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# A Stability Result[1]

## A Revised Proof of M. Kelemen's stability result (IEEE Transactions on Automatic Control, volume AC-31, No. 8, August 1986, pp.766-768)

Shahab Sheikholeslam and Charles A. Desoer

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*Abstract*

This note is a careful derivation of a result published by M. Kelemen,[Kel.], whose original contribution contains a number of obscurities.

Consider a smooth control system $\dot{x} = f(x, u)$ where for each <u>constant</u> input $u$ in some set the corresponding equilibrium point $q$ [hence $f(q, u) = 0$] is exponentially stable. Consider an input $u : [t_0, \infty) \rightarrow U$ and the corresponding equilibria $q(t)$. Let $x(t)$ be the solution corresponding to that $u(t)$ with $x(t_0)$ as initial condition. Roughly speaking, the following is established: if $x(t_0) - q(t_0)$ is sufficiently small and if $\dot{u}(t)$ is sufficiently small on $[t_0, \infty)$, then for some $\rho < \infty$, $\|x(.) - q(.)\|_\infty < \rho$ and $x(t)$ remains, for all $t$, in the basin of attraction of the sink $q(t)$.

# 1 Stability Result

Consider the dynamical system described as follows:

$$\dot{x} = f(x, u) \tag{1}$$

where $x$ belongs to $P$, an open subset of $R^n$ and $u$ belongs to $U$, an open subset of $R^m$.

**Definition** A point $q_0$ in $P$ is called a sink of (1) corresponding to the constant input $u_0$ in $U$ if $f(q_0, u_0) = 0$ and $Re\sigma[D_1 f(q_0, u_0)] < 0$; where $D_1 f(.,.)$ denotes the Jacobian matrix of $f(.,.)$ with respect to the first variable and $\sigma[.]$ denotes the spectrum of a matrix.

**Theorem** Suppose that $P \subset R^n$ is open, $U \subset R^m$ is open, and $P$ is convex; let $f : P \times U \rightarrow R^n$ be a $C^2$ function such that $M =$

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$\{(q,u) \in P \times U |\ q$ is a sink of (1) corresponding to $u\}$, is not empty. Let $Q$ be an open, connected subset of $M$, relatively compact in $M$. Let $u : [t_0, \infty) \to U$ and $q : [t_0, \infty) \to P$ be two given $C^1$ functions such that $(q(t), u(t)) \in Q$ for all $t \geq t_0$. Let $x(.)$ be the solution of (1) with the $u(.)$ defined above.

Then, for any $\rho > 0$, there exists $\delta_1 > 0$, $\delta_2 > 0$, independent of $t_0$, for all $u(.)$ and $q(.)$ defined as above and such that $|x(t_0) - q(t_0)| \leq \delta_1$ and $\max_{t \geq t_0} |\dot{u}(t)| \leq \delta_2$ we have:

i) $|x(t) - q(t)| < \rho$ for all $t \geq t_0$

ii) If in addition $\rho$ is sufficiently small, $x(t)$ belongs to domain of attraction of sink $q(t)$ with respect to input $u(t)$ for all $t \geq t_0$.

**Preliminary Analysis- Step I** Writing the integral formula for the Taylor's expansion of $f(x,u)$ about a sink $q$ corresponding to the constant input $u$ we obtain(since $f$ is $C^2$ and $P$ is convex):

$$\dot{x} = \int_0^1 D_1 f[q + \lambda(x - q), u] d\lambda \, (x - q) \tag{2}$$

where for convenience we suppress the explicit dependence of $x$ and $q$ on $t$.

Since $f(q,u) = 0$, differentiating both sides of this equation with respect to $t$ gives:

$$\frac{d}{dt} f(q,u) = D_1 f(q,u)\dot{q} + D_2 f(q,u)\dot{u} = 0 \tag{3}$$

Solving for $\dot{q}$ in terms of $\dot{u}$ in (3), we obtain:

$$\dot{q} = -[D_1 f(q,u)]^{-1} D_2 f(q,u)\dot{u} \tag{4}$$

where we noted that since $Re\sigma[D_1 f(q,u)] < 0$, $D_1 f(q,u)$ is invertible. Subtracting (4) from (2) we obtain:

$$\dot{x} - \dot{q} = \int_0^1 D_1 f[q + \lambda(x - q), u] d\lambda \, (x - q) + [D_1 f(q,u)]^{-1} D_2 f(q,u)\dot{u} \tag{5}$$

