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A NOTE ON ZADEH'S PROBABILISTIC DEFINITION

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This note presents a brief discussion regarding the interpretation of grades of membership describing fuzzy data from random experiments, when Zadeh's probabilistic definition is considered.

1. INTRODUCTION

Statistical problems often concern the drawing of conclusions about a random experiment, on the basis of the information supplied by the performance of that experiment. A *random experiment* is a process by which an observation is made, resulting in one outcome that cannot be previously predicted. In Statistics it is assumed that the random experiment can be repeated under (more or less) identical conditions, and there is a predictable long-run pattern (what is referred to as statistical regularity). According to its nature, to characterize a random experiment we need: i) to identify all experimental outcomes, ii) to identify all observable events, and iii) to assign probabilities to these events. An *observable event* is intended as a statement or question regarding the experimental outcome, and so that after the experiment has been conducted one can determine if it is true or false. Obviously, the observable events are determined from the ability of the person responsible for observing the experimental outcome.

In "traditional" Statistics it is supposed that the observer is able to perceive the exact outcome after each experimental performance. Then, the model describing a random experiment **X** is given by a probability space $(X,\mathcal{B}_X,P)$, where X is the sample space (or set of all possible experimental outcomes), $\mathcal{B}_X$ is the σ-field of all observable events, and P is a probability measure on $\mathcal{B}_X$. Furthermore, we hereafter assume that in the experiment we will consider a random variable (or vector) is to be observed, so that X will be the set of all variable (or vector) values and contained in a Euclidean space, and $\mathcal{B}_X$ will be the smallest Borel σ-field on X. Consequently, due to the assumption of the structure of σ-field for the set of all observable events, Probability Theory

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guarantees that every element in $\mathcal{B}_X$ will be identified with a "classical" subset of X, and the probability of an observable event $B \in \mathcal{B}_X$ is given by the Lebesgue-Stieltjes integral

$$P(B) = \int_B dP(x) = \int_X \chi_B(x)\, dP(x) \qquad (1)$$

(where χ_B = indicator or characteristic function of the Borel set B).

Nevertheless, one frequently encounters situations in which the ability of the observer does not allow him to express the available experimental information in terms of an exact outcome, but rather each observable event may be assimilated with a fuzzy subset of the sample space (or, alternatively, an observable event is intended in these situations as a statement or question regarding the experimental outcome, so that after the experiment has been conducted one can determine the "degree to which it is true") (cf. [10], [11]).

An illustrative situation is the following one: Water in a lake is examined to determine if it is drinkable. It is known that the water may contain a type of microorganisms, so that if the mean number of microorganisms per milliliter, θ, is greater than 7 water cannot be regarded as drinkable. Consider the random experiment consisting in observing the number of microorganisms per milliliter, and assume random distribution of microorganisms in lake water. Then, if exact information is available, we can describe this situation by means of a Poisson experiment with mean θ, $(\mathbb{N},\mathcal{B}_{\mathbb{N}},P_\theta)$, $P_\theta(x) = \theta^x e^{-\theta}/x!$ for $x \in \mathbb{N}$.

Suppose that a biologist is interested in concluding if lake water is drinkable or not, but the microorganisms are actually very difficult to be identified, and based on some features he can only decided after each experimental performance that there is a «very small number of micro-organisms», or a «moderate number of microorganisms», or «many microorganisms». This type of observable events can be easily described in terms of fuzzy subsets of the sample space, rather than in terms of a classical ones. Thus, a model for this new situation involves the mathematical identification of the available information (cf. [7], [9], [12]):

DEFINITION 1.1. A fuzzy event $\tilde{x}$ on X, characterized by a Borel-measurable membership function $\mu_{\tilde{x}}$ from X to [0,1], where $\mu_{\tilde{x}}(x)$ is the degree to which x belongs to $\tilde{x}$ (or degree to which x agrees with $\tilde{x}$), is called *fuzzy information associated with the experiment* **X**.

In particular, an observable event such as «a moderate number of microorganisms» could be assimilated, for instance, with the fuzzy event defined by the membership function in Figure 1.

