CONTROL AND STABILIZATION OF A FLEXIBLE BEAM ATTACHED TO A RIGID BODY

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Ö. Morgül

Department of Electrical Engineering and Computer Sciences

and the Electronics Research Laboratory

University of California, Berkeley, CA 94720

ABSTRACT

We consider a flexible spacecraft modeled as a rigid body which rotates arbitrarily in inertial space; a light flexible beam is clamped to the rigid body at one end and free at the other. The equations of motion are obtained by using free body diagrams. It is shown that suitable boundary controls applied to the free end of the beam and a control torque applied to the rigid body stabilize the system. The proof is obtained by using the energy of the system as a Lyapunov functional.

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introduction

Many mechanical systems, such as spacecrafts with flexible appendages, consist of coupled elastic and rigid parts. In such systems, if a good performance of overall system is desired, the dynamic effect of elastic members becomes important. Thus, over the last decade there has been a growing interest in obtaining new methods for the design, dynamics and control of the systems which have elastic parts. (e.g. see [Bal.1] and the references therein).

Consider a system which has rigid and elastic members.The motion of elastic members is usually described by a set of partial differential equations with appropriate boundary conditions. Since the motion of rigid parts is governed by a set of nonlinear ordinary differential equations and rigid members are coupled with elastic members, overall equations of motion are generally a set of coupled nonlinear partial and ordinary differential equations. These equations can be obtained by using standard methods in Mechanics, e.g. see [Gol.1].

After having obtained the equations of motion, the commonly used approach is to consider only finitely many modes of the elastic parts, which is called " modal analysis ". This approach reduces original equations to a set of coupled nonlinear *ordinary* differential equations. However, having established a control law for this reduced set of equations does not always guarantee that the same control law will work on the original set of equations, (e.g.one migh encounter so-called "spillover" problems , [Bal.2]). Also note that the actual number of modes of an elastic system,in theory, is infinite and the number of modes that should be retained is not known a priori.

Recently Biswas and Ahmed, [Bis.1], used a Lyapunov type approach to prove the stability of a rigid spacecraft with an elastic beam attached to it under appropriate forces and torques applied to the beam and the rigid spacecraft. Their proposed control laws contain *distributed* forces appied to the beam which are proportional to the beam deflection velocities. Implementation of such control laws might not be easy.

In recent years, boundary control of elastic systems (i.e. controls applied to the boundaries of elastic systems) has become an important research area. This idea is first applied to the systems governed by wave equation (e.g. strings), [Che.1], and recently extended to the beam equations. In particular Chen, [Che.2], proved that, in cantilever beam, a single actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam deflections.

In this paper, we consider the motion of a rigid body with a beam clamped to it, the other end of the beam is free. The rigid body is assumed to be rotating in an inertial space with its center of mass fixed in a given inertial frame. After having obtained the equations of motion, we define the rest state of the system. Then we state the contol problem, which is , if the system is perturbed from the rest state, to find appropriate control laws which drive the system to the rest state. We propose two different control laws, each of which consist of appropriate boundary force and moment controls applied to the beam at its free end and a torque control applied to the rigid body. We then show that the proposed control laws, rigid body angular velocities and beam deflections decay to the rest state.

In section 1, we explain the configuration under consideration and derive the equations of motion using free-body diagrams. Then we state the control problem and propose some feedback laws.

In section 2 and 3, we show that the proposed control laws solve the proposed control problem.

notation

boldface letters like **r** , **n** etc. denote vectors in R^3

$$\mathbf{L}^2 = \{ f : [0,L] \rightarrow R \mid \int_{x=0}^{x=L} f^2 \, dx < \infty \}$$

$$\mathbf{H}^k = \{ f \in \mathbf{L}^2 \mid f^i \in \mathbf{L}^2 \, , i=1,..,k \}.$$

$$\mathbf{H}^k{}_0 = \{ f \in \mathbf{H}^k \mid f(0) = f^1(0) = 0 \} .$$

f_x , f_t etc., denote $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, etc., resp.

x denotes standard cross-product in R^3.

$<,> : R^3 \, x \, R^3 \rightarrow R$ denotes the standard inner product in R^3.

section 1

1.1 equations of motion

We consider the following configuration : Figure 1 shows the rigid body (drawn as a square) and the beam ; P is a point on the beam.

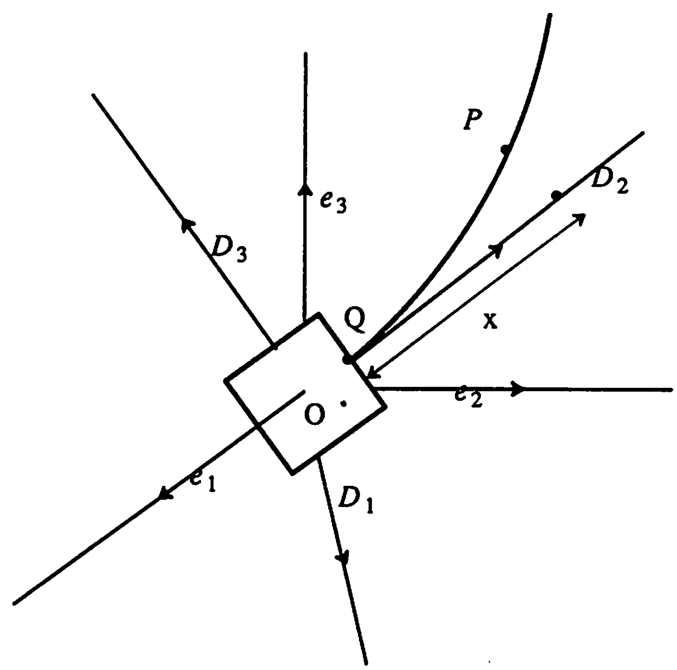

Figure 1 : Rigid body with flexible beam.

In figure 1, (O, e_1, e_2, e_3) denotes a dextral orthonormal *inertial* frame, which will be referred to as **N**, (O, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$) denotes a dextral orthonormal frame fixed in the rigid body, which will be referred as **B**, where O is also the center of mass of the rigid body and $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$ are the principal axes of inertia of the rigid body. The beam is clamped to the rigid body at the point Q at one end along the $\mathbf{D}_2$ axis and is free at the other end. Let L be the length of the beam. We assume that the mass of the rigid body is larger than the mass of the beam, so the center of mass of the rigid body is approximately the center of mass of the whole configuration. So the point O is fixed in the inertial space throughout the motion of the whole configuration and the rigid body may rotate arbitrarily in the inertial space.

The beam is initially straight, along the $\mathbf{D}_2$ axis. Let P be a typical beam element whose distance from Q in the undeformed configuration is x, let u_1 and u_3 be the displacement of P along the $\mathbf{D}_1$ and $\mathbf{D}_3$ axes, respectively. Let $\mathbf{r}(x,t) = \mathbf{OP}$ be the position vector of P. Let the beam be homogeneous with uniform cross-sections.

We define the contact force **n**(x,t) and the contact moment **m**(x,t) at the beam cross-sections as follows. Consider a beam cross-section C_x at x. The effect of the part of the beam which lies on the (x,L] segment of the beam to the materials which lies on the [0,x] segment is equivalent to a force applied to the cross-section C_x , which is called the contact force **n**(x,t), and a moment applied to the cross-section C_x, which is called the contact moment **m**(x,t). For further information, see [Ant.1].

Neglecting gravitation, surface loads and rotatory inertia of the beam cross-sections, we obtain the following equations that describe the motion of the whole configuration : $t \geq 0$

$$\frac{\partial \mathbf{n}}{\partial x} = \lambda \frac{\partial^2 \mathbf{r}}{\partial t^2} \qquad 0 < x < L \qquad t \geq 0 \tag{1.1}$$

$$\frac{\partial \mathbf{m}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x} \times \mathbf{n} = 0 \qquad 0 < x < L \qquad t \geq 0 \tag{1.2}$$

$$I_R \dot{\omega} + \omega \times I_R \omega = \mathbf{r}(0,t) \times \mathbf{n}(0,t) + \mathbf{m}(0,t) + \mathbf{N}_c(t) \tag{1.3}$$

where $\mathbf{n}(x,t)$ and $\mathbf{m}(x,t)$ are the contact force and the contact moment , respectively, λ is the mass per unit length of the beam, which is a constant by assumption, L is the length of the beam, I_R is the inertia operator of the rigid body, which is diagonal, ω is the angular velocity of the rigid body with respect to the inertial frame N and $\mathbf{N}_c(t)$ is the control torque applied to the rigid body.

