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ABSTRACT

In this paper we present a new canonical piecewise-linear circuit capable of realizing *every* member of the Chua's circuit family. It contains only six 2-terminal elements: five of them are linear resistors, capacitors and inductors and only one element is a three-segment piecewise-linear resistor. It is canonical in the sense that: (1) It can exhibit *all* possible phenomena associated with any three-region symmetric piecewise-linear continuous vector fields, including those defined in [1] and in [2], and more; (2) It contains the *minimum* number of circuit elements needed for such a circuit.

Using this circuit, we proved a theorem which specifies the constraint on the types of eigenvalue patterns associated with a piecewise-linear continuous vector field having three equilibrium points. This theorem has an explicit physical meaning and unified the corresponding theorems in [1] and [2]. We also present some computer simulation results of this circuit, including some new attractors which have not been observed before.

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1 Introduction

Among general piecewise-linear systems, the class of three-region symmetric (with respect to the origin) piecewise-linear continuous vector fields (henceforth denoted by **L**) is of particular interest and importance[1-10]. It is proved in [1] and [2] that any two vector fields ξ and ξ' in **L** are *linearly conjugate* if and only if their corresponding eigenvalues in each region are identical, and are *linearly equivalent* if and only if their corresponding normalized eigenvalues in each region are identical. Here, linear conjugacy implies the respective dynamic behaviors are identical, whereas linear equivalence implies *qualitatively* similar dynamic behaviors.

Therefore, if we can build a piecewise-linear circuit whose natural frequencies are equal to an arbitrarily prescribed set of eigenvalues, we can derive all possible phenomena in **L** by analyzing this one circuit alone. Such an attempt has been mentioned in[6], but no such circuit has been reported to date.

Although Chua's circuit displays rather rich nonlinear dynamic[8], many phenomena which can not be observed from Chua's circuit have been discovered from *other* piecewise-linear circuits[9][10]. However, we will prove that not one of these circuits is general enough to satisfy our above cited objective.

In this paper we will present a new piecewise-linear circuit. It contain the minimum number of circuit elements needed to generate all possible phenomena in any 3-dimensional, 3-region, continuous and symmetric piecewise-linear vector fields. In *Section 2* we demonstrate why no existing circuits can fulfill our purpose. In *Section 3* we give the structure of the canonical piecewise-linear circuit and the explicit formulas for calculating its elements parameters from an arbitrarily given set of eigenvalues. In *Section 4*, based on this circuit, we proved a theorem on the class of realizable eigenvalues, thereby unifying the corresponding theorems in[1] and [2]. In *Section 5* we present some simulation results, including a few attractors which have not been reported before. For certain eigenvalues where the canonical circuit in Section 3 requires *negative* dynamic elements, and/or too many *negative* resistors, other equivalent but more practical piecewise-linear circuits realizations are presented in *Section 6*.

2 Eigenvalue constraints for existing circuits

Consider the class of 3-dimensional, 3-region, and symmetric (with respect to the origin) piecewise-linear continuous vector fields. The eigenvalues in the inner region D_0 are denoted by μ_1, μ_2, and μ_3. The eigenvalues in the two outer regions D_{+1} and D_{-1} are equal, since the vector field is symmetric with respect to the origin. We denote them by ν_1, ν_2, and ν_3. Some of the μ's and the ν's may be complex conjugate numbers. In order to avoid complex numbers, let us define

$$p_1 = \mu_1 + \mu_2 + \mu_3, \quad p_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad p_3 = \mu_1\mu_2\mu_3 \tag{1}$$

$$q_1 = \nu_1 + \nu_2 + \nu_3, \quad q_2 = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1, \quad q_3 = \nu_1\nu_2\nu_3 \tag{2}$$

Let us first analyze the type of eigenvalue patterns that can be produced by Chua's circuit, as shown in Fig.1(a). The v-i relationship of the nonlinear resistor G_N is shown in Fig.1(b). The state equations of this circuit are:

$$\frac{dv_1}{dt} = \frac{1}{C_1}[G(v_2-v_1) - (G_b v_1 + \frac{1}{2}(G_a-G_b)(|v_1+1| - |v_1-1|))]$$

$$\frac{dv_2}{dt} = \frac{1}{C_2}[G(v_1-v_2) + i] \tag{3}$$

$$\frac{di}{dt} = -\frac{v_2}{L}$$

where we have chosen $v_1 = \pm 1$ as the break points for simplicity.

In the D_0 region, the state equations are linear:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-(G+G_a)}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & \frac{-G}{C_2} & \frac{1}{C_2} \\ 0 & \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} = \mathbf{M}_0 \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} \tag{4}$$

where $\mathbf{M}_0$ is a constant matrix.

The characteristic equation of $\mathbf{M}_0$ is:

$$|s\mathbf{1} - \mathbf{M}_0| = s^3 + s^2(\frac{G}{C_2} + \frac{G}{C_1} + \frac{G_a}{C_1}) + s(\frac{GG_a}{C_1C_2} + \frac{1}{LC_2}) + \frac{G+G_a}{LC_1C_2} = 0 \tag{5}$$

On the other hand, since μ_1, μ_2 and μ_3 are the eigenvalues we want this system to possess, we have

$$(s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 - p_1 s^2 + p_2 s - p_3 = 0 \tag{6}$$

Comparing (5) with (6), we obtain

$$\frac{G}{C_2} + \frac{G}{C_1} + \frac{G_a}{C_1} = -p_1 \tag{7}$$

$$\frac{GG_a}{C_1C_2} + \frac{1}{LC_2} = p_2 \tag{8}$$

$$\frac{G+G_a}{LC_1C_2} = -p_3 \tag{9}$$

Similarly, in the $D_{\pm 0}$ regions we have

$$\frac{G}{C_2}+\frac{G}{C_1}+\frac{G_b}{C_1}=-q_1 \tag{10}$$

$$\frac{GG_b}{C_1C_2}+\frac{1}{LC_2}=q_2 \tag{11}$$

$$\frac{G+G_b}{LC_1C_2}=-q_3 \tag{12}$$

Subtracing (10) from (7), (11) from (8) and (12) from (9), we obtain

$$\frac{G_a-G_b}{C_1}=-p_1+q_1 \tag{13}$$

$$\frac{G(G_a-G_b)}{C_1C_2}=p_2-q_2 \tag{14}$$

$$\frac{G_a-G_b}{LC_1C_2}=-p_3+q_3 \tag{15}$$

or,

$$\frac{1}{C_1}=\frac{-p_1+q_1}{G_a-G_b} \tag{16}$$

$$\frac{1}{C_2}=\frac{p_2-q_2}{G(q_1-p_1)} \tag{17}$$

$$\frac{1}{L}=\frac{G(q_3-p_3)}{p_2-q_2} \tag{18}$$

Substituting (16), (17) and (18) into (7), (8) and (9), and after some algebraic manipulations, we obtain a set of three linear algebraic equations in G, G_a and G_b:

$$\begin{bmatrix} 0 & (p_2-q_2)+q_1(q_1-p_1) & q_2-p_2-p_1(q_1-p_1) \\ (p_2-q_2)(q_1-p_1) & q_3-p_3-q_2(q_1-p_1) & p_2(q_1-p_1)-(q_3-p_3) \\ 0 & q_3 & -p_3 \end{bmatrix} \begin{bmatrix} G \\ G_a \\ G_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

This set of homogeneous algebraic equations has nonzero solutions only if the determinant of the matrix is equal to zero, i. e.

$$\det\begin{bmatrix} (p_2-q_2)+q_1(q_1-p_1) & (q_2-p_2)-p_1(q_1-p_1) \\ q_3 & -p_3 \end{bmatrix}=0 \tag{20}$$

or,

$$(p_2-q_2)(p_3-q_3)=(p_1-q_1)(p_3q_1-q_3p_1) \tag{21}$$

This is the main eigenvalue constraint on Chua's circuit. Only those eigenvalues which are subject to this constraint can be realized by Chua's circuit. In addition, the following obvious constraints must also be satisfied:

$$p_1 \neq q_1, \quad p_2 \neq q_2, \quad p_3 \neq q_3 \tag{22}$$

Otherwise, C_1, C_2 or L will tend to infinity in view of (16), (17) and (18).

Let us consider a numerical example using the following element parameters from Chua's circuit in [7] :

$$C_1 = \frac{1}{9}, \quad C_2 = 1, \quad G = 0.7, \quad G_a = -0.8, \quad G_b = -0.5, \quad L = \frac{1}{7} \tag{23}$$

Using to (7)-(12) we obtain

$$p_1 = 0.2, \quad p_2 = 1.96, \quad p_3 = 6.3, \quad q_1 = -2.5, \quad q_2 = 3.85, \quad q_3 = -12.6, \tag{24}$$

It is easy to verify that (24) does satisfy (21).

Consider next the piecewise-linear circuit in Fig.2. Since it is related to the torus attractors[9], we will refer to it as the "torus circuit". By an analysis similar to the above, we can show that the sets of eigenvalues of this circuit are subject to the following constraints:

$$p_2 - q_2 = 0 \tag{25a}$$

$$p_1 q_3 - p_3 q_1 = 0 \tag{25b}$$

It is easily to see that (25) is a special case of (21). However, (25a) violates (22). Therefore, all eigenvalue patterns produced by the torus circuit can *not* be produced by Chua's circuit, no matter how one adjusts the circuit parameters in Chua's circuit.

Consider next the piecewise-linear circuit in Fig.3. We will refer to it as the "double hook circuit"[10]. By a similar analysis we can show that this circuit is also subject to the eigenvalue constraints (21) and (22), as in Chua's circuit. Hence, from the point of view of computer simulation they are equivalent. However, the corresponding element parameters in these two circuits are different for a particular set of eigenvalues. From the point of view of experimental observation, one of these two circuits therefore would be a better choice if it requires fewer negative dynamic elements in a particular case.

3 The canonical piecewise-linear circuit

In this Section we will present a *universal* piecewise-linear circuit for realizing *any eigenvalue pattern* associated with any vector field in L.

Firstly we have to decide what is the minimum number of elements such a circuit needs. Since our objective is a 3-dimensional 3-region symmetric piecewise-linear continuous vector field, the circuit under consideration is allowed to have only one nonlinear resistor whose v-i characteristic is 3-segment piecewise-linear and symmetric with respect to the origin. The circuit must have three dynamic elements (capacitors and/or inductors) since the system is of third order. The rest are all linear resistors. Let us investigate next how many linear resistors are needed in general.

A linear autonomous R-C circuit has two circuit elements and can have only one natural frequency $\mu = 1/RC$. If we increase C to αC and decrease R to R/α, the natural frequency of the circuit will remain unchanged. Therefore, to produce a natural frequency μ, we can assign an arbitrary value to C or R (e.g. let $C = 1$) and find the value for the other parameter.

The situation is similar for Chua's circuit. In eqns. (7)-(12) there are six unknown parameters: C_1, C_2, G, G_a, G_b, and L. However, if we regard C_1, C_2, G, G_a, G_b, and $1/L$ as the unknown variables, the left hand side of eqns. (7)-(12) are homogeneous functions of the zero'th order. For any particular set of p's and q's, if (C_{10}, C_{20}, G_0, G_{a0}, G_{b0}, L_0) is a solution of (7)-(12), then (αC_{10}, αC_{20}, αG_0, αG_{a0}, αG_{b0}, L_0/α) (α is an arbitrary real number) will also be a solution. In other words, if (7)-(12) have solutions, we can assign an arbitrary value to any one of the six parameters (e.g. let $C_1 = 1$) and calculate the remaining five parameters. This means that out of the six circuit parameters the "degree of freedom" is only five. From the point of view of circuit theory, we can refer this observation as "impedance scaling".

It reveals the reason why Chua's circuit can not produce an arbitrary set of eigenvalues. There are not enough circuit parameters! Since we have six eigenvalues in our problem, we need at least seven parameters. Therefore, besides three dynamic elements and one nonlinear resistor (with two slopes in different regions counted as two circuit parameters), we need at least two linear resistor to build a canonical circuit.

Of course, not every circuit containing that many elements will qualify as a canonical circuit. Our canonical circuit is shown in Fig.4(a).

The state equations of this circuit are:

$$\frac{dv_1}{dt} = \frac{1}{C_1}[-f(v_1) + i_3]$$

$$\frac{dv_2}{dt} = \frac{1}{C_2}(-Gv_2 + i_3) \tag{26}$$

$$\frac{di_3}{dt} = \frac{-1}{L}(v_1 + v_2 + Ri_3)$$

where

$$f(v) = G_b v + \frac{1}{2}(G_a - G_b)(|v+1| - |v-1|) \tag{27}$$

is the v-i characteristic of the nonlinear resistor shown in Fig.4(b).

In the D_0 region (i.e. $|v_1| \le 1$), the state equations (26) become linear:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-G_a}{C_1} & 0 & \frac{1}{C_1} \\ 0 & \frac{-G}{C_2} & \frac{1}{C_2} \\ \frac{-1}{L} & \frac{-1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \mathbf{M}_0 \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} \tag{28}$$

where $\mathbf{M}_0$ is a constant matrix. The characteristic equation of $\mathbf{M}_0$ is:

$$|s1 - \mathbf{M}_0| = s^3 + s^2\left(\frac{G_a}{C_1} + \frac{G}{C_2} + \frac{R}{L}\right)$$
$$+ s\left(\frac{GG_a}{C_1C_2} + \frac{G_aR}{LC_1} + \frac{GR}{LC_2} + \frac{1}{LC_1} + \frac{1}{LC_2}\right) + \frac{G + G_a + GG_aR}{LC_1C_2} = 0 \tag{29}$$

Just as in Section 2, we obtain

$$\frac{G_a}{C_1} + \frac{G}{C_2} + \frac{R}{L} = -p_1 \tag{30}$$

$$\frac{GG_a}{C_1C_2} + \frac{G_aR}{LC_1} + \frac{GR}{LC_2} + \frac{1}{LC_1} + \frac{1}{LC_2} = p_2 \tag{31}$$

$$\frac{G + G_a + GG_aR}{LC_1C_2} = -p_3 \tag{32}$$

Similarly, from the equation in the $D_{\pm 0}$ regions (i.e. $|v_1| > 1$) we obtain

$$\frac{G_b}{C_1} + \frac{G}{C_2} + \frac{R}{L} = -q_1 \tag{33}$$

$$\frac{GG_b}{C_1C_2} + \frac{G_bR}{LC_1} + \frac{GR}{LC_2} + \frac{1}{LC_1} + \frac{1}{LC_2} = q_2 \tag{34}$$

$$\frac{G + G_b + GG_bR}{LC_1C_2} = -q_3 \tag{35}$$

Subtracting (33), (34) and (35) from (30), (31) and (32) respectively, we obtain

$$\frac{G_a - G_b}{C_1} = -p_1 + q_1 \tag{36}$$

$$\frac{G_a - G_b}{C_1}\left(\frac{G}{C_2} + \frac{R}{L}\right) = p_2 - q_2 \tag{37}$$

$$\frac{(G_a - G_b)(GR + 1)}{LC_1C_2} = -p_3 + q_3 \tag{38}$$

Substituting (36) into (37) and (38), we obtain

$$\frac{G}{C_2} + \frac{R}{L} = \frac{-p_2 + q_2}{p_1 - q_1} \tag{39}$$

and

$$\frac{(GR + 1)}{LC_2} = \frac{p_3 - q_3}{p_1 - q_1} \tag{40}$$

Substituting (39) into (30) and (33), we obtain

$$\frac{G_a}{C_1} = -p_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{41}$$

and

$$\frac{G_b}{C_1} = -q_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{42}$$

