \end{bmatrix} =: R. \quad (4.3.25)$$

The matrix R defined in equation (4.3.25) is H–unimodular since by assumption, Q_1 , Q_2 $\in \mathcal{M}(H)$ satisfy $\det(I_{n_{i2}} - Q_2 W_2 Q_1 W_1) \in J$. Therefore equation (4.2.7) is satisfied; equivalently,

$$\tilde{D}_c D_p R^{-1} + \tilde{N}_c N_p R^{-1} = I_{n_i} . \quad (4.3.26)$$

Note that $P \in \mathcal{M}(G_S)$ and N_p , $\tilde{N}_c N_p R^{-1} \in \mathcal{M}(G_S)$; hence equation (4.3.26) implies that $\det\tilde{D}_c = \det\tilde{D}_{c1}\det\tilde{D}_{c2} \in I$ and therefore, by Lemma 2.2.4.(ii), $\det\tilde{D}_{c1} \in I$ and $\det\tilde{D}_{c2} \in I$. By equation (4.3.23), the pair $(\tilde{D}_{c1}, \tilde{N}_{c1})$ is l.c. in $\mathcal{M}(H)$ since $\begin{bmatrix} \tilde{D}_{c1} \vdots \tilde{N}_{c1} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} \vdots Q_1 \end{bmatrix} E_1^{-1}$, where $E_1^{-1} \in \mathcal{M}(H)$ is H–unimodular. Similarly, $(\tilde{D}_{c2}, \tilde{N}_{c2})$ is also l.c. Therefore, using equation (4.3.26) and the same reasoning as in the (sufficiency) proof of Theorem 4.3.4R, we see that this C_d H–stabilizes P.

Any decentralized compensator C_d that H–stabilizes P is given by the expression in equation (4.3.23) for some unique pair $Q_1, Q_2 \in \mathcal{M}(H)$ such that $\det(I_{ni2} - Q_2W_2Q_1W_1) \in J$:

The pair (N_p, D_p) satisfies conditions (4.3.4)-(4.3.5); we have the generalized Bezout identity in equation (4.3.19). By assumption, $C_d = \begin{bmatrix} \tilde{D}_{c1}^{-1}\tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2}^{-1}\tilde{N}_{c1} \end{bmatrix}$ H–stabilizes P , where $\tilde{D}_{c1}, \tilde{D}_{c2}, \tilde{N}_{c1}, \tilde{N}_{c2} \in \mathcal{M}(H)$; hence by equation (4.2.7), $\det D_{H1} \in J$. By normalizing D_{H1}, with $Q_1 := \begin{bmatrix} \tilde{D}_{c1} & \vdots & \tilde{N}_{c1} \end{bmatrix} E_1 \begin{bmatrix} 0 \\ \cdots \\ I_{no1} \end{bmatrix} \in \mathcal{M}(H)$, $Q_2 := \begin{bmatrix} \tilde{D}_{c2} & \vdots & \tilde{N}_{c2} \end{bmatrix} E_2 \begin{bmatrix} 0 \\ \cdots \\ I_{no2} \end{bmatrix} \in \mathcal{M}(H)$, we obtain $\tilde{D}_c D_p + \tilde{N}_c N_p = I_{ni}$; hence

$$\begin{bmatrix} \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \end{bmatrix} \begin{bmatrix} E_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2 \end{bmatrix} \begin{bmatrix} I_{ni1} & 0 & 0 & 0 \\ 0 & W_1 & I_{no1} & 0 \\ 0 & I_{ni2} & 0 & 0 \\ W_2 & 0 & 0 & I_{no2} \end{bmatrix} = \begin{bmatrix} I_{ni1} & 0 & Q_1 & 0 \\ 0 & I_{ni2} & 0 & Q_2 \end{bmatrix}. \tag{4.3.27}$$

Post-multiplying both sides of equation (4.3.27) by the first H–unimodular matrix in equation (4.3.19), we obtain

$$\begin{bmatrix} \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_{ni1} & 0 & Q_1 & 0 \\ 0 & I_{ni2} & 0 & Q_2 \end{bmatrix} \begin{bmatrix} I_{ni1} & 0 & 0 & 0 \\ 0 & 0 & I_{ni2} & 0 \\ 0 & I_{no1} & -W_1 & 0 \\ -W_2 & 0 & 0 & I_{no2} \end{bmatrix} \begin{bmatrix} E_1^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} & \begin{bmatrix} -Q_1W_1 & 0 \end{bmatrix} E_2^{-1} \\ \begin{bmatrix} -Q_2W_2 & 0 \end{bmatrix} E_1^{-1} & \begin{bmatrix} I_{ni2} & Q_2 \end{bmatrix} E_2^{-1} \end{bmatrix} ; \tag{4.3.28}$$

By equation (4.3.28), $\begin{bmatrix} -Q_2W_2 & 0 \end{bmatrix} E_1^{-1} = 0$ implies that $Q_2W_2 = 0$; similarly, $\begin{bmatrix} -Q_1W_1 & 0 \end{bmatrix} E_2^{-1} = 0$ implies that $Q_1W_1 = 0$. Therefore $\det(I_{ni2} - Q_2W_2Q_1W_1) = 1$ for this choice of $Q_1, Q_2 \in \mathcal{M}(H)$. It is also clear that $Q_1W_1 = \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} E_1 \begin{bmatrix} 0 \\ W_1 \end{bmatrix} = 0$ and $Q_2W_2 = \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} E_2 \begin{bmatrix} 0 \\ W_2 \end{bmatrix} = 0$ from equation (4.3.27).

Finally, equation (4.3.28) shows that $(\tilde{D}_{c1}, \tilde{N}_{c1})$ and $(\tilde{D}_{c2}, \tilde{N}_{c2})$ are of the form given by the expression in (4.3.23).

Now we prove that the matrices $Q_1, Q_2 \in \mathcal{M}(H)$ define C_1 and C_2 uniquely:

Let $C_d = \begin{bmatrix} \tilde{D}_{c1} & 0 \\ 0 & \tilde{D}_{c2} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{c1} & 0 \\ 0 & \tilde{N}_{c2} \end{bmatrix} \in \mathcal{S}_d(P)$, and $\hat{C}_d = \begin{bmatrix} \hat{\tilde{D}}_{c1} & 0 \\ 0 & \hat{\tilde{D}}_{c2} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\tilde{N}}_{c1} & 0 \\ 0 & \hat{\tilde{N}}_{c2} \end{bmatrix} \in \mathcal{S}_d(P)$,

where $\mathcal{S}_d(P)$ is given in equation (4.3.23). By equation (4.3.23),

$$\begin{bmatrix} \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} E_1 & 0 \\ 0 & \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} E_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{ni1} & \vdots & Q_1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_{ni2} & \vdots & Q_2 \end{bmatrix} \end{bmatrix}, \tag{4.3.29}$$

and

$$\begin{bmatrix} \begin{bmatrix} \hat{\tilde{D}}_{c1} & \hat{\tilde{N}}_{c1} \end{bmatrix} E_1 & 0 \\ 0 & \begin{bmatrix} \hat{\tilde{D}}_{c2} & \hat{\tilde{N}}_{c2} \end{bmatrix} E_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{ni1} & \vdots & \hat{Q}_1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_{ni2} & \vdots & \hat{Q}_2 \end{bmatrix} \end{bmatrix}; \tag{4.3.30}$$

then $C_d = \hat{C}_d$ implies that $\begin{bmatrix} I_{ni1} & C_1 \end{bmatrix} E_1 = \tilde{D}_{c1}^{-1} \begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} = \hat{\tilde{D}}_{c1}^{-1} \begin{bmatrix} I_{ni1} & \hat{Q}_1 \end{bmatrix}$ and hence, $\tilde{D}_{c1}^{-1} = \hat{\tilde{D}}_{c1}^{-1}$; consequently, $Q_1 = \hat{Q}_1$. Similarly, $\begin{bmatrix} I_{ni2} & C_2 \end{bmatrix} E_2 = \tilde{D}_{c2}^{-1} \begin{bmatrix} I_{ni2} & Q_2 \end{bmatrix} = \hat{\tilde{D}}_{c2}^{-1} \begin{bmatrix} I_{ni2} & \hat{Q}_2 \end{bmatrix}$ and hence, $\tilde{D}_{c2}^{-1} = \hat{\tilde{D}}_{c2}^{-1}$; consequently, $Q_2 = \hat{Q}_2$.

Now let C_d be given by an l.c.f.r. as in equation (4.3.23) but $\hat{C}_d$ be given by an r.c.f.r. (N_c, D_c) as in equation (4.3.24); then

$$\begin{bmatrix} E_1^{-1}\begin{bmatrix} -N_{c1} \\ D_{c1} \end{bmatrix} & 0 \\ 0 & E_2^{-1}\begin{bmatrix} -N_{c2} \\ D_{c2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\hat{Q}_1 \\ I_{n_{o1}} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} -\hat{Q}_2 \\ I_{n_{o2}} \end{bmatrix} \end{bmatrix}. \tag{4.3.31}$$

By equations (4.3.29) and (4.3.31),

$$\begin{bmatrix} \begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} \end{bmatrix} E_1 & 0 \\ 0 & \begin{bmatrix} \tilde{D}_{c2} & \tilde{N}_{c2} \end{bmatrix} E_2 \end{bmatrix} \begin{bmatrix} E_1^{-1}\begin{bmatrix} -N_{c1} \\ D_{c1} \end{bmatrix} & 0 \\ 0 & E_2^{-1}\begin{bmatrix} -N_{c2} \\ D_{c2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\hat{Q}_1 + Q_1 & 0 \\ 0 & -\hat{Q}_2 + Q_2 \end{bmatrix} \tag{4.3.32}$$

By equation (4.3.32), $C_d = \hat{C}_d$, equivalently, $\tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1}$, ($-\tilde{D}_{c1}N_{c1} + \tilde{N}_{c1}D_{c1} = 0$ and $-\tilde{D}_{c2}N_{c2} + \tilde{N}_{c2}D_{c2} = 0$) if and only if $-\hat{Q}_1 + Q_1 = 0$ and $-\hat{Q}_2 + Q_2 = 0$. Therefore for the same $Q_1 \in H^{n_{i1} \mathrm{x} n_{o1}}$ and $Q_2 \in H^{n_{i2} \mathrm{x} n_{o2}}$, equations (4.3.23) and (4.3.24) give the same C_1 and C_2.

□

4.4. Application to stable rational functions

In this section we consider the case when $H = R_u(s)$ as in Example 2.2.2. Working in this principal ring allows us to show the connection between our results and those of [Wan., And.1]. The major result in this section is that P satisfies conditions (4.3.4)-(4.3.5) of Theorem 4.3.4R (equivalently, conditions (4.3.6)-(4.3.7) of Theorem 4.3.5L) if and only if the system has no *fixed-eigenvalues* in $\bar{\mathcal{U}}$. Therefore, for $H = R_u(s)$, Theorem 4.3.4R becomes equivalent to [Wan.1, Theorem 1]: P can be H-stabilized by a decentralized dynamic compensator if and only if it has no decentralized fixed-eigenvalues in $\bar{\mathcal{U}}$.

In [And.1], a rank test for fixed-eigenvalues was given in terms of a left-fraction representation of the plant P. We find that a similar test is useful in our approach; we give rank conditions in terms of an r.c.f.r., an l.c.f.r. and a b.c.f.r. of P. We start our discussion by considering real *constant* decentralized compensators.

Consider the system $S(P, K_d)$ in Figure 4.6; let $K_d := \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, $K_1 \in \mathbb{R}^{n_{i1} \times n_{o1}}$, $K_2 \in \mathbb{R}^{n_{i2} \times n_{o2}}$. Note that $S(P, K_d)$ in Figure 4.6 is the same as $S(P, C_d)$ in Figure 4.1, where $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ is replaced by the *real constant* matrix $K_d = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$.