The equation (1.1) is the balance of forces, the equation (1.2) is the balance of moments at the beam cross sections and the equation (1.3) is the rigid body angular momentum equation. Note that the first two terms in the right hand side of (1.3) represent the torque applied by the beam to the rigid body.

1.2 remark : Let $\mathbf{r} : R \rightarrow R^3$ denote a vector valued function of time, typically $\mathbf{r}(t)$ is the position of a particle. Let $\mathbf{r}^N = (r^N{}_1, r^N{}_2, r^N{}_3)^T$ and $\mathbf{r}^B = (r^B{}_1, r^B{}_2, r^B{}_3)^T$ denote the components of $\mathbf{r}$ in the dextral orthonormal frame **N** given by (O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) and in the dextral orthonormal frame **B** given by (O, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$) , respectively. Let ω denote the angular velocity of the frame **B** with respect to the frame **N**. Then we have the following (see [Kan.1]):

$$\sum_{i=1}^{i=3} \frac{dr_i^N}{dt}\mathbf{e}_i = \sum_{i=1}^{i=3} \frac{dr_i^B}{dt}\mathbf{D}_i + \omega \times \mathbf{r} .$$

If we define $(\frac{d\mathbf{r}}{dt})_N = \sum_{i=1}^{i=3} \frac{dr_i^N}{dt}\mathbf{e}_i$ and $(\frac{d\mathbf{r}}{dt})_B = \sum_{i=1}^{i=3} \frac{dr_i^B}{dt}\mathbf{D}_i$ then we obtain the following equation (see e.g. [Gol.1]) :

$$(\frac{d\mathbf{r}}{dt})_N = (\frac{d\mathbf{r}}{dt})_B + \omega \times \mathbf{r} . \quad \square \tag{1.4}$$

We use the Euler-Bernoulli beam model to give the component form of the contact force **n** and the contact moment **m** in terms of the beam deflections u_1, u_3. For more details, see [Mei.1]. Assuming that the beam is inextensible and neglecting the torsion, we express the contact force **n**, the contact moment **m**, and the position vector **r** in terms of u_1 and u_3 as follows: for $0 \le x \le L$, $t \ge 0$,

$$\mathbf{m} = m_1\mathbf{D}_1 + m_3\mathbf{D}_3 \qquad \mathbf{n} = n_1\mathbf{D}_1 + n_3\mathbf{D}_3 \tag{1.5}$$

$$m_1 = EI_3 u_{3xx} \qquad n_3 = -EI_3 u_{3xxx} \tag{1.6}$$

$$m_3 = -EI_1 u_{1xx} \qquad n_1 = -EI_1 u_{1xxx} \tag{1.7}$$

$$\mathbf{r} = u_1\mathbf{D}_1 + (b + x)\mathbf{D}_2 + u_3\mathbf{D}_3 \tag{1.8}$$

where EI_1 and EI_3 are the flexural rigidity of the beam deflectios along the axes $\mathbf{D}_1$ and $\mathbf{D}_3$, respectively, and b is the distance between the points O and Q.

Since the beam is clamped to the rigid body at the point Q, we have (see figure 1) :

$$u_i(0,t)=u_{ix}(0,t)=0 \;, \quad t\geq 0 \;, \; i=1,3 \tag{1.9}$$

The rest state of the system is by definition :

$$\omega=0 \tag{1.10.1}$$

$$u_1(x)=u_3(x)=0 \quad 0\leq x\leq L \tag{1.10.2}$$

$$u_{1t}(x)=u_{3t}(x)=0 \quad 0\leq x\leq L \tag{1.10.3}$$

We now state our

stabilization problem :

If the system given by the equations (1.1)-(1.9) is perturbed from the rest state defined by (1.10.1)-(1.10.2), find an appropriate control law that drives the system to the rest state.□

1.3 proposed control laws :

We propose two stabilizing control laws. Each law consist of appropriate forces and torques applied to the beam at the free end and a torque applied to the rigid body. We note that these two sets differ in the torque applied to the rigid body.

1.3.1 control law based on cancellation

This control scheme applies a force **n**(L,t) and a torque **m**(L,t) at the free end of the beam and a torque $\mathbf{N}_c$(t) applied to the rigid body. They are specified as follows : we choose $\alpha_i > 0$, $\beta_i > 0$, and a 3x3 positive definite constant matrix K, (which can be chosen diagonal); then for all $t\geq 0$, i = 1, 3, we require the following equations :

$$n_i(L,t)+\alpha_i u_{it}(L,t)=0 \tag{1.11}$$

$$m_i(L,t)+\beta_i u_{ixt}(L,t)=0 \tag{1.12}$$

$$\mathbf{N}_c(t)=-\mathbf{r}(0,t)\times\mathbf{n}(0,t)-\mathbf{m}(0,t)-K\omega(t) \tag{1.13.1}$$

Equation (1.11), {(1.12), resp.} represents a transversal force, { torque, resp. } applied at the free end of the beam in the direction, { around, resp.} the axis $\mathbf{D}_i$ whose magnitude is proportional to and whose sign is opposite to the end point deflection velocity, $u_{it}(L,t)$,{ end-point deflection angular velocity $u_{ixt}(L,t)$, resp. } of the beam along the direction of $\mathbf{D}_i$ axis, for i= 1, 3. Also note that to apply the control laws given by (1.11)-(1.13.1), the end point deflection velocities $u_{it}(L,t)$, the end point deflection angular velocities $u_{ix}(L,t)$, the rigid body angular velocity vector ω(t) and the moment applied by the beam to the rigid body must be measured. This moment consist of the effect of the contact force **n**(0,t) and the contact moment **m**(0,t) at the clamped end. Both can be measured by using strain rosettes and strain gauges, respectively,[Ana.1] .

The control law (1.13.1) cancels the effect of the beam on the rigid body. To see this, substitute (1.13.1) into (1.3), then the equation (1.3) becomes a set of nonlinear *ordinary* differential equations. Then substitute the solution ω(t) of (1.3) into the beam equation (1.1). Now the latter becomes a set of *linear* partial differential equations.

Equation (1.13.1) is reminiscent of a "computed torque" type control law in robotics, [Pau.1]. When substituted in (1.3), (1.13.1) cancels the effect of the beam on the rigid body. This type of control law recently has been applied to the attitude control of the flexible spacecraft, [Ana.1]. □

1.3.2 natural control law

This control sceheme applies the same boundary force **n**(L,t) and the moment **m**(L,t) as specified by the equations (1.11) and (1.12), respectively, but the torque applied to the rigid body is given by :

$$\mathbf{N}_c(t) = -\mathbf{r}(L,t) \times \mathbf{n}(L,t) - \mathbf{m}(L,t) - K\omega(t) \tag{1.13.2}$$

This control scheme is "natural" in the sense that it enables one to choose the total energy of the whole configuration as a Lyapunov function to study the stability of the system.

Unlike the control law (1.13.1), when (1.13.2) is substituted in (1.3), it does not cancel the effect of the beam on the rigid body. As a result of this, the equations (1.1)-(1.9), together with the control laws (1.11),(1.12) and (1.13.2) form a set of nonlinear ordinary and partial differential equations. The control law (1.11),(1.12),(1.13.2) requires that the end-point deflections u_i(L,t), the end-point deflection velocities u_{it}(L,t), the end-point deflection angular velocities u_{ixt}(L,t) and the rigid body angular velocity vector ω(t) be measured. The first two could be measured by optical means and the latter by gyros.

