PROPORTIONAL-PLUS-INTEGRAL STABILIZING COMPENSATORS FOR A CLASS OF MIMO FEEDBACK SYSTEMS WITH INFINITE-DIMENSIONAL PLANTS

by

Y-P. Harn and E. Polak

Memorandum No. UCB/ERL M88/57

22 August 1988

PROPORTIONAL-PLUS-INTEGRAL STABILIZING COMPENSATORS FOR A CLASS OF MIMO FEEDBACK SYSTEMS WITH INFINITE-DIMENSIONAL PLANTS

by

Y-P. Harn and E. Polak

Memorandum No. UCB/ERL M88/57

22 August 1988

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

PROPORTIONAL-PLUS-INTEGRAL STABILIZING COMPENSATORS FOR A CLASS OF MIMO FEEDBACK SYSTEMS WITH INFINITE-DIMENSIONAL PLANTS

by

Y-P. Harn and E. Polak

Memorandum No. UCB/ERL M88/57

22 August 1988

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Proportional-Plus-Integral Stabilizing Compensators for a Class of MIMO Feedback Systems with Infinite-Dimensional Plants†

by

Y-P. Harn and E. Polak

Department of Electrical Engineering
and Computer Sciences
University of California
Berkeley, Ca 94720

ABSTRACT It is shown that it is possible to design a proportional-plus-integral stabilizing compensator for a class of feedback systems with exponentially stable infinite dimensional plants. This simple compensator also enables the feedback system to track asymptotically polynomial inputs and to suppress asymptotically polynomial disturbances.

I. INTRODUCTION

Exponential stability and asymptotic tracking are among the most fundamental requirement in control system design. Not surprisingly, over the years these requirements have received a considerable amount of attention in the literature (see, e.g. [Bal.2, Bha.1, Ben.1, Che.2, Des.1, Gib.1, Mor.1]). Nevertheless, the existence of simple, *fiinite dimensional* stabilizing compensators which ensure asymptotic tracking of polynomial inputs, for feedback systems with *infinite dimensional* plants is still a largely unresolved question.

We will consider a class of systems which are described by a linear differential equation in a Hilbert space. Before proceeding further, we will define exponential stability for these systems in terms of the properties of a semigroup function and we will establish the relation between exponential stability of these systems and the spectrum of the semigroup generator. With these preliminaries out of the way, we will proceed to exhibit the existence of proportional-plus-integral (P-I) stabilizing compensators for a class of feedback systems with infinite dimensional plants. As is well known [Des.1], P-I compensators result in feedback systems which track asymptotically polynomial inputs and suppress asymptotically polynomial disturbances.

† The research reported herein was sponsored in part by the National Science Foundation under grant ECS-8121149, the Air Force Office of Scientific Research grant AFOSR-83-0361, the Office of Naval Research under grant N00014-83-K-0602, the State of California MICRO Program, and the General Electric Co.

2. PRELIMINARY RESULTS

Consider the multi-input multi-output feedback system $S(P,K)$, with infinite dimensional plant, shown in Fig. 1. We assume that the plant has n_i inputs and n_o outputs, and is described by a differential equation in a Hilbert space E:

$$\dot{x}_p = A_p x_p + B_p e_2, \qquad y_2 = C_p x_p + D_p e_2, \tag{2.1}$$

where $x_p \in E$, $e_2 \in \mathbb{R}^{n_i}$, $y_2 \in \mathbb{R}^{n_o}$. The operators $B_p : \mathbb{R}^{n_i} \to E$, $C_p : E \to \mathbb{R}^{n_o}$ and $D_p : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ are assumed to be bounded, while A_p may be an unbounded operator from E to E, with its domain dense in E.

For any $\alpha > 0$, we define a *stability region* $D_{-\alpha} \subset \mathbb{C}$, in the complex plane, by $D_{-\alpha} \triangleq \{s \in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}$. Let $U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -\alpha\}$ denote the complement of $D_{-\alpha}$ in $\mathbb{C}$, let $\partial U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = -\alpha\}$ denote its boundary, and let $U^o_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha\}$. Next, let $\sigma(A_p)$ be the *spectrum* of A_p and let $\rho(A_p)$ be the *resolvent set* of A_p which is defined to be the complement of $\sigma(A_p)$ in $\mathbb{C}$. We will denote the domain and the range of A_p by $D(A_p)$ and $R(A_p)$, respectively. The notation used in this paper follows the notation in [Bal.1] and [Kat.1].

Assumption 2.1: (i) A_p is a closed operator which generates an analytic semigroup. (ii) There exists an $\alpha_0 > 0$ such that the spectrum of A_p is a subset of $D_{-\alpha_0}$. ■

Assumption 2.2: The transfer function of the plant is given by $G_p(s) = C_p(sI - A_p)^{-1}B_p + D_p$, where I is the the identity operator in E. We assume that $\lim_{|s| \to \infty, \, s \in U_{-\alpha_0}} G_p(s) \to D_p$. ■

We assume that we are required to design a minimal, finite dimensional, proportional-plus-integral compensator, described by a differential equation of the form:

$$\dot{x}_c = A_c x_c + B_c e_1 \qquad y_1 = C_c x_c + D_c e_1 \ , \tag{2.2}$$

where $x_c \in \mathbb{R}^{n_c}$, $e_1 \in \mathbb{R}^{n_o}$, $y_1 \in \mathbb{R}^{n_i}$ and A_c, B_c, C_c and D_c are matrices of appropriate dimension, with all the eigenvalues of A_c equal to zero, for integral action. Since $\sigma(A_c) = \{0\}$, the compensator transfer function is $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c = \sum_{j=0}^{m} F_j / s^j$, where each $F_j \in \mathbb{R}^{n_i \times n_o}$ and m depends on A_c. To ensure well-posedness of the closed loop system, we assume that $\det(I_{n_i} + D_c D_p) \neq 0$.

