ROBUST STABILIZATION AND TRACKING FOR NONLINEAR SYSTEMS

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Committee Chairman

ABSTRACT

In this thesis the problems of local stabilization and output tracking are considered for the class of single-input single-output nonlinear control systems which are affine with respect to the control variable.

The local stabilization problem is considered for nonlinear systems with degenerate linearizations, i.e. with linearizations whose uncontrollable modes are on the $j\omega$-axis. A methodology for designing a stabilizing control law in this case is presented. It involves the following steps: 1) Reduction of the stability problem to the stability of the center manifold system. 2) Simplification of the vector field on the center manifold using the theory of normal forms. 3) Finding conditions under which the simplified vector field is asymptotically stable. Following these steps, three cases of degeneracies in the linearized system are treated and sufficient conditions for the existence of stabilizing controls are given in each case. In addition, a theorem is presented regarding the robustness of the foregoing control strategy to perturbations.

The notions of relative degree and minimum-phase for nonlinear systems are reviewed. Given a bounded desired tracking signal with bounded derivatives, a control law is designed for minimum-phase nonlinear systems which results in tracking of this signal by the output. This control law is modified in the presence of uncertainties associated with

the model vector fields in order to reduce the effects of these uncertainties on the tracking errors. Two types of uncertainties are considered: those satisfying a generalized matching condition but otherwise unstructured, and those arising from linear parametric uncertainties.

It is shown that for systems with the first type of uncertainty, high gain control laws can result in small tracking errors of $O(\bar{\varepsilon})$, where $\bar{\varepsilon}$ is a small design parameter. Furthermore, it is shown that such control schemes are robust with respect to unmodeled dynamics provided that $\bar{\varepsilon}$ is of $O(\varepsilon^{\frac{1}{3}})$, where ε represents the singular perturbation parameter characterizing the time scale separation between the unmodeled dynamics (fast) and the dynamics of the reduced model (slow) of the overall system. An alternate scheme which is based on the variable structure control strategy is presented as well. It is shown that in the absence of unmodeled dynamics this scheme results in zero tracking errors. In the presence of unmodeled dynamics, however, satisfactory tracking may not be achieved.

Adaptive control techniques are employed in dealing with systems having linear parametric uncertainties. A distinction is made between systems with relative degree one and those with higher relative degrees. For systems with relative degree larger than one, a new direct adaptive control scheme is presented which is considerably simpler than the augmented error scheme suggested previously by Narendra, Lin, and Valavani for linear systems and by Sastry and Isidori for nonlinear systems. Contrary to the augmented error scheme, however, in the absence of unmodeled dynamics this scheme results in small rather than zero tracking errors. But, in the presence of unmodeled dynamics the scheme is shown to be robust for systems with relative degree one and two, while the augmented error scheme is not known to be robust.

Acknowledgements

I would like to express my deepest gratitude to my adviser Professor Shankar Sastry for his guidance, support, encouragements, and friendship throughout the course of my studies at Berkeley.

I am very grateful to the members of my qualifying examination committee, Professor Charles A. Desoer, Professor Charles C. Pugh, and Professor Felix F. Wu, and to Professor Lawrence M. Grossman. I am particularly grateful to Professor Desoer for being the chairman of my committee, for reviewing this dissertation, and his enthusiasm and advice which he has given freely, to Professor Pugh for his review of the dissertation and his excellent teaching of mathematics, and to Professor Lawrence M. Grossman for his advising and encouragements particularly in my first year in Berkeley.

I also wish to express my appreciation to the sources of funding for this research - Joint Services Electronics Program under Contract F49620-87-C-0041, and the Army Research Office under Grant DAAL-03-88-0106.

During my stay at Berkeley, I have enjoyed the friendship of many of my fellow graduate students. They have been a major part of my life in the past few years and I have learned immensely from my discussions and interactions with them. I would like to thank all of them, particularly Homayoon Ansari, Farhad Ayrom, Erwei Bai, Mesut Baran, Hamid Bahadori, Arlene Cole, Curt Deno, Li-Chen Fu, Nazli Gundes, John Hauser, Greg Heinzinger, Ping Hsu, Paul Jacobs, Guntekin Kabuli, Zexiang Li, Jeff Mason, Bob Minnichelli, Omer Morgul, Richard Murray, Andreas Neyer, Niklas Nordstrom, Andy Packard, Brad Paden, Kris Pister, Pietro Perona, Omid Razavi, Shahram Shahruz, Shobana Venkatraman, and Hormoz Yaghutiel. I am also grateful to Wendy James, Celeste Lane, and Genevieve Thiebaut of the EECS department for their help on many occasions.

Finally, I am very thankful to my family - to my parents who have always been a source of inspiration and strength for me and whose boundless generosity and care has endured despite their distance and difficulties, and to Ramin and Sharon, Kaveh and Soheila, and especially Behzad for their affection and support over the years.

Contents

List of Symbols

$\mathbb{R}$	real line
$\mathbb{R}^n$	Euclidean n-space
$\mathbb{C}^o_-$, $\mathbb{C}^o_+$	open left half (resp. right half) of the complex plane
$\sigma(A)$	spectrum of the matrix A
$\sigma_{max}(A)$	maximum singular value of the matrix A
$\sigma_{min}(A)$	minimum singular value of the matrix A
C^r	differentiability class of r times differentiablly continuous
sgn(x)	function taking the value 1 for $x \geq 0$ and -1 for $x<0$
T_xU	tangent space to U at x
T_x^*U	dual space to the tangent space to U at x
D_xf	Jacobian of the vector field f with respect to x
$D_x^k\lambda$	kth derivative of the function λ with respect to x
$D^k_{x_1,..,x_k}\lambda$	derivative of the function λ with respect to x_1 through x_k
$d\lambda$	differential of the function λ
$d\lambda \cdot f$	directional derivative of λ with respect to the vector field f
$L_f\lambda$, L_fg	Lie derivative of function λ (resp. vector field g) with respect to vector field f
$\alpha \circ \lambda$	the composition of the functions α and λ
G^c	complement of the set G
Re(z)	real part of the complex number z
$\vert\ \vert$	norm
H^n_k	space of homogeneous polynomials of degree k in $\mathbb{R}^n$
H_k	H^n_k when the value of n is understood

Introduction

We study two important problems in connection with controller design for single input-single output (SISO) nonlinear systems, namely: local stabilization of an equilibrium point by a smooth state feedback control law, and stable output tracking of a desired trajectory via state feedback.

In connection to the first problem, it is well known that local stabilization of a nonlinear control system is immediate through a linear state feedback design based on the linearized system about an equilibrium point, provided the linearized system has all unstable modes in its controllable subspace. In particular, if the linearized system has uncontrollable modes in the right half plane, then the nonlinear system is not stabilizable by any smooth feedback. If, on the other hand, the uncontrollable modes are on the $j\omega$-axis, the linear information is inconclusive regarding the stabilizability of the nonlinear system. In this case, it may be possible to construct feedback controls of higher order than linear in the state variables which stabilize the nonlinear system. We will refer to systems whose linearizations have uncontrollable $j\omega$-axis eigenvalues as systems with *degenerate* linearizations. Thus in connection with the local stabilizability of nonlinear systems the only nontrivial cases are those of systems with degenerate linearizations. In such cases one is interested to know under what conditions the nonlinear system can be stabilized and given that those conditions are satisfied, how the actual feedback control law is constructed. These questions were partially answered in a recent paper of Aeyels [3] where he considered the case in which the degeneracy is of the Hopf type; i.e. the degenerate part consists of a pair of imaginary eigenvalues. Using the center manifold theory he reduced the stability problem of the original system to that of a two dimensional system on a center manifold of the system. On the center manifold system he used the results of Chow and Mallet-Paret, and Hassard and Wan (which are reported in Guckenheimer and Holmes [15]) to find conditions for stability. Taking a different approach Abed and Fu [1,2] also considered the Hopf degeneracy in addition to the case of a single zero eigenvalue. Their approach is based on the computations of bifurcation formulae using the projection method.

In this study, we will generalize the approach taken by Aeyels through a systematic use of the normal form theory of differential equations in order to find explicit stability conditions on the center manifold. These conditions on the normal form vector fields are then translated to explicit conditions on the original vector fields of the system which may be checked easily. Using this methodology, three new cases of degeneracies will be treated, namely: the cases of double zero eigenvalues with nonzero Jordan block, a pair of imaginary and single zero eigenvalues, and two pairs of imaginary eigenvalues. Stability conditions are derived through standard Lyapunov type arguments for the normal form vector fields on the center manifold.

The second problem that we will address in this thesis concerns the design of a control law which causes the output of a nonlinear system to follow a desired trajectory. This problem is now a classical one in linear control theory and has been studied from several viewpoints (see for example Callier and Desoer [10] for an algebraic approach, and Wonham [38] for a geometric approach). The extensive research effort of the past decade in nonlinear systems has had a profound effect on the solvability of this problem and a host of other problems for nonlinear control systems. In particular the works of Jakubczyk and

Respondek [25], Hunt, Su, and Meyer [21], and Su [34] in connection with state space linearization of nonlinear systems, allowed the application of all the state feedback techniques in linear theory to fully linearizable nonlinear systems. For the purpose of designing a tracking control law, however, full linearizability of the system is not necessary. It is, on the other hand, well known that the solvability of the tracking problem is closely related to the invertibility of the input-output map of the system. The invertibility problem was studied by Brocket and Mesarovic [8] and Silverman [33] in the context of linear systems, and by Hirschorn [19], Isidori et al [24], Claude [13], and Nijmeijer [30] in nonlinear systems. In his work, Hirschorn introduced some regularity conditions under which the techniques of Silverman for linear systems can be generalized to nonlinear systems. These conditions turn out to be less restrictive than those required for the state space linearization techniques. This is expected since the invertibility of the input-output map merely implies the input-output linearization of a nonlinear system, and not the state-space linearization of the system, as demonstrated by Byrnes and Isidori [9]. Similar developments in this area have been reported for discrete-time nonlinear systems as well. In particular Monaco and Normand-Cyrot [28] developed the discrete version of the concepts introduced in [24] in connection to the state-space linearization of a nonlinear system. In addition, there has been recent contributions by Jakubczyk [26] in relation to state-space linearization and by Nijmeijer [31] in input-output decoupling of nonlinear discrete-time systems.

As indicated above, the input-output linearization of a nonlinear system is based on the invertibility of the input-output map, and therefore it relies on exact cancellation of nonlinear terms. This presents a limitation to the theory in that in the presence of modeling errors and unmodeled dynamics exact cancellation is not possible. In this study we will consider those systems whose dynamics are not exactly known and/or contain dynamics which are neglected, possibly purposely in order to simplify the controller design task. We will identify two classes of uncertainties which will be dealt with: those uncertainties which satisfy a generalized matching condition, and those arising from linear parametric uncertainties in the model vector fields. For the first class of uncertainties we will employ high gain and sliding mode control strategies and for the second class of uncertainties we will use adaptive control techniques. Later we will examine the robustness of the high gain and adaptive schemes to unmodeled dynamics and make conclusions about the applicability of these techniques in a practical situation.

The organization of the thesis is as follows. In Chapter 1 we will briefly review the mathematical tools which will be needed. In Chapter 2 we will present the stabilizability results for systems with degenerate linearizations. Chapter 3 presents the basic output tracking theory for nonlinear systems along with the various schemes for dealing with modeling uncertainties in the vector fields. In Chapter 4 we will introduce the class of unmodeled dynamics as the dynamics evolving on a much faster time scale than the dominant dynamics of the system. We will use the standard singular perturbation model in order to represent the unmodeled dynamics. In this setting we will investigate the robustness properties of the high gain and adaptive control schemes with respect to the unmodeled dynamics. Finally in Chapter 5 we will give some concluding remarks along with directions for further research. In particular we will present a simple example in order to demonstrate some of the open problems in the linearization theory for nonlinear systems.

Chapter One
Preliminaries

In this chapter we review some basic definitions and analysis tools which will be used in subsequent chapters. We will first give some elementary definitions from differential geometry which set the notation as well. Next, we will review some basic facts from center manifold theory and normal form theory for differential equations, in the context that we need them here. The center manifold theorems are taken from Carr [11] and the normal form theorems from Guckenheimer and Holmes [15].

1. Some Basic Definitions

Let U be an open subset of $\mathbb{R}^n$. We will assume that the reader is familiar with the concept of a tangent space to U at a point $x \in U$, denoted by T_xU (see e.g. Boothby [7]).

Definition 1.1.1: A C^r vector field, f, on U is a C^r mapping which assigns to each point $x \in U$ a tangent vector $f(x) \in T_xU$.

□

Denoting the cotangent space at a point $x \in U$, the dual space to T_xU, by T_x^*U, we can define a covector field or a one-form in a similar fashion.

Definition 1.1.2: A C^r one-form, ω, on U is a C^r mapping which assigns to each point $x \in U$ a covector $\omega(x) \in T_x^*U$.

□

Now, let λ be a C^{r+1} function from U into $\mathbb{R}$. Then we define the differential of λ, denoted by $d\lambda$, to be the one-form:

$$d\lambda(f)|_x := \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i \,|_x \tag{1.1.1}$$

where f is any vector field on U. The right hand side of (1.1.1) is sometimes written as $d\lambda \bullet f(x)$. A one-form which is the differential of a function is referred to as an *exact* one-form. The expression in (1.1.1) also defines what is known as the "Lie derivative" of λ with respect to f, denoted by $L_f \lambda$, which is the directional derivative of λ along the vector field f. Note that $L_f \lambda$ is a function on U. Thus, if g is any other vector field on U we can define $L_g L_f \lambda$ as the Lie derivative of $L_f \lambda$ with respect to g. We can, therefore, compute successive Lie derivatives as permitted by the degree of differentiability of the function and the vector fields. Next, we define the Lie derivative of a vector field g with respect to a second vector field f.

Definition 1.1.3: Let f and g be vector fields defined on U of class C^r. Then the Lie derivative of g with respect to f, denoted by $L_f g$, is a C^{r-1} vector field on U which is uniquely determined by:

$$d\lambda \bullet L_f\, g(x) := d(d\lambda \bullet g) \bullet f(x) - d(d\lambda \bullet f) \bullet g(x) \tag{1.1.2}$$

where λ is any C^∞ function on U. $L_f\, g$ is sometimes denoted by $ad_f\, g$ which is the notation we will use in connection with the normal form theory of differential equations. It is easy to check that $L_f\, g$ is bilinear over $\mathbb{R}$, is skew commutative, and satisfies the Jacobi identity:

$$L_q L_f\, g = L_f\, L_q\, g - L_g L_q\, f \tag{1.1.3}$$

Next, we will present some results from center manifold theory.

1.1. Center Manifold Theory

Consider the following C^k dynamical system in $\mathbb{R}^n$:

$$\dot{x} = f(x) \tag{1.2.1}$$

A set $S \in \mathbb{R}^n$ is said to be a *local invariant set* if for all $x_0 \in S$ there exists $T>0$ such that the solution of the differential equation (1.2.1) passing through x_0 at $t=0$ remains in S for $|t|<T$. If T can be chosen to be ∞, then S is said to be an *invariant set.*

Now consider the following C^k dynamical system in $\mathbb{R}^{n+m}$:

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \qquad && x \in \mathbb{R}^n \\ \dot{y} &= By + g(x,y) \qquad && y \in \mathbb{R}^m \end{aligned} \tag{1.2.2}$$

where $(x=0,y=0)$ is an equilibrium point, that is:

$$f(0,0)=0 \quad ; g(0,0)=0 \tag{1.2.3}$$

Further f and g comprise only of quadratic and higher order terms , that is:

$$D_x f(0,0)=0 \quad ; D_y f(0,0)=0 \quad ; D_x g(0,0)=0 \quad ; D_y g(0,0)=0 \tag{1.2.4}$$

We also assume that $\sigma(B) \subset \mathbb{C}_-^o$ and $\sigma(A) \subset \{j\omega \mid \omega \in \mathbb{R}\}$. For this system we have:

Definition 1.2.1: A local invariant manifold M for the system (1.2.2) is called a *center manifold* if it contains the origin $(x=0,y=0)$ and is tangent to $y=0$ at the origin.

□

Remarks

1) $\{(x,0) \mid x \in R^n\}$ is the generalized eigenspace of the $j\omega$-axis eigenvalues of the linearization of the system (1.2.2). Thus a center manifold is a "nonlinear eigenspace" corresponding to the $j\omega$-axis eigenvalues.

2) If M is given locally as the graph of a function $y = h(x)$, then:

$$h(0) = 0$$

$$Dh(0) = 0$$

It is a basic theorem that center manifolds exist (though elementary examples show that they are not unique) and are locally given as the graph of a function $y = h(x)$.

Theorem 1.2.1: (Existence of Center Manifolds) If f and g in (1.2.2) are C^k vector fields for $k \geq 2$, then there exists a center manifold $y = h(x)$, $|x| < \varepsilon$, where h is of class C^k.

□

The flow on the center manifold is governed by

$$\dot{u} = Au + f(u, h(u)) \tag{1.2.5}$$

The following theorem connects the stability of the system (1.2.5) to that of the system (1.2.2).

Theorem 1.2.2: If the zero solution of (1.2.5) is stable (unstable, asymptotically stable), then the zero solution of (1.2.2) is stable (unstable, asymptotically stable).

□

Remark

In the instance that the zero solution of (1.2.5) is stable or asymptotically stable, we can relate the solutions of (1.2.5) to those of (1.2.2) for $(x(0), y(0))$ sufficiently small. Let $(x(t), y(t))$ be a solution of (1.2.2) with $(x(0), y(0))$ small enough. Then there exists a solution $u(t)$ of (1.2.5) such that:

$$x(t) = u(t) + O(e^{-\gamma t})$$

$$y(t) = h(u(t)) + O(e^{-\gamma t}) \qquad \text{as } t \to \infty$$

where the rate of convergence to the center manifold, γ, is related to the eigenvalues of B alone.

Thus we see that the study of stability (instability) of the system (1.2.2) may be reduced to the study of stability (instability) of (1.2.5), provided we have an expression for the function h. To solve for $h(x)$, we use the fact that $y = h(x)$ is invariant under the flow of (1.2.2), thus:

$$\dot{y} = \frac{d}{dt}h(x) = Dh(x)[Ax + f(x,h(x))]$$

$$= Bh(x) + g(x,h(x))$$

that is h satisfies the partial differential equation (PDE):

$$Dh(x)[Ax + f(x,h(x))] = Bh(x) + g(x,h(x)) \tag{1.2.6}$$

with $h(0) = 0$; $Dh(0) = 0$.

Any solution of the PDE (1.2.6) is a center manifold for (1.2.2). Typically, it is difficult to solve the PDE (1.2.6), consequently the following approximation theorem is of interest.

Theorem 1.2.3: Let ϕ be a C^1 mapping from a neighborhood of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that:

$$\phi(0) = 0 \quad ; D\phi(0) = 0$$

if ϕ satisfies the PDE (1.2.6) modulo terms of $O(|x|^k)$ then there exists a center manifold of (1.2.2) given by the graph of $y = h(x)$ such that as $x \to 0$, we have:

$$|h(x) - \phi(x)| = O(|x|^k)$$

□

Remark

In particular, Theorem 1.2.3 allows us to approximate $h(x)$ by polynomials in x to any desired accuracy.

Theorems 1.2.1, 1.2.2, and 1.2.3 allow us to explicitly compute an approximation of a center manifold system (1.2.5), whose stability properties coincide with those of the original system (1.2.2). Since the linear part of the vector field in (1.2.5) has all its eigenvalues on the $j\omega$-axis, we need to study the higher order terms of the vector field in order to determine the stability of the system. This is done next in a systematic way.

1.3. Normal Forms

To study the behavior of the solutions on the center manifold it is helpful to simplify the vector field but the simplifications should preserve the qualitative behavior of the solutions at least locally around the equilibrium point. In the following discussion a systematic procedure of simplifying the vector fields by means of repeated coordinate transformations is presented. The resulting simplified vector fields are called *normal forms*.

Define H_k to be the real vector space of vector fields whose coefficients are homogeneous polynomials of degree k. Given a linear vector field $L(x)$ we have the subspace:

$$ad\, L\, (H_k) := \{\, ad_L h(x) \mid h(x) \in H_k \,\}$$

and its complement G_k; i.e.,

$$H_k = ad\, L\, (H_k) \oplus G_k \tag{1.3.1}$$

Theorem 1.3.1: Let $\dot{x} = f(x)$ be a C^r dynamical system with $f(0) = 0$ and $Df(0)x = L(x)$. Then there exists an analytic change of coordinates in a neighborhood of the origin transforming the system to $\dot{y} = g(y)$ such that:

$$g(y) = g^1(y) + g^2(y) + \cdots + g^r(y) + R_r \tag{1.3.2}$$

where $g^1(y) = L(y)$; $g^k(y) \in G^k$, $2 \le k \le r$ and $R_r = o(|y|^r)$.

□

Proof: It suffices to show that for a given $k \geq 2$ the components of $ad\, L\, (H_k)$ can be locally removed from the vector field by an analytic change of coordinates. Performing this for $k=2,\ldots,r$ we obtain the desired coordinate transformation as the composition of the transformations for each k. Thus we let:

$$x = y + P(y)$$

where $P(y)$ is a homogeneous polynomial of degree k. We point out that $D_y\, x(0) = I$ so that we have a local diffeomorphism (thus preserving the local behavior of the flow of the vector field around the origin). Using this transformation, we get:

$$\dot{y} = (I+DP(y))^{-1}[f(y)+Df(y)P(y)+o(|y|^r)] \tag{1.3.3}$$

Now note that:

$$(I+DP(y))^{-1} = I-DP(y)+o(|y|^r) \tag{1.3.4}$$

and

$$\begin{aligned} Df(y)P(y) &= Df(0)P(y) + o(|y|^r) \\ &= DL\, P(y) + o(|y|^r) \end{aligned} \tag{1.3.5}$$

Using (1.3.4) and (1.3.5) in (1.3.3) we have:

$$\dot{y} = f(y) + DL\, P(y) - DP(y)\, L + o(|y|^r) \tag{1.3.6}$$

If we set:

$$f(y) = f_{k-1}(y) + f^{\,k}(y) + o(|y|^r)$$

with $f_{k-1}(y) = \{f^{\,j}(y) \mid j=1,\ldots,k-1\}$ and $f^k(y) \in H_k$ we get:

$$g(y) = g_{k-1}(y) + g^k(y)+o(|y|^r) \tag{1.3.7}$$

where:

$$g_{k-1}(y) = f_{k-1}(y) \tag{1.3.8}$$

$$g^k(y) = f^k(y) + ad\, L\, (P(y)) \tag{1.3.9}$$

By choosing the coefficients of $P(y)$, components of $ad\, L\, (H_k)$ can be removed from $f^k(y)$ while all lower order terms remain unchanged. Thus $g^k(y) \in G_k$.

□

Although the transformations for each k leave all lower order terms unchanged, they do alter the higher order terms. We have the following corollary to this effect.

Corollary 1.3.1: Let $\dot{x} = f(x)$ be a C^r dynamical system, such that $f^j(x) \in G_j$ for $j=2,...,k-1$, $k<r$. Let $\dot{y} = g(y)$ represent the transformed system after the removal of $O(k)$ terms in the span of $ad\, L\, (H_k)$. Then we have:

$$g^{j+k}(y) = f^{j+k}(y) + ad\, f^{j+1}(y)(P(y)) \qquad j=0,...,k-2 \tag{1.3.10}$$

$$g^{2k-1}(y) = f^{2k-1}(y) + ad\, f^k(y)(P(y)) - DP(y)[ad\, L(P(y))] \tag{1.3.11}$$

where (1.3.10) and (1.3.11) make sense provided $j+k \le r$ and $2k-1 \le r$ respectively.

□

Proof: Using the change of coordinates $x = y+P(y)$ we have:

$$\dot{y} = (I+DP(y))^{-1} f(y+P(y)) \tag{1.3.12}$$

We note:

$$(I+DP(y))^{-1} = I - DP(y) + (DP(y))^2 + O(2k-2) \tag{1.3.13}$$

$$f(y+P(y)) = f(y) + Df(y)P(y) + O(2k) \tag{1.3.14}$$

Now using (1.3.13) and (1.3.14) in (1.3.12) we get:

$$\dot{y} = f(y) + DfP - DPf - DP[DfP - DPf] + O(2k) \tag{1.3.15}$$

Collecting the $O(l)$ terms for various values of l we obtain (1.3.10) and (1.3.11) for $g(y)$.

□

Chapter Two

Stabilization of Nonlinear Systems with Degenerate Linearizations

2.1. Introduction

In this chapter we discuss the stabilization of nonlinear systems with degenerate linearizations, i.e. systems whose linearizations are not stabilizable by linear state feedback.

We consider systems of the form:

$$\dot{\xi}=\phi(\xi)+bu \tag{2.1.1}$$

where $\xi\in\mathbb{R}^n$, $u\in\mathbb{R}$, $\phi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is smooth, $b\in\mathbb{R}^n$, and 0 is an equilibrium point of the undriven system (2.1.1) i.e. $\phi(0) = 0$. The extension to systems of the form $\dot{\xi} = \phi(\xi) + \psi(\xi)\, u$ is straightforward and will be discussed in Section 2.6.

By way of notation, let $A = D_\xi\phi(0)$, the Jacobian of ϕ at $\xi = 0$. We partition the spectrum of A as:

$$\sigma(A) = \sigma^s \cup \sigma^u \cup \sigma^c$$

where $\sigma^s \subset \mathbb{C}^o_-$, $\sigma^u \subset \mathbb{C}^o_+$, and $\sigma^c \subset \{j\omega \mid \omega\in\mathbb{R}\}$. Using basis vectors for the (generalized) eigenspaces of σ^s,σ^u, and σ^c we may transform (2.1.1) to the form:

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix}=\begin{bmatrix}\hat{A}_{11} & 0 & 0\\ 0 & \hat{A}_{22} & 0\\ 0 & 0 & \hat{A}_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}\hat{\phi}_1(\xi_1,\xi_2,\xi_3)\\ \hat{\phi}_2(\xi_1,\xi_2,\xi_3)\\ \hat{\phi}_3(\xi_1,\xi_2,\xi_3)\end{bmatrix}+\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix}u \tag{2.1.2}$$

where $\sigma(\hat{A}_{11})=\sigma^c$, $\sigma(\hat{A}_{22})=\sigma^s$, $\sigma(\hat{A}_{33})=\sigma^u$, $\hat{\xi}_1 \in \mathbb{R}^{n_1}$, $\hat{\xi}_2 \in \mathbb{R}^{n_2}$, and $\hat{\xi}_3 \in \mathbb{R}^{n_3}$.

It is easy to see that (2.1.2) is locally stabilizable by linear state feedback when $(\hat{A}_{11},\hat{b}_1)$, and $(\hat{A}_{33},\hat{b}_3)$ are completely controllable. It is also easy to see that if $(\hat{A}_{33},\hat{b}_3)$ is not controllable, then no feedback law which is smooth at the origin can stabilize the system (2.1.2). Consequently we shall be interested in the case when $(\hat{A}_{33},\hat{b}_3)$ is controllable, and the critical eigenvalues (those of $\hat{A}_{11}$) are completely uncontrollable, i.e. $\hat{b}_1 = 0$. Now our objective is to construct a feedback law $u = F(\hat{\xi}_1,\hat{\xi}_2,\hat{\xi}_3)$ to stabilize the system. From the preceding discussion it is plausible that higher order (quadratic, cubic, etc) terms in $\hat{\xi}_1$ are needed to stabilize the system. By choosing u of the form $u = v + K_3\hat{\xi}_3$ such that $\sigma(\hat{A}_{33}+\hat{b}_3K_3) \subset \mathbb{C}^o_-$, we may rewrite (2.1.2), after a diagonalizing transformation, in the following form:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \phi_1(\xi_1,\xi_2) \\ \phi_2(\xi_1,\xi_2) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} v \qquad (2.1.3)$$

where $\xi_1 = \hat{\xi}_1$, $A_{11} = \hat{A}_{11}$, and $\sigma(A_{22}) \subset \mathbb{C}^o_-$. Then we must find $v = F(\xi_1,\xi_2)$, an analytic feedback law, such that the equilibrium point of (2.1.3) is asymptotically stable. It follows that when we set $v = F(\xi_1,\xi_2)$ with A_{22} stable, that any center manifold of (2.1.3) is tangent to $\{(\xi_1,0) \mid \xi_1 \in \mathbb{R}^{n_1}\}$ and is given locally by $\xi_2 = h(\xi_1)$. Further from (1.2.6) it follows that $h(\cdot)$ satisfies the following PDE:

$$Dh(\xi_1)[A_{11}\xi_1 + \phi_1(\xi_1,h)] = A_{22}h + \phi_2(\xi_1,h) + bF(\xi_1,h) \qquad (2.1.4)$$

and the flow on the center manifold is governed by:

$$\dot{\xi}_1 = A_{11}\xi_1 + \phi_1(\xi_1,h) \qquad (2.1.5)$$

Thus, we need to choose $F(\xi_1,\xi_2)$ in such a way that the resulting h produces an asymptotically stable equilibrium point on the center manifold. While a general solution is not available to this problem we consider several cases for the matrix A_{11}. The case where $A_{11} \in \mathbb{R}^{2\times 2}$ and has a pair of imaginary eigenvalues was considered by D. Aeyels [3]. In [1], Abed and Fu treat the same case using bifurcation formulae derived from the projection method. The same technique is also employed in [2] where, the case of a single critical mode is treated. The cases covered here, which have not been treated by Aeyels or Abed and Fu, are the following:

(i) Double zero eigenvalues with nonzero Jordan form.

