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Zero Dynamics of Regularly Perturbed Systems are Singularly Perturbed! †

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Abstract

In this paper we present new results on the structure of the zeros of linear and nonlinear systems under perturbation. In particular, we show that when state space descriptions of linear or nonlinear single-input single-output systems with relative degree ≥ 2 are regularly perturbed, then their zero dynamics are singularly perturbed and show a separation of time scales. In the SISO case, we give asymptotic formulas for the new *high frequency* zero dynamics arising from the regular perturbation.

1 Introduction

In this paper we show that regular perturbations of the state space descriptions of linear and nonlinear single-input single-output (SISO) systems of relative degree ≥ 2 result generically in the appearance of singularly perturbed (fast) zero dynamics. In other words, an effect of perturbations in the state space descriptions is the migration of some zeros from ∞ to finite locations in the complex plane. Depending on the sign of the regular perturbations, some of the perturbed zeros can migrate from ∞ to the right half of the complex plane. This leads to a reconsideration of minimum phase systems of high relative degree (pole-zero excess ≥ 2) as being only *dominantly* minimum phase since small perturbations result in right half plane zeros of large magnitude.

Our investigations in this direction were motivated in part by a study in [HSM88] of the linearization by nonlinear state feedback of a class of slightly non-minimum phase systems encountered in the flight control of VTOL aircraft. Indeed, in this work, the *true* system had a small *regular* perturbation in its equations caused by the way moments were generated on the aircraft. This, in turn, manifested itself as two zeros of large magnitude—one minimum phase and one *non-minimum* phase. We restrict ourselves to the SISO case here and postpone the considerably more technical MIMO case [HSK89].

This paper deals with both linear and nonlinear systems—definitions of zero dynamics for nonlinear systems have only recently become available [IM87,BI88]. The qualitative theory is similar for both classes of systems, though the techniques are rather different. The techniques also draw heavily from the literature on singular perturbation [KKO86,SOK84].

An outline of the paper is as follows: In Section 2, we develop explicit formulas for the locations of the large magnitude zeros of linear systems under perturbation. In Section 3, we repeat this development for the nonlinear case. Section 4 collects some concluding remarks.

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2 Linear Systems

In this section we will consider the effects of regular perturbations of b_0, c_0, A_0 on the zeros of a SISO linear system of the form

$$
\begin{aligned}
\dot{x} &= A_0 x + b_0 u \\
y &= c_0 x.
\end{aligned}
\tag{2.1}
$$

We will assume that the system (2.1) is *minimal* and has *relative degree* (excess of poles over finite zeros) γ_0, i.e.,

$$
\begin{gathered}
c_0 b_0 = c_0 A_0 b_0 = \cdots = c_0 A_0^{\gamma_0 - 2} b_0 = 0 \\
c_0 A_0^{\gamma_0 - 1} b_0 \neq 0.
\end{gathered}
\tag{2.2}
$$

To exhibit its $(n - \gamma_0)$ finite zeros, it is useful to use a normal form which will also prove convenient in the nonlinear case. To this end, we define

$$
\left[\begin{array}{c} \xi \\ \hline \eta \end{array}\right] := \left[\begin{array}{c} c_0 \\ c_0 A_0 \\ \vdots \\ c_0 A_0^{\gamma_0 - 1} \\ \hline H \end{array}\right] x =: \left[\begin{array}{c} \Xi \\ \hline H \end{array}\right] x
\tag{2.3}
$$

with $\xi \in \mathbf{R}^{\gamma_0}$, $\eta \in \mathbf{R}^{n-\gamma_0}$ such that $(\xi^T, \eta^T)^T$ is a change of coordinates on the original state space, i.e., $\begin{bmatrix} \Xi \\ H \end{bmatrix} \in \mathbf{R}^{n \times n}$ is nonsingular. Further, from the definition of relative degree in (2.2), we may choose H so that $Hb_0 = 0$. The linear system (2.1) can be rewritten in (ξ, η) coordinates as

$$
\left[\begin{array}{c} \dot{\xi} \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} & 0 \\ a_1^T & a_2^T \\ \hline P & Q \end{array}\right] \left[\begin{array}{c} \xi \\ \hline \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ c_0 A_0^{\gamma_0 - 1} b_0 \\ \hline 0 \end{array}\right] u.
\tag{2.4}
$$

Here, $a_1 \in \mathbf{R}^{\gamma_0}$, $a_2 \in \mathbf{R}^{n-\gamma_0}$, $P \in \mathbf{R}^{n-\gamma_0 \times \gamma_0}$, $Q \in \mathbf{R}^{n-\gamma_0 \times n-\gamma_0}$, and

$$
\begin{aligned}
c_0 A_0^{\gamma_0 - 1} x &= a_1^T \xi + a_2^T \eta \\
H A_0 x &= P\xi + Q\eta.
\end{aligned}
\tag{2.5}
$$

The form (2.4) is referred to as a *normal form* and it is well known that the $(n - \gamma_0)$ eigenvalues of Q are the zeros of the system (2.1). It is useful to note that the state feedback law

$$
u = -\frac{1}{c_0 A_0^{\gamma_0 - 1} b_0} (a_1^T \xi + a_2^T \eta)
\tag{2.6}
$$

renders the η variables unobservable and, furthermore, it *zeros the output* for all t, that is,

$$
y(t) = \dot{y}(t) = \cdots = y^{(\gamma_0 - 1)}(t) \equiv 0.
\tag{2.7}
$$

The subspace

$$
\begin{aligned}
\mathcal{V}_0 &= \left\{ x : c_0 x = c_0 A_0 x = \cdots = c_0 A_0^{\gamma_0 - 1} x = 0 \right\} \\
&= \{(0, \eta) : \eta \in \mathbf{R}^{n-\gamma_0}\}
\end{aligned}
\tag{2.8}
$$

(rendered invariant by (2.6)) is referred to as the *zero dynamics subspace* of (2.1).