As we mentioned before, among the seven parameters we can assign an arbitrary value to any one of them. Let us take

$$C_1 = 1 \tag{43}$$

Then we obtain the values of parameters G_a and G_b from (41) and (42):

$$G_a = -p_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{44}$$

$$G_b = -q_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{45}$$

Substituting (39), (40), (41) and (43) into (31), we obtain the following value of L:

$$L = \frac{1}{p_2 + \left(\frac{p_2 - q_2}{p_1 - q_1} - p_1\right)\left(\frac{p_2 - q_2}{p_1 - q_1}\right) - \frac{p_3 - q_3}{p_1 - q_1}} \tag{46}$$

Now from (32) we have

$$\frac{G}{C_2} = -L\left[p_3 + \frac{G_a(p_3 - q_3)}{C_1(p_1 - q_1)}\right] = k \tag{47}$$

where the constant k is introduced for simplicity. Substituting (47) into (39), we obtain the value of R:

$$R = -L\left(\frac{p_2 - q_2}{p_1 - q_1} + k\right) \tag{48}$$

From (40) we have

$$\frac{1}{LC_2} = \frac{p_3 - q_3}{p_1 - q_1} - \frac{GR}{LC_2} = \frac{p_3 - q_3}{p_1 - q_1} + k\left(k + \frac{p_2 - q_2}{p_1 - q_1}\right) \tag{49}$$

Hence, the value of C_2 is given by

$$C_2 = \frac{1}{L\left[\frac{p_3 - q_3}{p_1 - q_1} + k\left(k + \frac{p_2 - q_2}{p_1 - q_1}\right)\right]} \tag{50}$$

Finally, from (47) we obtain the value of G:

$$G = kC_2 \tag{51}$$

Equations (43), (44), (45), (46), (48), (50) and (51) are explicit formulas for calculating the seven parameters in our canonical circuit from *any given* set of eigenvalues. The only constraint is

$$p_1 \neq q_1 \tag{52}$$

Otherwise some of the parameters will tend to infinity. Constraint (52) is mainly academic since in the unlikely event that

$$p_1 = q_1 \tag{53}$$

for a given set of eigenvalues, we can usually eliminate this singular situation by perturbing one of the μ's or ν's without causing any qualitative change in the system's dynamic.

4 Some properties of the canonical circuit

The three-dimensional vector fields produced by our canonical circuit have the following properties:

1. The vector field is continuous everywhere and symmetric with respect to the origin.

2. The state space is partitioned into three regions D_{-1}, D_0 and D_{+1} by two parallel planes located at $v_1 = \pm 1$.

3. The vector field is *linear* in region D_0 and *affine* in regions D_{-1} and D_{+1}.

These properties are obvious from the structure of the canonical circuit. In [1] and [2], the vector fields under consideration are more restricted. They always have one equilibrium point in each region D_{-1}, D_0 and D_{+1}. Besides, the vector fields discussed in [1] all have one real eigenvalue and a pair of complex conjugate eigenvalues in each region (Henceforth called *type I* eigenvalues), while those discussed in [2] all have three real eigenvalues in the D_0 region, and one real and a pair of complex conjugate eigenvalues in the $D_{\pm 1}$ regions (Henceforth called *type II* eigenvalues). These two types of eigenvalue patterns are more interesting because their vector fields can exhibit *chaotic attractors*. However, the vector fields which can be produced by the canonical circuit have no such constraints. Their eigenvalues may be either all real, or one real plus a pair of complex conjugate values in each region. In the canonical circuit, the origin is of course an equilibrium point. However, the $D_{\pm 0}$ regions may or may not have equilibrium points. In the latter case, we say the regions $D_{\pm}$ have a *virtual* equilibrium point. In [1] and [2], there are theorems stating under what condition there will be equilibrium points in regions $D_{\pm 1}$. The proof of these theorems are quite involved and have little physical meaning. For our canonical circuit we can prove a similar theorem using a much simpler proof, which at the same time gives a much clearer physical meaning.

Theorem 1

The following three conditions are equivalent, each one giving a necessary and sufficient condition for a vector field realized by our canonical circuit to have equilibrium points in the $D_{\pm 1}$ region:

1. The canonical circuit has three dc operating points.

2. $(G + G_a(1 + GR))\,(G + G_b(1 + GR) < 0$ (54)

3. $p_3 q_3 < 0$ (55)

Proof

1. The equilibrium points of the nonlinear system described by (26) are obtained by solving

$$
\begin{aligned}
f(v_1) - i &= 0 \\
Gv_2 - i &= 0 \\
v_1 + v_2 + Ri &= 0
\end{aligned}
\tag{56}
$$

Equations (56) are exactly the KCL and KVL equations of the resistive circuit shown in Fig.5(a), which is obtained from the canonical circuit with the capacitors open circuited and the inductor short circuited. It is obvious from the state equations (28) that the canonical circuit has one and only one equilibrium point at the origin in the D_0 region (whenever $|\mathbf{M}_0| \neq 0$). Hence, if the circuit has three dc operating points, two of them must be located in the $D_{\pm 1}$ regions. The converse is true too. Figure 5(b) gives the physical interpretation: whenever the loadline has three intersection points with the 3-segment piecewise-linear characteristic, two of them must necessarily be located in the $D_{\pm 1}$ regions.

2. Segment B^+ in Fig.5(b) is described by the equation

$$i = (G_a - G_b) + G_b v \tag{57}$$

The loadline is described by

$$i = \frac{-v}{R + \frac{1}{G}} \tag{58}$$

The abscissa of the intersection point of (57) and (58) is given by

$$v_0 = \frac{(G_b - G_a)\,(1 + GR)}{G + G_b(1 + GR)} \tag{59}$$

Observe that the region D_{+1} has an equilibrium point if and only if

$$v_0 > 1 \tag{60}$$

It is easily to show that (60) is equivalent to (54) after some simple algebraic manipulations.

3. From (32) and (35) we have

$$\frac{(G + G_a(1 + GR))\,(G + G_b(1 + GR))}{L^2 C_1^2 C_2^2} = p_3 q_3 \tag{61}$$

This implies that (54) and (55) are equivalent. □

Remark

1. When the eigenvalues are of *type 1*, the sign of p_3 and q_3 are determined by the sign of γ_0 and γ_1 respectively, where γ_0 and γ_1 denote the real eigenvalues in the D_0 and $D_{\pm 1}$ regions. Therefore (55) is equivalent to the condition

$$\gamma_0 \gamma_1 < 0 \tag{62}$$

When the eigenvalues are of *type 2*, the sign of p_3 is determined by the sign of $\mu_1\mu_2\mu_3$ and the sign of q_3 is determined by the sign of γ_1. Therefore (55) is equivalent to the condition

$$\mu_1\mu_2\mu_3\gamma_1 < 0 \tag{63}$$

Conditions (62) and (63) are exactly the same conditions given in [1] and [2]. We have therefore unified Theorem 3.1 in [1] and Theorem 4.1 in [2], in addition to giving an explicit physical interpretation to the theorem.

2. It is obvious from (59) that the locations of the equilibrium points depend only on the values of G, G_a, G_b and R, and not on the values of C_1, C_2 and L.

5 Results of computer simulations

In this section we present a *sample* of some computer simulation results of our canonical piecewise-linear circuits, since the 6-dimensional eigenvalue parameter space is too huge to search thoroughly. Every point in this parameter space corresponds to one or more attractors. To determine and classify all possible attractors and their boundaries is a difficult project. Our simulation is by no means comprehensive. However, all attractors discovered in **L** and reported so far in the literatures are included here, in addition to some newly discovered ones.

Table 1 shows the simulation results when the inner region contains a pair of complex conjugate eigenvalues. Table 2 shows the simulation results when the inner region contains only *real* eigenvalues. Both Tables 1 and 2 contain some blanks, indicating that some attractors may be discovered in the future. Table 3 summarizes the values of the eigenvalues and the circuit parameters for a sample of attractors listed in Table 1 and 2. Since all vector fields with the same normalized eigenvalues are linearly equivalent[1][2], all examples in Table 3 have been normalized with $\omega_1 = 1$ for comparison purposes.

Table 1 Some attractors for type I eigenvalue patterns

	Eigenvalues				Type of stability			Observed attractors			
Class	γ_0	σ_0	γ_1	σ_1	stable at origin?	Does $P_\pm$ exist?	stable at $P_\pm$?	stable equilibrium	limit cycle	toroidal attractor	chaotic attractor
1	+	+	+	+	no	no		no	no	no	no
2	+	+	+	-	no	no		no	no	no	no
3	+	+	-	+	no	yes	no	no	*1		*2
4	+	+	-	-	no	yes	yes	yes	yes	no	no
5	+	-	+	+	no	no		no	no	no	no
6	+	-	+	-	no	no		no	no	no	no
7	+	-	-	+	no	yes	no	no	yes	*3	*4
8	+	-	-	-	no	yes	yes	yes		no	no
9	-	+	+	+	no	yes	no	no	no	no	no
10	-	+	+	-	no	yes	no	no	*5	*6	*7
11	-	+	-	+	no	no		no	no	no	no
12	-	+	-	-	no	no		no	yes	no	no
13	-	-	+	+	yes	yes	no	yes	no	no	no
14	-	-	+	-	yes	yes	no	yes	no	no	no
15	-	-	-	+	yes	no		yes	no	no	no
16	-	-	-	-	yes	no		yes	no	no	no

*1: see Fig.6 for an example of a periodic attractors.

*2: see Fig.7,8,9 for examples of 3 chaotic attractors.

*3: see Fig.10,11 for examples of toroidal attractors.

*4: see Fig.12,13,14,15,16,17 for examples of chaotic attractors.

*5: see Fig.18 for an example of a periodic attractor.

*6: see Fig.19,20 for examples of toroidal attractors.

*7: see Fig.21 for an example of a chaotic attractors.

Table 2 Some attractors for type II eigenvalue patterns

	Eigenvalues					Type of stability			Observed attractors			
Class	μ_1	μ_2	μ_3	γ_1	σ_1	stable at origin?	Does $P_\pm$ exist?	stable at $P_\pm$?	stable equil.	limit cycle	toroidal attractor	chaotic attractor
1	+	+	+	+	+	no	no		no	no	no	no
2	+	+	+	+	-	no	no		no	no	no	no
3	+	+	+	-	+	no	yes	no	no			
4	+	+	+	-	-	no	yes	yes	yes	yes	no	no
5	+	+	-	+	+	no	yes	no	no	no	no	no
6	+	+	-	+	-	no	yes	no	no	yes		*8
7	+	+	-	-	+	no	no		no	no	no	no
8	+	+	-	-	-	no	no		no	yes	no	no
9	+	-	-	+	+	no	no		no	no	no	no
10	+	-	-	+	-	no	no		no	no	no	no
11	+	-	-	-	+	no	yes	no	no			*9
12	+	-	-	-	-	no	yes	yes	yes		no	no
13	-	-	-	+	+	yes	yes	no	yes	no	no	no
14	-	-	-	+	-	yes	yes	no	yes	no	no	no
15	-	-	-	-	+	yes	no		yes	no	no	no
16	-	-	-	-	-	yes	no		yes	no	no	no

*8: see Fig.22 for an example of a chaotic attractor.

*9: see Fig.23,24 for examples of 2 chaotic attractors.

Table 3 Parameters of the canonical circuit associated with different attractors in Fig.6-24.

	Eigenvalues*					Circuit parameters*					
Fig.	μ_1	μ_2	μ_3	γ_1	σ_1	C_2	G	G_a	G_b	L	R
6	0.3	0.2	±j20	-20	0.2	1513	-492.4	19.35	39.65	.00128	-.0252
7	0.2	0.3	±j10	-3	0.3	1183	-2565	30.73	33.93	.00094	-.02947
8	.30	.0577	±j2.78	-1.33	0.29	170	-86	5.98	7.146	.0234	-.1376
9	.44	.0577	±j2.78	-1.33	0.29	105	-70.3	5.17	6.47	.0297	-0.15
10	.1474	-.0487	±j1.0	-.104	.0343	-.66	.071	-.12	-.036	-.5	-.09
11	.272	-.0635	±j.746	-.076	.0019	22	11	-2.3	-2.1	.22	.368
12	18.2	-.115	±j1.02	-3.5	.0055	-.6827	.01397	-18.16	3.298	-1.38	-.2916
13	.728	-.319	±j.892	-1.29	.061	-.632	-.0033	-.419	.839	-1.02	-.33
14	.272	-.136	±j.409	-.409	.0454	48.4	31.6	-2.68	-2.36	.171	.346
15	.728	-.319	±j.892	-1.29	.042	-.6	0.01	-.445	.85	-1.1	-0.4
16	1.474	-.0487	±j1.0	-1.04	.0343	-.154	.0285	-1.4	.94	-5.81	-1.25
17	.618	-.37	±j3.50	-2.30	.195	99.6	-89.4	6.72	8.50	.020	-.113
18	-0.33	.312	±j1.03	.186	-.167	-1.74	.030	-.324	.117	.442	.021
19	-.05	.08	±j1.0	.035	-.056	-3.2	.0042	-.114	.0726	.687	.0039
20	-1.62	.084	±j1.15	.922	-.412	-0.7	-.0015	1.034	-.515	-.685	-.285
21	-0.45	.0235	±j.346	.277	-.124	41.1	-18.5	2.36	1.93	.252	-.381
22	1.032	.1354	-.4425	.02	-0.2	-95.68	3.733	-2	.895	.4448	.5845
23	1.15	-2.98	-5.70	-.89	0.15	-1.35	.0014	6.63	-.31	.251	.226
24	.919	-.541	-3.64	-.353	.156	-15.6	-6.42	4.13	.906	.421	-.537

*: The parameters ω_1 and C_1 for all attractors are assumed to be equal to 1.

6 An alternate realization of the canonical circuit

We do not claim that the canonical circuit in Fig.4 is unique. However, so far we have not found any other circuit having the same degree of generality.

Figure 25(a) shows an alternate piecewise-linear circuit. It's state equations are given by:

$$
\begin{aligned}
\frac{dv_1}{dt} &= \frac{1}{C_1}(-G_1 v_1 + i_3) \\
\frac{dv_2}{dt} &= \frac{1}{C_2}(-G v_2 + i_3) \\
\frac{di_3}{dt} &= -\frac{1}{L}[v_1 + v_2 + f(i)]
\end{aligned}
\tag{64}
$$

where

$$f(i_3) = R_b i + \frac{1}{2}(R_a - R_b)\,(|i_3 + 1| - |i_3 - 1|) \tag{65}$$

is the v-i characteristic of the nonlinear resistor shown in Fig.25(b).