(ii) Pair of imaginary and a simple zero eigenvalue.

(iii) Two pairs of imaginary eigenvalues.

In the next three sections we will examine each of the above cases individually. We will assume, for simplicity, that $\xi_2 \in \mathbb{R}$. We will show in Section 2.6, by way of an example, that there is no loss of generality in this assumption. In Section 2.5 we will investigate the robustness of the control laws presented in Sections 2.2 through 2.4 with respect to perturbations in the vector fields. We will end the chapter with Section 2.6 where we will give some illustrative examples, and discuss our assumptions.

2.2. Case of Double Zero Eigenvalues

We consider here the case where $A_{11} \in \mathbb{R}^{2\times 2}$ and has the form:

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We let $[x,y]^T$ represent ξ_1 and we will drop the subscript from ξ_2 and represent it by ξ. We further let:

$$\phi_1(\xi_1,\xi) = [f(x,y,\xi),g(x,y,\xi)]^T$$

Now rewriting (2.1.3) with the above notation we get:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}f(x,y,\xi)\\ g(x,y,\xi)\end{bmatrix} \tag{2.2.1}$$

$$\dot{\xi} = -k\xi + \phi_2(x,y,\xi) + v$$

where $k > 0$. Since we will choose v to be of the form $F(x,y,\xi)$, we can assume that $\phi_2(x,y,\xi) = 0$. The center manifold is given locally by $\xi = h(x,y)$ with h satisfying:

$$Dh(x,y)\begin{bmatrix}y + f(x,y,h(x,y))\\ g(x,y,h(x,y))\end{bmatrix} = -kh(x,y) + F(x,y,h(x,y)) \quad ; h(0) = 0 \; ; Dh(0) = 0 \tag{2.2.2}$$

We now use Theorem 1.2.3 to approximate the center manifold up to terms of $O(3)$, that is:

$$h(x,y) = ax^2+bxy+cy^2 + O(3) \tag{2.2.3}$$

Note that the choice of h in (2.2.3) automatically gives $h(0,0) = 0$; $Dh(0,0) = 0$. Next we choose F to be of the form:

$$F(x,y,\xi) = \alpha x^2+\beta xy+\gamma y^2 \tag{2.2.4}$$

Using (2.2.3) and (2.2.4) in (2.2.2) we get:

$$(2ax+by)(y+f(x,y,h)) + (2cy+bx)g(x,y,h) = -kax^2-kbxy-kcy^2+\alpha x^2+\beta xy+\gamma y^2 + O(3) \tag{2.2.5}$$

Recalling that f and g are both of $O(2)$, we may equate terms of $O(2)$ in (2.2.5) to get:

$$\begin{bmatrix}k & 0 & 0\\ 2 & k & 0\\ 0 & 1 & k\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \tag{2.2.6}$$

For $k \neq 0$, we see from (2.2.6) that (a,b,c) can be arbitrarily assigned by a choice of (α,β,γ) in the control law (2.2.4). In other words, the control law determines a center manifold up to terms of $O(3)$. The remaining problem is to determine what choice of the parameters (a,b,c) in (2.2.3) stabilizes the flow on the center manifold, which is given by:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}f(x,y,h(x,y))\\ g(x,y,h(x,y))\end{bmatrix} \tag{2.2.7}$$

This program is continued using the normal form theory of Section 1.3 in the following theorem, where by a slight abuse of notation we let $f(x,y) = f(x,y,h(x,y))$ and $g(x,y) = g(x,y,h(x,y))$. Further in the sequel we use $D_x^k f$ to denote the kth partial derivative of f with respect to x.

Theorem 2.2.1: A necessary condition for the zero solution of the center manifold system (2.2.7) to be: stabilizable is:

$$\begin{aligned} &D_x^2 g = 0 \\ &D_{xy}^2 g + D_x^2 f = 0 \end{aligned} \tag{2.2.8}$$

Furthermore if (2.2.8) is satisfied, then the zero solution of (2.2.7) is locally asymptotically stable provided that:

$$\begin{aligned} &\frac{1}{3} D_x^3 g + (D_x^2 f)^2 < 0 \\ &D_{xxy}^3 g + D_x^3 f - D_x^2 f (D_{xy}^2 f + D_y^2 g) < 0 \end{aligned} \tag{2.2.9}$$

where all the derivatives above are evaluated at the origin.

□

Proof: For the vector fields in $\mathbb{R}^2$,

$$H_2 = span\left\{ \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \end{bmatrix} \begin{bmatrix} xy \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 \end{bmatrix} \begin{bmatrix} 0 \\ y^2 \end{bmatrix} \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\}$$

Further for the system in (2.2.7), $L(x,y) = [y,0]^T$. Then:

$$ad\, L\, (H_2) = \text{span}\left\{ \begin{bmatrix} -xy \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \end{bmatrix} \begin{bmatrix} x^2 \\ -2xy \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \end{bmatrix} \begin{bmatrix} xy \\ -y^2 \end{bmatrix} \right\}$$

Thus a complement to $ad\, L\, (H_2)$ is given by:

$$G_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ x^2 \end{bmatrix} \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\}$$

And, the normal form of (2.2.7) can be written as:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \delta x^2 + \varepsilon xy\end{aligned} \quad +O(3) \tag{2.2.10}$$

where $\delta = \tfrac{1}{2} D_x^2 g$ and $\varepsilon = D_{xy}^2 g + D_x^2 f$. It is easy to see that the zero solution of (2.2.10) is unstable for all nonzero values of δ and ε. Thus a necessary condition for stabilization is (2.2.8). Further with (2.2.8) holding, we may consider the $O(3)$ terms in the expansion of the vector field in (2.2.7). We have:

$$H_3 = \text{span}\left\{\begin{bmatrix}x^3\\0\end{bmatrix}\begin{bmatrix}x^2y\\0\end{bmatrix}\begin{bmatrix}xy^2\\0\end{bmatrix}\begin{bmatrix}y^3\\0\end{bmatrix},\begin{bmatrix}0\\x^3\end{bmatrix}\begin{bmatrix}0\\x^2y\end{bmatrix}\begin{bmatrix}0\\xy^2\end{bmatrix}\begin{bmatrix}0\\y^3\end{bmatrix}\right\}$$

Then:

$$ad\, L\, (H_3) = \text{span}\left\{\begin{bmatrix}3x^2y\\0\end{bmatrix}\begin{bmatrix}2xy^2\\0\end{bmatrix}\begin{bmatrix}y^3\\0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}x^3\\-3x^2y\end{bmatrix}\begin{bmatrix}x^2y\\2xy^2\end{bmatrix}\begin{bmatrix}xy^2\\-y^3\end{bmatrix}\begin{bmatrix}y^3\\0\end{bmatrix}\right\}$$

Therefore a complement to $ad\, L\, (H_3)$ is given by:

$$G_3 = \text{span}\left\{\begin{bmatrix}0\\x^3\end{bmatrix}\begin{bmatrix}0\\x^2y\end{bmatrix}\right\}$$

Thus we see that the normal form of (2.2.7) up to terms of $O(4)$ may be written as:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \lambda x^3 + \mu x^2 y\end{aligned} \quad + \quad O(4) \tag{2.2.11}$$

where: $\lambda = \dfrac{1}{6} D_x^3 g'$, and $\mu = \dfrac{1}{2}(D_{xxy}^3 g' + D_x^3 f')$. Here $[f',g']^T$ is the $O(3)$ vector field obtained from $(f,g)^T$ by removal of the $O(2)$ terms. Now using Corollary 1.3.1 to relate $[f',g']^T$ and $[f,g]^T$ we find for λ and μ:

$$\lambda = \frac{1}{6}D^3_x g + \frac{1}{2}(D^2_{xy}f)^2$$

$$\mu = \frac{1}{2}(D^3_{xxy}g + D^3_{xy}f) - \frac{1}{2}D^2_{xy}f\,(D^2_{xy} + D^2_y g)$$

Next consider the 3-jet of (2.2.11), that is the system obtained from (2.2.11) by neglecting the $O(4)$ terms. Takens [35] proved that local stability properties of this system coincide with those of (2.2.11) for nonzero values of λ and μ. Now using the following Lyapunov function candidate for the 3-jet of (2.2.11):

$$V = -\frac{1}{4}\lambda x^4 + \frac{1}{2}y^2$$

we have for $\dot{V}$:

$$\dot{V} = -\lambda x^3 y + \lambda x^3 y + \mu x^2 y^2$$

Thus for $\lambda < 0$ and $\mu < 0$, V is a positive definite function whose derivative $\dot{V}$ is negative semi definite. From LaSalle's theorem [37] it follows that the zero solution of the 3-jet is globally asymptotically stable, since the set $\Omega = \{\, x \mid \dot{V}=0 \,\}$ contains no nontrivial trajectories. Therefore for $\lambda < 0$ and $\mu < 0$, the equilibrium of (2.2.11) is locally asymptotically stable.

□

Corollary 2.2.1: The zero solution of (2.2.1) is stabilizable by a control of the form $v = \alpha x^2+\beta xy+\gamma y^2$, provided (2.2.8) is satisfied and $D^2_{x\xi}g \neq 0$.

□

Proof: By inspection, if $D^2_{x\xi}g \neq 0$, then the values of $D^3_x g$ and $D^3_{xxy}g$ can be assigned arbitrarily by a proper choice of (a,b,c) (and thus by (α,β,γ)). Thus through the control v the parameters λ and μ can be made negative.

□

2.3. Case of a Pair of Imaginary and a Simple Zero Eigenvalues

We consider here the case where $A_{11} \in \mathbb{R}^{3\times3}$ and is of the form:

$$A_{11} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case (2.1.3) may be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f(x,y,z,\xi) \\ g(x,y,z,\xi) \\ p(x,y,z,\xi) \end{bmatrix} \tag{2.3.1}$$

$$\dot{\xi} = -k\xi + v \qquad\qquad k > 0$$

The center manifold is represented by $\xi = h(x,y,z)$. Letting $v = F(x,y,z,\xi)$, we get that h satisfies the following:

$$Dh(x,y,z) \begin{bmatrix} -y+f(x,y,z,h(x,y,z)) \\ x+g(x,y,z,h(x,y,z)) \\ p(x,y,z,h(x,y,z)) \end{bmatrix} = -kh(x,y,z) + F(x,y,z,h(x,y,z)) \tag{2.3.2}$$

$$h(0) = 0 \; ; \; Dh(0) = 0$$

As before, using Theorem 1.2.3, we approximate the center manifold up to terms of $O(3)$, that is:

$$h(x,y,z) = ax^2+by^2+cz^2+dxy+exz+lyz + O(3) \tag{2.3.3}$$

Next we choose the following form for the feedback law F:

$$F(x,y,z,\xi) = \alpha x^2+\beta y^2+\gamma z^2+\sigma xy+\eta xz+\mu yz \tag{2.3.4}$$

Using (2.3.3) and (2.3.4) in (2.3.2) and equating the coefficients of the $O(2)$ terms we get:

$$\begin{bmatrix} k & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & -1 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 \\ -2 & 2 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 1 \\ 0 & 0 & 0 & 0 & -1 & k \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ l \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \sigma \\ \eta \\ \mu \end{bmatrix} \tag{2.3.5}$$

For $k \neq 0$, (2.3.5) implies that $[a,b,...]$ can be arbitrarily assigned by a choice of the control parameters $[\alpha,\beta,...]$. Thus the control law determines a center manifold up to $O(3)$ terms. Next we wish to determine what choice of $[a,b,...]$ renders an asymptotically stable equilibrium point for the center manifold system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f(x,y,z,h(x,y,z)) \\ g(x,y,z,h(x,y,z)) \\ p(x,y,z,h(x,y,z)) \end{bmatrix} \tag{2.3.6}$$

This is done in the next theorem, where as before $[f',g',p']^T$ denotes the vector field obtained from $[f,g,p]^T$ after removal of the $O(2)$ terms.

Theorem 2.3.1: The zero solution of (2.3.6) is not stabilizable unless:

$$D^2_{xz}f + D^2_{yz}g = 0 \tag{2.3.7a}$$

$$D^2_{x}p + D^2_{y}p = 0 \tag{2.3.7b}$$

$$D^2_{z}p = 0 \tag{2.3.7c}$$

Furthermore if (2.3.7) is satisfied, then the zero solution of (2.3.6) is asymptotically stable provided that:

$$D^3_{x}f' + D^3_{y}g' + D^3_{xyy}f' + D^3_{xxy}g' < 0 \tag{2.3.8a}$$

$$D^3_{z}p' < 0 \tag{2.3.8b}$$

$$D^3_{xxz}f' + D^3_{yzz}g' < 0 \tag{2.3.8c}$$

$$D^3_{xxz}p' + D^3_{yyz}p' < 0 \tag{2.3.8d}$$

Conditions (2.3.8c) and (2.3.8d) may be replaced by the single condition:

$$\text{sgn}(D^3_{xxz}f' + D^3_{yzz}g') = -\,\text{sgn}(D^3_{xxz}p' + D^3_{yyz}p') \qquad (2.3.8c)$$

□

Remark 2.3.1: We may use Corollary 1.3.1 to express (2.3.8) in terms of the vector field $[f,g,p]^T$. The resulting expressions, although extremely involved, would consist of the terms appearing in (2.3.8) plus additional terms involving various second order partial derivatives of $[f,g,p]^T$. The stabilizability conditions on $[f,g,p]^T$, however, can be determined from (2.3.8) alone, since the control can only affect the $O(3)$ terms in $[f,g,p]^T$.

Proof: In $\mathbb{R}^3$ we have:

$$H_2 = \text{span}\left\{ \begin{bmatrix} x^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} z^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} xz \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} yz \\ 0 \\ 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 \\ x^2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ z^2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xz \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ yz \\ 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xz \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ yz \end{bmatrix} \right\}$$

And for the system in (2.3.6):

$$L(x,y,z) = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

Thus:

$$ad\, L\, (H_2) = \text{span}\left\{ \begin{bmatrix} 2xy \\ x^2 \\ 0 \end{bmatrix} \begin{bmatrix} -2xy \\ y^2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ z^2 \\ 0 \end{bmatrix} \begin{bmatrix} y^2-x^2 \\ xy \\ 0 \end{bmatrix} \begin{bmatrix} yz \\ xz \\ 0 \end{bmatrix} \begin{bmatrix} -xz \\ yz \\ 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} -x^2 \\ 2xy \\ 0 \end{bmatrix} \begin{bmatrix} -y^2 \\ -2xy \\ 0 \end{bmatrix} \begin{bmatrix} -z^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -xy \\ y^2-x^2 \\ 0 \end{bmatrix} \begin{bmatrix} -xz \\ yz \\ 0 \end{bmatrix} \begin{bmatrix} -yz \\ -xz \\ 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ 2xy \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2xy \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^2-x^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ yz \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -zx \end{bmatrix} \right\}$$

It is easy to show that a complement to *ad L* (H_2) is given by:

$$G_2 = \text{span}\left\{\begin{bmatrix}xz\\yz\\0\end{bmatrix}\begin{bmatrix}yz\\-xz\\0\end{bmatrix}\begin{bmatrix}0\\0\\x^2+y^2\end{bmatrix}\begin{bmatrix}0\\0\\z^2\end{bmatrix}\right\}$$

Therefore the $O(2)$ normal form associated with (2.3.6) is of the form:

$$\begin{aligned}\dot{x} &= -y + \delta xz + \varepsilon yz \\ \dot{y} &= x + \delta yz - \varepsilon xz \qquad + O(3) \\ \dot{z} &= \lambda(x^2+y^2) + \rho z^2\end{aligned} \tag{2.3.9}$$

where, $\delta = \frac{1}{2}(D^2_{xz}f + D^2_{yz}g)$, $\varepsilon = \frac{1}{2}(D^2_{yz}f - D^2_{xz}g)$, $\lambda = \frac{1}{4}(D^2_{x}p + D^2_{y}p)$, and $\rho = \frac{1}{2}D^2_{z}p$. Now transforming the normal form in (2.3.9) to cylindrical coordinates we get:

$$\begin{aligned}\dot{r} &= \delta rz \qquad + O(|r,z|^3) \\ \dot{z} &= \lambda r^2 + \rho z^2 + O(|r,z|^3) \\ \dot{\theta} &= 1 \qquad + O(|r,z|^2)\end{aligned} \tag{2.3.10}$$

It is easy to show that the zero solution of (2.3.10) is not asymptotically stable for any nonzero values of δ, λ, and ρ. Therefore conditions (2.3.8) are necessary for stabilization. Note also that ε does not appear in (2.3.10).

Next, assuming (2.3.8) is satisfied, we consider the $O(3)$ terms in the expansion of the vector field in (2.3.6). We have:

$$H_3 = \text{span}\left\{\begin{bmatrix}x^3\\0\\0\end{bmatrix}\begin{bmatrix}y^3\\0\\0\end{bmatrix}\begin{bmatrix}z^3\\0\\0\end{bmatrix}\begin{bmatrix}x^2y\\0\\0\end{bmatrix}\begin{bmatrix}x^2z\\0\\0\end{bmatrix}\begin{bmatrix}y^2x\\0\\0\end{bmatrix}\begin{bmatrix}y^2z\\0\\0\end{bmatrix}\begin{bmatrix}z^2x\\0\\0\end{bmatrix}\begin{bmatrix}z^2y\\0\\0\end{bmatrix}\begin{bmatrix}xyz\\0\\0\end{bmatrix},\right.$$

$$\begin{bmatrix}0\\x^3\\0\end{bmatrix}\begin{bmatrix}0\\y^3\\0\end{bmatrix}\begin{bmatrix}0\\z^3\\0\end{bmatrix}\begin{bmatrix}0\\x^2y\\0\end{bmatrix}\begin{bmatrix}0\\x^2z\\0\end{bmatrix}\begin{bmatrix}0\\y^2x\\0\end{bmatrix}\begin{bmatrix}0\\y^2z\\0\end{bmatrix}\begin{bmatrix}0\\z^2x\\0\end{bmatrix}\begin{bmatrix}0\\z^2y\\0\end{bmatrix}\begin{bmatrix}0\\xyz\\0\end{bmatrix},$$

$$\left.\begin{bmatrix}0\\0\\x^3\end{bmatrix}\begin{bmatrix}0\\0\\y^3\end{bmatrix}\begin{bmatrix}0\\0\\z^3\end{bmatrix}\begin{bmatrix}0\\0\\x^2y\end{bmatrix}\begin{bmatrix}0\\0\\x^2z\end{bmatrix}\begin{bmatrix}0\\0\\y^2x\end{bmatrix}\begin{bmatrix}0\\0\\y^2z\end{bmatrix}\begin{bmatrix}0\\0\\z^2x\end{bmatrix}\begin{bmatrix}0\\0\\z^2y\end{bmatrix}\begin{bmatrix}0\\0\\xyz\end{bmatrix}\right\}$$

Then we get:

$$
ad\,L\,(H_3) = \text{span}\left\{ \begin{pmatrix} 3x^2y \\ x^3 \\ 0 \end{pmatrix} \begin{pmatrix} -3xy^2 \\ y^3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ z^3 \\ 0 \end{pmatrix} \begin{pmatrix} 2xy^2-x^3 \\ x^2y \\ 0 \end{pmatrix} \begin{pmatrix} 2xyz \\ x^2z \\ 0 \end{pmatrix} \right.
$$

$$
\begin{pmatrix} y^3-2x^2y \\ y^2x \\ 0 \end{pmatrix} \begin{pmatrix} -2xyz \\ y^2z \\ 0 \end{pmatrix} \begin{pmatrix} z^2y \\ z^2x \\ 0 \end{pmatrix} \begin{pmatrix} -z^2x \\ z^2y \\ 0 \end{pmatrix} \begin{pmatrix} y^2z-x^2z \\ xyz \\ 0 \end{pmatrix},
$$

$$
\begin{pmatrix} -x^3 \\ 3x^2y \\ 0 \end{pmatrix} \begin{pmatrix} -y^3 \\ -3xy^2 \\ 0 \end{pmatrix} \begin{pmatrix} -z^3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x^2y \\ 2xy^2-x^3 \\ 0 \end{pmatrix} \begin{pmatrix} -x^2z \\ 2xyz \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} -y^2x \\ y^3-2x^2y \\ 0 \end{pmatrix} \begin{pmatrix} -y^2z \\ -2xyz \\ 0 \end{pmatrix} \begin{pmatrix} -z^2x \\ z^2y \\ 0 \end{pmatrix} \begin{pmatrix} -z^2y \\ -z^2x \\ 0 \end{pmatrix} \begin{pmatrix} -xyz \\ y^2z-x^2z \\ 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 3x^2y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -3xy^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2xy^2-x^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2xyz \end{pmatrix}
$$

$$
\left. \begin{pmatrix} 0 \\ 0 \\ y^3-2x^2y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -2xyz \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z^2y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -z^2x \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y^2z-x^2z \end{pmatrix} \right\}
$$

It can be shown that a complement to $ad\,L\,(H_3)$ is given by:

$$
G_3 = \text{span}\left\{ \begin{pmatrix} x^3+xy^2 \\ y^3+x^2y \\ 0 \end{pmatrix} \begin{pmatrix} y^3+x^2y \\ -x^3-xy^2 \\ 0 \end{pmatrix} \begin{pmatrix} xz^2 \\ yz^2 \\ 0 \end{pmatrix} \begin{pmatrix} -yz^2 \\ xz^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^2z+y^2z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z^3 \end{pmatrix} \right\}
$$

Then the $O(3)$ normal form of (2.3.6) can be written in the form:

$$
\begin{aligned}
\dot{x} &= -y+\varepsilon yz+\alpha'(x^3+xy^2)+\beta'(y^3+x^2y)+\gamma' xz^2-\eta' yz^2 \\
\dot{y} &= x-\varepsilon xz+\alpha'(y^3+x^2y)-\beta'(x^3+xy^2)+\gamma' yz^2+\eta' xz^2 \; + \quad O(4) \\
\dot{z} &= \sigma'(x^2+y^2)z+\mu' z^3
\end{aligned}
\qquad (2.3.11)
$$

where with some algebra we may find:

$$\alpha' = \frac{1}{16}(D_x^3 f' + D_y^3 g' + D_{xyy}^3 f' + D_{xxy}^3 g')$$

$$\beta' = \frac{1}{16}(-D_x^3 g' + D_y^3 f' - D_{xyy}^3 g' + D_{xxy}^3 f')$$

$$\gamma' = \frac{1}{4}(D_{xzz}^3 f' + D_{yzz}^3 g')$$

$$\eta' = \frac{1}{4}(D_{xzz}^3 g' - D_{yzz}^3 f')$$

$$\sigma' = \frac{1}{4}(D_{xxz}^3 p' + D_{yyz}^3 p')$$

$$\mu' = \frac{1}{6}D_z^3 p'$$

Next transforming (2.3.11) into cylindrical coordinates we get:

$$\begin{aligned} \dot{r} &= \alpha' r^3 + \gamma' r z^2 + O(|r,z|^4) \\ \dot{z} &= \delta' r^2 z + \mu' z^3 + O(|r,z|^4) \\ \dot{\theta} &= 1 + O(|r,z|^2) \end{aligned} \tag{2.3.12}$$

Now we know that the local stability properties of (2.3.12) coincide with those of its 3-jet because the vector field in the 3-jet of (2.3.12) is a homogeneous polynomial vector field [17]. Therefore we only need to prove the theorem for the 3-jet of (2.3.12). To this end consider the following Lyapunov function candidate:

$$V = \frac{1}{2}Rr^2 + \frac{1}{2}Sz^2$$

where R and S are positive constants. Differentiating V we have:

$$\dot{V} = R\alpha' r^4 + R\gamma' r^2 z^2 + S\sigma' r^2 z^2 + S\mu' z^4$$

Therefore for $\alpha', \gamma', \sigma'$, and μ' all negative, or for α', μ' negative and $\mathrm{sgn}(\gamma') = -\mathrm{sgn}(\sigma')$, the equilibrium point of (2.3.12) is asymptotically stable.

□

Corollary 2.3.1: The zero solution of (2.3.1) is stabilizable by the control $v = \alpha x^2+\beta y^2+\gamma z^2+\sigma xy+\eta xz+\mu yz$ provided (2.3.7) is satisfied, $D^2_{z\xi}p \neq 0$, and either $D^2_{x\xi}f \neq 0$ and $D^2_{z\xi}g \neq 0$, or $D^2_{z\xi}f \neq 0$ and $D^2_{y\xi}g \neq 0$.

□

Proof: In view of the Remark 2.3.1, we see that with $D^2_{z\xi}p \neq 0$ (2.3.8b) and (2.3.8d) can be satisfied by making $D^3_z p$, and $D^3_{xxz}p$ or $D^3_{yyz}p$ arbitrarily negative with a proper choice of the parameters c , and a or b of the center manifold. Furthermore with $D^2_{x\xi}f \neq 0$ and $D^2_{z\xi}g \neq 0$, we see that (2.3.8a) and (2.3.8c) may be satisfied through the parameters a and l by making $D^3_x f$ and $D^3_{yzz}g$ arbitrarily negative. In addition (2.3.8a) and (2.3.8c) may be satisfied with $D^2_{z\xi}f \neq 0$ and $D^2_{y\xi}g \neq 0$ in a similar fashion.