2.1 Perturbations in b

To begin with, we study the effects of perturbations in the input channel alone, i.e., systems of the form

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u + \epsilon b_1 u \\ y &= c_0 x. \end{aligned} \tag{2.9}$$

Note that (2.9) remains minimal for ϵ small. Let the relative degree of the *perturbation system* (c_0, A_0, b_1) be γ_1. The case of greatest interest[‡] is $\gamma_1 < \gamma_0$, which, by the definition of γ_1, implies that

$$\begin{gathered} c_0 b_1 = c_0 A_0 b_1 = \cdots = c_0 A_0^{\gamma_1 - 2} b_1 = 0 \\ c_0 A_0^{\gamma_1 - 1} b_1 \neq 0 \end{gathered} \tag{2.10}$$

and that, for ϵ small enough, the relative degree of the *perturbed system* (2.9) will be γ_1. It is easy to obtain the form of (2.9) in the (ξ, η) coordinates defined in (2.3). It will, however, be convenient for us to decompose the ξ of (2.3) as $\xi^T = (\xi_1^T, \xi_2^T)$ with

$$\begin{aligned} \xi_1 &:= \begin{bmatrix} c_0 \\ c_0 A_0 \\ \vdots \\ c_0 A_0^{\gamma_1 - 1} \end{bmatrix} x \quad \in \mathbf{R}^{\gamma_1}, \\ \xi_2 &:= \begin{bmatrix} c_0 A_0^{\gamma_1} \\ \vdots \\ c_0 A_0^{\gamma_0 - 1} \end{bmatrix} x \quad \in \mathbf{R}^{\gamma_0 - \gamma_1}. \end{aligned} \tag{2.11}$$

Note that if $\gamma_0 = \gamma_1$, ξ_2 fails to exist. Also note that we have already used the assumption that $\gamma_1 \leq \gamma_0$ in the definition (2.11). The system (2.9) is now written as

$$\begin{bmatrix} \dot{\xi}_1 \\ \hline \dot{\xi}_2 \\ \hline \dot{\eta} \end{bmatrix} = \left[\begin{array}{cccc|cccc|c} 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & 0 & & & 0 \\ & & \ddots & 1 & & & & & \\ & & & 0 & 1 & & & & \\ \hline & & & & 0 & 1 & & & \\ & 0 & & & & \ddots & \ddots & & 0 \\ & & & & & & 0 & 1 & \\ & {}^1a_1^T & & & & {}^2a_1^T & & & a_2^T \\ \hline & {}^1P & & & & {}^2P & & & Q \end{array}\right] \begin{bmatrix} \xi_1 \\ \hline \xi_2 \\ \hline \eta \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \epsilon c_0 A_0^{\gamma_1 - 1} b_1 \\ \hline \epsilon c_0 A_0^{\gamma_1} b_1 \\ \vdots \\ \epsilon c_0 A_0^{\gamma_0 - 2} b_1 \\ c_0 A_0^{\gamma_0 - 1}(b_0 + \epsilon b_1) \\ \hline \epsilon H b_1 \end{bmatrix} u. \tag{2.12}$$

where ${}^1a_1 \in \mathbf{R}^{\gamma_1}$, ${}^2a_1 \in \mathbf{R}^{\gamma_0 - \gamma_1}$, ${}^1P \in \mathbf{R}^{(n-\gamma_0)\times\gamma_1}$, ${}^2P \in \mathbf{R}^{(n-\gamma_0)\times(\gamma_0-\gamma_1)}$, and

$$a_1^T = ({}^1a_1^T \; {}^2a_1^T)^T, \qquad P = [{}^1P \; {}^2P]. \tag{2.13}$$

Note that, in (2.12), the perturbations appear as input terms in the equations for $\dot{\xi}_{1\gamma_1}, \dot{\xi}_{21}, \ldots, \dot{\xi}_{2,\gamma_0-\gamma_1}$, and $\dot{\eta}$. To find the zero dynamics of (2.12), we use the state feedback

$$u = -\frac{1}{\epsilon c_0 A_0^{\gamma_1 - 1} b_1} \xi_{21} = -\frac{1}{\epsilon c_0 A_0^{\gamma_1 - 1} b_1} c_0 A_0^{\gamma_1} x \tag{2.14}$$

[‡]See the remarks after Theorem 2.1.

to *zero the output* making the subspace

$$\begin{aligned}\mathcal{V}_1 &= \left\{x : c_0x = c_0A_0x = \cdots = c_0A_0^{\gamma_1-1}x = 0\right\} \\ &= \{(0, \xi_2, \eta) : \xi_2 \in \mathbf{R}^{\gamma_0-\gamma_1}, \eta \in \mathbf{R}^{n-\gamma_0}\}\end{aligned} \tag{2.15}$$

invariant and the state variables $(\xi_2, \eta) \in \mathbf{R}^{n-\gamma_1}$ unobservable. The dynamics of the (ξ_2, η) variables on $\mathcal{V}_1$ are given by

$$\left[\begin{array}{c}\dot{\xi}_{21}\\ \vdots \\ \vdots \\ \dot{\xi}_{2,\gamma_0-\gamma_1} \\ \hline \dot{\eta}\end{array}\right] = \left[\begin{array}{cccccc|c} -\frac{c_0A_0^{\gamma_1}b_1}{c_0A_0^{\gamma_1-1}b_1} & 1 & & & & & \\ -\frac{c_0A_0^{\gamma_1+1}b_1}{c_0A_0^{\gamma_1-1}b_1} & 0 & 1 & & & & \\ \vdots & \vdots & \ddots & \ddots & & & 0 \\ -\frac{c_0A_0^{\gamma_0-2}b_1}{c_0A_0^{\gamma_1-1}b_1} & 0 & \cdots & 0 & & 1 & \\ -\frac{c_0A_0^{\gamma_0-1}b_1}{c_0A_0^{\gamma_1-1}b_1} - \frac{1}{\epsilon}\frac{c_0A_0^{\gamma_0-1}b_0}{c_0A_0^{\gamma_1-1}b_1} + {}^2a_{11} & {}^2a_{12} & \cdots & \cdots & & {}^2a_{1,\gamma_0-\gamma_1} & a_2^T \\ \hline & & {}^2\tilde{P} & & & & Q \end{array}\right] \left[\begin{array}{c}\xi_{21}\\ \vdots \\ \vdots \\ \xi_{2,\gamma_0-\gamma_1} \\ \hline \eta\end{array}\right] \tag{2.16}$$

where

$${}^2\tilde{P} = \left[\begin{array}{cc} -\frac{1}{c_0A_0^{\gamma_1-1}b_1}Hb_1 & 0 \end{array}\right] + {}^2P. \tag{2.17}$$