We define the Hilbert space H by $H = E \times \mathbb{R}^{n_c}$ and its inner product by

$$\left\langle \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \begin{bmatrix} z_p \\ z_c \end{bmatrix} \right\rangle_H = \langle x_p, z_p \rangle_E + \langle x_c, z_c \rangle_{\mathbb{R}^{n_c}} . \tag{2.3}$$

Since $e_1 = u_1 - y_2$ and $e_2 = y_1 + u_2$, we obtain the following state equations for the closed loop system

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = A \begin{bmatrix} x_p \\ x_c \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.4}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = C \begin{bmatrix} x_p \\ x_c \end{bmatrix} + D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{2.5}$$

where

$$A = \begin{bmatrix} A_p - B_p D_c (I_{n_o} + D_p D_c)^{-1} C_p & B_p (I_{n_i} + D_c D_p)^{-1} C_c \\ -B_c (I_{n_o} + D_p D_c)^{-1} C_p & A_c - B_c (I_{n_o} + D_p D_c)^{-1} D_p C_c \end{bmatrix} , \tag{2.6a}$$

$$B = \begin{bmatrix} B_p D_c (I_{n_o} + D_p D_c)^{-1} & B_p (I_{n_i} + D_c D_p)^{-1} \\ B_c (I_{n_o} + D_p D_c)^{-1} & -B_c (I_{n_o} + D_p D_c)^{-1} D_p \end{bmatrix} , \tag{2.6b}$$

$$C = \begin{bmatrix} -(I_{n_o} + D_p D_c)^{-1} C_p & -(I_{n_o} + D_p D_c)^{-1} D_p C_c \\ -D_c (I_{n_o} + D_p D_c)^{-1} C_p & (I_{n_i} + D_c D_p)^{-1} C_c \end{bmatrix} , \tag{2.6c}$$

$$D = \begin{bmatrix} (I_{n_o} + D_p D_c)^{-1} & -(I_{n_o} + D_p D_c)^{-1} D_p \\ D_c (I_{n_o} + D_p D_c)^{-1} & (I_{n_i} + D_c D_p)^{-1} \end{bmatrix} . \tag{2.6d}$$

The domain $D(A) = D(A_p) \times \mathbb{R}^{n_c} \subset H$; the operators B, C and D are easily seen to be bounded.

We will now show that because the operator A_p generates an analytic semigroup, the operator A also generates an analytic semigroup.

Proposition 2.1: The operator A generates an analytic semigroup, $T(\cdot)$.

Proof: We can decompose the matrix A in (2.6a) as follows:

$$A = F + Q \tag{2.7a}$$

where, for $\lambda_c \in \mathbb{R}$ arbitrary

$$F = \begin{bmatrix} A_p & 0 \\ 0 & \lambda_c I_{n_c} \end{bmatrix}, \quad Q = \begin{bmatrix} -B_p D_c (I_{n_o} + D_p D_c)^{-1} C_p & B_p (I_{n_i} + D_c D_p)^{-1} C_c \\ -B_c (I_{n_o} + D_p D_c)^{-1} C_p & A_c - B_c (I_{n_o} + D_p D_c)^{-1} D_p C_c - \lambda_c I_{n_c} \end{bmatrix}. \tag{2.7b}$$

It is easy to see that F generates the analytic semigroup

$$T_F(t) = \begin{bmatrix} T_p(t) & 0 \\ 0 & e^{\lambda_c t} I_{n_c} \end{bmatrix}, \tag{2.7c}$$

where $T_p(\cdot)$ is the semigroup generated by A_p. Since Q is a bounded operator, it follows from the perturbation theorem [Paz.1, p. 80] that A generates an analytic semigroup. ■

From Proposition 2.1 and [Tri.1], we obtain

Proposition 2.2: The operator A satisfies the *spectrum determined growth assumption*, i.e.,

$$\sup(Re(\sigma(A))) = \lim_{t \to \infty} \frac{\ln\|T(t)\|}{t}. \tag{2.8}$$

■

Next, from Proposition 2.2, we obtain the following result [Tri.1]:

Proposition 2.3: Given any $\beta > \sup(\mathrm{Re}(\sigma(A)))$, there exists an $M > 0$ such that

$$\|T(t)\|_H < M \cdot e^{\beta t}, \quad \forall\ t \geq 0. \tag{2.9}$$

■

Let $x = [x_p^t\ x_c^t]^t$. Then the formula

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau \tag{2.10}$$

defines a mild solution of (2.4) [Paz.1]. We can therefore define exponential stability of the feedback system $S(P,K)$ in terms of the semigroup $T(t)$ as follows.

Definition 2.1: The feedback system S(P , K) is exponentially stable if and only if there exists $\alpha > 0$ and $M > 0$ such that $\|T(t)\|_H < M \cdot e^{-\alpha t}$, $\forall\ t \geq 0$.

■

Propositions 2.2 and 2.3 yield the following result.

Proposition 2.4: The system S(P , K) is exponentially stable if and only if there exists $\alpha > 0$ such that $\sup(\mathrm{Re}(\sigma(A))) < -\alpha$. ■

We conclude this section by observing that it follows from Assumption 2.1 and Proposition 2.4, that the plant is exponentially stable.

3. EXISTENCE OF A STABILIZING PROPORTIONAL-PLUS-INTEGRATOR COMPENSATOR.

We will establish the existence of a proportional-plus-integral stabilizing compensator in two steps. First we will show that we can construct a proportional stabilizing compensator. Then we will show that we can construct an integral, stabilizing compensator. Finally we will combine these two results.

Definition 3.1: We say that a matrix transfer function $G: \mathbb{C} \to \mathbb{C}^{n_o \times n_i}$ is analytic in a region $U \subset \mathbb{C}$ if each of its element is analytic in U. ■

Assumption 2.1 is used in the Appendix to show that $G_p(s)$ is analytic in $U^o_{-\alpha_0}$.

We define the *characteristic function* $\chi(s)$ of the system $S(P,K)$, by

$$\chi(s) = \det(sI_{n_c} - A_c)\cdot\det(I_{n_i} + G_c(s)G_p(s)) = s^{n_c}\det(I_{n_i} + G_c(s)G_p(s)) = s^{n_c}\det(I_{n_o} + G_p(s)G_c(s)) \,. \quad (3.1)$$

To establish the next result, we will need the following Weinstein-Aronszajn formula.

Proposition 3.1 (The W-A Formula [Kat.1]): Let F be a closed operator in the Banach space X, let Q be an F-degenerate operator in X, let $R = R(Q)$, and let $y(s) = \det(I_R + (Q(F - sI)^{-1})|_R)$ be the associated W-A determinant, with I_R the identity operator in R and $(Q(F - sI)^{-1})|_R$ the restriction of $Q(F - sI)^{-1}$ to R. If Δ is a domain of the complex plane consisting of points of $\rho(F)$ and of isolated eigenvalues of F with finite multiplicities, then $y(s)$ is meromorphic in Δ and, for $A = F + Q$,

$$\tilde{\nu}(s;A) = \tilde{\nu}(s;F) + \nu(s;y), \quad s \in \Delta \,, \quad (3.2a)$$

where the *multiplicity function* $\nu(s;\phi)$ of ϕ in (3.2a) is defined by

$$\nu(s;\phi) = \begin{Bmatrix} k & \text{if } s \text{ is a zero of } \phi \text{ of order } k \\ -k & \text{if } s \text{ is a pole of } \phi \text{ of order } k \\ 0 & \text{for other } s \in \Delta \end{Bmatrix}, \quad (3.2b)$$

and the *multiplicity function* $\tilde{\nu}\,(s;F)$ for a closed operator F is defined by

$$\tilde{\nu}(s;F) = \begin{Bmatrix} 0 & \text{if } s\in\rho(F) \\ \dim(P) & \text{if } s \text{ is an isolated point of } \sigma(F) \\ +\infty & \text{in all other cases} \end{Bmatrix}, \tag{3.2c}$$

where P is the projection associated with an isolated point of $\sigma(F)$ (see [Kat.1, p.180]). ∎

Next, for any function f: $\mathbb{C} \to \mathbb{C}$, we define $Z(f(s)) \triangleq \{s\in \mathbb{C} \mid f(s) = 0\}$ to be its set of zeros.