Throughout our analysis, the initial conditions $u_i(x,0)$ and $u_{it}(x,0)$ are assumed to be sufficiently differentiable (i.e C^2 in t and C^4 in x) and compatible with the boundary conditions (1.9), (1.11), (1.12), for i=1, 3.

section 2

stability results for the control law based on cancellation :

After substituting (1.13.1) in (1.3), we obtain the following rigid body equation :

$$I_R \dot{\omega} + \omega \times I_R \omega = -K\ \omega \tag{2.1}$$

2.1 proposition : Consider the equation (2.1). There exist a $c > 0$ and an $\alpha > 0$ such that for all initial conditions $\omega(0) \in R^3$, the solution $\omega(t)$ of (2.1) satisfies

$$<\omega(t), \omega(t)> \leq c\ e^{-\alpha t} <\omega(0), \omega(0)> \qquad \text{for } all\ t \geq 0 \tag{2.2}$$

proof: Consider the following "energy function" for the rigid body :

$$E_R(t) = \frac{1}{2} <\omega(t), I_R \omega(t)> \tag{2.3}$$

$E_R(t)$ is the rotational kinetic energy of the rigid body with respect to the inertial frame N. Also note that since $I_R = \text{diag}(I_1, I_2, I_3)$, we have

$$I_{\min} <\omega, \omega> \leq 2\, E_R \leq I_{\max} <\omega, \omega> \qquad \text{for } all\ \omega \in R^3 \tag{2.4}$$

where $I_{\min} = \min(I_1, I_2, I_3)$ and $I_{\max} = \max(I_1, I_2, I_3)$.

Differentiating (2.3) and using (2.1) we obtain :

$$\begin{aligned} \dot{E}_R(t) &= <\omega, I_R \dot{\omega}> \\ &= -<\omega, \omega \times I_R \omega> - <\omega, K\ \omega> \\ &= -<\omega, K\ \omega> \end{aligned} \tag{2.5}$$

But, since K is positive definite, there exist positive, nonzero constants λ_1 and λ_2, which may be taken as the minimum and the maximum eigenvalues of $\frac{1}{2}(K + K^T)$, respectively , such that the following holds :

$$\lambda_1 < \omega, \omega > \le < \omega, K\ \omega > \le \lambda_2 < \omega, \omega > \tag{2.6}$$

Using (2.4)-(2.6), we obtain (2.2) where $c = \dfrac{\max(I_1, I_2, I_3)}{\min(I_1, I_2, I_3)}$ and $\alpha = \dfrac{2\lambda_1}{\max(I_1, I_2, I_3)}$. □

Next, we obtain the component form of equation (1.1). After applying (1.4) twice, we obtain the following :

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_N = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_B + \dot{\omega} \times \mathbf{r} + 2\,\omega \times \left(\frac{d\mathbf{r}}{dt}\right)_B + \omega \times (\omega \times \mathbf{r}). \tag{2.7}$$

Using (2.7) in (1.1)-(1.10), we obtain the following equations which govern the motion of transverse beam deflections in $\mathbf{D}_1$ and $\mathbf{D}_3$ directions :

$$EI_1\, u_{1xxxx} + \lambda\, u_{1tt} + 2\,\lambda\,\omega_2\, u_{3t} + \lambda\,(\dot{\omega}_2 + \omega_1\,\omega_3)\, u_3$$

$$- \lambda\,(\omega_2^2 + \omega_3^2)\, u_1 - \lambda\,(\dot{\omega}_3 - \omega_1\omega_2)\,(b + x) = 0 \qquad 0 \le x \le L,\ \ t \ge 0 \tag{2.8}$$

$$EI_3\, u_{3xxxx} + \lambda\, u_{3tt} - 2\,\lambda\,\omega_2\, u_{1t} - \lambda\,(\dot{\omega}_2 - \omega_1\,\omega_3)\, u_1$$

$$- \lambda\,(\omega_1^2 + \omega_2^2)\, u_3 + \lambda\,(\dot{\omega}_1 + \omega_2\omega_3)\,(b + x) = 0 \qquad 0 \le x \le L,\ \ t \ge 0 \tag{2.9}$$

Equations (2.8) and (2.9) can be rewritten in the following state space form :

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{EI_1}{\lambda}\dfrac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{EI_3}{\lambda}\dfrac{\partial^4}{\partial x^4} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^2 + \omega_3^2 & 0 & -(\dot{\omega}_2 + \omega_1\omega_3) & -2\omega_2 \\ 0 & 0 & 0 & 0 \\ \dot{\omega}_2 - \omega_1\omega_3 & 2\omega_2 & \omega_1^2 + \omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ (\dot{\omega}_3 - \omega_1\omega_2)\,(b+x) \\ 0 \\ -(\dot{\omega}_1 + \omega_2\omega_3)\,(b+x) \end{bmatrix} \tag{2.10}$$

whose solutions evolve in the following function space H :

$$H = \{ (u_1, u_{1t}, u_3, u_{3t}) \mid u_1 \in \mathbf{H}_0^2,\ u_3 \in \mathbf{H}_0^2, u_{1t} \in \mathbf{L}^2 , u_{3t} \in \mathbf{L}^2 \}. \tag{2.11}$$

where the function spaces $\mathbf{L}^2$, $\mathbf{H}^k$ and $\mathbf{H}^k{}_0$ are as defined below :

$$\mathbf{L}^2 = \{ f : [0,L] \rightarrow R \mid \int_{x=0}^{x=L} f^2 \, dx < \infty \}$$
$$\mathbf{H}^k = \{ f \in \mathbf{L}^2 \mid f^i \in \mathbf{L}^2 , i=1,..,k \}.$$
$$\mathbf{H}^k{}_0 = \{ f \in \mathbf{H}^k \mid f(0) = f^1(0) = 0 \} .$$

In H, we define the following inner product, which is called "energy" inner product

$$< z , \hat{z} >_E := \int_{x=0}^{x=L} (EI_1 u_{1xx} \hat{u}_{1xx} + EI_3 v_{1xx} \hat{v}_{1xx}) \, dx + \int_{x=0}^{x=L} \lambda (u_2 \hat{u}_2 + v_2 \hat{v}_2) \, dx \qquad \text{for } all \ z , \hat{z} \in H. \tag{2.12}$$

Note that, (2.12) induces a norm on H, which is called "energy norm". This norm is equivalent to a standard "Sobolev" type norm which makes H an Hilbert space.(for more details, see [Paz.1] and [Che.2])

To put (2.10) into an abstract equation form, we define the following operators $A : H \rightarrow H$, $B : R^+ x H \rightarrow H$ and function $f : R^+ \rightarrow H$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{EI_1}{\lambda} \frac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{EI_3}{\lambda} \frac{\partial^4}{\partial x^4} & 0 \end{bmatrix} \tag{2.13}$$

$$B(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^2 + \omega_3^2 & 0 & -(\dot{\omega}_2 + \omega_1 \omega_3) & -2\omega_2 \\ 0 & 0 & 0 & 0 \\ \dot{\omega}_2 - \omega_1 \omega_3 & 2\omega_2 & \omega_1^2 + \omega_3^2 & 0 \end{bmatrix} \tag{2.14}$$

$$f(t) = \begin{bmatrix} 0 \\ (\dot{\omega}_3 - \omega_1\omega_2)\,(b+x) \\ 0 \\ -(\dot{\omega}_1 + \omega_2\omega_3)\,(b+x) \end{bmatrix} \tag{2.15}$$

2.2 remark : A is an unbounded linear operator on H. $B(t)$ is bounded for all $t \in R^+$. Since $\omega(t)$ and $\dot{\omega}(t)$ are exponentially decaying functions of t, (see proposition 2.1 and equation (2.1)), so is $||B(t)||$, where the norm used here is the norm induced by the energy inner product given by (2.12). □

Using the above definitions, equation (2.10) can be put into the following abstract form :

$$\frac{dz}{dt} = A\ z + B(t)\ z + f(t) \qquad z(0) = z_0 \in H \tag{2.16}$$

where z=[u_1, u_{1t}, u_3, u_{3t}]^T. The domain $D(A)$ of the operator A is defined as follows :

$$\begin{aligned} D(A) = \{\ (u_1, u_{1t}, u_3, u_{3t}) : u_1 \in \mathrm{H}_0^4, u_{1t} \in \mathrm{H}_0^4, u_3 \in \mathrm{H}_0^2, u_{3t} \in \mathrm{H}_0^2, \\ -EI_1\, u_{1xxx}(L) + \alpha_1\, u_{1t}(L) = 0 \\ EI_1\, u_{1xx}(L) + \beta_1\, u_{1xt}(L) = 0 \\ -EI_3\, u_{3xxx}(L) + \alpha_2\, u_{3t}(L) = 0 \\ EI_3\, u_{3xx}(L) + \beta_2\, u_{3xt}(L) = 0\ \}. \end{aligned} \tag{2.17}$$

It is easy to show that $D(A)$ is dense in H, [Che.2].