In the D_0 region (i.e. $|i| \le 1$), the state equations (64) reduces to the linear equation:

$$
\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix}
=
\begin{bmatrix} \frac{-G_1}{C_1} & 0 & \frac{1}{C_1} \\ 0 & \frac{-G}{C_2} & \frac{1}{C_2} \\ -\frac{1}{L} & -\frac{1}{L} & \frac{-R_a}{L} \end{bmatrix}
\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}
= \mathbf{M}_0 \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}
\tag{66}
$$

where $\mathbf{M}_0$ is a constant matrix. The characteristic equation of $\mathbf{M}_0$ is given by

$$
\begin{aligned}
|s1 - \mathbf{M}_0| &= s^3 + s^2\left(\frac{G_1}{C_1} + \frac{G_2}{C_2} + \frac{R_a}{L}\right) \\
&+ s\left(\frac{G_1 G_2}{C_1 C_2} + \frac{G_1 R_a}{C_1 L} + \frac{G_2 R_a}{C_2 L} + \frac{1}{C_1 L} + \frac{1}{C_2 L}\right) + \frac{G_1 + G_2 + G_1 G_2 R_a}{C_1 C_2 L} = 0
\end{aligned}
\tag{67}
$$

It follows from (67) that

$$\frac{G_1}{C_1} + \frac{G_2}{C_2} + \frac{R_a}{L} = -p_1 \tag{68}$$

$$\frac{G_1 G_2}{C_1 C_2} + \frac{G_1 R_a}{L C_1} + \frac{G_2 R_a}{C_2 L} + \frac{1}{C_1 L} + \frac{1}{C_2 L} = p_2 \tag{69}$$

$$\frac{G_1 + G_2 + G_1 G_2 R_a}{C_1 C_2 L} = -p_3 \tag{70}$$

Similarly, from the equations for the $D_{\pm 0}$ regions (i.e. $|i| > 1$) we obtain

$$\frac{G_1}{C_1} + \frac{G_2}{C_2} + \frac{R_b}{L} = -q_1 \tag{71}$$

$$\frac{G_1 G_2}{C_1 C_2} + \frac{G_1 R_b}{C_1 L} + \frac{G_2 R_b}{C_2 L} + \frac{1}{C_1 L} + \frac{1}{C_2 L} = q_2 \tag{72}$$

$$\frac{G_1 + G_2 + G_1 G_2 R_b}{C_1 C_2 L} = -q_3 \tag{73}$$

Subtracting Eqs. (71), (72) and (73) from Eqs. (68), (69) and (70) respectively, we obtain

$$\frac{R_a - R_b}{L} = -p_1 + q_1 \tag{74}$$

$$\frac{R_a - R_b}{L}\left(\frac{G_1}{C_1} + \frac{G_2}{C_2}\right) = p_2 - q_2 \tag{75}$$

$$\frac{G_1 G_2 (R_a - R_b)}{C_1 C_2 L} = -p_3 + q_3 \tag{76}$$

Substituting Eq. (74) into Eqs. (75) and (76), we obtain

$$\frac{G_1}{C_1} + \frac{G_2}{C_2} = \frac{p_2 - q_2}{-p_1 + q_1} \tag{77}$$

and

$$\frac{G_1 G_2}{C_1 C_2} = \frac{p_3 - q_3}{p_1 - q_1} \tag{78}$$

Since one parameter can be assigned an arbitrary value, let us take

$$L = 1 \tag{79}$$

Substituting Eqs. (77) and (78) into Eqs. (68) and (71), we obtain the following values of R_a and R_b:

$$R_a = -p_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{80}$$

$$R_b = -q_1 + \frac{p_2 - q_2}{p_1 - q_1} \tag{81}$$

On the other hand, Eqs. (77) and (78) imply that G_1/C_1 and G_2/C_2 are the two roots of the following quadratic equation:

$$x^2 + x(\frac{p_2 - q_2}{p_1 - q_1}) + \frac{p_3 - q_3}{p_1 - q_1} = 0 \tag{82}$$

Solving for the values of G_1/C_1 and G_2/C_2, we obtain

$$\frac{G_1}{C_1} = \frac{-p_2 + q_2 + \sqrt{(p_2 - q_2)^2 - (p_1 - q_1)(p_3 - q_3)}}{2(p_1 - q_1)} \triangleq k_1 \tag{83}$$

$$\frac{G_2}{C_2} = \frac{-p_2 + q_2 - \sqrt{(p_2 - q_2)^2 - (p_1 - q_1)(p_3 - q_3)}}{2(p_1 - q_1)} \triangleq k_2 \tag{84}$$

Substituting Eqs. (78), (79), (80), (83) and (84) into Eqs. (69) and (70), we obtain

$$\frac{1}{C_1} + \frac{1}{C_2} = p_2 - \frac{p_3 - q_3}{p_1 - q_1} + \frac{p_2 - q_2}{p_1 - q_1}(\frac{p_2 - q_2}{p_1 - q_1} - p_1) \tag{85}$$

$$\frac{k_1}{C_1} + \frac{k_2}{C_2} = -p_3 - \frac{p_3 - q_3}{p_1 - q_1}(\frac{p_2 - q_2}{p_1 - q_1} - p_1) \tag{86}$$

Equations (85) and (86) constitute a system of 2 linear algebraic equations in $1/C_1$ and $1/C_2$, which is easy to solve. Finally, we can calculate G_1 and G_2 from (83) and (84):

$$G_1 = k_1 C_1 \tag{87}$$

$$G_1 = k_1 C_1 \tag{88}$$

Since all circuit parameters can be explicitly calculated from the given set of eigenvalues, this circuit qualifies as an alternate canonical circuit. However, it is subject to somewhat stronger restrictions than the canonical circuit proposed in Section 3. This is because Eqs. (83) and (84) have real solutions only if

$$(p_2 - q_2)^2 \geq 4(p_1 - q_1)(p_3 - q_3) \tag{89}$$

Therefore, it is not general enough to qualify as a canonical circuit. If we are only interested in computer simulation, the canonical circuit in Fig.4 is more than adequate and there is no need to search for alternate circuits. However, our canonical circuit may contain some *negative* dynamic elements for some sets of eigenvalues. In the laboratory negative C and L are usually harder to realize than negative R. Consequently, if another circuit can produce the same vector field but contains fewer negative dynamic elements, then it may be preferable to use such an alternate equivalent circuit for practical realization purposes.

7 Concluding remarks

We have developed a canonical circuit which is general enough for simulating all possible dynamics associated with any 3-dimensional, 3-region, and symmetric piecewise-linear continuous vector field. It contains only one 3-segment piecewise-linear resistor and the least number of two terminal linear elements and no controlled sources. All circuit parameters can be determined *explicitly* from any given set of eigenvalues with no constraints. It would of course be highly desirable to derive analogous canonical circuit for higher-dimensional systems.

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Figure captions

Fig.1 (a) Chua's circuit.

(b) The v-i characteristic of the nonlinear resistor G_N.

Fig.2 The torus circuit.

Fig.3 The double hook circuit.

Fig.4 (a) The canonical piecewise-linear circuit.

(b) The v-i characteristic of G_N.

Fig.5 (a) The dc circuit associated with the canonical piecewise-linear circuit.

(b) The dc operating points of the circuit.

Fig.6 An example of a limit cycle associated with a type I class 3 eigenvalues pattern.

Fig.7 An example of a chaotic attractor associated with a type I class 3 eigenvalues pattern.

Fig.8 An example of a chaotic attractor associated with a type I class 3 eigenvalues pattern.

Fig.9 An example of a chaotic attractor associated with a type I class 3 eigenvalues pattern.

Fig.10 An example of a toroidal attractor associated with a type I class 7 eigenvalues pattern.

Fig.11 An example of a toroidal attractor associated with a type I class 7 eigenvalues pattern.

Fig.12 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern. This attractor is identical to the double scroll in [7].

Fig.13 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern. This attractor is identical to the double scroll in [7].

Fig.14 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern. This attractor is different in shape from the double scroll.

Fig.15 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern. This attractor is identical to a Rossler band.

Fig.16 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern.

Fig.17 An example of a chaotic attractor associated with a type I class 7 eigenvalues pattern.

Fig.18 An example of a limit cycle associated with a type I class 10 eigenvalues pattern.

Fig.19 An example of a toroidal attractor associated with a type I class 10 eigenvalues pattern.

Fig.20 An example of a toroidal attractor associated with a type I class 10 eigenvalues pattern.

Fig.21 An example of a chaotic attractor associated with a type I class 10 eigenvalues pattern.

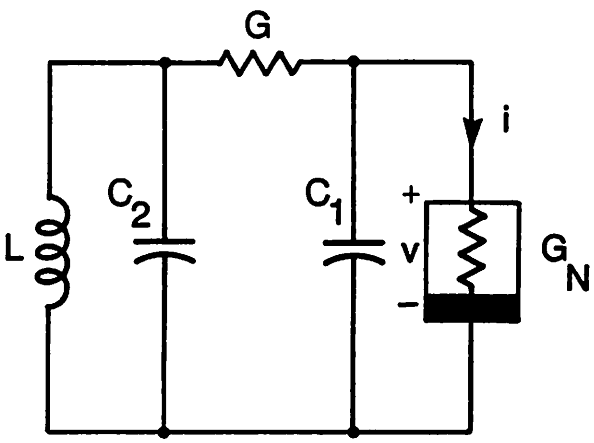
Fig.22 An example of a chaotic attractor associated with a type II class 6 eigenvalues pattern.

Fig.23 An example of a chaotic attractor associated with a type II class 11 eigenvalues pattern. This attractor is identical to the double hook in [10].

Fig.24 An example of a chaotic attractor associated with a type II class 11 eigenvalues pattern.

Fig.25 (a) An alternate but less general piecewise-linear circuit.

(b) The v-i characteristic of the nonlinear resistor R_N.



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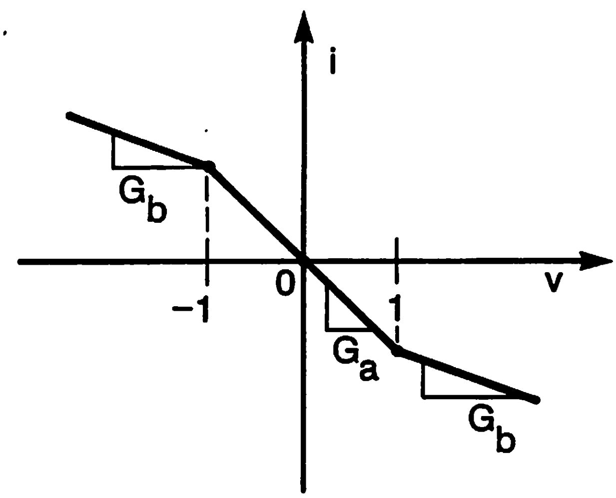

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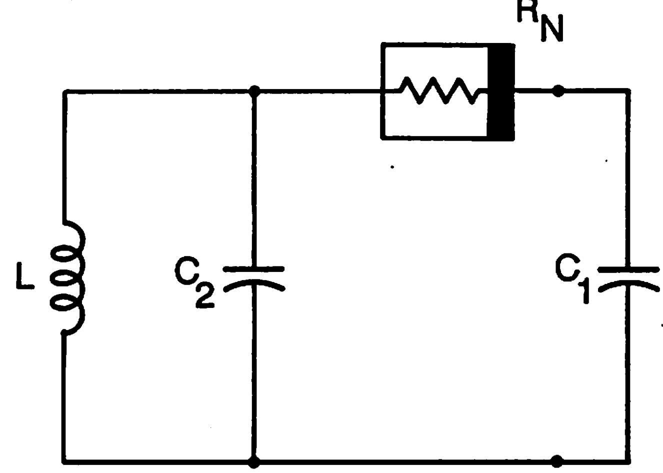