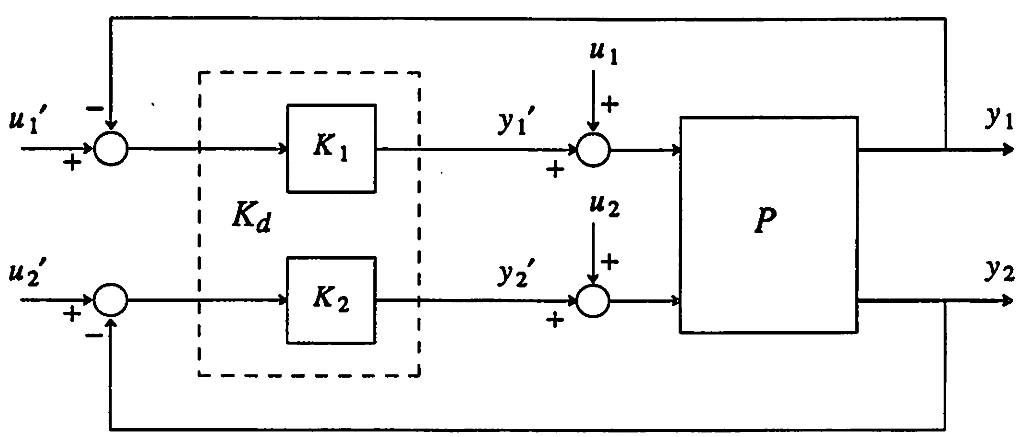

Figure 4.6: The constant output-feedback decentralized control system $S(P, K_d)$.

The plant P still satisfies Assumption 4.2.1 (A) of Section 4.2, where H is replaced by $R_u(s)$. Equations (4.2.4)-(4.2.5) are now replaced by equations (4.4.1)-(4.4.2) describing the system $S(P, K_d)$ with constant decentralized output-feedback control:

$$\begin{bmatrix} D_{p1}+K_1N_{p1} \\ D_{p2}+K_2N_{p2} \end{bmatrix} \xi_p = \begin{bmatrix} I_{ni1} & 0 & K_1 & 0 \\ 0 & I_{ni2} & 0 & K_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}, \tag{4.4.1}$$

$$\begin{bmatrix} N_{p1} \\ N_{p2} \\ D_{p1} \\ D_{p2} \end{bmatrix} \xi_p = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_{ni1} & 0 & 0 & 0 \\ 0 & I_{ni2} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_1' \\ u_2' \end{bmatrix}. \tag{4.4.2}$$

The closed-loop system $S(P, K_d)$, described by equations (4.4.1)-(4.4.2), is H-stable if and only if $\det\begin{bmatrix} D_{p1}+K_1N_{p1} \\ D_{p2}+K_2N_{p2} \end{bmatrix} \in J$. Furthermore, $s_o \in \bar{\mathcal{U}}$ is an eigenvalue of the closed-loop system if and only if

$$\det\begin{bmatrix} D_{p1}(s_o)+K_1N_{p1}(s_o) \\ D_{p2}(s_o)+K_2N_{p2}(s_o) \end{bmatrix} = 0. \tag{4.4.3}$$

4.4.1. Definition (Decentralized fixed-eigenvalue):

The plant P is said to have a *decentralized fixed-eigenvalue* (or *fixed-pole) at* $s_o \in \bar{\mathcal{U}}$ (with respect to $K_d = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$) iff $\det\begin{bmatrix} D_{p1}(s_o)+K_1N_{p1}(s_o) \\ D_{p2}(s_o)+K_2N_{p2}(s_o) \end{bmatrix} = 0$ *for all* $K_1, K_2 \in \mathcal{M}(\mathbb{R})$.

If $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue (fixed-pole), then obviously $s_o \in \bar{\mathcal{U}}$ is an eigenvalue of the open-loop system (i.e., with $K_1 = 0, K_2 = 0$, $\det\begin{bmatrix} D_{p1}(s_o) \\ D_{p2}(s_o) \end{bmatrix} = 0$ and hence, s_o is an eigenvalue of P); this eigenvalue $s_o \in \bar{\mathcal{U}}$ remains as a pole of the closed-loop system for *all* real constant decentralized feedback compensators. We prefer to call such $s_o \in \bar{\mathcal{U}}$ a *fixed-eigenvalue* rather than a *fixed-mode*; although the eigenvalue at $s_o \in \bar{\mathcal{U}}$ remains fixed irrespective of the constant decentralized compensator, the *eigenvector* v_o associated with the fixed-eigenvalue $s_o \in \bar{\mathcal{U}}$

depends on K_1 and K_2. Therefore the "mode" $v_o e^{s_o t}$ changes "direction" depending on the choice of constant decentralized control; equivalently, the initial condition that sets up the mode $v_o e^{s_o t}$ varies with K_1, K_2 although the eigenvalue at $s_o \in \bar{\mathcal{U}}$ does not move.

In Definition 4.4.1, fixed-eigenvalues are defined as those eigenvalues of the plant which cannot be moved by any *real constant* decentralized feedback. We will later establish that these fixed-eigenvalues remain fixed even under *dynamic* decentralized output-feedback, in particular under *complex* constant decentralized output-feedback.

Theorem 4.4.2R is the main result of this section. Theorem 4.4.3L and Theorem 4.4.4B are dual results for an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ and a b.c.f.r. (N_{pr}, D, N_{pl}), respectively.

4.4.2R. Theorem (Rank test on (N_p, D_p) for fixed-eigenvalues and $\mathcal{H}$–stabilizability):

Let $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$, $P = N_p D_p^{-1}$ satisfy Assumption 4.2.1 (A) where $\mathcal{H}$ is $R_u(s)$. Then statements (i), (ii), (iii), (iv) below are equivalent:

(i) The plant P has no decentralized fixed-eigenvalues in $\bar{\mathcal{U}}$;

(ii) for any r.c.f.r. (N_p, D_p) of P as in Assumption 4.2.1 (A),

$$rank \begin{bmatrix} D_{p1}(s) \\ N_{p1}(s) \end{bmatrix} \geq n_{i1}, \quad \text{for all } s \in \bar{\mathcal{U}}, \text{ and} \tag{4.4.4}$$

$$rank \begin{bmatrix} D_{p2}(s) \\ N_{p2}(s) \end{bmatrix} \geq n_{i2}, \quad \text{for all } s \in \bar{\mathcal{U}}; \tag{4.4.5}$$

(iii) conditions (4.3.4)-(4.3.5) of Theorem 4.3.4R hold; i.e., an r.c.f.r. (N_p, D_p) of P can be chosen so that

$$\begin{bmatrix} D_{p1}(s) \\ N_{p1}(s) \end{bmatrix} = E_1(s) \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1(s) \end{bmatrix}, \tag{4.4.6}$$

where $E_1(s) \in \mathcal{M}(R_u(s))$ is R_u–unimodular and $W_1(s) \in \mathcal{M}(R_u(s))$, and

$$\begin{bmatrix} D_{p2}(s) \\ N_{p2}(s) \end{bmatrix} = E_2(s) \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2(s) & 0 \end{bmatrix}, \tag{4.4.7}$$

where $E_2(s) \in \mathcal{M}(R_u(s))$ is R_u–unimodular and $W_2(s) \in \mathcal{M}(R_u(s))$;

(iv) there exists a dynamic decentralized compensator $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ (satisfying Assumption 4.2.1 (B)) which $\mathcal{H}$–stabilizes P.

4.4.3L. Theorem (Rank test on $(\tilde{D}_p, \tilde{N}_p)$ for fixed-eigenvalues and $\mathcal{H}$–stabilizability):

Let $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$, $P = \tilde{D}_p^{-1}\tilde{N}_p$ satisfy Assumption 4.2.1 (A) where $\mathcal{H}$ is $R_u(s)$; then statements (i), (ii), (iii), (iv) below are equivalent:

(i) The plant P has no decentralized fixed-eigenvalues in $\bar{\mathcal{U}}$;

(ii) for any l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ of P as in Assumption 4.2.1 (A),

$$rank\left[-\tilde{N}_{p1}(s) \;\vdots\; \tilde{D}_{p1}(s) \right] \geq n_{o1}, \quad \text{for all } s \in \bar{\mathcal{U}}, \text{ and} \tag{4.4.8}$$

$$rank\left[-\tilde{N}_{p2}(s) \;\vdots\; \tilde{D}_{p2}(s) \right] \geq n_{o2}, \quad \text{for all } s \in \bar{\mathcal{U}}; \tag{4.4.9}$$

(iii) conditions (4.3.6)-(4.3.7) of Theorem 4.3.5L hold; i.e., an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ of P can be chosen so that

$$\left[-\tilde{N}_{p1}(s) \;\vdots\; \tilde{D}_{p1}(s) \right] = \begin{bmatrix} 0 & I_{n_{o1}} \\ -W_2(s) & 0 \end{bmatrix} E_1(s)^{-1}, \tag{4.4.10}$$

where $E_1(s) \in \mathcal{M}(R_u(s))$ is R_u–unimodular and $W_2(s) \in \mathcal{M}(R_u(s))$, and

$$\left[-\tilde{N}_{p2}(s) \;\vdots\; \tilde{D}_{p2}(s) \right] = \begin{bmatrix} -W_1(s) & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} E_2(s)^{-1}, \tag{4.4.11}$$

where $E_2(s) \in \mathcal{M}(R_u(s))$ is R_u–unimodular and $W_1(s) \in \mathcal{M}(R_u(s))$,

(iv) there exists a dynamic decentralized compensator $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ (satisfying Assumption 4.2.1 (B)) which $\mathcal{H}$–stabilizes P.

4.4.4B. Theorem (Rank test on (N_{pr}, D, N_{pl}) for fixed-eigenvalues and H–stabilizability):

Let $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$, $P = N_{pr}D^{-1}N_{pl}$ satisfy Assumption 4.2.1 (A) where $H = R_{\mathcal{U}}(s)$; then statements (i), (ii), (iii) below are equivalent:

(i) The plant P has no decentralized fixed-eigenvalues in $\bar{\mathcal{U}}$;

(ii) for any b.c.f.r. (N_{pr}, D, N_{pl}) of P as in Assumption 4.2.1 (A),

$$rank\begin{bmatrix} D(s) & -N_{pl2}(s) \\ N_{pr1}(s) & 0 \end{bmatrix} \geq n \,, \quad \text{for all } s \in \bar{\mathcal{U}}, \text{ and} \qquad (4.4.4B)$$

$$rank\begin{bmatrix} D(s) & -N_{pl1}(s) \\ N_{pr2}(s) & 0 \end{bmatrix} \geq n \,, \quad \text{for all } s \in \bar{\mathcal{U}}; \qquad (4.4.5B)$$

(iii) there exists a dynamic decentralized compensator $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ (satisfying Assumption 4.2.1 (B)) which H–stabilizes P.

4.4.5S. Remark (State-space description of P):

Consider $P = C(sI_n - A)^{-1}B$, where (C, A, B) is $\bar{\mathcal{U}}$–stabilizable and $\bar{\mathcal{U}}$–detectable. Let $N_{pr} :=$ $\dfrac{C}{s+a} = \dfrac{1}{s+a}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, $D := \left(\dfrac{sI_n - A}{s+a}\right)$, $N_{pl} := B = \begin{bmatrix} B_1 \vdots B_2 \end{bmatrix}$, where $-a \in \mathbb{C}\backslash\bar{\mathcal{U}}$; then (N_{pr}, D, N_{pl}) as defined here is a b.c.f.r. of P. By Theorem 4.4.4B, the plant has no fixed-eigenvalues in $\bar{\mathcal{U}}$ if and only if conditions (4.4.4S)-(4.4.5S) below hold [And.1]:

$$rank\begin{bmatrix} sI_n - A & -B_2 \\ C_1 & 0 \end{bmatrix} \geq n \,, \quad \text{for all } s \in \bar{\mathcal{U}}, \text{ and} \qquad (4.4.4S)$$

$$rank\begin{bmatrix} sI_n - A & -B_1 \\ C_2 & 0 \end{bmatrix} \geq n \,, \quad \text{for all } s \in \bar{\mathcal{U}}; \qquad (4.4.5S)$$

we omitted the factor $\dfrac{1}{s+a}$ in equations (4.4.4S) and (4.4.5S) for simplicity.