Next, we state the existence and uniqueness theorem of the solutions of (2.16). [Paz.1].

2.3 fact : Consider the equation (2.16) with A, B, f defined in (2.13)-(2.15). Then :

i) The operator A generates an exponentially decaying C_0 semigroup $T(t)$ in H. That is, there exist a M > 0 and a $\delta > 0$ such that

$$||T(t)|| \le M\ e^{-\delta t} \qquad \text{for } all\ t \ge 0\ ; \tag{2.18}$$

ii) for all $z_0 \in D(A)$, (2.16) has unique classical solution , defined for all $t \ge 0$;

iii) in terms of $T(t)$, the solution z(t) of (2.16) may be written as :

$$z(t) = T(t)z_0 + \int_{x=0}^{x=L} T(t-s)\,B(s)\,z(s)\,ds + \int_{x=0}^{x=L} T(t-s)\,f(s)\,ds. \tag{2.19}$$

proof :

i) Due to the block diagonal form of A, it is an easy extention of theorem 3.1 of Chen, [Che.2].

ii) Since $B(t)$ is globally lipschitz on H and $||B(t)||$ is exponentially decaying due to the proposition 2.1, (also see remark 2.2), it follows that $A+B(t)$ defines a unique, globally defined semigroup on H, (e.g. see [Mar.1], [Paz.1]).

Since $f \in L^1[R, H]$ and is a C^∞ function of t, (see equation (2.15)), by standard theorems on nonhomogeneous partial differential equations (e.g. see pp. 105-110, [Paz.1]) it follows that (2.16) has unique solution defined for all $t \geq 0$.

iii) That the solution may be given as (2.19) can be verified by substitution, using $\frac{dT}{dt} = A\,T$. □

Next, we prove the exponential decay of the solutions of (2.16).

2.4 theorem : Consider the equation (2.16), where the operators A, $B(t)$ and the function $f(t)$ are defined in (2.13),(2.14) and (2.15) respectively. Then for all $z_0 \in D(A)$, the solution $z(t)$ of (2.16) decays exponentially to 0.

proof : By taking norms in (2.19) and using (2.18), we get :

$$\begin{aligned} ||z(t)|| \leq M\,e^{-\delta t}\,||z_0|| &+ \int_{s=0}^{s=t} M\,e^{-\delta(t-s)}\,||B(s)||\;||z(s)||\,ds \\ &+ \int_{s=0}^{s=t} M\,e^{-\delta(t-s)}\,||f(s)||\,ds \end{aligned} \tag{2.20}$$

But since $\omega(t)$ and $\dot{\omega}(t)$ are decaying exponentially, it follows that there exist positive constants $c_1 > 0, c_2 > 0, \delta_1 > 0, \delta_2 > 0$, such that for all $t \geq 0$

$$||B(t)|| \le c_1 e^{-\delta_1 t} \tag{2.21}$$

$$||f(t)|| \le c_2 e^{-\delta_2 t} \tag{2.22}$$

Using (2.21),(2.22) in (2.20), evaluating the last integral, and multiplying each side of (2.20) by $e^{\delta t}$, we get

$$||z(t)e^{\delta t}|| \le M\,||z_0|| + \frac{M\,c_2}{\delta-\delta_2}\,(e^{(\delta-\delta_2)t} - 1)$$

$$+ \int_{s=0}^{s=t} M\,c_1\,e^{-\delta_1 s}\,||z(s)e^{\delta s}||\;ds \tag{2.23}$$

Now applying a general form of Bellmann- Gronwall lemma ,(e.g. see [Des.1]), and using the following simple estimate

$$\int_{s=t_1}^{s=t} e^{-\delta_1 s}\,ds \le \int_{s=0}^{s=\infty} e^{-\delta_1 s}\,ds \le \frac{1}{\delta_1} \tag{2.24}$$

we obtain the following :

$$||z(t)e^{\delta t}|| \le M\,||z_0|| + \frac{M\,c_2}{\delta-\delta_2}\,(e^{(\delta-\delta_2)t} - 1)$$

$$+ \int_{s=0}^{s=t} M\,c_1\,e^{\frac{M\,c_1}{\delta_1}}\;[M\,||z_0|| + \frac{M\,c_2}{\delta-\delta_2}\,(e^{(\delta-\delta_2)s} - 1)]\;e^{-\delta_1 s}\;ds$$

$$\le M\,||z_0|| + \frac{M\,c_2}{\delta-\delta_2}\,(e^{(\delta-\delta_2)t} - 1) + \frac{M^2 c_1}{\delta_1}\,e^{\frac{M\,c_1}{\delta_1}}\;(||z_0|| - \frac{c_2}{\delta-\delta_2})\,(1 - e^{-\delta_1 t})$$

$$- \frac{M^2\,c_1\,c_2}{(\delta-\delta_2)(\delta-\delta_1-\delta_2)}\,e^{\frac{M\,c_1}{\delta_1}}\;(\,1 - e^{(\delta-\delta_1-\delta_2)t}). \tag{2.25}$$

Multiplying each side with $e^{-\delta t}$, we obtain the desired result. $\square$

section 3

stability results for the natural control scheme (1.3.2)

To prove the stability of the system given by equations (1.1)-(1.12) and (1.13.2), we first define the energy of the system as follows :

$$E(t) = \frac{1}{2} < \omega \, , I_R \, \omega > + \frac{1}{2} \int_{x=0}^{x=L} \lambda < \mathbf{r}_t , \mathbf{r}_t > dx$$

$$+ \frac{1}{2} \int_{x=0}^{x=L} (EI_1 u_{1xx}{}^2 + EI_3 u_{3xx}{}^2) \, dx \tag{3.1}$$

where < , > denotes the standard inner product in R^3 ; the first term in (3.1) is the rotational kinetic energy of the rigid body, the second term is the kinetic energy of the beam, both with respect to the inertial frame N, and the last term is the potential energy of the beam .

3.1 proposition : Consider the system given by the equations (1.1)-(1.12) and (1.13.2). Then the energy $E(t)$ defined by (3.1) is a nonincreasing function of t, along the solutions of (1.1)-(1.12) and (1.13.2).

proof : By differentiating $E(t)$ with respect to t, using (1.1) and (1,4), we get

$$\frac{d}{dt} E(t) = < \omega \, , I_R \, \dot{\omega} > + \int_{x=0}^{x=L} \lambda < \mathbf{r}_t \, , \mathbf{r}_{tt} > dx$$

$$+ \int_{x=0}^{x=L} (EI_1 u_{1xx} \, u_{1xxt} + EI_3 u_{3xx} \, u_{3xxt}) \, dx$$

$$= < \omega \, , I_R \, \dot{\omega} > + \int_{x=0}^{x=L} < \mathbf{r}_t , \mathbf{n}_x > dx + \int_{x=0}^{x=L} (EI_1 u_{1xx} \, u_{1xxt} + EI_3 u_{3xx} u_{3xxt}) \, dx$$

$$= < \omega \, , I_R \, \dot{\omega} > - EI_1 \int_{x=0}^{x=L} u_{1t} \, u_{1xxxx} \, dx - EI_3 \int_{x=0}^{x=L} u_{3t} \, u_{3xxxx} \, dx$$

$$+EI_1 \int_{x=0}^{x=L} u_{1xx}\, u_{1xxt}\, dx + EI_3 \int_{x=0}^{x=L} u_{3xx}\, u_{3xxt}\, dx \tag{3.2}$$

Using integration by parts we obtain the following equation , for i=1, 3 :

$$EI_i \int_{x=0}^{x=L} u_{it}\, u_{ixxxx}\, dx = EI_i u_{ixxx}(L,t).u_{it}(L,T)$$

$$-EI_i u_{ixx}(L,t).u_{ixt}(L,t) + EI_i \int_{x=0}^{x=L} u_{ixx}\, u_{ixxt}\, dx \quad . \tag{3.3}$$