□

2.4. Case of Two Pairs of Imaginary Eigenvalues

Here we consider the case where $A_{11} \in \mathbf{R}^{4\times4}$ and has the following form:

$$A_{11} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{bmatrix}$$

where we assume $\delta \notin \{\pm½,\pm1,\pm2,\pm3\}$. Rewritting (2.1.3) for this form of A_{11} we get:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} + \begin{bmatrix} f(x,y,z,w,\xi) \\ g(x,y,z,w,\xi) \\ p(x,y,z,w,\xi) \\ q(x,y,z,w,\xi) \end{bmatrix} \tag{2.4.1}$$

$$\dot{\xi} = -k\xi + v$$

Letting $v = F(x,y,z,w)$ and representing the center manifold by $\xi=h(x,y,z,w)$ we get:

$$Dh(x,y,z,w)\begin{bmatrix} -y+f(x,y,z,w,h(x,y,z,w)) \\ x+g(x,y,z,w,h(x,y,z,w)) \\ -\delta w+p(x,y,z,w,h(x,y,z,w)) \\ -\delta z+q(x,y,z,w,h(x,y,z,w)) \end{bmatrix} = -\,kh(x,y,z,w) + F(x,y,z,w) \tag{2.4.2}$$

$$h(0) = 0 \;;\; Dh(0) = 0$$

Choosing the following form for the control law F:

$$F(x,y,z,w) = \alpha x^2+\beta y^2+\gamma z^2+\delta w^2+\eta xy+\mu xz+\rho xw+$$

$$\lambda yz+\nu yw+\zeta zw \tag{2.4.3}$$

and approximating the center manifold up to terms of $O(3)$ as:

$$h(x,y,z,w) = ax^2+by^2+cz^2+dw^2+exy+lxz+mxw+$$

$$nyz+syw+tzw + O(3) \tag{2.4.4}$$

Using (2.4.3) and (2.4.4) in (2.4.2) and equating the $O(2)$ terms we find that for nonzero values of k there is a one to one correspondence between the control parameters $(\alpha,\beta,...)$ and the center manifold parameters $(a,b,...)$ regardless of the value of δ. Thus again the control law determines a center manifold completely up to terms of $O(3)$. Now in relation to the stability of the center manifold system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} + \begin{bmatrix} f(x,y,z,w,h(x,y,z,w)) \\ g(x,y,z,w,h(x,y,z,w)) \\ p(x,y,z,w,h(x,y,z,w)) \\ q(x,y,z,w,h(x,y,z,w)) \end{bmatrix} \tag{2.4.5}$$

we have the following theorem:

Theorem 2.4.1: The zero solution of (2.4.5) is asymptotically stable if:

$$D^3_x f' + D^3_y g' + D^3_{xyy} f' + D^3_{xxy} g' < 0 \tag{2.4.6a}$$

$$D^3_z p' + D^3_w q' + D^3_{zww} p' + D^3_{zzw} q' < 0 \tag{2.4.6b}$$

$$D^3_{xzz} f' + D^3_{xww} f' + D^3_{yzz} g' + D^3_{yww} g' < 0 \tag{2.4.6c}$$

$$D^3_{xxz}p' + D^3_{yyz}p' + D^3_{xxw}q' + D^3_{yyw}q' < 0 \tag{2.4.6d}$$

Conditions (2.4.6c) and (2.4.6d) may be replaced by the single condition:

$$\operatorname{sgn}\left(D^3_{xzz}f' + D^3_{xww}f' + D^3_{yzz}g' + D^3_{yww}g'\right) = -\operatorname{sgn}\left(D^3_{xxz}p' + D^3_{yyz}p' + D^3_{xxw}q' + D^3_{yyw}q'\right) \tag{2.4.6e}$$

Proof: We give a short sketch of the proof since the details, although quite similar to the proof of Theorem 2.3.1, are extremely lengthy and tedious. Calculating $ad\, L\,(H_2)$ and $ad\, L\,(H_3)$ for the system (2.4.5) we can show that all $O(2)$ terms of the vector field may be removed and that the $O(3)$ normal form can be written as:

$$\left.\begin{aligned}
\dot{x} &= -y+(\alpha'x+\beta'y)(x^2+y^2)+(\gamma'x+\sigma'y)(x^2+y^2)\\
\dot{y} &= x+(\alpha'y-\beta'x)(x^2+y^2)+(\gamma'y-\sigma'x)(x^2+y^2)\\
\dot{z} &= -\delta w+(\eta'z+\mu'w)(z^2+w^2)+(\rho'z+\nu'w)(x^2+y^2)\\
\dot{w} &= \delta z+(\eta'w-\mu'z)(z^2+w^2)+(\rho'w-\nu'z)(x^2+y^2)
\end{aligned}\right\} + O(4) \tag{2.4.7}$$

where:

$$\alpha' = \frac{1}{16}\left(D^3_x f' + D^3_y g' + D^3_{xyy} f' + D^3_{xxy} g'\right)$$

$$\beta' = \frac{1}{16}\left(D^3_y f' - D^3_x g' + D^3_{xxy} f' - D^3_{xyy} g'\right)$$

$$\gamma' = \frac{1}{8}\left(D^3_{xzz} f' + D^3_{xww} f' + D^3_{yzz} g' + D^3_{yww} g'\right)$$

$$\sigma' = \frac{1}{8}\left(D^3_{yzz} f' + D^3_{yww} f' - D^3_{xzz} g' - D^3_{xww} g'\right)$$

$$\eta' = \frac{1}{16}\left(D^3_z p' + D^3_w q' + D^3_{zww} p' + D^3_{zzw} q'\right)$$

$$\mu' = \frac{1}{16}\left(D^3_w p' - D^3_z q' + D^3_{zzw} p' - D^3_{zww} q'\right)$$

$$\rho' = \frac{1}{8}\left(D^3_{xxz} p' + D^3_{yyz} p' + D^3_{xxw} q' + D^3_{yyw} q'\right)$$

$$\nu' = \frac{1}{8}\left(D^3_{xxw} p' + D^3_{yyw} p' - D^3_{xxz} q' - D^3_{yyz} q'\right)$$

Now transforming (2.4.7) into cylindrical coordinates we obtain:

$$\begin{aligned}
\dot{r}_1 &= \alpha' r_1^3 + \gamma r_1 r_2^2 && + O(|r_1, r_2|^4) \\
\dot{r}_2 &= \rho' r_1^2 r_2 + \eta' r_2^3 && + O(|r_1, r_2|^4) \\
\dot{\theta}_1 &= 1 && + O(|r_1, r_2|^2) \\
\dot{\theta}_2 &= \delta && + O(|r_1, r_2|^2)
\end{aligned} \tag{2.4.8}$$

Conditions (2.4.6) are now clear by considering a Lyapunov function candidate of the form $V = Rr_1^2 + Sr_2^2$ for some positive constants R and S.

□

Corollary 2.4.1: The zero solution of (2.4.1) is stabilizable by the control law in (2.4.3) if $D^2_{x\xi} f$, $D^2_{y\xi} g$ are not both zero, and $D^2_{z\xi} p$, $D^2_{w\xi} q$ are not both zero.

□

The proof is very reminiscent of that of Corollary 2.3.1 and will, therefore, be omitted.

2.5. Robustness Considerations

In the previous sections we gave conditions under which the equilibrium point of a system with degenerate linearization can be stabilized. Since it is often the case that some inaccuracies exist in a model of a physical system, it is important to know what effects modeling errors have on the stability properties of the system. Thus, in this section we investigate the effects of perturbations in the vector fields on the stability properties of the system after a stabilizing control has been implemented. We consider unperturbed systems of the form:

$$\dot{x} = f(x) \qquad x \in U \subset \mathbb{R}^n \tag{2.5.1}$$

where we assume that $f(0) = 0$ and $x = 0$ is a locally asymptotically stable equilibrium point of the system with U as its domain of attraction. As the class of perturbed systems we consider all vector fields $\tilde{f}(x)$ which are ε–C^0 close to $f(x)$, that is all $\tilde{f}(x)$ satisfying:

$$\sup_{x \in U} |\hat{f}(x) - f(x)| < \varepsilon \tag{2.5.2}$$

Thus we may write any perturbed system as:

$$\dot{x} = f(x) + \varepsilon\, \delta(x) \tag{2.5.3}$$

where $\delta(x)$ is any vector field whose C^0 norm is less than one.

We point out that the class of perturbations considered here is quite large in that it contains all bounded perturbations of size ε. Since our system is not exponentially stable, however, we cannot expect to have bounded states for large values of ε. Nevertheless, we will show that for ε small enough the states will remain bounded. More precisely, we will show that there exists an open ball centered at the origin which remains attractive in the presence of perturbations and whose size depends on ε. The following theorem is to this effect.

Theorem 2.5.1: Let the equilibrium point of the system (2.5.1) be locally asymptotically stable with domain of attraction $U \subset \mathbf{R}^n$. Let the class of perturbed systems be given by (2.5.3). Then there exists $\varepsilon^* > 0$ and a monotone increasing function $r(\cdot)$ with $r(0) = 0$, such that for all $\varepsilon < \varepsilon^*$, and all $\delta(x)$ with $\sup_{x \in U} |\delta(x)| < 1$, there exists a ball of radius $r(\varepsilon)$ centered at the origin which contains the ω-limit set of all trajectories of (2.5.3) starting in U.

□

Proof: By assumption the equilibrium point of (2.5.1) is locally asymptotically stable. Therefore by a converse theorem of Lyapunov [16] there exists a locally positive definite function $V(x)$ whose derivative along the flow of (2.5.1), that is $\frac{\partial V}{\partial x}(x)\, f(x)$, is locally negative definite. Now using this function for the system (2.5.3) we have:

$$\dot{V} = \frac{\partial V}{\partial x}(x)\, f(x) + \varepsilon \frac{\partial V}{\partial x}(x)\delta(x) \tag{2.5.4}$$

Since $\frac{\partial V}{\partial x}(x)\, f(x)$ is locally negative definite, we can find a class-K function $\alpha(\cdot)$ such that:

$$\frac{\partial V}{\partial x}(x)\,f(x) \le -\,\alpha(|x|) \quad \forall \quad x \in U \tag{2.5.5}$$

Furthermore, let:

$$L := \sup_{x \in U} \frac{\left| \frac{\partial V}{\partial x}(x) \right|}{|x|} \tag{2.5.6}$$

Then using (2.5.5) and (2.5.6) in (2.5.4) we get:

$$\dot{V} \le -\,\alpha(|x|) + \varepsilon L|x| \tag{2.5.7}$$

where we have also used the fact the C^0 norm of $\delta(x)$ is less than one. Now, let R be the radius of the largest ball centered at the origin which is contained in U. Then define:

$$\varepsilon^* := \frac{\alpha(R)}{LR} \tag{2.5.8}$$

From the monotone increasing property of $\alpha(\cdot)$ we can see that for all $\varepsilon < \varepsilon^*$, there exists a monotone increasing function $r(\cdot)$ with $r(\varepsilon^*) = R$ and $r(0) = 0$, such that:

$$\alpha(|x|) - \varepsilon L|x| \ge 0 \quad \forall \quad r(\varepsilon) < |x| < R \tag{2.5.9}$$

Then it is clear that for all $\varepsilon < \varepsilon^*$ any trajectory of (2.5.3) starting in U will converge to the ball of radius $r(\varepsilon)$ centered at the origin.

□

2.6. Discussion and Examples

The previous sections were based on several seemingly restrictive assumptions. Here we will show the extension of the previous results to more general cases. In Sections 2.1 through 2.4 we considered systems of the form $\dot{\xi}=\phi(\xi)+bu$. Let us now consider the more general case of systems of the form:

$$\dot{\xi} = \phi(\xi) + \psi(\xi)\,u \tag{2.6.1}$$

where $\phi(0) = 0$. Letting $b := \psi(0)$, we may define $\Psi(\xi)$ by:

$$\Psi(\xi) := \psi(\xi) - b \tag{2.6.2}$$

Then, rewriting (2.6.1) using the above notation we have:

$$\dot{\xi} = \phi(\xi) + b\,u + \Psi(\xi)\,u \tag{2.6.3}$$

Now letting $A := D_\xi\phi(0)$ and transforming the system as in Section 2.1 we have:

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix} = \begin{bmatrix}A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} + \begin{bmatrix}\phi_1(\xi)\\ \phi_2(\xi)\\ \phi_3(\xi)\end{bmatrix} + \begin{bmatrix}0\\ b_2\\ b_3\end{bmatrix} u + \begin{bmatrix}\psi_1(\xi)\\ \psi_2(\xi)\\ \psi_3(\xi)\end{bmatrix} u \tag{2.6.4}$$

Representing the center manifold by $(\xi_2^T, \xi_3^T) = (h_2(\xi_1)^T, h_3(\xi_1)^T)$ we get:

$$Dh(\xi_1)[A_{11}\xi_1 + \phi_1(\xi_1, h_2(\xi_1), h_3(\xi_1)) + \psi_1(\xi_1, h_2(\xi_1), h_3(\xi_1))\,u] =$$

$$\begin{bmatrix}A_{22}h_2(\xi_1) + \phi_2(\xi_1, h_2(\xi_1), h_3(\xi_1)) + b_2\,u + \psi_2(\xi_1, h_2(\xi_1), h_3(\xi_1))\,u\\ A_{33}h_3(\xi_1) + \phi_3(\xi_1, h_2(\xi_1), h_3(\xi_1)) + b_3\,u + \psi_3(\xi_1, h_2(\xi_1), h_3(\xi_1))\,u\end{bmatrix} \tag{2.6.5}$$

Assuming the control u is a smooth function of the ξ's , it is clear that the terms $\psi_j u$'s are at least of $O(2)$ and will, therefore, have no effect on the $O(2)$ expansion of the center manifold. The flow on the center manifold, on the other hand, is now determined by

$$\dot{\xi}_1 = A_{11}\xi_1 + \phi_1(\xi_1, h_2(\xi_1), h_3(\xi_1)) + \psi_1(\xi_1, h_2(\xi_1), h_3(\xi_1))\,u \tag{2.6.6}$$

Since the stability of the zero solution of (2.6.6) is determined by the quadratic and higher order terms, the presence of $\psi_1 u$ will only relax the stabilizability conditions by adding more flexibility in satisfying the conditions of Theorems 2.2.1, 2.3.1, and 2.4.1 . In other words the special class of systems $\dot{\xi} = \phi(\xi) + b\,u$ represents a least controllable situation.

We next present two illustrative examples. The first example demonstrates the control design procedure and the effects of perturbations on the stabilized system. The second example involves the case where the controllable part of the system is not a scalar.

Example 1: Consider the system:

$$\begin{aligned} \dot{x} &= y - x^3 + xy^2 - 2y\xi \\ \dot{y} &= x^3 + x\xi \\ \dot{\xi} &= -5\xi + u \end{aligned} \qquad (2.6.7)$$

With $u \equiv 0$, a center manifold for (2.6.7) is given by $\xi \equiv 0$. The flow on this center manifold is governed by:

$$\begin{aligned} \dot{x} &= y - x^3 + xy^2 \\ \dot{y} &= x^3 \end{aligned} \qquad (2.6.8)$$

The origin of (2.6.8) is unstable as shown in Fig. 2.1 by the phase portrait of the system.

To stabilize the system we choose the control as in (2.2.4) and represent the center manifold by (2.2.3). Then the flow on the center manifold is governed by:

$$\left.\begin{aligned} \dot{x} &= y - x^3 + xy^2 - 2y(ax^2 + bxy + cy^2) \\ \dot{y} &= x^3 + x(ax^2 + bxy + cy^2) \end{aligned}\right. + O(4) \qquad (2.6.9)$$

From Theorem 2.2.1 we see that a choice parameters of the center manifold which stabilize the origin of (2.6.9) is given by: $a=-2$, $b=0$, and $c=0$. Using (2.2.6), the corresponding control parameters are given by: $\alpha=-10$, $\beta=-4$, and $\gamma=0$. Thus a stabilizing control law is given by:

$$u = -\,10x^2 - 4xy \qquad (2.6.10)$$

Fig. 2.2 shows the phase portrait of the stabilized system (2.6.9) for the above choice of parameters.

Next we introduce a linear perturbation in the original system. The perturbed system is given by:

$$\begin{aligned} \dot{x} &= \varepsilon x + y - x^3 + xy^2 - 2y\xi \\ \dot{y} &= \varepsilon y + x^3 + x\xi \qquad\qquad \varepsilon > 0 \\ \dot{\xi} &= -5\xi + u \end{aligned} \qquad (2.6.11)$$

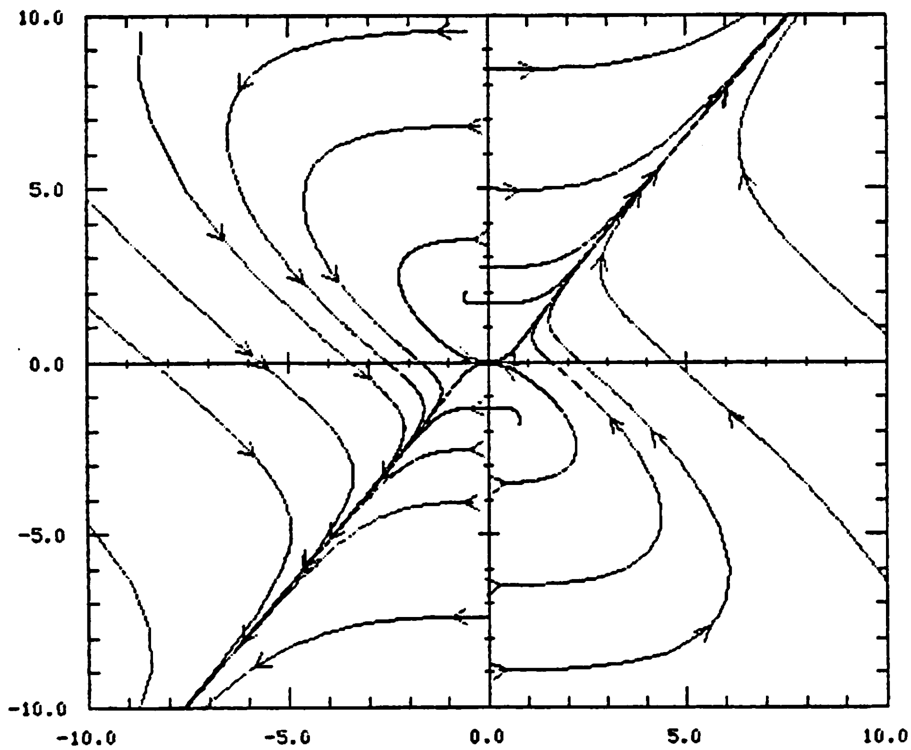

Figure 2.1 Phase portrait on the center manifold of the unstable system.

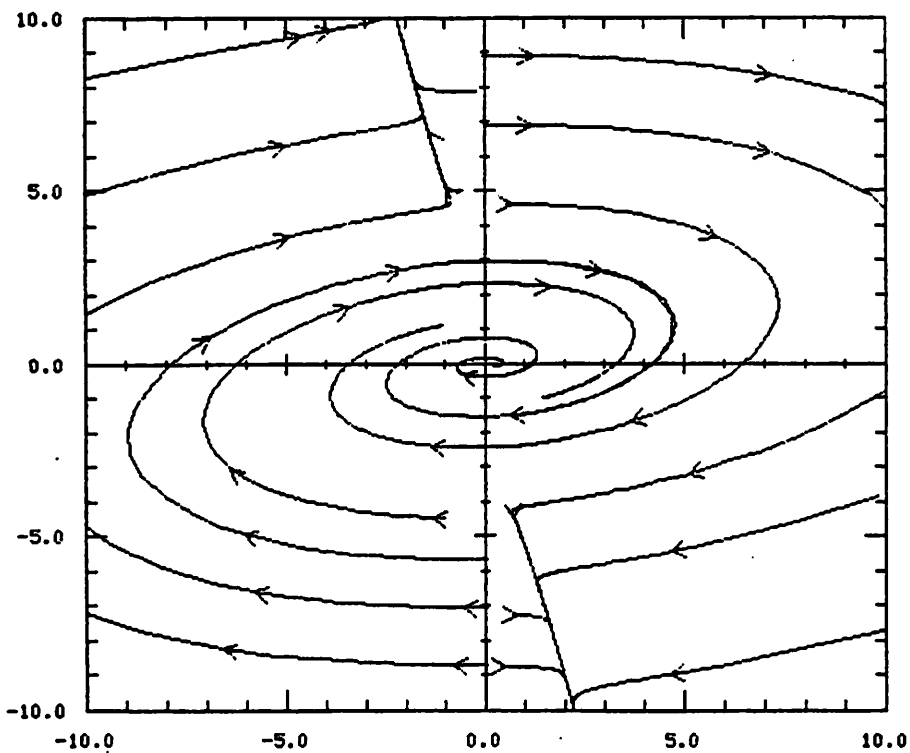

Figure 2.2 Phase portrait on the center manifold of the stabilized system.

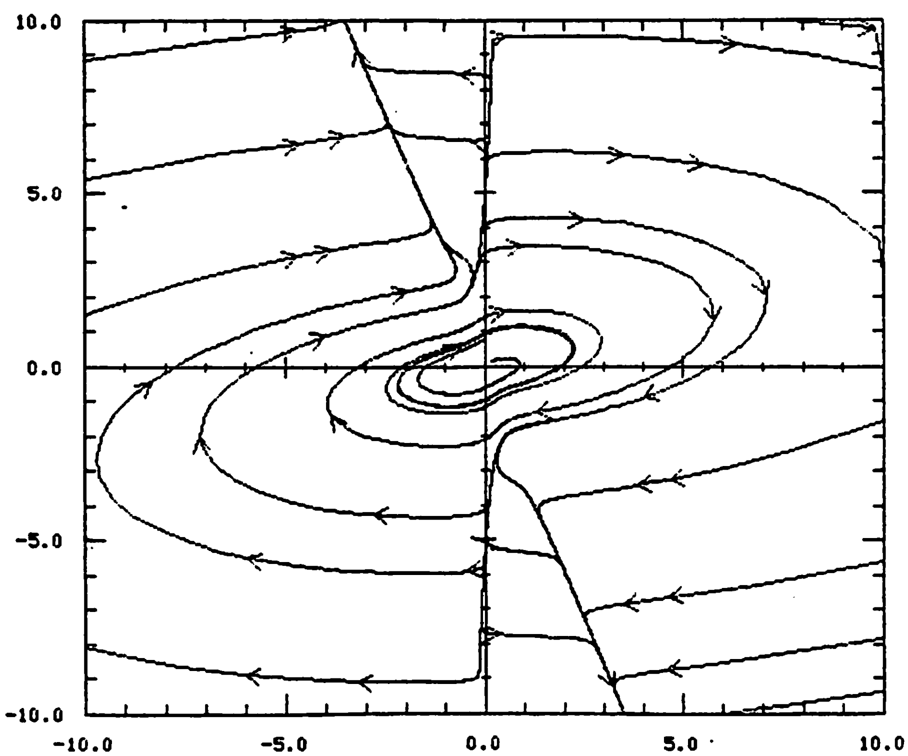

Figure 2.3 Phase portrait on the invariant manifold of the perturbed system.

Clearly the origin of (2.6.11) is unstable irrespective of the control u. From Theorem 2.5.1, however, we expect that for ε small, using the control (2.6.10) the trajectories of the system converge to a small ball centered at the origin. To demonstrate this we compute the center manifold of the suspended system obtained from (2.6.11). This is given by:

$$h(x,y,\varepsilon) = -2x^2 + 0.8\varepsilon x^2 - 0.32\varepsilon xy + 0.064\varepsilon y^2 + O(4) \tag{2.6.12}$$

Then the flow on this invariant manifold is determined by

$$\begin{aligned} \dot{x} &= \varepsilon x+y-x^3+xy^2-2yh(x,y,\varepsilon) \\ \dot{y} &= \varepsilon y+x^3+xh(x,y,\varepsilon) \end{aligned} \tag{2.6.13}$$

The phase portrait of (2.6.13) is shown in Fig. 2.3 for $\varepsilon = 0.5$. Comparison of Figs. 2.2 and 2.3 shows that as ε changes from zero, the stable equilibrium point at the origin bifurcates into a stable periodic orbit around the origin and an unstable equilibrium point at the origin.

Example 2: In this example we consider a system whose hyperbolic portion is not a scalar. Since the approach to all higher dimensional problems is identical, we consider a two dimensional example. Thus consider the system:

$$\begin{aligned} \dot{\xi}_1 &= \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}x^3+xy^2-x\xi_2\\ y\xi_2+x\xi_3+x^2y\end{bmatrix} \\ \xi_2 &= -\xi_2+u \\ \xi_3 &= -2\xi_3+u \end{aligned} \tag{2.6.14}$$

Representing the center manifold as:

$$\begin{bmatrix}\xi_2\\ \xi_3\end{bmatrix} = h(x,y) = \begin{bmatrix}h_2(x,y)\\ h_3(x,y)\end{bmatrix} = \begin{bmatrix}a_1x^2+b_1xy+c_1y^2\\ a_2x^2+b_2xy+c_2y^2\end{bmatrix} + O(3) \tag{2.6.15}$$

we have that with the control $u=\alpha x^2+\beta xy+\gamma y^2$, $h(x,y)$ satisfies:

$$Dh(x,y)\begin{bmatrix} y+x^3+xy^2-xh_2(x,y) \\ x^2y+yh_2(x,y)+xh_3(x,y) \end{bmatrix} =$$

$$\begin{bmatrix} -a_1x^2-b_1xy-c_1y^2+\alpha x^2+\beta xy+\gamma y^2 \\ -2a_2x^2-2b_2xy-2c_2y^2+\alpha x^2+\beta xy+\gamma y^2 \end{bmatrix} + O(3) \tag{2.6.16}$$

Now equating the $O(2)$ terms we get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}=\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{2.6.17}$$

The flow on the center manifold is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}=\begin{bmatrix} y+(1-a_1)x^3+(1-c_1)xy^2-b_1x^2y \\ a_2x^3+(a_1+b_2+1)x^2y+(b_1+c_{2)xy}^{2}+c_1y^3 \end{bmatrix}+O(4) \tag{2.6.18}$$

Then from Theorem 2.2.1 the zero solution of (2.6.18) is asymptotically stable if $a_2<0$ and $3(1-a_1)+a_1+b_2+1<0$. Thus for example choosing $(a_1,b_1,c_1)=(-2,-14,2)$ and $(a_2,b_2,c_2)=(-1,-8,-2)$ satisfies (2.6.17) and the above inequalities. Then we get that the control law:

$$u=-2x^2-18xy-12y^2 \tag{2.6.19}$$

stabilizes the zero solution of (2.6.18) and thus that of (2.6.14).

It is clear from (2.6.17) that although the control law does not determine the center manifold completely, it does give us the same number of degrees freedom in choosing the center manifold as was available in the case with a scalar hyperbolic state.

Chapter Three

Output Tracking in the Presence of Uncertainties

3.1. Introduction

The output tracking problem involves the design of a control law which causes the output of a system to follow a desired trajectory. This problem is now a classical one in linear control theory and has been studied from several viewpoints (see e.g. Callier and Desoer [10] for an algebraic approach, and Wonham [38] for a geometric approach). It is well known that the solvability of the tracking problem is closely related to the invertibility of the input-output map. The invertibility problem was studied by Brockett and Mesarovic [8], and Silverman [32], in the context of linear systems, and by Hirschorn [19], Isidori et al [24], Claude [13], and Nijmeijer [30], in nonlinear systems. In his work, Hirschorn introduced some regularity conditions under which the techniques of Silverman in linear systems can be generalized to nonlinear systems. Later, the same philosophy was adopted by Byrnes and Isidori [9] for input-output linearization of a nonlinear system. Invertibility of an input-output map, and thus the construction of a tracking control law, involves exact cancellations of nonlinear terms which requires complete knowledge of the dynamics of the system. This presents a limitation to the theory in that in the presence of modeling errors exact cancellations are no longer possible.

In this chapter we study the problem of output tracking of a desired trajectory for SISO nonlinear systems when the dynamics of the system are not completely known. In

Section 2 we review the basic theory and present the tracking control law in the ideal case where the dynamics are completely known. In Section 3 we will characterize the nature of the uncertainties that we will deal with in this chapter. We will introduce two classes of uncertainties: those satisfying a generalized matching condition, and those arising from linear parametric uncertainties. In Sections 4 and 5 we present modifications to the basic tracking control law of Section 2 in order to achieve robustness with respect to the first class of uncertainties. In Section 6 we present an adaptive control scheme in order to deal with the second class of uncertainties. Finally, in Section 7 we close the chapter with some concluding remarks.

3.2. Output Tracking in the Ideal Case

Consider a single input-single output nonlinear system of the form:

$$\begin{aligned} \dot{x} &= f(x)+g(x)u \\ y &= h(x) \end{aligned} \tag{3.2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f(\cdot)$ and $g(\cdot)$ are smooth vector fields on an open set $U \subset \mathbb{R}^n$, $f(0)=0$, and $h(\cdot)$ is a smooth function on U. We are interested in the problem of finding a control input u which would result in asymptotic convergence of the output y to a desired function of time y_d. In view of the results in linear control theory, it is clear that the problem may be solved if there exists a control of the form:

$$u = \alpha(x) + \beta(x)\, v \tag{3.2.2}$$

which results in a linear map from v to y. The form of the control (3.2.2) is attractive since the affine structure of the control in (3.2.1) is preserved under this form of feedback. The existence of such a linearizing feedback is in turn guaranteed if the system (3.2.1) possesses a so called "relative degree" from u to y. We will next define this term which was first introduced by Hirschorn [19] in the context of invertibility of nonlinear systems and later used by Isidori [23] in linearization theory.

Definition 3.2.1: The system (3.2.1) is said to have a ***relative degree*** at x_0 (from u to y) if there exists a positive integer $1 \leq \nu < \infty$ such that:

$$L_g h(x) = 0$$
$$L_g L_f h(x) = 0$$
$$\vdots$$
$$L_g L_f^{\nu-2} h(x) = 0$$

for all $x \in B(x_0)$, an open ball centered at x_0, and

$$L_g L_f^{\nu-1} h(x_0) \neq 0$$

In this case we say the system has relative degree ν. A point x_0 at which the system has a relative degree is called a ***regular point*** of the system.

□

Definition 3.2.2: The system (3.2.1) is said to have ***strong relative degree*** ν in an open set D, if there exists a positive integer $1 \leq \nu < \infty$ such that the system has relative degree ν at every point $x_0 \in D$.

□

Remark 3.2.1: The term "relative degree" is used here in analogy with the terminology used in linear systems theory. Let $H(s)$ represent the transfer function of an SISO linear system, then the relative degree of $H(s)$, the difference between the degrees of the denominator and the numerator polynomials in $H(s)$, is precisely the number of times the output must be differentiated before the first appearance of a control term..For the nonlinear system, it is clear from the above definitions that the control u first appears in the νth derivative of y, and hence ν is called the relative degree from u to y.

□

Remark 3.2.2: We point out that nonlinear systems for which no relative degree may be defined are considerably more complicated than those with relative degrees. Furthermore no general theory as yet exists for treating them.

□

We make the following assumption throughout; this assumption is central to the development of the theory presented here.

Assumption 3.2.1: (Regularity) The system (3.2.1) has a strong relative degree in its domain of definition, namely U.

□

Assumption 3.2.1 allows us to find a diffeomorphic coordinate transformation in U, resulting in a normal form for the system (3.2.1) which is particularly suited for the input-output linearization of the system. The following two propositions concern the construction of such a transformation.

Proposition 3.2.1: Let ν be the strong relative degree of the system (3.2.1). Then the set of 1-forms:

$$\begin{array}{l} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{\nu-1} h(x) \end{array}$$

is a linearly independent set for all $x \in U$.

□

To prove the proposition we need the following lemma.

Lemma 3.2.1: Let ν be the strong relative degree of the system (3.2.1). Then the following relations hold:

$$d\,(L_f^{\nu-l} h(x)) \cdot L_f^k g(x) = \begin{cases} (-1)^k\, L_g L_f^{\nu-1} h(x) & l = k+1 \\ 0 & k+1 < l \leq \nu \end{cases} \tag{3.2.3}$$

for all $k = 0,...,\nu-1$ and for all $x \in U$.