Thus, we see that the system (2.9) has $(n-\gamma_1) \geq (n-\gamma_0)$ zeros. To establish the structure of these zeros, we note that the ϵ-dependent term in (2.16) corresponding to $\dot{\xi}_{2,\gamma_0-\gamma_1}$ is of order $1/\epsilon$ and thus is certainly not a *regular* perturbation. This *singular* perturbation term is due to the *high-gain* form of the feedback control (2.14). The rich literature on high-gain systems (see, in particular, [Mar88,San83]) is therefore applicable to the study of perturbed zero dynamics.

Theorem 2.1 *The linear system (2.9) has $(n-\gamma_1)$ zeros which, according to their asymptotic behavior as $\epsilon \longrightarrow 0$, belong to two groups:*

- *The $(\gamma_0-\gamma_1)$ large zeros tend to ∞ asymptotically as*

$$\left(-\frac{1}{\epsilon}\frac{c_0A_0^{\gamma_0-1}b_0}{c_0A_0^{\gamma_1-1}b_1}\right)^{\frac{1}{\gamma_0-\gamma_1}}. \tag{2.18}$$

- *The remaining $(n-\gamma_0)$ zeros tend to the zeros of the* unperturbed *system (2.1).*

Proof To facilitate an asymptotic calculation of the eigenvalues of the matrix in (2.16), we transform the system into a standard singular perturbation form. To this end, we rescale ξ_2 as follows

$$\tilde{\xi}_{21} = \xi_{21},\ \tilde{\xi}_{22} = \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\xi_{22},\ \cdots,\ \tilde{\xi}_{2,\gamma_0-\gamma_1} = \epsilon^{\frac{\gamma_0-\gamma_1-1}{\gamma_0-\gamma_1}}\xi_{2,\gamma_0-\gamma_1} \tag{2.19}$$

and rewrite (2.16) as

$$\begin{aligned}\epsilon^{\frac{1}{\gamma_0-\gamma_1}}\dot{\tilde{\xi}}_2 &= W\tilde{\xi}_2 \qquad + \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\Gamma\tilde{\xi}_2 + O\left(\epsilon^{\frac{2}{\gamma_0-\gamma_1}}\right) \\ \dot{\eta} &= {}^2\tilde{P}\tilde{\xi}_2 + Q\eta\end{aligned} \tag{2.20}$$

where

$$W := \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \ddots \\ 0 & & \ddots & 1 \\ -\frac{c_0 A_0^{\gamma_0-1} b_0}{c_0 A_0^{\gamma_1-1} b_1} & 0 & \cdots & 0 \end{bmatrix} \text{ and } \Gamma := \begin{bmatrix} \left(-\frac{c_0 A_0^{\gamma_1} b_1}{c_0 A_0^{\gamma_1-1} b_1}\right) & 1 & & \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & {}^2a_{1,\gamma_0-\gamma_1} \end{bmatrix}. \tag{2.21}$$

To see that (2.20) is in the standard two-time-scale form of [KKO86], note that its right hand side is regularly perturbed by $\epsilon^{\frac{1}{\gamma_0-\gamma_1}}\Gamma$, while the matrix of the unperturbed part is block lower triangular. By inspection, the upper diagonal block is nonsingular as required for a standard form. It follows that the eigenvalues of (2.16) are asymptotically

$$\epsilon^{-\frac{1}{\gamma_0-\gamma_1}} \cdot \lambda(W) \cup \lambda(Q). \tag{2.22}$$

Clearly, the eigenvalues of Q are the $(n-\gamma_0)$ zeros of the unperturbed system. It is easy to see [Wil65, chapter 2] that the remaining $(\gamma_0-\gamma_1)$ eigenvalues are the $(\gamma_0-\gamma_1)$th roots of $(-\frac{c_0 A_0^{\gamma_0-1} b_0}{c_0 A_0^{\gamma_1-1} b_1})$ multiplied by $\epsilon^{-\frac{1}{\gamma_0-\gamma_1}}$, that is

$$\left(-\frac{1}{\epsilon}\frac{c_0 A_0^{\gamma_0-1} b_0}{c_0 A_0^{\gamma_1-1} b_1}\right)^{\frac{1}{\gamma_0-\gamma_1}}. \tag{2.23}$$

This is the asymptotic expression of the $(\gamma_0-\gamma_1)$ large zeros of the perturbed system which tend to ∞ as $\epsilon \to 0$. The remaining $(n-\gamma_0)$ tend as a set to the eigenvalues of Q (zeros of the unperturbed system). □

Remarks

1. Theorem 2.1 states that if the relative degree γ_1 of the perturbation ϵb_1 is less than that of the original b_0, then $(\gamma_0-\gamma_1)$ *of the original system's infinite zeros become finite* according to the asymptotic formula (2.23).

2. We leave it to the reader to verify (by direct calculation) that, if the relative degree of the perturbation b_1 is $\gamma_1 \geq \gamma_0$, then both the perturbed and unperturbed systems have the same number of zeros and the zero locations are a smooth function of ϵ.

