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A Method for Obtaining a Canonical Hamiltonian for Nonlinear LC Circuits*

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ABSTRACT

A method is given for obtaining by inspection the Hamiltonian and canonical variables for nonlinear lossless circuits composed of charge controlled capacitors, flux controlled inductors and independent voltage and current sources.

1. Introduction

With the current resurgence of interest in classical dynamics, a simple method for obtaining a Hamiltonian for nonlinear LC circuits would permit the application of new mathematical methods in classical dynamics [1,2] to circuits. Obtaining a Hamiltonian for the lossless part of an electronic system can be an important initial step in system analysis. In the study of weakly nonlinear oscillator circuits [3], all resistive elements are removed from the circuit, the lossless circuit is analyzed and a change of variables to action-angle coordinates is performed. As another example of this approach, in a study of a digital phase locked loop [4], an estimate of the average *hang time* of the loop due to transient chaos was obtained by studying the underlying chaotic Hamiltonian system.

Previous studies [5,6] have tended to focus on the Lagrangian formulation of circuit equations and give the Hamiltonian formulation of the circuit equations as a special case. In [5] this resulted in the restriction that the nonlinear capacitors and inductors have bijective element relations; in addition, the formulation used capacitor flux and inductor charge as state variables, which in some sense seems unnatural. In [6] the result was a *generalized* Hamiltonian formulation; although the variables used were capacitor charge and inductor flux, a generalized notion of derivative was used, which results in the Hamiltonian not having the usual canonical form. In this work we show how to construct, based on the circuit topology, canonical

*Research sponsored by Office of Naval Research Contract N00014-84-K-0367 and National Science Foundation Grant ECS-8517364.

variables that are simple linear combinations of capacitor charges and inductor fluxes, so that the circuit equations have the canonical form:

$$\dot{p}_k = \frac{\partial H(\mathbf{p},\mathbf{q})}{\partial q_k}, \tag{1a}$$

$$\dot{q}_k = \frac{-\partial H(\mathbf{p},\mathbf{q})}{\partial p_k}, \tag{1b}$$

for $k = 1, ..., r$.

2. Basic equations

Let $\mathcal{N}$ be a circuit composed of n_C, two-terminal, nonlinear, charge-controlled capacitors and n_L, two-terminal, nonlinear, flux-controlled inductors. The constitutive relations for the capacitors and the inductors are written as follows:

$$v_{C_k} = \hat{v}_{C_k}(q_{C_k}), \quad k = 1, ..., n_C \tag{2}$$

and

$$i_{L_k} = \hat{i}_{L_k}(\varphi_{L_k}), \quad k = 1, ..., n_L. \tag{3}$$

Assume that there are no capacitor loops or inductor cutsets. If capacitor loops or inductor cutsets exist and the *offending* elements have bijective relations between charge and voltage or flux and current then the circuit can be transformed to an equivalent circuit with no capacitor loops or inductor cutsets. See [7], pages 304 - 305, for details. From this assumption we can pick a tree containing all of the capacitors and no inductors. Partition the branch voltage and current vectors as follows:

$$\mathbf{i} = [\mathbf{i}_L^T, \mathbf{i}_C^T]^T, \quad \text{and} \quad \mathbf{v} = [\mathbf{v}_L^T, \mathbf{v}_C^T]^T.$$

Where $\mathbf{i}_L$, $\mathbf{v}_L \in \mathbf{R}^{n_L}$ and $\mathbf{i}_C$, $\mathbf{v}_C \in \mathbf{R}^{n_C}$. With the above partition the fundamental loop matrix has the form:

$$\mathbf{B} = (\mathbf{I}_{n_L \times n_L} \quad \mathbf{B}_{LC}).$$

Kirchhoff's voltage law, $\mathbf{Bv} = \mathbf{0}$, gives:

$$\mathbf{v}_L = -\mathbf{B}_{LC}\mathbf{v}_C. \tag{4}$$

Kirchhoff's current law, $\mathbf{i} = \mathbf{B}^T\mathbf{i}_{links}$, gives:

$$\mathbf{i}_C = \mathbf{B}_{LC}^T\mathbf{i}_L. \tag{5}$$

The state equations for this circuit, using capacitor charge and inductor flux as state variables, can be written as follows:

$$\dot{\mathbf{q}}_C = \mathbf{B}_{LC}^T \hat{\mathbf{i}}(\varphi_L), \qquad \dot{\varphi}_L = -\mathbf{B}_{LC} \hat{\mathbf{v}}(\mathbf{q}_C) \tag{6}$$

where

$$\hat{\mathbf{v}}(q_C) = \left[\hat{v}_{C_1}(q_{C_1}), \ldots, \hat{v}_{C_{n_C}}(q_{C_{n_C}})\right]^T, \qquad \hat{\mathbf{i}}(\varphi_L) = \left[\hat{i}_{L_1}(\varphi_{L_1}), \ldots, \hat{i}_{L_{n_L}}(\varphi_{L_{n_L}})\right]^T. \tag{7}$$

To obtain a Hamiltonian, we give the energy function for the circuit and use this energy function to formulate the state equations of the circuit. Define the energy, $E(\mathbf{q}_C, \varphi_L)$, of the circuit to be the sum of the energies in all the inductors and capacitors, that is

$$E(\mathbf{q}_C, \varphi_L) = \sum_{k=1}^{n_C} \int_0^{q_{C_k}} \hat{v}_{C_k}(u)du + \sum_{k=1}^{n_L} \int_0^{\varphi_{L_k}} \hat{i}_{L_k}(u)du. \tag{8}$$

We can write the state equations in terms of this energy function as follows:

$$\dot{\mathbf{q}}_C = \mathbf{B}_{LC}^T \left[\frac{\partial E(\mathbf{q}_C, \varphi_L)}{\partial \varphi_L}\right]^T, \qquad \dot{\varphi}_L = -\mathbf{B}_{LC} \left[\frac{\partial E(\mathbf{q}_C, \varphi_L)}{\partial \mathbf{q}_C}\right]^T. \tag{9}$$

For the above equations to be in canonical Hamiltonian form the number of capacitors and the number of inductors should be equal and $\mathbf{B}_{LC}$ must be the identity matrix. Since this would only be true for a trivial circuit, the above equation is not generally in canonical form.

3. Change of variables

Let $\mathbf{N} \in \mathbf{R}^{n_L \times n_L}$ and $\mathbf{M} \in \mathbf{R}^{n_C \times n_C}$ be nonsingular matrices, $\mathbf{x} \in \mathbf{R}^{n_C}$ and $\mathbf{y} \in \mathbf{R}^{n_L}$. Define a change of variables as follows:

$$\mathbf{q}_C = \mathbf{M}\mathbf{x}, \qquad \varphi_L = \mathbf{N}\mathbf{y}. \tag{10}$$

Applying this change of variables to equation (9) gives:

$$\dot{\mathbf{x}} = \mathbf{M}^{-1}\mathbf{B}_{LC}^T(\mathbf{N}^T)^{-1}\left[\frac{\partial E(\mathbf{q}_C, \varphi_L)}{\partial \mathbf{y}}\right]^T \tag{11a}$$

$$\dot{\mathbf{y}} = -\mathbf{N}^{-1}\mathbf{B}_{LC}(\mathbf{M}^T)^{-1}\left[\frac{\partial E(\mathbf{q}_C, \varphi_L)}{\partial \mathbf{x}}\right]^T \tag{11b}$$

Our goal is to find $\mathbf{M}$ and $\mathbf{N}$ such that

$$\mathbf{N}^{-1}\mathbf{B}_{LC}(\mathbf{M}^T)^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times(n_C-r)} \\ \mathbf{0}_{(n_L-r)\times r} & \mathbf{0}_{(n_L-r)\times(n_C-r)} \end{pmatrix}. \tag{12}$$

With this choice of $\mathbf{M}$ and $\mathbf{N}$ we see that

(1) for $k > r$, $\dot{x}_k = 0$ and $\dot{y}_k = 0$; therefore, x_k and y_k are constant. Because of this, we can restrict our analysis to the $2r$ *active* variables, x_k, y_k for $k = 1, \ldots, r$. For these variables, $E(\mathbf{q}_C, \varphi_L)$ is the canonical Hamiltonian.

(2) $E(\mathbf{q}_C, \varphi_L)$, the Hamiltonian for the $2r$ *active* variables, still represents the total energy in the capacitors and inductors.

4. Calculation of the M and N matrices

Although there are many choices for **M** and **N** that satisfy equation (12), we will restrict ourselves to two alternative choices that can be easily obtained by inspection of the circuit topology. Let $\mathcal{C}$ denote the set of all capacitors in the circuit and $\mathcal{L}$ the set of all inductors in the circuit. Our topological restriction of no capacitor loops or inductor cutsets implied that the set $\mathcal{C}$ forms a tree and $\mathcal{L}$ forms a cotree for the circuit and using the $\mathcal{C}$ tree and $\mathcal{L}$ cotree we wrote the state equation (9) of our circuit. The essence of our method is to look at an alternate tree (cotree) containing as many inductors (capacitors) as possible.

