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SUFFICIENCY AND FUZZINESS IN RANDOM EXPERIMENTS

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In previous papers, the consequences of the "presence of fuzziness" in the experimental information on which statistical inferences are based were discussed. Thus, the intuitive assertion «Fuzziness entails a Loss of Information» was formalized, by comparing the information in the "exact case" with that in the "fuzzy case". This comparison was carried out on the basis of different criteria to Compare Experiments (e.g., that based on Sufficiency). The question we are now interested in is the following: how will different degrees of Fuzziness in the experimental information affect the Sufficiency?. In this paper, a study of this question for Bernoulli experiments is first developed. The immediate generalization of that study to other experiments indicates that two fuzzy data associated with the same experiment become comparable only whenever a restrictive condition is verified. In all cases, and under very general conditions, the comparison is coherent with the axiomatic requirements for measures of fuzziness.

Keywords and phrases: Fuzziness, Fuzzy Information, Random Experiment, Sufficiency, Zadeh's Probability of a Fuzzy Event.

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1. Comparing Traditional Experiments through Sufficiency

The essential element in statistical problems is the *random experiment,* that is a process by which an observation is made, resulting in an outcome that cannot be previously predicted. In addition, it is often assumed that the experiment can be repeated under more or less identical conditions and there is statistical regularity. In such a situation, the components of a model for a random experiment are: i) the identification of all experimental outcomes; ii) the identification of all observable events (or statements regarding the experimental outcome, so that after the experiment has been performed one can determine if each one of them is true or false); iii) the assignment of probabilities to these events.

According to the second component, or more precisely according to the ability to observe the experimental outcomes, the traditional approach often admits that the observer is able to perceive the exact outcome after each experimental performance. The model associated with this "traditional experiment" is then given by a probability space $(X,\mathcal{B}_X,P_\theta)$, $\theta \in \Theta$, where X is the sample space (or set of all possible exact outcomes), $\mathcal{B}_X$ is the σ-field of all events of interest (so that, each observable event may be mathematically identified with a measurable subset of the sample space X), and θ is the state or parameter value governing the experimental distribution P_θ. Furthermore, it is usually supposed that X is a set of real numbers and $\mathcal{B}_X$ is the smallest Borel σ-field on X (in other words, the elementary observable events are all the singletons of exact outcomes).

Given two experiments, $\mathbf{E} = (X,\mathcal{B}_X,P_\theta)$, $\theta \in \Theta$, $\mathbf{F} = (Y,\mathcal{B}_Y,Q_\theta)$, $\theta \in \Theta$, whose distributions are governed by the same state of nature or parameter value θ, the idea of comparing such experiments was introduced by Bohnenblust, Shapley and Sherman, in a private communication whose basic results are collected by Blackwell (1951), and developed into a theory by Blackwell (1951, 1953). Many preference relations to compare experiments have been then examined and connected with the previous ones (see, for instance, papers referenced in Lehmann, 1988).

Blackwell's (1951, 1953) method for comparing experiments states that the experiment $\mathbf{E}$ is *sufficient for* the experiment $\mathbf{F}$ if there exists a nonnegative function h on $X \times Y$, so that the density function associated with Q_θ with respect to a σ-finite measure ν on $\mathcal{B}_X \times \mathcal{B}_Y$ is given by

$$g_\theta(y) = \int_X h(x,y)\, f_\theta(x)\, d\nu(x), \quad \text{for all } \theta \in \Theta$$

where

$$\int_Y h(x,y)\, dv(y) = 1, \quad \text{for all} \quad x \in X$$

and h is integrable with respect to x ($f_\theta(x)$ being the density function associated with P_θ with respect to the σ-finite measure ν).

Since the function h (called stochastic transformation) does not depend on θ, the above sufficiency condition indicates intuitively that an outcome from **F** could be generated from an observation on **E** and an auxiliary randomization according to h. Consequently, to observe **F** does not add any probabilistic information about θ to what is contained in **E**.

A second traditional approach, which is eventually considered in the statistical literature, admits that the observer is not able to perceive the exact outcome after each experimental performance, since the required degree of accuracy to specify a measured quantity exactly is lacking, but the available experimental information is grouped in classes or intervals. The problem of the loss of information due to grouping of observations has been examined from different viewpoints (cf., Kale, 1964, Kullback, 1968, and Ferentinos and Papaioannou, 1979).