Note that conditions (4.4.4S)-(4.4.5S) need to be checked only for those $s \in \bar{\mathcal{U}}$ such that $\det(sI_n - A) = 0$. The derivation of conditions (4.4.4S)-(4.4.5S) is very simple due to Theorem 4.4.4B.

4.4.6. Comments:

(i) Theorem 4.4.2R states that $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue if and only if *either* $rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} < n_{i1}$ *or* $rank\begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} < n_{i2}$. Note that conditions (4.4.4) and (4.4.5) cannot *both* fail at the same time: if both conditions were not satisfied, then $rank\begin{bmatrix} D_p(s_o) \\ N_p(s_o) \end{bmatrix} \le rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} + rank\begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} < n_{i1} + n_{i2}$, which contradicts that (N_p, D_p) is a r.c. pair. Therefore, if $rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} = \alpha < n_{i1}$, then $rank\begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} \ge n_{i2} + \alpha$ so that $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue but *not* an eigenvalue associated with a hidden-mode.

Similarly, conditions (4.4.8) and (4.4.9), conditions (4.4.4B)-(4.4.5B) or conditions (4.4.4S)-(4.4.5S) cannot fail at the same time.

(ii) Theorem 4.4.2R states that if the system has no fixed-eigenvalues in $\bar{\mathcal{U}}$, then the Smith form of $\begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix}$ is $\begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix}$ (here we assume that W_1 is also put in the Smith form), *and* at the same time the Smith form of $\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix}$ is $\begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix}$ (here W_2 is also put in the Smith form and appropriate column permutations are made). Equivalently, the first n_{i1} invariant factors of $\begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix}$ are equal to 1 and the first n_{i2} invariant factors of $\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix}$ are equal to 1. Hence, $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue of P if and only if *either* the $n_{i1}th$ invariant factor of $\begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix}$ is zero at $s_o \in \bar{\mathcal{U}}$ *or* the $n_{i2}th$ invariant factor of $\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix}$ is zero at $s_o \in \bar{\mathcal{U}}$.

(iii) Let $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$; then in equations (4.4.6)-(4.4.7), since $N_{p1}, N_{p2} \in \mathcal{M}(\mathbb{R}_{sp}(s))$, W_1 and $W_2 \in \mathcal{M}(\mathbb{R}_{sp}(s))$; hence, for $k = 1, 2$, $rank\begin{bmatrix} D_{pk}(\infty) \\ N_{pk}(\infty) \end{bmatrix} \le n_{ik}$. Hence if conditions (4.4.4)-(4.4.5) hold, then $\begin{bmatrix} D_{pk} \\ N_{pk} \end{bmatrix}$ has *exactly* n_{ik} invariant factors that are equal to 1.

(iv) Conditions (4.4.4B)-(4.4.5B) of Theorem 4.4.4B need to be checked only for all $s \in \bar{\mathcal{U}}$ such that $\det D(s) = 0$ (in Remark 4.4.5S, for all $s \in \bar{\mathcal{U}}$ such that $\det(sI_n - A) = 0$) since $rank D(s) = n$ for all other $s \in \bar{\mathcal{U}}$. In other words, if $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue, then s_o is an $\bar{\mathcal{U}}$–pole of $P = N_{pr} D^{-1} N_{pl}$.

(v) From conditions (4.4.4B)-(4.4.5B) of Theorem 4.4.4B, we obtain the following conditions on fixed-eigenvalues: Rewrite P as $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} N_{pr1} D^{-1} N_{pl1} & N_{pr1} D^{-1} N_{pl2} \\ N_{pr2} D^{-1} N_{pl1} & N_{pr2} D^{-1} N_{pl2} \end{bmatrix}$.

(a) (*A sufficient condition for no fixed-eigenvalues in* $\bar{\mathcal{U}}$): If (N_{pr1}, D, N_{pl1}) is a b.c.f.r. of P_{11}, then the plant P has no fixed-eigenvalues in $\bar{\mathcal{U}}$; (the same holds if (N_{pr2}, D, N_{pl2}) is a b.c.f.r. of P_{22}). This claim follows from noting that $rank \begin{bmatrix} D(s) \\ N_{pr1}(s) \end{bmatrix} = n$ for all $s \in \bar{\mathcal{U}}$ since (N_{pr1}, D) is an r.c. pair (hence condition (4.4.4B) holds) *and* that $rank \begin{bmatrix} D(s) \vdots -N_{pl1}(s) \end{bmatrix} = n$ for all $s \in \bar{\mathcal{U}}$ (hence condition (4.4.5B) holds). We can state this same condition in the state-space setting of Remark 4.4.5S where $P_{11} = C_1 (sI_n - A)^{-1} B_1$: if $(C_1, (sI_n - A), B_1)$ is $\bar{\mathcal{U}}$ –stabilizable and $\bar{\mathcal{U}}$–detectable (sometimes referred to as single-channel minimality), then P has no fixed-eigenvalues in $\bar{\mathcal{U}}$.

(b) (*Some necessary conditions on the transmission-zeros of the partial maps* P_{ij} *if* $s_o \in \bar{\mathcal{U}}$ *is a fixed-eigenvalue):* (1) Let $s_o \in \bar{\mathcal{U}}$ be a fixed-eigenvalue; then either condition (4.4.4B) fails (and hence $s_o \in \bar{\mathcal{U}}$ is a transmission-zero (t.z.) of P_{12}) *or* condition (4.4.5B) fails (and hence $s_o \in \bar{\mathcal{U}}$ is a t.z. of P_{21}). (2) Let $n_{o1} = n_{i1}$ and $n_{o2} = n_{i2}$; if $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue, then s_o is a t.z. of P_{11}, P_{22} and of the plant P. To justify this claim, without loss of generality, let condition (4.4.4B) fail at $s_o \in \bar{\mathcal{U}}$; then $rank \begin{bmatrix} D(s_o) \\ N_{pr1}(s_o) \end{bmatrix} < n$ implies that $rank \begin{bmatrix} D(s_o) & -N_{pl1}(s_o) \\ N_{pr1}(s_o) & 0 \end{bmatrix} < n + n_{i1}$ (and hence $s_o \in \bar{\mathcal{U}}$ is a t.z. of P_{11}), and $rank \begin{bmatrix} D(s_o) \vdots -N_{pl2}(s_o) \end{bmatrix} < n$ implies that $s_o \in \bar{\mathcal{U}}$ is a t.z. of P_{22}. Finally, since condition

(4.4.4B) fails, $rank\begin{bmatrix} D(s_o) & -N_{pl2}(s_o) & \vdots & -N_{pl1}(s_o) \\ N_{pr1}(s_o) & 0 & \vdots & 0 \end{bmatrix} < n + n_{i1}$ implies that

$$rank\begin{bmatrix} D(s_o) & -N_{pl2}(s_o) & -N_{pl1}(s_o) \\ N_{pr1}(s_o) & 0 & 0 \\ \cdots & \cdots & \cdots \\ N_{pr2}(s_o) & 0 & 0 \end{bmatrix} < n + n_{i1} + n_{o2}$$ and hence, $s_o \in \bar{\mathcal{U}}$ is a t.z. of the plant P.

Existence of these transmission-zeros is similarly proved if we start by assuming that condition (4.4.5B) fails.

□

4.4.7. Corollary:

If $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue, then the system $S(P, C_d)$ also has a mode associated with $s_o \in \bar{\mathcal{U}}$ for *all dynamic* decentralized compensators C_d (in particular, for all complex constant decentralized compensators).

Proof:

By Theorem 4.4.2R, $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue if and only if either condition (4.4.4) or (4.4.5) fails. Suppose, without loss of generality, that $rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} < n_{i1}$. Then $rank(\begin{bmatrix} \tilde{D}_{c1}(s_o) & \tilde{N}_{c1}(s_o) \end{bmatrix} \cdot \begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix}) < n_{i1}$, for all $\tilde{D}_{c1}(s_o)$, $\tilde{N}_{c1}(s_o)$.

Therefore $rank\begin{bmatrix} (\tilde{D}_{c1}D_{p1} + \tilde{N}_{c1}N_{p1})(s_o) \\ (\tilde{D}_{c2}D_{p2} + \tilde{N}_{c2}N_{p2})(s_o) \end{bmatrix} \le rank\begin{bmatrix} (\tilde{D}_{c1}D_{p1} + \tilde{N}_{c1}N_{p1})(s_o) \end{bmatrix} + rank\begin{bmatrix} (\tilde{D}_{c2}D_{p2} + \tilde{N}_{c2}N_{p2})(s_o) \end{bmatrix} < n_{i1} + n_{i2}$, for all $\tilde{D}_{c1}(s_o)$, $\tilde{N}_{c1}(s_o)$, $\tilde{D}_{c2}(s_o)$, $\tilde{N}_{c2}(s_o)$; consequently, $s_o \in \bar{\mathcal{U}}$ is always a closed-loop eigenvalue of $S(P, C_d)$.

□

We only prove Theorem 4.4.2R in detail; the proof of Theorem 4.4.3L is very similar and follows from Theorem 4.3.5L. The proof of Theorem 4.4.4B follows from Theorem 4.4.2R (equivalently, Theorem 4.4.3L) and Comment 4.3.6.(iii).

Proof of Theorem 4.4.2R:

The equivalence of statements (iii) and (iv) was already established in Theorem 4.3.4R for any principal ring H; here we take $H = R_u(s)$. Now we prove the first three statements:

(i) <=> (ii)

(=>) Without loss of generality, suppose that condition (4.4.4) fails for some $s_o \in \bar{\mathcal{U}}$, i.e., let, for some $s_o \in \bar{\mathcal{U}}$,

$$rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} < n_{i1}; \qquad (4.4.12)$$

then $rank\begin{bmatrix} D_{p1}(s_o) + K_1 N_{p1}(s_o) \end{bmatrix} = rank\left(\begin{bmatrix} I_{n_{i1}} & K_1 \end{bmatrix}\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix}\right) \le rank\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \end{bmatrix} < n_{i1}$ for all $K_1 \in \mathcal{M}(\mathbb{R})$. So, $rank\begin{bmatrix} D_{p1}(s_o) + K_1 N_{p1}(s_o) \\ D_{p2}(s_o) + K_2 N_{p2}(s_o) \end{bmatrix} \le rank\begin{bmatrix} D_{p1}(s_o) + K_1 N_{p1}(s_o) \end{bmatrix} + rank\begin{bmatrix} D_{p2}(s_o) + K_2 N_{p2}(s_o) \end{bmatrix} < n_{i1} + n_{i2}$, for all $K_1, K_2 \in \mathcal{M}(\mathbb{R})$; therefore, by Definition 4.4.1, $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue.

The proof would be entirely similar if we started by assuming that condition (4.4.5) fails at some $s_o \in \bar{\mathcal{U}}$.

