ON THE DESIGN OF FINITE DIMENSIONAL STABILIZING COMPENSATORS FOR INFINITE DIMENSIONAL FEEDBACK-SYSTEMS VIA SEMI-INFINITE OPTIMIZATION

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ABSTRACT The design of stabilizing compensators for linear, time invariant feedback systems, by means of semi-infinite optimization algorithms, requires a stability test in the form of a finite or infinite set of differentiable inequalities. In a recent paper, Polak and Wuu have presented a set of easily solvable, differentiable inequalities, which are related to the classical Nyquist stability criterion, and which constitute a necessary and sufficient condition of stability for finite dimensional systems. In this paper it is shown that a similar set of easily solvable inequalities can be used to design finite dimensional stabilizing compensators for a class of infinite dimensional feedback systems. Computational aspects of the new stability test are discussed.

1. INTRODUCTION

Exponential stability is the most fundamental requirement in control system design and hence, over the years, a considerable amount of effort has been expended in developing efficient techniques for the design of stabilizing controllers. At present, the advent of computer-aided design is necessitating the development of new approaches. Thus, although the Nyquist stability criterion [Nyq.1] has served for many years as a principal "manual" tool in the design of stabilizing compensators for linear time-invariant systems, it cannot be used in conjunction with computer-aided design techniques which make use of semi-infinite optimization [Pol.1]. This is due to the fact that the Nyquist criterion defines an integer valued encirclement function, while semi-infinite optimization requires, at a minimum, that constraint and cost functions be locally Lipschitz continuous. Similarly, because eigenvalues are not differentiable at points of multiplicity and because they can be extremely sensitive to design parameter changes elsewhere, the use of inequalities involving closed loop system eigenvalues is also not a good

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idea.

The first attempt to produce a frequency domain stability test which is compatible with the requirements of semi-infinite optimization, was presented in [Pol.1], in the form of a differentiable semi-infinite, frequency-domain inequality which constitutes a *sufficient condition* of stability for *finite-dimensional*, linear, time-invariant systems. A significant improvement was presented in [Pol.2] where an alternate differentiable semi-infinite inequality was proposed which constitutes a *necessary and sufficient condition* of stability for finite dimensional, linear time invariant systems.

The design technique proposed in [Pol.2] is based on the following observation. Suppose that $\chi(s)$ is a characteristic polynomial. Then all the zeros of $\chi(s)$ are in $\overset{o}{\mathbb{C}}_-$ if and only if there exists a polynomial $d(s)$, of the same degree as $\chi(s)$ and whose zeros are in $\overset{o}{\mathbb{C}}_-$, such that

$$\mathrm{Re}\,[\chi(j\omega) \,/\, d(j\omega)] > 0, \quad \forall \;\; \omega \in (-\infty\, ,\, \infty\,)\,. \tag{1.1}$$

The proof of this result is simple. If all the zeros of $\chi(s)$ are in $\overset{o}{\mathbb{C}}_-$, then set $d(s) = \chi(s)$ and hence (1.1) holds. Alternatively, if (1.1) holds then the origin is not encircled by the locus of $\chi(j\omega) \,/\, d(j\omega)$ and hence the conclusion holds as for the Nyquist stability criterion. When used in design, the characteristic polynomial is also a differentiable function of compensator designable parameters $x \in \mathbb{R}^n$, and has the form $\chi(x,s)$; and the normalizing polynomial $d(s)$ is written in a factored form, such as $d(s,q) = \Pi_{j=1}^{n/2}\,(s^2 + a_j s + b_j)$, which makes it simple to ensure that the zeros of $d(s)$ are in $\overset{o}{\mathbb{C}}_-$ (q is a vector whose components are the a_j, b_j).

In this paper we extend the computational stability criterion presented in [Pol.2], to a form that can be used in the design of *finite dimensional* controllers for a class of feedback systems with infinite dimensional plants, to be described in Sec. 2. Since in this case the characteristic function is not a polynomial, there is no simple way to define a normalizing polynomial (of finite degree) for a test of the form (1.1). Hence approximation theory has to be brought into play as well as some aspects of semigroup theory as it applies to partial differential equations.

In Section 2, we will describe a class of feedback systems with infinite dimensional plants and define their exponential stability in terms of the properties of a semigroup function. Then we will establish the relation between exponential stability of the closed-loop system and its spectrum. We will define the characteristic function of the closed-loop system in Sec. 3, which will be seen to be of the same form as for the finite dimensional case. Finally, a necessary and sufficient computational stability criterion will be presented, in the form of a tractable semi-infinite inequality, which can be used in the design of stabilizing controllers for flexible structures.

2. PRELIMINARY RESULTS

Consider the feedback system $S(P,K)$, with n_i inputs and n_o outputs in Fig. 1. We assume that the plant is described by

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p e_2 \\ y_2 &= C_p x_p + D_p e_2, \end{aligned} \tag{2.1}$$

where $x_p \in E$, is a Hilbert space, $e_2 \in \mathbb{R}^{n_i}$, $y_2 \in \mathbb{R}^{n_o}$. The operators $B_p : \mathbb{R}^{n_i} \to E$, $C_p : E \to \mathbb{R}^{n_o}$ and $D_p : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ are assumed to be bounded, while A_p may be an unbounded operator from E to E, with its domain dense in E.

Let $\alpha > 0$ be a given positive constant. We define a stability region in the complex plane by $U_{-\alpha} \triangleq \{s \in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}$. Let $U_{-\alpha}^c = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -\alpha\}$, $\partial U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = -\alpha\}$ and $U_{-\alpha}^{c,o} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha\}$. Let $\sigma(A_p)$ be the spectrum of A_p and let $\rho(A_p)$ be the resolvent set of A_p which is the complement set of $\sigma(A_p)$ in $\mathbb{C}$. We will denote the domain and the range of A_p by $D(A_p)$ and $R(A_p)$, respectively. The notation used in this paper is that found in [Bal.1] and [Kat.1].

Assumption 1: (i) A_p is a closed operator which generates an analytic semigroup. (ii) The spectrum of A_p is a subset of $U_{-\alpha}$ and $\sup(\mathrm{Re}(\sigma(A_p))) < -\alpha$. ■

The transfer function of the plant is given by $G_p(s) = C_p(s - A_p)^{-1}B_p + D_p$. We assume that

$$\lim_{\substack{|s| \to \infty \\ s \in U_{-\alpha}^c}} G_p(s) \to D_p \,. \tag{2.2}$$

The convergence in (2.2) is understood to be componentwise.