□

Proof: Clearly for $k=0$ the assertion is true by the definition of ν. Now suppose (3.2.3) is true for $k = k_0 < \nu-1$. We will show that it is true for $k = k_0+1$. Now, for $k = k_0+1$ we have:

$$\begin{aligned} d\,(L_f^{\nu-l}h(x)) \bullet L_f^{k_0+1}g(x) &= d\,(L_f^{\nu-l}h(x)) \bullet L_f L_f^{k_0}g(x) \\ &= d\,[d\,(L_f^{\nu-l}h(x)) \bullet L_f^{k_0}g(x)] \bullet f(x) \\ &\quad - d\,[d\,(L_f^{\nu-l}h(x)) \bullet f(x)] \bullet L_f^{k_0}g(x) \end{aligned} \tag{3.2.4}$$

where the second equality follows from the definition of a Lie bracket of two vector fields given in (1.1.2). Rewriting (3.2.4) we have:

$$\begin{aligned} d\,(L_f^{\nu-l}h(x)) \bullet L_f^{k_0+1}g(x) &= d\,[d\,(L_f^{\nu-l}h(x)) \bullet L_f^{k_0}g(x)] \bullet f(x) \\ &\quad - d\,(L_f^{\nu-l+1}h(x)) \bullet L_f^{k_0}g(x) \end{aligned} \tag{3.2.5}$$

Since, by assumption, (3.2.3) is true for $k = k_0$, the right hand side of (3.2.5) is zero for all $l > k_0+2$. And for $l = k_0+2$ we have:

$$\begin{aligned} d\,(L_f^{\nu-k_0-2}h(x)) \bullet L_f^{k_0+1}g(x) &= -\,d\,(L_f^{\nu-k_0-1}h(x)) \bullet L_f^{k_0}g(x) \\ &= -\,(-1)^{k_0}L_gL_f^{\nu-1}h(x) = (-1)^{k_0+1}L_gL_f^{\nu-1}h(x) \end{aligned} \tag{3.2.6}$$

where the second equality in (3.2.6) follows from (3.2.3) for $l = k_0+1$. This proves the lemma.

□

Proof of Proposition 3.2.1: Consider the following matrix multiplication performed point-wise for each x:

$$\begin{bmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{\nu-2}h(x) \\ dL_f^{\nu-1}h(x) \end{bmatrix} \begin{bmatrix} (-1)^{\nu-1}L_f^{\nu-1}g(x) \,, (-1)^{\nu-2}L_f^{\nu-2}g(x) \,, \ldots \,, g(x) \end{bmatrix} =: \Lambda(x)$$

It is easy to see from Lemma 3.2.1 that $\Lambda(x)$ has the following form:

$$\Lambda(x) = \begin{bmatrix} \lambda(x) & 0 & \cdot & 0 \\ * & \lambda(x) & \cdot & 0 \\ * & * & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdot & \lambda(x) \end{bmatrix}$$

where $\lambda(x) := L_g L_f^{\nu-1}h(x)$ and *'s denote other terms which are possibly nonzero. Since the system (3.2.1) has strong relative degree ν in the set U, the matrix $\Lambda(x)$ is nonsingular for all $x \in U$. This proves the proposition since the nonsingularity of $\Lambda(x)$ implies the linear independence of the 1-forms $dL_f^k h(x)$, k=0, . . . ,ν−1.

□

Proposition 3.2.2: Let ν be the strong relative degree of the system (3.2.1). Then we can find $n-\nu$ functions $\eta_j(x)$, j=1, . . . ,$n-\nu$, with the following properties:

i) $d\eta_j(x) \bullet g(x) = 0 \quad \forall\, x \in U, \forall\, j=1, \ldots, n-\nu$.

ii) The set of n 1-forms $\{\, d\eta_j(x), dL_f^k h(x) \mid j=1, \ldots, n-\nu \,,\, k=0, \ldots, \nu-1 \}$ is a linearly independent set for all $x \in U$.

Therefore defining:

$$\begin{aligned} \xi_1 &:= h(x) \\ \xi_2 &:= L_f h(x) \\ &\vdots \\ \xi_\nu &:= L_f^{\nu-1}h(x) \end{aligned}$$

we have that $\Phi(x) := (\xi,\eta)$ is a diffeomorphic change of coordinates in the set U, where $\xi := [\xi_1, \ldots, \xi_\nu]^T$ and $\eta := [\eta_1, \ldots, \eta_{n-\nu}]^T$.

□

Proof: By assumption $g(x)$ is a nonzero vector field for all $x \in U$. Therefore by Frobenius' theorem [7], the annihilator of $g(x)$, denoted by $\Omega(x)$, has dimension $n-1$ and is spanned by a set of $n-1$ exact 1-forms. That is:

$$\Omega(x) = \text{span}\{\, d\eta_i(x) \mid i=1, \ldots, n-1\}$$

In addition, from the definition of ν we know that:

$$dL_f^k h(x) \in \Omega(x) \qquad k=0, \ldots, \nu-2$$

Since $dL_f^k h(x)$'s are in fact linearly independent, they form a partial basis for $\Omega(x)$ which may be completed with the addition of $n-\nu$ of $d\eta_i(x)$'s which are linearly independent of the $dL_f^k h(x)$'s. Relabeling these η_i's from 1 to $n-\nu$ we have that:

$$\Omega(x) = \text{span}\{\, dL_f^k h(x), d\eta_j(x) \mid k=0, \ldots, \nu-2, j=1, \ldots, n-\nu\}$$

Finally since $dL_f^{\nu-1} h(x) \bullet g(x) \neq 0 \ \forall x \in U$, we conclude that $dL_f^{\nu-1} h(x)$ is linearly independent of $\Omega(x)$ and this proves the proposition.

□

Proposition 3.2.2 enables us to transform the system (3.2.1) into the following normal form:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{\nu-1} &= \xi_\nu \\ \dot{\xi}_\nu &= b(\xi,\eta) + a(\xi,\eta)\, u \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{3.2.7}$$

where:

$$b(\xi,\eta) := L_f^{\nu} h \circ \Phi^{-1}(\xi,\eta)$$
$$a(\xi,\eta) := L_g L_f^{\nu-1} h \circ \Phi^{-1}(\xi,\eta)$$
$$q(\xi,\eta) := d\eta \cdot f(x) \circ \Phi^{-1}(\xi,\eta)$$

moreover $a(\xi,\eta) \neq 0,\ \forall (\xi,\eta) \in \Phi(U)$.

The choice of the linearizing control law is now apparent from the normal form (3.2.7). Since $a(\xi,\eta)$ is bounded away from zero, its inverse is well defined and the control law:

$$u = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta) + v] \tag{3.2.8}$$

which is of the form (3.2.2), results in the system:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_\nu &= v \\
\dot{\eta} &= q(\xi,\eta) \\
y &= \xi_1
\end{aligned} \tag{3.2.9}$$

Thus the dynamics governing the state variables ξ are that of a linear system. In fact, the new control v affects the output $y=\xi_1$ through a chain of ν integrators. This new control can now be designed in a number of ways to achieve output tracking.

The control law (3.2.8) also makes the state vector η completely unobservable at the output. Since we are interested in achieving stable tracking, we require that η remains bounded for bounded ξ. However, we can see from (3.2.9) that ξ can be thought of as an external input vector with respect to the dynamics of η. Since ξ is expected to track arbitrary time functions, it is clear that boundedness of η is entirely dependent on the vector field $q(\cdot\ ,\ \cdot)$, which belongs to the tangent space to $\mathbb{R}^{n-\nu}$. The dynamics

$$\dot{\eta} = q(0,\eta) \tag{3.2.10}$$

is referred to as the *zero dynamics* [9], since it corresponds to the dynamics of the system (3.2.1) restricted to the submanifold

$$S := \{(\xi,\eta) \mid \xi = 0\} = \{x \mid h(x) = L_f h(x) = \cdots = L_f^{\gamma-1} h(x) = 0\}$$

The stability properties of the zero dynamics are extremely crucial since they determine whether or not η remains bounded when a tracking control is applied. At this point we will introduce some further terminology.

Definition 3.2.3: The system (3.2.1) is said to be a *minimum-phase* nonlinear system if the equilibrium point $\eta = 0$ of the zero dynamics (3.2.10) is asymptotically stable.

□

Remark 3.2.3: The above definition was made in analogy with linear systems and was first introduced by Byrnes and Isidori [9]. In the case of linear systems it can be shown that the zero dynamics are linear with eigenvalues equal to the zeros of the transfer function from u to y. If the transfer function zeros are all in the left half plane, the system is called minimum-phase. Thus for minimum-phase linear systems the dynamics (3.2.10) are asymptotically stable, and the above definition for nonlinear systems parallels this property of linear systems.

□

For purposes of stable tracking, however, we require that the dynamics:

$$\dot{\eta} = q(\xi,\eta) \tag{3.2.11}$$

be bounded-input bounded-state (BIBS) stable. It is easily shown that asymptotic stability of (3.2.10) is not sufficient to guarantee BIBS stability of (3.2.11), and stronger stability criteria are needed. The difficulty with extending the asymptotic stability of (3.2.10) to BIBS stability of (3.2.11) arises from some cases in which the linearization of (3.2.10) has eigenvalues on the $j\omega$-axis. A sufficient condition for the BIBS stability of (3.2.11) is the exponential stability of (3.2.10). We have the following definition to this effect.

Definition 3.2.4: The system (3.2.1) is said to be a *hyperbolically* minimum-phase nonlinear system, if its corresponding zero dynamics are exponentially stable.

□

The following theorem is our basic stable tracking theorem when we have complete knowledge of the vector fields $f(x)$ and $g(x)$ in (3.2.1). In the theorem ξ_k^d denotes the $(k-1)$st derivative of the desired tracking signal $y_d(t)$ for $k=1, \ldots, \nu$, and $\xi^d := (\xi_1^d, \ldots, \xi_\nu^d)^T$.

Theorem 3.2.1: Let the system (3.2.1) be hyperbolically minimum-phase, and the control u chosen according to (3.2.8) with:

$$v = \dot{\xi}_\nu^d + a_1(\xi_\nu^d - \xi_\nu) + a_2(\xi_{\nu-1}^d - \xi_{\nu-1}) + \cdots + a_\nu(\xi_1^d - \xi_1) \tag{3.2.12}$$

such that the polynomial:

$$s^\nu + a_1 s^{\nu-1} + \cdots + a_\nu \tag{3.2.13}$$

is a Hurwitz polynomial.

Then there exists a positive constant c^* and an open set $\Omega \subset \Phi(U)$, such that if $|\xi^d| < c^*$, then for all initial conditions in Ω asymptotic output tracking is achieved with the trajectories remaining in the set $\Phi(U)$ for all time.

□

Proof: With the control v given by (3.2.12), the system (3.2.9) can be written as:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_\nu &= -a_1 e_\nu - a_2 e_{\nu-1} - \cdots a_\nu e_1 \\
\dot{\eta} &= q(\xi, \eta)
\end{aligned} \tag{3.2.14}$$

where $e_k := \xi_k - \xi_k^d$. Since the polynomial in (3.2.13) was chosen to be Hurwitz, it is clear that $e_k \to 0$ as $t \to \infty$ $\forall k$. Thus it remains to show that there exists a constant $c^* > 0$ such that for $|\xi^d| < c^*$, the vector η is also bounded and in addition $(\xi(t),\eta(t)) \in \Phi(U)$, $\forall\, t$. The system is hyperbolically minimum-phase by assumption, therefore by a converse theorem of Lyapunov [16] there exists a positive definite function $V_0(\eta)$ satisfying the following inequalities:

$$\sigma_2\, |\eta|^2 \le V_0(\eta) \le \sigma_1\, |\eta|^2 \tag{3.2.15a}$$

$$\frac{\partial V_0}{\partial \eta} q(0,\eta) \le -\lambda_1\, |\eta|^2 \tag{3.2.15b}$$

$$\left|\frac{\partial V_0}{\partial \eta}\right| \le \lambda_2\, |\eta| \tag{3.2.15c}$$

where $\sigma_1, \sigma_2, \lambda_1$, and λ_2 are positive constants depending on $q(0,\eta)$. Differentiating $V_0(\eta)$ along the trajectories of (3.2.14) yields:

$$\begin{aligned} \dot{V}_0(\eta) &= \frac{\partial V_0}{\partial \eta}\, q(\xi,\eta) \\ &= \frac{\partial V_0}{\partial \eta}\, q(0,\eta) + \frac{\partial V_0}{\partial \eta}\, [q(\xi,\eta) - q(0,\eta)] \end{aligned} \tag{3.2.16}$$

Using (3.2.15) in (3.2.16) we obtain:

$$\dot{V}_0(\eta) \le -\lambda_1\, |\eta|^2 + \lambda_2\, |\eta|\, |q(\xi,\eta) - q(0,\eta)| \tag{3.2.17}$$

Recall that the original system is defined in an open set $U \subset \mathbb{R}^n$. Thus the system (3.2.9) is defined in $\Phi(U)$. Then define:

$$L := \sup_{(\xi,\eta)\in\Phi(U)} \frac{|q(\xi,\eta) - q(0,\eta)|}{|\xi|} \tag{3.2.18}$$

We will assume that L is finite. This is certainly true if the system is analytic and U is bounded. We will say more about this later. From (3.2.17) and (3.2.18) we conclude that for all $(\xi,\eta) \in \Phi(U)$ we have:

$$\dot{V}_0(\eta) \le -\lambda_1\, |\eta|^2 + \lambda_2\, L\, |\eta|\, |\xi| \tag{3.2.19}$$

Thus $\dot{V}_0 \leq 0$, whenever

$$|\eta| > \frac{\lambda_2 L}{\lambda_1} |\xi| \tag{3.2.20}$$

Therefore, we can see that η is bounded whenever ξ is bounded. On the other hand, we observe that ξ is bounded if ξ^d is bounded. Thus, we conclude that η remains bounded for all bounded tracking signals ξ^d. However, we also need to guarantee that both ξ and η remain in $\Phi(U)$. So let:

$$\Omega_C := \{\eta \mid V_0(\eta) \leq C\} \tag{3.2.21}$$
$$\underline{C}(r) := \inf\{C \mid B_r \subset \Omega_C\} \tag{3.2.22}$$

where B_r denotes a ball of radius r centered at the origin. It is easily seen that if $\dot{V}_0 \leq 0$ for $|\eta| > K$, and if $\eta(0) \in \Omega_{\underline{C}(K)}$, then:

$$\eta(t) \in \Omega_{\underline{C}(K)}, \quad \forall t \tag{3.2.23}$$

On the other hand from the error dynamics we know that there exists a constant $M \geq 1$ such that:

$$|e(t)| < M\, |e(0)|, \quad \forall t \tag{3.2.24}$$

In addition, we can write:

$$|\xi(t)| \leq |e(t)| + |\xi^d(t)| < M|e(0)| + |\xi^d(t)| \tag{3.2.25}$$

where the second inequality follows from (3.2.24). Now letting $|\xi^d| < c$ and $|e(0)| < 2c$, we define:

$$\begin{aligned} c_1 &:= (2M+1)c \\ c_2 &:= \frac{\lambda_2 L}{\lambda_1}(2M+1)c \\ \overline{G}(c) &:= \{(\xi,\eta) \mid |\xi| < c\ ;\eta \in \Omega_{\underline{C}(c_2)}\} \end{aligned} \tag{3.2.26}$$

Then from (3.2.24) and (3.2.25) we conclude that for all initial conditions in $\overline{G}(c)$, we have that:

$$\begin{aligned} |\xi(t)| &\le c_1 , \quad \forall t \\ \eta(t) &\in \Omega_{\underline{C}(c_2)} , \quad \forall t \end{aligned} \tag{3.2.27}$$

Finally let:

$$c^* := \sup \{ \ c \ | \ \{(\xi,\eta) \ | \ |\xi|<c_1 \ ; \eta \in \Omega_{\underline{C}(c_2)}\} \ \subset \ \Phi(U) \ \} \tag{3.2.28}$$

Then the theorem is proved with $|\xi^d|<c^*$ and $\Omega := \overline{G}(c^*)$.

□

Remark 3.2.4: We point out that if the domain of definition of the system (3.2.1), namely the open set U, is the entire $\mathbb{R}^n$ or an unbounded subset, then L as defined in (3.2.18) may in fact be infinity. However, in that case for every open and bounded subset of U, $\overline{U}$, containing the origin we may give an upper bound for $|\xi^d|$ such that output tracking is possible with the trajectories of the system remaining in the set $\overline{U}$.

□

3.3. Characterization of Uncertainties

In this section assume that the system (3.2.1) is not completely known and that the true system is a perturbation of the known model in the following form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \Delta f(x) + \Delta g(x)u \\ y &= h(x) \end{aligned} \tag{3.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f(\cdot)$, $g(\cdot)$, $\Delta f(\cdot)$, and $\Delta g(\cdot)$ are smooth vector fields on an open set $U \subset \mathbb{R}^n$, $f(0) = 0$, and $h(\cdot)$ is a smooth function on U. We wish to modify the tracking control laws of the previous section in order to reduce or eliminate the effects of the perturbation vector fields, $\Delta f(x)$ and $\Delta g(x)$, on the tracking error. The extent to which we will be successful in robustifying our tracking scheme will, of course, depend on the characteristics of the perturbation vector fields. The following definition introduces a terminology for classification of perturbations.

Definition 3.3.1: Let the unperturbed system (3.2.1) have a strong relative degree ν. A vector field $\phi(x)$ is said to have an index γ with respect to the system (3.2.1) if

$$\phi(x) \in Ker \{dh(x), dL_f h(x), \cdots, dL_f^{\gamma-1} h(x)\} \qquad (3.3.2)$$

□

It is clear from the above definition that the index of a perturbation vector field with respect to an unperturbed system is simply the number of times the system output must be differentiated with respect to time before the first appearance of the perturbation terms. Thus the following facts are rather obvious.

Fact 3.3.1: Let the unperturbed system (3.2.1) have strong relative degree ν. Then this relative degree is unchanged by the addition of perturbations if the perturbation vector fields $\Delta f(x)$ and $\Delta g(x)$ have indices larger or equal to $\nu-1$.

□

Fact 3.3.2: Let the unperturbed system (3.2.1) have strong relative degree ν. If the perturbation vector fields $\Delta f(x)$ and $\Delta g(x)$ have indices equal to $\nu+1$, the system with the control law (3.2.8) and (3.2.12) is completely robust with respect to these perturbations provided that the stability properties of the zero dynamics are preserved in the face of the perturbations.

□

In addition to index considerations, we will differentiate between those perturbations arising from linear parametric uncertainties and all other perturbations. Hence in this study, we consider perturbations which satisfy one of the following two assumptions.

Assumption 3.3.1: (Generalized Matching Assumption) The perturbations $\Delta f(x)$ and $\Delta g(x)$ are smooth vector fields with indices γ_1 and γ_2 and

$$\min\{\gamma_1,\gamma_2\} \geq \nu-1 \tag{3.3.3}$$

□

Assumption 3.3.2: The perturbation vector fields are the result of linear parametric uncertainties in the vector fields of the model (3.2.1), and are of the form:

$$\begin{aligned} \Delta f(x) &= \sum_{i=1}^{N}(\alpha_i^* - \alpha_i)f_i(x) \\ \Delta g(x) &= \sum_{j=1}^{M}(\beta_j^* - \beta_i)g_j(x) \end{aligned} \tag{3.3.4}$$

In addition:

$$\begin{aligned} f(x) &= \sum_{i=1}^{N}\alpha_i f_i(x) \\ g(x) &= \sum_{j=1}^{M}\beta_j g_j(x) \end{aligned} \tag{3.3.5}$$

where $f_i(x)$'s and $g_j(x)$'s are known vector fields and the scalar quantities α_i's and β_j's are our estimates of the parameters α_i^*'s and β_j^*'s, respectively. The parameters α_i^*'s and β_j^*'s are constant unknowns which lie in known open intervals. That is:

$$\begin{aligned} &\alpha_i^* \in (l_{\alpha_i}, h_{\alpha_i}) \qquad i=1,\ldots,N \\ &\beta_j^* \in (l_{\beta_j}, h_{\beta_j}) \qquad j=1,\ldots,M \end{aligned} \tag{3.3.6}$$

for some known scalars l_{α_i}'s, h_{α_i}'s, l_{β_j}'s, and h_{β_j}'s.

□

It is worth pointing out that Assumption 3.3.1 is a generalization of the so called "matching condition" which is the basis for the current state of robust tracking and regulation for nonlinear systems. We shall state the matching assumption here for completeness and comparison with our assumption.

Matching Assumption: The system (3.3.1) is said to satisfy the matching condition if the unperturbed system is ***completely feedback linearizable*** and the perturbation vector fields satisfy:

$$\Delta f(x) \quad \text{and} \quad \Delta g(x) \in \text{span}\{g(x)\} \tag{3.3.7}$$

□

In our Assumption 3.3.1, the system need not be completely feedback linearizable; it is only required to have a strong relative degree. Moreover, the condition (3.3.3) is satisfied if (3.3.7) holds. However, it is easy to see that (3.3.7) is a much stronger condition than (3.3.3). In fact the two conditions coincide only when the relative degree of the system, ν, is equal to n, that is the system is feedback linearizable. For all relative degrees $\nu < n$, the set of perturbations satisfying (3.3.7) are a proper subset of the set of perturbations satisfying (3.3.3). Thus Assumption 3.3.1 is a significant generalization of the matching condition and is much less restrictive.

As robustifying techniques, we will use high gain and the sliding mode control methodologies for perturbations satisfying Assumption 3.3.1. For perturbations which satisfy Assumption 3.3.2, we will use adaptive control. In high gain and sliding mode control strategies we will treat the perturbations as disturbances and will try to reduce their effect on the tracking error. For systems satisfying Assumption 3.3.2, on the other hand, due to the parametric nature of the uncertainties we are able to use adaptive control techniques to update our model of the system in order to achieve zero or small tracking errors. These tasks will be carried out in the next three sections.

3.4. Robust Tracking Using High Gain Control

Consider the system (3.3.1) and let the perturbation vector fields satisfy Assumption 3.3.1. Then using Proposition 3.2.2 we may transform the system to the following form:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_\nu &= b(\xi,\eta) + a(\xi,\eta)u + \tilde{\delta}_1(\xi,\eta) + \tilde{\delta}_2(\xi,\eta)u \\
\dot{\eta} &= q(\xi,\eta) \\
y &= \xi_1
\end{aligned} \tag{3.4.1}$$

where:

$$\begin{aligned}
b(\xi,\eta) &:= L_f^\nu h \circ \Phi^{-1}(\xi,\eta) \\
a(\xi,\eta) &:= L_g L_f^{\nu-1} h \circ \Phi^{-1}(\xi,\eta) \\
q(\xi,\eta) &:= d\eta \bullet (f+\Delta f) \circ \Phi^{-1}(\xi,\eta) \\
\tilde{\delta}_1(\xi,\eta) &:= L_{\Delta f} L_f^{\nu-1} h \circ \Phi^{-1}(\xi,\eta) \\
\tilde{\delta}_2(\xi,\eta) &:= L_{\Delta g} L_f^{\nu-1} h \circ \Phi^{-1}(\xi,\eta)
\end{aligned} \tag{3.4.2}$$

Now using the control (3.2.8) in (3.4.1) we have:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_\nu &= v + \delta_1(\xi,\eta) + \delta_2(\xi,\eta)v \\
\dot{\eta} &= q(\xi,\eta) \\
y &= \xi_1
\end{aligned} \tag{3.4.3}$$

where:

$$\begin{aligned}
\delta_1(\xi,\eta) &:= \tilde{\delta}_1(\xi,\eta) - \tilde{\delta}_2(\xi,\eta)\,\frac{b(\xi,\eta)}{a(\xi,\eta)} \\
\delta_2(\xi,\eta) &:= \tilde{\delta}_2(\xi,\eta)\,\frac{1}{a(\xi,\eta)}
\end{aligned}$$

Let $y_d(t)$ be the desired tracking signal and let $\xi^d(t)$ denote the ν dimensional vector whose kth component is the $(k-1)$th derivative of $y_d(t)$. Define $e_k := \xi_k - \xi_k^d$. Then rewriting

(3.4.2) in the error coordinates, we obtain:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_\nu &= v + \delta_1(\xi,\eta) + \delta_2(\xi,\eta)v - \dot{\xi}_\nu^d \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{3.4.4}$$

In order to reduce the effects of the unknown perturbation terms $\delta_1(\xi,\eta)$ and $\delta_2(\xi,\eta)$ on the tracking error e, we will choose a large gain in the control v which decomposes the error dynamics into a fast and slow part. Then it will be seen that the perturbations formally appear only in the fast dynamics and the slow dynamics will be independent of the perturbations. Thus in the overall dynamics the effect of the perturbations on the tracking errors will be of the order of the time scale separation between the slow and the fast dynamics. This design is carried out in detail in the following theorem. We will give a direct proof here which does not rely on the standard results in singular perturbation theory. This has the advantage of providing estimates on the size of the domain of attraction and bounds on the norm of the error vector as time tends to infinity. Before stating the theorem we give the following definitions:

$$\begin{aligned} \varepsilon_1 &:= \sup_{(\xi,\eta)\in\Phi(U)} |\delta_1(\xi,\eta)| \\ \varepsilon_2 &:= \sup_{(\xi,\eta)\in\Phi(U)} |\delta_2(\xi,\eta)| \end{aligned} \tag{3.4.5}$$

Theorem 3.4.1: Let the system (3.3.1) be hyperbolically minimum-phase and the control law chosen according to (3.2.8). Let ε_2, as defined in (3.4.5), be less than 1. Let the control v be chosen as:

$$v = \dot{\xi}_\nu^d(t) - \frac{1}{\varepsilon}\,[e_\nu + a_1 e_{\nu-1} + \cdots + a_{\nu-1}e_1] \tag{3.4.6}$$

where ε is a small positive constant and the polynomial $s^{\nu-1} + a_1 s^{\nu-2} + \cdots + a_{\nu-1}$ is Hurwitz. Then there exists a positive constant ε^*, a set $D \subset \mathbb{R}^2$ containing the origin, and an open set $\Omega \subset \Phi(U)$ such that for all $\varepsilon < \varepsilon^*$ and $|\xi^d(t)| < c$, and $|\dot{\xi}_\nu^d(t)| < d$ for all t with $(c,d) \in D$ and all initial conditions in Ω, the trajectories of the system remain in the open set $\Phi(U)$ and the tracking errors converge to an ε-neighborhood of the origin.

□

Proof: With the control v given by (3.4.6), the system (3.4.4) becomes:

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_\nu &= \frac{-1}{\varepsilon}\,[e_\nu + a_1 e_{\nu-1} + \cdots + a_{\nu-1}e_1] + \delta_1(\xi,\eta) + \delta_2(\xi,\eta)v \\
\dot{\eta} &= q(\xi,\eta)
\end{aligned}
\tag{3.4.7}
$$

To prove the theorem, we will first assume that the system trajectories remain in the open set $\Phi(U)$ so that the bounds given by (3.4.5) are valid. With this assumption we then show that the tracking errors converge to an ε-neighborhood of the origin. Later we will show that for a proper choice of ε and tracking signal $y_d(t)$ the trajectories do indeed remain in the set $\Phi(U)$.