Using (3.3) and boundary conditions (1.11) and (1.12) in (3.2), we get

$$\dot{E}(t) = -<\omega, K\omega> - \alpha_1 u_{1t}{}^2(L,t) - \alpha_3 u_{3t}{}^2(L,t)$$

$$-\beta_1 u_{1xt}{}^2(L,t) - \beta_3 u_{3xt}{}^2(L,t) \le 0 \tag{3.4}$$

Since the rate of change of the energy is nonpositive, it follows that the energy is a nonincreasing function of time. for all $z \in H$. □

3.2 remark : If one sets $\alpha_i = \beta_i = 0$, for i=1,3, and $K = 0$,(i.e no control applied to the system), one gets $\dot{E}$(t) = 0: as expected, the total energy (given by the equation (3.1)) is conserved. □

3.3 remark : We need an estimate, which states that if the energy given by (3.1) stays bounded, then so does the beam deflections $u_i(x,t)$ and their derivatives $u_{ix}(x,t)$,(hence also so does $\mathbf{r}(x,t)$), for all x∈ [0,L], for i=1,3. Using the boundary conditions and the fundamental theorem of calculus, for i=1,3 we get for all $0 \le x \le L$, for all $t \ge 0$:

$$u_i(x,t) = \int_{s=0}^{s=x} u_{is}(s,t)\, ds \tag{3.5}$$

therefore, using Jensen's inequality, [Mit.1] , we get :

$$(u_i(x,t))^2 \le L \int_{s=0}^{s=L} u_{is}{}^2(s,t)\, ds \tag{3.6}$$

By using the same arguments, we get, for all $x \in [0,L]$

$$(u_{ix}(x,t))^2 \le L \int_{s=0}^{s=L} u_{iss}{}^2(s,t)\,ds \tag{3.7}$$

hence, combining (3.5) and (3.6), we get :

$$(u_i(x,t))^2 \le L \int_{s=0}^{s=L} u_{is}{}^2(s,t)\,ds \le L^2 \int_{s=0}^{s=L} u_{iss}{}^2(s,t)\,ds \quad . \square \tag{3.8}$$

Next, we will show that the rate of decay of the energy is at least $\frac{1}{t}$ for large t.

3.4 theorem : Consider the system described by the equations (1.1)-(1.12)and (1.13.2). Then there exists a $T \ge 0$ such that the energy given by (3.1) is bounded above by $O\,(\frac{1}{t})$ for all $t \ge T$.

proof : To show that $E(t)$ decreases at least as $O\,(\frac{1}{t})$, we first define the following function $V(t)$: for any $0 < \varepsilon < 1$,

$$V(t) = 2(1-\varepsilon)\,t\,E(t) + 2\int_{x=0}^{x=L} \lambda x < \mathbf{r}_t , \mathbf{r}_x > dx \tag{3.9}$$

Next, we need the following estimate on V(t). Note that :

$$-\int_{x=0}^{x=L} \lambda x < \mathbf{r}_t , \mathbf{r}_x > dx \le \lambda L \int_{x=0}^{x=L} (< \mathbf{r}_x , \mathbf{r}_x > + < \mathbf{r}_t , \mathbf{r}_t >)\,dx \tag{3.10}$$

Now by using Remark 3.3, we can find a $M_1 > 0$ and a $M_2 > 0$ such that

$$-2\int_{x=0}^{x=L} \lambda x < \mathbf{r}_t , \mathbf{r}_x > dx \le M_1 E(t) + M_2 \tag{3.11}$$

Therefore, using the last inequality in (3.9), we get

$$(2\,(1-\varepsilon)\,t - M_1)\,E(t) - M_2 \le V(t) \qquad t \ge 0 \tag{3.12}$$

Now, differentiating (3.9) and using equations (1.1)-(1.12), we get :

$$\dot{V}(t) = 2\,(1-\varepsilon)\,E(t) + 2\,(1-\varepsilon)\,t\,\dot{E}(t)$$

$$+\,2\int_{x=0}^{x=L} \lambda\, x < \mathbf{r}_{tt}\,,\mathbf{r}_x > dx + 2\int_{x=0}^{x=L} \lambda\, x < \mathbf{r}_t\,,\mathbf{r}_{tx} > dx \tag{3.13}$$

Using integration by parts, the third and fourth integrals in (3.13) can be evaluated as follows :

$$2\int_{x=0}^{x=L} \lambda\, x < \mathbf{r}_t\,,\mathbf{r}_{tx} > dx = \lambda L < \mathbf{r}_t(L,t), \mathbf{r}_t(L,t) > - \int_{x=0}^{x=L} \lambda < \mathbf{r}_t\,,\mathbf{r}_t > dx \tag{3.14}$$

$$\int_{x=0}^{x=L} \lambda\, x < \mathbf{r}_{tt}\,,\mathbf{r}_x > dx = \int_{x=0}^{x=L} x < \mathbf{n}_x\,,\mathbf{r}_x > dx$$

$$= -\int_{x=0}^{x=L} x\, u_{1x}\, EI_1 u_{1xxxx}\, dx - \int_{x=0}^{x=L} x\, u_{3x}\, EI_3 u_{3xxxx}\, dx \tag{3.15}$$

To evaluate the last to integral, we need the following :

$$\int_{x=0}^{x=L} x\, u_x\, u_{xxxx}\, dx = L\, u_x(L,t)\, u_{xxx}(L,t) - u_x(L,t)\, u_{xx}(L,t)$$

$$-\frac{L}{2} u_{xx}^{\,2}(L,t) + \frac{3}{2}\int_{x=0}^{x=L} u_{xx}^{\,2}\, dx \tag{3.16}$$

After using (3.14)-(3.16) in (3.13), we get

$$\dot{V}(t) = (1-\varepsilon) < \omega, I_R\, \omega > + (1-\varepsilon)\int_{x=0}^{x=L} \lambda < \mathbf{r}_t, \mathbf{r}_t > dx$$

$$+\,(1-\varepsilon)\int_{x=0}^{x=L} (\,EI_1 u_{1xx}^{\,2} + EI_3 u_{3xx}^{\,2}\,)\, dx - 2\,(1-\varepsilon) < \omega, K\,\omega >$$

$$-\,2\,(1-\varepsilon)\,\alpha_1\, u_{1t}^{\,2}(L,t) - 2\,(1-\varepsilon)\,\alpha_3\, u_{3t}^{\,2}(L,t)$$

$$-\,2\,(1-\varepsilon)\,\beta_1\, u_{1xt}^{\,2}(L,t) - 2\,(1-\varepsilon)\,\beta_3\, u_{3xt}^{\,2}(L,t)$$

$$+\lambda L < \mathbf{r}_t(L,t), \mathbf{r}_t(L,t) > - \int_{x=0}^{x=L} \lambda < \mathbf{r}_t, \mathbf{r}_t > dx$$

$$-2\,L\; u_{1x}(L,t)\;\alpha_1 u_{1t}(L,t) - 2\,u_{1x}(L,t)\;\beta_1 u_{1xt}(L,t)$$

$$+L\;\frac{\beta_1^{\,2}}{EI_1} u_{1xt}^{\;2}(L,t) - 3\,EI_1 \int_{x=0}^{x=L} u_{1xx}^{\;2}\,dx$$

$$-2\,L\; u_{3x}(L,t)\;\alpha_3 u_{3t}(L,t) - 2\,u_{3x}(L,t)\;\beta_3 u_{3xt}(L,t)$$

$$+L\;\frac{\beta_3^{\,2}}{EI_3} u_{3xt}^{\;2}(L,t) - 3\,EI_3 \int_{x=0}^{x=L} u_{3xx}^{\;2}\,dx \tag{3.17}$$

To estimate some of the terms in (3.17), we need the following inequalities :

$$(a+b)^2 \le 2(a^2+b^2) \qquad a\in R,\; b\in R. \tag{3.18}$$

$$a\,b \le \delta^2 a^2 + \frac{1}{\delta^2} b^2 \qquad \delta\in R, \delta\neq 0\;, a\in R,\; b\in R. \tag{3.19}$$

Finally using Remark 3.1 and Remark 3.3, we get the following estimate on the end point velocities of the beam in the inertial frame.

$$< \mathbf{r}_t(L,t), \mathbf{r}_t(L,t) > \le k_1\,(u_{1t}^{\;2}(L,t) + u_{3t}^{\;2}(L,t)) + k_2 < \omega, \omega > \tag{3.20}$$

for some $k_1 > 0$ and $k_2 > 0$.