3. Theorem 2.1 has important implications for the concepts of non-minimum phase and minimum phase. In particular, if $\gamma_0-\gamma_1 > 2$, it follows that arbitrarily small perturbations of the form (2.9) result in non-minimum phase systems since, for $\gamma_0-\gamma_1 > 2$, at least one of the roots of (2.18) is in the right half plane. Of course, for ϵ small enough, the non-minimum phase zeros are far off in the right half plane prompting us to think of the perturbed system as being *slightly non-minimum phase*. Nevertheless, numerous system theory results based on a *strict minimum phase assumption* should be reexamined in this light.

4. Even when $\gamma_0-\gamma_1 = 1$, the relative signs of the quantities in (2.18) *may result in right half plane zeros.* In particular, some zeros *will* be in the right half plane either when ϵ is positive or when ϵ is negative.

5. Note that, if a perturbation resulting in direct feed-through ($y = c_0x + \epsilon d_1 u$) were allowed, then γ_1 would be 0 and the asymptotes would coincide with the familiar *root locus* asymptotes.

2.2 Perturbations in c

Consider now the effects of perturbations of (2.1) in the output channel, i.e.,

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u \\ y &= c_0 x + \epsilon c_1 x. \end{aligned} \tag{2.24}$$

As before, if γ_1 represents the relative degree of the perturbation system (c_1, A_0, b_0), it follows from considerations dual to those given above that if $\gamma_1 < \gamma_0$ the system (2.24) has $(\gamma_0 - \gamma_1)$ extra zeros given asymptotically by the $(\gamma_0 - \gamma_1)$ roots of

$$\left(-\frac{1}{\epsilon} \frac{c_0 A_0^{\gamma_0 - 1} b_0}{c_1 A_0^{\gamma_1 - 1} b_0} \right)^{\frac{1}{\gamma_0 - \gamma_1}}. \tag{2.25}$$

2.3 Perturbations in A

The qualitative effects of perturbations in A_0 are similar in that some of the $(n - \gamma_0)$ zeros at ∞ may become finite; the details of the proof are however more subtle.

Consider

$$\begin{aligned} \dot{x} &= (A_0 + \epsilon A_1) x + b_0 u \\ y &= c_0 x. \end{aligned} \tag{2.26}$$

Further, let the *perturbed* system (2.26) have relative degree γ_1 ($\gamma_1 < \gamma_0$, as before, is the case of interest), i.e.,

$$\begin{aligned} c_0 b_0 = c_0(A_0 + \epsilon A_1) b_0 = \cdots = c_0(A_0 + \epsilon A_1)^{\gamma_1 - 2} b_0 = 0 \qquad & \forall \epsilon \\ c_0(A_0 + \epsilon A_1)^{\gamma_1 - 1} b_0 \neq 0 \qquad & \text{for } \epsilon \text{ small.} \end{aligned} \tag{2.27}$$

From (2.27), it is easy to see that the relative degree γ_1 depends on A_1 in a complicated fashion. For the purpose of this paper, we will restrict our attention to the class of perturbations A_1 satisfying assumptions (2.29) and (2.30) below. Define the subpaces

$$\Delta_i := \text{ span } \left\{ b_0, A_0 b_0, \ldots, A_0^i b_0 \right\}. \tag{2.28}$$

Assume that

$$A_1 \Delta_i \subset \Delta_i \subset \text{ Ker } c_0 \qquad \text{for } i = 1, \ldots, \gamma_1 - 3 \tag{2.29}$$

and

$$A_1 \Delta_{\gamma_1 - 2} \not\subset \Delta_{\gamma_1 - 2} \text{ and } A_1 \Delta_{\gamma_1 - 2} \not\subset \text{ Ker } c_0. \tag{2.30}$$

If $\gamma_1 < 3$, assumption (2.29) is vacuous and if $\gamma_1 < 2$, then (2.30) is vacuous. We conjecture that, if the assumptions (2.29), (2.30) are violated, then the fast zero dynamics occur at a *multiplicity* of time scales. The assumptions (2.29), (2.30) guarantee that

$$\begin{aligned} c_0(A_0 + \epsilon A_1)^{\gamma_1 - 1} b_0 &= \epsilon c_0 A_1 A_0^{\gamma_1 - 2} b_0 + O(\epsilon^2) \\ &=: \epsilon \alpha_0(\epsilon). \end{aligned} \tag{2.31}$$

Note that $\alpha_0(0) = c_0 A_1 A_0^{\gamma_1 - 2} b_0$.

The normal form for the system (2.26) is not easily obtained in the (ξ, η) coordinates of (2.3); consequently, we define

$$\begin{bmatrix} \xi^\epsilon \\ \\ \hline \eta \end{bmatrix} := \begin{bmatrix} c_0 \\ c_0(A_0 + \epsilon A_1) \\ \vdots \\ c_0(A_0 + \epsilon A_1)^{\gamma_0 - 1} \\ \hline H \end{bmatrix} x. \tag{2.32}$$

The matrix in (2.32) is a perturbation of that in (2.3) and is therefore nonsingular for *small* ϵ. We partition ξ^ϵ into

$$\begin{aligned}
\xi_1^\epsilon &:= \begin{bmatrix} c_0 \\ c_0(A_0+\epsilon A_1) \\ \vdots \\ c_0(A_0+\epsilon A_1)^{\gamma_1-1} \end{bmatrix} x \quad \in \mathbf{R}^{\gamma_1}, \\
\xi_2^\epsilon &:= \begin{bmatrix} c_0(A_0+\epsilon A_1)^{\gamma_1} \\ \vdots \\ c_0(A_0+\epsilon A_1)^{\gamma_0-1} \end{bmatrix} x \quad \in \mathbf{R}^{\gamma_0-\gamma_1}.
\end{aligned} \tag{2.33}$$