Adding and subtracting $D_1 f(q,u)(x - q)$ from the right-hand side of (5) gives:

$$\begin{aligned} \dot{x} - \dot{q} = D_1 f(q,u)(x - q) \quad &+ \quad \int_0^1 \{D_1 f[q + \lambda(x - q), u] - D_1 f(q,u)\} \, d\lambda (x - q) \\ &+ \quad [D_1 f(q,u)]^{-1} D_2 f(q,u)\dot{u} \end{aligned} \tag{6}$$

With a slight abuse of notation we write:

$$A(t) := A(q(t), u(t)) := D_1 f(q(t), u(t)) \tag{7}$$

$$R(t) := R(q(t), u(t), x(t)) := \int_0^1 \{D_1 f[q(t) + \lambda(x - q)(t), u(t)] - D_1 f(q(t), u(t))\} \, d\lambda \tag{8}$$

$$B(t) := B(q(t), u(t)) := [D_1 f(q(t), u(t))]^{-1} D_2 f(q(t), u(t)) \tag{9}$$

Using these notations we rewrite (6) as follows:

$$\dot{x} - \dot{q} = A(t)(x - q) + R(t)(x - q) + B(t)\dot{u} \tag{10}$$

Using (10) we can write an implicit relation for $(x - q)$ as follows:

$$(x - q)(t) = \Phi(t, t_0)(x - q)(t_0) + \int_{t_0}^{t} \Phi(t, s) \{R(s)(x - q)(s) + B(s)\dot{u}(s)\} \, ds \tag{11}$$

where $\Phi(t, t_0)$ is the state transition matrix of the linear system:

$$\dot{z} = A(t)z \tag{12}$$

Since $(q(t), u(t)) \in Q$, $Q$ is relatively compact in $M$, and $D_1 f(.,.)$ is continuous(since $f$ is $C^2$), we note from (7) that

$$A(.) \text{ is bounded on } [t_0, \infty). \tag{13}$$

Since $\sigma(A(t)) = \sigma[A(q(t), u(t))]$ is a continuous function of its entries, $(q(t), u(t)) \in Q$ with $Q$ relatively compact in $M$ and $q(t)$ is a sink of (1), for all $t \geq t_0$, it can be shown that:

$$\text{there exists a } \mu < 0 \text{ such that } Re\sigma(A(t)) \leq \mu < 0 \text{ for all } t \geq t_0 \tag{14}$$

From (13) and (14), it is well known [Brock., Theorem2, sec.32] that there exists an $\epsilon > 0$ such that:

$$\text{if } |\dot{A}(t)| \leq \epsilon \text{ then for some } k \geq 1 \quad \text{and some } \eta > 0 \quad \text{and for all } t \geq s \geq t_0, \quad |\Phi(t, s)| \leq k e^{-\eta(t-s)}. \tag{15}$$

To obtain a relation between $\dot{A}(t)$ and $\dot{u}(t)$, differentiate both sides of (7) with respect to $t$ and use the chain rule :

$$\dot{A}(t) = D_1 D_1 f[q(t), u(t)]\dot{q}(t) + D_2 D_1 f[q(t), u(t)]\dot{u}(t) \tag{16}$$

Writing $\dot{q}(t)$ in terms of $\dot{u}(t)$ using (4) and (9) in (16) we get:

$$\begin{aligned}\dot{A}(t) &= \{-D_1 D_1 f[q(t), u(t)]B(t) + D_2 D_1 f[q(t), u(t)]\}\,\dot{u}(t) \\ &:= D(q(t), u(t))\dot{u}(t)\end{aligned} \tag{17}$$

Since $(q(t), u(t)) \in Q$, $Q$ is relatively compact in $M$, and $D(.,.)$ is continuous(since $f$ is $C^2$), $D(.,.)$ is bounded on $Q$. Hence if we let $a := max_Q|D(q,u)|$, then $0 \leq a < \infty$.
Now if

$$max_{t \geq t_0}|\dot{u}(t)| \leq \delta_2' := \frac{\epsilon}{a} \tag{18}$$

then $|\dot{A}(t)| \leq |D(q(t), u(t))||\dot{u}(t)| \leq \epsilon$ and (15) is satisfied.