Thus define:

$$
\begin{aligned}
E_1 &:= a_1 e_{\nu-1} + \cdots + a_{\nu-1}e_1 \\
E_2 &:= a_2 e_{\nu-1} + \cdots + a_{\nu-1}e_2 - a_1 E_1 \\
\zeta &:= e_\nu + E_1 \\
\bar{e} &:= (e_1, e_2, \cdots, e_{\nu-1})^T \\
e &:= (e_1, e_2, \cdots, e_{\nu-1}, e_\nu)^T
\end{aligned}
\tag{3.4.8}
$$

Then (3.4.7) can be written as:

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_{\nu-1} &= \zeta - E_1 \\
\varepsilon\dot{\zeta} &= -[1 - \varepsilon a_1 - \delta_2(\xi,\eta)]\zeta + \varepsilon\delta_1(\xi,\eta) + \varepsilon\delta_2(\xi,\eta)\dot{\xi}_\nu^d + \varepsilon E_2 \\
\dot{\eta} &= q(\xi,\eta)
\end{aligned}
\tag{3.4.9}
$$

Consider the following positive definite function in the $(\bar{e},\zeta)$ coordinates:

$$V_1(\bar{e},\zeta) = \bar{e}^T P \bar{e} + \frac{1}{2}\,\varepsilon\gamma\zeta^2 \tag{3.4.10}$$

where P is chosen such that

$$PA + A^T P = -I \tag{3.4.11}$$

and

$$A := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{\nu-1} & -a_{\nu-2} & \cdots & -a_1 \end{bmatrix}$$

and γ is a positive scalar to be determined later. Differentiating V_1 along the flow of (3.4.9) we obtain:

$$\dot{V}_1 = -|\bar{e}|^2 + 2\bar{e}^T Pb\zeta - \gamma(1-\varepsilon a_1-\delta_2(\xi,\eta))\zeta^2$$

$$+\,\varepsilon\gamma[\delta_1(\xi,\eta) + \delta_2(\xi,\eta)\dot{\xi}_\nu^d(t) + E_2]\zeta \tag{3.4.12}$$

where $b := [0,0,..,1]^T \in \mathbb{R}^{\nu-1}$. With the assumption that the trajectories of the system remain in the open set U, we can use (3.4.5) in (3.4.12) to obtain:

$$\dot{V}_1 \le -|\bar{e}|^2 - \gamma\,(1-\varepsilon_2-\varepsilon a_1)|\zeta|^2 + \rho_1|\bar{e}||\zeta| + \varepsilon\gamma(\varepsilon_1+\varepsilon_2 d+\rho_2|\bar{e}|)|\zeta| \tag{3.4.13}$$

where ρ_1 and ρ_2 are positive constants such that $|2\bar{e}^T Pb| \le \rho_1|\bar{e}|$ and $|E_2| \le \rho_2|\bar{e}|$, and we have also used the bound $|\dot{\xi}_\nu^d(t)| < d$. Define:

$$\begin{aligned} c_1 &:= 1-\varepsilon_2-\varepsilon a_1 \\ c_2 &:= \gamma(\varepsilon_1+\varepsilon_2 d) \\ \rho &:= \rho_1+\varepsilon\gamma\rho_2 \\ \varepsilon^* &:= \frac{1-\varepsilon_2}{2a_1} \end{aligned} \tag{3.4.14}$$

By assumption, $\varepsilon_2 < 1$, thus $\varepsilon^* > 0$ and for all $\varepsilon < \varepsilon^*$ we have that $c_1 > 0$. Now using the inequality:

$$\rho|\bar{e}||\zeta| \leq \frac{1}{4}|\bar{e}|^2+\rho^2|\zeta|^2 \tag{3.4.15}$$

in (3.4.13) we conclude that:

$$\dot{V}_1 \leq -\frac{3}{4}|\bar{e}|^2 - (\gamma c_1-\rho^2)|\zeta|^2 + \varepsilon c_2|\zeta| \tag{3.4.16}$$

If γ is chosen such that $\gamma c_1 > \rho^2$, then it is easy to see from (3.4.16) that there exists a positive constant R such that for all $(\bar{e},\zeta)$ outside of the ball of radius εR, denoted by $B_{\varepsilon R}$, we have $\dot{V}_1 < 0$. Now let:

$$\Omega_c := \{(\bar{e},\zeta) \mid V_1 \leq c\}$$
$$\underline{c} := \inf\{c \mid \Omega_c \supset B_{\varepsilon R}\}$$

Then it is clear that $\underline{c}$ is of $O(\varepsilon)$ and that the trajectories will converge to the set $\Omega_{\underline{c}}$. This ends the first part of the proof. To complete the proof we need to show that for a proper choice of the tracking signal the trajectories in fact remain in the open set $\Phi(U)$. Thus consider the following Lyapunov function candidate for the system in (3.4.9):

$$V(\bar{e},\zeta,\eta) = \bar{e}^T P\bar{e} + \frac{1}{2}\varepsilon\gamma\zeta^2 + \mu V_0(\eta) \tag{3.4.17}$$

where P is the solution of (3.4.11), V_0 is a Lyapunov function for the zero dynamics and satisfies the inequalities (3.2.15), and γ and μ are positive constants to be determined later. Differentiating V we have:

$$\begin{aligned}\dot{V} = &-|\bar{e}|^2 + 2\bar{e}^T Pb\zeta - \gamma(1-\varepsilon a_1 - \delta_2(\xi,\eta))\zeta^2 \\ &+ \varepsilon\gamma[\delta_1(\xi,\eta) + \delta_2(\xi,\eta)\dot{\xi}_v^d(t)+E_2]\zeta + \mu\frac{\partial V_0}{\partial \eta}q(\xi,\eta)\end{aligned} \tag{3.4.18}$$

where b is defined as in (3.4.12). Using (3.2.15), (3.2.18), and (3.4.14) in (3.4.18) yields:

$$\begin{aligned}\dot{V} \leq &-|\bar{e}|^2 - \gamma c_1\zeta^2 + \rho|\bar{e}||\zeta| + \varepsilon c_2|\zeta| - \mu\lambda_1|\eta|^2 \\ &+ \mu\lambda_2 LK|\bar{e}||\eta| + \mu\lambda_2 L|\eta||\zeta| + \mu\lambda_2 Lc|\eta|\end{aligned} \tag{3.4.19}$$

where K is a positive constant such that $|e| \le K|\bar{e}|+|\zeta|$, and we have used $|\xi^d| < c$. From (3.4.19) and the inequalities:

$$\begin{aligned} \mu\lambda_2 LK|\bar{e}||\eta| &< \frac{1}{4}|\bar{e}|^2 + \mu^2\lambda_2^2 L^2 K^2|\eta|^2 \\ \mu\lambda_2 L|\zeta||\eta| &< \frac{1}{4}|\zeta|^2 + \mu^2\lambda_2^2 L^2|\eta|^2 \end{aligned} \tag{3.4.20}$$

we can write:

$$\begin{aligned} \dot{V} \le &-\frac{1}{2}|\bar{e}|^2 - (\gamma c_1 - \rho^2 - \frac{1}{4})|\zeta|^2 - \mu(\lambda_1 - \mu\lambda_2^2 L^2(1+K^2))|\eta|^2 \\ &+ \varepsilon c_2|\zeta| + \mu\lambda_2 Lc|\eta| \end{aligned} \tag{3.4.21}$$

Now it is clear that we can choose γ and μ such that the square terms in (3.4.21) are all negative. Thus for example take:

$$\gamma = \frac{1}{c_1}(1.25+\rho^2)$$
$$\mu = \frac{\lambda_1}{2\lambda_2^2 L^2(1+K^2)}$$

Then (3.4.21) becomes:

$$\dot{V} \le -\frac{1}{2}|\bar{e}|^2 - |\zeta|^2 - \mu\frac{\lambda_1}{2}|\eta|^2 + \varepsilon c_2|\zeta| + \mu\lambda_2 Lc|\eta| \tag{3.4.22}$$

Recalling from (3.4.14) that c_2 depends on d (the bound for $\dot{\xi}_v^d(t)$), we conclude from (3.4.22) that for every ε, c, and d there exists an $\bar{R}(\varepsilon,c,d)$ such that $\dot{V}$ is negative outside the ball of radius $\bar{R}$. Let:

$$\Omega_C := \{(\bar{e},\zeta,\eta) \mid V \le C\}$$
$$\underline{C}(c,d) := \inf\{C \mid \Omega_C \supset B_{\bar{R}(\varepsilon^*,c,d)}\}$$

Then we can find a set $D \subset \{(c,d) \mid c \ge 0\,;\, d \ge 0\}$ such that whenever $(c,d) \in D$ we have that

$$(\bar{e},\zeta,\eta) \in \Omega_{\underline{C}(c,d)} \Longrightarrow (\xi,\eta) \in \Phi(U)$$

Define:

$$\Omega := \{(\xi,\eta) \mid (\xi,\eta)\in\Phi(U) \Longrightarrow (\bar{e},\zeta,\eta)\in\Omega_{\underline{C}(c,d)}\,,\ \forall\ (c,d)\in D\}$$

Then with $\varepsilon < \varepsilon^*$, $|\xi^d(t)| < c$, $\dot{\xi}_v^{d(t)} < d$, and $(c,d) \in D$, we have that for all initial conditions in Ω the system trajectories stay in the open set $\Phi(U)$ and this completes the proof.

□

3.5. Robust Tracking Via Sliding Mode Control

Consider the system (3.3.1) with the perturbations satisfying Assumption 3.3.1. For a given desired tracking signal $y_d(t)$ we have shown in the previous section that the system (3.3.1) can be reduced to the system:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_v &= v + \delta_1(\xi,\eta) + \delta_2(\xi,\eta)v - \dot{\xi}_v^d \\
\dot{\eta} &= q(\xi,\eta)
\end{aligned} \qquad (3.4.3)$$

In this section we remove the restriction that the control input be smooth. Rather we consider piecewise smooth inputs in order to achieve zero tracking errors despite the uncertainties present in our model of the system. Our objective is to choose a $(v-1)$ dimensional subspace in $\mathbb{R}^v$ with the property that the error dynamics restricted to this subspace are asymptotically stable. It is clear that with the presence of uncertainties no subspace can be made invariant with a smooth control. We will therefore use the discontinuities in the control in order to force the dynamics to evolve on the chosen subspace. We will refer to this subspace, made invariant via discontinuous control, as a sliding surface. Thus define:

$$S := e_v + a_1 e_{v-1} + \cdots + a_{v-1} e_1 \qquad (3.5.1)$$

where the scalars a_1 through a_{v-1} are chosen so that the polynomial $s^{v-1} + a_1 s^{v-2} + \cdots + a_{v-1}$ is Hurwitz. Clearly if the dynamics were forced to evolve on the surface characterized by $S = 0$, then $e_1(t)$ through $e_{v-1}(t)$ would converge to zero as t tends to infinity. Thus we will choose $S = 0$ as our sliding surface. The following theorem states our result.

Theorem 3.5.1: Let the system (3.3.1) be hyperbolically minimum-phase and the control law be chosen according to (3.2.7) with v given by:

$$v = \dot{\xi}_v^d - a_1 e_v - \cdots - a_{v-1} e_2 - K\,\mathrm{sgn}(S) \tag{3.5.2}$$

where K is a positive constant. Let ε_2, as defined in (3.4.5), be less than 1. Let $\dot{\xi}_v^d$ be bounded. Then there exist positive constants K^* and c^*, and a set $\Omega \subset \Phi(U)$ such that for $K > K^*$ and $|\xi^d| < c^*$, we have that for all initial conditions in Ω the trajectories of the system remain in the set $\Phi(U)$ and the tracking errors tend to zero as $t \to \infty$.

□

Proof: Using (3.5.1) and (3.5.2) we can, through a coordinate change, write (3.4.3) as:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_{v-1} &= -a_1 e_{v-1} - \cdots - a_{v-1} e_1 + S \\
\dot{S} &= -K\,\mathrm{sgn}(S) + \delta_1(\xi,\eta) + \delta_2(\xi,\eta) v \\
\dot{\eta} &= q(\xi,\eta)
\end{aligned} \tag{3.5.3}$$

We will now show that if the states ξ and η remain in the set $\Phi(U)$, then there exists a constant K^* such that for $K > K^*$ we have that $\bar{e} \to 0$ as $t \to \infty$, where $\bar{e}$ is defined in (3.4.8). To this end consider the positive definite function of S:

$$V_1(S) := \frac{1}{2} S^2 \tag{3.5.4}$$

Differentiating V_1 along (3.5.3) we have:

$$\dot{V}_1 = S[-K\,\mathrm{sgn}(S) + \delta_1(\xi,\eta) + \delta_2(\xi,\eta) v] \tag{3.5.5}$$

Then using the bounds for the perturbations inside the set $\Phi(U)$, as given by (3.4.5), in (3.5.5) gives:

$$\dot{V}_1 \leq -K|S| + (\varepsilon_1+\varepsilon_2|v|)\,|S| \tag{3.5.6}$$

From (3.5.2) we have that:

$$|v| \leq d+K+A|e| \tag{3.5.7}$$

where d is the bound on $|\dot{\xi}^d_v|$, and A is the norm of the vector $[a_1,\ldots, a_{v-1}]$. Let C be the maximum of $|\xi|$ in $\Phi(U)$. Then since, by assumption, the trajectories stay in $\Phi(U)$, we conclude that $|\xi^d| < C$. This implies that $|e| < 2C$ for all $\xi\in\Phi(U)$. Then we have from (3.5.7) that:

$$|v| \leq d+K+2AC \tag{3.5.8}$$

Using this bound in (3.5.6) we obtain:

$$\dot{V}_1 \leq -[K-\varepsilon_1-\varepsilon_2(d+2AC)-\varepsilon_2 K]\,|S| \tag{3.5.9}$$

Since $\varepsilon_2 < 1$, we can define:

$$K^* := \frac{\varepsilon_1+\varepsilon_2(d+2AC)}{1-\varepsilon_2} \tag{3.5.10}$$

It is clear from (3.5.9) that for $K > K^*$, $\dot{V}_1$ will be negative definite and thus S will tend to zero. In fact we can see from (3.5.3) that S reaches zero in finite time. To conclude that $\bar{e}\to 0$ as $t\to\infty$, we need only to look at (3.5.1). With S=0, (3.5.1) describes a stable $(n-1)$st order homogeneous differential equation which implies that $\bar{e}$ tends to zero assymptotically. Thus in the error coordinates, the trajectories first reach the sliding surface S=0 in finite time, and then converge to zero exponentially on the sliding surface.

To show that the state trajectories of the system remain in the set U for all time, we consider the following positive definite function as a Lyapunov function candidate:

$$V(\bar{e},S,\eta) = \bar{e}^TP\bar{e} + \gamma(\frac{S^2}{2} + \frac{S^4}{4}) + \mu V_0(\eta) \tag{3.5.11}$$

where P satisfies (3.4.11), $V_0(\eta)$ satisfies (3.2.15), and the positive constants γ and μ will be specified later. Then we have:

$$\begin{aligned}\dot{V} = &-|\bar{e}|^2 + 2\bar{e}^TPbS + \gamma(S+S^3)\ [-K\text{sgn}(S)+\delta_1(\xi,\eta)+\delta_2(\xi,\eta)v] \\ &+ \mu\frac{\partial V_0}{\partial\eta}q(\xi,\eta)\end{aligned} \tag{3.5.12}$$

where $b := [0,0,...,1]^T \in \mathbb{R}^{\nu-1}$. Then using (3.2.15) and (3.5.10) in (3.5.12) we conclude that in the set $\Phi(U)$:

$$\begin{aligned}\dot{V} \le &-|\bar{e}|^2 + \rho|\bar{e}||S| - \gamma(1-\varepsilon_2)(K-K^*)(|S|+|S|^3) - \mu\lambda_1|\eta|^2 \\ &+ \mu\lambda_2L|\eta||S| + \mu\lambda_2L(A+1)|\bar{e}||\eta| + \mu\lambda_2Lc|\eta|\end{aligned} \tag{3.5.13}$$

where ρ satisfies $|2\bar{e}^TPb|<\rho|\bar{e}|$. Next employing the inequalities:

$$\begin{aligned}&\rho|\bar{e}||S| \le \frac{|\bar{e}|^2}{4} + \rho^2|S|^2 \\ &\mu\lambda^2L(A+1)|\bar{e}||\eta| \le \frac{|\bar{e}|^2}{4} + \mu^2\lambda_2^2L^2(A+1)^2|\eta|^2 \\ &\mu\lambda_2L|\eta||S| \le \frac{|S|^2}{4} + \mu^2\lambda_2^2L^2|\eta|^2\end{aligned} \tag{3.5.14}$$

we obtain:

$$\begin{aligned}\dot{V} \le &-\frac{1}{2}|\bar{e}|^2 - \mu[\lambda_1-\mu\lambda_2^2L^2((A+1)^2+1)] + (\rho^2+\frac{1}{4})|S|^2 \\ &-\gamma(1-\varepsilon_2)(K-K^*)(|S|+|S|^3) + \mu\lambda_2Lc|\eta|\end{aligned} \tag{3.5.15}$$

It is easy to see that:

$$-\gamma(1-\varepsilon_2)(K-K^*)(|S|+|S|^3) + (\rho^2+\frac{1}{4})|S|^2 < 0 \quad \forall\ S\neq 0$$

if

$$\gamma(1-\varepsilon_2)(K-K^*) > \frac{1}{2}(\rho^2+\frac{1}{4})$$

Thus choose:

$$\gamma = \frac{(\rho^2+\frac{1}{4})}{(1-\varepsilon_2)(K-K^*)}$$
$$\mu = \frac{\lambda_1}{2\lambda_2^2 L^2(1+(A+1)^2)} \tag{3.5.16}$$

Then letting $\Omega_C := \{(\bar{e},S,\eta) \mid V \le C\}$, it is clear that with the above choices for γ and μ, we can find $\underline{C}(c) > 0$ such that $\dot{V}$ is negative outside of $\Omega_{\underline{C}}$. Furthermore $\underline{C}(0)=0$. Thus letting:

$$c^* := \sup \{c \mid (\bar{e},S,\eta) \in \Omega_{\underline{C}(c)} \Longrightarrow (\xi,\eta) \in \Phi(U)\} \tag{3.5.17}$$

we can find an open set $\Omega \subset \Phi(U)$ such that for all $c<c^*$ and all initial conditions in Ω the trajectories of the system remain in $\Phi(U)$.

□

3.6. Adaptive Tracking in the Presence of Parametric Uncertainties

We will now turn to the class of perturbations satisfying Assumption 3.3.2. Because the perturbations in this case arise from parametric uncertainties in the vector fields, we can see that the true system has the form:

$$\dot{x} = \sum_{i=1}^{N} \alpha_i^* f_i(x) + \sum_{j=1}^{M} \beta_j^* g_j(x) u$$
$$y = h(x) \tag{3.6.1}$$

where $f_i(\cdot)$'s and $g_j(\cdot)$'s are smooth vector fields in the open set $U \subset \mathbb{R}^n$, $f_i(0)=0$, $\forall\, i$, and $h(\cdot)$ is a smooth function. Moreover the scalars α_i^*'s and β_j^*'s are constant parameters which are assumed to be unknown. Our model of the system then, is based on our estimates of these parameters, i.e., α_i's and β_j's. Our approach here in dealing with the uncertainties is fundamentally different from those of the previous two sections. Because of the special structure of the system, namely linearity in the unknown parameters, we will be able to update our estimates of the parameters based on the observations of the tracking errors. Thus the controller parameters in this case are time varying rather than fixed. This is in contrast to the controllers studied in Sections 3.4 and 3.5.

In our development we will first consider systems with relative degree one, and later generalize the approach to higher relative degrees.

3.6.1. Relative Degree One Systems:

The system (3.6.1) can be transformed to its normal form according to Proposition 3.3.2. Thus in the new coordinates we can write:

$$\begin{aligned} \dot{y} &= \sum_{i=1}^{N} \alpha_i^* L_{f_i}h(x) + \sum_{j=1}^{M} \beta_j^* L_{g_j}h(x)u \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{3.6.2}$$

If the parameters were known, the control law:

$$u^* = \frac{1}{\sum_{j=1}^{M} \beta_j^* L_{g_j}h(x)} \left[-\sum_{i=1}^{N} \alpha_i^* L_{f_i}h(x) + \dot{y}_d - a(y-y_d(t))\right] \tag{3.6.3}$$

would result in asymptotic tracking of the desired signal $y_d(t)$ by the output, for $a > 0$. In the absence of perfect knowledge of the parameters, however, we will replace them in (3.6.3) with their estimates. Thus the actual control law is given by

$$u = \frac{1}{\sum_{j=1}^{M} \beta_j L_{g_j}h(x)} \left[-\sum_{i=1}^{N} \alpha_i L_{f_i}h(x) + \dot{y}_d - a(y-y_d(t))\right] \tag{3.6.4}$$

Because the system is assumed to have strong relative degree one, we know that the function:

$$\sum_{j=1}^{M} \beta_j^* L_{g_j}h(x) \tag{3.6.5}$$

is bounded away from zero, and therefore its inverse is well defined for all $x \in U$. Thus the control law (3.6.3) is well defined for all $x \in U$. However, the control law (3.6.4) may not be well defined, since the function:

$$\sum_{j=1}^{M} \beta_j \, L_{g_j} h(x) \tag{3.6.6}$$

may become zero for some choice of the parameter estimates β_j's. To eliminate such a possibility we shall restrict the parameter estimates to take on values which guarantee boundedness of (3.6.6) away from zero. We make the following assumption to this effect.

Assumption 3.6.1: Let the bounds (3.3.6) be given. There exists a constant $\delta > 0$, such that the function in (3.6.6) is bounded away from zero for all parameter estimates satisfying:

$$\beta_j \in (l_{\beta_j} - \delta \, , \, h_{\beta_j} + \delta) =: I_{\beta_j} \tag{3.6.7}$$

□

Thus Assumption 3.6.1 guarantees that whenever our estimates of the parameters lie within the bounds (3.6.7), the control law (3.6.4) is bounded for all $x \in U$. Now using this control law in (3.6.2) we have:

$$\dot{y} = \sum_{i=1}^{N} \alpha_i^* \, L_{f_i} h(x) + \frac{\sum_{j=1}^{M} \beta_j^* \, L_{g_j} h(x)}{\sum_{j=1}^{M} \beta_j \, L_{g_j} h(x)} [- \sum_{i=1}^{N} \alpha_i \, L_{f_i} h(x) + \dot{y}_d - a(y - y_d)] \tag{3.6.8}$$

which can be written in the following form:

$$\dot{y} = \dot{y}_d - a(y - y_d) + \sum_{i=1}^{N} (\alpha_i^* - \alpha_i) \, L_{f_i} h(x) + \sum_{j=1}^{M} (\beta_j^* - \beta_j) \, L_{g_j} h(x) \, u \tag{3.6.9}$$

Define:

$$\begin{aligned} \Phi^T &:= [\alpha_1^* - \alpha_1, \ldots, \alpha_N^* - \alpha_N, \beta_1^* - \beta_1, \ldots, \beta_M^* - \beta_M] \\ W^T(x,t) &:= [L_{f_1} h(x), \ldots, L_{f_N} h(x), L_{g_1} h(x) u, \ldots, L_{g_M} h(x) u] \end{aligned} \tag{3.6.10}$$

Then (3.6.9) becomes:

$$\dot{e} = -ae + \Phi^T W(x,t) \tag{3.6.11}$$

where $e := y - y_d(t)$.

Theorem 3.6.1: Let the system (3.6.1) have strong relative degree one and be hyperbolically minimum-phase. Let the control law (3.6.4) be chosen with the parameter update law given by:

$$\dot{\Phi} = -We \quad \forall \ \beta_j \in I_{\beta_j} \tag{3.6.12}$$

and the parameter resetting law given by:

$$\beta_j(t^+) = \begin{cases} l_{\beta_j} & \text{if } \ \beta_j(t) = l_{\beta_j} - \delta \\ h_{\beta_j} & \text{if } \ \beta_j(t) = h_{\beta_j} + \delta \end{cases} \tag{3.6.13}$$

Let $\dot{y}_d(t)$ be bounded. Then there exists a positive constant c^* and an open set $\Omega \subset \Phi(U)$, such that if $|y_d| < c^*$, then for all initial conditions in Ω output tracking of the desired signal $y_d(t)$ is achieved with the trajectories remaining in the set $\Phi(U)$.

□

Remark 3.6.1: The parameter resetting law (3.6.13) is adopted from [14], where it was first introduced in the context of adaptive control of robotic manipulators. The purpose of parameter resetting is to keep the parameter estimates within the bounds (3.6.7) so as to ensure the boundedness of the control u.

□

Proof: We consider the positive definite function:

$$V = e^2 + \Phi^T \Phi \tag{3.6.14}$$

We will first compute the change in V due to resetting of a single parameter, say β_k, at time t. We have:

$$\Delta V(t) = V(t^+) - V(t)$$

$$= e^2(t^+) + \sum_{i=1}^{N}(\alpha_i^* - \alpha_i(t^+))^2 + \sum_{j=1}^{M}(\beta_j^* - \beta_j(t^+))^2$$
$$- e^2(t) - \sum_{i=1}^{N}(\alpha_i^* - \alpha_i(t))^2 - \sum_{j=1}^{M}(\beta_j^* - \beta_j(t))^2 \tag{3.6.15}$$

It is clear from (3.6.11) that there will be no discontinuity in e as a result of a parameter jump; thus $e(t^+) = e(t)$. In addition all parameters other than β_k will remain unchanged at t^+. Therefore (3.6.15) becomes:

$$\Delta V(t) = (\beta_k^* - \beta_k(t^+))^2 - (\beta_k^* - \beta_k(t))^2 \tag{3.6.16}$$

Now using the resetting law (3.6.13) in (3.6.16) gives:

$$\Delta V(t) = \begin{cases} -\delta^2 - 2(\beta_k^* - l_{\beta_k})\,\delta & \text{if} \quad \beta_k(t) = l_{\beta_k} - \delta \\ -\delta^2 - 2(h_{\beta_k} - \beta_k^*)\,\delta & \text{if} \quad \beta_k(t) = h_{\beta_k} + \delta \end{cases} \tag{3.6.17}$$

Therefore the change in V due to a parameter resetting is always negative since $\beta_k^* - l_{\beta_k} > 0$ and $h_{\beta_k} - \beta_k^* > 0$ by definition. Furthermore from (3.6.17) we can conclude that for all parameter resettings:

$$\Delta V(t) \leq -\delta^2 \tag{3.6.18}$$

Now, differentiating V along (3.6.11) and (3.6.12) we have:

$$\dot{V} = -a\,e^2 + \sum_{i=1}^{s}\delta(t - t_i)\Delta V(t_i) \tag{3.6.19}$$

where t_i's are the instants of time at which resettings take place, s is the total number of resettings, and $\delta(\cdot)$ is a delta function. From (3.6.19) it is clear that $\dot{V}$ is always negative for $e \neq 0$. Therefor from (3.6.14) we conclude that, e and Φ will be bounded. In addition, with e and y_d bounded, ξ is bounded and thus from the minimum-phase property of the system we know that η remains bounded as well. Moreover from inequalities (3.2.15) and the bounds on Φ we may find a constant c^* and an open set $\Omega \subset \Phi(U)$ such that if $|y_d| < c^*$, then for all initial conditions in Ω we have that (ξ,η) remains in the open set

$\Phi(U)$.

To show that $e \to 0$ as $t \to \infty$, we note that boundedness of e, Φ, η, and $\dot{y}_d(t)$ implies that W is bounded. This in turn implies that $\dot{e}$ is bounded. Therefor e is uniformly continuous. Furthermore, since by (3.6.18) we know that $|\Delta V(t_i)|$ is bounded away from zero, and since V is positive, we may conclude that at most a finite number of resettings may occur. Thus $s < \infty$. So after all parameter resettings have occurred, that is $t > t_s$, we have $\dot{V} = -a\, e^2$. Therefore $\int_{t_s}^{\infty} e^2\, dt < \infty$. This and the uniform continuity of e establish convergence of e to zero.

□

3.6.2. Systems With Relative Degree Larger Than One

Assume that the system (3.6.1) has strong relative degree $\nu > 1$. Then by Proposition 3.3.2 it can be transformed to:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_\nu &= L_{f^*}^{\nu} h(x) + L_{g^*} L_{f^*}^{\nu-1} h(x)\, u \\
\dot{\eta} &= q(\xi,\eta)
\end{aligned}
\qquad (3.6.20)
$$

where $f^*(x) := \sum_{i=1}^{N} \alpha_i^* f_i(x)$ and $g^*(x) := \sum_{j=1}^{M} \beta_j^*\, g_j(x)$.

In extending the straight forward approach of the previous subsection to the current situation we encounter two sources of difficulties. First, the functions $L_{f^*}^{\nu} h(x)$ and $L_{g^*} L_{f^*}^{\nu-1} h(x)$ are no longer linear in the unknown parameters when $\nu > 1$. They involve various products of these parameters. This can be seen from the following expressions:

$$
L_{f^*}^{\nu} h(x) = \sum_{i_1,\dots,i_\nu}^{N} \alpha_{i_1}^* \alpha_{i_2}^* \cdots \alpha_{i_\nu}^*\, L_{f_{i_1}} L_{f_{i_2}} \cdots L_{f_{i_\nu}} h(x) \qquad (3.6.21)
$$

$$L_g L_f^{\nu-1} h(x) = \sum_{i_1,\ldots,i_{\nu-1}}^{N} \sum_{j=1}^{M} \alpha_{i_1}^* \cdots \alpha_{i_{\nu-1}}^* \beta_j^* \, L_{g_j} L_{f_{i_1}} \cdots L_{f_{i_{\nu-1}}} h(x) \tag{3.6.22}$$

This problem can be resolved by defining each of the products $\alpha_{i_1}^* \alpha_{i_2}^* \cdots \alpha_{i_\nu}^*$ and $\alpha_{i_1}^* \cdots \alpha_{i_{\nu-1}}^* \beta_j^*$ as a new parameter which can be updated separately. Such a definition would allow us to parallel the development of the previous subsection for the new parameters which now appear linearly in (3.6.20). Therefore, we define a new parameter vector as follows:

$$\theta^{1*} := [(\alpha_1^*)^\nu, (\alpha_1^*)^{\nu-1}\alpha_2^*, \ldots, (\alpha_N^*)^\nu, (\alpha_1^*)^{\nu-1}\beta_1^*, \ldots, (\alpha_N^*)^{\nu-1}\beta_M^*]^T \tag{3.6.23}$$

So that (3.6.20) may be written as:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_\nu &= \theta^{1*T} [W_1(x) + W_2(x)\, u] \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{3.6.24}$$

where $W_1(x)$ and $W_2(x)$ are vectors whose components are the various successive Lie derivatives of $h(x)$ along the vector fields $f_i(x)$ and $g_j(x)$, as they appear in (3.6.21) and (3.6.22). Now, we use the control law:

$$u = \frac{1}{\theta^{1^T} W_2(x)} [-\theta^{1^T} W_1(x) + v] \tag{3.6.25}$$

where θ^1 is the estimate of θ^{1*}. Then with this control (3.6.24) becomes:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_\nu &= v + \Phi^{1^T} [W_1(x) + W_2(x)\, u] \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{3.6.26}$$

where $\Phi^1 := \theta^{1*} - \theta^1$.