Using these estimates in (3.17), we obtain :

$$\dot{V}(t) \le -(1-\varepsilon)\,(\,2\,t < \omega, K\;\omega > - < \omega, I_R\;\omega > - \lambda L k_2 < \omega, \omega >\,)$$

$$-\varepsilon \int_{x=0}^{x=L} \lambda < \mathbf{r}_t, \mathbf{r}_t > dx - (\,\varepsilon + 2)\int_{x=0}^{x=L} (\,EI_1 u_{1xx}^{\;2} + EI_3 u_{3xx}^{\;2}\,)\,dx$$

$$-2\,(1-\varepsilon)\,\alpha_1\,t\;u_{1t}(L,t)^2 - 2\,(1-\varepsilon)\,\alpha_3\,t\;u_{3t}(L,t)^2$$

$$-2\,(1-\varepsilon)\,\beta_1\, t\, u_{1xt}(L,t)^2 - 2\,(1-\varepsilon)\,\beta_3\, t\, u_{3xt}(L,t)^2$$

$$+\lambda L k_1 (u_{1t}{}^2(L,t) + u_{3t}{}^2(L,t)) + 2\,L\,\alpha_1\,(\delta_1{}^2\, u_{1x}{}^2(L,t) + \frac{1}{\delta_1{}^2}\, u_{1t}{}^2(L,t)\,)$$

$$+2\,L\;\alpha_3\,(\delta_2{}^2\, u_{3x}{}^2(L,t) + \frac{1}{\delta_2{}^2}\, u_{3t}{}^2(L,t)) + \delta_3{}^2\, u_{1x}{}^2(L,t)$$

$$+2\,\beta_1\,(\delta_3{}^2\, u_{1x}{}^2(L,t)\frac{1}{\delta_3{}^2}\, u_{1xt}{}^2(L,t)\,)$$

$$+2\,\beta_3\,(\delta_4{}^2\, u_{3x}{}^2(L,t)\frac{1}{\delta_4{}^2}\, u_{3xt}{}^2(L,t)\,)$$

$$+\frac{L}{EI_1}\,\beta_1{}^2 u_{1xt}{}^2(L,t) + \frac{L}{EI_3}\,\beta_3{}^2 u_{3xt}{}^2(L,t) \tag{3.21}$$

where $\delta_i \in R$ are any nonzero real numbers, for i=1,2,3,4. Now, collecting likewise terms, we rearrange (3.21) as follows :

$$\dot{V}(t) \le -(1-\varepsilon)\,(\,2\,t < \omega, K\,\omega - < \omega, I_R\,\omega > - \lambda L k_2 < \omega, \omega >\,)$$

$$-\varepsilon \int_{x=0}^{x=L} \lambda < \mathbf{r}_t, \mathbf{r}_t > dx - (2\,(1-\varepsilon)\,\alpha_1\, t - \lambda L k_1 - \frac{2\,L\;\alpha_1}{\delta_1{}^2}) u_{1t}{}^2(L,t)$$

$$-(2\,(1-\varepsilon)\,\alpha_3\, t - \lambda L k_1 - \frac{2\,L\;\alpha_3}{\delta_2{}^2}\,) u_{3t}{}^2(L,t)$$

$$-(2\,(1-\varepsilon)\,\beta_1\, t - \frac{2\,\beta_1}{\delta_3{}^2} - L\;\frac{\beta_1^2}{EI_1}\,)\, u_{1xt}{}^2(L,t) -$$

$$-(2\,(1-\varepsilon)\,\beta_3\, t - 2\;\frac{\beta_3}{\delta_4^2} - L\;\frac{\beta_3^2}{EI_3}\,)\, u_{3xt}{}^2(L,t)$$

$$-(\,\varepsilon + 2)\int_{x=0}^{x=L} (\,EI_1 u_{1xx}{}^2 + EI_3 u_{3xx}{}^2\,)\, dx$$

$$+ (2L\alpha_1\delta_1^2 + 2\beta_1\delta_3^2)\, u_{1x}^2(L,t) + (2L\ \alpha_3\delta_2^2 + 2\beta_3\delta_4^2)\, u_{3x}^2(L,t)\,) \tag{3.22}$$

By Remark 3.3 (e.g. using (3.7)) and choosing δ_i sufficiently small, i=1,2,3,4 , the sum of the last two lines in (3.22) can be made negative. Then, we conclude that after some $T \in R$,

$$\dot{V}(t) \le 0 \qquad t \ge T \tag{3.23}$$

hence

$$V(t) \le V(T) \qquad t \ge T \tag{3.24}$$

Using (3.12) and (3.24), we get the following estimate, which proves the theorem 3.4 :

$$E(t) \le \frac{V(T) + M_2}{2(1-\varepsilon)\,t - M_1} \qquad t \ge T \ \square \tag{3.25}$$

For the sake of brevity, the existence, the uniqueness and the exponential decay of the solutions of the equations given by (1.1)-(1.12) and (1.13.2) are presented in the appendix.

appendix

In this appendix, first we give an existence and uniqueness theorem, A.1, to the linear part of the equations (1.1)-(1.12) and (1.13.2) (i.e. the "natural" control scheme). Then including the nonlinear terms, in the theorem A.2 we prove the exponential decay of the solutions of the same equations.

For simplicity, we'll take the positive definite matrix $K = diag\,(k_1, k_2, k_3)$. Then equations (1.1)-(1.12) and (1.13.2) can be written as:

$$EI_1\, u_{1xxxx} + \lambda\, u_{1tt} + 2\,\lambda\, \omega_2\, u_{3t} + \lambda\,(\dot{\omega}_2 + \omega_1\, \omega_3)\, u_3$$

$$- \lambda\,(\omega_2^2 + \omega_3^2)\, u_1 - \lambda\,(\dot{\omega}_3 - \omega_1\omega_2)\,(b + x) = 0 \qquad 0 \le x \le L,\ \ t \ge 0 \qquad \text{(A.1)}$$

$$EI_3\, u_{3xxxx} + \lambda\, u_{3tt} - 2\,\lambda\, \omega_2\, u_{1t} - \lambda\,(\dot{\omega}_2 - \omega_1\, \omega_3)\, u_1$$

$$- \lambda\,(\omega_1^2 + \omega_2^2)\, u_3 + \lambda\,(\dot{\omega}_1 + \omega_2\omega_3)\,(b + x) = 0 \qquad 0 \le x \le L,\ \ t \ge 0 \qquad \text{(A.2)}$$

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + k_1\omega_1 = EI_3 \int_{x=0}^{x=L} (b + x)u_{3xxxx}\, dx \qquad \text{(A.3)}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 + k_2\omega_2 = EI_3 \int_{x=0}^{x=L} u_1 u_{3xxxx}\, dx - EI_1 \int_{x=0}^{x=L} u_3 u_{1xxxx}\, dx \qquad \text{(A.4)}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + k_3\omega_3 = EI_1 \int_{x=0}^{x=L} -(b + x)u_{1xxxx}\, dx \qquad \text{(A.5)}$$

together with the boundary conditions (1.9),(1.11) and (1.12).

Let the function space H be the same as defined in (2.11). Define a new function space $\hat{H}$ = H xR^3 . Then, separating the linear and nonlinear parts, the equations (A.1)-(A.5) can be put into the following matrix form :

$$\frac{dz}{dt} = \hat{A}\ z + T_l(z) + g(z)\ . \qquad \text{(A.6)}$$

where $z = [u_1, u_{1t}, u_3, u_{3t}, \omega_1, \omega_2, \omega_3]^T$.