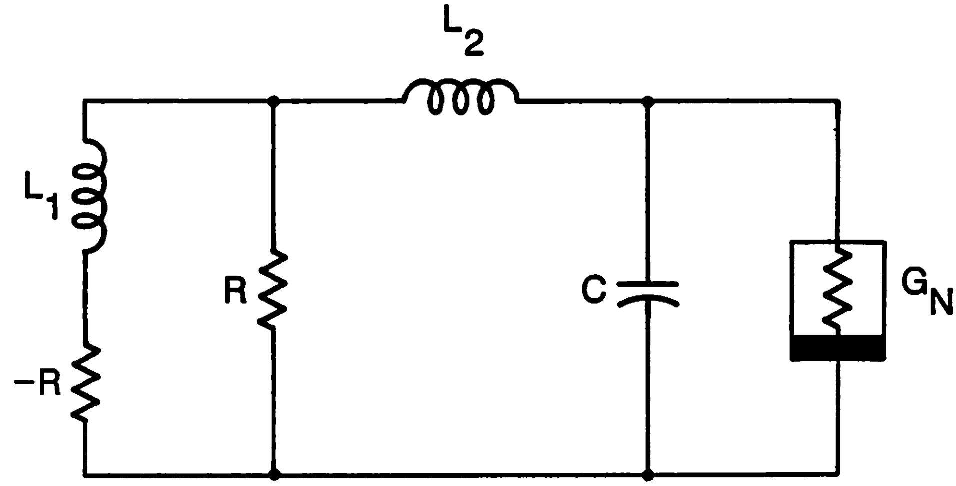

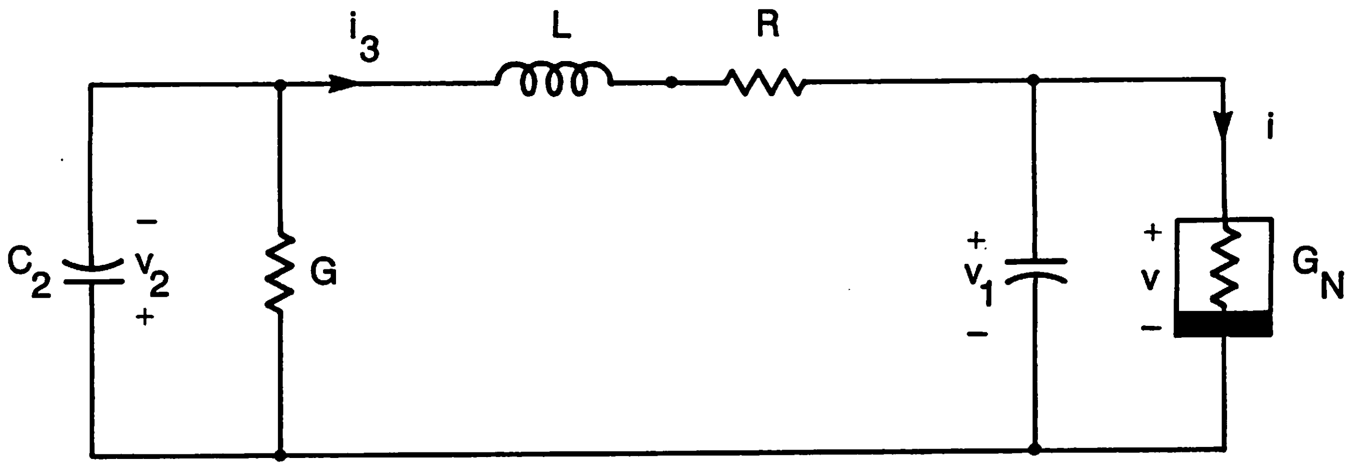

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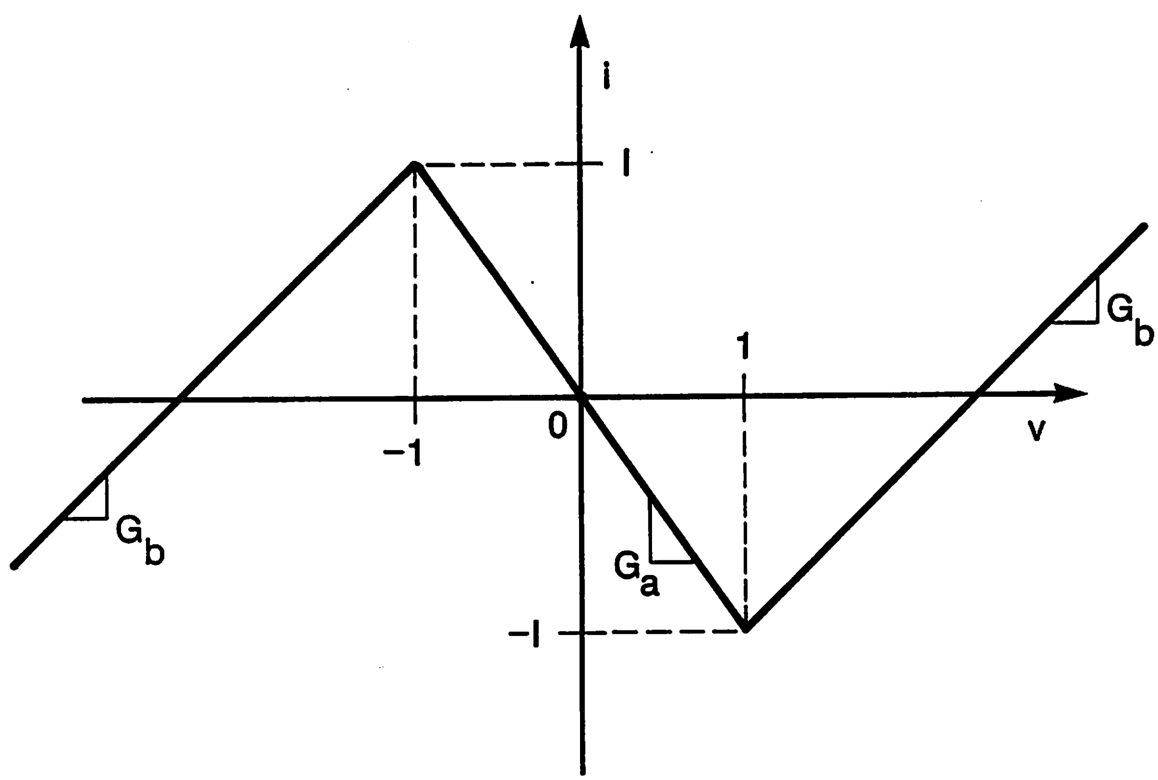

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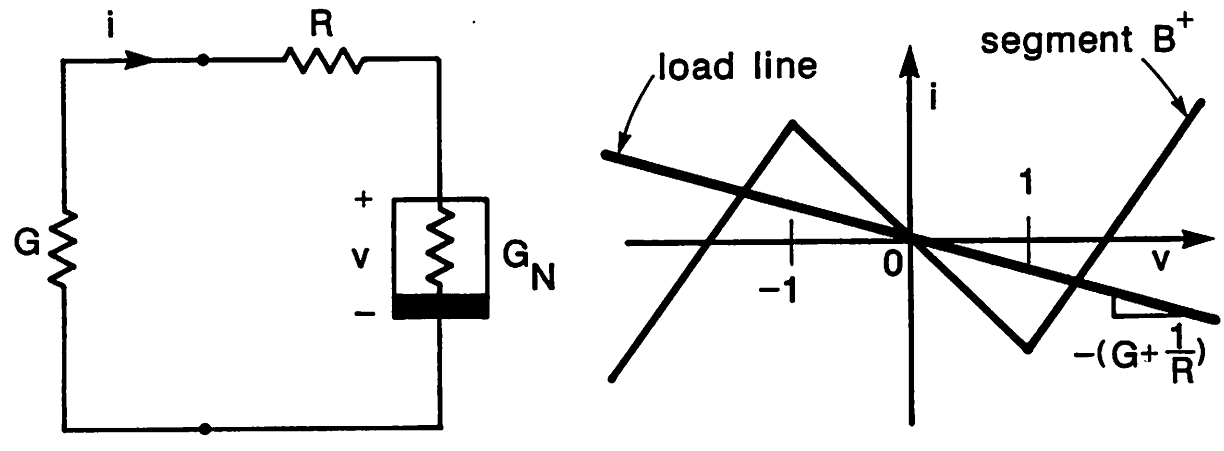
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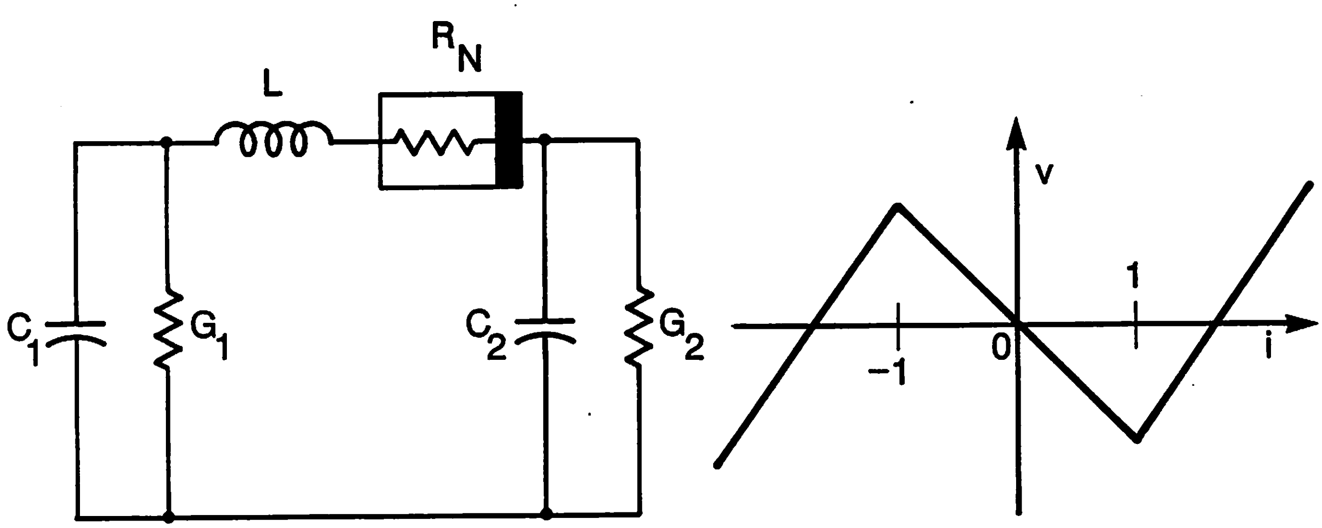
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Fig.6