The notion of sufficiency could be also applied in this second approach. Thus, if the available experimental information from an experiment is grouped in accordance with a partition $\mathbb{X} = \{E_i\}_{i\in I}$ (where $E_i \in \mathcal{B}_X$), then we can define a new probability space $\mathbb{E} = (\mathbb{X},\sigma(\mathbb{X}),\mathbb{P}_\theta)$, $\theta\in\Theta$, so that $\sigma(\mathbb{X})$ = σ-field generated by $\mathbb{X}$ and

$$\mathbb{P}_\theta(E_i) = \int_X \chi_{E_i}(x)\, f_\theta(x)\, dv(x), \quad i \in I$$

(where χ_{E_i} is the indicator function of E_i) and, on the basis of this induced probability $\mathbb{P}_\theta$, we can easily compare partitions associated with the same or different experiments.

2. Fuzziness in the Experimental Information

In previous papers (1987, 1988abc), we have analyzed an intermediate approach, in which we admit that the ability of the observer does not allow him to express the available information from the performance of $E = (X,\mathcal{B}_X,P_\theta)$, $\theta \in \Theta$, in terms of an exact outcome, but rather each observable event may be mathematically identified with a measurable fuzzy subset of the space X (or, alternatively, an observable event is intended in this approach as a statement regarding the experimental outcome, so that after the experiment has been conducted one can determine the "degree to which it is true").

It is worth remarking that in this approach one assumes that the empirical evidence contained in the present experimental observation conveys no information concerning the probability distribution of the exact outcomes, but the "degree of compatibility"of these outcomes with it. So, all probabilistic information regarding P_θ (and θ, in the Bayesian context) had been obtained from previous experimental observations (or, degrees of belief). Consequently, we suppose that, after the experiment has been conducted, there is still an imprecision associated with the identification of the obtained outcome, and this imprecision is non-probabilistic but possibilistic in nature (see, Zadeh, 1978).

A model for this new situation starts with the mathematical identification of the available information, (Okuda *et al.*, 1978, Tanaka *et al.*, 1979, Zadeh, 1978),

DEFINITION 2.1. A fuzzy event e on X, characterized by a Borel-measurable membership function μ_e from X to $[0,1]$, where $\mu_e(x)$ represents the "degree of compatibility" of x with e (or degree to which e is satisfied when x is the outcome in the performance of E), is called *fuzzy information associated with the experiment* E.

As an illustrative example of this notion, we can consider the following one: Suppose that the time of attention (in minutes) to a concrete game in a population of ten-year children has an exponential distribution with unknown parameter θ (θ = inverse of the population mean time). A psychologist want to draw conclusions about θ, but as the loss of interest in a game does not usually happen in an instantaneous way, he cannot measure the time of attention exactly. Assume that he express the outcome after a measurement by means of propositions such as "too much time", or "around 20 minutes", or "a moderate time".

The uncertainty associated with these propositions is non-probabilistic in nature (since it is an uncertainty regarding concepts, not regarding exact events), but it could be easily described by means of fuzzy information. Thus: the proposition "too much time" could be assimilated with the fuzzy information e characterized, for instance, by the membership function $\mu_e(x) = (x-50)/10$ if $x \in (50,60)$, $= 1$ if $x \geq 60$, $= 0$ otherwise; the proposition "around 20 minutes" could be assimilated with the fuzzy information e' characterized, for instance, by the membership function $\mu_{e'}(x) = (x-10)/10$ if $x \in (10,20]$, $= (30-x)/10$ if $x \in (20,30)$, $= 0$ otherwise; the proposition "a moderate time" could be assimilated with the fuzzy information e'' characterized, for instance, by the membership function