The compensator is assumed to be finite dimensional, as follows:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c e_1 \\ y_1 &= C_c x_c + D_c e_1 \,, \end{aligned} \tag{2.3}$$

where $x_c \in \mathbb{R}^{n_c}$, $e_1 \in \mathbb{R}^{n_o}$, $y_1 \in \mathbb{R}^{n_i}$ and A_c, B_c, C_c and D_c are matrices of appropriate dimension. The compensator transfer function is $G_c(s) = C_c(s - A_c)^{-1}B_c + D_c$. To ensure well-posedness, we assume that $\det(I_{n_i} + D_c D_p) \neq 0$.

We define a Hilbert space $H = E \times \mathbb{R}^{n_c}$ with inner product

$$\left\langle \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \begin{bmatrix} z_p \\ z_c \end{bmatrix} \right\rangle_H = \langle x_p, z_p \rangle_E + \langle x_c, z_c \rangle_{\mathbb{R}^{n_c}} . \tag{2.4}$$

Since $e_1 = u_1 - y_2 - d_o - d_s$ and $e_2 = y_1 + u_2$, we obtain the following state equations for the closed loop system

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = A \begin{bmatrix} x_p \\ x_c \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \\ d_o \\ d_s \end{bmatrix} \tag{2.5a}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ z_2 \end{bmatrix} = C \begin{bmatrix} x_p \\ x_c \end{bmatrix} + D \begin{bmatrix} u_1 \\ u_2 \\ d_o \\ d_s \end{bmatrix} \tag{2.5b}$$

where

$$A = \begin{bmatrix} A_p - B_p D_c (1+D_p D_c)^{-1} C_p & B_p (1+D_c D_p)^{-1} C_c \\ -B_c (1+D_p D_c)^{-1} C_p & A_c - B_c (1+D_p D_c)^{-1} D_p C_c \end{bmatrix} \tag{2.6a}$$

$$B = \begin{bmatrix} B_p D_c (1+D_p D_c)^{-1} & B_p (1+D_c D_p)^{-1} & -B_p D_c (1+D_p D_c)^{-1} & -B_p D_c (1+D_p D_c)^{-1} \\ B_c (1+D_p D_c)^{-1} & -B_c (1+D_p D_c)^{-1} D_p & -B_c (1+D_p D_c)^{-1} & -B_c (1+D_p D_c)^{-1} \end{bmatrix} , \tag{2.6b}$$

$$C = \begin{bmatrix} -(1+D_p D_c)^{-1} C_p & -(1+D_p D_c)^{-1} D_p C_c \\ -D_c (1+D_p D_c)^{-1} C_p & (1+D_c D_p)^{-1} C_c \\ (1+D_p D_c)^{-1} C_p & D_p (1+D_c D_p)^{-1} C_c \end{bmatrix} \tag{2.6c}$$

$$D = \begin{bmatrix} (1+D_p D_c)^{-1} & -(1+D_p D_c)^{-1} D_p & -(1+D_p D_c)^{-1} & -(1+D_p D_c)^{-1} \\ D_c (1+D_p D_c)^{-1} & (1+D_c D_p)^{-1} & -D_c (1+D_p D_c)^{-1} & -D_c (1+D_p D_c)^{-1} \\ D_p D_c (1+D_p D_c)^{-1} & D_p (1+D_c D_p)^{-1} & (1+D_p D_c)^{-1} & -D_p D_c (1+D_p D_c)^{-1} \end{bmatrix} . \tag{2.6d}$$

The domain $D(A) = D(A_p) \times \mathbb{R}^{n_c} \subset H$; the operators B, C and D are easily seen to be bounded.

Because the operator A_p generates an analytic semigroup, so does the operator A:

Proposition 1: The operator A generates an analytic semigroup, $T(\cdot)$.

Proof: We can decompose the matrix A in Eq. (2.6a) as follows:

$$A = F + Q \tag{2.7a}$$

where, for $\lambda_c \in \mathbb{R}$ arbitrary

$$F = \begin{bmatrix} A_p & 0 \\ 0 & \lambda_c I_{n_c} \end{bmatrix}, \quad Q = \begin{bmatrix} -B_p D_c (I_{n_o} + D_p D_c)^{-1} C_p & B_p (I_{n_i} + D_c D_p)^{-1} C_c \\ -B_c (I_{n_o} + D_p D_c)^{-1} C_p & A_c - B_c (I_{n_o} + D_p D_c)^{-1} D_p C_c - \lambda_c I_{n_c} \end{bmatrix}. \tag{2.7b}$$

It is easy to see that F generates an analytic semigroup

$$T_F(t) = \begin{bmatrix} T(t) & 0 \\ 0 & e^{-\lambda_c t} I_{n_c} \end{bmatrix}. \tag{2.7c}$$

Note that Q is a bounded operator. By applying the perturbation theorem [Paz.1, p.80], we conclude that A generates an analytic semigroup. ■

From Proposition 1 and [Tri.1], we obtain

Proposition 2: The operator A satisfies the *spectrum determined growth assumption*, i.e.,

$$\sup(Re(\sigma(A))) = \lim_{t \to \infty} \frac{\ln\|T(t)\|}{t}. \quad ■ \tag{2.8}$$

From Proposition 2, we obtain the following result [Tri.1]:

Proposition 3: Given any $\beta > \sup(\mathrm{Re}(\sigma(A)))$, there exists an $M > 0$ such that

$$\|T(t)\|_H < M \cdot e^{\beta t}, \quad \forall\, t \geq 0. \quad ■ \tag{2.9}$$

Let $x = [x_p^t\ x_c^t]^t$. Then the formula

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau) B u(\tau) d\tau \tag{2.10}$$

defines a mild, strong or classical solution of Eq. (2.5a), depending on the initial state x_0 and input $u(t)$ [Paz.1]. Therefore we can define the exponential stability of the feedback system $S(P,K)$ in terms of the semigroup $T(t)$.

Definition 1: The feedback system S(P , K) is α–stable if and only if there exists $M > 0$ such that

$$\|T(t)\|_H < M \cdot e^{-\alpha t}, \quad \forall\, t \geq 0 . \qquad \blacksquare \tag{2.11}$$

Propositions 2 and 3 yield the following result.