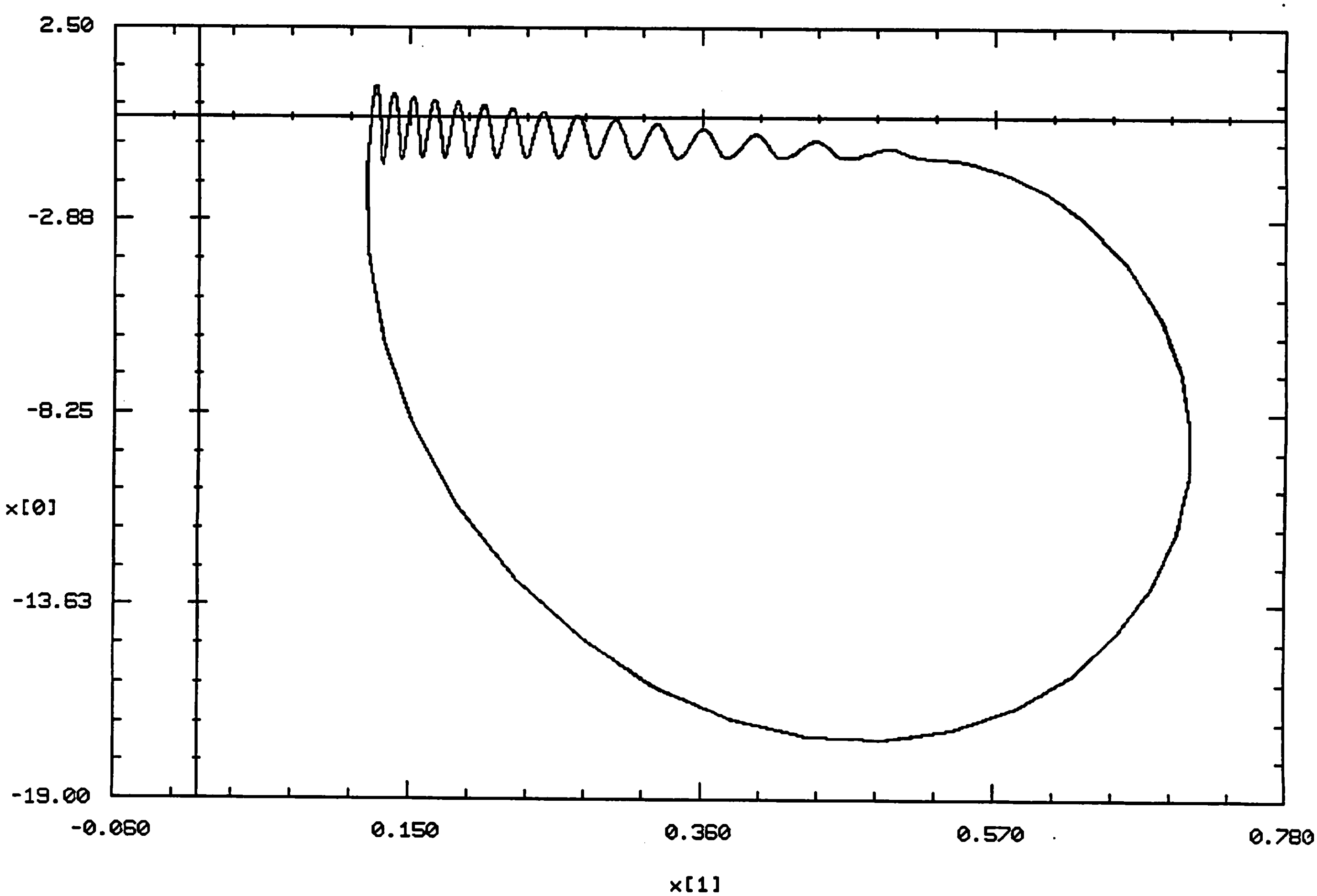


Fig.7

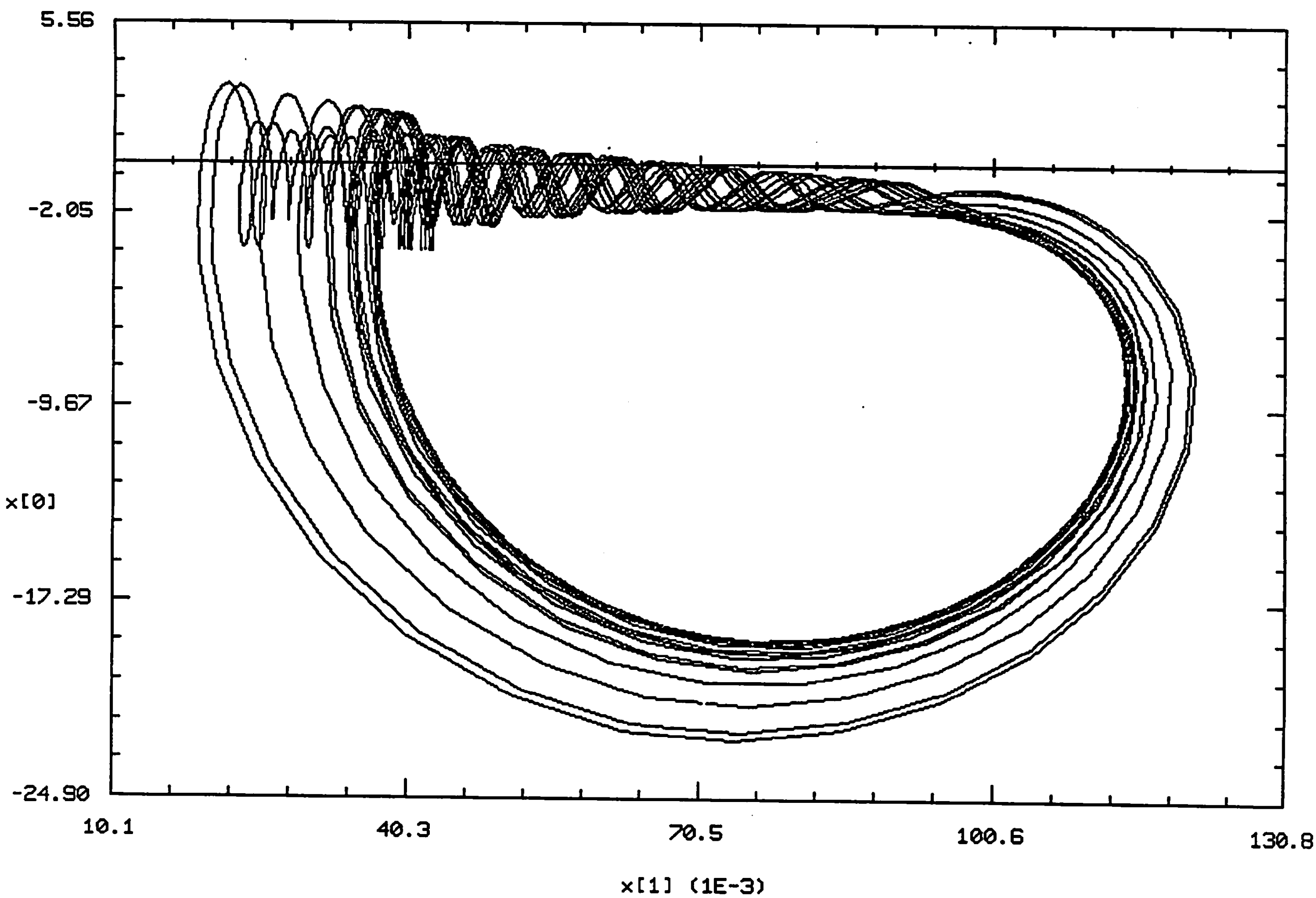


Fig.8

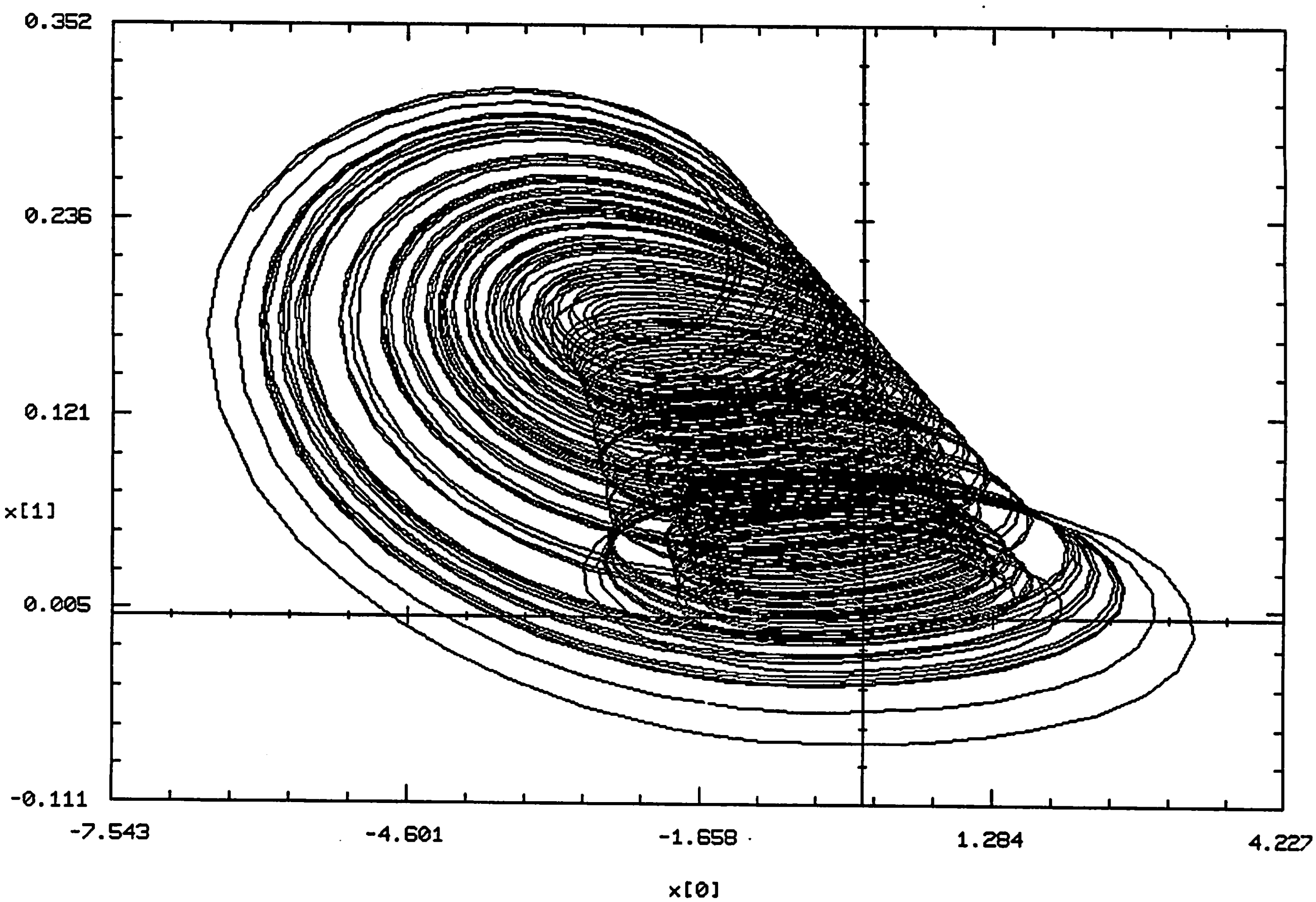


Fig.9

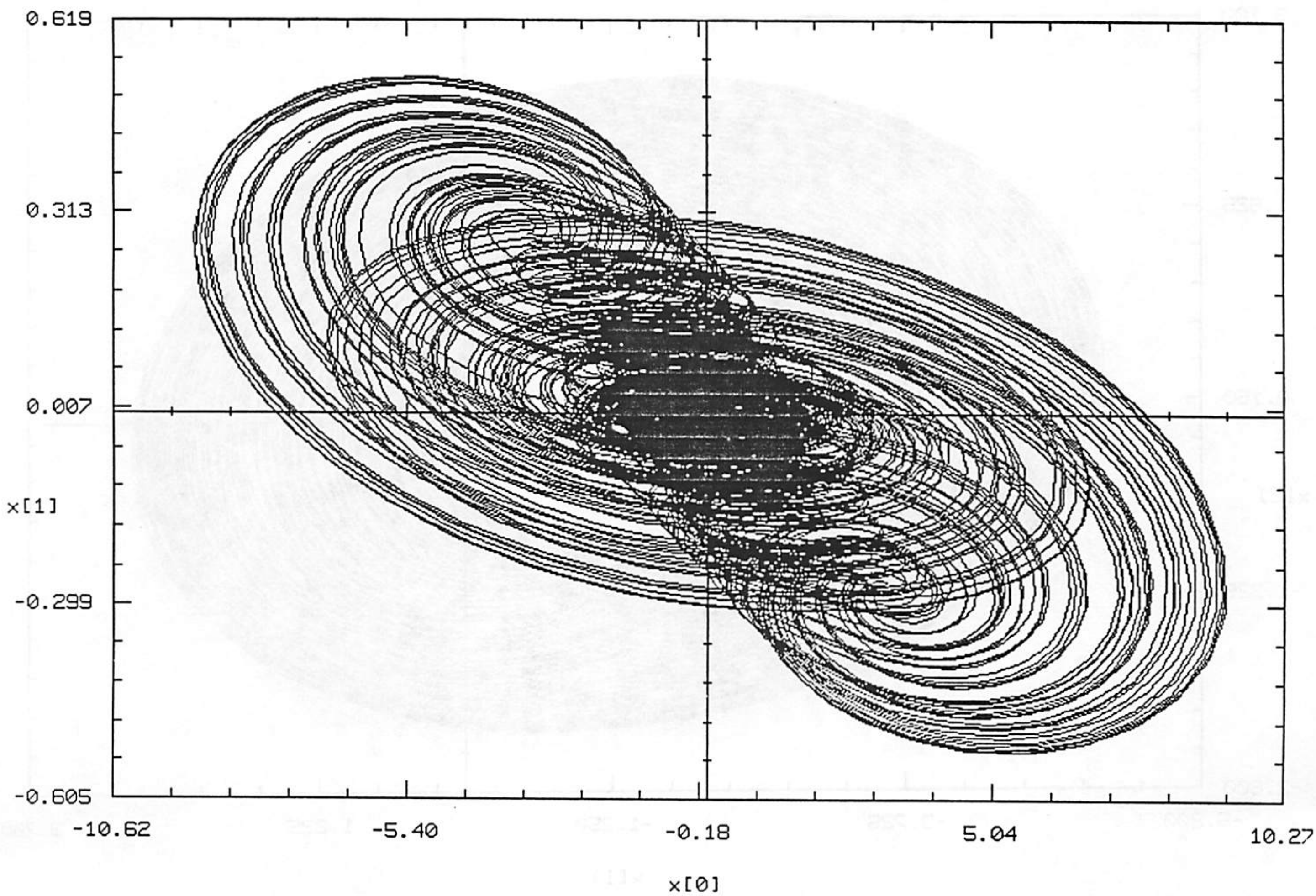


Fig.10

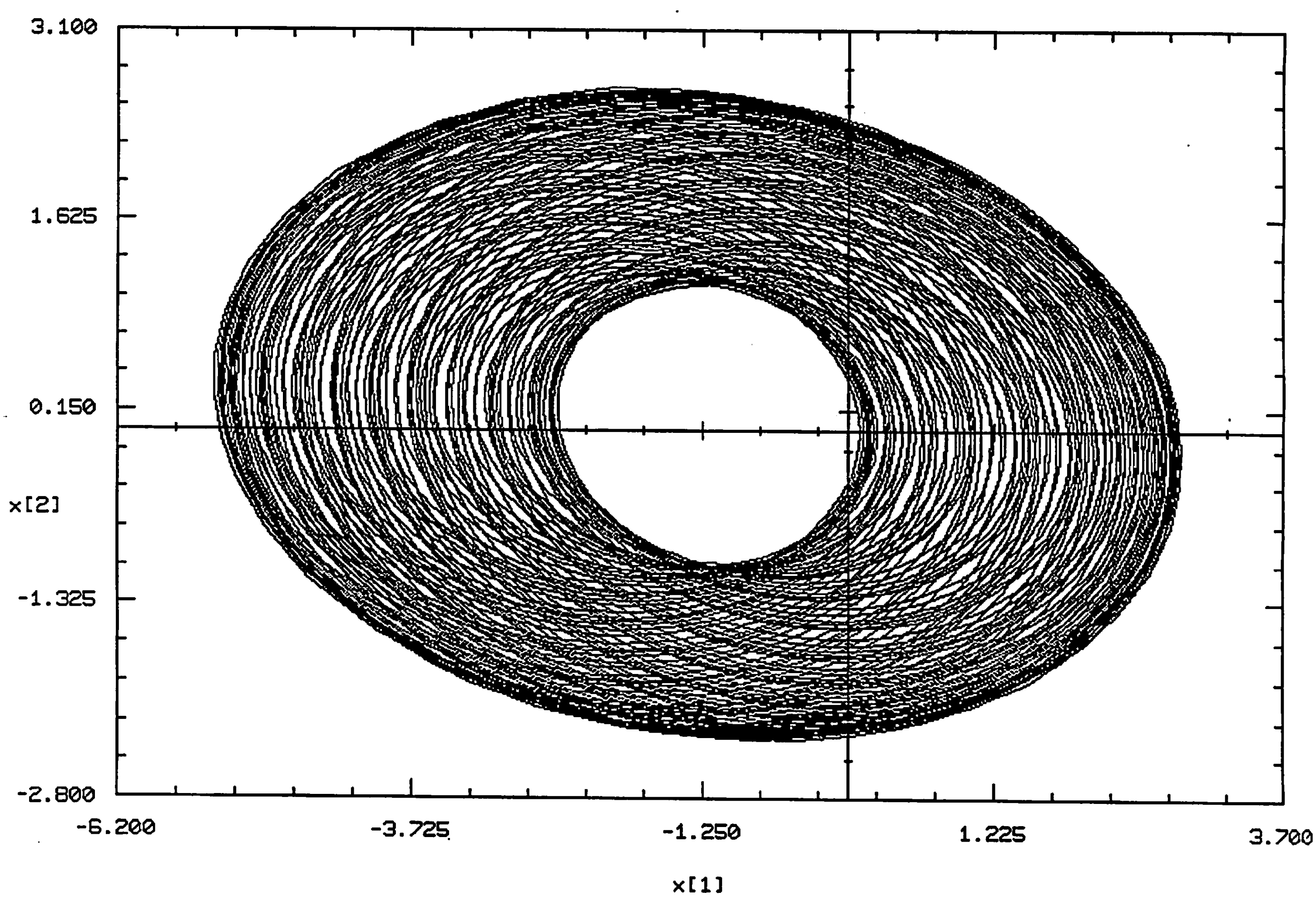
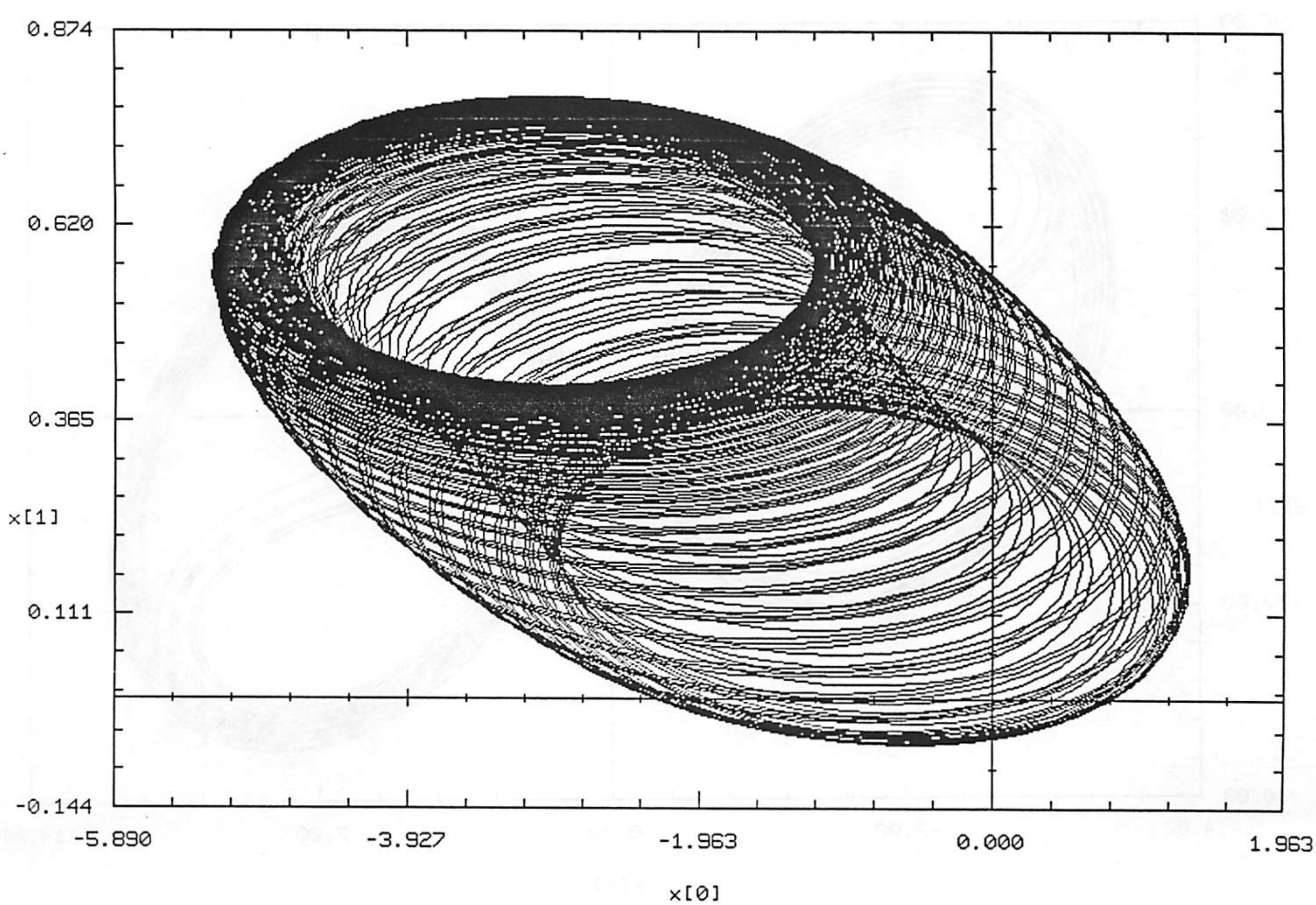