(<=) Let equations (4.4.4)-(4.4.5) hold but suppose, for a contradiction, that $s_o \in \bar{\mathcal{U}}$ is a fixed-eigenvalue. Then

$$\max_{K_1, K_2 \in \mathcal{M}(\mathbb{R})} rank\begin{bmatrix} D_{p1}(s_o) + K_1 N_{p1}(s_o) \\ D_{p2}(s_o) + K_2 N_{p2}(s_o) \end{bmatrix} < n_i, \qquad (4.4.13)$$

for $K_1, K_2 \in \mathcal{M}(\mathbb{R})$. Let $\hat{\mathbb{K}}_1 := \{ \hat{K}_1 \in \mathcal{M}(\mathbb{R}) : rank(D_{p1}(s_o) + \hat{K}_1 N_{p1}(s_o)) = n_{i1} \}$; by Lemma 2.7.1, $\hat{\mathbb{K}}_1$ is nonempty since (4.4.4) holds.

Choose $\hat{K}_1 \in \hat{\mathbb{K}}_1$; then there are R_u–unimodular matrices L_1, R_1 such that $L_1(D_{p1}(s_o) + \hat{K}_1 N_{p1}(s_o))R_1 = \begin{bmatrix} I_{n_{i1}} & 0 \end{bmatrix}$, where the 0 matrix on the right is $n_{i1} \times n_{i2}$. Let

$$\begin{bmatrix} L_1(D_{p1}(s_o)+\hat{K}_1N_{p1}(s_o)) \\ D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} R_1 =: \begin{bmatrix} I_{ni1} & 0 \\ \hat{D}_{21}(s_o) & \hat{D}_{22}(s_o) \\ \hat{N}_{21}(s_o) & \hat{N}_{22}(s_o) \end{bmatrix} ; \tag{4.4.14}$$

then $\max_{K_2 \in \mathcal{M}(\mathbb{R})} rank \begin{bmatrix} D_{p1}(s_o)+\hat{K}_1N_{p1}(s_o) \\ D_{p2}(s_o)+K_2N_{p2}(s_o) \end{bmatrix} = \max_{K_2 \in \mathcal{M}(\mathbb{R})} rank \begin{bmatrix} L_1(D_{p1}(s_o)+\hat{K}_1N_{p1}(s_o))R_1 \\ (D_{p2}(s_o)+K_2N_{p2}(s_o))R_1 \end{bmatrix} =$

$$\max_{K_2 \in \mathcal{M}(\mathbb{R})} rank\left(\begin{bmatrix} I_{ni1} & 0 & 0 \\ 0 & I_{ni2} & K_2 \end{bmatrix} \begin{bmatrix} I_{ni1} & 0 \\ \hat{D}_{21}(s_o) & \hat{D}_{22}(s_o) \\ \hat{N}_{21}(s_o) & \hat{N}_{22}(s_o) \end{bmatrix} \right) = n_{i1} + \max_{K_2 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{22}(s_o)+K_2\hat{N}_{22}(s_o));$$

and hence, by equation (4.4.13), $\max_{K_2 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{22}(s_o)+K_2\hat{N}_{22}(s_o)) < n_{i2}$. Therefore, by Lemma 2.7.1, with $A := \hat{N}_{22}(s_o)$, $B := \hat{D}_{22}(s_o)$, $K_2 \in \mathbb{R}^{n_{i2} \times n_{o2}}$, $\rho = \gamma := n_{i2}$, $\eta := n_{o2}$,

$$rank \begin{bmatrix} \hat{D}_{22}(s_o) \\ \hat{N}_{22}(s_o) \end{bmatrix} = \max_{K_2 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{22}(s_o)+K_2\hat{N}_{22}(s_o)) < n_{i2};$$

hence $rank \begin{bmatrix} I_{ni1} & 0 \\ \hat{D}_{21}(s_o) & \hat{D}_{22}(s_o) \\ \hat{N}_{21}(s_o) & \hat{N}_{22}(s_o) \end{bmatrix} = n_{i1} + rank \begin{bmatrix} \hat{D}_{22}(s_o) \\ \hat{N}_{22}(s_o) \end{bmatrix} < n_{i1}+n_{i2}$. Consequently, since equation (4.4.14) holds for all $\hat{K}_1 \in \hat{\mathrm{K}}_1$,

$$\max_{\hat{K}_1 \in \hat{\mathrm{K}}_1} rank \begin{bmatrix} D_{p1}(s_o)+\hat{K}_1N_{p1}(s_o) \\ D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} = \max_{K_1 \in \mathcal{M}(\mathbb{R})} rank \begin{bmatrix} D_{p1}(s_o)+K_1N_{p1}(s_o) \\ D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} < n_i . \tag{4.4.15}$$

Let $rank \begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} =: r_2$; by equation (4.4.5), $r_2 \geq n_{i2}$; then there are R_u–unimodular matrices L_2 , R_2 such that $L_2 \begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} R_2 = \begin{bmatrix} 0 & I_{r2} \\ 0 & 0 \end{bmatrix}$, where the 0 in the bottom left is $(n_{i2}+n_{o2}-r_2) \times (n_i-r_2)$. Let $\begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \\ L_2 \begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} \end{bmatrix} R_2 =: \begin{bmatrix} \hat{D}_{11}(s_o) & \hat{D}_{12}(s_o) \\ \hat{N}_{12}(s_o) & \hat{N}_{12}(s_o) \\ 0 & I_{r2} \\ 0 & 0 \end{bmatrix}$; then equation (4.4.15) implies that

$$\max_{K_1 \in \mathcal{M}(\mathbb{R})} rank \begin{bmatrix} D_{p1}(s_o)+K_1N_{p1}(s_o) \\ L_2\begin{bmatrix} D_{p2}(s_o) \\ N_{p2}(s_o)\end{bmatrix}\end{bmatrix} R_2 = \max_{K_1 \in \mathcal{M}(\mathbb{R})} rank \left(\begin{bmatrix} I_{n_{i1}} & K_1 & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 \end{bmatrix} \begin{bmatrix} \hat{D}_{11}(s_o) & \hat{D}_{12}(s_o) \\ \hat{N}_{11}(s_o) & \hat{N}_{12}(s_o) \\ 0 & I_{r_2} \\ 0 & 0 \end{bmatrix} \right)$$

$= r_2 + \max_{K_1 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{11}(s_o)+K_1\hat{N}_{11}(s_o)) < n_i$. But $n_i - r_2 \le n_{i1}$ since $r_2 \ge n_{i2}$; hence $\min\{ n_i-r_2 , n_{i1} \} = n_i-r_2$; therefore, $\max_{K_1 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{11}(s_o)+K_1\hat{N}_{11}(s_o)) < n_i-r_2$. Once again by Lemma 2.7.1, with $A := \hat{N}_{11}(s_o)$, $B := \hat{D}_{11}(s_o)$, $\rho = \gamma := n_{i1}$, $\eta := n_{o1}$, we obtain

$$rank \begin{bmatrix} \hat{D}_{11}(s_o) \\ \hat{N}_{11}(s_o) \end{bmatrix} = \max_{K_1 \in \mathcal{M}(\mathbb{R})} rank(\hat{D}_{11}(s_o)+K_1\hat{N}_{11}(s_o)) < n_i-r_2 . \qquad (4.4.16)$$

Finally, by equation (4.4.16), $rank \begin{bmatrix} D_{p1}(s_o) \\ N_{p1}(s_o) \\ D_{p2}(s_o) \\ N_{p2}(s_o) \end{bmatrix} = rank \begin{bmatrix} \hat{D}_{11}(s_o) & \hat{D}_{12}(s_o) \\ \hat{N}_{11}(s_o) & \hat{N}_{12}(s_o) \\ 0 & I_{r_2} \\ 0 & 0 \end{bmatrix} = rank \begin{bmatrix} \hat{D}_{11}(s_o) \\ \hat{N}_{11}(s_o) \end{bmatrix} + r_2$ $< n_i$; but this is a contradiction to $rank \begin{bmatrix} D_p(s) \\ N_p(s) \end{bmatrix} = n_i$, for all $s \in \bar{\mathcal{U}}$. Therefore, if (N_p, D_p) is an r.c. pair and if equations (4.4.4) and (4.4.5) hold, $s_o \in \bar{\mathcal{U}}$ cannot be a fixed-eigenvalue.

(ii) <=> (iii)

(<=) If conditions (4.4.6)-(4.4.7) hold for some r.c.f.r. (N_p, D_p) of P , then any other r.c.f.r. of P is of the same form as in these conditions except for $\mathcal{R}_u$–unimodular right-factors $R(s)$. Clearly then the rank conditions in (4.4.4)-(4.4.5) are satisfied for all r.c.f.r.'s of P since the matrices $E_1(s)$, $E_2(s)$ and the right-factors $R(s)$ are $\mathcal{R}_u$–unimodular.

(=>) Condition (4.4.4) implies that there are $\mathcal{R}_u$–unimodular matrices L_1 , $R_1 \in \mathcal{M}(\mathcal{R}_u(s))$ such that

$$L_1 \begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix} R_1 = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & \tilde{N}_{12} \end{bmatrix} , \tilde{N}_{12} \in \mathcal{R}_u(s)^{n_{o1} x n_{i2}} . \qquad (4.4.17)$$

Furthermore, by equation (4.4.5), there is an R_u–unimodular matrix $L_2 \in \mathcal{M}(R_u(s))$ such that

$$L_2(\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix} R_1) =: \begin{bmatrix} -\tilde{D}_{21} & \tilde{D}_{22} \\ \tilde{N}_{21} & 0 \end{bmatrix}, \text{ where } \tilde{D}_{22} \in R_u(s)^{n_{i2} x n_{i2}}, \tag{4.4.18}$$

$$\text{and } rank\begin{bmatrix} \tilde{D}_{21} & \vdots & \tilde{D}_{22} \end{bmatrix} = n_{i2}, \text{ for all } s \in \bar{\mathcal{U}}. \tag{4.4.19}$$

By equation (4.4.19), the pair $(\tilde{D}_{22}, \tilde{D}_{21})$ is l.c.; hence there are matrices V_{2l}, U_{2l}, X_2, Y_2, U_2, $V_2 \in \mathcal{M}(R_u(s))$ such that

$$\begin{bmatrix} V_2 & U_2 \\ -\tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} Y_2 & -U_{2l} \\ X_2 & V_{2l} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & I_{n_{i2}} \end{bmatrix}. \tag{4.4.20}$$

Now since (N_p, D_p) is an r.c. pair,

$$rank(\begin{bmatrix} L_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & L_2 \end{bmatrix} \begin{bmatrix} D_{p1}(s) \\ N_{p1}(s) \\ \cdots \\ D_{p2}(s) \\ N_{p2}(s) \end{bmatrix} R_1 = rank \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & \tilde{N}_{12}(s) \\ \cdots & \cdots \\ \tilde{D}_{21}(s) & \tilde{D}_{22}(s) \\ \tilde{N}_{21}(s) & 0 \end{bmatrix} = n_i, \text{ for all } s \in \bar{\mathcal{U}}. \tag{4.4.21}$$

Equation (4.4.21) implies that $rank\begin{bmatrix} \tilde{N}_{12}(s) \\ \tilde{D}_{22}(s) \end{bmatrix}(s) = n_{i2}$, for all $s \in \bar{\mathcal{U}}$; equivalently, $(\tilde{N}_{12}, \tilde{D}_{22})$ is an r.c. pair, and hence (recalling the Bezout identity) there are matrices V_{2r}, U_{2r}, $\tilde{X}_2$, $\tilde{Y}_2$, $\tilde{U}_2$, $\tilde{V}_2 \in \mathcal{M}(R_u(s))$ such that

$$\begin{bmatrix} V_{2r} & U_{2r} \\ -\tilde{X}_2 & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} \tilde{D}_{22} & -\tilde{U}_2 \\ \tilde{N}_{12} & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} I_{n_{i2}} & 0 \\ 0 & I_{n_{o1}} \end{bmatrix}. \tag{4.4.22}$$