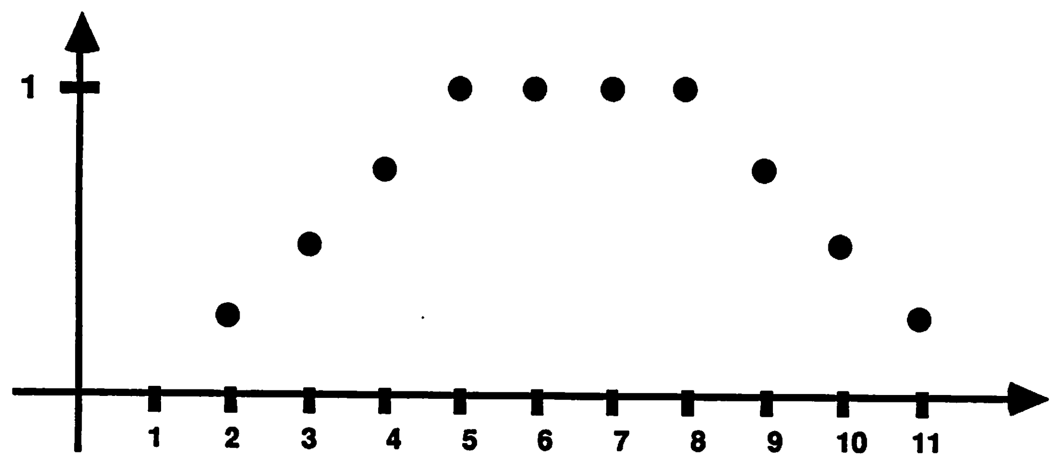

Fig. 1. Membership function describing the fuzzy information «a moderate number of microorganisms».

This assimilation would indicate that the biologist regarded at 9 as compatible with the perception «a moderate number of microorganisms» according to a degree equal to .75, and so on.

2. ZADEH'S PROBABILISTIC DEFINITION

For statistical purposes, one can be interested in establishing the probabilities of all observable events, even for the case in which they are characterized by means of fuzzy subsets.

Zadeh, [11], suggested to quantify the "induced probability" of the fuzzy information associated with an experiment as follows:

DEFINITION 2.1. The *probability of* $\mathcal{X}$ *induced by* P is given by the Lebesgue-Stieltjes integral

$$\mathcal{P}(\mathcal{X}) = \int_X \mu_{\mathcal{X}}(x)\, dP(x) \qquad (2)$$

According to Zadeh, [12], the value $\mathcal{P}(\mathcal{X})$ could be viewed as the "degree of consistency" of the probability distribution P with the possibility distribution associated with the membership function $\mu_{\mathcal{X}}$

Although (2) is introduced as a definition (not a result), it may be justified through different arguments:

* it is the most *immediate extension* of the non-fuzzy case (in which we replace the indicator or characteristic function by the membership function),

and·

* it is coherent with *Le Cam's definition* of the probability of bounded numerical functions in a *single stage experiment* (cf. [5], [6]). Thus, Le Cam suggested to replace the structure of a classical experiment, by a weaker structure $(X,\mathcal{V}_X,P)$, called single stage experiment, where $\mathcal{V}_X$ is a vector lattice for the usual operations (sum, product by real numbers, pointwise supremum and infimum) that contains the indicator or characteristic function of X and complete for the norm $\sup|\cdot|$, P is a normalized linear functional on $\mathcal{V}_X$, and for $f \in \mathcal{V}_X$ the value of P at f may be considered as the Lebesgue-Stieltjes integral given by $P(f) = \int_X f(x)\, dP(x)$. Consequently, if the fuzzy event χ is such that μ_χ belongs to a vector lattice $\mathcal{V}_X$ in a single stage experiment, then $P(\mu_\chi) = \mathcal{P}(\chi)$.

3. DISCUSSING THE INTERPRETATION OF GRADES OF MEMBERSHIP

Let $\mathbf{X} = (X,\mathcal{B}_X,P)$ be a random experiment and let χ denote a fuzzy event associated with $\mathbf{X}$. Given $x^\circ \in X$, we can define a new experiment $\mathbf{X}(x^\circ)$ in which a random variable (or vector) degenerated at x° is to be observed. This experiment would be characterized by the probability space $(\{x^\circ\},\mathcal{B}_{\{x^\circ\}},P_{x^\circ})$, where $P_{x^\circ}(\{x^\circ\}) = 1$. If we now consider the restriction of the fuzzy event χ to $\{x^\circ\}$ (or, more precisely, the restriction of the mapping μ_χ from X to $\{x^\circ\}$), then