Fig.11



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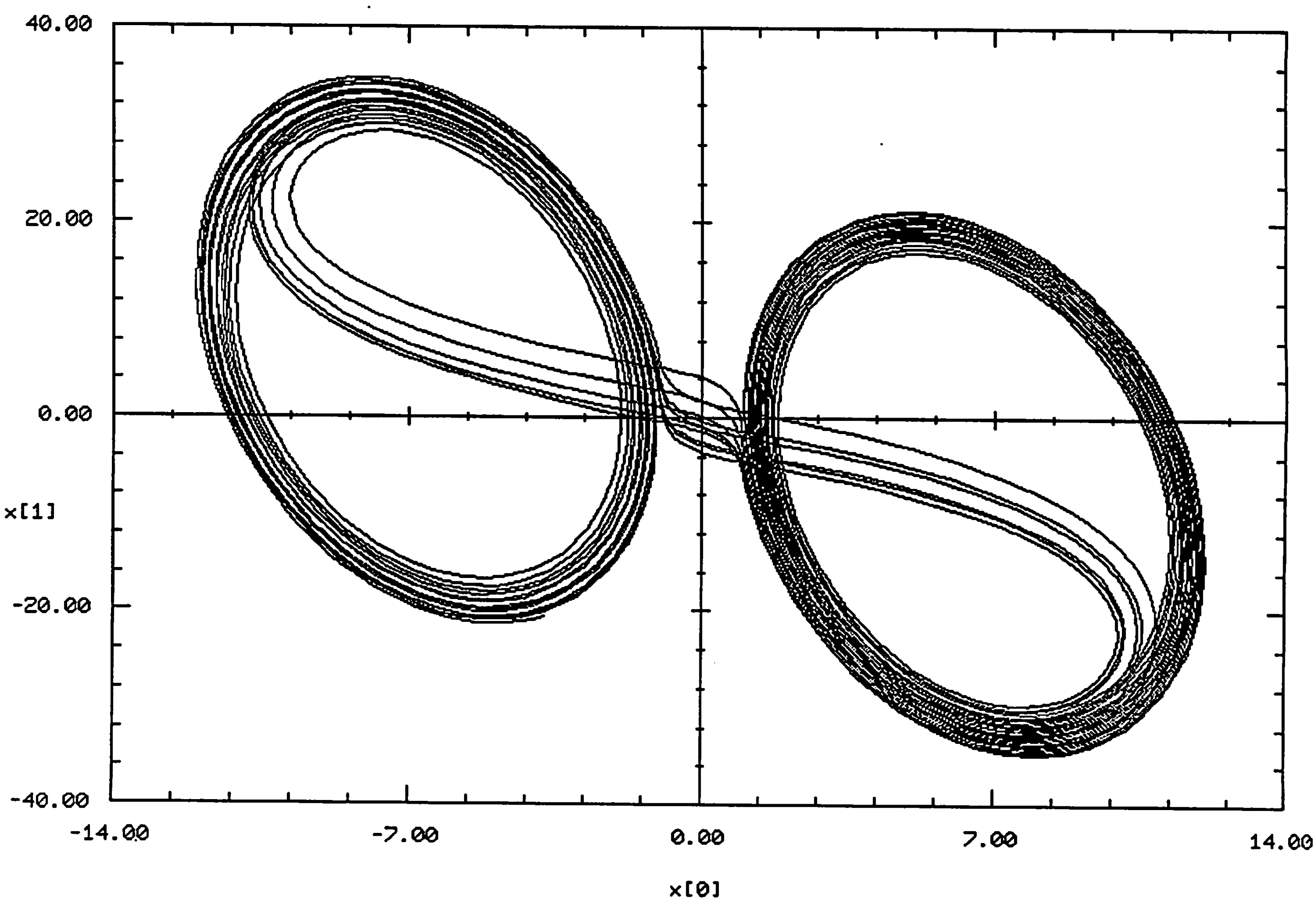

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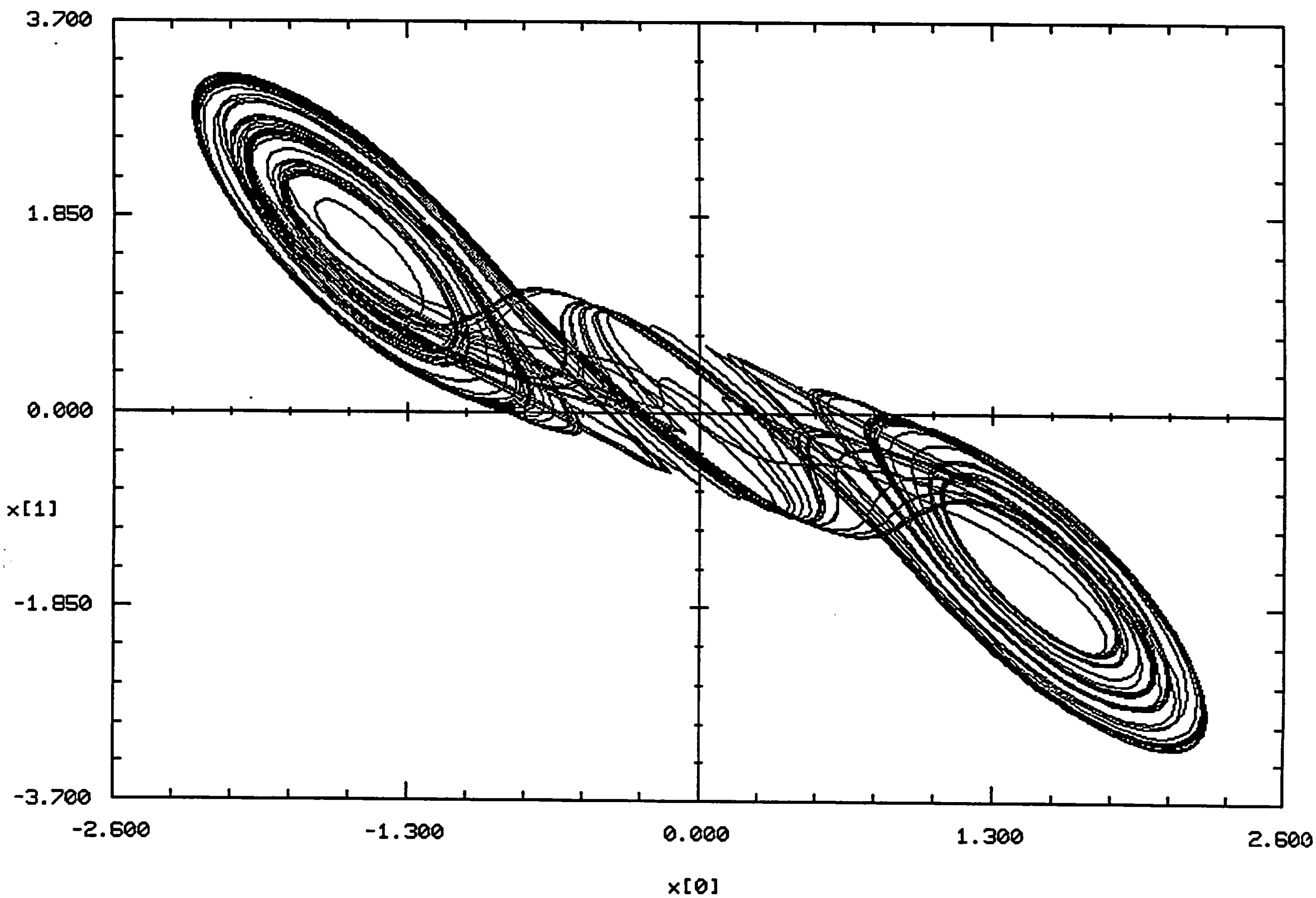

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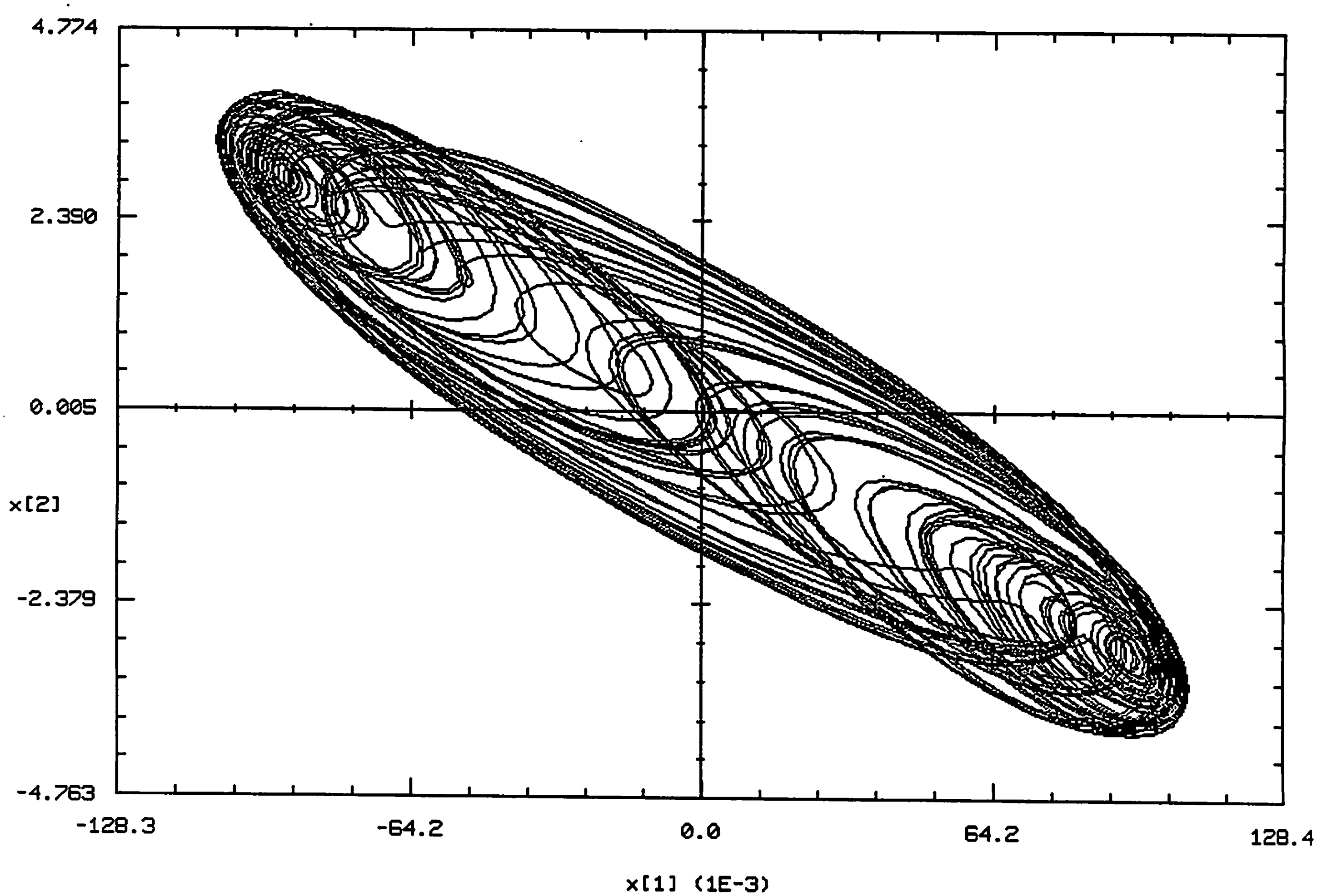
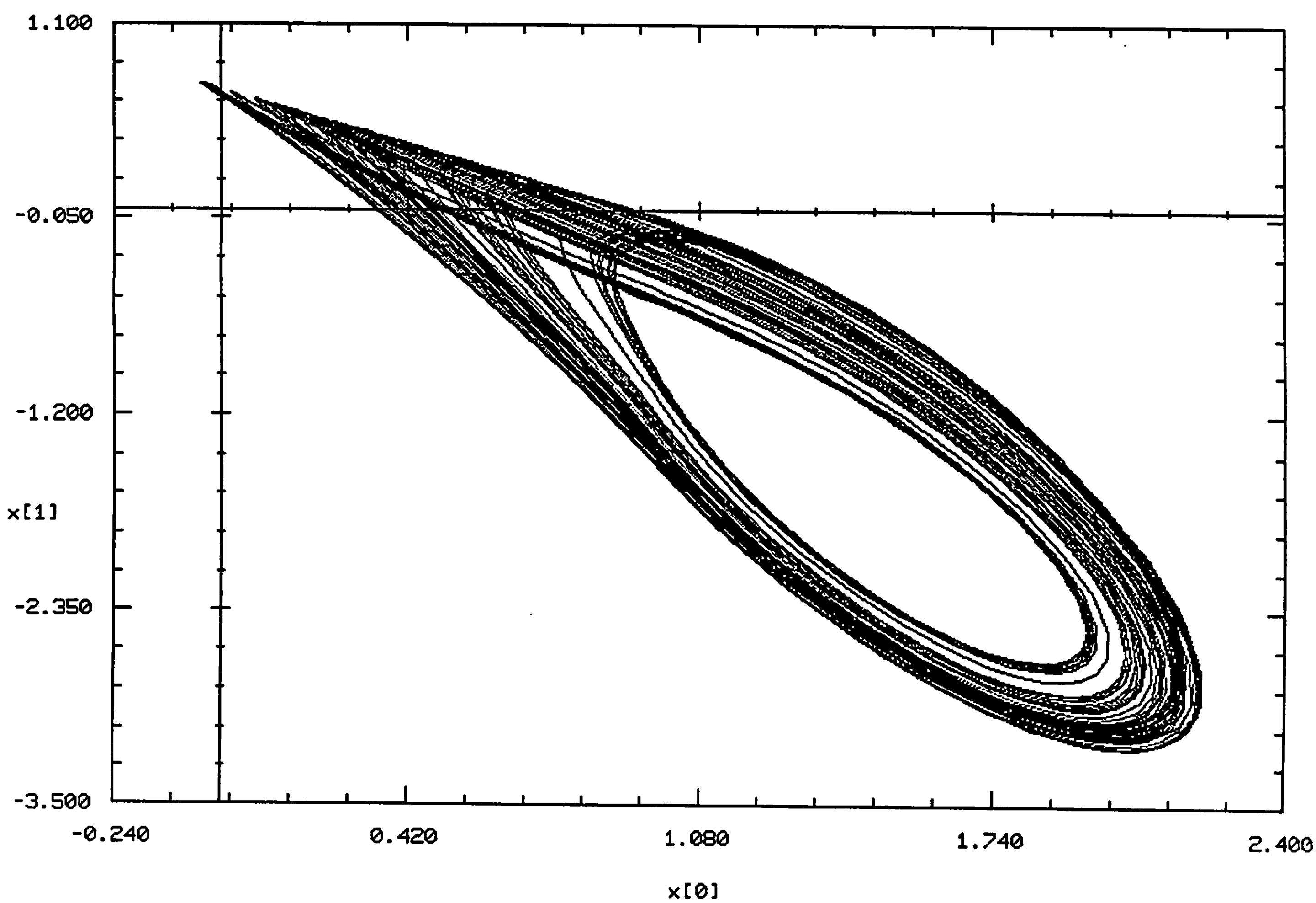