Let l denote the number of independent, inductor-only loops and s the number of independent, capacitor-only cutsets. The colored branch theorem [8], corollary 6, allows us to partition $\mathcal{C}$ and $\mathcal{L}$ as follows:

$\mathcal{L}_1$ is the largest subset of $\mathcal{L}$ that can be put into a tree for the circuit; $\mathcal{L}_1$ contains $n_L - l$ elements.

$\mathcal{C}_1$ is the largest subset of $\mathcal{C}$ that can be put into a cotree for the circuit; $\mathcal{C}_1$ contains $n_C - s$ elements.

$\mathcal{C}_2$ is the subset of $\mathcal{C}$ needed to complete a tree for the circuit with $\mathcal{L}_1$; $\mathcal{C}_2$ contains s elements.

$\mathcal{L}_2$ is the subset of $\mathcal{L}$ neeeded to complete a cotree for the circuit with $\mathcal{C}_1$; $\mathcal{L}_2$ contains l elements.

We now have a tree, $\mathcal{L}_1 \cup \mathcal{C}_2$, containing as many inductors as possible and a cotree, $\mathcal{C}_1 \cup \mathcal{L}_2$, containing as many capacitors as possible. Since the number of tree branches is a constant of the circuit, that is, independent of the chosen tree, we have $n_C = n_L - l + s$ and thus $n_C - s = n_L - l$. Let $r = n_C - s = n_L - l$. We partition and order our branch voltage and current vectors as follows:

$$\mathbf{i} = \left[\mathbf{i}_{\mathcal{L}_1}^T, \mathbf{i}_{\mathcal{L}_2}^T, \mathbf{i}_{\mathcal{C}_1}^T, \mathbf{i}_{\mathcal{C}_2}^T\right]^T, \qquad \mathbf{v} = \left[\mathbf{v}_{\mathcal{L}_1}^T, \mathbf{v}_{\mathcal{L}_2}^T, \mathbf{v}_{\mathcal{C}_1}^T, \mathbf{v}_{\mathcal{C}_2}^T\right]^T, \tag{13}$$

where $\mathbf{i}_{\mathcal{L}_1}$, $\mathbf{i}_{\mathcal{C}_1}$, $\mathbf{v}_{\mathcal{L}_1}$ and $\mathbf{v}_{\mathcal{C}_1}$ are elements of $\mathbf{R}^r$; $\mathbf{i}_{\mathcal{L}_2}$ and $\mathbf{v}_{\mathcal{L}_2}$ are elements of $\mathbf{R}^l$; $\mathbf{i}_{\mathcal{C}_2}$ and $\mathbf{v}_{\mathcal{C}_2}$ are elements of $\mathbf{R}^s$.

Based on the $\mathcal{C}$ tree, the $\mathcal{L}$ cotree and the partitioning of the variables in (13), the fundamental loop matrix can be written as follows:

$$\mathbf{B} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times l} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0}_{l\times r} & \mathbf{I}_l & \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \tag{14a}$$

where $\mathbf{B}_{LC}$ is partitioned as:

$$\mathbf{B}_{LC} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}. \tag{14b}$$

The corresponding fundamental cutset matrix is

$$\mathbf{Q} = \begin{pmatrix} -\mathbf{B}_{11}^T & -\mathbf{B}_{21}^T & \mathbf{I}_r & \mathbf{0}_{r\times s} \\ -\mathbf{B}_{12}^T & -\mathbf{B}_{22}^T & \mathbf{0}_{s\times r} & \mathbf{I}_s \end{pmatrix}. \tag{15}$$

The colored branch theorem along with the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree and $\mathcal{C}_1 \cup \mathcal{L}_2$ cotree gives us a fundamental loop matrix with the following form:†

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{\mathbf{B}}_{11} & \mathbf{0}_{r\times l} & \mathbf{I}_r & \tilde{\mathbf{B}}_{12} \\ \tilde{\mathbf{B}}_{21} & \mathbf{I}_l & \mathbf{0}_{l\times r} & \mathbf{0}_{l\times s} \end{pmatrix}. \tag{16}$$

Note that $\tilde{\mathbf{B}}_{22} = \mathbf{0}$. This is because we defined $\mathcal{L}_1$ to be the largest set of inductors that can be put into a tree, that is, if $L_i \in \mathcal{L}_2$ then L_i forms a loop exclusively with branches of $\mathcal{L}_1$ and not with branches of $\mathcal{C}_2$. The corresponding fundamental cutset matrix is

$$\tilde{\mathbf{Q}} = \begin{pmatrix} \mathbf{I}_r & -\tilde{\mathbf{B}}_{21}^T & -\tilde{\mathbf{B}}_{11}^T & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times l} & -\tilde{\mathbf{B}}_{12}^T & \mathbf{I}_s \end{pmatrix}. \tag{17}$$

Tellegen's theorem tells us that $\tilde{\mathbf{B}}\mathbf{Q}^T = \mathbf{0}$ and $\tilde{\mathbf{Q}}\mathbf{B}^T = \mathbf{0}$. Applying Tellegen's theorem to equations (14a) – (17) we get:

$$-\tilde{\mathbf{B}}_{11}\mathbf{B}_{11} + \mathbf{I}_r = \mathbf{0}, \qquad -\tilde{\mathbf{B}}_{11}\mathbf{B}_{12} + \tilde{\mathbf{B}}_{12} = \mathbf{0}, \tag{18a}$$

$$-\tilde{\mathbf{B}}_{21}\mathbf{B}_{11} - \mathbf{B}_{21} = \mathbf{0}, \qquad -\tilde{\mathbf{B}}_{21}\mathbf{B}_{12} - \mathbf{B}_{22} = \mathbf{0}, \tag{18b}$$

and

$$\mathbf{I}_r - \tilde{\mathbf{B}}_{11}^T\mathbf{B}_{11}^T = \mathbf{0}, \qquad -\tilde{\mathbf{B}}_{21}^T - \tilde{\mathbf{B}}_{11}^T\mathbf{B}_{21}^T = \mathbf{0}, \tag{19a}$$

$$-\tilde{\mathbf{B}}_{12}^T\mathbf{B}_{11}^T + \mathbf{B}_{12}^T = \mathbf{0}, \qquad -\tilde{\mathbf{B}}_{12}^T\mathbf{B}_{21}^T + \mathbf{B}_{22}^T = \mathbf{0}. \tag{19b}$$

We are now ready to give two different change of variable schemes based on the components of the $\mathbf{B}$ and $\tilde{\mathbf{B}}$ matrices.

† The form of the fundamental loop and fundamental cutset matrix for the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree is nonstandard because of the ordering of the circuit variables, that is, we are not listing links before tree branches.

Change of variables I

Let

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times s} \\ \tilde{\mathbf{B}}_{12}^T & -\mathbf{I}_s \end{pmatrix}, \qquad \mathbf{N}^{-1} = \begin{pmatrix} \tilde{\mathbf{B}}_{11} & \mathbf{0}_{r\times l} \\ \tilde{\mathbf{B}}_{21} & \mathbf{I}_l \end{pmatrix}. \tag{20a}$$

Then

$$\mathbf{N}^{-1}\mathbf{B}_{LC}(\mathbf{M}^{-1})^T = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times s} \\ \mathbf{0}_{l\times r} & \mathbf{0}_{l\times s} \end{pmatrix}, \tag{20b}$$

and furthermore

$$\mathbf{M} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times s} \\ \tilde{\mathbf{B}}_{12}^T & -\mathbf{I}_s \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{0}_{r\times l} \\ \mathbf{B}_{21} & \mathbf{I}_l \end{pmatrix}. \tag{20c}$$

Proof Multiply the matrices and apply equations (18a) and (18b).

Change of variables II

Let

$$\mathbf{M}^{-1} = \begin{pmatrix} \tilde{\mathbf{B}}_{11}^T & \mathbf{0}_{r\times s} \\ \tilde{\mathbf{B}}_{12}^T & -\mathbf{I}_s \end{pmatrix}, \qquad \mathbf{N}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times l} \\ \tilde{\mathbf{B}}_{21} & \mathbf{I}_l \end{pmatrix}. \tag{21a}$$

Then

$$\mathbf{M}^{-1}\mathbf{B}_{LC}^T(\mathbf{N}^{-1})^T = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times l} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times l} \end{pmatrix}, \tag{21b}$$

and furthermore

$$\mathbf{M} = \begin{pmatrix} \mathbf{B}_{11}^T & \mathbf{0}_{r\times s} \\ \mathbf{B}_{12}^T & -\mathbf{I}_s \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times l} \\ -\tilde{\mathbf{B}}_{21} & \mathbf{I}_l \end{pmatrix}. \tag{21c}$$

Proof Multiply the matrices and apply equations (19a) and (19b).