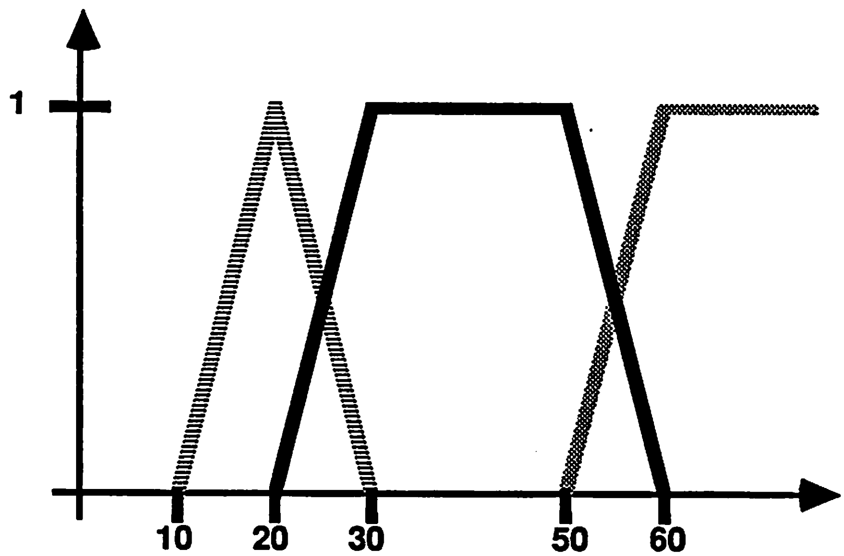

Fig. 1. Membership functions of the fuzzy data "too much time" (▒) "around 20 minutes" (☰), and "a moderate time" (▬).

$\mu_{e''}(x) = (x-20)/10$ if $x \in (20,30]$, $= 1$ if $x \in (30,50]$, $(60-x)/10$, if $x \in (50,60)$, $= 0$ otherwise (see Figure 1).

Another relevant element to model the new situation is the assignment of "probabilities" to the observable (fuzzy) events. Zadeh (1968) suggested to quantify the "induced probability" of a fuzzy event as follows:

DEFINITION 2.2. The *probability of* e *induced by* P_θ is given by

$$\mathcal{P}_\theta(e) = \int_X \mu_e(x)\; f_\theta(x)\, dv(x)$$

According to Zadeh (1978), the value $\mathcal{P}_\theta(e)$ could be interpreted as the "degree of consistency" of the probability distribution P_θ with the possibility distribution (Zadeh, 1978) assimilated with the membership function μ_e.

The use of the preceding definition could be justified by means of the following arguments: i) it is the most immediate extension of the second approach (grouping of data), in which we replace the indicator function of a grouped datum by the membership function of a fuzzy datum; ii) when $\mu_e(x)$ is interpreted as a kind of "probability" with which the observer perceives e when he really has obtained x in the performance of $\mathbf{E}$, Zadeh's definition is coherent with the *Total Probability Rule* of the Probability Theory; iii) Zadeh's definition is coherent with Le Cam's (1964, 1986) definition of the "probability" of bounded numerical functions in a *single stage experiment*.

Remark. It should be emphasized, that when we accept to use Zadeh's probabilistic definition we are implicitly accepting the second interpretation, ii), described in the previous paragraph. More precisely, the "induced probability" of e given x (defined in accordance with Definition 2.2) would coincide with $\mu_e(x)$. Nevertheless, this "induced probability" could not be appropriately defined within the probabilistic context, since $e \notin \mathcal{B}_X$ (i.e., e cannot be identified with a classical Borel set). Consequently, the interpretation in ii) means only an intuitive but not a formal approximation to quantify μ_e.

The question of how to measure the fuzziness of a particular fuzzy observation (or, in general, of a fuzzy subset) has been exhaustively studied in the literature of Fuzzy Sets (see, for instance, Klir and Folger, 1988). Formally,

DEFINITION 2.3. A *measure of fuzziness* is a real function f defined on $\tilde{\mathcal{F}}(X)$ (set of all fuzzy subsets of X) satisfying the following requirements:

Axiom 1. $f(e) = 0$ if and only if e is a crisp set.

Axiom 2. If $e, e' \in \tilde{\mathcal{F}}(X)$ and e is "sharper" than e', then $f(e) \leq f(e')$.

Axiom 3. $f(e)$ assumes the maximum value if and only if e is "maximally fuzzy".