Fig.15



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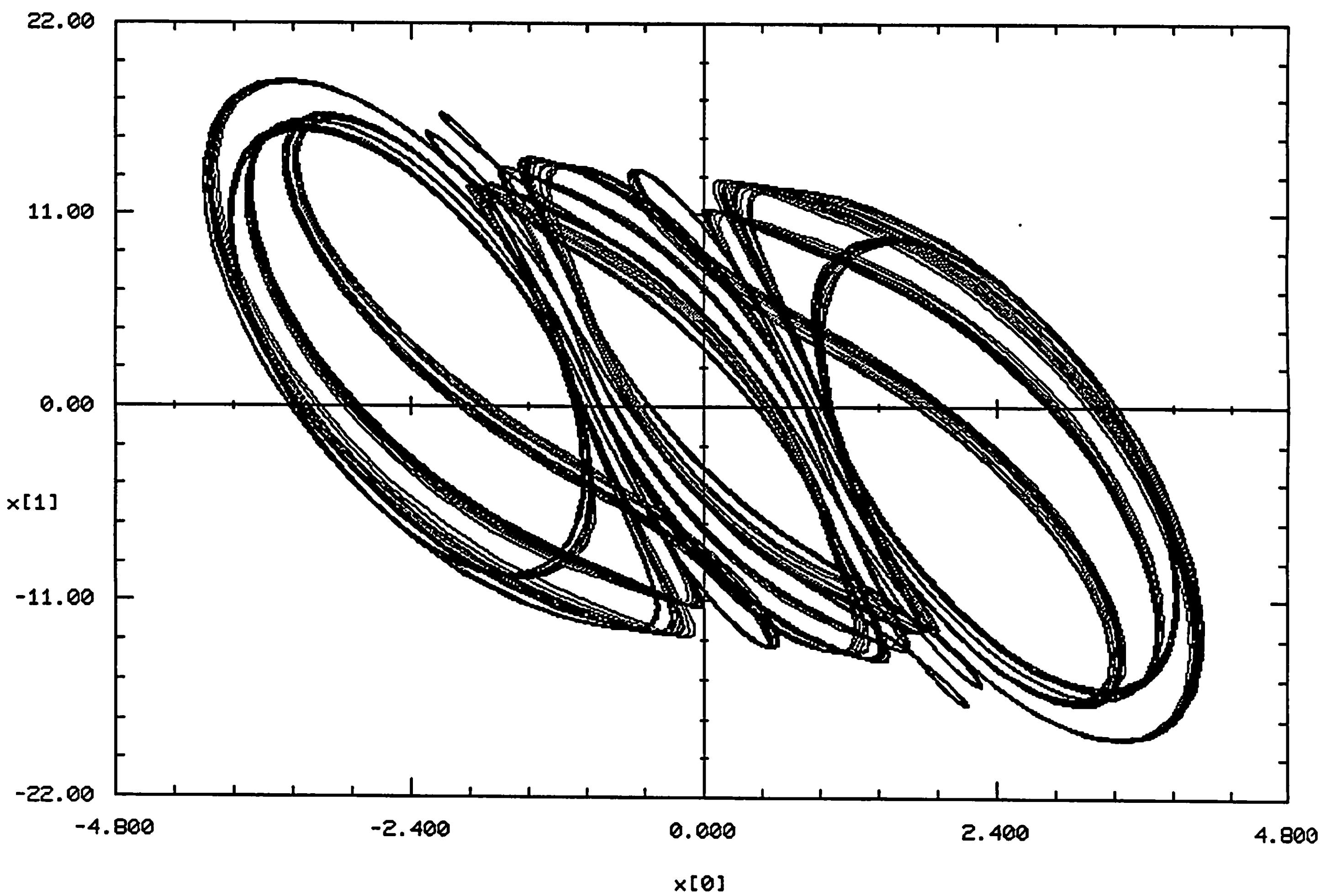

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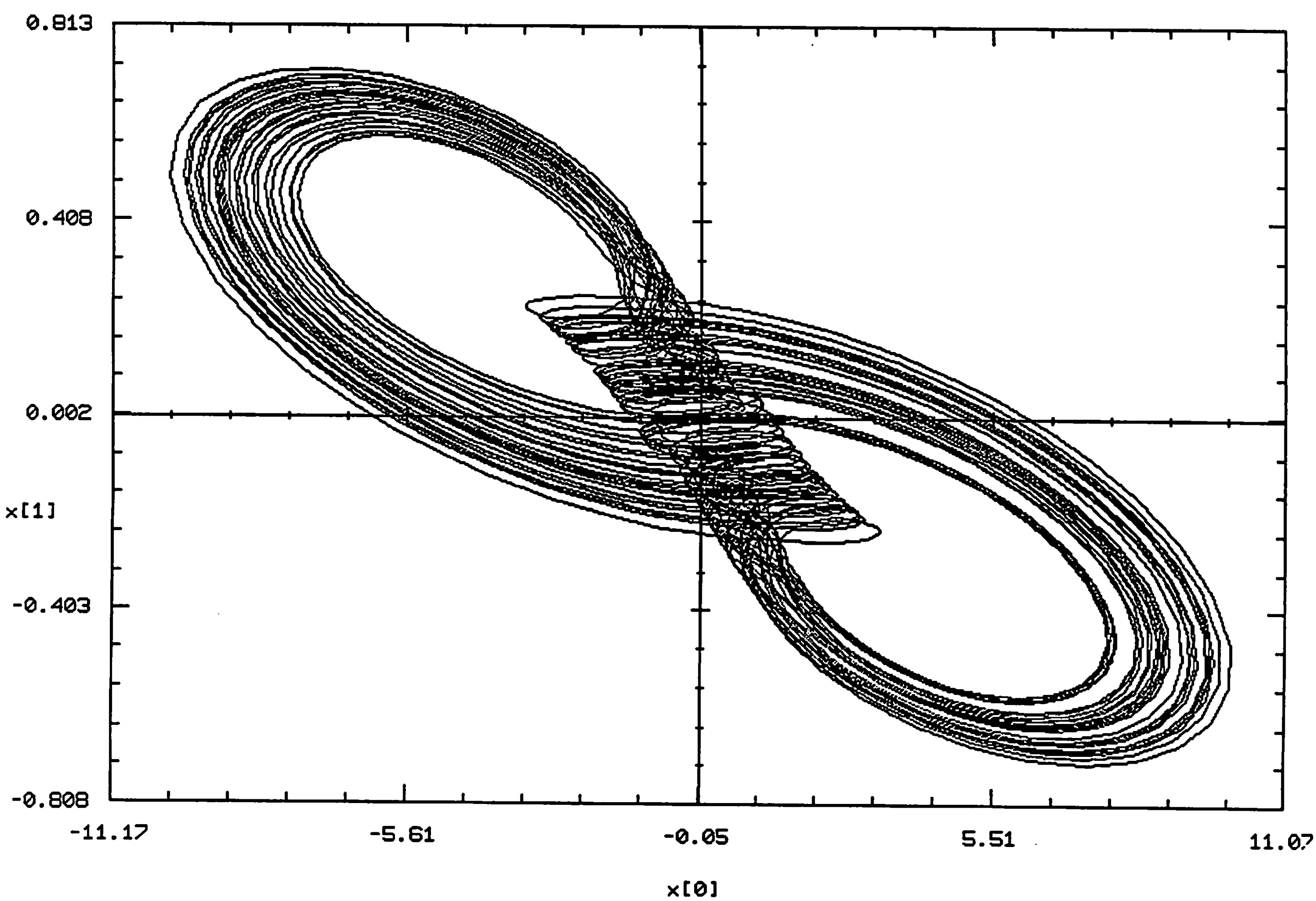

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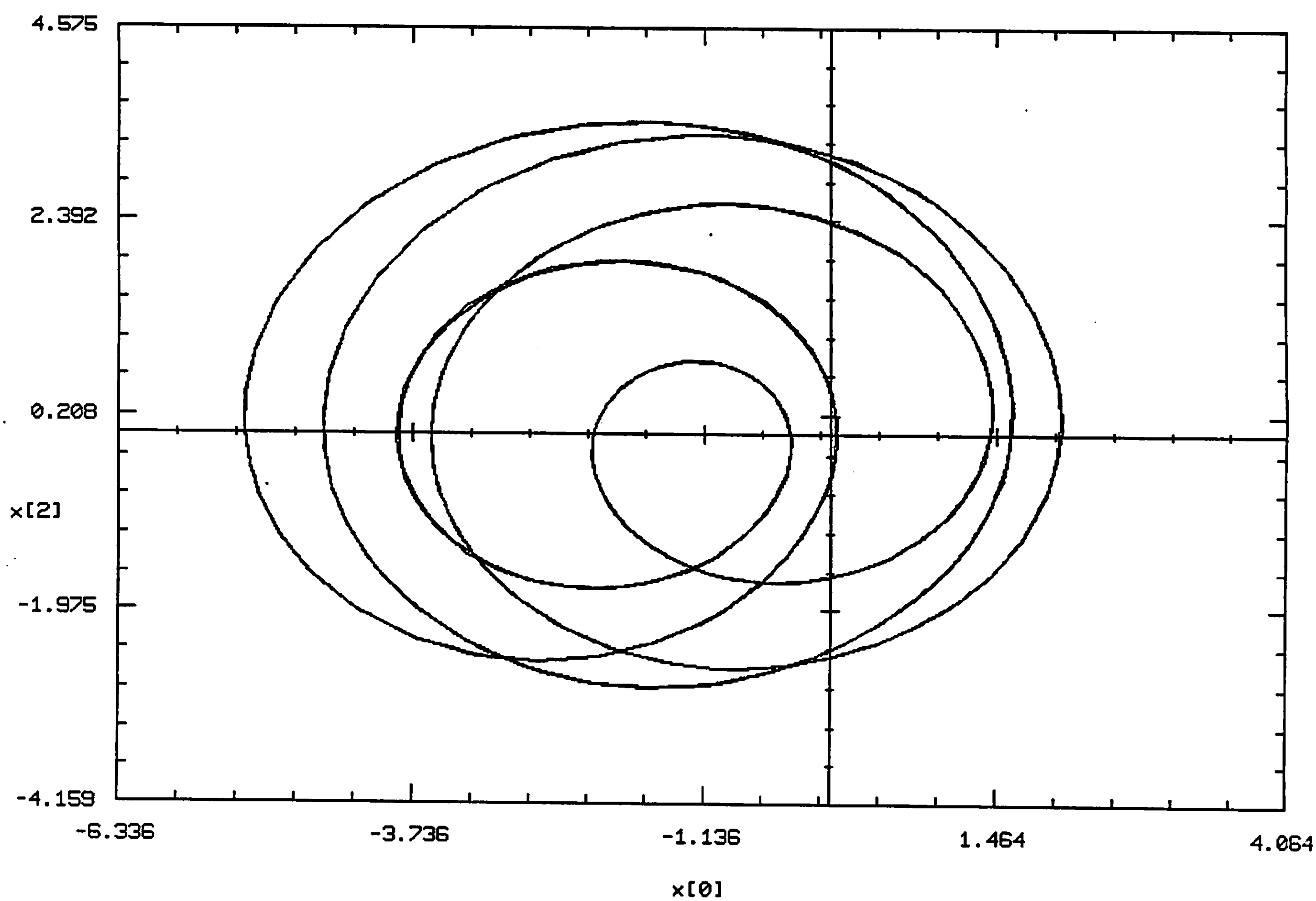

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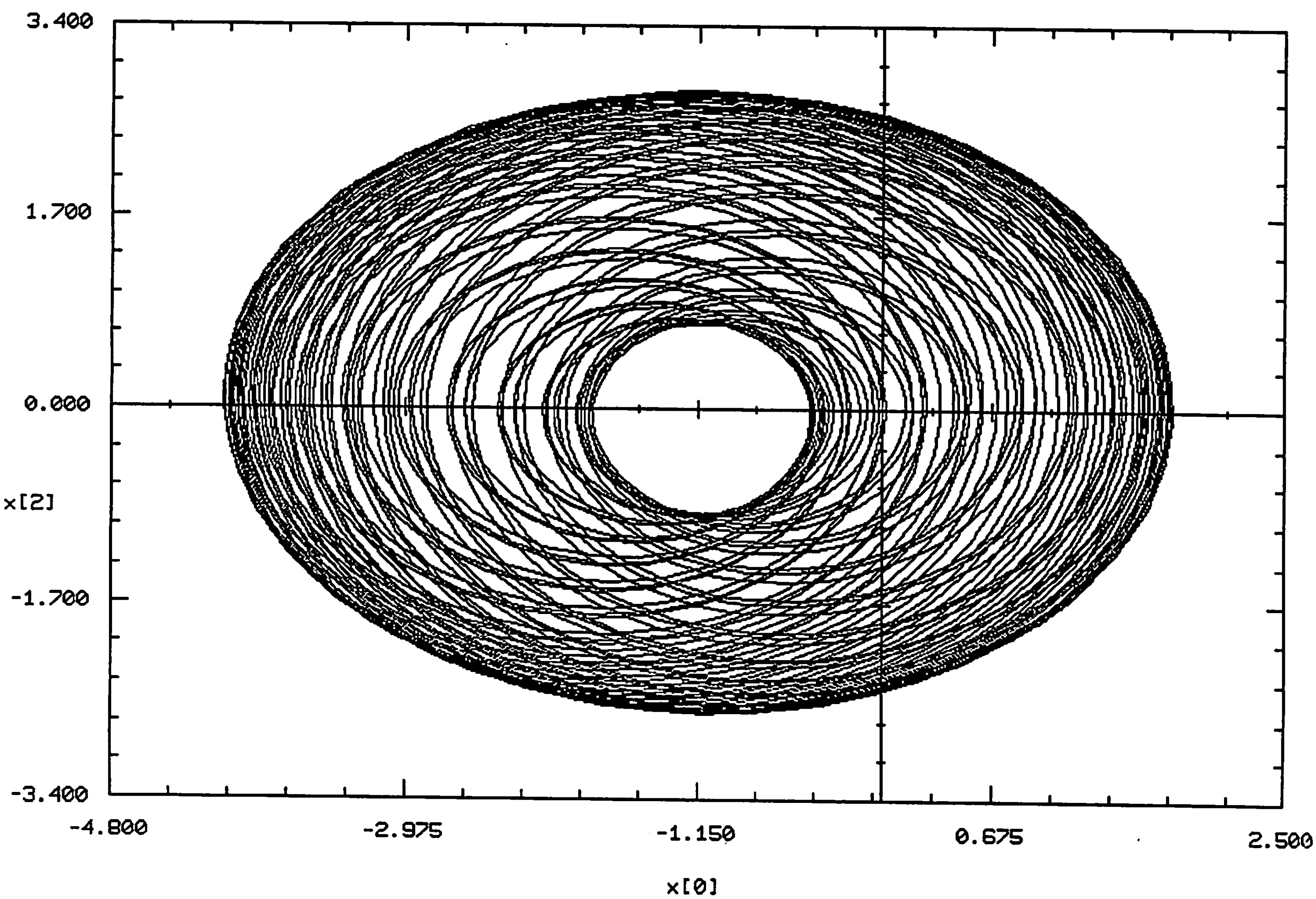