In these coordinates we have

$$\begin{bmatrix} \dot{\xi}_1^\epsilon \\ \hline \dot{\xi}_2^\epsilon \\ \hline \dot{\eta} \end{bmatrix} = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \begin{matrix} \\ 0 \\ \\ 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} & 0 \\ {}^1a_1^T(\epsilon) & {}^2a_1^T(\epsilon) & a_2^T(\epsilon) \\ \hline {}^1P(\epsilon) & {}^2P(\epsilon) & Q(\epsilon) \end{array}\right] \begin{bmatrix} \xi_1^\epsilon \\ \hline \xi_2^\epsilon \\ \hline \eta \end{bmatrix} + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \epsilon\alpha_0(\epsilon) \\ \hline \epsilon\alpha_1(\epsilon) \\ \vdots \\ \epsilon\alpha_{\gamma_0-\gamma_1-1}(\epsilon) \\ \epsilon\alpha_{\gamma_0-\gamma_1}(\epsilon) + c_0A_0^{\gamma_0-1}b_0 \\ \hline 0 \end{array}\right] u. \tag{2.34}$$

In (2.34) the vectors, ${}^1a_1(\epsilon)$, ${}^2a_1(\epsilon)$, $a_2(\epsilon)$, and matrices, ${}^1P(\epsilon)$, ${}^2P(\epsilon)$, $Q(\epsilon)$, are all perturbations of the corresponding entries in (2.12) and the α_i ($i \geq 1$) are smooth functions of ϵ. Note that, with the exception of $c_0A_0^{\gamma_0-1}b_0$, all the input coefficients are multiplied by ϵ. We now leave it to the interested reader to verify that the unbounded (as functions of ϵ) zeros have the asymptotic form of the $(\gamma_0 - \gamma_1)$ roots of

$$\left(-\frac{1}{\epsilon}\frac{c_0A_0^{\gamma_0-1}b_0}{\alpha_0(0)}\right)^{\frac{1}{\gamma_0-\gamma_1}}. \tag{2.35}$$

Equation (2.35) is very similar to (2.18) except that $\epsilon c_0A_0^{\gamma_1-1}b_1$ is replaced by $\epsilon\alpha_0(0)$ $(= c_0A_1A_0^{\gamma_1-2}b_0)$, the control coefficient for $\dot{\xi}_{1\gamma_1}^\epsilon$.

3 Nonlinear Systems

We briefly review the definition of zero dynamics for SISO nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u \\ y &= h_0(x). \end{aligned} \tag{3.1}$$

Let x_0 be an equilibrium point of the undriven system (i.e., $f_0(x_0) = 0$) and let $U \subset \mathbf{R}^n$ be an open neighborhood of x_0. We will assume the the system has *strict relative degree* γ_0 *at* x_0, i.e.,

$$\begin{gathered} L_{g_0}h_0(x) = L_{g_0}L_{f_0}h_0(x) = \cdots = L_{g_0}L_{f_0}^{\gamma_0-2}h_0(x) \equiv 0 \quad \forall x \in U \\ L_{g_0}L_{f_0}^{\gamma_0-1}h_0(x) \neq 0. \end{gathered} \tag{3.2}$$

Note that we implicitly assume that the system *has* a relative degree! We will further assume (w.l.o.g.) that $h_0(x_0) = 0$.

To find a convenient normal form for the nonlinear system (3.1), we begin by defining

$$\begin{aligned} \xi &:= \begin{bmatrix} h_0(x) \\ L_{f_0}h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_0-1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0} \\ \eta &:= \eta(x) \qquad \in \mathbf{R}^{n-\gamma_0} \end{aligned} \tag{3.3}$$

such that (ξ, η) is a diffeomorphism of x in U. From the definition of relative degree, we may choose $\eta(x)$ so that

$$L_{g_0}\eta_i(x) \equiv 0 \quad i = 1, \ldots, n - \gamma_0. \tag{3.4}$$

The *normal form* of the nonlinear system (3.1) is then written (using (ξ, η) coordinates) as

$$\left[\begin{array}{c} \dot{\xi} \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ & \ddots & \ddots & & 0 \\ & & \ddots & 1 & \\ & & & 0 & \\ \hline & & 0 & & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ \hline \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ b(\xi,\eta) \\ \hline q(\xi,\eta) \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ a(\xi,\eta) \\ \hline 0 \end{array}\right] u. \tag{3.5}$$

Here, $a(\xi, \eta) = L_{g_0}L_{f_0}^{\gamma_0-1}h_0(x)$ and $b(\xi, \eta) = L_{f_0}^{\gamma_0}h_0(x)$ in the (ξ, η) coordinates and $q_i(\xi, \eta) = L_{f_0}\eta_i(x)$ in the (ξ, η) coordinates.

The *zero dynamics* of a nonlinear system are the dynamics of (3.5) consistent with the constraint that the output is held identically zero, i.e., $y(t) \equiv 0$. From the normal form (3.5), it is clear that the nonlinear state feedback

$$u = -\frac{1}{a(\xi, \eta)}b(\xi, \eta) = -\frac{1}{L_{g_0}L_{f_0}^{\gamma_0-1}h_0(x)}L_{f_0}^{\gamma_0}h_0(x) \tag{3.6}$$

results in $y(t) \equiv 0$. Furthermore, the control law (3.6) renders the manifold

$$\begin{aligned} \mathcal{M}_0 &= \left\{x : h_0(x) = L_{f_0}h_0(x) = \cdots = L_{f_0}^{\gamma_0-1}h_0(x) = 0\right\} \\ &= \{(0, \eta) : \eta \in \mathbf{R}^{n-\gamma_0}\} \end{aligned} \tag{3.7}$$

invariant and makes the η variables unobservable. Since $y(t) \equiv 0$ is equivalent to $\xi \equiv 0$, we find that the zero dynamics of (3.5) (hence (3.1)) evolve on the *zero dynamics manifold* $\mathcal{M}_0$ and are described by

$$\dot{\eta} = q(0, \eta). \tag{3.8}$$

Let η_0 be the η component of x_0 (i.e., $x_0 \mapsto (0, \eta_0)$ under the change of coordinates). Then η_0 is an equilibrium point of (3.8). Further, we may associate with η_0 the (Jacobian) linearization of $q(0, \eta)$ at $\eta = \eta_0$, i.e.,

$$\frac{\partial q(0, \eta_0)}{\partial \eta} \tag{3.9}$$

with its eigenvalues referred to as the *nonlinear zeros* of the system (3.1).