Theorem 3.1 The system $S(P,K)$ is exponentially stable if and only if there exists an $\alpha > 0$ such that $Z(\chi(s)) \subset D_{-\alpha}$.

Proof: (The notation and the definitions used in this proof follow [Kat.1].) We begin by decomposing the matrix A as in (2.7a), (2.7b) with $\mathrm{Re}(\lambda_c) < -\alpha_0$. Therefore $(sI - F)$ is invertible for $s\in U_{-\alpha_0} \subset \rho(A_p)$, and Q is an F-degenerate operator because it is bounded. Consider $s\in U_{-\alpha_0}$. Since $(sI - F)^{-1}$ exists and is bounded, we can define $V(s)$ by

$$V(s) = Q(sI - F)^{-1}$$
$$= \begin{bmatrix} -B_pD_c(I_{n_o} + D_pD_c)^{-1}C_p(sI - A_p)^{-1} & B_p(I_{n_i} + D_cD_p)^{-1}C_c(s - \lambda_c)^{-1} \\ -B_c(I_{n_o} + D_pD_c)^{-1}C_p(sI - A_p)^{-1} & (A_c-B_c(I_{n_o} + D_pD_c)^{-1}D_pC_c - \lambda_cI_{n_c})(s - \lambda_c)^{-1} \end{bmatrix}. \tag{3.3}$$

Let $B_0 \triangleq R(B_p) \times \mathbb{R}^{n_c}$ and let $V_{B_0}(s)$ denote the restriction of $V(s)$ to B_0. Then $\det(I + V(s)) \triangleq \det(I_{B_0} + V_{B_0})$ is well defined. We will show that $\det(I_{B_0} + V_{B_0}) = \chi(s)$ and then apply the W-A formula.

Let $b_j \triangleq B_pe_j$, $j = 1,2,\dots,n_i$, where $\{e_j\}_{j=1}^{n_i}$ is the standard unit basis in $\mathbb{R}^{n_i}$. Suppose that $\bar{n} \le n_i$ is the largest positive interger such that any $\bar{n}$+1 elements of the set $\{b_j\}_{j=1}^{n_i}$ are linearly dependent in the Hilbert space H. Without loss of generality, we can assume that $\{b_j\}_{j=1}^{\bar{n}}$ is a basis for $R(B_p)$. Under this basis, the linear operator B_p assumes the form $B_p = (I_{\bar{n}\times\bar{n}} \mid \tilde{B}_p)$ where the *i–th* column of $\tilde{B}_p$ is obtained by expressing $b_{\bar{n}+i}$ in terms of the basis $\{b_j\}_{j=1}^{\bar{n}}$. Let $\bar{B} \triangleq (b_1, b_2, \dots, b_{\bar{n}})$. Then it is easy to show that

$$V_{B_0}(s) = \begin{bmatrix} -B_pD_c(I_{n_o} + D_pD_c)^{-1}C_p(sI - A_p)^{-1}\bar{B} & B_p(I_{n_i} + D_cD_p)^{-1}C_c(s - \lambda_c)^{-1} \\ -B_c(I_{n_o} + D_pD_c)^{-1}C_p(sI - A_p)^{-1}\bar{B} & (A_c - B_c(I_{n_o} + D_pD_c)^{-1}D_pC_c - \lambda_cI_{n_c})(s - \lambda_c)^{-1} \end{bmatrix} \tag{3.4a}$$

$$= \begin{bmatrix} -B_pD_c(I_{n_o} + D_pD_c)^{-1}M & B_p(I_{n_i} + D_cD_p)^{-1}C_c(s - \lambda_c)^{-1} \\ -B_c(I_{n_o} + D_pD_c)^{-1}M & (A_c - B_c(I_{n_o} + D_pD_c)^{-1}D_pC_c - \lambda_cI_{n_c})(s - \lambda_c)^{-1} \end{bmatrix}, \tag{3.4b}$$

where $M \triangleq [r_1, r_2, \ldots, r_{\bar{n}}] \in \mathbb{R}^{n_o \times \bar{n}}$ with $r_i \triangleq C_p(sI - A_p)^{-1} b_i$, $1 \le i \le \bar{n}$. Because each element in (3.4b) is in matrix form, it is easy to show that

$$\chi(s) = \det(I_{B_0} + V_{B_0}(s)) = s^{n_c} \cdot \det(I_{n_i} + G_c(s)G_p(s)) . \tag{3.5}$$

Let $\Delta = U_{-\alpha}$, where $0 < \alpha \le \alpha_0$ is such that $\Delta \subset U_{-\alpha_0} \subset \rho(A_p)$. Then from the W-A formula (3.2a), we have that

$$\{s \in U_{-\alpha} \mid s \in \sigma(A)\} = \{s \in U_{-\alpha} \mid s \in Z(\chi(s))\} . \tag{3.6}$$

If the system $S(P,K)$ is exponentially stable, then, from Proposition 2.4, we can find some $0 < \alpha < \alpha_0$ such that $\sup(Re(\sigma(A))) < -\alpha$, i.e., $\sigma(A) \subset D_{-\alpha}$. Therefore, from (3.6) $\{s \in U_{-\alpha} \mid s \in Z(\chi(s))\}$ is an empty set, i.e., $Z(\chi(s)) \subset D_{-\alpha}$. On the other hand, if there exists $\beta > 0$ such that $Z(\chi(s)) \subset D_{-\beta}$, then, setting $\alpha = \min\{\beta, \alpha_0\}$, we obtain from (3.6) that $\sup(\mathrm{Re}(\sigma(A))) \le -\alpha$, which implies that the system $S(P,K)$ is exponentially stable. This completes the proof. ■

In the proofs to follow, we will make use of Rouche's theorem, stated below [Chu.1].