**Step II** Denote $P_Q = \{q \in P | (q,u) \in Q\}$ (i.e., $P_Q$ is the projection of $Q$ on $P$). Let $Z$ be a compact set such that $\overline{P}_Q \subset Z^0 \subset Z \subset P$ where $Z^0 :=$ interior of $Z$. Such a $Z$ exists because $\overline{P}_Q$ is a compact subset of open set $P$. Let $W := Q \times Z$. Since $f$ is $C^2$, $R(.,.,.)$, defined in (8), is a continuous function. Since $Q$ is relatively compact in $M$, $Z$ is compact, and $R(.,.,.)$ is continuous, it follows that $R(.,.,.)$ is *uniformly continuous* on $W$.
Note that when $x(t) = q(t)$ in (8) we obtain $R(t) = 0$; also $q(t) \in P_Q \subset Z$. Thus, using the uniform continuity of $R(.,.,.)$ on $W$, we note that:
Given any $c > 0$, there exists a $\delta' := \delta'(c) > 0$ such that for all $t \geq t_0$,

$$\text{if } x(t) \in Z \text{ and } |x(t) - q(t)| \leq \delta' \text{ then } |R(t)| \leq c. \tag{19}$$

Taking norms of (11), and using (15) and (19), we conclude that:
if a) $max_{t \geq t_0}|\dot{u}(t)| \leq \delta_2'$, b) for all $t \geq t_0$, $x(t) \in Z$ and c) for all $t \geq t_0$, $|x(t) - q(t)| \leq \delta'$ then for all $t \geq t_0$

$$\begin{aligned}|x(t) - q(t)| &\leq ke^{-\eta(t-t_0)}|x(t_0) - q(t_0)| + k\int_{t_0}^{t} e^{-\eta(t-s)}|B(s)||\dot{u}(s)|ds \\ &+ \int_{t_0}^{t} ke^{-\eta(t-s)}c|x(s) - q(s)|ds\end{aligned} \tag{20}$$

Using Bellman-Gronwall inequality[Hal., ch. I, Lemma I.6, consequence 1], we note that if the hypotheses of (20) are satisfied we obtain for all $t \geq t_0$:

$$|x(t)-q(t)| \leq ke^{(-\eta+kc)(t-t_0)}|x(t_0)-q(t_0)| + k\int_{t_0}^{t} e^{(-\eta+kc)(t-s)}|B(s)||\dot{u}(s)|ds \tag{21}$$

Let $d :=$ distance between $\overline{P}_Q$ and $\partial Z$ where $\partial Z$ denotes boundary of $Z$. Since $\overline{P}_Q$ is a proper subset of $Z$, $d > 0$.
Let $b := max_Q|B(q,u)|$, where $B(.,.)$ is defined in (9). Since $Q$ is relatively compact in $M$, and $B(.,.)$ is continuous(since $f$ is $C^2$ and (13) and (14) hold), we conclude that $b < \infty$.
Choose $c > 0$ such that $-\eta + kc < 0$. Choose $\delta' := \delta'(c) > 0$ such that (19) is satisfied. Let $\delta := min\,\{\delta'(c), d\}$ , and choose constants $l$ and $r$ such that $0 < l < 1$, $0 \leq r \leq 1$. Denote $\delta_1 := \frac{l\delta r}{k}$ and $\delta_2 := min\left\{\delta_2', -\frac{(-\eta+kc)(1-r)l\delta}{kb}\right\}$.
Note that $\delta > 0$, $\delta_1 \geq 0$, and $\delta_2 \geq 0$.

**Lemma 1** If $c$, $\delta$, $\delta_1$, and $\delta_2$ are chosen as above and if $x(t_0)$ and $u(.)$ are such that $|x(t_0) - q(t_0)| \leq \delta_1$, and $max_{t\geq t_0}|\dot{u}(t)| \leq \delta_2$ then the hypotheses required for (20) and (21) are satisfied.