We will now study the effects of perturbations on the normal form (3.5).

3.1 Perturbations in g

Consider, as in the previous section, perturbations in the input channel alone, i.e.,

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)u + \epsilon g_1(x)u \\
y &= h_0(x).
\end{aligned}
\tag{3.10}
$$

We will assume that the *perturbation system* (h_0, f_0, g_1) has a strict relative degree of γ_1, i.e.,

$$
\begin{gathered}
L_{g_1}h_0(x) = L_{g_1}L_{f_0}h_0(x) = \cdots = L_{g_1}L_{f_0}^{\gamma_1-2}h_0(x) \equiv 0 \quad \forall x \in U \\
L_{g_1}L_{f_0}^{\gamma_1-1}h_0(x) \neq 0.
\end{gathered}
\tag{3.11}
$$

As before, the case of greatest interest is when $\gamma_1 \leq \gamma_0$. Following the previous development, we partition ξ as

$$
\begin{aligned}
\xi_1 &= \begin{bmatrix} h_0(x) \\ L_{f_0}h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_1-1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_1}, \\
\xi_2 &= \begin{bmatrix} L_{f_0}^{\gamma_1}h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_0-1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0-\gamma_1}.
\end{aligned}
\tag{3.12}
$$

The perturbed system (3.10) expressed in (ξ_1, ξ_2, η) coordinates looks like

$$
\left[\begin{array}{c} \dot{\xi}_1 \\ \hline \dot{\xi}_2 \\ \hline \dot{\eta} \end{array}\right]
=
\left[\begin{array}{cccc|cccc|c}
0 & 1 & & & & & & & \\
 & \ddots & \ddots & & & 0 & & & 0 \\
 & & \ddots & 1 & & & & & \\
 & & & 0 & 1 & & & & \\
\hline
 & & & & 0 & 1 & & & \\
 & 0 & & & & \ddots & \ddots & & 0 \\
 & & & & & & \ddots & 1 & \\
 & & & & & & & 0 & \\
\hline
 & 0 & & & & 0 & & & 0
\end{array}\right]
\left[\begin{array}{c} \xi_1 \\ \hline \xi_2 \\ \hline \eta \end{array}\right]
+
\left[\begin{array}{c} 0 \\ \vdots \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ b(\xi,\eta) \\ \hline q(\xi,\eta) \end{array}\right]
+
\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \epsilon L_{g_1}L_{f_0}^{\gamma_1-1}h_0 \\ \hline \epsilon L_{g_1}L_{f_0}^{\gamma_1}h_0 \\ \vdots \\ \epsilon L_{g_1}L_{f_0}^{\gamma_0-2}h_0 \\ a(\xi,\eta) + \epsilon L_{g_1}L_{f_0}^{\gamma_0-1}h_0 \\ \hline \epsilon L_{g_1}\eta_1 \\ \vdots \\ \epsilon L_{g_1}\eta_{n-\gamma_0} \end{array}\right] u.
\tag{3.13}
$$

Note that, in (3.13), we have deliberately chosen not to write the $L_{g_1}L_{f_0}^i h_0$ terms in the (ξ, η) coordinates.

Using the nonlinear state feedback

$$
u = -\frac{1}{\epsilon L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\xi_{21} = -\frac{1}{\epsilon L_{g_1}L_{f_0}^{\gamma_1-1}h_0(x)}L_{f_0}^{\gamma_1}h_0(x)
\tag{3.14}
$$

to zero the output, we make the manifold

$$
\begin{aligned}
\mathcal{M}_1 &= \left\{x : h_0(x) = L_{f_0}h_0(x) = \cdots = L_{f_0}^{\gamma_1-1}h_0(x) = 0\right\} \\
&= \{(0, \xi_2, \eta) : \xi_2 \in \mathbf{R}^{\gamma_0-\gamma_1}, \eta \in \mathbf{R}^{n-\gamma_0}\}
\end{aligned}
\tag{3.15}
$$

invariant and the $(\xi_2, \eta) \in \mathbb{R}^{n-\gamma_1}$ variables unobservable. Thus, the zero dynamics of (3.10) are precisely the dynamics of (ξ_2, η) on $\mathcal{M}_1$ given by

$$
\left[\begin{array}{c}
\dot{\xi}_{21} \\ \vdots \\ \vdots \\ \dot{\xi}_{2,\gamma_0-\gamma_1} \\ \hline \dot{\eta}
\end{array}\right]
=
\left[\begin{array}{ccccc|c}
-\frac{L_{g_1}L_{f_0}^{\gamma_1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 1 & & & & \\
-\frac{L_{g_1}L_{f_0}^{\gamma_1+1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 0 & 1 & & & \\
\vdots & \vdots & \ddots & \ddots & & 0 \\
-\frac{L_{g_1}L_{f_0}^{\gamma_0-2}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 0 & \cdots & 0 & 1 & \\
-\frac{L_{g_1}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} - \frac{1}{\epsilon}\frac{L_{g_0}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 0 & \cdots & \cdots & 0 & \\
\hline
-\frac{L_{g_1}\eta_1}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & & & \\
\vdots & & & 0 & & 0 \\
-\frac{L_{g_1}\eta_{n-\gamma_0}}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & & &
\end{array}\right]
\left[\begin{array}{c}
\xi_{21} \\ \vdots \\ \vdots \\ \xi_{2,\gamma_0-\gamma_1} \\ \hline \eta
\end{array}\right]
+
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ b(\xi,\eta) \\ \hline q(\xi,\eta)
\end{array}\right].
\tag{3.16}
$$