Change of variables I leaves the r active capacitor charges *untouched* while the change of variables II leaves the r active inductor fluxes *untouched.*

5. Change of Variables by Inspection

To ease application of the above changes of variables, relabel the capacitors and inductors so that the first r capacitors are in $\mathcal{C}_1$ and the first r inductors are in $\mathcal{L}_1$, that is: $C_k \in \mathcal{C}_1$ for $k = 1, ..., r$; $C_k \in \mathcal{C}_2$ for $k = r + 1, ..., n_C$; $L_k \in \mathcal{L}_1$ for $k = 1, ..., r$; and $L_k \in \mathcal{L}_2$ for $k = r + 1, ..., n_L$. Interpreting the above changes of variables in terms of loops and cutsets we get the following:

Change of Variables I

For $k = 1, ..., r$:

$q_k = x_k$.

For $k = r + 1, ..., n_C$:

$q_k = -x_k$ − Algebraic sum of x_j such that $C_j \in \mathcal{C}_1$ is in a fundamental cutset defined by C_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

For $k = 1, ..., r$:

$\varphi_k =$ Algebraic sum of y_j such that $C_j \in \mathcal{C}_1$ is in a fundamental loop defined by L_k with $\mathcal{C}$ as tree.

For $k = r + 1, ..., n_L$:

$\varphi_k = y_k$ + Algebraic sum of y_j such that $C_j \in \mathcal{C}_1$ is in a fundamental loop defined by L_k with $\mathcal{C}$ as tree.

Inverse Change of Variables I

For $k = 1, ..., r$:

$x_k = q_k$.

For $k = r + 1, ..., n_C$:

$x_k = -q_k$ − Algebraic sum of q_j such that $C_j \in \mathcal{C}_1$ is in a fundamental cutset defined by C_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

For $k = 1, ..., r$:

y_k = Algebraic sum of φ_j such that $L_j \in \mathcal{L}_1$ is in a fundamental loop defined by C_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

For $k = r+1, ..., n_L$:

$y_k = \varphi_k$ + Algebraic sum of φ_j such that $L_j \in \mathcal{L}_1$ is in a fundamental loop defined by L_k with with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

Change of Variables II

For $k = 1, ..., r$:

$q_k = -$ Algebraic sum of x_j such that $L_j \in \mathcal{L}_1$ is in a fundamental cutset defined by C_k with $\mathcal{C}$ as tree.

For $k = r+1, ..., n_C$:

$q_k = -x_k -$ Algebraic sum of x_j such that $L_j \in \mathcal{L}_1$ is in a fundamental cutset defined by C_k with $\mathcal{C}$ as tree.

For $k = 1, ..., r$:

$\varphi_k = y_k$.

For $k = r+1, ..., n_L$:

$\varphi_k = y_k -$ Algebraic sum of y_j such that $L_j \in \mathcal{L}_1$ is in a fundamental loop defined by L_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

Inverse Change of Variables II

For $k = 1, ..., r$:

$x_k = -$ Algebraic sum of q_j such that $C_j \in \mathcal{C}_1$ is in a fundamental cutset defined by L_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

For $k = r+1, ..., n_C$:

$x_k = -q_k -$ Algebraic sum of q_j such that $C_j \in \mathcal{C}_1$ is in a fundamental cutset defined by C_k with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

For $k = 1, ..., r$:

$y_k = \varphi_k$.

For $k = r+1, ..., n_L$:

$y_k = \varphi_k$ + Algebraic sum of φ_j such that $L_j \in \mathcal{L}_1$ is in a fundamental loop defined by L_k with with $\mathcal{L}_1 \cup \mathcal{C}_2$ as tree.

We obtain the linear transformations between the x_k's and the q_k's by looking at various cutsets; similarly, we obtain the linear transformations between the y_k's and the φ_k's by looking at various loops. When we obtain either the ***inverse change of variables I*** or the ***inverse change of variables II*** we only need to consider the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree. However, when we obtain the forward ***change of variables I*** or ***change of variables II*** we need to use both the $\mathcal{C}$ tree and the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree.

Example 1

For the circuit shown in Fig. 1(a), we show with thick lines the elements in the capacitor tree, $\mathcal{C} = \{C_1, C_2, C_3\}$. The corresponding cotree is $\mathcal{L} = \{L_1, L_2\}$. We redraw the circuit in Fig. 1(b), and show with thick lines the tree containing as many inductors as possible $\{L_1, L_2, C_3\}$. For this circuit, the number of capacitors $n_C = 3$, the number of inductors $n_L = 2$, the number of independent inductor loops $l = 0$ and the number of independent capacitor cutsets $s = 1$. We note that $r = n_C - s = n_L = 2$. We partition $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$, and $\mathcal{L}$ into $\mathcal{L}_1$ and $\mathcal{L}_2$ where $\mathcal{C}_1 = \{C_1, C_2\}$, $\mathcal{C}_2 = \{C_3\}$, $\mathcal{L}_1 = \{L_1, L_2\}$ and $\mathcal{L}_2 = \emptyset$.

The capacitor element relations are:

$$\mathbf{v}_C = \hat{\mathbf{v}}(\mathbf{q}_C) = \begin{pmatrix} a q_1^2 \\ \frac{1}{C_2} q_2 \\ \frac{1}{C_3} q_3 \end{pmatrix}. \tag{22}$$

The inductor element relations are:

$$\mathbf{i}_L = \hat{\imath}(\varphi_L) = \begin{pmatrix} \frac{1}{L_1}\varphi_1 \\ k \sin \varphi_2 \end{pmatrix}. \tag{23}$$

The energy function for the circuit is:

$$E(\mathbf{q}_C, \varphi_L) = \frac{a}{3} q_1^3 + \frac{1}{2C_2} q_2^2 + \frac{1}{2C_3} q_3^2 + \frac{1}{2L_1}\varphi_1^2 - k \cos \varphi_2. \tag{24}$$

The $\mathbf{B}_{LC}$ matrix, obtained from Fig. 1(a), is

$$\mathbf{B}_{LC} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \tag{25}$$

In this example we apply the *change of variables I*. We obtain the inverse *change of variables I* from Fig. 1(b), that is, we use the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree. To obtain y_1 and y_2 in terms of φ_1 and φ_2 we look at the fundamental loops involving the cotree branches C_1 and C_2. C_1 forms a fundamental loop with L_1, orientation opposing the loop, and L_2, orientation opposing the loop. Hence, $y_1 = -\varphi_1 - \varphi_2$. C_2 form a loop with L_2, orientation opposing the loop. Hence, $y_2 = -\varphi_2$. Since the *change of variables I* leaves the first $r = 2$ capacitor charges alone we have: $x_1 = q_1$, $x_2 = q_2$. Capacitor C_3 form a cutset by itself and not with any other capacitors in the $\mathcal{C}_1 \cup \mathcal{L}_2$ cotree, hence $x_3 = -q_3$.

For the forward *change of variables I* we use both Fig. 1(a) and Fig. 1(b). To obtain φ_1 and φ_2 in terms of y_1 and y_2 we look at the fundamental loops formed by L_1 and L_2 with C as tree, that is Fig. 1(a). L_1 forms a fundamental loop with C_2, orientation the same, and with C_1, orientation opposing. Hence, $\varphi_1 = -y_1 + y_2$. L_2 forms a fundamental loop with C_2, orientation opposing. Hence, $\varphi_2 = -y_2$. By definition: $q_1 = x_1$ and $q_2 = x_2$. Looking at the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree, Fig. 1(b), we see that C_3 forms a cutset by itself. Hence, $q_3 = -x_3$

Our transformation matrices are:

$$\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{N}^{-1} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \tag{26}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

The transformed energy function is:

$$E(\mathbf{Mx}, \mathbf{Ny}) = \frac{a}{3}x_1^3 + \frac{1}{2C_2}x_2^2 + \frac{1}{2C_3}x_3^2 + \frac{1}{2L_1}(-y_1 + y_2)^2 - k\cos y_2. \tag{28}$$

With this transformed energy function we have the following state equation:

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{y}}\right]^T \tag{29a}$$

$$\dot{\mathbf{y}} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{x}}\right]^T \tag{29b}$$

For our four *active* variables x_1, x_2, y_1 and y_2 the transformed energy can be used as a Hamiltonian.