Rouche's theorem: Let $f(\cdot)$ and $g(\cdot)$ be functions which are analytic inside and on a positively oriented simple closed contour C in the complex plane. If $|f(s)| > |g(s)|$ at each point s on C, then the functions $f(s)$ and $f(s) + g(s)$ have the same number of zeros, counting multiplicities, inside C. ■

Theorem 3.2: Consider the feedback system $S(P,K)$ in Fig. 1 and suppose that $A_c = 0$, $B_c = 0$, $C_c = 0$ and n_c=0. Then there exists a matrix $D_c \ne 0$ such that the closed loop system is exponentially stable.

Proof: By Theorem 3.1, the system $S(P,K)$ is exponentially stable if and only if there exists an $\alpha > 0$ such that $Z[\det(I_{n_i} + D_c G_p(s))] \subset D_{-\alpha}$. Suppose that $G_p(s) = [g_{ij}(s)]$ and $D_c = [d_{ij}]$. Then

$$\det(I_{n_c} + D_c G_p(s)) = \det([\Delta_{ij} + \sum_{k=1}^{n_0} d_{ik} g_{kj}(s)]_{ij}]$$
$$= 1 + \sum_{l=1}^{n_i}\sum_{k=1}^{n_0} d_{lk} g_{kl}(s) + o(s) \triangleq 1 + H(s) , \tag{3.7}$$

where $o(s)$ represents the second and higher order terms in d_{ij} and $g_{ij}(s)$, and $\Delta_{ij} = 1$ when $i = j$, and $\Delta_{ij} = 0$ otherwise. Because $G_p(s)$ is analytic on $U^o_{-\alpha_0}$ and because of Assumption 2.2, there exists an positive $\alpha < \alpha_0$ and $M > 0$ such that $|g_{ij}(s)| < M, \forall\ s \in \partial U_{-\alpha}$. It is clear that we can always choose a matrix $D_c \neq 0$, with sufficiently small components, d_{ij}, to ensure that $|H(s)| < 1, \forall\ s \in \partial U_{-\alpha}$. Setting $C = \partial U_{-\alpha}$, $f(s) \equiv 1$ and $g(s) = H(s)$, we obtain from Rouche's theorem that $\det(I_{n_c} + D_c G_p(s)) = 1 + H(s)$ has the same number of zeros in $U_{-\alpha}$ as $f(\cdot)$, which is zero. Therefore $\det(I_{n_c} + D_c G_p(s))$ has no zeros on $U_{-\alpha}$, i.e., $Z(I_{n_c} + D_c G_p(s)) \subset D_{-\alpha}$, which completes the proof. ■

Assumption 3.1: The matrix $G_p(0)$ has maximum rank, i.e., 0 is not a transmission zero of $G_p(s)$. ■

Theorem 3.3: Suppose that $D_c = 0$ and $A_c = 0$, so that $G_c(s) = \frac{1}{s} C_c B_c$. Then there exists an $n_i \times n_o$ maximum-rank matrix F_I such that for any B_c, C_c such that $C_c B_c = F_I$, the closed loop system is exponentially stable.

Proof **Case I:** $n_i = n_o$, i.e, the plant and the compensator transfer matrices are square. Let $n_c = n_i = n_o$, $B_c = F_I \in \mathbb{R}^{n_c \times n_c}$, $C_c = I_{n_c}$. From Theorem 3.1, we know that the system is exponentially stable if there exists an $\alpha > 0$ such that

$$Z(\det(sI_{n_c}) \det(I_{n_c} + G_p \frac{F_I}{s})) = Z(\det(sI_{n_c} + G_p F_I)) \subset D_{-\alpha} . \tag{3.8}$$

Suppose that $G_p(s) = [g_{ij}(s)]$ and $F_I = [f_{ij}]$. Hence

$$\det(sI_{n_c} + G_p F_I) = s^{n_c} + s^{n_c - 1} \sum_{l=1}^{n_c} \sum_{k=1}^{n_c} f_{lk} g_{kl}(s) + s^{n_c - 2}(\cdots) + \cdots + \det G_p \cdot \det F_I . \tag{3.9}$$

Let $f(s) = s^{n_c}$ and let $g(s) = s^{n_c - 1} \sum_{l=1}^{n_c} \sum_{k=1}^{n_c} f_{lk} g_{kl}(s) + s^{n_c - 2}(\cdots) + \cdots + \det G_p \cdot \det F_I$. Suppose that $|f_{ij}| < \delta, \forall\ ij$. Let $0 < \alpha < \alpha_0$ and suppose that $s \in U_{-\alpha}$. Then

$$\begin{aligned} |g(s)| &\leq |s^{n_c - 1}| |\sum_{l=1}^{n_c} \sum_{k=1}^{n_c} f_{lk} g_{kl}| + |s^{n_c - 2}| (\cdots) + \cdots + \det G_p \cdot \det F_I \\ &\leq |s|^{n_c - 1} N_1 M \delta + |s|^{n_c - 2} N_2 M^2 \delta^2 + \cdots + N_{n_c} M^{n_c} \delta^{n_c} \\ &\leq N |s|^{n_c} (|s|^{-1} M \delta + |s|^{-2} M^2 \delta^2 + |s|^{-3} M^3 \delta^3 + \cdots + |s|^{-n_c} M^{n_c} \delta^{n_c}), \end{aligned} \tag{3.10}$$

where N_i is the number of product terms in the coefficients of $|s|^{n-i}$, $N = \max_i N_i$, and $M > 0$ is such that $|g_{ij}(s)| < M, \forall\ s \in \partial U_{-\alpha}$. Hence, since $|s| \geq \alpha$ for any $s \in \partial U_{-\alpha}$, if $\delta < \frac{\alpha}{2NM}$,

$$\left|\frac{g(s)}{f(s)}\right| = \frac{|g(s)|}{|f(s)|} \leq N(|s|^{-1}M\delta + |s|^{-2}M^2\delta^2 + \cdots + |s|^{-n_c}M^{n_c}\delta^{n_c})$$

$$\leq \frac{NM\delta}{|s|-M\delta} \leq \frac{NM\delta}{\alpha - M\delta} \leq \frac{2NM\delta}{\alpha} < 1 \,. \tag{3.11}$$

Setting $C = \partial U_{-\alpha}$ and applying Rouche's theorem, we conclude that $\det(sI_{n_c} + G_p(s)F_I)$ has n_c zeros in $U_{-\alpha}$.