PROPOSITION 3.1. The probability of the restriction of χ induced by P_{x° is given by

$$\mathcal{P}_{x^\circ}(\chi) = \mu_\chi(x^\circ) \qquad (3) \qquad \square$$

This last result indicates that, when we use Zadeh's probabilistic definition, $\mu_\chi(x^\circ)$ could be intuitively interpreted as "a kind" of (induced) probability with which the observer gets χ when he really has obtained x°. That interpretation was previously considered by Ruspini, [8], Tanaka *et al.*, [9], and Hisdal (see [4] and some papers referenced in it), and for the former example would indicate that in 75 % of milliliters of the lake water in which there are 9 microorganisms, the observer would conclude that there would be a moderate number of them.

Nevertheless, it should be emphasized that such an interpretation does not mean a rigourous approximation to quantify the grade of membership, because of the following reasons:

i) The (induced) probability in Eq.(3) cannot be well-defined in a classical probabilistic framework. Thus, according to Probability Theory, to define the probability of χ this event has to belong to the corresponding σ-field, and χ cannot be identified with a classical (Borel) set.

ii) the source of uncertainty is in this case fuzziness, instead of randomness. In this way, the scheme in Figure 2 explains the mechanism leading to obtain fuzzy information.

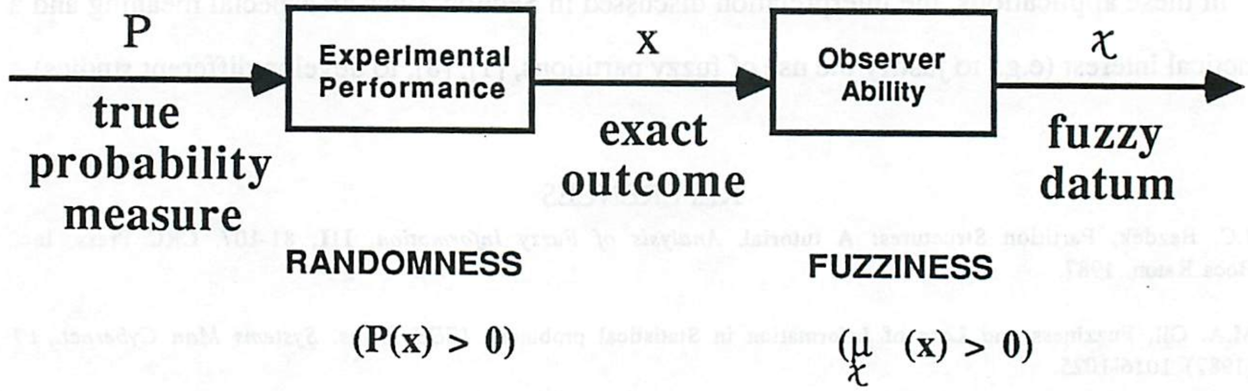

Fig. 2. Process leading to obtain fuzzy information

In this scheme we easily distinguish between these two types of uncertainty. So, after the experiment has been conducted, randomness disappears, but sometimes fuzziness remains (since a concrete experimental outcome will have been obtained, although the observer ability does not permit him to perceive it exactly, or the nature of the event itself does not permit the observer to conclude if it is true or false). In other words, the uncertainty associated with propositions we assimilate with fuzzy events is non-probabilistic in nature, since it regards to concepts, not exact events.

iii) The process of restricting the fuzzy event χ from $\mathbf{X}$ to $\mathbf{X}(x^\circ)$ does not mean an intuitive step in computing $\mu_\chi(x^\circ)$, since χ makes only sense when it is defined over all X.

The preceding arguments indicate that when we try to use Eq.(3) as a way to approximate $\mu_\chi(x)$, we have first to formalize χ and its probability. This formalization can be carried out in a framework involving Probability Theory and Fuzzy Sets Theory, but it is not possible to develop in a classical probabilistic framework.

Consequently, we can conclude that Eq.(3) can be viewed as an intuitive but not a formal way to assign grades of membership to fuzzy information associated with a random experiment.

4. CONCLUDING REMARKS

Applications of Zadeh's probabilistic definition can be found in the literature of Statistics with fuzzy data (see, for instance, [3], [4], [7], [8], and [9]).

In these applications, the interpretation discussed in Section 3 has an especial meaning and a practical interest (e.g., to justify the use of fuzzy partitions, [1], [8], to develop different studies).

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