The second difficulty concerns the availibility of the ξ_k's for use in feedback. At this point, to parallel the development of the relative degree one case, we need to use the ξ_k's both in the control v and in the update law for the parameters. Recall that $\xi_k := L_{f^*}^{k-1}h(x)$, $k=1,...,\nu$. Since the vector field $f^*(x)$ is a function of the parameters, so are the ξ_k's. Therefore, in the absence of exact knowledge of the parameters, the ξ_k's are also unknown and can not be used in feedback.

As far as the control v is concerned, we can use the estimates of the states ξ_k's in the control law and in doing so we will augment the parameter vector θ^{1*} with additional parameters to be updated. The states ξ_k's can be written as:

$$\xi_k := L_{f^*}^{k-1}h(x) = \sum_{i_1,\ldots,i_{k-1}}^{N} \alpha_{i_1}^* \cdots \alpha_{i_{k-1}}^* L_{f_{i_1}} L_{f_{i_2}} \cdots L_{f_{i_{k-1}}} h(x)\ ,\ \ k=2,...,\nu \tag{3.6.27}$$

Therefore we define the parameter vector:

$$\theta^{2*} := [\alpha_1^*, \ldots, \alpha_N^*, (\alpha_1^*)^2, \alpha_1^*\alpha_2^*, \ldots, (\alpha_N^*)^2, \ldots, (\alpha_N^*)^{\nu-1}]^T \tag{3.6.28}$$

Using this definition in (3.6.27) we can write:

$$\xi_k = \theta^{2*^T}\overline{W}_k(x)\ ,\ \ k=2,...,\nu \tag{3.6.29}$$

where $\overline{W}_2(x)$ through $\overline{W}_\nu(x)$ are vectors whose components are zeros and various Lie derivatives of $h(x)$ along the vector fields $f_i(x)$, $i=1,...,n$.

The control v is chosen to be:

$$v = -\,\theta^{2^T}[a_1\overline{W}_\nu(x) + a_2\overline{W}_{\nu-1}(x) + \cdots + a_{\nu-1}\overline{W}_2(x) + a_\nu\xi_1]$$

$$+\, a_1\xi_\nu^d(t) + a_2\xi_{\nu-1}^d(t) + \cdots + a_\nu\xi_1^d(t) \tag{3.6.30}$$

where θ^2 is the estimate of the parameter vector θ^{2*} and the coefficients a_1 through a_ν are chosen so that the polynomial $s^\nu + a_1 s^{\nu-1} + \cdots + a_1$ is Hurwitz. Defining $\Phi^2 := \theta^{2*} - \theta^2$, we can write (3.6.30) as:

$$v = -a_1 e_\nu - a_2 e_{\nu-1} - \cdots - a_\nu e_1 + \Phi^{2^T} W_3(x) \tag{3.6.31}$$

where we have defined $W_3(x)$ by:

$$W_3(x) := a_1 \overline{W}_\nu(x) + \cdots + a_{\nu-1} \overline{W}_2(x) \tag{3.6.32}$$

Using (3.6.31) in (3.6.26) we obtain:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_{\nu-1} &= e_\nu \\
\dot{e}_\nu &= -a_1 e_\nu - a_2 e_{\nu-1} - \cdots - a_\nu e_1 + \Phi^{1^T}[W_1(x) + W_2(x)\, u] + \Phi^{2^T} W_3(x) \\
\dot{\eta} &= q(\xi, \eta)
\end{aligned} \tag{3.6.33}$$

Defining:

$$\begin{aligned}
\theta^{*^T} &:= [\theta^{1*^T}, \theta^{2*^T}] \\
\theta^T &:= [\theta^{1^T}, \theta^{2^T}] \\
\Phi &:= \theta^* - \theta \\
W^T(x,t) &:= [W_1^T(x) + W_2^T(x)\, u\, ,\, W_3^T(x)]
\end{aligned} \tag{3.6.34}$$

we can write (3.6.33) as:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_{\nu-1} &= e_\nu \\
\dot{e}_\nu &= -a_1 e_\nu - a_2 e_{\nu-1} - \cdots - a_\nu e_1 + \Phi^T W(x,t) \\
\dot{\eta} &= q(\xi, \eta)
\end{aligned} \tag{3.6.35}$$

Therefore with the new parameter vector we can repeat the development of the relative degree one case if we are able to update the parameters. However, the parameter update law requires the use of the e_k's $k=2,\ldots,\nu$, which are not available. We note, however, that the e_k's are the successive derivatives of e_1, which is available. Thus we can generate approximate derivatives of the output ξ_1 in order to construct approximate e_k's for use in the parameter update law which would result in approximate output tracking of the desired signal. The approximate derivatives will be generated by a ν dimensional filter which is discussed in the lemma below.

Lemma 3.6.1: Suppose the function $y(t)$ and its first ν derivatives are bounded, so that $|y^{(k)}| < Y_k$, k=0,...,ν where Y_k's are positive constants. Consider the following linear system:

$$\begin{aligned}
\varepsilon\dot{\zeta}_1 &= \zeta_2 \\
\varepsilon\dot{\zeta}_2 &= \zeta_3 \\
&\vdots \\
\varepsilon\dot{\zeta}_{\nu-1} &= \zeta_\nu \\
\varepsilon\dot{\zeta}_\nu &= -b_1\zeta_\nu - b_2\zeta_{\nu-1} - \cdots - \zeta_1 + y(t)
\end{aligned} \tag{3.6.36}$$

where the parameters b_1 through $b_{\nu-1}$ are chosen so that $s^\nu + s^{\nu-1} + ... + 1$ is Hurwitz. Then there exist positive constants K_k, k=2,...,ν and t^* such that for all $t > t^*$ we have:

$$|\frac{1}{\varepsilon^k}\zeta_{k+1} - y^{(k)}| \leq \varepsilon K_{k+1} \qquad k=1,\ldots,\nu-1 \tag{3.6.37}$$

Further the constants K_k are decreasing functions of the Y_k's.

□

Proof: We can use the last equation in (3.6.36) to find an expression for $\dot{y}$. So that:

$$\frac{1}{\varepsilon}\zeta_2 - \dot{y} = \frac{1}{\varepsilon}\zeta_2 - \varepsilon\ddot{\zeta}_\nu - b_1\dot{\zeta}_\nu - \cdots - \dot{\zeta}_1 \tag{3.6.38}$$

Next using (3.6.36) in (3.6.38) yields:

$$\frac{1}{\varepsilon}\zeta_2 - \dot{y} = -\varepsilon(\ddot{\zeta}_\nu + b_1\ddot{\zeta}_{\nu-1} + \cdots + b_{\nu-1}\ddot{\zeta}_1) \tag{3.6.39}$$

Now differentiating (3.6.39) and using (3.6.36) we have:

$$\frac{1}{\varepsilon^k}\zeta_{k+1} - y^{(k)} = -\varepsilon S^{(k+1)} \qquad k=1,\ldots,\nu-1 \tag{3.6.40}$$

where $S := \zeta_\nu + b_1\zeta_{\nu-1} + \cdots + b_{\nu-1}\zeta_1$. The proof is complete if there exist constants K_k

such that $|S^{(k)}| \leq K_k$. To show this, we note that the derivatives of the vector ζ may be computed as follows:

$$\zeta^{(k)}(t) = \frac{1}{\varepsilon^k} A^k e^{\frac{At}{\varepsilon}} [\zeta(0) + A^{-1}by(0) + \varepsilon A^{-2}b\dot{y}(0) + \cdots + \varepsilon^{k-1}A^{-k}by^{(k-1)}(0)]$$
$$+ \frac{1}{\varepsilon} e^{\frac{At}{\varepsilon}} \int_0^t e^{-\frac{A\tau}{\varepsilon}} by^{(k)}(\tau)d\tau \qquad (3.6.41)$$

where A is the matrix corresponding to the homogeneous part of (3.6.36) and $b := [0,0,...,1]^T$. Therefore for any $\delta > 0$, we may find a t^* so that for all $t > t^*$ the first term in (3.6.41) is bounded by δY_k for each k. Further since $y^{(k)}$ is bounded by Y_k, there are constants D_k such that the second term in (3.6.41) is bounded by $D_k Y_k$ for each k. Now fix an arbitrarily small δ^*. Then for $t > t^*$ we have that:

$$|S^{(k)}| \leq B(D_k + \delta^*)Y_k \doteq: K_k \qquad (3.6.42)$$

where B is the norm of the vector $[1, b_1, ..., b_{\nu-1}]$. This completes the proof.

□

Having the filter in (3.6.36) we now define the approximate error vector, denoted by e^a, as follows:

$$e^a := [\ \xi_1 - y_d(t), \frac{1}{\varepsilon} \zeta_2 - \dot{y}_d(t), \ldots, \frac{1}{\varepsilon^{\nu-1}} \zeta_\nu - y_d^{(\nu-1)}(t)\ ]^T \qquad (3.6.43)$$

We will use e^a in the parameter update law in place of e, the true error vector. Before stating our result, however, we make a further assumption regarding the boundedness of the control defined in (3.6.25) and the boundedness of the parameter estimates in the course of adaptation. This assumption is the counterpart of Assumption 3.6.1 for the relative degree one case. In this assumption, we use the bounds in (3.3.6) to compute bounds on the new parameters θ_i's. That is we find scalar constants h_{θ_i} and l_{θ_i} so that $\theta_i^* \in (l_{\theta_i}, h_{\theta_i})$.

Assumption 3.6.2: Let the parameters θ_i's be known to lie in open intervals $(l_{\theta_i}, h_{\theta_i})$.

There exists a positive constant δ such that the function $\theta^{1^T} W_2(x)$ is bounded away from zero for all parameter estimates satisfying:

$$\theta_i \in (l_{\theta_i} - \delta \; , \; h_{\theta_i} + \delta) =: I_{\theta_i} \tag{3.6.44}$$

□

We now present the main result of this subsection in the following theorem. In the theorem b denotes the column vector $[0,0,\ldots,1]^T$ and P is the symmetric positive definite matrix which satisfies the Lyapunov equation $PA + A^T P = -I$ with A being the matrix corresponding to the error vector field in (3.6.35) when Φ is zero.

Theorem 3.6.2: Let the system (3.6.1) have relative degree ν and be hyperbolically minimum-phase. Let the control law (3.6.25) be given with v specified by (3.6.31). Choose the parameter update law:

$$\dot{\Phi} = -\gamma \, e^{a^T} PbW \quad \forall \quad \theta_i \in I_{\theta_i} \tag{3.6.45}$$

where γ is a positive gain and the filter in (3.6.36) is turned on at $t < -t^*$. Let the parameter resetting law be given by:

$$\theta_i(t^+) = \begin{cases} l_{\theta_i} & \text{if } \; \theta_i(t) = l_{\theta_i} - \delta \\ h_{\theta_i} & \text{if } \; \theta_i(t) = h_{\theta_i} + \delta \end{cases} \tag{3.6.46}$$

Then there exist an open set $\Omega \subset \Phi(U)$ and positive constants c^*, d_1, d_2, and t_1 such that for $|\xi^d| < c^*$ we have that for all initial conditions in Ω the trajectories remain in the set $\Phi(U)$ and:

$$|e(t)| < d_1 \sqrt{\varepsilon} \; , \; \forall \; t > t_1 \tag{3.6.47}$$

provided that $\gamma > \dfrac{d_2}{\varepsilon}$ and $\dot{\xi}_\nu^d$ is bounded.

□

Proof: We will first assume that the trajectories remain in the open set $\Phi(U)$. Then we consider the following positive definite function of e and Φ:

$$V = e^T P e + \frac{1}{\gamma} \Phi^T \Phi \tag{3.6.48}$$

We know from the proof of Theorem 3.6.1 that V will always decrease as a result of a parameter resetting event. Now differentiating V yields:

$$\dot{V} = -|e|^2 + 2e^T Pb\Phi^T W - 2e^{a^T} Pb\Phi^T W \tag{3.6.49}$$

Now we use the fact that $e^a = e - \varepsilon p$, where p is defined by:

$$p^T := (0, \ddot{S}, \ldots, S^{(v)}) \tag{3.6.50}$$

and S is defined in (3.6.40). This results in:

$$\dot{V} = -|e|^2 + 2\varepsilon\, p^T Pb\Phi^T W \tag{3.6.51}$$

We know from Lemma 3.6.1 that $S^{(k)}$ are bounded if ξ and $\dot{\xi}_v$ are bounded. Now in the region $\Phi(U)$, ξ is bounded and by (3.6.26), $\dot{\xi}_v$ is bounded if $\dot{\xi}_v^d$ is bounded. Furthermore Φ remains bounded by the resetting mechanism. Therefore, we know that there exists a positive constant D, such that:

$$|\, 2p^T Pb\Phi^T W \,| < D\,, \quad \forall \;\; (\xi,\eta) \in \Phi(U) \tag{3.6.52}$$

Thus in $\Phi(U)$ we may write:

$$\dot{V} \leq -|e|^2 + \varepsilon\, D \tag{3.6.53}$$

Then adding on the effects of parameter resettings we have:

$$\dot{V} \leq -|e|^2 + \varepsilon\, D - \frac{1}{\gamma}\sum_{i=1}^{s} \delta(t - t_i)\, \delta^2 \tag{3.6.54}$$

Since the parameters are bounded due to resetting, we can see from (3.6.54) that there is a constant d_3 such that outside the set $G := \{(e,\Phi) \mid |e|^2 \leq \varepsilon\, D\,,\, |\Phi|^2 \leq d_3\}$, $\dot{V}$ is strictly negative. Thus we conclude that the system trajectories will enter and remain in the smallest set of the form $\Omega_c := \{(e,\Phi) \mid V \leq c\}$ which contains G. Thus define:

$$\underline{c} := \inf \{c \mid \Omega_c \supset G\} \tag{3.6.55}$$

To ensure that $|e|$ is small in the set $\Omega_{\underline{c}}$ we must have $\underline{c}$ small. Now define:

$$\begin{aligned} d_1^2 &:= 2\, D\, \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)} \\ d_2 &:= \frac{d_3}{D\, \sigma_{\max}(P)} \end{aligned} \tag{3.6.56}$$

where $\sigma_{\max}(P)$ and $\sigma_{\min}(P)$ are the maximum and the minimum singular values of the matrix P respectively. If $\gamma > \frac{d_2}{\varepsilon}$, then we can conclude that $\underline{c} < 2\varepsilon\sigma_{\max}(P)D$ and thus we have that:

$$|e| \leq d_1\sqrt{\varepsilon}\,, \quad \forall\ e \in \Omega_{\underline{c}} \tag{3.6.57}$$

To prove that the trajectories will remain in the open set $\Phi(U)$ we use the Lyapunov function:

$$V_1 = e^TPe + \frac{1}{\gamma}\Phi^T\Phi + \mu V_0(\eta) \tag{3.6.58}$$

where P is as in (3.6.48) and $V_0(\eta)$ is the Lyapunov function for the zero dynamics and satisfies the inequalities in (3.2.15). Differentiating V_1 along the flow of (3.6.35) and using (3.2.15) and (3.6.52) we obtain:

$$\dot{V}_1 \leq -\,|e|^2 - \mu\lambda_1|\eta|^2 + \mu\lambda^2L|\eta||\xi| + \varepsilon D \tag{3.6.59}$$

Then using $|\xi| \leq |e| + |\xi^d|$ and $|\xi^d| < c$, we conclude from (3.6.59):

$$\dot{V}_1 \leq -\,|e|^2 - \mu\lambda_1|\eta|^2 + \mu\lambda_2L|\eta||e| + \mu\lambda_2Lc|\eta| + \varepsilon D \tag{3.6.60}$$

Using the inequalities:

$$\begin{aligned} \mu\lambda_2L|\eta||e| &\leq \frac{1}{4}|e|^2 + \mu^2\lambda_2^2L^2|\eta|^2 \\ \mu\lambda_2Lc|\eta| &\leq \frac{1}{4}c^2 + \mu^2\lambda_2^2L^2|\eta|^2 \end{aligned} \tag{3.6.61}$$

in (3.6.60) we have:

$$\dot{V}_1 \le -\frac{3}{4}|e|^2 - \mu\lambda_1(1 - 2\mu\frac{\lambda_2^2 L^2}{\lambda_1})|\eta|^2 + \frac{1}{4}c^2 + \varepsilon D \tag{3.6.62}$$

Then choosing μ small enough results in a negative coefficient in front of $|\eta|^2$. Thus for example choose:

$$\mu = \frac{\lambda_1}{4\lambda_2^2 L^2} \tag{3.6.63}$$

By virtue of the resetting mechanism we have $|\Phi|^2 \le d_3$. Then with μ chosen in (3.6.63) we can see that outside the set:

$$G_1 := \{(e,\Phi,\eta) \mid \frac{3}{4}|e|^2 + \frac{1}{2}\mu\lambda_1|\eta|^2 \le \varepsilon D + \frac{1}{4}c^2 \ ; \ |\Phi|^2 \le d_3\} \tag{3.6.64}$$

$\dot{V}_1$ is strictly negative. Now define:

$$\begin{aligned} &\Omega_C := \{(e,\Phi,\eta) \mid V_1 \le C\} \\ &\underline{C} := \inf\{C \mid \Omega_C \supset G_1\} \end{aligned} \tag{3.6.65}$$

Then clearly all trajectories starting in $\Omega_{\underline{C}}$ will remain in $\Omega_{\underline{C}}$ for all time. From (3.6.65) we can also see that $\underline{C}$ is a function of ε and c. Therefore for ε small we can find c^* such that for all $c < c^*$ we have that:

$$(e,\Phi,\eta) \in \Omega_{\underline{C}(c)} \Longrightarrow (\xi,\eta) \in \Phi(U)$$

Therefore we can find an open set $\Omega \subset \Phi(U)$ such that for all $\xi^d(t)$ with $|\xi^d| < c^*$ and all initial conditions in Ω we have $(\xi(t),\eta(t)) \in \Phi(U)$, $\forall t$.

□

3.7. Concluding Remarks

In this chapter we have presented the basic output tracking control scheme for SISO nonlinear control systems along with several techniques for robustifying the scheme with respect to modeling errors. In Sections 3.4 and 3.5 we used high gain and sliding mode control schemes, respectively, in order to deal with the class of uncertainties satisfying the generalized matching condition (3.3.3). Although this class of uncertainties was shown to be significantly broader than the class of uncertainties satisfying the matching condition (3.3.7), it does not encompass all uncertainties of interest. Thus, it is extremely desirable to design schemes for dealing with uncertainties excluded by (3.3.3) as well. More work in this direction needs to be done.

In Section 3.6 we presented a new adaptive scheme which is based on computing approximate derivatives. This scheme should be compared with the augmented error scheme of Narendra, Lin, and Valavani [29] for linear systems, and Sastry and Isidori [36] for nonlinear systems. Although our scheme is considerably simpler to implement than the augmented error scheme (the augmented error scheme requires as many filters as there are parameters), it results in small rather than zero tracking errors. In addition, in our scheme the filter which computes approximate derivatives must be turned on before adaptation starts. This is done to keep the initial transient error in the filter from entering the adaptation loop.

In the following chapter we will focus on the issue of robustness to unmodeled dynamics. The analysis of the robustness properties of a tracking system is extremely important from a practical point of view, in that it results in control design criteria which must be met in order to ensure stability and adequate performance for the system.

Chapter Four

Robust Tracking with Unmodeled Dynamics

4.1. Introduction

In choosing a model of a physical system for the purpose of control design, one often tries to find the least complicated model which is adequate for the control task at hand. It is inevitable that the simplicity of the model comes at the expense of neglecting some characteristics of the system in the model. Often the neglected effects correspond to the part of the dynamics which is, in some sense, secondary with respect to the control task. Such effects typically arise from time scale separations inherent in the system. In this case, if the control variations are on the slow time scale of the system, the faster dynamics of the system constitute a secondary effect and may be neglected in the control design process. In linear systems, this corresponds to neglecting high frequency effects of the system which lie outside the bandwidth of the control input. The neglected part of the dynamics is generally referred to as "unmodeled dynamics" in the control literature.

Although by neglecting part of the dynamics of the system the control design process is greatly simplified, we are left with the additional task of guaranteeing robustness of the design with respect to the unmodeled dynamics. That is, we must guarantee that the use of the control input which was designed on the basis of a simplified model, results in stability and adequate performance in the true system.

This chapter is devoted to the analysis of the controllers of the previous chapter when they are implemented in the presence of unmodeled dynamics. One method of introducing the unmodeled dynamics is through the use of a small singular perturbation parameter, denoted by ε, which multiplies the derivative of the state variables corresponding to the fast dynamics in the system. In this case, the simplified model, which is referred to as the reduced model, is computed formally by setting ε equal to zero. We, therefore, assume that the system (3.2.1) of the previous chapter corresponds to the reduced model of the following singularly perturbed system [12]:

$$\begin{aligned} \dot{x} &= f_1(x) + F_1(x)\, z + g_1(x)\, u \\ \varepsilon \dot{z} &= f_2(x) + F_2(x)\, z + g_2(x)\, u \\ y &= h(x) \end{aligned} \tag{4.1.1}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f_1(\cdot)$, $f_2(\cdot)$, $g_1(\cdot)$, and $g_2(\cdot)$ are smooth vector fields on an open set $U \subset \mathbb{R}^n$, $f_1(0) = 0$, $f_2(0) = 0$, $F_1(\cdot)$ and $F_2(\cdot)$ are $n \times m$ and $m \times m$ matrices whose columns are smooth vector fields on U, and $h(\cdot)$ is a smooth function on U. In (4.1.1) the state variables z contain the fast dynamics of the system.

The basic assumption in singularly perturbed systems is the nonsingularity of the matrix $F_2(x)$ for all $x \in U$. This assumption is necessary for the system (4.1.1) to exhibit a two time scale behavior and possess a reduced model of dimension n. In our study, however, we make a stronger assumption which also asserts the stability of the unmodeled dynamics.

Assumption 4.1.1: (Stability of unmodeled dynamics) Let $\sigma(F_2(x))$ denote the spectrum of $F_2(x)$ for each $x \in U$. Then there exists $\sigma^* > 0$ such that

$$\mathrm{Re}(\sigma(F_2(x))) \leq -\sigma^* \quad \forall \; x \in U$$

□

With this assumption we may compute the reduced model of (4.1.1) as follows: letting $\varepsilon = 0$, we obtain:

$$0 = f_2(x) + F_2(x)\, z + g_2(x)\, u \tag{4.1.2}$$

Solving for z and denoting the solution by z_s we have:

$$z_s := -F_2^{-1}(x)\, [f_2(x) + g_2(x)\, u] \tag{4.1.3}$$

The subscript s in z_s refers to the fact that (4.1.3) represents the slow component of the state variables z. Substituting (4.1.3) in the first equation in (4.1.1) we obtain the following reduced model of the system:

$$\dot{x} = f_1(x) - F_1(x)F_2^{-1}(x)\, [f_2(x) + g_2(x)\, u] + g_1(x)\, u \tag{4.1.4}$$

which can be written as (3.2.1) if we define:

$$\begin{aligned} f(x) &:= f_1(x) - F_1(x)F_2^{-1}(x)f_2(x) \\ g(x) &:= g_1(x) - F_1(x)F_2^{-1}(x)g_2(x) \end{aligned} \tag{4.1.5}$$

Following [12], we will define the fast component of the state z by:

$$z_f := z - z_s \tag{4.1.6}$$

Then rewriting (4.1.1) in terms of z_f, we obtain:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\, u + F_1(x)\, z_f \\ \varepsilon \dot{z}_f &= F_2(x)\, z_f - \varepsilon \frac{d}{dt}(z_s(x)) \\ y &= h(x) \end{aligned} \tag{4.1.7}$$

We note that in (4.1.7) the reduced model of the system appears explicitly and can be obtained as the limit of (4.1.7) when ε tends to zero. It is also clear from (4.1.7) that z_f represents the fast dynamics of the system and if Assumption 4.1.1 holds, it will converge to zero as ε tends to zero. To show the explicit dependence of the dynamics in (4.1.7) on the control input u, we compute the derivative of $z_s(x)$ with respect to time. Therefore, we have:

$$\frac{d}{dt}(z_s(x)) = -\frac{\partial}{\partial x}(F_2^{-1}(x)f_2(x))\,[f(x) + g(x)\,u + F_1(x)\,z_f]$$
$$-\frac{\partial}{\partial x}(F_2^{-1}(x)g_2(x))\,[f(x) + g(x)\,u + F_1(x)\,z_f]\,u$$
$$-F_2^{-1}(x)g_2(x)\,\dot{u} \tag{4.1.8}$$

To simplify the presentation throughout this chapter we define the following quantities:

$$\begin{aligned} r_1(x) &:= \frac{\partial}{\partial x}(F_2^{-1}(x)f_2(x))\,f(x) \\ r_2(x) &:= \frac{\partial}{\partial x}(F_2^{-1}(x)f_2(x))\,g(x) + \frac{\partial}{\partial x}(F_2^{-1}(x)g_2(x))\,f(x) \\ r_3(x) &:= \frac{\partial}{\partial x}(F_2^{-1}(x)g_2(x))\,g(x) \\ r_4(x) &:= F_2^{-1}(x)g_2(x) \\ R_1(x) &:= \frac{\partial}{\partial x}(F_2^{-1}(x)f_2(x))\,F_1(x) \\ R_2(x) &:= \frac{\partial}{\partial x}(F_2^{-1}(x)g_2(x))\,F_1(x) \end{aligned} \tag{4.1.9}$$

Using (4.1.8) and (4.1.9) in (4.1.7) yields:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u + F_1(x)\,z_f \\ \varepsilon\dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\,u]\,z_f + \varepsilon[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}] \\ y &= h(x) \end{aligned} \tag{4.1.10}$$

The system in (4.1.10) is in the appropriate form for our subsequent analysis of the robustness properties of the controllers introduced in the previous chapter. It contains the reduced model of the system, for which the controllers were designed, and it is explicitly written in terms of the slow and the fast modes of the system.

4.2. Robustness of High Gain Control to Unmodeled Dynamics:

In Section 3.4 we applied high gain control to the system (3.3.1) in order to reduce the effects of the uncertainties of the system on the tracking errors. Here, we will assume that (3.3.1) is the reduced model of a full order system of the form (4.1.10). Our goal is to analyze the performance of the high gain controller when it is implemented in the full order system. Therefore, we start with the following variation of (4.1.10) which contains the uncertainties in the reduced model.