Now we let $C = C_\varepsilon \triangleq \{s \in \mathbb{C} \mid |s + \varepsilon| < \varepsilon/2\}$ where $\varepsilon > 0$. Clearly, there is an $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$, then $C_\varepsilon \subset U_{-\alpha}$. Since by Assumption 3.1 $\det G_p(0) \neq 0$, it follows by continuity that there exists an $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\det G_p(-\varepsilon) \neq 0$ for all $\varepsilon \leq \varepsilon_1$. Finally, there is an $\varepsilon_2 \in (0, \varepsilon_1)$ such that for all $\varepsilon \in (0, \varepsilon_2)$, if $F_I \triangleq G_p^{-1}(-\varepsilon)\varepsilon$, then $|f_{ij}| < \alpha / 2MN$, $\forall\ ij$, is satisfied, and, in addition,

$$\begin{aligned}
\det(sI_{n_c} + G_p(s)F_I) &= \det((s+\varepsilon)I_{n_c} + G_p(s)F_I - \varepsilon I_{n_c}) \\
&= \det\left\{(s+\varepsilon)I_{n_c} + \left[G_p(-\varepsilon) + (s+\varepsilon)\int_0^1 G_p'(-\varepsilon + t(s+\varepsilon))dt\right]G_p^{-1}(-\varepsilon)\varepsilon - \varepsilon I_{n_c}\right\} \\
&= \det\left\{(s+\varepsilon)I_{n_c} + \varepsilon(s+\varepsilon)\left[\int_0^1 G_p'(-\varepsilon + t(s+\varepsilon))dt\right]G_p^{-1}(-\varepsilon)\right\} \\
&= (s+\varepsilon)^{n_c} + (s+\varepsilon)^{n_c-1}\varepsilon(s+\varepsilon)Q_1(s) + (s+\varepsilon)^{n_c-2}\varepsilon^2(s+\varepsilon)^2Q_2(s) + \\
&\qquad \cdots + \varepsilon^{n_c}(s+\varepsilon)^{n_c}Q_{n_c}(s),
\end{aligned} \tag{3.12}$$

where the $Q_i(s)$ are determined from the expansion of the determinant. It is easy to see that the $Q_i(s)$ are analytic on $U_{-\alpha}$ and therefore they are analytic on and inside C. Let $W_i = \max_{s \in C} |Q_i(s)|$ and let $W = \max_i W_i$. Let $f(s) \triangleq (s+\varepsilon)^{n_c}$ and $g(s) \triangleq (s+\varepsilon)^{n_c-1}\varepsilon(s+\varepsilon)Q_1(s) + (s+\varepsilon)^{n_c-2}\varepsilon^2(s+\varepsilon)^2Q_2(s) + \cdots + \varepsilon^{n_c}(s+\varepsilon)^{n_c}Q_{n_c(s)}$. Then $|f(s)| = \varepsilon^{n_c} / 2^{n_c}$, $\forall\ s \in C$, and, if $\varepsilon < 1/2W$,

$$|g(s)| \leq \left(\frac{\varepsilon}{2}\right)^{n_c-1}\varepsilon\frac{\varepsilon}{2}W_1 + \left(\frac{\varepsilon}{2}\right)^{n_c-2}\varepsilon^2\left(\frac{\varepsilon}{2}\right)^2W_2 + \cdots + \varepsilon^{n_c}\left(\frac{\varepsilon}{2}\right)^{n_c}W_{n_c}$$

$$\leq W\left[\frac{\varepsilon^{n_c+1}}{2^{n_c}} + \frac{\varepsilon^{n_c+2}}{2^{n_c}} + \cdots + \frac{\varepsilon^{2n_c}}{2^{n_c}}\right]$$

$$\leq W\frac{\varepsilon^{n_c+1}}{2^{n_c}}\frac{1}{1-\varepsilon} < \frac{\varepsilon^{n_c}}{2^{n_c}} . \tag{3.13}$$

Therefore we obtain that $|f(s)| > |g(s)| \ \forall \ s\in C$. It now follow from Rouche's theorem that $\det(sI_{n_c} + G_p(s)F_I) = f(s) + g(s)$ has the same number of zeros, n_c, inside C as $f(s)$. Therefore we have shown that $Z(\det(sI_{n_c} + G_p(s)F_I)) \subset D_{-\varepsilon/2}$ with $\varepsilon > 0$.

Case II: $n_o < n_i$. Because of Assumption 3.1, we may assume the determinant of the first n_o columns of $G_p(0)$ is not equal to 0. Let $n_c = n_o$, $B_c = I_{n_o}$, $C_c = F_I \in \mathbb{R}^{n_i \times n_o}$. It follows from Theorem 3.1, that the system is exponentially stable if there exists $\alpha > 0$ such that $Z[\det(sI_{n_c} + G_p(s)F_I)] \subset D_{-\alpha}$. The proof for Case II follows the same arguments as for Case I, except for the way in which we choose F_I. Let

$$G_{p,\varepsilon}(s) \triangleq \left[\begin{array}{ccc|ccc} g_{1,1}(s) & \cdots & g_{1,n_o}(s) & g_{1,(n_o+1)}(s) & \cdots & g_{1,n_i}(s) \\ g_{2,1}(s) & \cdots & g_{2,n_o}(s) & g_{2,(n_o+1)}(s) & \cdots & g_{2,n_i}(s) \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ g_{n_o,1}(s) & \cdots & g_{n_o,n_o}(s) & g_{n_o,(n_o+1)}(s) & \cdots & g_{n_o,n_i}(s) \\ \hline & 0_{(n_i-n_o)\times n_o} & & & I_{(n_i-n_o)\times(n_i-n_o)} & \end{array}\right] \tag{3.14}$$

Then by the above assumption, $\det G_{p,\varepsilon}(0) \neq 0$. Let $\varepsilon > 0$ be such that $\det G_{p,\varepsilon}(-\varepsilon) \neq 0$ and let $F_\varepsilon = G_{p,\varepsilon}^{-1}(-\varepsilon)\cdot\varepsilon \in \mathbb{R}^{n_i \times n_i}$, and let $F_I \in \mathbb{R}^{n_i \times n_o}$ consist of the first n_o columns of F_ε. Then, since $G_p(-\varepsilon)F_I = \varepsilon I_{n_o}$, (3.12) becomes

$$\det(sI_{n_c} + G_p(s)F_I)$$

$$= \det\left[(s+\varepsilon)I_{n_c} + \left[G_p(-\varepsilon) + (s+\varepsilon)\int_0^1 G_p'(-\varepsilon + t(s+\varepsilon))dt\right]F_I - \varepsilon I_{n_c}\right]$$

$$= \det\left[(s+\varepsilon)I_{n_c} + \varepsilon(s+\varepsilon)\left[\int_0^1 G_p'(-\varepsilon + t(s+\varepsilon))dt\right]F_I\right] . \tag{3.15}$$

The rest of the proof follows that for Case I.