From the two generalized Bezout identities (4.4.20) and (4.4.22) we obtain

$$\begin{bmatrix} V_2 + U_2 V_{2r} \tilde{D}_{21} & U_2 U_{2r} \\ -\tilde{X}_2 \tilde{D}_{21} & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} Y_2 & -U_{2l} \tilde{U}_2 \\ \tilde{N}_{12} X_2 & \tilde{V}_2 + \tilde{N}_{12} V_{2l} \tilde{U}_2 \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & I_{n_{o1}} \end{bmatrix}. \tag{4.4.23}$$

Now let

$$R_2 := \begin{bmatrix} Y_2 & -U_{2l} \\ X_2 & V_{2l} \end{bmatrix} \in R_u(s)^{n_i x n_i} ; \qquad (4.4.24)$$

and let

$$R := R_1 R_2 \begin{bmatrix} I_{n_{i1}} & U_2 V_{2r} \\ 0 & I_{n_{i2}} \end{bmatrix} \in R_u(s)^{n_i x n_i} . \qquad (4.4.25)$$

By equations (4.4.22), (4.4.24), and (4.4.17), R is R_u–unimodular. Let

$$E_1^{-1} := \begin{bmatrix} V_2 + U_2 V_{2r} \tilde{D}_{21} & U_2 U_{2r} \\ -\tilde{X}_2 \tilde{D}_{21} & \tilde{Y}_2 \end{bmatrix} L_1 \in R_u(s)^{(n_{i1}+n_{o1}) x (n_{i1}+n_{o1})} . \qquad (4.4.26)$$

By equations (4.4.23) and (4.4.17), E_1^{-1} is R_u–unimodular. Let

$$E_2^{-1} := \begin{bmatrix} I_{n_{i2}} & 0 \\ \tilde{N}_{21} U_{2l} \tilde{U}_2 \tilde{X}_2 & I_{n_{o2}} \end{bmatrix} L_2 \in R_u(s)^{(n_{i2}+n_{o2}) x (n_{i2}+n_{o2})} . \qquad (4.4.27)$$

By (4.4.18), E_2^{-1} is also R_u–unimodular. Finally let $W_1 := \tilde{X}_2 \in R_u(s)^{n_{o1} x n_{i2}}$ and let $W_2 := \tilde{N}_{21} Y_2 \in R_u(s)^{n_{o2} x n_{i1}}$. Then from equations (4.4.25), (4.4.26) and (4.4.27), we obtain

$$\begin{bmatrix} E_1^{-1} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & E_2^{-1} \end{bmatrix} \begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix} R = \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \\ \cdots & \cdots \\ 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} ; \qquad (4.4.28)$$

and hence, with R an R_u–unimodular matrix, we have shown that *some* r.c.f.r. $(N_p R , D_p R)$ of P satisfies

$$\begin{bmatrix} D_{p1} \\ N_{p1} \\ \cdots \\ D_{p2} \\ N_{p2} \end{bmatrix} R = \begin{bmatrix} E_1 \begin{bmatrix} I_{n_{i1}} & 0 \\ 0 & W_1 \end{bmatrix} \\ \cdots \\ E_2 \begin{bmatrix} 0 & I_{n_{i2}} \\ W_2 & 0 \end{bmatrix} \end{bmatrix} . \qquad (4.4.29)$$

Therefore, any r.c.f.r. of P can be put in the form in equations (4.4.6) and (4.4.7), except that they may have some R_u–unimodular right-factor.

□

Proof of Theorem 4.4.3L:

Conditions (4.4.10)-(4.4.11) are equivalent to conditions (4.4.6)-(4.4.7) by Theorem 4.3.5L. Therefore Theorem 4.4.3L follows from Theorem 4.4.2R.

□

Proof of Theorem 4.4.4B:

We only need to prove that conditions (4.4.4B)-(4.4.5B) are equivalent to conditions (4.4.4)-(4.4.5); the rest follows by Theorem 4.4.2R:

Following equation (4.3.4B) in Comment 4.3.6.(iii), condition (4.4.4) of Theorem 4.4.2R holds if and only if

$$\begin{bmatrix} Y_1 \\ N_{pr1}X \end{bmatrix}(s) = E_1(s)\begin{bmatrix} I_{ni1} & 0 \\ 0 & W_1(s) \end{bmatrix} R(s) \tag{4.4.30}$$

for some R_u–unimodular $R \in R_u(s)^{n_i \times n_i}$ and R_u–unimodular $E_1 \in H^{(n_{i1}+n_{o1})\times(n_{i1}+n_{o1})}$, where $W_1(s) \in R_u(s)^{n_{o1}\times n_{i2}}$. By Theorem 4.4.2R, condition (4.4.4) is equivalent to condition (4.4.6); hence condition (4.4.4B) holds if and only if

$$rank\begin{bmatrix} Y_1(s) \\ N_{pr1}X(s) \end{bmatrix} \geq n_{i1} \text{ , for all } s \in \bar{\mathcal{U}} . \tag{4.4.31}$$

From the Bezout identity (4.2.2) we obtain

$$\begin{bmatrix} D & -N_{pl1} & -N_{pl2} \\ 0 & I_{ni1} & 0 \\ N_{pr1} & 0 & 0 \end{bmatrix}\begin{bmatrix} V_{pl} & X \\ -U_{pl1} & Y_1 \\ -U_{pl2} & Y_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -U_{pl1} & Y_1 \\ N_{pr1}V_{pl} & N_{pr1}X \end{bmatrix} . \tag{4.4.32}$$

Condition (4.4.31) holds if and only if the matrix on the right of equation (4.4.32) has $rank \geq n + n_{i1}$, for all $s \in \bar{\mathcal{U}}$; since the second matrix on the left of equation (4.4.32) is R_u–unimodular, condition (4.4.31) holds if and only if

$$rank\begin{bmatrix} D(s) & -N_{pl1}(s) & -N_{pl2}(s) \\ 0 & I_{ni1} & 0 \\ N_{pr1}(s) & 0 & 0 \end{bmatrix} \geq n + n_{i1}\,,\ \text{for all } s \in \bar{\mathcal{U}}\,,$$

if and only if $rank\begin{bmatrix} D(s) & -N_{pl1}(s) \\ N_{pr2}(s) & 0 \end{bmatrix} + n_{i1} \geq n + n_{i1}$, for all $s \in \bar{\mathcal{U}}$. We conclude that condition (4.4.31) holds if and only if condition (4.4.4B) holds.

The equivalence of condition (4.4.5B) to condition (4.4.5) can be established similarly.

□

4.4.8. Algorithm (Decentralized compensator design):

Theorem 4.3.7 and the proof of Theorem 4.4.2R ((ii) => (iii)) suggest the following algorithm for finding the set of all H–stabilizing decentralized compensators based on *any* r.c.f.r. of P .

Given: $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$ satisfying Assumption 4.2.1 (A) and conditions (4.4.4)-(4.4.5) in Theorem 4.4.2R.

Step 1: Find R_u–unimodular matrices L_1, R_1 such that

$$L_1\begin{bmatrix} D_{p1} \\ N_{p1} \end{bmatrix} R_1 = \begin{bmatrix} I_{ni1} & 0 \\ 0 & \tilde{N}_{12} \end{bmatrix}. \tag{4.4.33}$$

Step 2: Find an R_u–unimodular matrix $L_2 \in \mathcal{M}(R_u(s))$ such that

$$L_2(\begin{bmatrix} D_{p2} \\ N_{p2} \end{bmatrix} R_1) = \begin{bmatrix} -\tilde{D}_{21} & \tilde{D}_{22} \\ \tilde{N}_{21} & 0 \end{bmatrix}, \text{ where } \tilde{D}_{22} \in R_u(s)^{n_{i2} \times n_{i2}}, \tag{4.4.34}$$

$$\text{and} \qquad (\tilde{D}_{22}, \tilde{D}_{21}) \quad \text{is an l.c. pair}. \tag{4.4.35}$$

Step 3: Find a generalized Bezout identity for the l.c. pair $(\tilde{D}_{22}, \tilde{D}_{21})$:

$$\begin{bmatrix} V_2 & U_2 \\ -\tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}\begin{bmatrix} Y_2 & -U_{2l} \\ X_2 & V_{2l} \end{bmatrix} = \begin{bmatrix} I_{ni1} & 0 \\ 0 & I_{ni2} \end{bmatrix}. \tag{4.4.36}$$

Find a generalized Bezout identity for the r.c. pair $(\tilde{N}_{12}, \tilde{D}_{22})$:

$$\begin{bmatrix} V_{2r} & U_{2r} \\ -\tilde{X}_2 & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} \tilde{D}_{22} & -\tilde{U}_2 \\ \tilde{N}_{12} & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} I_{ni2} & 0 \\ 0 & I_{no1} \end{bmatrix} . \tag{4.4.37}$$

Step 4: Let

$$E_1^{-1} := \begin{bmatrix} V_2 + U_2 V_{2r} \tilde{D}_{21} & U_2 U_{2r} \\ -\tilde{X}_2 \tilde{D}_{21} & \tilde{Y}_2 \end{bmatrix} L_1 , \quad E_2^{-1} := \begin{bmatrix} I_{ni2} & 0 \\ \tilde{N}_{21} U_{2l} \tilde{U}_2 \tilde{X}_2 & I_{no2} \end{bmatrix} L_2 , \tag{4.4.38}$$

and let

$$W_1 := \tilde{X}_2 , \qquad W_2 := \tilde{N}_{21} Y_2 . \tag{4.4.39}$$

Step 5: $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{c1}^{-1} \tilde{N}_{c1} & 0 \\ 0 & \tilde{D}_{c2}^{-1} \tilde{N}_{c1} \end{bmatrix}$ *H*–stabilizes the given $P \in \mathcal{M}(\mathbb{R}_{sp}(s))$,

where

$$\left[\tilde{D}_{c1} \vdots \tilde{N}_{c1} \right] = \left[I_{ni1} \vdots Q_1 \right] E_1^{-1} , \tag{4.4.40}$$

$$\left[\tilde{D}_{c2} \vdots \tilde{N}_{c2} \right] = \left[I_{ni2} \vdots Q_2 \right] E_2^{-1} , \tag{4.4.41}$$

for some $Q_1 , Q_2 \in \mathcal{M}(R_u(s))$ such that

$$\det(I_{ni2} - Q_2 W_2 Q_1 W_1) \in J . \tag{4.4.42}$$

□

4.5. Extension to multi-channel decentralized control systems

In this section we extend the results of Section 4.3 to m-channel decentralized systems ($m > 2$), and study the implications of the rational functions case of Section 4.4. We do not give complete proofs here since the two-channel case was studied in detail; the clues we give for each proof should suffice.

We only analyze the m-channel decentralized system as in Analysis 4.2.3.(i); the other cases are also easy to extend.

Consider the m-channel decentralized control system $S(P, C_d)_m$ shown in Figure 4.7; the subscript m is added in $S(P, C_d)_m$ to emphasize that this is an m-channel system.