The notions "sharper" and "maximally fuzzy" above, are usually interpreted as follows:

1) e is intended as "sharper" than e' if $\mu_e(x) \leq \mu_{e'}(x)$ for $\mu_{e'}(x) \leq 1/2$, and $\mu_e(x) \geq \mu_{e'}(x)$ for $\mu_{e'}(x) \geq 1/2$, for all $x \in X$.

2) e is intended as "maximally fuzzy" if and only if $\mu_e(x) = 1/2$, for all $x \in X$.

3. Connections between Sufficiency and Fuzziness

In previous papers (1987, 1988a), we discussed the consequences of the presence of fuzziness in the experimental information, by comparing the "information" in the "exact case" (in which fuzziness is completely absent) with that in the "fuzzy case". This comparison was carried out through different criteria to compare experiments, such as those based on Sufficiency (Blackwell, 1951, 1953), Shannon's Information Measure (Lindley, 1956), Expected Value of Sample Information (Raiffa and Schlaifer, 1961), Fisher's Amount of Information (Stone, 1961), and others.

The aim of this section is to develop a similar study by comparing through Sufficiency two situations, associated with the same population, in both of which different degrees of fuzziness can be present. To connect fuzziness in the experimental observations with the probabilistic notion of sufficiency, it should be first noted that the degree of fuzziness of a fuzzy set is usually expressed (Klir and Folger, 1988), in the most natural way, in terms of the lack of distinction between the set and its complement, since the less a set differs from its complement, the fuzzier it is. (Although the definition of the complement of a fuzzy set it not unique, we herein will employ that most commonly used, the fuzzy set e^c described by the membership function $\mu_{e^c}(x) = 1 - \mu_e(x)$, for all $x \in X$). Then, the degree of fuzziness of the fuzzy information e could be interpreted as the lack of distinction in the fuzzy 2-partition (Bezdek, 1987), $\mathcal{X} = \{e, e^c\}$.

On the other hand, on the basis of this fuzzy 2-partition and Zadeh's probabilistic definition, (Definition 2.2) it is possible to construct a new "probability" space, $\mathcal{E} = (\mathcal{X}, \mathcal{F}(\mathcal{X}), \mathcal{P}_\theta)$, $\theta \in \Theta$, (where $\mathcal{F}(\mathcal{X})$ = parts of $\mathcal{X}$), which may be regarded as a "probability space induced by $\mathcal{X}$".

Let $\mathbf{E} = (X, \mathcal{B}_X, P_\theta)$, $\theta \in \Theta$, be a random experiment and let e and e' denote two fuzzy observations associated with $\mathbf{E}$. Let $\mathcal{E} = (\mathcal{X}, \mathcal{F}(\mathcal{X}), \mathcal{P}_\theta)$, $\mathcal{E}' = (\mathcal{X}', \mathcal{F}(\mathcal{X}'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\mathcal{X} = \{e, e^c\}$, $\mathcal{X}' = \{e', e'^c\}$. Then, the notion of sufficiency may be immediately applied as follows:

DEFINITION 3.1.We will say that $\mathcal{E}$ is *sufficient for* $\mathcal{E}'$ if there exists a nonnegative function h on $\mathcal{X}\mathrm{x}\mathcal{X}'$ such that

$$\mathcal{P}_\theta(e') = h(e,e')\mathcal{P}_\theta(e) + h(e^c,e')\mathcal{P}_\theta(e^c), \quad \mathcal{P}_\theta(e'^c) = h(e,e'^c)\mathcal{P}_\theta(e) + h(e^c,e'^c)\mathcal{P}_\theta(e^c)$$

where

$$h(e,e') + h(e,e'^c) = 1, \quad h(e^c,e') + h(e^c,e'^c) = 1$$

Obviously, the conditions concerning $\mathcal{P}_\theta(e')$ and $\mathcal{P}_\theta(e'^c)$ are equivalent.

As the two fuzzy observations we have just compared in Definition 3.1 are associated with the same experiment (and, consequently, with the same probabilistic information), the comparison *via* sufficiency must be mainly dependent on the fuzziness in those observations. We are now going to formalize this assertion. Thus, we first consider the simplest case in which the referential experiment is Bernoulli.