Case III: $n_o > n_i$. Let $n_c = n_i$, $B_c = F_I$ and $C_c = I_{n_i}$. It follows from Theorem 3.1 that the system $S(P,K)$ is exponentially stable if there exists $\alpha > 0$ such that

$$Z(\det(sI_{n_c})\det(I_{n_o} + G_p\frac{F_I}{s})) = Z(\det(sI_{n_i})\det(I_{n_i} + \frac{F_I}{s}G_p(s))) = Z(\det(sI_{n_c} + F_IG_p(s))) \subset D_{-\alpha}. \tag{3.16}$$

Because of Assumption 3.1, we can assume that the determinant of the first n_i rows of $G_p(0)$ is not equal to 0. Let

$$G_{p,\epsilon}(s) \triangleq \left[\begin{array}{ccc|c} g_{1,1}(s) & \cdots & g_{1,n_i}(s) & \\ g_{2,1}(s) & \cdots & g_{2,n_i}(s) & \\ \cdot & & \cdot & \\ \cdot & & \cdot & 0_{n_i\times(n_o-n_i)} \\ \cdot & & \cdot & \\ g_{n_i,1}(s) & \cdots & g_{n_i,n_i}(s) & \\ \hline g_{n_i+1,1}(s) & \cdots & g_{n_i+1,n_i}(s) & \\ \cdot & & \cdot & \\ \cdot & & \cdot & I_{(n_o-n_i)\times(n_o-n_i)} \\ \cdot & & \cdot & \\ g_{n_o,1}(s) & \cdots & g_{n_o,n_i}(s) & \end{array}\right]. \tag{3.17}$$

Then by assumption, $\det G_{p,\epsilon}(0) \neq 0$. Let $\varepsilon > 0$ be such that $\det G_{p,\epsilon}(-\varepsilon) \neq 0$, let $F_\epsilon = G_{p,\epsilon}^{-1}(-\varepsilon)\cdot\varepsilon$, and let $F_I \in \mathbb{R}^{n_i\times n_o}$ consist of the first n_i rows of F_ϵ. The rest of the proof proceeds as for Case I. This completes the proof. ∎

We are now ready to establish our main result.

Theorem 3.4: Suppose that Assumption 3.1 is satisfied. Then for any integer $m \geq 1$, there exist m+1 $n_i \times n_o$ matrices F_j, $0 \leq j \leq m$, with F_m of maximum rank, such that, if $[A_c,B_c,C_c,D_c]$ is a minimal realization of the matrix transfer function $\sum_{j=0}^{m} F_j/s^j$, then the closed loop system is exponentially stable.

Proof: Case I: $n_i \geq n_o$. We will prove this theorem by induction. Since the only requirement on D_c in the proof of Theorem 3.2, is that its components be sufficiently small, it is clear that there exists a matrix $F_0(= D_c)$ with maximum rank such that I+$G_p(0)F_0$ and $G_p(0)F_0$ are both invertible. Hence the Theorem is true for $m = 0$. Now suppose that $m \geq 1$ and that we can construct a minimal stabilizing compensator $[A_c',B_c',C_c',D_c']$, with transfer function $\sum_{i=0}^{m-1} F_i'/s^i$, where F_{m-1}' has maximum rank and

$G_p(0)F'_{m-1}$ is invertible. Now, see Fig.2a, consider this closed loop system as a "new plant" with transfer function $G'_p(s) \triangleq [(I_{n_o} + G_p(s)\sum_{i=0}^{m-1} F'_i / s^i)^{-1} G_p(s) \sum_{i=0}^{m-1} F'_i / s^i]$. Then $G'_p(0) = [G_p(0)F'_{m-1}]^{-1}G_p(0)F'_{m-1} = I_{n_o}$ for $m > 1$ and $G'_p(0) = (I_{n_o} + G_p(0)F'_0)^{-1} G_p(0)F'_0$ for m=1. In either case, Assumption 3.1 is satisfied. According to Theorem 3.3, for this new plant, we can find a stabilizing compensator K, whose transfer function is of the form $F'_m I$, with $F'_m \in \mathbb{R}^{n_o \times n_o}$ of maximum rank. For this compensator, there exists $\alpha > 0$ such that

$$
\begin{aligned}
&Z\Big[\det(sI_{n_o} + G'_p(s)F'_m)\Big] \\
&= Z\Big[\det(sI_{n_o} + (I_{n_o} + G_p(s)\sum_{i=0}^{m-1} F'_i / s^i)^{-1}G_p(s)(\sum_{i=0}^{m-1} F'_i / s^i)F'_m)\Big] \\
&= Z\Big[\det[sI_{n_o} + (s^{m-1}I_{n_o} + G_p(s)\sum_{i=0}^{m-1}F'_i s^{m-1-i})^{-1}G_p(s)(\sum_{i=0}^{m-1}F'_i s^{m-1-i})F'_m]\Big] \\
&= Z\Big[(\det[s^{m-1}I_{n_o} + G_p(s)\sum_{i=0}^{m-1}F'_i s^{m-1-i}])^{-1}\det[s^m I_{n_o} + G_p(s)\sum_{i=0}^{m}(F'_{i-1}F'_m + \bar{F}_i)s^{m-i}]\Big] \subset D_{-\alpha},
\end{aligned}
\tag{3.18}
$$