Proposition 4: The system S(P , K) is α–stable if and only if

$$\sup(\mathrm{Re}(\sigma(A))) < -\alpha . \qquad \blacksquare \tag{2.12}$$

Remark 2.1: It follows from Proposition 4 and Assumption 1, that the plant is α–stable. ■

3. A COMPUTATIONAL STABLITY CRITERION

We define the characteristic function $\chi\colon \mathbb{C} \to \mathbb{C}$, of the closed-loop system $S(P,K)$, by

$$\chi(s) \triangleq \det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s)) , \tag{3.1}$$

and, for any function $f\colon \mathbb{C} \to \mathbb{C}$, we define $Z(f(s)) \triangleq \{s \in \mathbb{C} \mid f(s) = 0\}$.

Theorem 1: The system $S(P,K)$ is α–stable if and only if $Z(\chi(s)) \subset U_{-\alpha}$.

Proof: We have to use the Weinstein-Aronszajn (W-A) formula in this proof. The W-A formula and the all related definitions and notations which we use can be found in Appendix 1 or [Kat.1].

We begin by decomposing the matrix A as in (2.7a), (2.7b) with $\mathrm{Re}(\lambda_c) < -\alpha$. Therefore $(F - sI)$ is invertible for $s \in U^c_{-\alpha}$, and Q is an F-degenerate operator because it is bounded. Consider $s \in U^c_{-\alpha} \subset \rho(A_p)$. Since $(F - sI)^{-1}$ exists and is bounded, we can define $V(s)$ as follows

$$V(s) = Q(F - sI)^{-1}$$
$$= \begin{bmatrix} -B_pD_c(I_{n_o} + D_pD_c)^{-1}C_p(A_p - sI)^{-1} & B_p(I_{n_i} + D_cD_p)^{-1}C_c(\lambda_c - s)^{-1} \\ -B_c(I_{n_o} + D_pD_c)^{-1}C_p(A_p - sI)^{-1} & (A_c - B_c(I_{n_o} + D_pD_c)^{-1}D_pC_c - \lambda_cI_{n_c})(\lambda_c - s)^{-1} \end{bmatrix} . \tag{3.2}$$

Let $B_0 \triangleq R(B_p) \times \mathbb{R}^{n_c}$ and let $V_{B_0}(s)$ denote the restriction of $V(s)$ to B_0. Then $\det(I + V(s)) \triangleq \det(I_{B_0} + V_{B_0})$ is well defined (see Appendix 1). We will show that $\det(I_{B_0} + V_{B_0}) = \chi(s)$ and then apply the W-A formula.

Let $b_j \triangleq B_pe_j$, $j = 1,2,\ldots,n_i$, where $\{e_j\}_{j=1}^{n_i}$ is the standard unit basis in $\mathbb{R}^{n_i}$. Suppose that $\bar{n} \leq n_i$ is the largest positive interger such that, without loss of generality, $\{b_j\}_{j=1}^{\bar{n}}$ is a linearly independent subset in the Hilbert space H. Now we take $\{b_j\}_{j=1}^{\bar{n}}$ as a basis for $R(B_p)$. Under this basis, the linear operator B_p assumes the form $B_p = (I_{\bar{n}x\bar{n}} \mid \tilde{B}_p) \in \mathbb{R}^{\bar{n}xn_i}$ where the i-th column of $\tilde{B}_p$ is obtained by expressing $b_{\bar{n}+i}$ in terms of the basis $\{b_j\}_{j=1}^{\bar{n}}$. Let $\bar{B} \triangleq (b_1, b_2, \ldots, b_{\bar{n}})$. Then it is easy to show that

$$V_{B_0}(s) = \begin{bmatrix} -B_pD_c(I_{n_o}+D_pD_c)^{-1}C_p(A_p-sI)^{-1}\bar{B} & B_p(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_c(I_{n_o}+D_pD_c)^{-1}C_p(A_p-sI)^{-1}\bar{B} & (A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_cI_{n_c})(\lambda_c-s)^{-1} \end{bmatrix} \quad (3.3a)$$

$$= \begin{bmatrix} -B_pD_c(I_{n_o}+D_pD_c)^{-1}M & B_p(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_c(I_{n_o}+D_pD_c)^{-1}M & (A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_cI_{n_c})(\lambda_c-s)^{-1} \end{bmatrix}, \quad (3.3b)$$

where $M \triangleq [r_1, r_2, \ldots, r_{\bar{n}}] \in \mathbb{R}^{n_o x \bar{n}}$ with $r_i \triangleq C_p(A_p - sI)^{-1}b_i$, $1 \le i \le \bar{n}$. It is clear that $G_p(s) = -(C_p(A_p - sI)^{-1}B_p) + D_p = -M \cdot B_p + D_p$. Because each element in (3.3b) is in matrix form, it is easy to show that

$$\begin{aligned} \det(I_{B_0} + V_{B_0}(s)) &= \det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s)) \\ &= \chi(s) . \end{aligned} \quad (3.4)$$

Now we make use of the W-A formula. Let $\Delta = U^c_{-(\alpha+\varepsilon)}$, where $\varepsilon > 0$ and, by Assumption 1, can be chosen small enough to ensure that $U^c_{-(\alpha+\varepsilon)}$ still belongs to the resolvent set of A_p. Let F be defined as in (2.7). From the W-A formula, we have that

$$\{s \in U^c_{-(\alpha+\varepsilon)} \mid s \in \sigma(A)\} = \{s \in U^c_{-(\alpha+\varepsilon)} \mid s \in Z(\chi(s))\} . \quad (3.5)$$

If the system $S(P,K)$ is exponentially stable, $\sup(Re(\sigma(A))) < \alpha$, by Proposition 3, i.e., $\sigma(A) \subset U_{-\alpha}$. Therefore, from (3.5), with $\varepsilon = 0$, $\{s \in U^c_{-\alpha} \mid s \in Z(\chi(s))\}$ is an empty set i.e., $Z(\chi(s)) \subset U_{-\alpha}$.