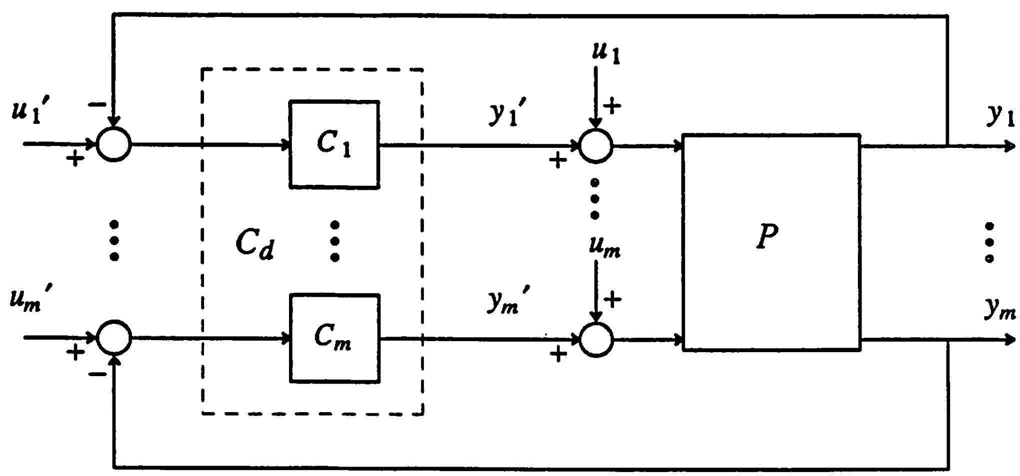

Figure 4.7: The m-channel decentralized control system $S(P, C_d)_m$.

4.5.1. Assumptions:

Extend Assumption 4.2.1 to m-channels:

(A) Let $P \in G^{n_o \times n_i}$ be an m-channel plant, where $n_o = n_{o1} + \cdots + n_{om}$, $n_i = n_{i1} + \cdots + n_{im}$. Let (N_p, D_p) be an r.c.f.r. of P, where $N_p =: \begin{bmatrix} N_{p1} \\ \vdots \\ N_{pm} \end{bmatrix} \in H^{n_o \times n_i}$,

$D_p =: \begin{bmatrix} D_{p1} \\ \vdots \\ D_{pm} \end{bmatrix} \in H^{ni \times ni}$, $N_{pj} \in H^{noj \times ni}$, $D_{pj} \in H^{nij \times ni}$, $j = 1, \cdots, m$. Let $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of P , where $\tilde{D}_p =: \begin{bmatrix} \tilde{D}_{p1} & \cdots & \tilde{D}_{pm} \end{bmatrix} \in H^{no \times no}$, $\tilde{N}_p =: \begin{bmatrix} \tilde{N}_{p1} & \cdots & \tilde{N}_{p2} \end{bmatrix} \in H^{no \times ni}$, $\tilde{D}_{pj} \in H^{no \times noj}$, $\tilde{N}_{pj} \in H^{no \times nij}$, $j = 1, \cdots, m$. The b.c.f.r. (N_{pr}, D, N_{pl}) of P is similarly partitioned into m-channels.

(B) Let $C_d = diag\begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$, $C_j \in G^{nij \times noj}$. Let $(\tilde{D}_{cj}, \tilde{N}_{cj})$ be an l.c.f.r. of C_j , where $\tilde{D}_{cj} \in H^{nij \times nij}$, $\tilde{N}_{cj} \in H^{nij \times noj}$. Let (N_{cj}, D_{cj}) be an r.c.f.r. of C_j , where $N_{cj} \in H^{nij \times noj}$, $D_{cj} \in H^{noj \times noj}$, $j = 1, \cdots, m$. Then $(\tilde{D}_c, \tilde{N}_c)$ is an l.c.f.r. and (N_c, D_c) is an r.c.f.r. of C_d, where $\tilde{D}_c = diag\begin{bmatrix} \tilde{D}_{c1} \cdots \tilde{D}_{cm} \end{bmatrix}$, $\tilde{N}_c = diag\begin{bmatrix} \tilde{N}_{c1} \cdots \tilde{N}_{cm} \end{bmatrix}$, $N_c = diag\begin{bmatrix} N_{c1} \cdots N_{cm} \end{bmatrix}$, $D_c = diag\begin{bmatrix} D_{c1} \cdots D_{cm} \end{bmatrix}$.

4.5.2. Analysis:

Let $P = N_p D_p^{-1}$ and let $C = \tilde{D}_c^{-1} \tilde{N}_c$, where (N_p, D_p) is an r.c. pair as in Assumption 4.5.1 (A), and $(\tilde{D}_c, \tilde{N}_c)$ is an l.c. pair as in Assumption 4.5.1 (B). The m-channel system $S(P, C_d)_m$ is then described by equations (4.5.1)-(4.5.2).

$$\begin{bmatrix} \tilde{D}_{c1} D_{p1} + \tilde{N}_{c1} N_{p1} \\ \vdots \\ \tilde{D}_{cm} D_{pm} + \tilde{N}_{cm} N_{pm} \end{bmatrix} \xi_p = \begin{bmatrix} \tilde{D}_{c1} & \cdots & 0 & \tilde{N}_{c1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \tilde{D}_{cm} & 0 & \cdots & \tilde{N}_{cm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \\ u_1' \\ \vdots \\ u_m' \end{bmatrix} , \tag{4.5.1}$$

$$\begin{bmatrix} N_{p1} \\ \vdots \\ N_{pm} \\ D_{p1} \\ \vdots \\ D_{pm} \end{bmatrix} \xi_p = \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_1' \\ \vdots \\ y_m' \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ I_{ni} & \vdots & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \\ u_1' \\ \vdots \\ u_m' \end{bmatrix} . \tag{4.5.2}$$

The system $S(P, C_d)_m$ is H–stable if and only if

$$\begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{D}_{cm} & \tilde{N}_{cm} \end{bmatrix} \begin{bmatrix} D_{p1} \\ N_{p1} \\ \vdots \\ D_{pm} \\ N_{pm} \end{bmatrix} \text{ is } H\text{–unimodular} . \qquad (4.5.3)$$

4.5.3. Theorem (Conditions on P for decentralized H–stabilizability):

Let $P \in \mathcal{m}(G_S)$ satisfy Assumption 4.5.1 (A); then there exists an H–stabilizing decentralized compensator C_d (satisfying Assumption 4.5.1 (B)) for P if and only if P has an r.c.f.r. (N_p, D_p) which satisfies condition (4.5.4R) and equivalently, an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ which satisfies condition (4.5.5L) below:

$$\begin{bmatrix} D_{p1} \\ N_{p1} \\ \\ D_{p2} \\ N_{p2} \\ \vdots \\ D_{pm} \\ N_{pm} \end{bmatrix} = \begin{bmatrix} E_1 \begin{bmatrix} I_{ni1} & 0 & \cdots & 0 \\ 0 & W_{12} & \cdots & W_{1m} \end{bmatrix} \\ \\ E_2 \begin{bmatrix} 0 & I_{ni2} & \cdots & 0 \\ W_{21} & 0 & \cdots & W_{2m} \end{bmatrix} \\ \vdots \\ E_m \begin{bmatrix} 0 & 0 & \cdots & I_{nim} \\ W_{m1} & W_{m2} & \cdots & 0 \end{bmatrix} \end{bmatrix} , \qquad (4.5.4R)$$

$$\begin{bmatrix} -\tilde{N}_{p1} & \tilde{D}_{p1} & \vdots & -\tilde{N}_{p2} & \tilde{D}_{p2} & \cdots & -\tilde{N}_{pm} & \tilde{D}_{pm} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} 0 & I_{no1} \\ -W_{21} & 0 \\ \vdots & \vdots \\ -W_{m1} & 0 \end{bmatrix} E_1^{-1} & \vdots & \begin{bmatrix} -W_{12} & 0 \\ 0 & I_{no2} \\ \vdots & \vdots \\ -W_{m2} & 0 \end{bmatrix} E_2^{-1} & \cdots & \begin{bmatrix} -W_{1m} & 0 \\ -W_{2m} & 0 \\ \vdots & \vdots \\ 0 & I_{nom} \end{bmatrix} E_m^{-1} \end{bmatrix} , \qquad (4.5.5L)$$

where, for $j = 1, 2, \cdots, m$, $E_j \in H^{(nij+noj)\mathrm{x}(nij+noj)}$ is H–unimodular and $W_{jk} \in H^{noj \mathrm{x} nik}$, $k = 1, 2, \cdots, m$ (note that $W_{jk} = 0$ when $k = j$).

Proof:

We only prove condition (4.5.4R); the proof of condition (4.5.5L) is similar:

(<=) By assumption, condition (4.5.4R) holds. For $j = 1, \cdots, m$, consider the compensators $C_j = \tilde{D}_{cj}^{-1}\tilde{N}_{cj}$, where

$$\left[\tilde{D}_{cj} \vdots \tilde{N}_{cj} \right] = \left[I_{n_{ij}} \vdots 0 \right] E_j^{-1} . \tag{4.5.6}$$

It can be shown that the C_j's satisfy Assumption 4.5.1 (B) the same way as in the proof of Theorem 4.3.4R. Now substitute equation (4.5.6) into equation (4.5.3); clearly, the m-channel system $S(P, C_d)_m$ is H-stable.

(=>) For $j = 1, \cdots, m$, partition the matrices $D_{pj} =: \left[D_{j1} \cdots D_{jm} \right]$, $N_{pj} =: \left[N_{j1} \cdots N_{jm} \right]$. The system $S(P, C_d)_m$ is H-stable by assumption; therefore, by normalizing equation (4.5.3), we have

$$\begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{D}_{cm} & \tilde{N}_{cm} \end{bmatrix} \begin{bmatrix} D_{11} & \cdots & D_{1m} \\ N_{11} & \cdots & N_{1m} \\ \vdots & & \vdots \\ D_{m1} & \cdots & D_{mm} \\ N_{m1} & \cdots & N_{mm} \end{bmatrix} = \begin{bmatrix} I_{n_{i1}} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & I_{n_{im}} \end{bmatrix} . \tag{4.5.7}$$

As in the proof of Theorem 4.3.4R, for $j = 1, \cdots, m$, (N_{jj}, D_{jj}) is an r.c. pair and $\det D_{jj} \in \mathcal{I}$; let $(\tilde{D}_{jj}, \tilde{N}_{jj})$ be an l.c.f.r. of $N_{jj}D_{jj}^{-1}$. Then there exist matrices $\tilde{U}_{jj}$, $\tilde{V}_{jj} \in \mathfrak{m}(H)$ such that the following generalized Bezout identity can be written for each $j = 1, \cdots, m$:

$$\begin{bmatrix} \tilde{D}_{cj} & \tilde{N}_{cj} \\ -\tilde{N}_{jj} & \tilde{D}_{jj} \end{bmatrix} \begin{bmatrix} D_{jj} & -\tilde{U}_{jj} \\ N_{jj} & \tilde{V}_{jj} \end{bmatrix} = \begin{bmatrix} I_{n_{ij}} & 0 \\ 0 & I_{n_{oj}} \end{bmatrix} . \tag{4.5.8}$$

For $j = 1, \cdots, m$, let $E_j^{-1} := \begin{bmatrix} \tilde{D}_{cj} & \tilde{N}_{cj} \\ -\tilde{N}_{jj} & \tilde{D}_{jj} \end{bmatrix}$; E_j is H-unimodular by equation (4.5.8). Let $W_{jk} := -\tilde{N}_{jj}D_{jk} + \tilde{D}_{jj}N_{jk}$, $k = 1, \cdots, j-1, j+1, \cdots, m$ (note that $W_{jj} = 0$). Then by equations (4.5.7)-(4.5.8) we obtain

$$E_j^{-1}\begin{bmatrix} D_{j1} & \cdots & D_{jm} \\ N_{j1} & \cdots & N_{jm} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & I_{n_{ij}} & 0 & \cdots & 0 \\ W_{j1} & \cdots & 0 & W_{jj+1} & \cdots & W_{jm} \end{bmatrix} . \tag{4.5.9}$$

For $j = 1, \cdots, m$, pre-multiplying both sides of equation (4.5.9) by E_j, we get condition (4.5.4R).