where $F'_{-1} \triangleq 0$, $\bar{F}_i = F'_i$ for $0 \le i \le m-1$ and $\bar{F}_m$=0. Let $X \triangleq \det(s^{m-1}I_{n_o} + G_p(s)\sum_{i=0}^{m-1}F'_i s^{m-1-i}) = \det(s^{m-1}I_{n_o})\cdot\det(I_{n_o} + G_p(s)\sum_{i=0}^{m-1}F'_i s^{-i})$ and let $Y \triangleq \det(s^m I_{n_o} + G_p(s)\sum_{i=0}^{m}(F'_{i-1}F'_m + \bar{F}_i)s^{m-i}) = \det(s^m I_{n_o})\cdot\det(I_{n_o} + G_p(s)\sum_{i=0}^{m}(F'_{i-1}F'_m + \bar{F}_i)s^{-i})$. By assumption, $[A'_c,B'_c,C'_c,D'_c]$ is a minimal realization for $\sum_{i=0}^{m-1} F'_i / s^i = (\sum_{i=0}^{m-1}F'_i s^{m-1-i})\cdot(s^{m-1}I_{n_o})^{-1}$. Since F'_{m-1} has maximum rank, it can be shown that $\sum_{i=0}^{m-1}F'_i s^{m-1-i}$ and $s^{m-1}I_{n_o}$ are coprime. From [Che.1, Chap. 6], it follows that A'_c is a square matrix of dimension $n_c = deg(\det(s^{m-1}I_{n_o}))=(m-1)\cdot n_o$. Because $\sum_{i=0}^{m-1}F'_i / s^i$ is a stabilizing compensator for the plant $G_p(s)$, it follows from Theorem 3.1, that there exists a $\beta > 0$ such that $Z(X) \in D_{-\beta}$. It now follows from (3.18) that $Z(Y) \subset D_{-\gamma}$, where $\gamma = \min(\alpha,\beta)$. We now set $F_i = F'_{i-1}F'_m + \bar{F}_i$, for $0 \le i \le m$. Then $F_m = F'_{m-1}\cdot F'_m$ has maximum rank because F'_{m-1} has maximum rank and F'_m is invertible. Also $G_p(0)F_m = (G_p(0)F'_{m-1})F'_m$ is invertible because $G_p(0)F'_{m-1}$ and F'_m are invertible. Hence we conclude that any minimal realization for the transfer function $\sum_{i=0}^{m} F_i / s^i$ is a stabilizing compensator for the plant P.

Case II: $n_i < n_o$. We proceed again by induction, as for Case I, except now we reason in terms of the configuration shown in Fig. 2b. Thus we now set

$G_p'(s) = \sum_{i=0}^{m-1} F_i' / s^i \, G_p(s)[I_{n_i} + \sum_{i=0}^{m-1} F_i' / s^i \, G_p(s)]^{-1}$ and $F_m' \in \mathbb{R}^{n_i \times n_i}$ and we examine the set $Z[\det(sI_{n_i} + F_m' G_p'(s))]$ instead of $Z[\det(sI_{n_o} + G_p'(s) F_m')]$. The rest of the proof continues as for Case I and is therefore omitted. This completes the proof. ■

4. CONCLUSION

Since it is possible to both stabilize and ensure asymptotic tracking of polynomial inputs and asymptotic reejection of polynomial disturbances by means of very simple finite dimensional compensators, it is clear that fairly complex design specifications may be possible to be satisfied by fairly low dimensional compensators. Such compensators are best designed using nonsmooth optimization techniques, as outlined in [Pol.1].

APPENDIX: ANALYTICITY RESULTS

Theorem A.1: The matrix transfer function $G_p(s) = C_p(sI - A_p)^{-1}B_p + D_p$ is a componentwise analytic function over $U^o_{-\alpha_0}$.

Proof: First, we will prove that each component of $G_p(s) = C_p(sI - A_p)^{-1}B_p + D_p$ is an analytic function over $U^o_{-\alpha_0}$. We denote the (i,j)–*th* component of $G_p(s)$ by $G_{ij}(s)$. Then

$$G_{ij}(s) = C_{p,i\cdot}(sI - A_p)^{-1}B_{p,\cdot j} + D_{ij} \, , \tag{A.1}$$

where $C_{p,i\cdot}$ is the *i–th* row of C_p , $B_{p,\cdot j}$ is the *j–th* column of B_p and D_{ij} is the (ij)-th component of D. We will prove that $G_{ij}(s)$ is differentiable by showing that

$$\lim_{\Delta s \to 0} \frac{G_{ij}(s + \Delta s) - G_{ij}(s)}{\Delta s} = -C_{p,i\cdot}(sI - A_p)^{-2}B_{p,\cdot j} \, . \tag{A.2}$$

Consider Δs small enough such that both s and $s+\Delta s$ belong to $U^o_{-\alpha_0} \subset \rho(A_p)$. Then we have

$$\left| \frac{(C_{p,i\cdot}((s + \Delta s)I - A_p)^{-1}B_{p,\cdot j} + D_{ij}) - (C_{p,i\cdot}(sI - A_p)^{-1}B_{p,\cdot j} + D_{ij})}{\Delta s} + C_{p,i\cdot}(sI - A_p)^{-2}B_{p,\cdot j} \right|_{\mathbb{C}}$$

$$= \left| C_{p,i\cdot} \left[\frac{((s + \Delta s)I - A_p)^{-1} - (sI - A_p)^{-1}}{\Delta s} + (sI - A_p)^{-2} \right] B_{p,\cdot j} \right|_{\mathbb{C}}$$

$$\leq \|C_{p,i\cdot}\| \, \left\| \frac{((s + \Delta s)I - A_p)^{-1} - (sI - A_p)^{-1}}{\Delta s} + (sI - A_p)^{-2} \right\| \, \|B_{p,\cdot j}\|$$

$$\leq \|C_p\| \, \| \frac{((s + \Delta s)I - A_p)^{-1} - (sI - A_p)^{-1}}{\Delta s} + (sI - A_p)^{-2} \| \, \|B_p\|$$

$$= \|C_p\| \, \| \frac{((s + \Delta s)I - A_p)^{-1}\Big[(sI - A_p) - ((s + \Delta s)I - A_p)\Big](sI - A_p)^{-1}}{\Delta s} + (sI - A_p)^{-2} \| \, \|B_p\|$$

$$= \|C_p\| \, \| - ((s + \Delta s)I - A_p)^{-1}(sI - A_p)^{-1} + (sI - A_p)^{-2} \| \, \|B_p\|$$

$$= \|C_p\| \, \| ((s + \Delta s)I - A_p)^{-1}\Big[-(sI - A_p) + ((s + \Delta s)I - A_p)\Big](sI - A_p)^{-1}(sI - A_p)^{-1} \| \, \|B_p\|$$

$$= |\Delta s| \, \|C_p\| \, \| ((s + \Delta s)I - A_p)^{-1}(sI - A_p)^{-1}(sI - A_p)^{-1} \| \, \|B_p\| \to 0 \text{ as } |\Delta s| \to 0. \quad \text{(A.3)}$$

Therefore (A.2) is proved and $G_{ij}(s)$ is an analytic function on $U^o_{-\alpha_0}$. ■

REFERENCES

[Bal.1] A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, 1981.

[Bal.2] M. J. Balas, "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams", *IEEE Trans. on Automatic Control* Vol. AC-27, No. 3, pp. 522-535, 1982.

[Bha.1] A. Bhaya and C. A. Desoer, "On the Design of Large Flexible Structures", *IEEE Trans. on Automatic Contr.*, Vol. AC-30, No. 11, pp 1118-1120, 1985.

[Ben.1] R. J. Benhabib, R. P. Iwens, and R. Jackson, "Stability of Large Space Structure Control Systems Using Positivity Concepts", *J. Guidance Contr.*, Vol. 4, No. 5, pp. 487-494, 1981.