Example 2

For the circuit shown in Fig. 2(a), we show with thick lines the elements in the capacitor tree $\mathcal{C} = \{C_1, C_2, C_3\}$. The corresponding inductor cotree is $\mathcal{L} = \{L_1, L_2, L_3, L_4\}$. We redraw the circuit in Fig. 2(b) and show with thick lines the tree containing as many inductors as possible $\{L_1, L_2, L_3\}$. For this circuit, the number of capacitors $n_C = 3$, the number of inductors $n_L = 4$, the number of independent inductor loops $l = 1$ and the number of independent capacitor cutsets $s = 0$. We note that $r = n_L - l = n_C - s = 3$ and we partition $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$, and $\mathcal{L}$ into $\mathcal{L}_1$ and $\mathcal{L}_2$ where $\mathcal{C}_1 = \{C_1, C_2, C_3\}$, $\mathcal{C}_2 = \emptyset$, $\mathcal{L}_1 = \{L_1, L_2, L_3\}$ and $\mathcal{L}_2 = \{L_4\}$.

The capacitor element relations are:

$$\mathbf{v}_C = \hat{\mathbf{v}}(\mathbf{q}_C) = \begin{pmatrix} bq_1^3 \\ \frac{1}{C_2}q_2 \\ dq_3^5 \end{pmatrix} \tag{30}$$

The inductor element relations are:

$$\mathbf{i}_L = \hat{\mathbf{i}}(\varphi_L) = \begin{pmatrix} \frac{1}{L_1}\varphi_1 \\ k\cos\varphi_2 \\ g_3\varphi_3^3 \\ g_4\varphi_4^3 \end{pmatrix} \tag{31}$$

The energy function for the circuit is:

$$E(\mathbf{q}_C, \varphi_L) = \frac{b}{4}q_1^4 + \frac{1}{2C_2}q_2^2 + \frac{d}{6}q_3^6 + \frac{1}{2L_1}\varphi_1^2 + k\sin\varphi_2 + \frac{g_3}{4}\varphi_3^4 + \frac{g_4}{4}\varphi_4^4 \tag{32}$$

The $\mathbf{B}_{LC}$ matrix, obtained from Fig. 2(a), is

$$\mathbf{B}_{LC} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \tag{33}$$

In this example we apply the *change of variables II*. We obtain the *inverse change of variables II* from Fig. 2(b), that is, we use the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree. To obtain x_1, x_2 and x_3 in terms of q_1, q_2 and q_3 we look at the fundamental cutsets involving the tree branches L_1, L_2 and L_3. L_1 forms a cutset with C_1, orientation opposing, and L_4 orientation the same. Recall, we are only concerned with the elements of $\mathcal{C}_1$ in the cutset for the purposes of obtaining the transformation. Hence, $x_1 = q_1$. L_2 forms a cutset with C_1, orientation opposing, C_2, orientation opposing and C_3 orientation opposing. Hence, $x_2 = q_1 + q_2 + q_3$. L_3 form a cutset with C_3, orientation the same, and L_4, orientation the same. Hence, $x_3 = -q_3$. By definition of *change of variables II* we have: $y_1 = \varphi_1$, $y_2 = \varphi_2$ and $y_3 = \varphi_3$. L_4, an element of the $C_1 \cup \mathcal{L}_2$ cotree, forms a loop with L_3, orientation opposing and L_1, orientation opposing. Hence, $y_4 = \varphi_4 - \varphi_1 - \varphi_3$.

For the forward *change of variables II* we use both Fig. 2(a) and Fig. 2(b). To obtain q_1, q_2 and q_3 in terms of x_1, x_2 and x_3, we look at the fundamental cutsets formed by C_1, C_2 and C_3 with $\mathcal{C}$ as tree. C_1 forms a cutset with L_1, orientation opposing and $L_4 \in \mathcal{L}_2$, orientation opposing. Hence, $q_1 = x_1$. C_2 forms a cutset with L_1, orientation the same, L_2, orientation opposing and L_3, orientation opposing. Hence, $q_2 = -x_1 + x_2 + x_3$. C_3 forms a cutset with L_3, orientation the same and $L_4 \in \mathcal{L}_2$, orientation the same. Hence, $q_3 = -x_3$. By definition of *change of variables II* we have: $\varphi_1 = y_1$, $\varphi_2 = y_2$, $\varphi_3 = y_3$. To obtain φ_4 we refer to Fig. 2(b), that is, the $\mathcal{L}_1 \cup \mathcal{C}_2$ tree. L_4 forms a fundamental loop with L_3, orientation opposing and L_1 orientation opposing, from the inpection rules we get $\varphi_4 = y_4 + y_1 + y_3$.

Our transformation matrices are:

$$\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{N}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}. \tag{34}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \tag{35}$$

The transformed energy function is:

$$E(\mathbf{Mx}, \mathbf{Ny}) = \frac{b}{4}x_1^4 + \frac{1}{2C_2}(-x_1 + x_2 + x_3)^2 + \frac{d}{6}x_3^6 + \frac{1}{2L_1}y_1^2 + k \sin y_2 + \frac{g_3}{4}y_3^4 + \frac{g_4}{4}(y_1 + y_3 + y_4)^4 \tag{36}$$

With this transformed energy function we have the following state equation:

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{y}} \right]^T \tag{37a}$$

$$\dot{y} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{x}}\right]^T \tag{37b}$$

For our six *active* variables x_1, x_2, x_3, y_1, y_2 and y_3 the transformed energy can be used as a Hamiltonian.

6. LC Circuits with Independent Sources

Let $\mathcal{N}$ be a circuit containing n_E independent voltage sources and n_J independent current sources, in addition to n_C two terminal, charge-controlled capacitors and n_L two-terminal, flux-controlled inductors. We now show how to obtain a canonical Hamiltonian formulation for this circuit.

Assume that the circuit has no loops formed by capacitors and voltage sources and no cutsets formed by inductors and current sources. Let $\mathcal{E}$ denote the set of independent voltage sources and $\mathcal{J}$ the set of independent current sources. The above condition implies that $\mathcal{C} \cup \mathcal{E}$ forms a tree for the circuit and $\mathcal{L} \cup \mathcal{J}$ forms a cotree for the circuit. Partition the branch voltage and current vectors as follows:

$$\mathbf{i} = [\mathbf{i}_L^T, \mathbf{i}_J^T, \mathbf{i}_C^T, \mathbf{i}_E^T]^T$$

and

$$\mathbf{v} = [\mathbf{v}_L^T, \mathbf{v}_J^T, \mathbf{v}_C^T, \mathbf{v}_E^T]^T.$$

With the above partition the fundamental loop matrix has the form:

$$\mathbf{B} = \begin{pmatrix} \mathbf{I}_{n_L} & \mathbf{0} & \mathbf{B}_{LC} & \mathbf{B}_{LE} \\ \mathbf{0} & \mathbf{I}_{n_J} & \mathbf{B}_{JC} & \mathbf{B}_{JE} \end{pmatrix}$$

Kirchhoff's voltage law gives:

$$\mathbf{v}_L = -\mathbf{B}_{LC}\mathbf{v}_C - \mathbf{B}_{LE}\mathbf{v}_E \tag{38a}$$

$$\mathbf{v}_J = -\mathbf{B}_{JC}\mathbf{v}_C - \mathbf{B}_{JE}\mathbf{v}_E. \tag{38b}$$

Kirchhoff's current law gives:

$$\mathbf{i}_C = \mathbf{B}_{LC}^T\mathbf{i}_L + \mathbf{B}_{JC}^T\mathbf{i}_J \tag{39a}$$

$$\mathbf{i}_E = \mathbf{B}_{LE}^T\mathbf{i}_L + \mathbf{B}_{JE}^T\mathbf{i}_J \tag{39b}$$

Note that $\mathbf{v}_E \in \mathbf{R}^{n_E}$ and $\mathbf{i}_J \in \mathbf{R}^{n_J}$ are given functions of time, since they are the voltages across the independent voltage sources and the currents throught the independent current sources, respectively. From (38a) - (39b) we see that once we know $\mathbf{v}_C$ and $\mathbf{i}_L$ we can determine all other circuit variables. The state equations for this circuit are:

$$\dot{\varphi}_L = -\mathbf{B}_{LC}\hat{\mathbf{v}}_C(\mathbf{q}_C) - \mathbf{B}_{LE}\mathbf{v}_E(t) \tag{40a}$$

$$\dot{\mathbf{q}}_C = \mathbf{B}_{LC}^T\hat{\mathbf{i}}_L(\varphi_L) + \mathbf{B}_{JC}^T\mathbf{i}_J(t) \tag{40b}$$

Consider the circuit $\mathcal{N}_1$ obtained from $\mathcal{N}$ by short circuiting the voltage sources and open circuiting the current sources. $\mathcal{N}_1$ has the following fundamental loop matrix:

$$\mathbf{B}_{\mathcal{N}_1} = (\, I_{n_L} \quad \mathbf{B}_{LC}\,).$$

Order and partition $\mathcal{C}$ and $\mathcal{L}$ as in Sec. 4 and obtain the transformation matrices M and N for $\mathcal{N}_1$. Applying the change of variables (10) to (40a) and (40b) gives:

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times(n_L-r)} \\ \mathbf{0}_{(n_C-r)\times r} & \mathbf{0}_{(n_C-r)\times(n_L-r)} \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx},\mathbf{Ny})}{\partial \mathbf{y}}\right]^T + \mathbf{M}^{-1}\mathbf{B}_{JC}^T\mathbf{i}_J(t) \tag{41a}$$

$$\dot{\mathbf{y}} = -\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times(n_C-r)} \\ \mathbf{0}_{(n_L-r)\times r} & \mathbf{0}_{(n_L-r)\times(n_C-r)} \end{pmatrix} \left[\frac{\partial E(\mathbf{Mx},\mathbf{Ny})}{\partial \mathbf{x}}\right]^T - \mathbf{N}^{-1}\mathbf{B}_{LE}^T\mathbf{v}_E(t) \tag{41b}$$

or in component form:

$$\dot{x}_k = \frac{\partial E(\mathbf{Mx},\mathbf{Ny})}{\partial y_k} + \alpha_k(t), \qquad 1 \le k \le r \tag{42a}$$

$$\dot{y}_k = -\frac{\partial E(\mathbf{Mx},\mathbf{Ny})}{\partial x_k} - \beta_k(t), \qquad 1 \le k \le r \tag{42b}$$

and

$$\dot{x}_k = \alpha_k(t) \qquad r < k \le n_C, \tag{42c}$$

$$\dot{y}_k = -\beta_k(t) \qquad r < k \le n_L, \tag{42d}$$

where

$$\alpha_k(t) = \sum_{j=1}^{n_J}\sum_{l=1}^{n_C}[\mathbf{M}^{-1}]_{kl}\mathbf{B}_{JC_{lj}} i_{J_j}(t), \tag{43a}$$

$$\beta_k(t) = \sum_{j=1}^{n_E}\sum_{l=1}^{n_L}[\mathbf{N}^{-1}]_{kl}\mathbf{B}_{LE_{lj}} v_{E_j}(t). \tag{43b}$$

The variables x_k for $r < k \leq n_C$ and y_k for $r < k \leq n_L$ are *trivial* in the sense that their dynamics can be obtained simply by integrating equations (42c)-(42d) directly; therefore, they will be omitted from the canonical Hamiltonian formulation.

Define

$$H(x,y) = E(\mathbf{M}x, \mathbf{N}y) + \sum_{k=1}^{r}[\alpha_k(t)y_k + \beta_k(t)x_k]. \tag{44}$$

Then for our $2r$ *active* variables we have the following canonical Hamiltonian formulation:

$$\dot{x}_k = \frac{\partial H(x,y,t)}{\partial y_k} \tag{45a}$$

$$\dot{y}_k = \frac{\partial H(x,y,t)}{\partial x_k} \tag{45b}$$

for $1 \leq k \leq r$.

Example 3.

In Fig. 3 we show a circuit $\mathcal{N}$ that contains independent voltage and current sources. Let the capacitor and inductor constitutive relations be the same as in Example 1. We show with thick lines the elements in the capacitor-voltage source tree, $\mathcal{C} \cup \mathcal{E} = \{C_1, C_2, C_3, v_{s3}\}$. The corresponding inductor-current source cotree is $\mathcal{L} \cup \mathcal{J} = \{L_1, L_2, i_{s1}, i_{s2}\}$. The fundamental loop matrix is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note:

$$\mathbf{B}_{LC} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{B}_{LE} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{46a}$$

$$\mathbf{B}_{JC} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_{JE} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{46b}$$

If we short circuit the voltage sources and open circuit the current sources we get the circuit of Example 1 (Fig. 1). Using the M and N matrices (26) and (27) obtained in Example 1, we obtain the following transformed state equations:

$$\dot{\mathbf{x}} = \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial y}\right]^T + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i_{s1}(t) \\ i_{s2}(t) \end{pmatrix} \tag{47a}$$

$$\dot{\mathbf{y}} = -\left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial x}\right]^T - \begin{pmatrix} -1 \\ -1 \end{pmatrix} v_{s3}(t), \tag{47b}$$

where $E(\mathbf{Mx}, \mathbf{Ny})$ is given in (28). Define

$$\begin{aligned} H(\mathbf{x}, \mathbf{y}) &= E(\mathbf{Mx}, \mathbf{Ny}) + (\,x_1 \quad x_2\,) \begin{pmatrix} -1 \\ -1 \end{pmatrix} v_{s3}(t) + (\,y_1 \quad y_2\,) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} i_{s1}(t) \\ i_{s2}(t) \end{pmatrix} \\ &= E(\mathbf{Mx}, \mathbf{Ny}) + (x_1 + x_2)v_{s3}(t) + y_1[i_{s1}(t) + i_{s2}(t)] \end{aligned} \tag{48}$$

Then

$$\dot{x}_k = \frac{\partial H(\mathbf{x}, \mathbf{y})}{\partial y_k} \tag{49a}$$

$$\dot{y}_k = -\frac{\partial H(\mathbf{x}, \mathbf{y})}{\partial x_k} \tag{49b}$$

for $k = 1, 2$ is the canonical Hamiltonian formulation for our four *active* variables and

$$\dot{x}_3 = i_{s1}(t) \tag{50}$$

is the equation for our *inactive* variable.

7. Separation of Variables for Linear Systems

When the nonlinearities in the capacitors and inductors are small, so that the unperturbed circuit $\mathcal{N}$ contains linear capacitors and inductors, then a change of coordinates to the *principal axes* of the system is desirable in order to reveal the oscillatory modes of the circuit, to put the equations in a form suitable for canonical perturbation analysis, or to obtain the action-angle variables for the circuit. In this section we give a method for obtaining this transformation starting from the canonical form derived earlier. In addition, this transformation only involves the active variables, which are already separated into two canonically conjugate sets. This contrasts with the technique used in [3] where the transformation to principal axes was made directly for the entire set of active and inactive variables.

Let $\mathcal{N}$ be the circuit of Sec. 2 or Sec. 6, with the matices M and N giving the transformation to canonical form. Partition the x and y vectors as follows:

$$\mathbf{x} = [\mathbf{x_r}^T, \mathbf{x_s}^T]^T \quad \text{and} \quad \mathbf{y} = [\mathbf{y_r}^T, \mathbf{y_l}^T]^T \tag{51}$$

where $\mathbf{x_r}, \mathbf{y_r} \in \mathbf{R}^r$ are the *active* variables and $\mathbf{x_s} \in \mathbf{R}^s$, $\mathbf{y_l} \in \mathbf{R}^l$ are the *inactive* variables. For the *active* variables we have

$$\dot{\mathbf{x}}_\mathbf{r} = \left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{y_r}}\right], \tag{52a}$$

$$\dot{\mathbf{y}}_\mathbf{r} = -\left[\frac{\partial E(\mathbf{Mx}, \mathbf{Ny})}{\partial \mathbf{x_r}}\right]. \tag{52b}$$

Suppose the capacitors and inductors have the following constitutive relations:

$$v_{C_k} = \frac{1}{C_k} q_{C_k} + \epsilon \hat{v}_{C_k}(q_{C_k}) \quad k = 1, ..., n_C \tag{53a}$$

and

$$i_{L_k} = \frac{1}{L_k} \varphi_{L_k} + \epsilon \hat{i}_{L_k}(\varphi_{L_k}) \quad k = 1, ..., n_L. \tag{53b}$$

Note that $\epsilon = 0$ is the linear case.