□

We now extend Theorem 4.3.7, which gives the set of all H–stabilizing compensators, to m-channels: For future reference, we define

$$Q := diag\left[Q_1 \cdots Q_m \right] , \tag{4.5.10}$$

$$W := \begin{bmatrix} 0 & W_{12} & \cdots & W_{1m} \\ W_{21} & 0 & \cdots & W_{2m} \\ \vdots & \vdots & & \vdots \\ W_{m1} & W_{m2} & \cdots & 0 \end{bmatrix} . \tag{4.5.11}$$

4.5.4. Theorem (Set of all H–stabilizing decentralized compensators for $S(P, C_d)_m$):

Let $P \in \mathcal{M}(G_S)$ satisfy Assumption 4.5.1 (A); let in addition an r.c.f.r. (N_p, D_p) of P satisfy condition (4.5.4R) and equivalently, let an l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ of P satisfy condition (4.5.5L) of Theorem 4.5.3. Under these conditions, the set $\mathbf{S}_d(P)$ of *all* H–stabilizing decentralized compensators for P is given by

$$\mathbf{S}_d(P) := \Big\{ \; C_d = diag\left[C_1 \cdots C_m \right] = diag\left[\tilde{D}_{c1}^{-1}\tilde{N}_{c1} \cdots \tilde{D}_{cj}^{-1}\tilde{N}_{cj} \right] :$$

$$\text{for } j = 1, \cdots, m, \left[\tilde{D}_{cj} \vdots \tilde{N}_{cj} \right] = \left[I_{n_{ij}} \vdots Q_j \right] E_j^{-1},$$

$$\text{for some } Q_j \in H^{n_{ij} \times n_{oj}} \text{ such that } \det(I_{n_i} + QW) \in J \; \Big\} ; \tag{4.5.12}$$

equivalently,

$$\mathbf{S}_d(P) := \Big\{ \; C_d = diag\left[C_1 \cdots C_m \right] = diag\left[N_{c1}D_{c1}^{-1} \cdots N_{cj}D_{cj}^{-1} \right] :$$

$$\text{for } j = 1, \cdots, m , \begin{bmatrix} -N_{cj} \\ D_{cj} \end{bmatrix} = E_j \begin{bmatrix} -Q_j \\ I_{n_{oj}} \end{bmatrix} ,$$

$$\text{for some } Q_j \in H^{n_{ij} \times n_{oj}} \text{ such that } \det(I_{n_i} + QW) \in J \; \Big\} . \tag{4.5.13}$$

Furthermore, for $j = 1, \cdots, m$, the matrices Q_j, subject to $\det(I_{ni} + QW) \in J$, determine C_j uniquely.

Proof:

The proof is similar to that of Theorem 4.3.7. We write a generalized Bezout identity which extends equation (4.3.19) to m-channels:

$$\begin{bmatrix} \begin{bmatrix} I_{ni1} & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} E_1^{-1} & \cdots & \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ I_{nim} & 0 \end{bmatrix} E_m^{-1} \\ \vdots & \vdots & \vdots \\ \begin{bmatrix} 0 & I_{no1} \\ -W_{21} & 0 \\ \vdots & \vdots \\ -W_{m1} & 0 \end{bmatrix} E_1^{-1} & \cdots & \begin{bmatrix} -W_{1m} & 0 \\ -W_{2m} & 0 \\ \vdots & \vdots \\ 0 & I_{nom} \end{bmatrix} E_m^{-1} \end{bmatrix} \cdot$$

$$\begin{bmatrix} E_1 \begin{bmatrix} I_{ni1} & \cdots & 0 \\ 0 & \cdots & W_{1m} \end{bmatrix} & \cdots & E_1 \begin{bmatrix} 0 & \cdots & 0 \\ I_{no1} & \cdots & 0 \end{bmatrix} \\ \vdots & \vdots & \vdots \\ E_m \begin{bmatrix} 0 & \cdots & I_{nim} \\ W_{m1} & \cdots & 0 \end{bmatrix} & \cdots & E_m \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & I_{nom} \end{bmatrix} \end{bmatrix} = I_{ni+no}. \quad (4.5.14)$$

Equation (4.5.14) is obtained from conditions (4.5.4R)-(4.5.5L) and is of the form

$$\begin{bmatrix} V_{p1} & U_{p1} & \cdots & V_{pm} & U_{pm} \\ -\tilde{N}_{p1} & \tilde{D}_{p1} & \cdots & -\tilde{N}_{pm} & \tilde{D}_{pm} \end{bmatrix} \begin{bmatrix} D_{p1} & -\tilde{U}_{p1} \\ N_{p1} & \tilde{V}_{p1} \\ \vdots & \vdots \\ D_{pm} & -\tilde{U}_{pm} \\ N_{pm} & \tilde{V}_{pm} \end{bmatrix} = I_{ni+no} .$$

Using standard methods, by normalizing equation (4.5.3), it is easy to show that if C_d H-stabilizes P then $C_d = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p)$, where $\tilde{D}_c := (V_p - Q\tilde{N}_p)$ and $\tilde{N}_c :=$ $(U_p + Q\tilde{D}_p)$ and Q are block-diagonal. For $j = 1, \cdots, m$, let $\tilde{D}_{cj} := V_{pj} - Q_j\tilde{N}_{pj}$ and $\tilde{N}_{cj} :=$ $U_{pj} + Q_j\tilde{D}_{pj}$. From equation (4.5.14),

$$\begin{bmatrix} \tilde{D}_{c1} & \tilde{N}_{c1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{D}_{cm} & \tilde{N}_{cm} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} & \cdots & 0 \\ 0 & \cdots & \begin{bmatrix} I_{nim} & Q_m \end{bmatrix} E_m^{-1} \end{bmatrix} + QW ; \qquad (4.5.15)$$

substituting equations (4.5.10)-(4.5.11) into (4.5.15), $QW = 0$, and hence, $\det(I_{n_i} + QW) \in J$. Therefore, C_d is given by the expression in equation (4.5.12).

Conversely, if C_d is given by the expression in equation (4.5.12), then the matrix in equation (4.5.3) becomes $I_{n_i} + QW$, which is H–unimodular due to the condition in (4.5.12).

The proof of equation (4.5.13) is similar.

□

4.5.6. Comments (The rational functions case): [And.1, Xie.1]

Let H be $R_u(s)$ as in Section 4.4. The definition of decentralized fixed-eigenvalues is extended to m-channels as follows: The plant P has a decentralized fixed-eigenvalue at $s_o \in \bar{\mathcal{U}}$ (with respect to $K_d = diag[\, K_1 \cdots K_m \,]$) iff $\det \begin{bmatrix} D_{p1}(s_o) + K_1 N_{p1}(s_o) \\ \vdots \\ D_{pm}(s_o) + K_m N_{pm}(s_o) \end{bmatrix} = 0$ for all K_1, ..., $K_m \in \mathcal{M}(\mathbb{R})$. Extending Theorems 4.4.2R, 4.4.3L, 4.4.4B to m-channels, we state six equivalent conditions below:

(i) The plant P has no decentralized fixed-eigenvalues in $\bar{\mathcal{U}}$;

(ii) for $k = 1, \cdots m-1$, for all nonempty subsets $\alpha = \{\, \alpha_1 , \cdots , \alpha_k \,\}$ of $\{\, 1 , \cdots , m \,\}$,

$$rank \begin{bmatrix} D_{p\alpha_1}(s) \\ N_{p\alpha_1}(s) \\ \vdots \\ D_{p\alpha_k}(s) \\ N_{p\alpha_k}(s) \end{bmatrix} \geq \sum_{\alpha_j \in \alpha} n_{i\alpha_j} \text{, for all } s \in \bar{\mathcal{U}} ; \qquad (4.5.16R)$$

(iii) for $k = 1, \cdots m-1$, for all nonempty subsets $\alpha = \{\, \alpha_1 , \cdots , \alpha_k \,\}$ of $\{\, 1 , ..., m \,\}$,

$$rank \begin{bmatrix} -\tilde{N}_{p\alpha_1}(s) & \tilde{D}_{p\alpha_1}(s) & \cdots & \tilde{N}_{p\alpha_k}(s) & \tilde{D}_{p\alpha_k}(s) \end{bmatrix} \geq \sum_{\alpha_j \in \alpha} n_{i\alpha_j} \text{, for all } s \in \bar{\mathcal{U}} ; \qquad (4.5.17L)$$

(iv) conditions (4.5.4R) and (4.5.5L) of Theorem 4.5.3 hold;

(v) for $k = 1, \cdots m-1$, for all partitions of the set $\{1, \cdots, m\}$ into two disjoint subsets $\{\alpha_1, \cdots, \alpha_k\}$ and $\{\alpha_{k+1}, \cdots, \alpha_m\}$,

$$rank \begin{bmatrix} D(s) & -N_{pl\alpha_{k+1}}(s) & \cdots & -N_{pl\alpha_m}(s) \\ N_{pr\alpha_1}(s) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ N_{pr\alpha_k}(s) & 0 & \cdots & 0 \end{bmatrix} \geq n \, , \quad \text{for all } s \in \bar{\mathcal{U}} \, ; \tag{4.5.18B}$$

(vi) there exists a dynamic decentralized compensator $C_d = diag \begin{bmatrix} C_1 \cdots C_m \end{bmatrix}$ which H–stabilizes P .

In conditions (4.5.16R) and (4.5.17L), the set α is a strictly proper subset of $\{ 1, \cdots, m \}$ because (N_p, D_p) is r.c. implies $rank \begin{bmatrix} D_p(s) \\ N_p(s) \end{bmatrix} = n_i$, for all $s \in \bar{\mathcal{U}}$ and $(\tilde{D}_p, \tilde{N}_p)$ is l.c. implies $rank \begin{bmatrix} \tilde{N}_p(s) & \tilde{D}_p(s) \end{bmatrix} = n_i$, for all $s \in \bar{\mathcal{U}}$. In condition (4.5.18B) the two disjoint subsets are strictly proper subsets of $\{ 1, \cdots, m \}$ because if either one was equal to $\{ 1, \cdots, m \}$, then condition (4.5.18B) is automatically satisfied since (N_{pr}, D, N_{pl}) is bicoprime.

Condition (4.5.18B) can also be written in the state-space setting as in Remark 4.4.5S.

□

4.5.7. Achievable I/O maps of $S(P, C_d)_m$:

The set

$$\mathrm{A}_d(P) := \{ \; H_{\bar{y}\bar{u}} \; : \; C_d \; H\text{–stabilizes } P \; \} \tag{4.5.19}$$

is called the set of *all achievable I/O maps* of the m-channel decentralized feedback system $S(P, C_d)_m$.

Since the class $\mathrm{S}_d(P)$ is a subset of the class $\mathrm{S}(P)$ of all stabilizing decentralized compensators for P in the configuration $S(P, C)$, the class $\mathrm{A}_d(P)$ is also a subset of the set $\mathrm{A}(P)$ of all achievable maps of the unity-feedback system $S(P, C)$.

Let an r.c.f.r. (N_p, D_p) of P satisfy condition (4.5.4R); then from equation (4.2.3), we obtain

$$\mathrm{A}_d(P) = \{ \; H_{\overline{yu}} = \begin{bmatrix} N_p(I_{ni} + QW)^{-1}\tilde{D}_c & N_p(I_{ni} + QW)^{-1}\tilde{N}_c \\ D_p(I_{ni} + QW)^{-1}\tilde{D}_c - I_{ni} & D_p(I_{ni} + QW)^{-1}\tilde{N}_c \end{bmatrix} \tag{4.5.20}$$

$$: Q \in \mathcal{m}(H) \text{ such that } \det(I_{ni} + QW) \in J \;\; \} ,$$

where

$$\tilde{D}_c = diag \left[\begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} \begin{bmatrix} I_{ni1} \\ 0 \end{bmatrix} \cdots \begin{bmatrix} I_{nim} & Q_m \end{bmatrix} E_m^{-1} \begin{bmatrix} I_{nim} \\ 0 \end{bmatrix} \right] ,$$

$$\tilde{N}_c = diag \left[\begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} \begin{bmatrix} 0 \\ I_{no1} \end{bmatrix} \cdots \begin{bmatrix} I_{nim} & Q_m \end{bmatrix} E_m^{-1} \begin{bmatrix} 0 \\ I_{nom} \end{bmatrix} \right] .$$

4.6. $\Sigma(\hat{P}, \hat{C})$ with a decentralized feedback compensator

In order to summarize the results of Chapters Three and Four, we now combine the H-stabilizing compensator design procedure using $\Sigma(\hat{P}, \hat{C})$ with decentralized control: Suppose that the general configuration $\Sigma(\hat{P}, \hat{C})$ requires the additional restriction that the 2-2 block C of $\hat{C}$ is block-diagonal, i.e., $\hat{C}$ is replaced by $\hat{C}_d := \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_d \end{bmatrix}$, where $C_d = diag\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ is block-diagonal as in Assumption 4.2.1' (B) (see Figure 4.8).