3.1. Sufficiency and Fuzziness in Bernoulli Experiments

In particular, when $\mathbf{E} = (X, \mathcal{B}_X, P_\theta)$, $\theta \in \Theta$, is a Bernoulli experiment, it involves only two outcomes (often assimilated with the real values 0 and 1). The probability measure is then defined by $P_\theta(0) = 1 - \theta$, $P_\theta(1) = \theta$, $[0,1] \supset \Theta$. Many experiments are of this type: a vaccine is effective or it is not; a patient has a symptom or does not have it; a pathological condition is present or absent.

A fuzzy observation e associated with the Bernoulli experiment $\mathbf{E}$ may be described by means of a pair (μ_0,μ_1), where $\mu_0 = \mu_e(0)$ and $\mu_1 = \mu_e(1)$. The induced "probability" in this case would be given by $\mathcal{P}_\theta(e) = \mu_0 + \theta(\mu_1 - \mu_0)$. Examples of fuzzy observations associated with a Bernoulli experiment are, for instance, the following ones: a given patient sometimes cannot be diagnosed as having a particular malady or not; some organisms cannot be classified as belonging to a certain species or not.

If $e' = e^c$, then $\mathcal{E}$ and $\mathcal{E}'$ would be indifferent (that is, $\mathcal{E}$ would be sufficient for $\mathcal{E}'$ and, conversely, $\mathcal{E}'$ would be sufficient for $\mathcal{E}$), and hence we can constraint our study to the case in which $0 \le \mu_0 \le \mu_1 \le 1$ and $0 \le \mu_0' \le \mu_1' \le 1$.

The following theorem establishes equivalent conditions for the sufficiency,

THEOREM 3.1.1. Let $\mathbf{E} = (X,\mathcal{B}_X,P_\theta)$, $\theta \in \Theta$, be a random experiment and let e and e' denote two fuzzy observations associated with $\mathbf{E}$. Let $\mathcal{E} = (\mathcal{X},\mathcal{F}(\mathcal{X}),\mathcal{P}_\theta)$, $\mathcal{E}' = (\mathcal{X}',\mathcal{F}(\mathcal{X}'),\mathcal{P}_\theta)$, $\theta \in \Theta$, where $\mathcal{X} = \{e,e^c\}$, $\mathcal{X}' = \{e',e'^c\}$. Then,

i) $\mathcal{E}$ is sufficient for $\mathcal{E}'$ if and only if

(3.1.1) $\quad \mu_0\mu_1' \le \mu_0'\mu_1 \le \mu_0'\mu_1 + (\mu_1' - \mu_0') \le \mu_0\mu_1' + (\mu_1 - \mu_0)$

(where $\mu_0 = \mu_e(0)$, $\mu_1 = \mu_e(1)$ and $\mu_0' = \mu_{e'}(0)$ and $\mu_1' = \mu_{e'}(1)$).

Consequently, if $\mathcal{E}$ and $\mathcal{E}'$ are comparable, then

ii) $\mathcal{E}$ is sufficient for $\mathcal{E}'$ if and only if $\mu_1' - \mu_0' \le \mu_1 - \mu_0$. ❑

Although theoretically we have $0 \le \mu_0 \le 1, 0 \le \mu_1 \le 1$, we can constraint with no loss of generality our study to the cases in which $0 \le \mu_0 \le 1/2$ and $1/2 \le \mu_1 \le 1$, or $1/2 \le \mu_0 \le 1$ and $0 \le \mu_1 \le 1/2$. Thus, in the observation from a Bernoulli trial one could: a) obtain quite fuzzy information (so that the outcomes 0 and 1 are equally compatible with the observation), that could be often represented by $\mu_0 = \mu_1 = 1/2$; b) obtain information so that 0 is less compatible with the observation than 1, that could be often represented by $0 \le \mu_0 < 1/2$ and $1/2 < \mu_1 \le 1$; c) obtain information so that 0 is more compatible with the observation than 1, that could be often represented by $1/2 < \mu_0 \le 1$ and $0 \le \mu_1 < 1/2$. Due to the indifference between $\mathcal{E}$ and $\mathcal{E}'$ whenever $e' = e^c$, we constraintourselves to the case in which $0 \le \mu_0 \le 1/2$ and $1/2 \le \mu_1 \le 1$.