On the other hand, if $Z(\chi(s)) \subset U_{-\alpha}$, then $\sup(Re(\sigma(A))) \le -\alpha$ by (3.5) with $\varepsilon = 0$. Suppose $\sup(Re(\sigma(A))) = -\alpha$. Then, from (3.5), there exists a sequence $\{s_i\}_{i=1}^{\infty}$ such that $s_i \in U^c_{-(\alpha+\varepsilon)}$, $\forall\ i \in \mathbb{N}$, $Re(s_i) \to -\alpha$, and $\chi(s_i) = 0$. Since $\det(sI_{n_c} - A_c)$ is a polynomial of finite order and $\lim_{\substack{s \in \mathbb{C} \\ |s| \to \infty}} \det(I + G_c(s)G_p(s)) = \det(I + D_cD_p) \neq 0$, $\{s_i\}_{i=0}^{\infty}$ is a bounded set. Hence there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to s_0$ and $Re(s_0) = -\alpha$. We will prove that $\chi(s)$ is analytic on $U^{c,o}_{-(\alpha+\varepsilon)}$ in Appendix II. Consequently $\chi(s)$ is continuous at s_0. Therefore, $\chi(s_0) = 0$, i.e. $s_0 \in Z(\chi(s))$. This contradicts $Z(\chi(s)) \subset U_{-\alpha}$. Hence $\sup(Re(\sigma(A))) < -\alpha$ and the system is α–stable. This completes the proof. ■

Theorem 2: $Z(\chi(s))\subset U_{-\alpha}$ <=> there exists $N_n > 0$ and polynomials $d_0(s)$ and $n_0(s)$ with degrees of $N_d = N_n+n_c$ and N_n, respectively, such that

$$(i)\ Z(d_0(s))\subset U_{-\alpha}\,,\quad Z(n_0(s))\subset U_{-\alpha}\,; \tag{3.6a}$$

$$(ii)\ \mathrm{Re}\left[\frac{\chi(s)n_0(s)}{d_0(s)}\right] > 0 \quad \forall\ s\in\partial U_{-\alpha}\,. \tag{3.6b}$$

Proof: (<=) This is clear from the Nyquist Criterion.

(=>) Since $\chi(s)$ is analytic on $U^c_{-(\alpha+\varepsilon)}$, by a straightforward generalization of an approximation result in [Sal.1, p.30], we can find a rational function $d_0(s)/n_0(s)$ with all zeros of $n_0(s)\subset U_{-\alpha}$ for any $\delta > 0$ such that

$$\|\chi(s) - d_0(s)/n_0(s)\| \triangleq \sup_{s\in\partial U_{-\alpha}} |\chi(s) - d_0(s)/n_0(s)| < \delta\,. \tag{3.7}$$

Because the degree of $\det(sI_{n_c} - A_c)$ is n_c, $\lim_{|s|\to\infty}|\chi(s)/s^{n_c}| = |\det(I + D_cD_p)|$. We conclude that the degree of $d_0(s)$ must exceed the degree of $n_0(s)$ by n_c. Also $\inf_{s\in\partial U_{-\alpha}}|\chi(s)| = c_0 > 0$ must hold. Suppose that it does not. Then $\inf_{s\in\partial U_{-\alpha}}|\chi(s)| = 0$. Since $|\chi(s)| \to \infty$ as $|s| \to \infty$, there exists a bounded sequence $\{s_i\}$ such that $s_i\in U^c_{-(\alpha+\varepsilon)}$, $Re(s_i) \to -\alpha$ and $\chi(s_i) \to 0$. Since $\{s_i\}$ is bounded, there exists a subsequence $\{s_{i_k}\}$ and an $\overset{*}{s}$ such that $s_{i_k} \to \overset{*}{s}$. From the continuity of $\chi(s)$ on $U^{c,o}_{-(\alpha+\varepsilon)}$, it follows that $\chi(\overset{*}{s}) = 0$. This contradicts the assumption that $Z(\chi(s))\subset U_{-\alpha}$. Therefore $1/\chi(s)$ is analytic on $U^{c,o}_{-\alpha}$ and continuous on $\partial U_{-\alpha}$ and $\|1/\chi(s)\| = 1/c_0$. So $|\chi(s)| \geq c_0 > 0$, for all $s\in U^c_{-\alpha}$. Hence $Z(d_0(s))\subset U_{-\alpha}$, if δ is chosen less than c_0. Otherwise, say, there exists $s_0\in U^c_{-\alpha}$ such that $d_0(s_0) = 0$. Since $|\chi(s_0)-d_0(s_0)/n_0(s_0)| < \delta$, we have that $|\chi(s_0)| < \delta < c_0$. This results in a contradiction.

Since

$$|\chi(s) - d_0(s)/n_0(s)| < \delta\,, \quad \forall\ s\in\partial U_{-\alpha}\,, \tag{3.8}$$

we obtain that for all $s\in\partial U_{-\alpha}$

$$\Big[|\chi(s) - d_0(s)/n_0(s)|\Big] / \Big[|d_0(s)/n_0(s)|\Big] < \delta|n_0(s)/d_0(s)|$$

$$< \delta\|n_0(s)/d_0(s)\|$$

$$< 1 \qquad \text{if } \delta < 1/\|n_0(s)/d_0(s)\| \,. \tag{3.9}$$

It follows from the above that for all $s \in \partial U_{-\alpha}$

$$|\chi(s) n_0(s)/d_0(s) - 1| < 1 \,, \tag{3.10}$$

which implies that

$$\mathrm{Re}[\, \chi(s) n_0(s)/d_0(s) \,] > 0 \,, \ \forall \ s \in \partial U_{-\alpha} \,, \tag{3.11}$$

which completes our proof. ■

Remark: Theorem 1 and 2 can also be applied to the case where the plant has a finite number of unstable poles (counting multiplicities) located in $U^c_{-\alpha}$. To prove this, we only have to replace the definition of the characteristic function $\chi(s)$ in (3.1) by the following expression:

$$\chi(s) = \left(\prod_{i=1}^{m}(s - s_i)\right)\det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s)) \,, \tag{3.12}$$

where $\{s_i\}_{i=1}^{m}$ is the set of poles of the plant located in $U^c_{-\alpha}$.

4. NUMERICAL IMPLEMENTATION OF THE STABILITY CRITERION

When used to design stabilizing controllers for the system $S(P,K)$, the test (3.6a) and (3.6b) becomes only a sufficient condition of stability, because one is forced to choose in advance the degree N_d of the polynomial $d_0(s)$. We shall now sketch out the numerical aspects of using the inequalities (3.6a), (3.6b) in the design of a stabilizing controller for the system $S(P,K)$. We assume that the order n_c of the controller (2.3) has been selected and that the elements of the matrices in (2.3) are continuously differentiable in the design parameter vector p_c.