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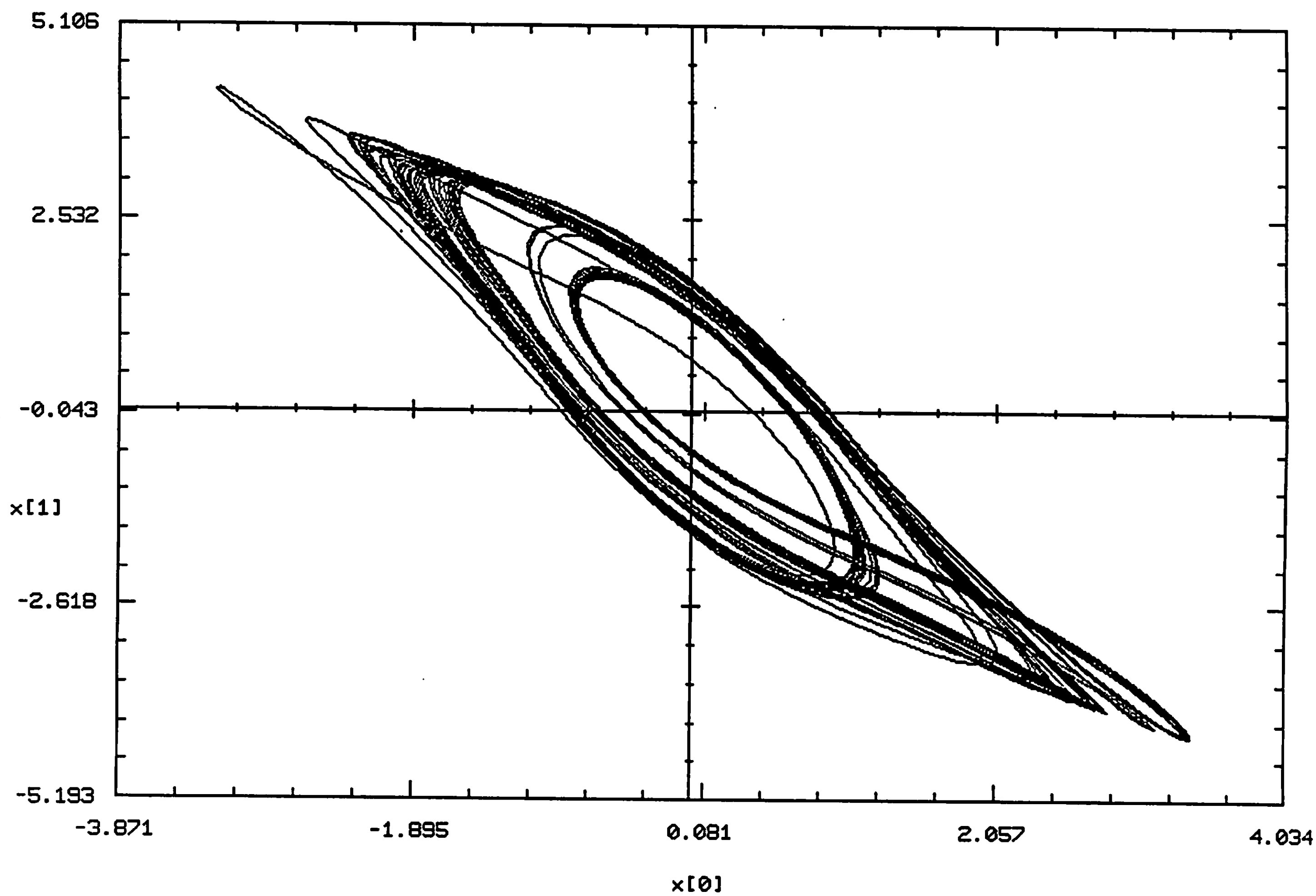

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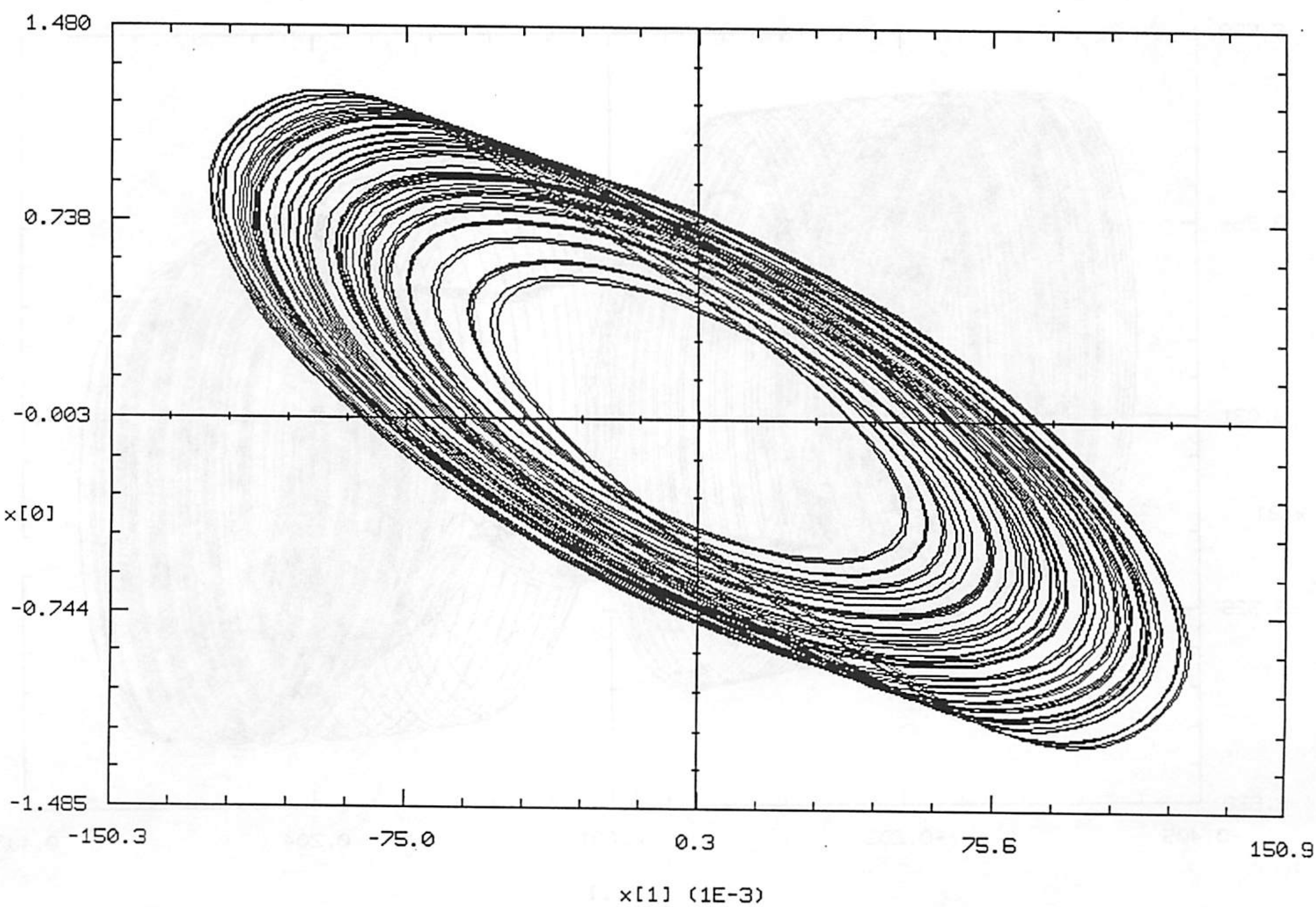
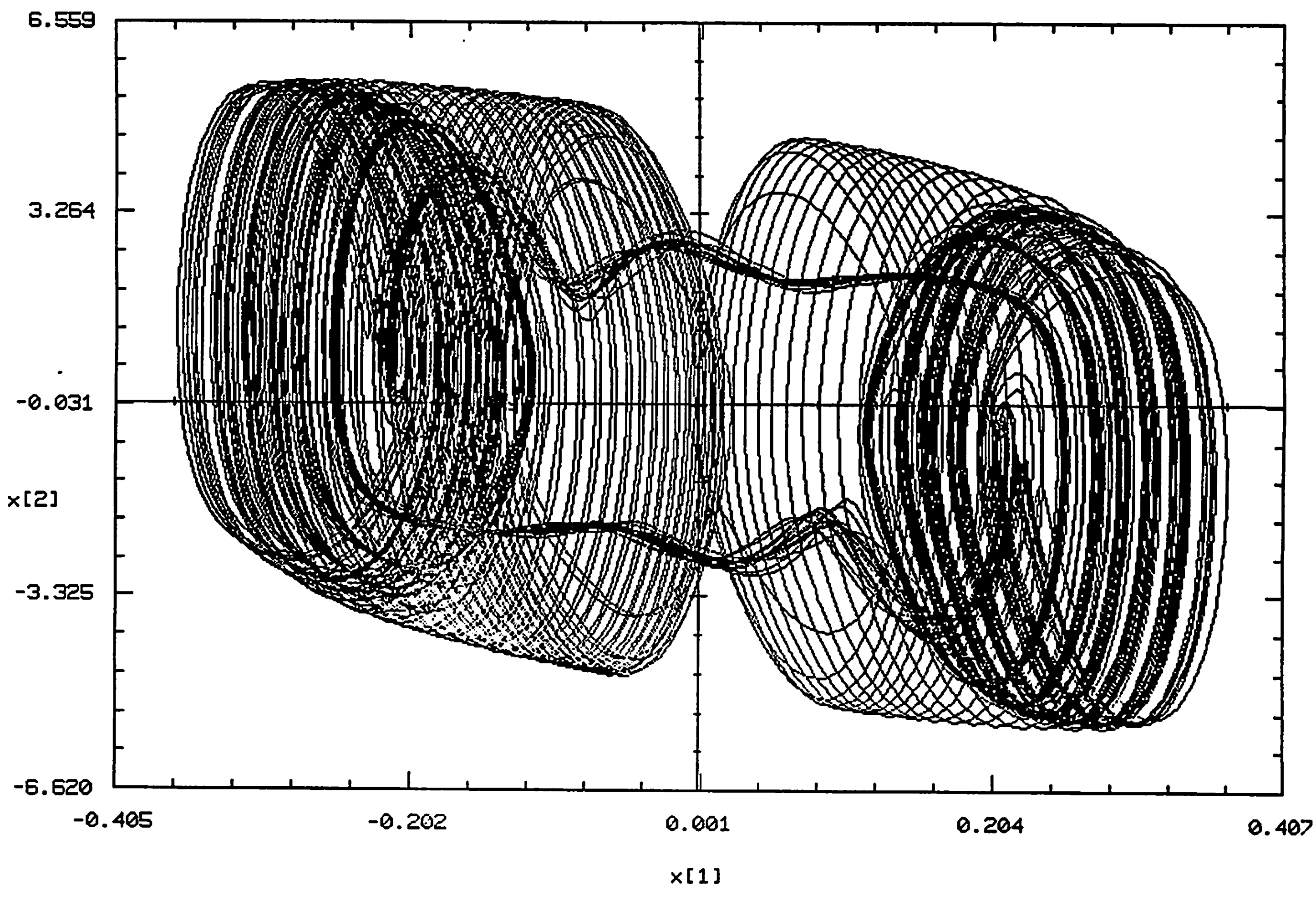


Fig.22



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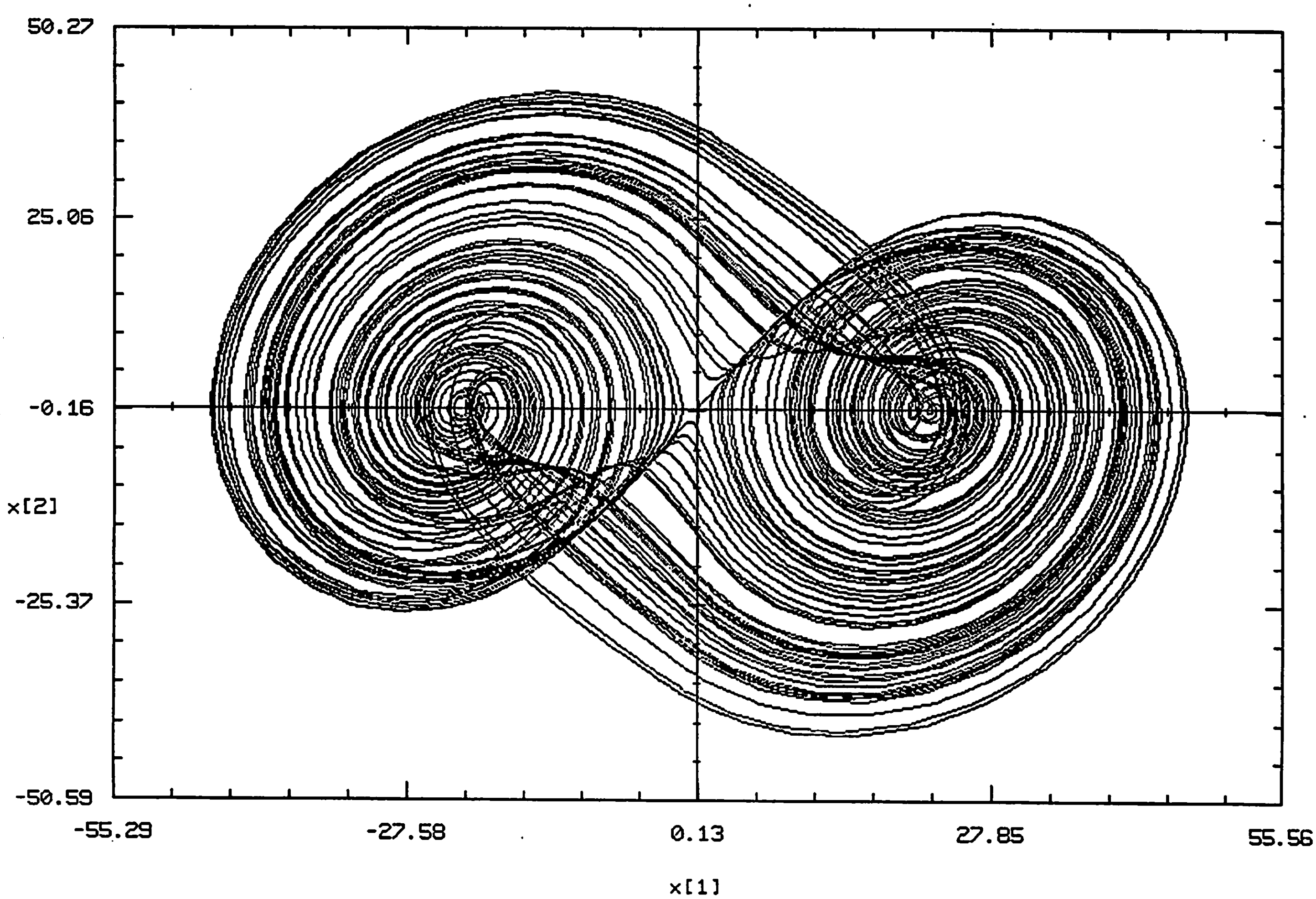

Fig.24