Before we state a nonlinear counterpart to Theorem 2.1, we apply the scaling specified by equation (2.19) to (3.16), namely

$$
\tilde{\xi}_{21} = \xi_{21},\ \tilde{\xi}_{22} = \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\xi_{22},\ \cdots,\ \tilde{\xi}_{2,\gamma_0-\gamma_1} = \epsilon^{\frac{\gamma_0-\gamma_1-1}{\gamma_0-\gamma_1}}\xi_{2,\gamma_0-\gamma_1}. \tag{3.17}
$$

The transformation of (3.17) renders terms in the first column of the matrix in (3.16) potentially unbounded as ϵ tends to zero. Now rewrite (3.16) as

$$
\left[\begin{array}{c}
\epsilon^{\frac{1}{\gamma_0-\gamma_1}}\dot{\tilde{\xi}}_2 \\ \hline \dot{\eta}
\end{array}\right]
=
\left[\begin{array}{cccc|c}
0 & 1 & & & \\
\vdots & \ddots & \ddots & & 0 \\
0 & & \ddots & 1 & \\
-\frac{L_{g_0}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 0 & \cdots & 0 & \\
\hline
-\frac{L_{g_1}\eta_1}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & & \\
\vdots & & 0 & & 0 \\
-\frac{L_{g_1}\eta_{n-\gamma_0}}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & &
\end{array}\right]
\left[\begin{array}{c}
\tilde{\xi}_2 \\ \hline \eta
\end{array}\right]
+ \epsilon^{\frac{1}{\gamma_0-\gamma_1}} k(\tilde{\xi}_2, \eta, \epsilon). \tag{3.18}
$$

In (3.18), We will *assume* that $k(\tilde{\xi}_2, \eta, \epsilon) \in \mathbb{R}^{n-\gamma_1}$ is a smooth function of $\tilde{\xi}$, η, and ϵ. Thus, for example, we may require that for some K that

$$
\begin{aligned}
\left|\frac{L_{g_1}L_{f_0}^{\gamma_1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\right| &\leq K|\xi_{21}|, \\
\left|\frac{L_{g_1}L_{f_0}^{\gamma_1+1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\right| &\leq K|\xi_{21}| + K|\xi_{22}|, \\
&\text{etc.}
\end{aligned}
\tag{3.19}
$$

With this assumption, which is unique to the nonlinear case, it may be verified that the second term in (3.18) is multiplied by $\epsilon^{\frac{1}{\gamma_0-\gamma_1}}$ in analogy with the second term on the right hand side of (2.20). Equation (3.18) shows the two time scale nature of the zero dynamics. We mention in passing that if conditions of the form of (3.19) do not hold then there may be more than two time scales in the zero dynamics. The $\tilde{\xi}_2$ variables are the fast variables and the slow manifold of the zero dynamics of (3.18) is the subspace corresponding to the η variables (up to zeroth order in ϵ). The fast dynamics of the $\tilde{\xi}_2$ variables are determined by studying the scaled variables $\tilde{\xi}_2$ in the fast time, i.e., set $\tau = t/\epsilon^{1/\gamma_0 - \gamma_1}$ and then set $\epsilon = 0$ to get

$$\begin{aligned} \frac{d\tilde{\xi}_2}{d\tau} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & & & 0 \end{bmatrix} \tilde{\xi}_2 \\ \eta(\tau) &\equiv \eta_0. \end{aligned} \tag{3.20}$$

Note that $\frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0}$ in (3.20) should be expressed as a function of $(\xi_1, \tilde{\xi}_2, \eta)$ with ξ_1 and η set to 0 and η_0, respectively. From this, the stability properties of the fast system are easy to deduce. Clearly, $\tilde{\xi}_2 = 0$ is an equilibrium point of (3.20) and the eigenvalues of (the linearization of) (3.20) are the $(\gamma_0 - \gamma_1)$ different roots of the lower left term of (3.20), i.e.,

$$\left(-\frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} \right)^{\frac{1}{\gamma_0 - \gamma_1}} \quad \text{at } (\xi_1, \tilde{\xi}_2, \eta) = (0, 0, \eta_0). \tag{3.21}$$

In the original time scale, these eigenvalues are of order $1/\epsilon^{1/\gamma_0 - \gamma_1}$.

The *remarkable new feature* of the fast zeros of the *nonlinear system* is that they *vary with* η_0 *on the base slow manifold* $\mathcal{M}_0$. Pictorially then, we have the following figure for the foliation of $\mathcal{M}_1$.

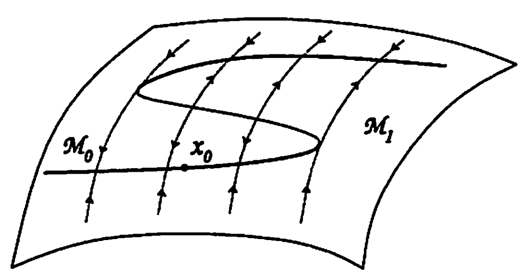

Figure 1: Showing the foliation of the zero dynamics manifold of $\mathcal{M}_1$ (with $\mathcal{M}_0$ embedded) by the fast zero dynamics of (3.20).

Collecting these observations, we have the following counterpart to Theorem 2.1.

Theorem 3.1 *The nonlinear system (3.10) has* $(n - \gamma_1)$ *zeros. According to their asymptotic behavior as* $\epsilon \to 0$*, the zeros belong to two groups:*

- *As* $\epsilon \to 0$, $(\gamma_1 - \gamma_0)$ *zeros arise from the dynamics of*

$$\epsilon^{\frac{1}{\gamma_0 - \gamma_1}} \frac{d\tilde{\xi}_2}{dt} = \left. \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & & & 0 \end{bmatrix} \right|_{(\xi_1, \tilde{\xi}_2, \eta) = (0, 0, \eta_0)} \tilde{\xi}_2 \tag{3.22}$$

- *The remaining* $(n - \gamma_0)$ *zeros tend to the zeros of the* unperturbed *system (3.1) as* $\epsilon \to 0$.