First we will describe a computationally efficient parametrization for $d_0(s)$ and $n_0(s)$ which is based on the following observation. When $a,b \in \mathbb{R}$, $Z[(s+\alpha) + a)] \subset U_{-\alpha}$ if and only if $a > 0$, and $Z[(s+\alpha)^2 + a(s + \alpha) + b] \subset U_{-\alpha}$ if and only if $a > 0$, $b > 0$. Hence, assuming that the degree of $d_0(s)$ is odd, we set

$$d_0(s,q_d) \triangleq ((s+\alpha) + a_0)\prod_{i=1}^{m}((s + \alpha)^2 + a_i(s + \alpha) + b_i), \tag{4.1}$$

where $q_d \triangleq [a_0,a_1,a_2, \cdots ,b_1,b_2, \cdots ,b_m]^T \in \mathbb{R}^{2m+1}$ and $N_d = 2m+1$. The polynomial $n_0(s)$, which is of degree $N_n = N_d - n_c$ can be parametrized similarly, with corresponding parameter vector q_n.

It follows from Theorem 2 that a stabilizing controller can be obtained by solving the following set of inequalities:

$$q_d^i - \varepsilon \geq 0, \quad \text{for } i = 1,2, \cdots ,N_d , \tag{4.2a}$$

$$q_n^i - \varepsilon \geq 0, \quad \text{for } i = 1,2, \cdots ,N_n, \tag{4.2b}$$

$$\mathrm{Re}(\frac{\chi(-\alpha + j\omega,p_c)n_0(-\alpha+j\omega,q_n)}{d_0(-\alpha + j\omega,q_d)}) - \varepsilon \geq 0, \quad \forall\ \omega \in [0,\infty) , \tag{4.2c}$$

where q^i is the ith element of q.

When a minimax type algorithm, such as can be found in [Pol.3], is used to solve the system (4.2a) - (4.2c), subject to a box constraint on p_c, it must evaluate the characteristic function $\chi(s,p_c)$ and its partial derivatives with respect to p_c^i for $s = \alpha + j\omega$ for many values of ω. Hence one must try to perform these operations as efficiently as possible.

To evaluate $\chi(s,p_c)$, we have to calculate $\det(sI_{n_c} - A_c(p_c))$ and $\det(I_{n_i} + G_c(s,p_c)G_p(s))$. The simplest situation occurs when the matrix $A_c(p_c)$ is diagonalizable, i.e., there exists a matrix of eigenvectors $V(p_c)$ such that

$$\Lambda(p_c) = V(p_c)^{-1}A_c(p_c)V(p_c) , \tag{4.3}$$

where $\Lambda(p_c)$ is a diagonal matrix whose diagonal elements are the eigenvalues $\lambda_j(p_c)$ of the matrix $A_c(p_c)$. In this case, considerable computational savings result from the use of the formula

$$\det[sI_{n_c} - A_c(p_c)] = \det[sI_{n_c} - \Lambda(p_c)] = \prod_{j=1}^{n_c}[s - \lambda_j(p_c)], \tag{4.4a}$$

and of the formula

$$G_c(s,p_c) = C_c(p_c)(sI_{n_c} - A_c(p_c))^{-1}B_c(p_c) + D_c(p_c) \tag{4.4b}$$

When diagonalization cannot be used, one can simplify the computation of the required determinants by first reducing $A_c(p_c)$ to upper Hessenberg form $H_c(p_c)$ by means of an orthongonal similarity transformation:

$$H_c(p_c) = U(p_c)^T A_c(p_c) U(p_c) \,, \tag{4.5}$$

where $U(p_c)$ is a Hermitian matrix. This leads to the formula

$$\det[sI_{n_c} - A_c(p_c)] = \det[sI_{n_c} - H(p_c)] \tag{4.6a}$$

and

$$G_c(s,p_c) = C_c(p_c)U(p_c)(sI_{n_c} - H(p_c))^{-1}U(p_c)^T B_c(p_c) + D_c(p_c) \,. \tag{4.6b}$$

Next we need to deal with the evaluation of the plant transfer function $G_p(-\alpha + j\omega)$ for many values of ω. The infinite dimensional form (2.1) can be used to evaluate this transfer function if one is willing to resort to mode truncation. A better way is to go back to the physical plant whose original describing partial differential equations were transcribed into the form (2.1). For this we must consider an example.

Consider the planar bending motion of a flexible beam with one end fixed and another end attached a particle with mass M, as shown in Fig. 2. Note that the x-axis in Figure 2 is the beam undeformed centroidal line and the y-axis is the cross section principal axis. The associated control system is required to damp out vibrations as well as to position the tip of the beam. Without loss of generality, we assume that the beam is of unit length. Then its bending motion can be described by a partial differential equation of the form,

$$m\frac{\partial^2 w(t,x)}{\partial t^2} + cI\frac{\partial^5 w(t,x)}{\partial x^4 \partial t} + EI\frac{\partial^4 w(t,x)}{\partial x^4} = \sum_{j=1}^{n} f^j(t)\zeta^j(x,x^j), \;\; t \geq 0, \;\; 0 \leq x \leq 1 \,, \tag{4.7a}$$

with boundary conditions

$$w(t,0) = 0 \,, \; \frac{\partial w(t,0)}{\partial x} = 0, \tag{4.7b}$$

$$J\frac{\partial^3 w(t,1)}{\partial x \partial t^2} + cI\frac{\partial^3 w(t,1)}{\partial x^2 \partial t} + EI\frac{\partial^2 w(t,1)}{\partial x^2} = 0 \,, \tag{4.7c}$$

$$M\frac{\partial^2 w(t,1)}{\partial t^2} - cI\frac{\partial^4 w(t,1)}{\partial x^3 \partial t} - EI\frac{\partial^3 w(t,1)}{\partial x^3} = 0, \tag{4.7d}$$

where $w(t,x)$ is the vibration along the y-axis, $f^j(t)$ is a control force, $\zeta^j(x,x^j)$ is the influence function of the j–th actuator, which is determined by the location, x^j, and the physical characteristics of the actuator. The constants in (4.7a) - (4.7d) are as follows: m is the distributed mass per unit length of the beam, c is the material viscous damping coefficient, A is the beam cross sectional area, E is Young's modulus, EA is the beam extensional stiffness, M is the end mass, I is the beam sectional moment of inertia with respect to y-axis, EI is the beam flexural stiffness in the direction of y-axis, J is the inertia of the end mass in the direction of y-axis.