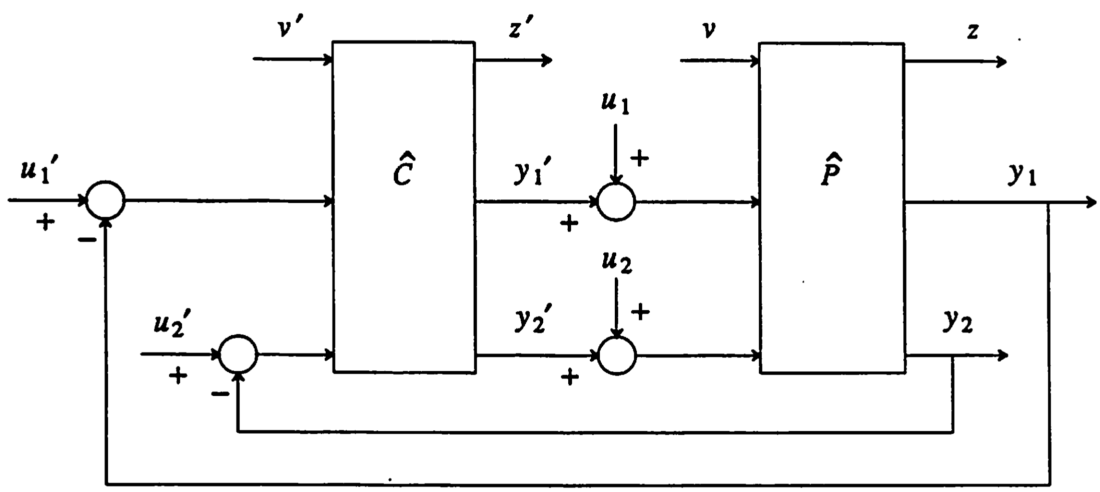

Figure 4.8: $\Sigma(\hat{P}, \hat{C})$ with a decentralized feedback-loop.

The class of all $\hat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P \end{bmatrix} \in G^{(\eta_o+n_o)\times(\eta_i+n_i)}$ that can be H-stabilized by some $\hat{C} \in G^{(\eta_o'+n_i)\times(\eta_i'+n_o)}$, is given in Theorem 3.3.9. The class of all two-channel $P \in G^{n_o \times n_i}$ that can be H-stabilized by some decentralized compensator $C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$, $C_1 \in G^{n_{i1} \times n_{o1}}$, $C_2 \in G^{n_{i2} \times n_{o2}}$ in the configuration $S(P, C_d)$, is given in Theorem 4.3.4R.

Combining these two, the class of all $\hat{P} \in G^{(\eta_o+n_o)\times(\eta_i+n_i)}$ that can be H–stabilized by some $\hat{C}_d$ is given by the set

$$\left\{ \begin{bmatrix} \hat{N}_{11} & N_{12} \\ \tilde{V}_p \tilde{N}_{21} & \begin{bmatrix} \begin{bmatrix} 0 & I_{no1} \end{bmatrix} E_1 \begin{bmatrix} I_{ni1} & 0 \\ 0 & W_1 \end{bmatrix} \\ \begin{bmatrix} 0 & I_{no2} \end{bmatrix} E_2 \begin{bmatrix} 0 & I_{ni2} \\ W_2 & 0 \end{bmatrix} \end{bmatrix} \end{bmatrix} \begin{bmatrix} I_{\eta_i} & 0 \\ -\tilde{U}_p \tilde{N}_{21} & \begin{bmatrix} \begin{bmatrix} I_{ni1} & 0 \end{bmatrix} E_1 \begin{bmatrix} I_{ni1} & 0 \\ 0 & W_1 \end{bmatrix} \\ \begin{bmatrix} I_{ni2} & 0 \end{bmatrix} E_2 \begin{bmatrix} 0 & I_{ni2} \\ W_2 & 0 \end{bmatrix} \end{bmatrix} \end{bmatrix}^{-1} : \right.$$

$$\left. E_1, E_2 \in \mathcal{M}(H) \text{ are } H\text{–unimodular and } W_1 \in H^{no1 \times ni2}, W_2 \in H^{no2 \times ni1} \right\}. \tag{4.6.1}$$

Let $\hat{\mathbf{S}}_d(P)$ denote the set of all H–stabilizing compensators $\hat{C}_d = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_d \end{bmatrix}$; i.e.,

$$\hat{\mathbf{S}}_d(P) := \left\{ \hat{C}_d = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_d \end{bmatrix} : \hat{C}_d \ \ H\text{–stabilizes } \hat{P} \right\}. \tag{4.6.2}$$

Combining $\hat{\mathbf{S}}(\hat{P})$ given in Theorem 3.3.11 and $\mathbf{S}_d(P)$ given in Theorem 4.3.7, the class $\hat{\mathbf{S}}_d(P)$ of all H–stabilizing compensators $\hat{C}_d$ is given by:

$$\hat{S}_d(P) := \Bigg\{ \; \hat{C}_d = \begin{bmatrix} I_{\eta_o'} & -Q_{12}\tilde{N}_p \\ 0 & \begin{bmatrix} \begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} \begin{bmatrix} I_{ni1} \\ 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_{ni2} & Q_2 \end{bmatrix} E_2^{-1} \begin{bmatrix} I_{ni2} \\ 0 \end{bmatrix} \end{bmatrix} \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} Q_{11} & Q_{12}\tilde{D}_p \\ Q_{21} & \begin{bmatrix} \begin{bmatrix} I_{ni1} & Q_1 \end{bmatrix} E_1^{-1} \begin{bmatrix} 0 \\ I_{no1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_{ni2} & Q_2 \end{bmatrix} E_2^{-1} \begin{bmatrix} 0 \\ I_{no2} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$: Q_{11}, Q_{12}, Q_{21} \in m(H),$$

$$Q_1 \in H^{ni1 \times no1}, Q_2 \in H^{ni2 \times no2} \text{ such that } \det(I_{ni2} - Q_2 W_2 Q_1 W_1) \in J \; \Bigg\} . \quad (4.6.3)$$

Note that the subblock P could have m-local channels instead of two channels; the extension to this case follows from Theorem 4.5.3.

Chapter Five

Conclusions

A unified algebraic theory for full output-feedback and decentralized output-feedback schemes is presented in Chapters Three and Four, using the fundamental tools of the factorization approach presented in Chapter Two. For each compensation scheme, the main objectives are H–stability, the class of all H–stabilizable plants, the class of all H–stabilizing compensators, and all achievable closed-loop I/O maps.

In Section 3.2, H–stabilizing compensators for the standard unity-feedback system $S(P, C)$ are parametrized starting with right-coprime, left-coprime and bicoprime factorizations of the plant (see equations (3.2.27)-(3.2.30) for the class of all one-parameter H–stabilizing compensators). Each closed-loop I/O map of $S(P, C)$ in equation (3.2.38) is an affine function of the compensator parameter matrix Q. The conditions for H–stability of the general system configuration $\Sigma(\hat{P}, \hat{C})$ are given in Section 3.3; this system allows full feedback from one of two (vector-)outputs of the plant $\hat{P}$ to one of two (vector-)inputs of the compensator $\hat{C}$. The class of all $\hat{P}$ that can be H–stabilized by some $\hat{C}$ in the configuration $\Sigma(\hat{P}, \hat{C})$ is parametrized in Theorem 3.3.9. The class of all H–stabilizing two-input two-output compensators $\hat{C}$ is given in Theorem 3.3.11; this class is parametrized by four parameter matrices Q_{11}, Q_{12}, Q_{21}, $Q \in \mathcal{m}(H)$. Each closed-loop I/O map of $\Sigma(\hat{P}, \hat{C})$ in equation (3.3.58) is an affine function of one of these four compensator parameters, which can be chosen to satisfy several performance requirements. The map $H_{zv'} : v' \mapsto z$ from the external-input v' of $\hat{C}$ to the actual output z of $\hat{P}$ is diagonalized in Section 3.4 by choosing the matrix Q_{21} as in equation (3.4.7). The class of all achievable maps $H_{zv'}$ which are diagonal and nonsingular is given in Theorem 3.4.2.

The two-channel decentralized feedback system $S(P, C_d)$ is studied in Chapter Four. This system is the same as the unity-feedback system $S(P, C)$ except that the compensator is restricted to be block-diagonal. Clearly, not all plants P can be H–stabilized by a decentralized compensator; the class of all decentralized H–stabilizable plants P is given in Theorem 4.3.4R. The class of all H–stabilizing decentralized compensators C_d is given in Theorem 4.3.7; this class is parametrized by two matrices which satisfy a unimodularity condition. In Section 4.4, the general algebraic results are applied to the case of proper stable rational functions $R_u(s)$; decentralized H–stabilizability conditions are interpreted in terms of fixed-eigenvalues in Theorem 4.4.2R. See Algorithm 4.4.8 for designing an H–stabilizing decentralized compensator, starting with any right-coprime factorization $N_p D_p^{-1}$ of P. In Section 4.5, the parametrization of H–stabilizing compensators is extended to m-channel decentralized control systems. In Section 4.6, the compensation schemes of $\Sigma(\hat{P}, \hat{C})$ and $S(P, C_d)$ are combined; the two-channel plant P is considered as the 2-2 subblock of a plant $\hat{P}$ in the configuration $\Sigma(\hat{P}, \hat{C})$ and the 2-2 subblock of $\hat{C}$ is restricted to be block-diagonal. The class of all compensators $\hat{C}_d$ such that $\Sigma(\hat{P}, \hat{C})$ has a decentralized feedback-loop is given in equation (4.6.3).

The parametrization of all H–stabilizing compensators is a key concept in all compensator design problems. The constrained optimization design approach is formulated in terms of these parametrizations: the optimization algorithm chooses the compensator parameter matrices (Q for $S(P, C)$; Q_{11}, Q_{12} , Q_{21} , Q for $\Sigma(\hat{P}, \hat{C})$; Q_1 , ... , Q_m , where $\det(I_{n_i} + QW) \in J$, for $S(P, C_d)_m$), that satisfy performance criteria as well as time-domain or frequency-domain constraints (see for example [Gus.1]). H^∞–norm minimization problems rely on the parametrization of all H–stabilizing compensators and the achievable I/O maps (see for example [Sal.1] and the references therein). The four independent parameter matrices of $\Sigma(\hat{P}, \hat{C})$ would be extremely useful in minimizing the H^∞–norm of I/O maps, each of which are affine functions in only one of the four parameter matrices; in this configuration, minimizing the H^∞–norm of the disturbance-to-output map would not result in undesirable responses in the map from the control-input to the actual output since these maps are decoupled from each other. Computer-aided design algorithms for one-parameter compensation schemes like $S(P, C)$ are already used extensively. The parametrizations of all H–stabilizing compensators presented in this work forms the basis of the development of numerical algorithms and software for computer-aided design.

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