The energy has the form $E_\epsilon(\mathbf{q}_C, \varphi_L) = E_0(\mathbf{q}_C, \varphi_L) + \epsilon E_{nl}(\mathbf{q}_C, \varphi_L)$, where

$$E_0(\mathbf{q}_C, \varphi_L) = 1/2 \mathbf{q}_C^T \mathbf{C}^{-1} \mathbf{q}_C + 1/2 \varphi_L^T \mathbf{L}^{-1} \varphi_L \tag{54}$$

and

$$E_{nl}(\mathbf{q}_C, \varphi_L) = \sum_{k=1}^{n_C} \int_0^{q_{C_k}} \hat{v}_{C_k}(u) du + \sum_{k=1}^{n_L} \int_0^{\varphi_{L_k}} \hat{i}_{L_k}(u) du, \tag{55}$$

$$\mathbf{C} = \text{diag}[C_1, C_2, ..., C_{n_C}], \tag{56a}$$

$$\mathbf{L} = \text{diag}[L_1, L_2, ..., L_{n_L}]. \tag{56b}$$

Under the transformation, **M** and **N**, to canonical form the energy becomes:

$$E_\epsilon(\mathbf{Mx}, \mathbf{Ny}) = (1/2)\mathbf{x}^T\mathbf{Ax} + (1/2)\mathbf{y}^T\mathbf{Dy} + \epsilon E_{nl}(\mathbf{Mx}, \mathbf{Ny}) \tag{57a}$$

where

$$\mathbf{A} = \mathbf{M}^T\mathbf{C}^{-1}\mathbf{M} \quad \text{and} \quad \mathbf{D} = \mathbf{N}^T\mathbf{L}^{-1}\mathbf{N} \tag{57b}$$

For change of variables I:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{C_r}^{-1} + \tilde{\mathbf{B}}_{12}\mathbf{C_s}^{-1}\tilde{\mathbf{B}}_{12}^T & -\tilde{\mathbf{B}}_{12}\mathbf{C_s}^{-1} \\ -\mathbf{C_s}^{-1}\tilde{\mathbf{B}}_{12}^T & \mathbf{C_s}^{-1} \end{pmatrix}, \tag{58a}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{12}^T & \mathbf{D}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{11}^T\mathbf{L_r}^{-1}\mathbf{B}_{11} + \mathbf{B}_{21}^T\mathbf{L_l}^{-1}\mathbf{B}_{21} & \mathbf{B}_{21}^T\mathbf{L_l}^{-1} \\ \mathbf{L_l}^{-1}\mathbf{B}_{21} & \mathbf{L_l}^{-1} \end{pmatrix}. \tag{58b}$$

For change of variables II:

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}_{11}\mathbf{C_r}^{-1}\mathbf{B}_{11}^T + \mathbf{B}_{12}\mathbf{C_s}^{-1}\mathbf{B}_{12}^T & -\mathbf{B}_{12}\mathbf{C_s}^{-1} \\ \mathbf{C_s}^{-1}\mathbf{B}_{12}^T & \mathbf{C_s}^{-1} \end{pmatrix}, \tag{59a}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{L_r}^{-1} + \tilde{\mathbf{B}}_{21}^T\mathbf{L_l}^{-1}\tilde{\mathbf{B}}_{21} & -\tilde{\mathbf{B}}_{21}\mathbf{L_l}^{-1} \\ -\mathbf{L_l}^{-1}\tilde{\mathbf{B}}_{21} & \mathbf{L_l}^{-1} \end{pmatrix}. \tag{59b}$$

Where

$$\mathbf{C_r} = \text{diag}[C_1, ..., C_r], \qquad \mathbf{C_s} = \text{diag}[C_{r+1}, ..., C_{n_C}], \tag{60a}$$

$$\mathbf{L_r} = \text{diag}[L_1, ..., L_r] \quad \text{and} \quad \mathbf{L_l} = \text{diag}[L_{r+1}, ..., L_{n_L}]. \tag{60b}$$

To achieve the transformation to *principal axes* it suffices to separate only the active variables in $E_0(\mathbf{Mx}, \mathbf{Ny})$. To do this we need a transformation that diagonalizes $\mathbf{A}_{11}$ and $\mathbf{D}_{11}$ while preserving the canonical structure of (52a) and (52b). Consider the change of variables

$$\mathbf{x_r} = \mathbf{Fu}, \qquad \mathbf{y_r} = \mathbf{Gw} \tag{61}.$$

Applying this change of variables to (52a) and (52b) we get the following necessary and sufficient condition on the matrices **F** and **G** to preserve the canonical form

$$\mathbf{F}^{-1}(\mathbf{G}^{-1})^T = \mathbf{I} \quad \text{or} \quad \mathbf{FG}^T = \mathbf{I} \tag{62}$$

Since $\mathbf{A}_{11}$ and $\mathbf{D}_{11}$ are positive definite we can take their square root, i.e., Cholesky decomposition [9],

$$\mathbf{A}_{11} = \mathbf{A}_{11}^{1/2^T} \mathbf{A}_{11}^{1/2} \tag{63a}$$

$$\mathbf{D}_{11} = \mathbf{D}_{11}^{1/2^T} \mathbf{D}_{11}^{1/2} \tag{63b}$$

Let $\mathbf{P}$ and $\mathbf{Q}$ be the singular value decomposition [9] of $\mathbf{A}_{11}^{1/2}\mathbf{B}_{11}^{1/2^T}$, i.e.,

$$\mathbf{Q}^T \left[\mathbf{A}_{11}^{1/2}\mathbf{B}_{11}^{1/2^T} \right] \mathbf{P} = \boldsymbol{\Omega}, \tag{64a}$$

where

$$\boldsymbol{\Omega} = \mathrm{diag}[\omega_1, \omega_2, ..., \omega_r], \tag{64b}$$

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \quad \text{and} \quad \mathbf{Q}^T\mathbf{Q} = \mathbf{I}. \tag{64c}$$

Define

$$\mathbf{F} = \mathbf{A}_{11}^{-1/2}\mathbf{Q}\boldsymbol{\Omega}^{1/2} \quad \text{and} \quad \mathbf{G} = \mathbf{D}_{11}^{-1/2}\mathbf{P}\boldsymbol{\Omega}^{1/2} \tag{65a}$$

where

$$\mathbf{A}_{11}^{-1/2} = \left[\mathbf{A}_{11}^{1/2}\right]^{-1}, \qquad \mathbf{D}_{11}^{-1/2} = \left[\mathbf{D}_{11}^{1/2}\right]^{-1} \tag{65b}$$

and

$$\boldsymbol{\Omega}^{1/2} = \mathrm{diag}[\sqrt{\omega_1}, \sqrt{\omega_2}, ..., \sqrt{\omega_r}]. \tag{65c}$$

One can check that (62) is satisfied and that

$$E_0(\mathbf{u},\mathbf{w};\mathbf{x}_s,\mathbf{y}_l) = (1/2)(\mathbf{u}^T,\mathbf{x}_s{}^T)\begin{pmatrix} \boldsymbol{\Omega} & \mathbf{F}^T\mathbf{A}_{12} \\ \mathbf{A}_{12}^T\mathbf{F} & \mathbf{A}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{x}_s \end{pmatrix} + (1/2)(\mathbf{w}^T,\mathbf{y}_l{}^T)\begin{pmatrix} \boldsymbol{\Omega} & \mathbf{G}^T\mathbf{D}_{12} \\ \mathbf{D}_{12}^T\mathbf{G} & \mathbf{D}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{w} \\ \mathbf{y}_l \end{pmatrix}. \tag{66}$$

For the linear case, $\epsilon = 0$, the equations are

$$\dot{\mathbf{u}} = \boldsymbol{\Omega}\mathbf{w} + \mathbf{G}^T\mathbf{D}_{12}\mathbf{y}_l, \tag{67a}$$

$$\dot{\mathbf{w}} = -\boldsymbol{\Omega}\mathbf{u} + \mathbf{F}^T\mathbf{A}_{12}\mathbf{x}_s. \tag{67b}$$

Recall, $\mathbf{y}_l$ and $\mathbf{x}_s$ are *inactive* or *trivial* variables and are either constants or independent functions of time. For the weakly nonlinear case, $\epsilon \neq 0$, we have

$$\dot{\mathbf{u}} = \boldsymbol{\Omega}\mathbf{w} + \mathbf{G}^T\mathbf{D}_{12}\mathbf{y}_l + \epsilon\left[\frac{\partial E_{nl}(\mathbf{u},\mathbf{w};\mathbf{x}_s,\mathbf{y}_l)}{\partial \mathbf{w}}\right]^T, \tag{68a}$$

$$\dot{\mathbf{w}} = -\boldsymbol{\Omega}\mathbf{u} - \mathbf{F}^T\mathbf{A}_{12}\mathbf{x}_s - \epsilon\left[\frac{\partial E_{nl}(\mathbf{u},\mathbf{w};\mathbf{x}_s,\mathbf{y}_l)}{\partial \mathbf{u}}\right]^T. \tag{68b}$$

Example 4.