The output sensors can be assumed to be modeled by

$$y_i(t) = \int_0^1 \kappa_i(\nu, z_i) w(t,\nu) d\nu \ , \quad t \geq 0 \ , \quad 1 \leq i \leq n_o, \tag{4.8}$$

where n_o is the number of the sensors, and $\kappa_i(\nu, z_i)$ is the distribution function of the i-th sensor and z_i is the location of the i-th sensor.

Taking the Laplace transforms of the equations (4.7a) - (4.7d) and (4.8) with respect to time, we obtain, for each value of $s = -\alpha + j\omega$, the *ordinary differential equation*

$$(cIs + EI)\frac{d^4W(s,x)}{dx^4} + ms^2W(s,x) = \sum_{j=1}^{n} F^j(s)\zeta^j(x,x^j), \quad 0 \leq x \leq 1 \ , \tag{4.9a}$$

with boundary conditions

$$W(s,0) \ = \ 0 \ , \ \frac{dW(s,0)}{dx} = 0 \ , \tag{4.9b}$$

$$(cIs + EI)\frac{d^2W(s,1)}{dx^2} + Js^2\frac{dW(s,1)}{dx} = 0 \ , \tag{4.9c}$$

$$(cIs + EI)\frac{d^3W(s,1)}{dx^3} - Ms^2W(s,1) = 0 \ , \tag{4.9d}$$

and

$$Y_i(s) = \int_0^1 \kappa_i(\nu, z_i) W(s,\nu) d\nu \ , \quad 1 \leq i \leq n_o, \tag{4.10}$$

where $W(s,x)$, $F^j(s)$ and $Y_i(s)$ are the Laplace transforms of $w(t,x)$, $f^j(t)$ and $y_i(t)$, respectively. It follows that the (i,j)-th element of $G_p(s)$ can be obtained by setting $F^j(s) = 1$ and $F^k(s) = 0$ for all other k and then solving (4.9a) - (4.9d) and (4.10).

Next, we turn to the computation of the partial derivatives of $\chi(s,p_c)$. This requires the calculation of the partial derivatives of $\det[s - A_c(p_c)]$ and $\det[I_{n_i} + G_c(p_c)G_p(s)]$. When the eigenvalues $\lambda_j(p_c)$ of

$A_c(p_c)$ are distinct, they are differentiable [Kat.1] and their partial derivatives are given by

$$\frac{\partial \lambda_j(p_c)}{\partial p_c^i} = \langle u_j, \frac{\partial A_c(p_c)}{\partial p_c^i} v_j \rangle / \langle u_j, v_j \rangle \tag{4.11}$$

where v_j and u_j are the right and left eigenvectors, respectively, of $A_c(p_c)$, corresponding to the eigenvalue $\lambda_j(p_c)$. In this case, the partial derivatives of $\det[sI_{n_c} - A_c(p_c)]$ can be computed making use of the following formula [Pol.2]:

$$\frac{\partial \det[sI_{n_c} - A_c(p_c)]}{\partial p_c^i} = \sum_{j=1}^{n_c} \{ - \frac{\partial \lambda_j(p_c)}{\partial p_c^i} \prod_{\substack{k=1 \\ k \neq j}}^{n_c} [s - \lambda_k(p_c)] \} = \det[sI_{n_c} - A_c(p_c)] \sum_{j=1}^{n_c} - \frac{\partial \lambda_j(p_c)}{\partial p_c^i} \frac{1}{s - \lambda_j(p_c)} . \tag{4.12}$$

When the eigenvalues of $A_c(p_c)$ are not distinct, the computation of its partial derivative requires a more general formula which can be found in [Pol.2].

The computation of the partial derivatives of $\det[I_{n_i} + G_c(s,p_c)G_p(s)]$ can also be carried out by making use of a formula analogous to (4.12), provided that the matrix $\left[I_{n_i} + G_c(s,p_c)G_p(s)\right]$ has distinct eigenvalues.

When the eigenvalues of $(I_{n_i} + G_c(s,p_c)G_p(s))$ are not distinct, the computation of its partial derivative becomes considerably more difficult. Fortunately, this is not very likely to be the case in practice.

Remark: It is important to observe that the evaluation of the plant frequency response can be carried out without resorting to modal truncation. Thus, in the range of critical frequencies, the plant frequency response can be computed with very high precision, simply by integrating the differential equation (4.9a) - (4.9d), without incurring the serious problems that are associated with "spill-over" when modal truncation is used. ■

5. CONCLUSION

We have presented a necessary and sufficient computational stability criterion and have shown how it can be used in the design of stabilizing controllers for infinite dimensional feedback systems. A major advantage of our approach is that it avoids common "spill-over" effects which result from modal truncation of partial differential equation models. We expect that our approach will be useful in the design of finite dimensional controllers for flexible structures.

APPENDIX I: THE WEINSTEIN-ARONSZAJN FORMULA

The following material was extracted from [Kat.1]. Let Q and F be operators in the Banach space X. We say F is a *closed operator* in X if for any sequence $\{u_i\} \subset D(F)$ such that $u_i \to u$ and $Fu_i \to v$, u belongs to $D(F)$ and $Fu = v$. The operator Q is F–bounded if $D(F) \subset D(Q)$ and

$$\|Qu\| \le a\|u\| + b\|Fu\|, \ \forall \ u \in D(F) , \tag{A.1.1}$$

where a,b are nonnegative constants. The operator Q is a *degenerate operator* if $R(Q)$ is finite-dimensional; Q is F-degenerate if Q is F-bounded and $R(Q)$ is finite-dimensional.

Suppose that Q is F-degenerate. Then $Q(F - sI)^{-1}$ is a degenerate operator if $s \in \rho(F)$. Let $R \triangleq R(Q)$. For any $s \in \rho(F)$, the *W-A (Weinstein-Aronszajn) determinant* $y(s;F,Q) = \det(1 + Q(F - sI)^{-1})$ is defined by [Kat.1, p.161]

$$y(s;F,Q) = \det(I_R + (Q(F - sI)^{-1})|_R) \tag{A.1.2}$$

where I_R is the identity operator in R and $(Q(F - sI)^{-1})|_R$ is the restriction of $Q(F - sI)^{-1}$ to R.