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u + \Delta f(x) + \Delta g(x)\,u + F_1(x)\,z_f \\ \varepsilon\dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\,u]\,z_f + \varepsilon[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}] \\ y &= h(x) \end{aligned} \tag{4.2.1}$$

where, as in Section 3.4, we assume that $\Delta f(x)$ and $\Delta g(x)$ have indices larger or equal to the relative degree of the unperturbed reduced system, namely ν. By Proposition 3.2.2, we can perform the change of coordinates $(\xi,\eta,z_f) = (\Phi(x),z_f)$ to obtain:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \chi_1(x)\,z_f \\ &\vdots \\ \dot{\xi}_{\nu-1} &= \xi_\nu + \chi_{\nu-1}(x)\,z_f \\ \dot{\xi}_\nu &= b(\xi,\eta) + a(\xi,\eta)\,u + \bar{\delta}_1(\xi,\eta) + \bar{\delta}_2(\xi,\eta)\,u + \chi_\nu(x)\,z_f \\ \dot{\eta} &= q(\xi,\eta) + \bar{\chi}(x)\,z_f \\ \varepsilon\dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\,u]\,z_f + \varepsilon[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}] \\ y &= \xi_1 \end{aligned} \tag{4.2.2}$$

where $\chi_k(x) := d\xi_k(x) \cdot F_1(x)$ for $k=1,\dots,\nu$, $\bar{\chi}(x) := d\eta(x) \cdot F_1(x)$, and all other quantities are as defined in (3.4.2). Next, given a desired tracking signal $y_d(t)$, we choose the control law given in Section 3.4, that is:

$$u = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta) + \dot{\xi}_\nu^d - \frac{1}{\bar{\varepsilon}}[e_\nu + a_1 e_{\nu-1} + \cdots + a_{\nu-1}e_1]\,] \tag{4.2.3}$$

In (4.2.3), $\bar{\varepsilon}$ is the high gain parameter (which was denoted by ε in Section 3.4) and $e_k := \xi_k - \xi_k^d$ for $k=1,\dots,\nu$ where ξ_k^d denotes the $(k-1)$st derivative of the signal $y_d(t)$. The parameters a_1 through $a_{\nu-1}$ are chosen such that the polynomial $s^{\nu-1} + a_1 s^{\nu-2} + \cdots + a_{\nu-1}$ is Hurwitz. Using (4.2.3) in (4.2.2) yields:

$$
\begin{aligned}
&\dot{e}_1 = e_2 + \chi_1(x)\, z_f \\
&\vdots \\
&\dot{e}_{v-1} = e_v + \chi_{v-1}(x)\, zf \\
&\dot{e}_v = -\frac{1}{\overline{\varepsilon}}[e_v + a_1 e_{v-1} + \cdots + a_{v-1}e_1] + \delta_1(\xi,\eta) + \delta_2(\xi,\eta)\, v + \chi_v(x)\, zf \\
&\dot{\eta} = q(\xi,\eta) + \overline{\chi}(x)\, zf \\
&\varepsilon \dot{z}_f = [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\, u]\, z_f + \varepsilon[r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}] \\
&y = \xi_1
\end{aligned}
\tag{4.2.4}
$$

where v is defined by:

$$
v := \dot{\xi}_v^d - \frac{1}{\overline{\varepsilon}}[e_v + a_1 e_{v-1} + \cdots + a_{v-1}e_1] \tag{4.2.5}
$$

We now use the definitions given in (3.4.8) to transform (4.2.4) to the following system:

$$
\begin{aligned}
&\dot{e}_1 = e_2 + \chi_1(x)\, z_f \\
&\vdots \\
&\dot{e}_{v-1} = \zeta - E_1 + \chi_{v-1}(x)\, zf \\
&\overline{\varepsilon}\dot{\zeta} = -\,[1 - \overline{\varepsilon}a_1 - \delta_2(\xi,\eta)]\zeta + \overline{\varepsilon}[\,\delta_1(\xi,\eta) + \delta_2(\xi,\eta)\, \dot{\xi}_v^d + \chi_v(x)\, zf\,] \\
&\dot{\eta} = q(\xi,\eta) + \overline{\chi}(x)\, zf \\
&\varepsilon \dot{z}_f = [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\, u]\, z_f + \varepsilon[r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}] \\
&y = \xi_1
\end{aligned}
\tag{4.2.6}
$$

Theorem 4.2.1: Let the reduced system of (4.2.1) have relative degree v and be hyperbolically minimum-phase. Let the perturbation vector fields in (4.2.1) have indices larger or equal to v. Let ε_2, as defined in (3.4.5), be less than 1. Let the desired tracking signal and its first $(v+1)$ derivatives be bounded. Let the control law be specified by (4.2.3). Then there exists a positive constant $\overline{\varepsilon}^*$, a monotone increasing function $\varepsilon^*(\overline{\varepsilon})$, a set $D \subset \{\, (c,d,d_1) \in \mathbb{R}^3 \mid c \geq 0\,,\, d \geq 0\,, d_1 \geq 0\}$, and an open set $\Omega \subset \Phi(U) \times \mathbb{R}^m$ such that for all $\overline{\varepsilon} \leq \overline{\varepsilon}^*$, all $\varepsilon \leq \varepsilon^*(\overline{\varepsilon})$, all desired tracking signals satisfying $|\xi(t)| \leq c$, $|\dot{\xi}_v(t)| \leq d$, and $|\ddot{\xi}_v^d| \leq d_1$ for all t with $(c,d) \in D$, and all initial conditions in Ω, the trajectories remain bounded and in $\Phi(U) \times \mathbb{R}^m$, $\forall\, t$ and the tracking error will be of $O(\overline{\varepsilon})$.

□

Proof: To prove the boundedness of the solutions, we consider the following positive definite function as a Lyapunov function candidate:

$$V = \bar{e}^T P \bar{e} + \frac{1}{2}\gamma\zeta^2 + \mu V_0(\eta) + z_f^T \bar{P}(x) z_f \tag{4.2.7}$$

where $\bar{e} := [e_1, \ldots, e_{v-1}]^T$, P is defined in (3.4.11), $V_0(\eta)$ is a Lyapunov function for the zero dynamics of (3.3.1) and satisfies (3.2.15), the scalars γ and μ are positive constants to be determined later, and $\bar{P}(x)$ is a positive definite matrix for all $x \in U$ and satisfies the following Lyapunov equation:

$$\bar{P}(x)F_2(x) + F_2^T(x)\bar{P}(x) = -I \tag{4.2.8}$$

where I is the $m \times m$ identity matrix. The existence of $\bar{P}(x)$ is guaranteed by Assumption 4.1.1. Differentiating V along the flow of (4.2.6) yields:

$$\begin{aligned}\dot{V} = &- |\bar{e}|^2 + 2\bar{e}^T P b\zeta + 2\bar{e}^T P\chi(x)z_f - \gamma[1 - \varepsilon a_1 - \delta_2(\xi,\eta)]\zeta^2 \\ &+ \frac{\gamma}{\varepsilon}[\delta_1(\xi,\eta) + \delta_2(\xi,\eta)\dot{\xi}_v^d + E_2 + \chi_v(x)z_f]\zeta + \mu\frac{\partial V_0}{\partial \eta}q(\xi,\eta) + \mu\frac{\partial V_0}{\partial \eta}\bar{\chi}(x)z_f \\ &- \frac{1}{\varepsilon}|z_f|^2 + z_f^T[\bar{P}(x)(R_1(x) + R_2(x)\,u) + (R_1(x) + R_2(x)\,u)^T\bar{P}(x)]z_f \\ &+ z_f^T\bar{P}(x)[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}] + z_f^T\dot{\bar{P}}(x)z_f\end{aligned} \tag{4.2.9}$$

In (4.2.9) b denotes the column vector $[0,0,\ldots,1]^T \in \mathbb{R}^{v-1}$ and $\chi(x)$ denotes the matrix $[\chi_1^T(x),\ldots,\chi_{v-1}^T(x)]^T$. At this point we give the following definitions:

$$\begin{aligned}\rho_1 &:= |2Pb| \\ \rho_2 &:= \sup_{|\bar{e}| \le 1} \frac{|E_2|}{|\bar{e}|} \\ \rho_3 &:= \sup_{x \in U} \sigma_{\max}[2P\chi(x)] \\ \rho_4 &:= \sup_{x \in U} |\chi_v(x)| \\ \rho_5 &:= \sup_{x \in U} \sigma_{\max}[\bar{\chi}(x)] \\ \sigma_k &:= \sup_{x \in U} \sigma_{\max}[\bar{P}(x)R_k(x) + R_k^T(x)\bar{P}(x)]\,, \quad k=1,2\end{aligned} \tag{4.2.10}$$

Using (3.2.15), (3.2.18), (3.4.5), and (4.2.10) in (4.2.9) we obtain:

$$
\begin{aligned}
\dot{V} \le &- |\bar{e}|^2 + \rho_1|\bar{e}||\zeta| + \rho_3|\bar{e}||z_f| - \frac{\gamma}{\bar{\varepsilon}}(1-\bar{\varepsilon}a_1-\varepsilon_2)|\zeta|^2 \\
&+ \gamma(\varepsilon_1+\varepsilon_2 d) + \gamma\rho_2|\bar{e}||\zeta| + \gamma\rho_4|z_f||\zeta| - \mu\lambda_1|\eta|^2 + \mu\lambda^2 L|\xi||\eta| + \mu\lambda_2\rho_5|\eta||z_f| \\
&- \frac{1}{\varepsilon}[1-\varepsilon\sigma_1-\varepsilon\sigma_2\nu-\varepsilon|\dot{\bar{P}}(x)|]|z_f|^2 \\
&+ |2\bar{P}(x)\,[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}]\,|\,|z_f| \qquad (4.2.11)
\end{aligned}
$$

Now, from (4.2.3) we can conclude that there are positive constants $\bar{l}_1$ and $\bar{l}_2$ such that

$$
|u| \le \bar{l}_1(|\xi|+|\eta|) + \bar{l}_2 d + \frac{\bar{l}_2}{\bar{\varepsilon}}|\zeta| \qquad (4.2.12)
$$

where d is the bound on $|\dot{\xi}_v^d|$. Also, from (4.2.3) and (4.2.1) we can find positive constants $\bar{l}_3$ through $\bar{l}_{10}$ such that:

$$
\begin{aligned}
|\dot{u}| \le &\bar{l}_3(|\xi|+|\eta|) + \bar{l}_4 d + \bar{l}_5 d^2 + \bar{l}_6 d_1 + \bar{l}_7(\varepsilon_1+\varepsilon_2 d) \\
&+ \frac{\bar{l}_9}{\bar{\varepsilon}^2}|\zeta| + \frac{\bar{l}_{10}}{\bar{\varepsilon}}|\bar{e}| \qquad (4.2.13)
\end{aligned}
$$

where d_1 is the bound on $|\ddot{\xi}_v^d|$. From (4.2.12) and (4.2.13) we can conclude that for all $x \in U$ we have:

$$
\begin{aligned}
|2\bar{P}(x)[r_1(x) + r_2(x)\,u + r_3(x)\,u^2 + r_4(x)\,\dot{u}]| \le &\, l_1|\xi| + l_2|\eta| + l_3 d + l_4 d^2 \\
&+ l_5 d_1 + l_6(\varepsilon_1+\varepsilon_2 d) + \frac{l_8}{\bar{\varepsilon}}|z_f| + \frac{l_9}{\bar{\varepsilon}^2}|\zeta| + \frac{l_{10}}{\bar{\varepsilon}}|\bar{e}| \qquad (4.2.14)
\end{aligned}
$$

for some positive constants l_1 through l_{10}. From similar computations we can find positive constants p_1, p_2, and p_3 such that for all $x \in U$ we have:

$$
\sigma_1 + \sigma_2|u| + |\dot{\bar{P}}(x)| \le \frac{p_1}{\bar{\varepsilon}} + p_2 d + p_3|z_f| \qquad (4.2.15)
$$

We define K by the inequality $|\xi| \le K|\bar{e}|+|\zeta|+|\xi^d|$. Then using this inequality, (4.2.14), (4.2.15), and the bound $|\xi^d| \le c$ in (4.2.11) yields:

$$\dot{V} \le -|\bar{e}|^2 + (\rho_1+\gamma\rho_2)|\bar{e}||\zeta| + (\rho_3+l_1K+\frac{l_{10}}{\bar{\varepsilon}})|\bar{e}||z_f| - \frac{\gamma}{\bar{\varepsilon}}(1-\bar{\varepsilon}a_1-\varepsilon_2)|\zeta|^2$$
$$+ (\gamma\rho_4+\frac{l_9}{\bar{\varepsilon}^2}+l_1)|\zeta||z_f| - \mu\lambda_1|\eta|^2 + \mu\lambda_2LK|\bar{e}||\eta| + (\mu\lambda_2\rho_5+l_2)|\eta||z_f|$$
$$+ \mu\lambda_2L|\zeta||\eta| + \gamma(\varepsilon_1+\varepsilon_2d)|\zeta| + (l_1c+l_3d+l_4d^2+l_5d_1+l_6(\varepsilon_1+\varepsilon_2d))|z_f|$$
$$+ \mu\lambda_2Lc - \frac{1}{\varepsilon}(1-\varepsilon\frac{p_1}{\bar{\varepsilon}} - \varepsilon p_2d - \varepsilon p_3|z_f|)|z_f|^2 \tag{4.2.16}$$

To simplify (4.2.16) we make the following definitions:

$$\begin{aligned} c_1 &:= 1 - \bar{\varepsilon}a_1 - \varepsilon_2 \\ c_2 &:= \rho_1 + \gamma\rho_2 \\ c_3 &:= l_{10} + \bar{\varepsilon}\,\rho_3 + \bar{\varepsilon}\,l_1K \\ c_4 &:= l_9 + \bar{\varepsilon}^{\,2}\gamma\rho_4 + \bar{\varepsilon}\,2l_1 \\ c_5 &:= \lambda_2\rho_5 + \frac{1}{\mu}l_2 \\ c_6 &:= l_1c + l_3d + l_4d^2 + l_5d_1 + l_6(\varepsilon_1+\varepsilon_2d) \end{aligned} \tag{4.2.17}$$

Then using (4.2.17) in (4.2.16) we obtain:

$$\dot{V} \le -|\bar{e}|^2 - \frac{\gamma}{\bar{\varepsilon}}c_1|\zeta|^2 - \mu\lambda_1|\eta|^2 - \frac{1}{\varepsilon}(1 - \frac{\varepsilon}{\bar{\varepsilon}}p_1 - \varepsilon p_2d - \varepsilon p_3|z_f|\,)|z_f|^2$$
$$+ c_2|\bar{e}||\zeta| + \frac{c_3}{\bar{\varepsilon}}|\bar{e}||z_f| + \frac{c_4}{\bar{\varepsilon}^2}|\zeta||z_f| + \mu\lambda_2LK|\bar{e}||\eta| + \mu c_5|\eta||z_f|$$
$$+ \mu\lambda_2L|\zeta||\eta| + \gamma(\varepsilon_1+\varepsilon_2d)|\zeta| + c_6|z_f| + \mu\lambda_2Lc \tag{4.2.18}$$

In order to obtain an estimate of the region outside which $\dot{V}$ is strictly negative, we use the following set of inequalities:

$$
\begin{aligned}
&c_2|\bar{e}||\zeta| \le \frac{1}{4}|\bar{e}|^2 + c_2^2|\zeta|^2 \\
&\frac{c_3}{\bar{\varepsilon}}|\bar{e}||z_f| \le \frac{1}{4}|\bar{e}|^2 + \frac{c_3^2}{\bar{\varepsilon}^2}|z_f|^2 \\
&\frac{c_4}{\bar{\varepsilon}^2}|z_f||\zeta| \le \frac{1}{4\bar{\varepsilon}}|\zeta|^2 + \frac{c_4^2}{\bar{\varepsilon}^3}|z_f|^2 \\
&\mu\lambda_2 LK|\bar{e}||\eta| \le \frac{1}{4}|\bar{e}|^2 + \mu^2\lambda_2^2L^2K^2|\eta|^2 \\
&\mu\lambda_2 L|\zeta||\eta| \le \frac{1}{4}|\zeta|^2 + \mu^2\lambda_2^2L^2|\eta|^2 \\
&\mu c_5|\eta||z_f| \le \frac{1}{4}|z_f|^2 + \mu^2c_5^2|\eta|^2 \\
&\gamma(\varepsilon_1+\varepsilon_2 d)|\zeta| \le \bar{\varepsilon}\gamma^2(\varepsilon_1+\varepsilon_2 d)^2 + \frac{1}{4\bar{\varepsilon}}|\zeta|^2 \\
&c_6|z_f| \le \varepsilon c_6^2 + \frac{1}{4\varepsilon}|z_f|^2
\end{aligned}
\tag{4.2.19}
$$

to obtain:

$$
\begin{aligned}
\dot{V} \le &-\frac{1}{4}|\bar{e}|^2 - \frac{1}{\bar{\varepsilon}}(\gamma c_1 - \bar{\varepsilon}c_2^2 - \frac{1}{2} - \frac{\bar{\varepsilon}}{4})|\zeta|^2 - \mu\lambda_1[1 - \frac{\mu}{\lambda_1}(\lambda_2^2L^2(1+K^2) + c_5^2)]\,|\eta|^2 \\
&- \frac{1}{\varepsilon}[\frac{3}{4} - \frac{\varepsilon}{\bar{\varepsilon}^3}(p_1\bar{\varepsilon}^2 + \bar{\varepsilon}^3p_2d + \bar{\varepsilon}c_3^2 + c_4^2 + \frac{\bar{\varepsilon}^3}{4}) - \varepsilon p_3|z_f|]\,|z_f|^2 \\
&+ \mu\lambda_2Lc + \varepsilon c_6^2 + \bar{\varepsilon}\gamma^2(\varepsilon_1+\varepsilon_2 d)^2
\end{aligned}
\tag{4.2.20}
$$

From (4.2.17) we can observe that there exists an $\bar{\varepsilon}_1$ such that for all $\bar{\varepsilon} < \bar{\varepsilon}_1$ we have that $c_1 > \frac{1-\varepsilon_2}{2}$. Then we can choose $\gamma = \frac{4}{1-\varepsilon_2}$. With this choice, we can find $\bar{\varepsilon}_2$ such that for all $\bar{\varepsilon} < \bar{\varepsilon}_2$ we have that:

$$
\gamma c_1 - \bar{\varepsilon}c_2^2 - \frac{1}{2} - \frac{\bar{\varepsilon}}{4} > \frac{1}{2} \tag{4.2.21}
$$

Then letting:

$$
\bar{\varepsilon}^* = \min\{\bar{\varepsilon}_1\ ,\ \bar{\varepsilon}_2\} \tag{4.2.22}
$$

we conclude that (4.2.21) holds for all $\bar{\varepsilon} < \bar{\varepsilon}^*$. Furthermore, we can find a monotone increasing function $\varepsilon^*(\bar{\varepsilon})$ such that for all $\varepsilon < \varepsilon^*(\bar{\varepsilon})$ we have:

$$\frac{3}{4} - \frac{\varepsilon}{\bar{\varepsilon}^3}[p_1\bar{\varepsilon}^2 + \bar{\varepsilon}^3 p_2 d + \bar{\varepsilon}c_3^2 + c_4^2 + \bar{\varepsilon}\frac{3}{4}) - \varepsilon p_3|z_f| > \frac{1}{2} \tag{4.2.23}$$

for all $(\xi,\eta)\in\Phi(U)$ and z_f with $|z_f| < \frac{1}{8\varepsilon p_3}$. Finally we can choose μ small enough so that the coefficient of $|\eta|^2$ in (4.2.20) will be negative as well. So we choose:

$$\mu = \frac{\lambda_1}{2[\lambda_2^2 L^2(1+K^2) + c_5^2]} \tag{4.2.24}$$

Then, with the above choices of the parameters, we conclude from (4.2.20) that:

$$\begin{aligned} \dot{V} \le & -\frac{1}{2}|\bar{e}|^2 - \frac{1}{2\bar{\varepsilon}}|\zeta|^2 - \frac{1}{2}\mu\lambda_1|\eta|^2 - \frac{1}{2\varepsilon}|z_f|^2 \\ & + \mu\lambda_2 Lc + \varepsilon c_6^2 + \bar{\varepsilon}\gamma^2(\varepsilon_1+\varepsilon_2 d)^2 \end{aligned} \tag{4.2.25}$$

for all $(\xi,\eta)\in\Phi(U)$ and z_f with $|z_f| < \frac{1}{8\varepsilon p_3}$. Now we define the following sets:

$$G := \{(\bar{e},\zeta,\eta,z_f) \mid \frac{1}{2}|\bar{e}|^2 + \frac{1}{2\bar{\varepsilon}}|\zeta|^2 + \frac{1}{2}\mu\lambda_1|\eta|^2 + \frac{1}{2\varepsilon}|z_f|^2 \le \mu\lambda_2 Lc + \varepsilon c_6^2 + \bar{\varepsilon}\gamma^2(\varepsilon_1+\varepsilon_2 d)^2 \}$$

$$O := \{(\bar{e},\zeta,\eta,z_f) \mid |z_f| < \frac{1}{8\varepsilon p_3} \}$$

$$\Psi := \{z_f \mid |z_f| < \frac{1}{8\varepsilon p_3} \}$$

Then we observe from (4.2.25) that $\dot{V}$ is strictly negative in $O \cap G^c$, where G^c denotes the complement of G. Let:

$$\Omega_C := \{(\bar{e},\zeta,\eta,z_f) \mid V \le C \}$$

$$\underline{C}(\varepsilon,\bar{\varepsilon},c,d,d_1) := \inf \{C \mid \Omega_C \supset G \}$$

Clearly $\underline{C}$ is an increasing function of ε, $\bar{\varepsilon}$, c, d, and d_1. Assuming that ε and $\bar{\varepsilon}$ are small enough so that for $c=d=d_1=0$ we can write:

$$(\bar{e},\zeta,\eta,z_f) \in \Omega_{\underline{C}(\varepsilon,\bar{\varepsilon},0,0,0)} \Longrightarrow (\xi,\eta,z_f) \in \Phi(U)\times\Psi$$

we can find a set $D \subset \{(c,d,d_1) \mid c\geq 0\ ;\ d\geq 0\ ;\ d_1\geq 0\}$ such that:

$$(\bar{e},\zeta,\eta,z_f) \in \Omega_{\underline{C}(\varepsilon,\bar{\varepsilon},c,d,d_1)} \Longrightarrow (\xi,\eta,z_f) \in \Phi(U)\times\Psi\ ,\ \forall\ (c,d,d_1) \in D \qquad (4.2.26)$$

From (4.2.26), it is easy to see that there exists a set $\Omega \subset \Phi(U)\times\Psi$ such that for all initial conditions in Ω and all $(c,d,d_1) \in D$ the trajectories remain $\Phi(U)\times\Psi$ for all time.

To conclude that the size of the tracking error vector, namely $|\bar{e}|$, will be of $O(\bar{\varepsilon})$, we need only to look at (4.2.6). From the dynamics of z_f we conclude that $|z_f|$ will be of $O(\varepsilon)$. Therefore the dominant driving term in the dynamics of $\bar{e}$ is ζ which is of $O(\bar{\varepsilon})$. This implies that $|\bar{e}|$ will be of $O(\bar{\varepsilon})$. More precisely, we conclude that there are positive constants t_1 and E such that:

$$|\bar{e}(t)| < \bar{\varepsilon} E\ ,\ \forall\ t > t_1$$

□

Remark 4.2.1: Theorem 4.2.1 indicates that reducing the effects of uncertainties by the use of high gain can only be done to the extent that the gain is not too high to excite the unmodeled dynamics. Thus if we have a fixed ε for the unmodeled dynamics, the gain of the controller (given by $\bar{\varepsilon}$) must be small enough so that we have: $\varepsilon \leq \varepsilon^*(\bar{\varepsilon})$.

□

4.3. Robustness of Adaptive Controllers to Unmodeled Dynamics:

Adaptive control of linear systems in the presence of modeling errors has been the subject of investigation by several authors. Rohrs, et al [32] demonstrated that an adaptive control scheme based on the reduced model of a plant can become unstable in the presence of stable unmodeled dynamics which are even outside the bandwidth of the control input. The prime instability mechanism was shown to be the slow parameter drift to infinity. It was observed that errors initially converged to a small neighborhood of the origin and then slowly drifted to infinity along with the parameter estimates. Bodson and Sastry [6] later showed the stability of the adaptive scheme is preserved if the control input satisfies the persistency of excitation conditions. In the absence of persistency of excitation conditions, Ioannou and Kokotovic [22] suggested a modification to the adaptive scheme by the addition of a small term in the parameter update law proportional to the negative of the parameter estimates. This modification guarantees the boundedness of the parameter estimates and results in the stability of the adaptive scheme. It was shown, however, that the modified scheme no longer resulted in zero tracking errors in the ideal case.

All previous robustness results (in linear systems) in connection to output-error (the input-error scheme of Bodson and Sastry [6] for linear systems does not rely on the Strict Positive Realness of the system and can be applied to higher relative degrees than one) adaptive schemes have been formulated for systems with relative degree one. Here we will define an index, $\gamma_u \geq 0$ for the unmodeled dynamics and we will show that if $\gamma_u \geq \nu - 2$, where ν is the relative degree of the reduced model of the system, then our adaptive scheme of Section 3.6 is robust with respect to unmodeled dynamics and the tracking errors converge to a small neighborhood of the origin. To ensure the boundedness of our parameter estimates we rely on the a priori bounds on the true parameters and the resetting mechanism.

Following Assumption 3.2.2 for the reduced model of the system (4.1.1), we consider the following system:

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{N} \alpha_i^* f_i(x) + \sum_{j=1}^{M} \beta_j^* g_j(x)\, u + F_1(x) z_f \\
\varepsilon \dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\, u] z_f + \varepsilon [r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}] \qquad (4.3.1) \\
y &= h(x)
\end{aligned}$$

where all the terms appearing in (4.3.1) have been previously defined in. We have the following definition as a way of classifying the unmodeled dynamics.

Definition 4.3.1: The unmodeled dynamics in (4.3.1) are said to have index γ_u if $d(L_f^k h(x)) \bullet F_1(x)$ is zero for all $x \in U$ and $k < \gamma_u$ and is nonzero for $k = \gamma_u$.

□

Based on the developments of Section 3.6 we can transform (4.3.1) into the following system:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \chi_1(x) z_f \\
&\vdots \\
\dot{\xi}_{\nu-1} &= \xi_\nu + \chi_{\nu-1}(x) z_f \\
\dot{\xi}_\nu &= \theta^{1*T}[W_1(x) + W_2(x)\, u] + \chi_\nu(x) z_f \\
\dot{\eta} &= q(\xi,\eta) + \overline{\chi}(x) z_f \\
\varepsilon \dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\, u] z_f + \varepsilon[r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}] \\
y &= \xi_1
\end{aligned} \tag{4.3.2}$$

where θ^{1*} is defined in (3.6.23), $W_1(x)$ and $W_2(x)$ are as defined in (3.6.24), $\chi_k(x) := d\xi_k \bullet F_1(x)$, and $\overline{\chi}(x) := d\eta \bullet F_1(x)$. Now, given a desired tracking signal $y_d(t)$, we use the control law of Section 3.6, that is:

$$u = \frac{1}{\theta^{1T} W_2(x)}[-\theta^{1T} W_1(x) - a_1 e_\nu - a_2 e_{\nu-1} - \cdots - a_\nu e_1 + \Phi^{2T} W_3(x) - \dot{\xi}_\nu^d] \tag{4.3.3}$$

where $e_k := \xi_k - \xi_k^d$ for $k=1,\ldots,\nu$, ξ_k^d denotes the $(k-1)$st derivative of the tracking signal $y_d(t)$, a_1 through a_ν are chosen so that the polynomial $s^\nu + a_1 s^{\nu-1} + \cdots + a_\nu$ is Hurwitz, and Φ^2 and $W_3(x)$ are defined in (3.6.34). With this control law (4.3.2) can be written as:

$$\begin{aligned}
\dot{e}_1 &= e_2 + \chi_1(x) z_f \\
&\vdots \\
\dot{e}_{\nu-1} &= e_\nu + \chi_{\nu-1}(x) z_f \\
\dot{e}_\nu &= -a_1 e_\nu - a_2 e_{\nu-1} - \cdots - a_\nu e_1 + \Phi^T W(x,t) + \chi_\nu(x) z_f \\
\dot{\eta} &= q(\xi,\eta) + \overline{\chi}(x) z_f \\
\varepsilon \dot{z}_f &= [F_2(x) + \varepsilon R_1(x) + \varepsilon R_2(x)\, u] z_f + \varepsilon[r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}]
\end{aligned} \tag{4.3.4}$$

We now recall Lemma 3.6.1 and the definition of e^a from (3.6.43) which will be used in the following theorem.

Theorem 4.3.1: Let the reduced system of (4.3.1) have relative degree ν and be hyperbolically minimum-phase. Let the unmodeled dynamics have index $\gamma_u \geq \nu-2$. Let the desired tracking signal and its first $(\nu+1)$ derivatives be bounded. Choose the control law (4.3.3) with the parameter update law:

$$\dot{\theta} = -\gamma e^{a^T} PbW \tag{4.3.5}$$

where γ is a positive gain and the filter in (3.6.36) is turned on at $t < -t^*$. Choose the parameter resetting law:

$$\theta_i(t^+) = \begin{cases} l_{\theta_i} & \text{if } \theta_i(t) = l_{\theta_i} - \delta \\ h_{\theta_i} & \text{if } \theta_i(t) = h_{\theta_i} + \delta \end{cases} \tag{4.3.6}$$

Then there exists a positive scalar ε^*, a set $D \subset \mathbb{R}^2$, a positive constant d_2, and an open set $\Omega \subset \Phi(U)\times\mathbb{R}^m$, such that for all $\varepsilon < \varepsilon^*$, all tracking signals with $|\xi(t)| < c$ and $|\dot{\xi}_\nu(t)| < d$ for all t with $(c,d)\in D$, and all initial conditions in Ω, the trajectories remain bounded in $\Phi(U)\times\mathbb{R}^m$ for all time and the tracking error will be of $O(\varepsilon)$ provided that $\gamma = d_2\varepsilon^{-\frac{2}{3}}$.