Consider the circuit of Fig. 2(a) with linear capacitors and inductors, i.e.,

$$v_{C_k} = \frac{1}{C_k} q_{C_k} \quad \text{for} \quad k = 1, 2, 3 \tag{69a}$$

and

$$i_{L_k} = \frac{1}{L_k} \varphi_{L_k} \quad \text{for} \quad k = 1, 2, 3, 4 \tag{69b}$$

with the following normalized capacitor and inductor values: $C_1 = 1F$, $C_2 = 1F$, $C_3 = (1/2)F$, $L_1 = (1/2)H$, $L_2 = (1/2)H$, $L_3 = 1H$, and $L_4 = (1/3)H$. Using the $\mathbf{M}$ and $\mathbf{N}$ matrices obtained from Example 2, we have:

$$\mathbf{A} = \mathbf{M}^T\mathbf{C}^{-1}\mathbf{M} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \tag{70a}$$

$$\mathbf{D} = \mathbf{N}^T\mathbf{L}^{-1}\mathbf{N} = \begin{pmatrix} 5 & 0 & 3 & 3 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 3 \\ 3 & 0 & 3 & 3 \end{pmatrix} \tag{70b}$$

Hence,

$$\mathbf{A}_{11} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{11} = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix} \tag{71}$$

The Cholesky factorization of $\mathbf{A}_{11}$ and $\mathbf{D}_{11}$ is

$$\mathbf{A}_{11}^{1/2} = \begin{pmatrix} 1.4142 & -0.7071 & -0.7071 \\ 0.0000 & 0.7071 & 0.7071 \\ 0.0000 & 0.0000 & 2.0976 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{11}^{1/2} = \begin{pmatrix} 2.2361 & 0.0000 & 1.3416 \\ 0.0000 & 1.4142 & 0.0000 \\ 0.0000 & 0.0000 & 1.4832 \end{pmatrix} \tag{72}$$

The singular value decomposition of $\mathbf{A}_{11}^{1/2}\mathbf{B}_{11}^{1/2^T}$ is

$$\mathbf{Q} = \begin{pmatrix} 0.4082 & -0.8991 & 0.1581 \\ 0.4082 & 0.3347 & 0.8493 \\ 0.8165 & 0.2822 & -0.5037 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0.8563 & -0.4505 & 0.2524 \\ 0.0000 & 0.4888 & 0.8724 \\ 0.5164 & 0.7471 & -0.4185 \end{pmatrix} \tag{73a}$$

and

$$\boldsymbol{\Omega} = \mathrm{diag}[3.3166, 2.5243, 0.7923] \tag{73b}$$

Finally,

$$\mathbf{F} = \mathbf{A}_{11}^{-1/2}\mathbf{Q}\boldsymbol{\Omega}^{1/2} = \begin{pmatrix} 1.0514 & -0.6340 & 0.6340 \\ 0.0000 & 0.4350 & 1.3861 \\ 1.0514 & 0.3170 & -0.3170 \end{pmatrix} \tag{74a}$$

and

$$\mathbf{G} = \mathbf{D}_{11}^{-1/2}\mathbf{P}\boldsymbol{\Omega}^{1/2} = \begin{pmatrix} 0.3170 & -0.8003 & 0.2512 \\ 0.0000 & 0.5491 & 0.5491 \\ 0.6340 & 0.8003 & -0.2512 \end{pmatrix} \tag{74b}$$

The Hamiltonian for the *active* variables is

$$-E(\mathbf{u}, \mathbf{w}; y_4) = (1/2)(u_1 \quad u_2 \quad u_3)\begin{pmatrix} 3.3166 & 0.0000 & 0.0000 \\ 0.0000 & 2.5243 & 0.0000 \\ 0.0000 & 0.0000 & 0.7923 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$+ (1/2)(w_1 \quad w_2 \quad w_3 \quad y_4)\begin{pmatrix} 3.3166 & 0.0000 & 0.0000 & 2.8532 \\ 0.0000 & 2.5243 & 0.0000 & 0.000 \\ 0.0000 & 0.0000 & 0.7923 & 0.0000 \\ 2.8532 & 0.0000 & 0.0000 & 3.0000 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ y_4 \end{pmatrix} \tag{75}.$$

Recall $y_4 = \varphi_4 - \varphi_1 - \varphi_3 = const$ is the *inactive* variable for this circuit. The equation of motion for the *active* variables is

$$\dot{\mathbf{u}} = \mathrm{diag}[3.3166, 2.5243, 0.7923]\mathbf{w} + \begin{pmatrix} 2.8532 \\ 0.0000 \\ 0.0000 \end{pmatrix} y_4, \tag{76a}$$

$$\dot{\mathbf{w}} = -\,\mathrm{diag}[3.3166, 2.5243, 0.7923]\mathbf{u} \tag{76b}$$

and the final change of variables is

$$\begin{pmatrix} q_{C_1} \\ q_{C_2} \\ q_{C_3} \end{pmatrix} = \begin{pmatrix} 1.0514 & -0.6340 & 0.6340 \\ 0.0000 & 1.3860 & 0.4351 \\ -1.0514 & -0.3170 & 0.3170 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{77a}$$

$$
\begin{pmatrix} \varphi_{L_1} \\ \varphi_{L_2} \\ \varphi_{L_3} \\ \varphi_{L_4} \end{pmatrix} = \begin{pmatrix} 0.3170 & -0.8003 & 0.2512 & 0.0000 \\ 0.0000 & 0.5491 & 0.5491 & 0.0000 \\ 0.6340 & 0.8003 & -0.2512 & 0.0000 \\ 0.9510 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ y_4 \end{pmatrix}. \tag{77b}
$$

8. Conclusion

In this paper we have given a method for obtaining the Hamiltonian and the proper canonical variables that is both very general and simple. A tree of capacitors was used to write the state equations for an LC circuit with independent sources, and transformations to canonical variables were generated from the circuit topology by considering an alternate tree (cotree) containing of as many inductors (capacitors) as possible. A method was given to obtain these transformations byy inspection, based upon fundamental loops and cutsets of both the capacitor and alternate trees. Furthermore, when the nonlinearities in the capacitors and inductors are small, a change of coordinates was given to achieve complete separation of the canonical variables of the unperturbed (linear) system. These results should allow the application of techniques from Hamiltonian dynamics, such as canonical perturbation theory, to circuits.

Acknowledgment

One of the authors would like to thank Christopher Patrick Silva whose research interest in lossless circuits motivated this paper and Professor Leon O. Chua for reviewing some results that eventually lead to this paper.

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Figure captions

Fig. 1. (a) Nonlinear LC circuit with capacitor tree highlighted. (b) Same circuit with $\mathcal{L}_1 \cup \mathcal{C}_2$ alternate tree highlighted.

Fig. 2. (a) Nonlinear LC circuit with capacitor tree highlighted. (b) Same circuit with $\mathcal{L}_1 \cup \mathcal{C}_2$ alternate tree highlighted.

Fig. 3. Nonlinear LC circuit with independent voltage and current sources. Capacitor-voltage source tree highlighted.

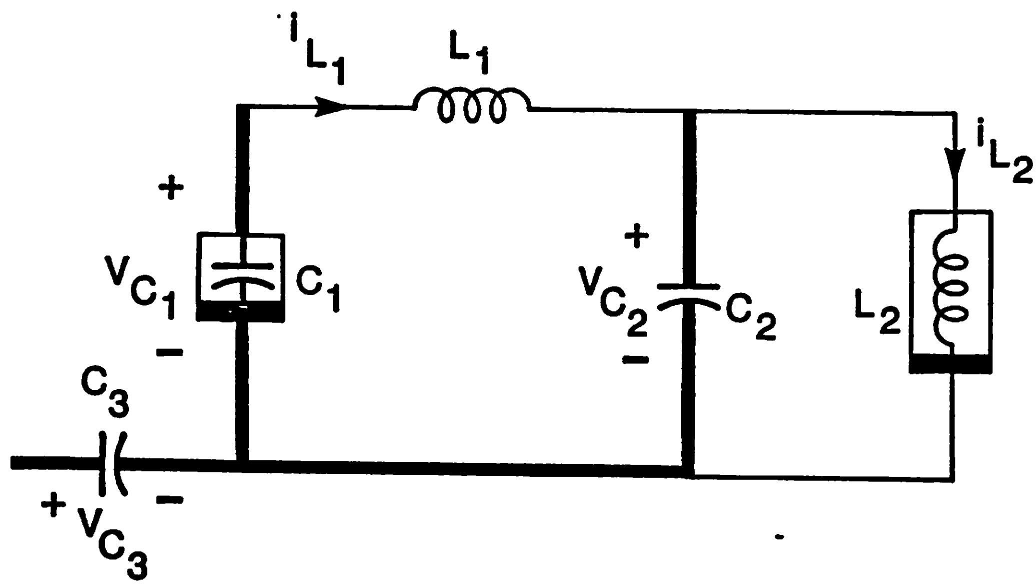

Figure 1(a)