**Proof of Lemma 1** First note that $max_{t\geq t_0}|\dot{u}(t)| \leq \delta_2'$ from the definition of $\delta_2$. Next we will show that

$$|x(t) - q(t)| < \delta' \text{ for all } t \geq t_0 \tag{22}$$

Suppose (22) is false. Then there exists $t_2 \in (t_0, \infty)$ such that

$$|x(t) - q(t)| < \delta' \text{ for all } t \in [t_0, t_2) \text{ and } |x(t_2) - q(t_2)| = \delta'. \tag{23}$$

**Claim 1:**

$$x(t) \in Z \text{ for all } t \in [t_0, t_2]. \tag{24}$$

Suppose (24) is false. Then there exists a $t_3 \in (t_0, t_2)$ such that:

$$x(t) \in Z \text{ for all } t \in [t_0, t_3) \text{ and } x(t_3) \in \partial Z. \tag{25}$$

From (23) and (25) we note that:
$x(t) \in Z$ and $|x(t) - q(t)| < \delta'$ for all $t \in [t_0, t_3)$.
Thus, hypotheses of (21) are satisfied for all $t \in [t_0, t_3)$ and we obtain from (21):

$$|x(t) - q(t)| < k\delta_1 + kb\delta_2(-\frac{1}{-\eta + kc}) = l\delta \text{ for all } t \in [t_0, t_3) \tag{26}$$

By continuity of $x(.)-q(.)$, and using the last inequality in (26) we obtain: $|x(t_3) - q(t_3)| \leq l\delta < \delta \leq d$ which contradicts (25) in that $x(t_3) \in \partial Z$ (i.e., $|x(t_3) - q(t_3)| \geq d$). Hence, (24) is true and Claim 1 is established.

From (23) and (24), we note that:
$x(t) \in Z$ and $|x(t) - q(t)| \leq \delta'$ for all $t \in [t_0, t_2]$.
Thus, hypotheses of (21) are satisfied for all $t \in [t_0, t_2]$ and (26) is true for all $t \in [t_0, t_2]$. In particular, we have $|x(t_2) - q(t_2)| \leq l\delta < \delta \leq \delta'$ which contradicts (23) in that $|x(t_2) - q(t_2)| = \delta'$. Hence, (22) is true.

Finally, to complete the proof of Lemma 1 we will show that:

$$x(t) \in Z^0 \text{ for all } t \geq t_0 \tag{27}$$

Suppose (27) is false. Then there exists $t_1 \in (t_0, \infty)$ such that

$$x(t) \in Z^0 \text{ for all } t \in [t_0, t_1) \text{ and } x(t_1) \in \partial Z. \tag{28}$$

From (22) and (28) we note that the hypotheses of (21) are satisfied for all $t \in [t_0, t_1)$ and (26) is true for all $t \in [t_0, t_1)$. Namely, $|x(t) - q(t)| \leq l\delta < \delta \leq d$ for all $t \in [t_0, t_1)$. By continuity of $x(.)-q(.)$ we get $|x(t_1) - q(t_1)| \leq l\delta < d$ which contradicts (28) in that $x(t_1) \in \partial Z$ which implies $|x(t_1) - q(t_1)| \geq d$. Hence, (27) is true. This completes the proof of Lemma 1.

**Proof of theorem, part (i):** Now given $\rho > 0$, choose $c > 0$, $\delta_1 > 0$, $\delta_2 > 0$, and $0 < l < min\{1, \frac{\rho}{\delta}\}$ so that hypotheses of Lemma 1 are satisfied. Then, using (21) we obtain
$|x(t) - q(t)| \leq l\delta < \rho$ for all $t \in [t_0, \infty)$. Hence, part (i) of the theorem is established.
(Note: $\delta_1$, $\delta_2$ depend only on $f$ and $Q$ not on $t_0$, $u(.)$, and $q(.)$.)

**Proof of part (ii):** If $\rho \leq \frac{\delta}{k}$ (let $r = 1$ and $0 < l < 1$), then $\delta_1 = \frac{l\delta}{k} < \frac{\delta}{k}$ and we have

$$|x(t_0) - q(t_0)| < \frac{\delta}{k} \tag{29}$$

In addition, since $r = 1$, $\delta_2 = 0$ and we get

$$\dot{u}(t) = 0 \text{ for all } t \geq t_0 \tag{30}$$

Hence, using the inequality in (21), (29), and (30) we get:

$$|x(t') - q(t')| \leq \delta e^{(-\eta + kc)(t' - t_0)}$$

and

$$|x(t') - q(t')| \leq \delta \leq d \text{ for all } t' \geq t_0$$

Hence, $x(t')$ belongs to the domain of attraction of $(q(t'), u(t'))$ for all $t' \geq t_0$. Hence, part (ii) of the theorem is established.

# 2 References


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