[Che.1] C. T. Chen, *Linear System Theory and Design*, CBS College Publishing, 1984.

[Che.2] M. J. Chen and C. A. Desoer, "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems", *Int. J. Contr.*, Vol. 35, No.2, pp. 255-267, 1982.

[Chu.1] R. V. Churchill, J. W. Brown, R. F. Verhey, *Complex Variables and Applications*, McGRAW-Hill, 1974.

[Des.1] C. A. Desoer and Y. T. Wang, "Linear Time Invariant Robust Servomechanism Problem: A Self Contained Exposition", in Advances in Control and Dynamical Systems, Vol. 16, C. T. Leondes (Ed.), Academic Press, New York, pp. 81-129, 1980.

[Gib.1] J. S. Gibson, "An Analysis of Optimal Modal Regulation: Convergence and Stability", *SIAM J. Control and Optimization*, Vol 19, No. 5, pp. 686-707, 1981.

[Kat.1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[Mor.1] M. Morari, "Robust Stability with Integral Control", *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, Tx, Dec. 14-16, pp. 865-869, 1983.

[Paz.1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[Pol.1] E. Polak, D. Q. Mayne and D. M. Stimler, "Control System Design via Semi-Infinite Optimization", *Proceedings of the IEEE*, pp. 1777-1795, December 1984.

[Tri.1] R. Triggiani, "On the Stabilizability Problem in Banach Space", *Journal of Mathematical Analysis and Applications*, 52, pp. 383-403, 1975.

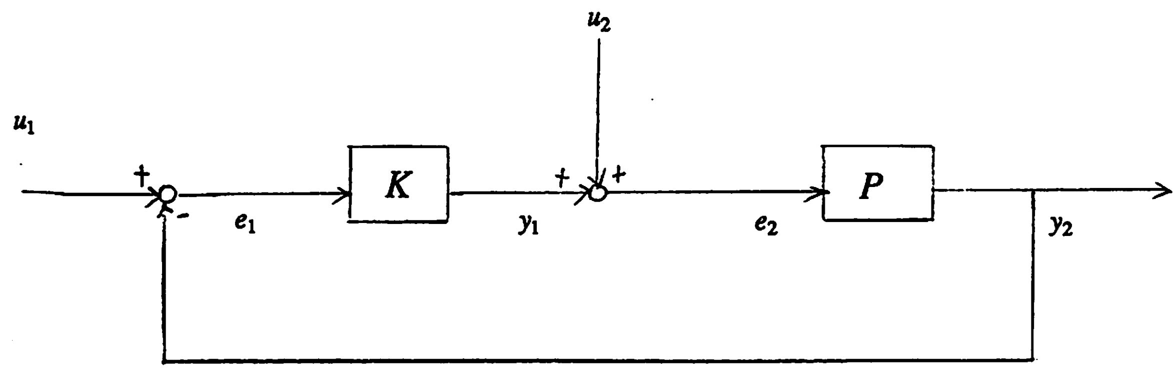

Figure 1: The Feedback system $S(P, K)$

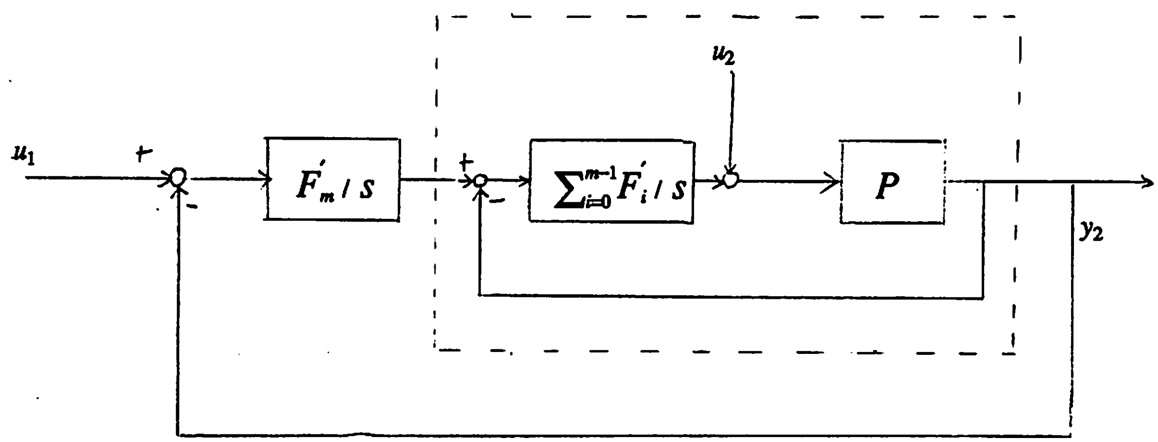

Figure 2.a: Feedback compensator structure for the proof of Case I in Theorem 3.4

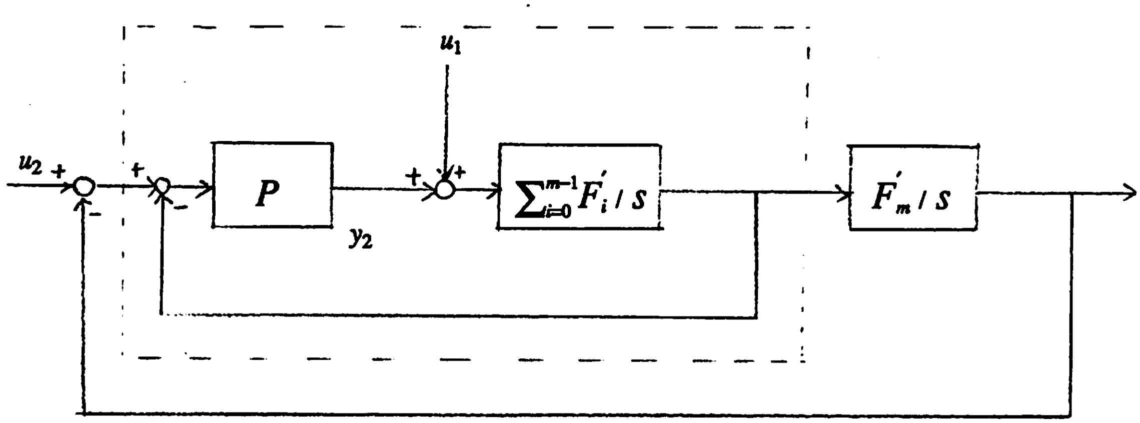

Figure 2.b: Feedback compensator structure for the proof of Case II in Theorem 3.4