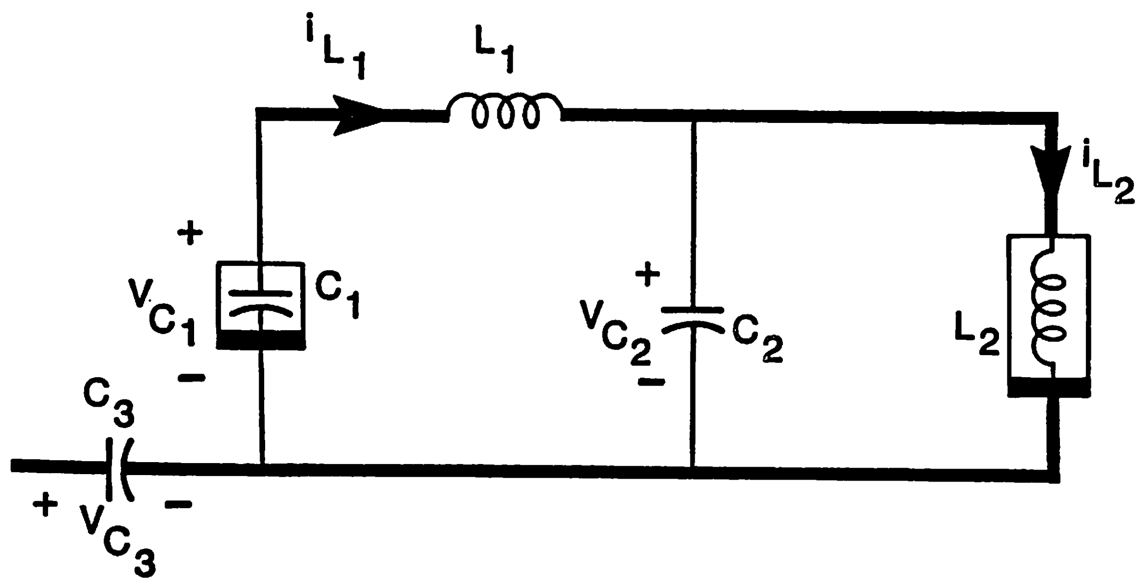

Figure 1(b)

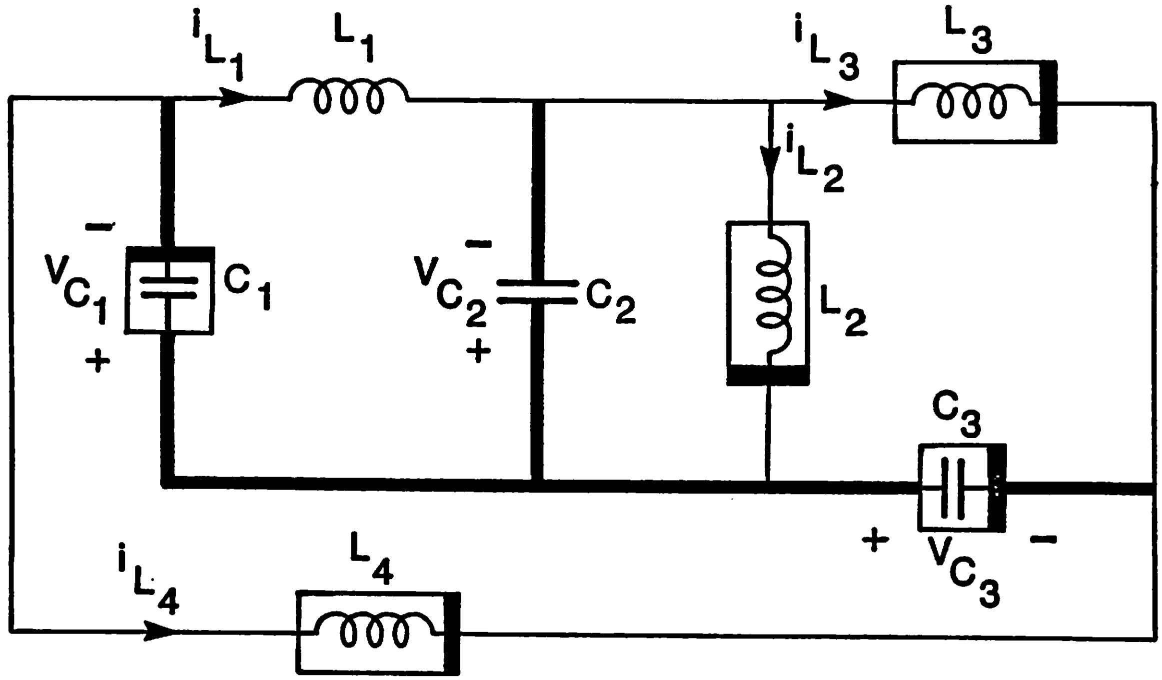

Figure 2(a)

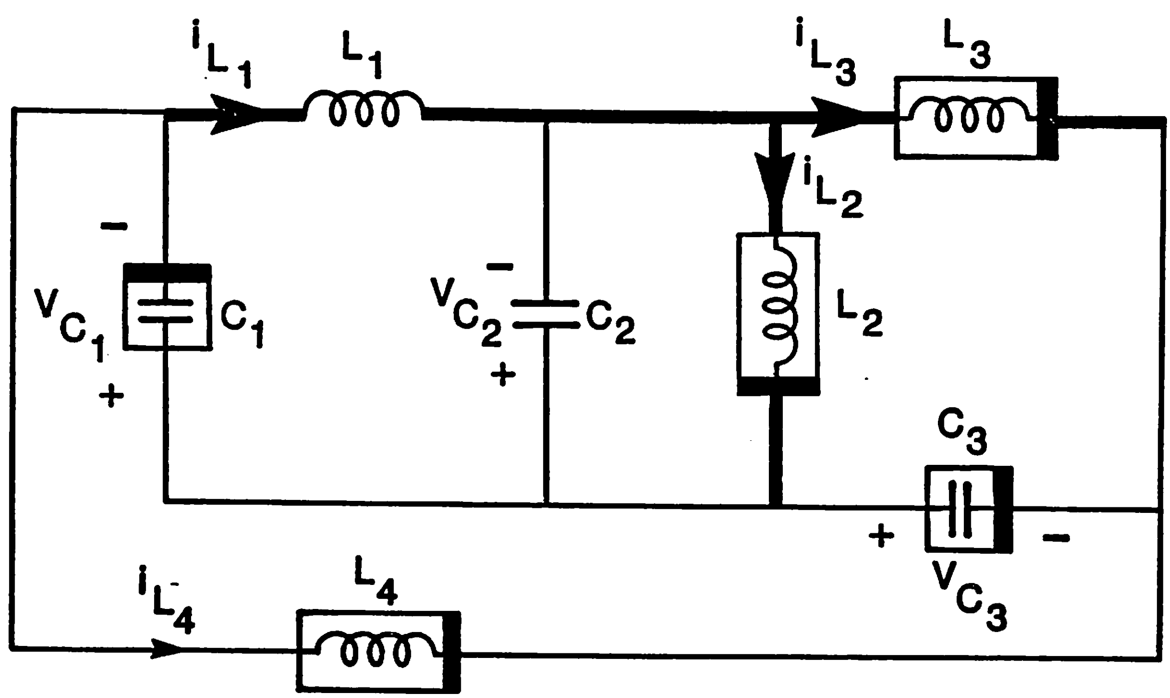

Figure 2(b)

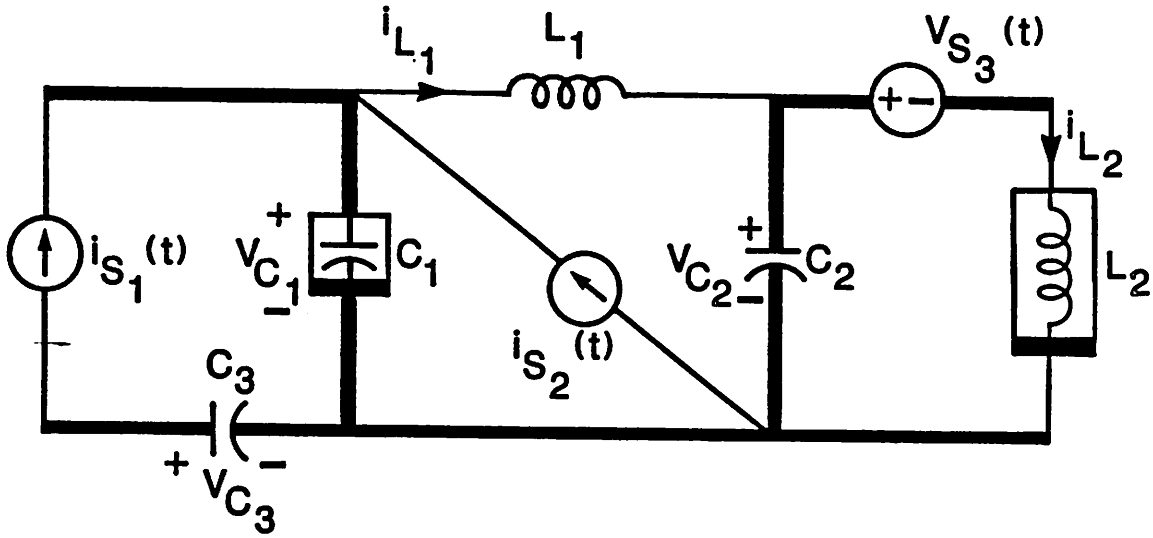

Figure 3