Let ϕ be a numerical meromorphic function defined on a domain Δ of the complex plane. By meromorphic function, we mean a single-valued function having no singularities other than (at most) poles in the domain in which the function is defined [Kno.1]. We define the *multiplicity function* $\nu(s;\phi)$ of ϕ by

$$\nu(s;\phi) = \begin{cases} k & \text{if } s \text{ is a zero of } \phi \text{ of order } k, \\ -k & \text{if } s \text{ is a pole of } \phi \text{ of order } k, \\ 0 & \text{for other } s \in \Delta \end{cases} . \tag{A.1.3}$$

Thus $\nu(s;\phi)$ takes the values $0,\pm 1,\pm 2,\ldots$ or $+\infty$. We define the *multiplicity function* $\tilde{\nu}(s;F)$ for a closed operator F by

$$\tilde{\nu}(s;F) = \begin{cases} 0 & \text{if } s \in \rho(F) , \\ \dim(P) & \text{if } s \text{ is an isolated point of } \sigma(F), \\ +\infty & \text{in all other cases} \end{cases} , \tag{A.1.4}$$

where P is the projection associated with an isolated point of $\sigma(F)$ [Kat.1, p.180]. If the isolated point s is an eigenvalue of F, then $\dim(P)$ is finite and equal to the multiplicity of the eigenvalue. Thus

$\bar{\nu}(s;F)$ is defined for all complex numbers s and takes on the values $0, 1, 2, \cdots$ or $+\infty$.

Theorem A.1 (The W-A Formula): Let F be a closed operator in the Banach space X, let Q be a F-degenerate operator in X and let $y(s) = y(s;F,Q)$ be the associated W-A determinant. If Δ is a domain of the complex plane consisting of points of $\rho(F)$ and of isolated eigenvalues of F with finite multiplicities, then $y(s)$ is meromorphic in Δ and, for $A = F + Q$,

$$\bar{\nu}(s;A) = \bar{\nu}(s;F) + \nu(s;y), \quad s \in \Delta . \quad \blacksquare \tag{A.1.5}$$

APPENDIX II: ANALYTICITY OF THE CHARACTERISTIC FUNCTION

Theorem A.2: The characteristic function $\chi(s) = \det(sI - A_c)\det(1 + G_c(s)G_p(s))$ is an analytic function on $U^{c,\rho}_{-(\alpha+\varepsilon)}$.

Proof: (i) First, we will prove that each component of $G_p(s) = C_p(s - A_p)^{-1}B_p + D_p$ is an analytic function over $U^{c,\rho}_{-(\alpha+\varepsilon)}$. We denote the (i , j)-th component of $G_p(s)$ by $G_{i,j}(s)$. Then

$$G_{i,j}(s) = C_{p,i\cdot}(s - A_p)^{-1}B_{p,\cdot j} + D_{i,j} , \tag{A.2.1}$$

where $C_{p,i\cdot}$ is the i-th row of C_p , $B_{p,\cdot j}$ is the j-th column of B_p and $D_{i,j}$ is the (i,j)-th component of D_p. We will prove that $G_{i,j}(s)$ is differentiable by showing that

$$\lim_{\Delta s \to 0} \frac{G_{i,j}(s+\Delta s) - G_{i,j}(s)}{\Delta s} = -C_{p,i\cdot}(s - A_p)^{-2}B_{p,\cdot j} , \tag{A.2.2}$$

which follows from:

$$\lim_{\Delta s \to 0} \left| \frac{(C_{p,i\cdot}(s + \Delta s - A_p)^{-1}B_{p,\cdot j} + D_{i,j}) - (C_{p,i\cdot}(s - A_p)^{-1}B_{p,\cdot j} + D_{i,j})}{\Delta s} + C_{p,i\cdot}(s - A_p)^{-2}B_{p,\cdot j} \right|$$

$$= \lim_{\Delta s \to 0} \left| C_{p,i\cdot} \left[\frac{(s + \Delta s - A_p)^{-1} - (s - A_p)^{-1}}{\Delta s} + (s - A_p)^{-2} \right] B_{p,\cdot j} \right|$$

$$\leq \lim_{\Delta s \to 0} \|C_{p,i\cdot}\| \left\| \frac{(s + \Delta s - A_p)^{-1} - (s - A_p)^{-1}}{\Delta s} + (s - A_p)^{-2} \right\| \|B_{p,\cdot j}\|$$

$$\leq \lim_{\Delta s \to 0} \|C_p\| \left\| \frac{(s + \Delta s - A_p)^{-1} - (s - A_p)^{-1}}{\Delta s} + (s - A_p)^{-2} \right\| \|B_p\|$$

$$= \lim_{\Delta s \to 0} \|C_p\| \left\| \frac{(s + \Delta s - A_p)^{-1} \left[(s - A_p) - (s + \Delta s - A_p) \right] (s - A_p)^{-1}}{\Delta s} + (s - A_p)^{-2} \right\| \|B_p\|$$

$$= \lim_{\Delta s \to 0} \|C_p\| \, \| - (s + \Delta s - A_p)^{-1}(s - A_p)^{-1} + (s - A_p)^{-2}\| \, \|B_p\|$$

$$= \lim_{\Delta s \to 0} \|C_p\| \, \|(s + \Delta s - A_p)^{-1}\Big[-(s - A_p) + (s + \Delta s - A_p)\Big](s - A_p)^{-1}(s - A_p)^{-1}\| \, \|B_p\|$$

$$= \lim_{\Delta s \to 0} |\Delta s| \, \|C_p\| \, \|(s + \Delta s - A_p)^{-1}(s - A_p)^{-1}(s - A_p)^{-1}\| \, \|B_p\|$$

$$= 0 \, . \tag{A.2.3}$$

Therefore $G_{i,j}(s)$ is an analytic function on $U^{c,o}_{-(\alpha + \varepsilon)}$.