$\hat{A}:\hat{H} \to \hat{H}$ is a linear operator whose matrix form is specified by the following :

$$\hat{A} = \{m_{ij} : i = 1,..,7\ ,\ j = 1,..,7\ \} \qquad \text{(A.7)}$$

where all m_{ij} are zero except :

$$m_{12} = m_{34} = 1$$

$$m_{21} = -\frac{EI_1}{\lambda}\frac{\partial^4}{\partial x^4} - \frac{EI_1}{I_3}(b+x)\int_{x=0}^{x=L}(b+x)\frac{\partial^4}{\partial x^4}\,dx$$

$$m_{27} = -\frac{k_3}{I_3}(b+x)$$

$$m_{43} = -\frac{EI_3}{\lambda}\frac{\partial^4}{\partial x^4} - \frac{EI_3}{I_1}(b+x)\int_{x=0}^{x=L}(b+x)\frac{\partial^4}{\partial x^4}\,dx$$

$$m_{45} = \frac{k_1}{I_1}(b+x)$$

$$m_{53} = \frac{EI_3}{I_1}\int_{x=0}^{x=L}(b+x)\frac{\partial^4}{\partial x^4}\,dx$$

$$m_{55} = -\frac{k_1}{I_1}$$

$$m_{66} = -\frac{k_2}{I_2}$$

$$m_{71} = -\frac{EI_1}{I_3}\int_{x=0}^{x=L}(b+x)\frac{\partial^4}{\partial x^4}\,dx$$

$$m_{77} = -\frac{k_3}{I_3}$$

$T_l : \hat{H} \to \hat{H}$ is a nonlinear integral operator defined as :

$$T_l(z) = \begin{bmatrix} 0 \\ u_3 \int_{x=0}^{x=L} \left(-\frac{EI_1}{I_2} u_3 u_{1xxxx} + \frac{EI_3}{I_2} u_1 u_{3xxxx} \right) dx \\ 0 \\ u_1 \int_{x=0}^{x=L} \left(-\frac{EI_3}{I_2} u_1 u_{3xxxx} + \frac{EI_1}{I_2} u_3 u_{1xxxx} \right) dx \\ 0 \\ \int_{x=0}^{x=L} \left(-\frac{EI_3}{I_2} u_1 u_{3xxxx} + \frac{EI_1}{I_2} u_3 u_{1xxxx} \right) dx \\ 0 \end{bmatrix} \qquad \text{(A.8)}$$

$g : \hat{H} \rightarrow \hat{H}$ is a nonlinear operator defined as :

$$g(z) = [g_1(z),...,g_7(z)]^T \tag{A.9}$$

where all $g_i(z)$ are defined as follows :

$$g_1(z) = g_3(z) = 0$$

$$g_2(z) = \frac{I_1 - I_3}{I_2}\omega_1\omega_3 u_3 + \frac{I_1 - I_2}{I_3}\omega_1\omega_2 (b+x)$$

$$+ \frac{k_2}{I_2}\omega_2 u_3 - 2\,\omega_2\, u_{3t} + (\omega_2^2 + \omega_3^2)u_1 - \omega_1\omega_3 u_3 - \omega_1\omega_2(b+x)$$

$$g_4(z) = + \frac{I_3 - I_1}{I_2}\omega_1\omega_3 u_1 - \frac{I_2 - I_3}{I_1}\omega_2\omega_3 (b+x)$$

$$- \frac{k_2}{I_2}\omega_2 u_1 + 2\,\omega_2\, u_{1t} + (\omega_1^2 + \omega_2^2)u_3 - \omega_1\omega_3 u_1 - \omega_2\omega_3(b+x)$$

$$g_5(z) = \frac{I_2 - I_3}{I_1}\omega_2\omega_3$$

$$g_6(z) = \frac{I_3 - I_1}{I_2}\omega_1\omega_3$$

$$g_7(z) = \frac{I_1 - I_2}{I_3}\omega_1\omega_2$$

Note that $\hat{A} : \hat{H} \rightarrow \hat{H}$ is an unbounded linear operator and its domain $D(\hat{A})$ is defined as $D(\hat{A}) = D(A) \times R^3$ where $D(A)$ is defined in (2.17) and is dense in $\hat{H}$, since $D(A)$ is dense in H.

In $\hat{H}$ we define the following "energy" inner product:

$$< z\ , \hat{z} >_1 = I_1\omega_1\hat{\omega}_1 + I_2\omega_2\hat{\omega}_2 + I_3\omega_3\hat{\omega}_3$$

$$+ \int_{x=0}^{x=L} \lambda[u_{1t} - \omega_3(b+x)]\,[\hat{u}_{1t} - \hat{\omega}_3(b+x)]\,dx$$

$$+ \int_{x=0}^{x=L} \lambda[u_{3t} + \omega_1(b+x)]\,[\hat{u}_{3t} + \hat{\omega}_1(b+x)]\,dx$$

$$+ \int_{x=0}^{x=L} (EI_1 u_{1xx}\hat{u}_{1xx} + EI_3 u_{3xx}\hat{u}_{3xx})\,dx$$

This inner product induces a norm on Ĥ, which is given below :

$$\| z \|^2 = 2\hat{E}(t) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \tag{A.10}$$

$$+ \int_{x=0}^{x=L} \lambda([u_{1t} - \omega_3(b+x)]^2 + [u_{3t} + \omega_1(b+x)]^2)\, dx$$

$$+ \int_{x=0}^{x=L} (EI_1 u_{1xx}^2 + EI_3 u_{3xx}^2)\, dx$$

Note that the usual "Sobolev" type norm which makes $\hat{H}$ a Banach space is given by :

$$\| z \|_1^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \int_{x=0}^{x=L} (u_1^2 + u_{1x}^2 + u_{1xx}^2)\, dx \tag{A.11}$$

$$+ \int_{x=0}^{x=L} (u_3^2 + u_{3x}^2 + u_{3xx}^2)\, dx + \int_{x=0}^{x=L} (u_{1t}^2 + u_{3t}^2)\, dx$$

But, by remark 1.3 and inequalities (3.18)-(3.19) it can be shown that the norms given by (A.11) and (A.10) are equivalent to each other.

A.1 theorem : Consider the linear operator $\hat{A} : \hat{H} \to \hat{H}$ given by (A.7). Then :

i) $\hat{A}$ generates a C_0 semigroup $\hat{T}(t)$;

ii) there exist positive constants $M > 0$ and $\delta > 0$ such that the following holds :

$$\| \hat{T}(t) \| \le M\, e^{-\delta t} \qquad t \ge 0 . \tag{A.12}$$

proof :

i) We will use the Lumer-Phillips theorem to prove (i),(see p.14, [Paz.1]). So we have to show that $\hat{A}$ is dissipative and the operator $(\lambda I - \hat{A}) : \hat{H} \to \hat{H}$ is onto for some $\lambda > 0$.

To prove that $\hat{A}$ is dissipative, consider the following equation :

$$\frac{dz}{dt} = \hat{A}z \qquad z(0) \in D(\hat{A}). \tag{A.13}$$

Then ,differentiating (A.10) and using (A.13) and (A.7), we get the following :

$$\frac{d\hat{E}}{dt} = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 + \int_{x=0}^{x=L} \lambda[u_{1t} - \omega_3(b+x)]\,[u_{1tt} - \dot{\omega}_3(b+x)]\, dx$$

$$+ \int_{x=0}^{x=L} \lambda[u_{3t} + \omega_1(b+x)]\,[u_{3tt} + \dot{\omega}_1(b+x)]\, dx$$

$$+ \int_{x=0}^{x=L} (EI_1 u_{1xx} u_{1xxt} + EI_3 u_{3xx} u_{3xxt}) \, dx$$

$$= -k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2 - \alpha_1 u_{1t}^2(L,t)$$

$$- \alpha_2 u_{3t}^2(L,t) - \beta_1 u_{1xt}^2(L,t) - \beta_2 u_{3xt}^2(L,t) \quad \leq 0 \qquad \text{(A.24)}$$

This proves that $\hat{A}$ is dissipative.