On the basis of the preceding theorem, and under the preceding assumptions, the following results state that the comparison of fuzzy observations by means of the notion of sufficiency is coherent with the axiomatic requirements that every measure of fuzziness must satisfy.

THEOREM 3.1.2. Let $\mathbf{E} = (X, \mathcal{B}_X, P_\theta)$, $\theta \in \Theta$, be a random experiment and let e and e' denote two fuzzy observations associated with $\mathbf{E}$. Let $\mathcal{E} = (\mathcal{X}, \mathcal{F}(\mathcal{X}), \mathcal{P}_\theta)$, $\mathcal{E}' = (\mathcal{X}', \mathcal{F}(\mathcal{X}'), \mathcal{P}_\theta)$, $\theta \in \Theta$, where $\mathcal{X} = \{e, e^c\}$, $\mathcal{X}' = \{e', e'^c\}$. Then,

i) if e is a crisp set, then $\mathcal{E}$ is sufficient for $\mathcal{E}'$, whatever the fuzzy set e' may be (that is, exact experimental information is always sufficient for fuzzy experimental information);

ii) if e is sharper than e', then $\mathcal{E}$ is sufficient for $\mathcal{E}'$;

iii) if e' is maximally fuzzy, then $\mathcal{E}$ is sufficient for $\mathcal{E}'$ (that is, fuzzy experimental information is always sufficient for uniformly fuzzy information). ❑

The result ii) in Theorem 3.1.2 is now illustrated by means of an example:

Example. Consider a population of mice, a fraction θ of which have a character C.

Assume that the character C may be recognized through two different symptoms A and B, each one of which determines the presence of character C.

However, suppose that after examining each mouse for presence of C, the accessible mechanisms of detection of A and B do not allow us to state them exactly, but it is only possible to conclude a: "the mouse has A quite sharply" or b: "the mouse seems more or less to have B". If these imprecise propositions are, for instance, assimilated with the fuzzy events characterized by the membership functions $\mu_a(1) = 0.9$, $\mu_a(0) = 0.2$, $\mu_b(1) = 0.6$, $\mu_b(0) = 0.3$ (quantifying the degree to which the perception agree with having or not each symptom with the available propositions, where $0 =$ C is absent, $1 =$ C is present), and we are interested in drawing conclusions about θ, it is preferred to try to detect A than B. Thus, if we define $h(a,b) = 45/7$, $h(a^c,b) = 15/7$, then $\mathcal{P}_\theta(b) = h(a,b)\mathcal{P}_\theta(a) + h(a^c,b)\mathcal{P}_\theta(a^c)$, whence $\mathcal{E} = (\mathcal{A}, \mathcal{F}(\mathcal{A}), \mathcal{P}_\theta)$, $\theta \in \Theta = [0,1]$, (where $\mathcal{A} = \{a, a^c\}$) is sufficient for $\mathcal{E}' = (\mathcal{B}, \mathcal{F}(\mathcal{B}), \mathcal{P}_\theta)$, $\theta \in \Theta = [0,1]$, (where $\mathcal{B} = \{b, b^c\}$).

3.2. Sufficiency and Fuzziness in other Experiments

Difficulties in the extension of this study for other experiments arise because of the non-comparability of the fuzzy data, unless some restricted conditions are satisfied. Thus, the immediate generalization of the present study to Binomial, Poisson, Exponential or Normal experiments indicates that two fuzzy data associated with the same experiment become comparable only in a special situation (in which the membership functions describing these data coincide with a particular linear transformation). Thus, by using the Inversion Formula Theorem of the Fourier Integral, we can verify that $\mathcal{E}$ is sufficient for $\mathcal{E}'$ if and only if there exist $\alpha,\beta \in [0,1]$ (independent of θ) such that

$$\mu_{e'}(x_1) - \mu_{e'}(x_0) = (\alpha - \beta)[\mu_e(x_1) - \mu_e(x_0)], \text{ for all } x_0, x_1 \in X$$

and

$$\mu_{e'}(x) = \beta + (\alpha - \beta)\, \mu_e(x), \text{ for all } x \in X$$

Therefore, if $\mu_0 = \mu_e(x_0)$, $\mu_1 = \mu_e(x_1)$, $\mu_0' = \mu_{e'}(x_0)$, $\mu_1' = \mu_{e'}(x_1)$ and $0 \le \mu_0 \le \mu_1 \le 1$, $0 \le \mu_0' \le \mu_1' \le 1$, condition (3.1.1) must be also satisfied. So, under the conditions, $0 \le \mu_0 \le 1/2$, $1/2 \le \mu_1 \le 1$, $0 \le \mu_0' \le 1/2$, $1/2 \le \mu_1' \le 1$, a result similar to that in Theorem 3.2, connecting sufficiency and fuzziness, can be obtained for that special situation.