□

Proof: We will first prove the boundedness of solutions. Since $\gamma_u \geq \nu-2$, we know that $\chi_k(x) = 0$ for all $x\in U$ and $k=1,\dots,\nu-2$. We consider the following Lyapunov function candidate:

$$V = e^T Pe + \frac{1}{\gamma}\Phi^T\Phi + \mu V_0(\eta) + \pi z_f^T \overline{P}(x) z_f \tag{4.3.7}$$

where $V_0(\eta)$ is the Lyapunov function for the dynamics $\dot{\eta} = q(0,\eta)$ and satisfies the inequalities (3.2.15), $\overline{P}(x)$ satisfies (4.2.8), and μ and π will be determined later. We know from the proof of Theorem (3.6.1) that the change in V due to a parameter resetting event is always negative. Now, differentiating V along the flow of (4.3.4) we have:

$$\begin{aligned}\dot{V} = &- |e|^2 + 2e^T P b_1 \chi_{v-1}(x) z_f + 2e^T P b \chi_v(x) z_f + 2e^T P b \Phi^T W \\ &- 2e^{a T} P b \Phi^T W + \mu \frac{\partial V_0}{\partial \eta} q(\xi,\eta) + \mu \frac{\partial V_0}{\partial \eta} \overline{\chi}(x) z_f \\ &- \frac{\pi}{\varepsilon}|z_f|^2 + \pi z_f^T [\overline{P}(x)(R_1(x) + R_2(x)\, u) + (R_1(x) + R_2(x)\, u)^T \overline{P}(x)] z_f \\ &+ 2\pi z_f^T \overline{P}(x)[r_1(x) + r_2(x) + r_3(x)\, u^2 + r_4(x)\, \dot{u}] + \pi z_f^T \dot{\overline{P}}(x) z_f \end{aligned} \tag{4.3.8}$$

where $b_1 := [0,0,\ldots,1,0]^T \in \mathbb{R}^v$.

From Lemma (3.6.1) and (4.3.4) we know that:

$$e^a = e - \overline{\varepsilon} p + b \chi_{v-1}(x) z_f \tag{4.3.9}$$

where $\overline{\varepsilon}$ is the filter parameter from (3.6.36), and $p := (0, \ddot{S}, S^{(3)}, \ldots, S^{(v)})$ with S defined by (3.6.40). Using (4.3.9) in (4.3.8) we obtain:

$$\begin{aligned}\dot{V} = &- |e|^2 + 2e^T P[b_1 \chi_{v-1}(x) + b \chi_v(x)] z_f + 2\overline{\varepsilon} p^T P b \Phi^T W \\ &- 2b^T P b \Phi^T W \chi_{v-1}(x) z_f + \mu \frac{\partial V_0}{\partial \eta} q(\xi,\eta) + \mu \frac{\partial V_0}{\partial \eta} \overline{\chi}(x) z_f \\ &- \frac{\pi}{\varepsilon}|z_f|^2 + \pi z_f^T [\overline{P}(x)(R_1(x) + R_2(x)\, u) + (R_1(x) + R_2(x)\, u)^T \overline{P}(x)] z_f \\ &+ 2\pi z_f^T \overline{P}(x)[r_1(x) + r_2(x) + r_3(x)\, u^2 + r_4(x)\, \dot{u}] + \pi z_f^T \dot{\overline{P}}(x) z_f \end{aligned} \tag{4.3.10}$$

By virtue of the resetting mechanism, we know that Φ is bounded. Therefore from (4.3.3) we can write:

$$|u| \le \overline{l}_1 |e| + \overline{l}_2 |\eta| + \overline{l}_3 c + \overline{l}_4 d \quad \forall\ x \in U \tag{4.3.11}$$

for some positive constants $\overline{l}_1$ through $\overline{l}_4$. In (4.3.11) c and d denote the bounds on $|\xi^d|$ and $|\dot{\xi}_v^d|$ respectively. From (4.3.3) we can also conclude that:

$$|\dot{u}| \le \gamma(l_1 + l_2|z_f|) + l_3|\xi| + l_4|\eta| + l_5|z_f| + l_6(c,d,d_1) \quad \forall\ x \in U \tag{4.3.12}$$

where l_1 through l_5 are positive constants, d_1 is the bound on $\ddot{\xi}_v^d$, and $l_6(c,d,d_1)$ is an increasing function of its arguments with $l_6(0,0,0) = 0$.

From (4.3.10) and (4.3.11) we can write:

$$|2\bar{P}(x)[r_1(x) + r_2(x)\, u + r_3(x)\, u^2 + r_4(x)\, \dot{u}]| \le$$
$$(k_1+\gamma k_2)|\xi| + (k_3+\gamma k_4)|\eta| + (k_5+\gamma k_6)|z_f| + (k_7(c,d,d_1)+\gamma k_8(c,d)) \qquad (4.3.13)$$

where k_1 through k_6 are positive constants and k_7 and k_8 are increasing functions of their arguments with $k_7(0,0,0) = 0$ and $k_8(0,0) = 0$.

By assumption the filter (3.6.36) is turned on before the controller loop is closed. Therefore, we know that during the control process p is bounded by the bounds given in (3.6.42) if $(\xi,\eta) \in \Phi(U)$ and $\dot{\xi}_v^d$ is bounded. Thus, we can find a positive constant D_1 such that:

$$|2p^T P b \Phi^T W| \le D_1 \quad \forall \quad (\xi,\eta) \in \Phi(U) \qquad (4.3.14)$$

Similarly, we can find a positive constant D_2 such that:

$$|2b^T P b \Phi^T W \chi_{v-1}(x)| \le D_2 \quad \forall \quad (\xi,\eta) \in \Phi(U) \qquad (4.3.15)$$

Next, we define:

$$\rho_1 := \sup_{x \in U} |2P(b_1\chi_{v-1}(x) + b\chi_v(x))|$$
$$\rho_2 := \sup_{x \in U} \sigma_{max}(\bar{\chi}(x)) \qquad (4.3.16)$$
$$\sigma_k := \sup_{x \in U} \sigma_{max}(\bar{P}(x)R_k(x) + R_k^T(x)\bar{P}(x)) \quad k=1,2$$

where $\sigma_{max}(A)$ denotes the maximum singular value of the matrix A. Further from (4.3.1) we can write:

$$|\dot{\bar{P}}(x)| \le p_1+p_2|u|+p_3|z_f| \qquad (4.3.17)$$

for some positive constants p_1 through p_3. Using (4.3.13) through (4.3.17) and the inequality $|\xi| \le |e| + c$ in (4.3.10) yields:

$$\dot{V} \le -|e|^2 + \rho_1|e||z_f| + \bar{\varepsilon}D_1 + D_2|z_f| - \mu\lambda_1|\eta|^2 + \mu\lambda_2 L|e||\eta|$$
$$+ \mu\lambda_2 Lc + \mu\lambda_2\rho_2|\eta||z_f| - \frac{\pi}{\varepsilon}[1-\varepsilon(\sigma_1+p_1+(\sigma_2+p_2)|u|+p_3|z_f|)]|z_f|^2$$
$$+ \pi(k_1+\gamma k_2)|e||z_f| + \pi(k_3+\gamma k_4)|\eta||z_f| + \pi(k_5+\gamma k_6)|z_f|^2$$
$$+ \pi(k_7+\gamma k_8)|z_f| + \pi(k_1+\gamma k_2)c|z_f| \qquad (4.3.18)$$

To simplify the following development we define:

$$c_1 := \frac{D_2}{\pi}+k_7+\gamma k_8+k_1 c+\gamma k_2 c$$
$$c_2 := \frac{\mu\lambda_2\rho_2}{\pi}+k_3+\gamma k_4 \qquad (4.3.19)$$
$$c_3 := \frac{\rho_1}{\pi}+k_1+\gamma k_2$$

Then using the following inequalities:

$$\pi c_1|z_f| \le \frac{\pi}{12\varepsilon}|z_f|^2 + 3\varepsilon\pi c_1^2$$
$$\pi c_2|\eta||z_f| \le \frac{\pi}{12\varepsilon}|z_f|^2 + 3\varepsilon\pi c_2^2|\eta|^2$$
$$\pi c_3|e||z_f| \le \frac{\pi}{12\varepsilon}|z_f|^2 + 3\varepsilon\pi c_3^2|e|^2 \qquad (4.3.20)$$
$$\mu\lambda_2 L|e||\eta| \le |e\frac{1}{4} + \mu^2\lambda_2^2 L^2|\eta|^2$$

in (4.3.18) we obtain:

$$\dot{V} \le -(\frac{3}{4} - 3\varepsilon\pi c_3^2)|e|^2 - \mu\lambda_1(1 - \mu\frac{\lambda_2^2 L^2}{\lambda_1} - 3\varepsilon\pi c_2^2)|\eta|^2$$
$$-\frac{\pi}{\varepsilon}[\frac{3}{4} - \varepsilon(\sigma_1+p_1+(\sigma_2+p_2)\nu+p_3|z_f|+k_5+\gamma k_6)]|z_f|^2$$
$$+ \bar{\varepsilon}D_1 + 3\varepsilon\pi c_1^2 + \mu\lambda_2 Lc \qquad (4.3.21)$$

It is clear from (4.3.21) that if $\gamma << \frac{1}{\varepsilon}$, then for ε small enough the coefficient of $|z_f|^2$ will be negative. On the other hand, we recall from Section 3.6 that in order for the tracking errors to converge to a small neighborhood of the origin, we need γ to be as large as possible. Thus with this fact in mind we choose:

$$\gamma = d_2\,\varepsilon^{-\frac{2}{3}} \tag{4.3.22}$$

where d_2 is a positive constant to be determined later in the proof. This choice of γ will be justified shortly. Now it is easy to see that we can find $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ we have:

$$\frac{3}{4} - \varepsilon(\sigma_1 + p_1 + (\sigma_2 + p_2)\nu + p_3|z_f| + k_5 + \gamma k_6) > \frac{1}{2} \tag{4.3.23}$$

for all $(\xi, \eta) \in \Phi(U)$ and z_f with $|z_f| < \dfrac{1}{8\varepsilon p_3}$.

From (4.3.19) we can see that with the above choice for γ the constants c_1 through c_3 will be of $O(\varepsilon^{-\frac{2}{3}})$. Therefore we need to choose π small enough (of $O(\varepsilon^{\frac{1}{3}})$) so that the products πc_k^2 for k=1,2,3 are small of $O(1)$. Thus choose π so that:

$$\begin{aligned} 3\varepsilon\pi c_2^2 &< \frac{1}{4} \\ 3\varepsilon\pi c_3^2 &< \frac{1}{4} \end{aligned} \tag{4.3.24}$$

In addition we choose μ such that we have:

$$\mu\frac{\lambda_2^2 L^2}{\lambda_1} < \frac{1}{4} \tag{4.3.25}$$

With the above choices for π and μ we can conclude from (4.3.21) that:

$$\dot{V} \le -\frac{1}{2}|e|^2 - \frac{\mu\lambda_1}{2}|\eta|^2 - \frac{\pi}{2\varepsilon}|z_f|^2 + \overline{\varepsilon}D_1 + 3\varepsilon\pi c_1^2 + \mu\lambda_2 Lc \tag{4.3.26}$$

By the resetting mechanism we know that Φ remains bounded. Thus we can find a positive constant d_3 such that $|\Phi|^2 < d_3$ for all time. Now we define the following sets:

$$G := \{(e,\eta,z_f,\Phi) \mid \frac{1}{2}|e|^2+\frac{\mu\lambda_1}{2}|\eta|^2+\frac{1}{2\varepsilon}|z_f|^2 \le \overline{\varepsilon}D_1+3\varepsilon\pi c_1^2+\mu\lambda_2 Lc \ ; \ |\Phi|^2 \le d_3\}$$
$$O := \{(e,\eta,z_f,\Phi) \mid |z_f| < \frac{1}{8\varepsilon p_3}\}$$
$$\Psi := \{(z_f) \mid |z_f| < \frac{1}{8\varepsilon p_3}\}$$

Then clearly $\dot{V}$ is strictly negative in $O \cap G^c$ where G^c denotes the complement of G. Letting:

$$\Omega_C := \{(e,\eta,z_f,\Phi) \mid V \le C\}$$
$$\underline{C}(\varepsilon,\overline{\varepsilon},c,d) := \inf\ \{C \mid \Omega_C \supset G\}$$

it is clear that $\underline{C}$ is an increasing function of ε, $\overline{\varepsilon}$, c, and d. Now we assume that ε and $\overline{\varepsilon}$ are small enough so that when c=d=0, for all the points inside $\Omega_{\underline{C}(\varepsilon,\overline{\varepsilon},0,0)}$ we have that $(\xi,\eta)\in\Phi(U)$. Then there exists a set $D \subset \{(c,d) \mid c \ge 0\ ,\ d \ge 0\}$ such that:

$$(e,\eta,z_f,\Phi)\in\Omega_{\underline{C}(\varepsilon,\overline{\varepsilon},c,d)} \Longrightarrow (\xi,\eta)\in\Phi(U) \quad \forall\ (c,d)\in D \tag{4.3.27}$$

From (4.3.27) we can conclude that there exists an set $\Omega \subset \Phi(U)\times\Psi$ such that for all initial conditions in Ω and all $(c,d)\in D$ the trajectories of the system remain in the set $\Phi(U)\times\Psi$ for all time.

To find an estimate of the size of the tracking errors, we consider the following positive definite function of e and Φ:

$$V_1 = e^T P e + \frac{1}{\gamma}\Phi^T\Phi \tag{4.3.28}$$

Differentiating V_1 along the flow of (4.3.4) we can write:

$$\dot{V}_1 \le -|e|^2 + \rho_1|e||z_f| + \overline{\varepsilon}D_1 \tag{4.3.29}$$

By the boundedness of states we can conclude from (4.3.4) that there is a constant d_4 such that as t tends to infinity we have:

$$|z_f| \le d_4\gamma\varepsilon \tag{4.3.30}$$

Using (4.3.30) in (4.3.29) we have:

$$\dot{V}_1 \leq -|e|^2 + d_4\gamma\bar{\varepsilon}|e| + \bar{\varepsilon}D_1 \tag{4.3.31}$$

From (4.3.31) we can conclude that:

$$\dot{V}_1 \leq -\frac{3}{4}|e|^2 + \gamma^2\bar{\varepsilon}^2 d_4^2 + \bar{\varepsilon}D_1 \tag{4.3.32}$$

From (4.3.28) and (4.3.32), it is easy to see that when $\bar{\varepsilon} < \varepsilon$ in order to minimize the value of V_1 on the smallest level set of V_1 which contains the set $\{(e,\Phi) \mid |e|^2 \leq 2\gamma^2\bar{\varepsilon}^2 d_4^2 + \bar{\varepsilon}D_1 \,;\, |\Phi|^2 \leq d_3\}$, we must choose γ according to (4.3.22) for some positive constant d_2. With this choice it is easy to see that $|e|$ will be of $O(\varepsilon^{\frac{1}{3}})$. That is, there are constants t_1 and E such that:

$$|e(t)| < \varepsilon^{\frac{1}{3}} E\,, \qquad \forall\, t > t_1$$

□

4.4. Concluding Remarks

In this chapter we have studied the robustness of the high gain and adaptive control schemes, which were presented in Chapter 3, to unmodeled dynamics. The unmodeled dynamics were represented by singular perturbations of the model of the system used in the control design process. The result presented in Theorem 4.2.1 indicates that high gain control can be used in order to suppress the effects of uncertainties, but the gain must be small enough so that the controller action will not excite the unmodeled dynamics. The proof of this theorem also gives specific upper and lower bounds on the high gain parameter, $\bar{\varepsilon}$, when the singular perturbation parameter, ε, is known. In Section 4.3 we gave conditions on the vector fields corresponding to the unmodeled dynamics under which the adaptive scheme was shown to be robust. Theorem 4.3.1 indicated that in order to guarantee that the controller will result in asymptotically small errors of $O(\varepsilon^{\frac{1}{3}})$, we must resort to fast adaptation, that is large γ. But at the same time γ must be small enough so that the adaptation process does not excite the unmodeled dynamics.

It is clear from Theorems 4.2.1 and 4.3.1 that a robustness result based on the singular perturbation technique requires that the control action be on the time scale of the reduced model, so that it does not excite the fast part of the dynamics corresponding to the unmodeled dynamics (though it is easily seen that this condition is not in general sufficient to guarantee robustness; e.g. the adaptive scheme is not robust without parameter resetting). This is precisely why a result similar to the high gain and adaptive cases cannot be formulated for the sliding mode control scheme. The difficulty in formulating such a result stems from the jumps in the control law, which corresponds to very fast action. Thus, in this case we cannot argue on the basis of time scale separation, and the Lyapunov arguments of Sections 4.2 and 4.3 will not be adequate. This does not, however, indicate that the sliding mode scheme is not robust. In fact the results of Theorems 4.2.1 and 4.3.1 are merely sufficient conditions.

Chapter Five

Conclusions and Open Problems

This thesis has dealt with two problems of importance in the design of controllers for nonlinear systems, namely: local stabilization of an equilibrium point by smooth state feedback, and local stable output tracking of a desired trajectory using state feedback. In each case particular emphasis was placed on the robustness properties of controllers to modeling errors and uncertainties. In particular, in cases where it was possible the controller structure was modified in order to reduce (or eliminate) the effects of uncertainties; in other cases explicit bounds were given on performance degradation due to modeling errors.

In the study of local stabilization of nonlinear systems, it was pointed out that the only nontrivial cases are those of systems whose linearizations about the equilibrium point of interest are degenerate. In such cases we know from the center manifold theory that the stability properties of the system coincide with that of a smaller dimensional system defined on the center manifold of the system. Thus the control design process consisted of the following steps: 1) computation of an approximate center manifold for the system and the subsequent reduction of the stability problem to the stability of the system defined on the center manifold, 2) simplification of the vector field on the center manifold using the theory of normal forms, 3) finding conditions under which the simplified vector field is asymptotically stable and the construction of a stabilizing control law under these conditions. In connection to the robustness of these control laws, a theorem was presented which stated that although the stability of the equilibrium point can be destroyed in the presence of perturbations, there exists a small neighborhood of the equilibrium point (whose size depends on the size of the perturbations) which remains attractive.

As was seen in Chapter 2, computing normal forms of the vector fields on the center manifold requires working in the space of homogeneous polynomials of degree k in $\mathbf{R}^n$, denoted by H_k^n, where n is the dimension of the center manifold. The dimension of H_k^n grows rapidly with n. It can be shown that the dimension of H_k^n is given by the expression $n\sum_{i_{k-1}}^{n}\sum_{i_{k-2}}^{i_{k-1}}\cdots\sum_{i_1}^{i_2} i_1$. Thus for example the dimensions of H_3^3, H_3^4, and H_3^5 are 30, 80, and 175, respectively. Therefore, extending the results of Chapter 2 to systems in which the dimension of the degenerate part is large proves to be quite tedious and difficult. Thus, in such cases we believe that it is more reasonable to apply the design methodology of Chapter 2 to the specific system of interest rather than deriving general conditions such as the ones given in Chapter 2.

The results of Chapter 2 can also be extended to lightly damped systems. In this case, if a system has uncontrollable modes which are weakly stable (i.e. having eigenvalues with small negative real parts), then by the use of nonlinear terms in the control law it may be possible to enlarge the domain of attraction of the equilibrium point.

In Chapter 3, following the presentation of the basic tracking theorem for minimum-phase nonlinear systems, we introduced an index for the uncertainties which gave a measure of their contribution to the input-output map of the system. We were then able to determine what class of uncertainties can be dealt with by the use of high gain and sliding mode control techniques. This class was identified as the set of uncertainties satisfying a generalized matching condition. This condition was shown to be a significant generalization of the well known matching condition in that the class of uncertainties satisfying the generalized matching condition is typically much larger than the class of uncertainties satisfying the matching condition and the latter always contains the former.

Extensions to the cases where the uncertainties do not satisfy any matching conditions are highly desirable. Some relevant work in this direction (although the problem considered is that of disturbance decoupling) is reported by Marino et al [27] where they construct a multi-time scale system in order to reduce the effects of uncertainties.

Linear parametric uncertainties were also treated in Chapter 3. Adaptive control techniques were employed in this case to update the estimates of the unknown parameters which were used in the control law. Parameter resetting was used in order to ensure the boundedness of the linearizing control law. In this setting, a new parameter update law was presented for systems with relative degree larger than one. This scheme was shown to result in $O(\varepsilon)$ tracking errors where ε is a small design parameter.

The analysis in Chapter 4 dealt with the robustness properties of the high gain and adaptive control schemes presented in Chapter 3 with respect to unmodeled dynamics. The unmodeled dynamics were represented by parasitic dynamics evolving on a time scale which is much faster than the dominant dynamics of the system on which the control design is based. Thus, it was assumed that the model used for the purpose of control design is the reduced model of a singularly perturbed system. In the case of the high gain control law it was concluded that the gain of the controller can not be arbitrarily large and must be within a bound specified by the unmodeled dynamics. In the case of adaptive control schemes an index, γ_u, was defined for the unmodeled dynamics. Then it was shown that the adaptive scheme of Chapter 3 is robust with respect to unmodeled dynamics if the index of the unmodeled dynamics is larger or equal to $\nu-2$, where ν is the relative degree of the reduced model. In particular, the scheme is robust for all systems with relative degree one or two.

A worthwhile extension of the results in Chapter 4 would be a robustness result for the sliding mode control scheme. Due to the switching nature of the control law the arguments which are based on the time scale separation between the reduced model and the unmodeled dynamics are no longer valid. Therefore, a robustness proof similar to the high gain and adaptive cases can not be used and other techniques must be employed to prove robustness if in fact the scheme is robust.

In the remainder of this chapter we will present an example of a simple physical system for which the a controller can not be designed using the theory presented in Chapter 3. Our objective in presenting this example is to point out two major shortcomings of the theory presented in Chapter 3. This example involves the control of a two segment robot arm which is pinned at one end about which it can rotate. The control input is torque applied to the arm at the end which is pinned. The two segments of the arm are connected by a torsional spring which produces a torque proportional to the angle between the two segments. Such a system can be thought of as the crudest finite element approximation of a flexible arm. Higher order approximations can be achieved by increasing the number of rigid segments connected by torsional springs.

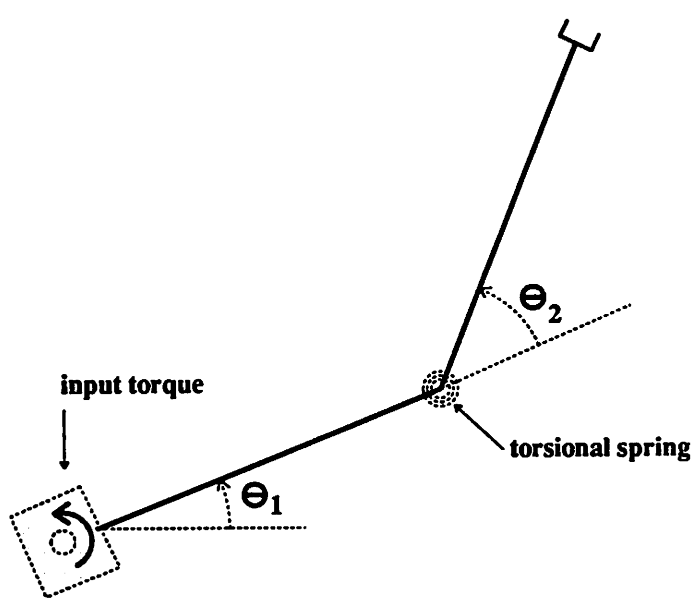

Figure 5.1 Arm configuration.

Figure 5.1 shows the configuration of the arm. The objective is to control the angle of the end effector through the torque applied at the base. With the angles θ_1 and θ_2 identified in Figure

5.1 we can write the equations of motion of the arm as follows:

$$\begin{bmatrix} \frac{5}{3}+\cos(\theta_2) & \frac{1}{3}+\cos(\theta_2) \\ \frac{1}{3}+\cos(\theta_2) & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 + \frac{1}{2}\sin(\theta_2)(2\dot{\theta}_1+\dot{\theta}_2)\,\dot{\theta}_2 \\ -k\theta_2 - \frac{1}{2}\sin(\theta_2)\dot{\theta}_1^2 \end{bmatrix} \tag{5.1}$$

where we have assumed, for simplicity, that the two segments are symmetric uniform rods of unit mass and length. Also k denotes the proportionality constant of the torsional spring. We can write (5.1) as a set of first order differential equations in the standard form: $\dot{x} = f(x) + g(x)\,u$. Thus letting x_1, x_2, x_3, and x_4 denote θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$ respectively, and letting u denote τ_1, we obtain the following expressions for the vector fields $f(x)$ and $g(x)$:

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{1}{\Delta}[\frac{1}{6}\sin(x_2)(2x_3+x_4) + (\frac{1}{3}+\frac{1}{2}\cos(x_2))(kx_2+\frac{1}{2}\sin(x_2)x_3^2] \\ \frac{-1}{\Delta}[(\frac{1}{6}+\frac{1}{4}\cos(x_2))\sin(x_2)(2x_3+x_4)x_4 - (\frac{5}{3}+\cos(x_2))(kx_2+\frac{1}{2}\sin(x_2)x_3^2)] \end{bmatrix} \tag{5.2}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3\Delta} \\ -\frac{2+3\cos(x_2)}{6\Delta} \end{bmatrix} \tag{5.3}$$

where Δ denotes the determinant of the inertia matrix in (5.1) and is given by $\Delta = \frac{7}{36} + \frac{1}{4}\sin^2(\theta_2) > 0$. We are interested in controlling the angle of the end effector. Thus define the output y to be:

$$y = h(x) := x_1 + \frac{x_2}{2} \tag{5.4}$$

Then computing the Lie derivative of $h(x)$ with respect to $f(x)$ and $g(x)$ we obtain:

$$L_g h(x) = 0$$
$$L_g L_f h(x) = \frac{2 - 3\cos(x_2)}{12\Delta} \tag{5.5}$$

The first difficulty is now apparent from (5.5). The function $L_g L_f h(x)$ is neither identically zero nor is it nonzero for all values of x. In fact, this function goes through zero at values of x_2 corresponding to $\cos(x_2) = \frac{2}{3}$. More generally in a system in which the first control term appears in the vth derivative of the output, the singularities of $L_g L_f^{\nu-1} h(x)$ present a major limitation to the present theory in that as the trajectories approach the points of singularity, the linearizing control becomes unbounded. It is, therefore, necessary to devise an alternative control strategy in a neighborhood of singular points (or the singular manifold) of $L_g L_f^{\nu-1} h(x)$. We believe that this problem is a rather challenging one and a solution would be a major contribution to the theory. An attempt in this direction has been made by Hirschorn and Davis [20]. Their approach relies on identifying the class of tracking signals for which the control input remains bounded as the system trajectories pass through the singularities. This implies that the initial condition can be chosen exactly so that the output of the system starts on the desired path. Such a scheme, however, is practically unstable since the slightest perturbation or noise in the system, which is unavoidable, results in instability.

Continuing with our example, we note that we are interested in keeping x_2 at zero, which is far from the singular points of $L_g L_f h(x)$. Thus, we can apply the construction in Chapter 3 locally around $x_2 = 0$. To this end, define the new coordinates:

$$\xi_1 = x_1 + \frac{1}{2}x_2$$
$$\xi_2 = x_3 + \frac{1}{2}x_4$$
$$\eta_1 = x_2 \tag{5.6}$$
$$\eta_2 = (1 + \frac{3}{2}\cos(x_2))x_3 + x_4$$

where η_1 and η_2 are chosen so that $L_g \eta_i = 0, i=1,2$. The inverse of this transformation is given by:

$$
\begin{aligned}
x_1 &= \xi_1 - \frac{1}{2}\eta_1 \\
x_2 &= \eta_1 \\
x_3 &= \frac{4\xi_2 - 2\eta_2}{2 - 3\cos(\eta_1)} \\
x_4 &= \frac{4\eta_2 - 2(2 + 3\cos(\eta_1))\xi_2}{2 - 3\cos(\eta_1)}
\end{aligned} \tag{5.7}
$$

Clearly, the above transformation is a diffeomorphism for all x_2 with $\cos(x_2) > \frac{2}{3}$. Now computing the zero dynamics of the system we have:

$$
\begin{aligned}
\dot{\eta}_1 &= \frac{4\eta_2}{2 - 3\cos(\eta_1)} \\
\dot{\eta}_2 &= -3k\eta_1 + \frac{6\sin(\eta_1)\eta_2^2}{(2 - 3\cos(\eta_1))^2}
\end{aligned} \tag{5.8}
$$

Checking the stability of (5.8) we find that the linearization of (5.8) about the origin has eigenvalues at $\pm 2\sqrt{3k}$. Therefore, the zero dynamics of the system with respect to the output function (5.4) are always unstable. Thus, we can not apply a tracking control law based on the input-output linearization of the system since the internal dynamics of the system will be unstable. This is the second limitation of the theory, that is the non-applicability of the current theory to non-minimum-phase nonlinear systems. Development of a theory for non-minimum-phase systems would be a major advancement of the present theory.

The foregoing example has pointed out some directions for future research. There are many other aspects of nonlinear systems which require close attention as well (e.g. extension of the present theory to discrete time systems and multi input-multi output systems), and we feel that the present work is part of an ongoing effort towards a better understanding and the development of a comprehensive set of tools for nonlinear systems.

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