To prove that the linear operator $(\lambda I - \hat{A}):\hat{H} \to \hat{H}$ is onto for some $\lambda > 0$, we decompose the operator $\hat{A}$ as follows :

$$\hat{A} = A_1 + T_D \qquad \text{(A.15)}$$

where A_1: $\hat{H} \to \hat{H}$ is defined as :

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EI_1}{\lambda}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{EI_3}{\lambda}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_1}{I_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{k_2}{I_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_3}{I_3} \end{bmatrix} \qquad \text{(A.16)}$$

and the operator $T_D : \hat{H} \to \hat{H}$ is defined as

$$T_D = \hat{A} - A_1 \qquad \text{(A.17)}$$

We first note the following remarks :

1) $A_1:\hat{H} \to \hat{H}$ is an linear unbounded operator . Its domain $D(A_1)$ is equal to $D(\hat{A})$. By using theorem 3.1 of Chen, [Che.2], it can be shown that A_1 generates an C_0 contraction semigroup. Hence, $(\lambda I - A_1):\hat{H} \to \hat{H}$ is an invertible operator for all $\lambda > 0$. In fact the range of $(\lambda I - A_1)^{-1}$ is

equal to $D(A_1)$ and by Hille-Yosida theorem, (see e.g. p.8, [Paz.1]), we have :

$$\|(\lambda I - A_1)^{-1}\| \le \frac{1}{\lambda} \qquad \lambda > 0,\ \lambda \in R.$$

2) $T_D : \hat{H} \to \hat{H}$ is a degenerate linear operator relative to the A_1.(see p. 245, [Kat.1]).By definition, the range space of T_D is finite dimensional and there exist positive constants a and b such that :

$$\|T_D z\| \le a\|z\| + b\|A_1 z\| \qquad \text{for all } z \in D(A_1). \tag{A.18}$$

That the operator T_D has a finite dimensional range follows from (A.17),(A.7) and (A.16).

By using (A.17) and (A.11), it can be shown that (A.18) holds for some positive a and b.

From the remarks 1 and 2 above it follows that $T_D(\lambda I - A_1)^{-1} : \hat{H} \to \hat{H}$ is a bounded linear operator with finite dimensional range ; hence $\|T_D(\lambda I - A_1)^{-1}\| \le M$ for some M > 0 and $T_D(\lambda I - A_1)^{-1}$ is a compact operator, (see p.245, [Kat.1]).

Next we need the following fact :

Fact : for all $\lambda > 0$, 1 is not an eigenvalue of the compact operator $T_D(\lambda I - A_1)^{-1}$.

Proof : Suppose not. Then there exists a $\lambda > 0$ and a $y \in \hat{H}$, $y \neq 0$ such that the following holds :

$$y = T_D(\lambda I - A_1)^{-1} y. \tag{A.19}$$

Define $x \in D(A_1)$ as

$$x = (\lambda I - A_1)^{-1} y.$$

Then (A.19) implies that the following equation also holds :

$$(\lambda I - A_1 - T_D)\, x = 0.$$

But since $\hat{A} = A_1 + T_D$ is dissipative and $\lambda > 0$, it follows that x = 0, which implies y = 0, which is a contradiction. □

From the above fact it follows that the operator $I - T_D(\lambda I - A_1)^{-1}$ is invertible for all $\lambda > 0$. Hence we conclude that $(\lambda I - A_1 - T_D) : \hat{H} \to \hat{H}$ is invertible for all $\lambda > 0$ and its inverse is given by :

$$(\lambda I - A_1 - T_D)^{-1} = (\lambda I - A_1)^{-1}(I - T_D(\lambda I - A_1)^{-1})^{-1}$$

This shows that $(\lambda I - A_1 - T_D) : \hat{H} \to \hat{H}$ is onto for all $\lambda > 0$. Then , the assertion (i) follows from the Lumer-Phillips theorem, [Paz.1].

ii) To prove that the semigroup $\hat{T}(t)$ generated by $\hat{A}$ is exponentially decaying, we first follow a similar argument we made in proving the theorem 3.1. We first define the following function $\hat{V}$(t)

$$\hat{V}(t) = 2\,(1-\varepsilon)\,t\,\hat{E}(t) + 2\int_{x=0}^{x=L} \lambda x\,(u_{1t} - \omega_3(b+x))u_{1x}\,dx + 2\int_{x=0}^{x=L} \lambda x\,(u_{3t} - \omega_1(b+x))u_{3x}\,dx \qquad \text{(A.20)}$$

where $\varepsilon \in (0,1)$ is arbitrary.

Applying Schwartz's inequality to the integrals in (A.20) and using $x \le L$, it can be shown that there exists a $K > 0$ such that the following estimate holds :

$$(2\,(1-\varepsilon)t - K)\hat{E}(t) \le \hat{V}(t)$$

Differentiating $\hat{V}$(t) with respect to t, using equations (A.1)-(A.5) and following the line of the proof of the theorem 3.1, we can conclude that there exists a $T > 0$ such that $\hat{V}$(t) is bounded above for all $t \ge T$. Therefore $\hat{E}(t)$ is bounded above by $O(\frac{1}{t})$, for all $t \ge T$. Hence for some M > 0

$$\int_{t=0}^{t=\infty} \hat{E}^2(t)\,dt \le M$$

The assertion (ii) then follows from a theorem due to Pazy, (thm. 4.1, [Paz.1]). □

We now show the existence and uniqueness of the solutions of (A.6). The main difficulty is the fact that the nonlinear operator $T_I(z) : \hat{H} \to \hat{H}$ defined by (A.8) is also unbounded, that is not defined for all $z \in \hat{H}$. But, with an appropriate norm defined on D($\hat{A}$), (see (A.21) below), $T_I(z) : D(\hat{A}) \to \hat{H}$ becomes an C^∞ operator.

A.2 theorem : Consider the equation (A.6), where the operators $\hat{A}$, T_D and g are defined in the equations (A.7)-(A.9), respectively. Then:

i) for all initial conditions z(0) $\in D(\hat{A})$, (A.6) has unique classical solution z(t) defined for all t > 0 ;

ii) in terms of the semigroup $\hat{T}(t)$ generated by the linear operator $\hat{A}$, this solution can be written as :

$$z(t) = \hat{T}(t)z(0) + \int_{=0}^{s=t} \hat{T}(t-s)T_I(z(s))\,ds + \int_{s=0}^{s=t} \hat{T}(t-s)g(z(s))\,ds \ ;$$

iii) the solutions of (A.6) are exponentially decaying.

proof :

i) We define the following norm on D($\hat{A}$) :

$$|||z||| = ||\hat{A}z||_1 \qquad z \in D(\hat{A}), \tag{A.21}$$

where $||\ .||_1$ is defined in (A.11).

A simple calculation shows that this norm is equivalent to the norm given by (A.10), hence D($\hat{A}$) with this norm becomes a Banach space. Let us call this space [D($\hat{A}$)]. Then $T_I : [D(\hat{A})] \to \hat{H}$ becomes an C^∞ operator, since its components are linear combinations of products and integrals of the components of z over [0,L], (see the equations (A.6) and (A.8)).

Also note that g : $\hat{H} \to \hat{H}$,as defined by the equation (A.9), is a C^∞ map, since its components are products of the components of z. Therefore it follows from a theorem due to Segal that ,(thm.2, [Seg.1]), equation (A.6) has unique classical solution for all initial conditions z(0) $\in D(\hat{A})$, defined in [0, ε] for some $\varepsilon > 0$. But since the theorem 3.1 shows that the solutions are decaying to 0, this local existence theorem can be extended globally (i.e. for all $t > 0$).

ii) This may be proven by substitution in (A.6);

iii) Since by the theorem 3.1 the solutions of (A.6) are decaying to 0 in $\hat{H}$, it follows that the positive orbits $O_0^+(t) = \{ z(t) \in \hat{H} \mid z(0) = z_0 ,\ t > 0\}$ belong to a compact set in $\hat{H}$. Therefore by a generalization of LaSalle's invariance argument to the infinite dimensional spaces ,[Hal.1], and by the energy decay estimate (3.4) it follows that asymptotically the rate of change of the energy defined by (3.4) decays to 0. That is $u_{it}(L,t)$, $u_{ixt}(L,t)$, i=1,3 and $\omega(t)$ decay to 0, as $t \rightarrow \infty$.

Using integration by parts in (A.8) and the above conclusion, and the techniques used in thm. 3.4, we obtain the following estimates :

$$|| T_I(z(t)) || \leq \gamma_1(t) || z(t) || .$$
$$|| g(z(t)) || \leq \gamma_2(t) || z(t) || .$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are asymptotically decaying to 0.

Using these estimates and following the arguments made in the proof of the theorem 2.4, we conclude that the solutions of (A.6) are decaying exponantially to 0. □

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