4. Concluding Remarks

As in the non-fuzzy case, the inconveniences in the extension of this study for general experiments arise because of non-comparability problems.

It should be hence interesting to analyze in the near future questions similar to those discussed in this paper, but based on comparisons avoiding non-comparability inconveniences (that is, establishing complete preorderings) such as, for instance, that based on the Expected Value of Sample Information, or the expected Fisher's Amount of Information and so on, for a particular prior distribution.

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REFERENCES

Bezdek, J.C. (1987). Partition Structures: A tutorial, *Analysis of Fuzzy Information,* **III**, 81-107, CRC Press, Inc., Boca Raton.

Blackwell, D.A. (1951). Comparison of Experiments, *Proc. 2nd Berkeley Symposium on Mathematical Statistics and Probability*, 93-102, University of California Press.

Blackwell, D.A. (1953). Equivalent Comparison of Experiments, *Ann. Math. Statist.*, **24**, 265-272.

Ferentinos, K. and Papaioannou, T. (1979). Loss of Information due to Groupings. *Trans.8th Prague Conf. on Inf. Theo., Stat. Dec. Func. and Random Proc.*, C, 87-94.

Gil, M.A. (1987). Fuzziness and Loss of Information in Statistical problems, *IEEE Trans. Systems Man Cybernet.*, **17**, 1016-1025.

Gil, M.A. (1988a). On the Loss of Information due to Fuzziness in Experimental Observations. *Ann. Inst. Statis. Math.*, **40**, 4, 627-639.

Gil, M.A. (1988b). Probabilistic-Possibilistic approach to some Statistical Problems with Fuzzy Experimental Observations. In *Lecture Notes in Economics, No. 310 - Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making*, (eds. J. Kacprzyk and M. Fedrizzi), 286-306, Springer-Verlag, Berlin.

Gil, M.A., Corral, N. and Gil, P. (1988c). The Minimum Inaccuracy Estimates in χ^2 Tests for Goodness of Fit with Fuzzy Observations, *J. Statist. Plann. Inference*, **19**, 95-115.

Kale, K. (1964). A note on the Loss of Information due to Grouping of Observations, *Biometrika*, **51**, 495-497.

Klir, G.J. and Folger, T.A. (1988). *Fuzzy Sets, Uncertainty and Information.* Prentice Hall. New Jersey.

Kullback, S. (1968). *Information Theory and Statistics.* Dover. New York.

Le Cam, L. (1964). Sufficiency and Approximate Sufficiency. *Ann. Math. Statist.*, **35**, 1419-1455.

Le Cam, L. (1986). *Asymptotic Methods in Statistical Decision Theory.* Springer-Verlag. Berlin.

Lehmann, E.L. (1988). Comparing Location Experiments. *Ann. Statist.*, **16**, 521-533.

Lindley, D.V. (1956). On a Measure of the Information provided by an Experiment. *Ann. Math. Statist.* **27**, 986-1005.

Okuda, T., Tanaka, H. and Asai, K. (1978). A Formulation of Fuzzy Decision Problems with Fuzzy Information, using Probability Measures of Fuzzy Events, *Inform. and Control*, **38**, 135-147.

Tanaka, H., Okuda, T. and Asai, K. (1979). Fuzzy Information and Decision in Statistical Model. In *Advances in Fuzzy Sets Theory and Applications*, 303-320, North-Holland. Amsterdam.

Zadeh, L.A. (1968). Probability Measures of Fuzzy Events. *J. Math. Anal. Appl.*, **23**, 421-427.

Zadeh, L.A. (1978). Fuzzy Sets as a basis for a Theory of Possibility. *Fuzzy Sets and Systems*, **1**, 3-28.