(ii) Let (D_c, N_c) be the left coprime factorization pair for $G_c(s)$, i.e., $G_c(s) = D_c^{-1}(s)N_c(s)$, with $D_c(s)$ and $N_c(s)$ are analytic in $U^{c,o}_{-(\alpha + \varepsilon)}$. Consider

$$\begin{aligned}
\chi(s) &= \det(sI_{n_c} - A_c)\det(I + G_c(s)G_p(s)) \\
&= \det(sI_{n_c} - A_c)\det(I + D_c^{-1}(s)N_c(s)G_p(s)) \\
&= \det(sI_{n_c} - A_c)\det(D_c + N_c(s)G_p(s))\det(D_c^{-1}(s)) \\
&= \frac{\det(sI_{n_c} - A_c)}{\det D_c(s)}\det(D_c + N_c(s)G_p(s)) \, .
\end{aligned} \tag{A.2.4}$$

We can choose $D_c(s) = I_{n_i} - C_c(sI_{n_c} - A_c + FC_c)^{-1}F$ and $N_c(s) = C_c(sI_{n_c} - A_c + FC_c)^{-1}(B_c - FC_c) + D_c$, where F is such that $(sI_{n_c} - A_c + FC_c)$ has poles over $U_{-\alpha}$ (We assume the compensator is detectable and stabilizable). Then

$$\begin{aligned}
D_c(s) &= I_{n_i} - C_c(I_{n_c} + (sI_{n_c} - A_c)^{-1}FC_c)^{-1}(sI_{n_c} - A_c)^{-1}F \\
&= I_{n_i} - (I_{n_i} + C_c(sI_{n_c} - A_c)^{-1}F_c)^{-1}C_c(sI_{n_c} - A_c)^{-1}F_c \\
&= (I_{n_i} + C_c(sI - A_c)^{-1}F_c)^{-1} \, ,
\end{aligned} \tag{A.2.5}$$

and

$$\chi(s) = \frac{\det(sI_{n_c} - A_c)}{\det(I_{n_i} + C_c(sI_{n_c} - A_c)^{-1}F_c)^{-1}}\det(D_c(s) + N_c(s)G_p(s))$$

$$= \det(sI_{n_c} - A_c)\det(I_{n_i} + C_c(sI_{n_c} - A_c)^{-1}F_c)\det(D_c(s) + N_c(s)G_p(s))$$

$$= \det(sI_{n_c} - A_c)\det(I_{n_c} + (sI_{n_c} - A_c)^{-1}F_cC_c)\det(D_c(s) + N_c(s)G_p(s))$$

$$= \det(sI_{n_c} - A_c + F_cC_c)\det(D_c(s) + N_c(s)G_p(s)) \ . \qquad \text{(A.2.6)}$$

Since $\det(sI_{n_c} - A_c + F_cC_c)$ is a polynomial and since each component of $D_c(s) + N_c(s)G_p(s)$ is analytic over $U^{c,o}_{-(\alpha+\varepsilon)}$, it follows that $\chi(s)$ is analytic over $U^{c,o}_{-(\alpha+\varepsilon)}$. ■

REFERENCES

[Bal.1] A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, 1981.

[Kat.1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[Kno.1] K. Knopp, *Theory of Functions, Part II*, Dover, 1947.

[Nyq.1] H. Nyquist, "Regeneraton Theory", *Bell Syst. Tech. J.* vol.2, pp.126-147, Jan. 1932.

[Paz.1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[Pol.1] E. Polak, "A Modified Nyquist Stability Test for Use in Computer Aided Design", *IEEE Trans. on Control*, Vol. AC-29, No.1, pp.91-93, 1984.

[Pol.2] E. Polak and T. L. Wuu, "On the Design of Stabilizing Compensators Via Semi-infinite Optimization", *IEEE Trans. on Control*, in press.

[Pol.3] E. Polak, "On the Mathematical Foundations of Nondifferentiable Optimization in Engineering Design", *SIAM Review*, pp. 21-91, March 1987.

[Sal.1] S. E. Salcudean, *Algorithms for Optimal Design of Feedback Compensators*, Ph.D. Thesis, Department of EECS, University of California, Berkeley, 1986.

[Tri.1] R. Triggiani, "On the Stabilizability Problem in Banach Space", *Journal of Mathematical Analysis and Applications*, 52, pp. 383-403, 1975.

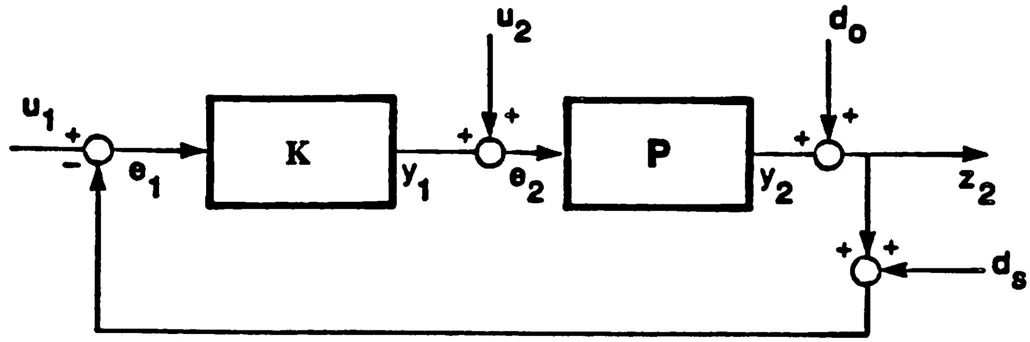

Figure 1: The feedback system *S(P,K)*.

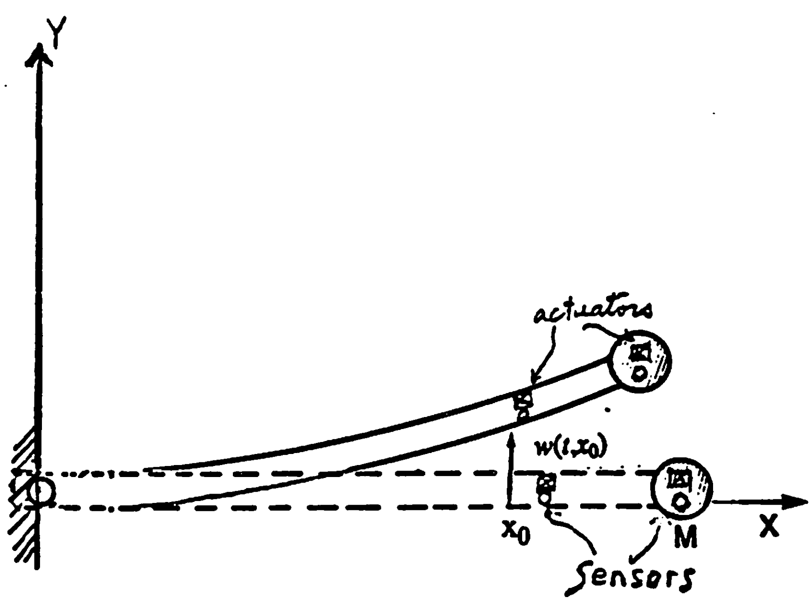

Figure 2: Planar bending motion of a flexible beam.