Proof The preceding observations yield the first part of the theorem. The verification that as $\epsilon \to 0$ the dynamics of η in (3.18) tend to those of $\dot{\eta} = q(0,\eta)$ follows from setting $\epsilon = 0$ and $\tilde{\xi}_2 = 0$ in (3.18). (Actually, this happens in quite a subtle fashion since some terms appear as multiples of $\tilde{\xi}_{21}$ and others appear as multiplies of $\epsilon^{1/\gamma_0-\gamma_1}$!) □

Remarks

1. As in the remarks after Theorem 2.1, we leave it to the reader to verify that, if the relative degree of the perturbation g_1 (i.e., γ_1) is $\geq \gamma_0$, then the zero dynamics of the perturbed and unperturbed systems have the same dimension and qualitative properties (i.e., the perturbation in the zero dynamics is *regular*).
2. The remarkable additional feature found in nonlinear systems that is not present in linear systems is that the locations of the asymptotically unbounded zeros in the complex plane, i.e., $1/\epsilon^{1/\gamma_0-\gamma_1}$ times the quantities in (3.21), vary with η. Thus, as suggested by figure 1, some parts of the zero dynamics manifold $\mathcal{M}_0$ of the unperturbed system may be attractive to the fast dynamics of the $\tilde{\xi}_2$ variables while other parts may be repulsive. Of course, it is easy to show that if the equilibrium point x_0 of the original system corresponds to $(0,\eta_0)$, then the fast zeros will be given by
$$\left(-\frac{1}{\epsilon}\frac{L_{g_0}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\right)^{\frac{1}{\gamma_0-\gamma_1}}\Bigg|_{(\xi_1,\tilde{\xi}_2,\eta)=(0,0,\eta_0)} \tag{3.23}$$
In figure 1, for example, x_0 has been shown belonging to a region where the manifold $\mathcal{M}_0$ is attractive.

3.2 Perturbations in h

As in Section 2.2, the qualitative results of Section 3.1 hold when the nonlinear system (3.1) is perturbed in h_0. To this end, we consider the perturbed system
$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u \\ y &= h_0(x) + \epsilon h_1(x) \end{aligned} \tag{3.24}$$
with relative degree $\gamma_1 \leq \gamma_0$. It is no longer possible to invoke duality but one may use a new set of coordinates for the normal form given by
$$\begin{aligned} \xi^\epsilon &:= \begin{bmatrix} h_0(x)+\epsilon h_1(x) \\ L_{f_0}(h_0(x)+\epsilon h_1(x)) \\ \vdots \\ L_{f_0}^{\gamma_0-1}(h_0(x)+\epsilon h_1(x)) \end{bmatrix} \in \mathbb{R}^{\gamma_0} \\ \eta &:= \eta(x) \qquad\qquad \in \mathbb{R}^{n-\gamma_0} \end{aligned} \tag{3.25}$$
Note that the diffeomorphism of (3.25) is a perturbation of that in (3.3). By partitioning ξ^ϵ into ξ_1^ϵ and ξ_2^ϵ and scaling ξ_2^ϵ as above, it can be shown that the $\tilde{\xi}_2^\epsilon$ variables in the time scale $\tau = t/\epsilon^{1/\gamma_0-\gamma_1}$ satisfy
$$\frac{d\tilde{\xi}_2^\epsilon}{d\tau} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{a_0(0,\tilde{\xi}_2^\epsilon,\eta)}{\alpha_0(0,\tilde{\xi}_2^\epsilon,\eta)} & & & 0 \end{bmatrix} \tilde{\xi}_2^\epsilon \tag{3.26}$$
where $\alpha_0(\xi_1^\epsilon,\tilde{\xi}_2^\epsilon,\eta)$ and $a_0(\xi_1^\epsilon,\tilde{\xi}_2^\epsilon,\eta)$ are $L_{g_0}L_{f_0}^{\gamma_1-1}h_1$ and $L_{g_0}L_{f_0}^{\gamma_0-1}h_0$ in $(\xi_1^\epsilon,\tilde{\xi}_2^\epsilon,\eta)$ coordinates.

3.3 Perturbations in f

The situation here is delicate and analogous to that in Section 2.3. Consider the *perturbed* system

$$\begin{aligned} \dot{x} &= f_0(x) + \epsilon f_1(x) + g_0(x)u \\ y &= h_0(x) \end{aligned} \tag{3.27}$$

and assume that it has relative degree $\gamma_1 < \gamma_0$. The class of perturbations $f_1(x)$ satisfy a nonlinear analog of assumptions (2.29) and (2.30). Define the distributions

$$\Delta_i := \text{span}\left\{g_0, ad_{f_0}g_0, \ldots, ad^i_{f_0}g_0\right\}. \tag{3.28}$$

Assume that (the notation $\perp$ means the orthogonal distribution)

$$ad_{f_1}\Delta_i \subset \Delta_i \subset \{dh_0\}^{\perp} \quad \text{for } i = 1, \ldots, \gamma_1 - 3 \tag{3.29}$$

and

$$ad_{f_1}\Delta_{\gamma_1-2} \not\subset \Delta_{\gamma_1-2} \text{ and } ad_{f_1}\Delta_{\gamma_1-2} \not\subset \{dh_0\}^{\perp}. \tag{3.30}$$

If $\gamma_1 < 3$, assumption (3.29) is vacuous and if $\gamma_1 < 2$, (3.30) is vacuous. As in the linear case, we conjecture that if these assumptions are violated then the fast zero dynamics may occur at a multiplicity of time scales. These assumptions guarantee that

$$\begin{aligned} L_{g_0}L^{\gamma_1-1}_{f_0+\epsilon f_1}h_0 &= \epsilon L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0 + O(\epsilon^2) \\ &=: \epsilon\alpha_0(\epsilon). \end{aligned} \tag{3.31}$$

Note that $\alpha_0(0) = L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0$.

For coordinates, one uses

$$\begin{aligned} \xi^{\epsilon} &:= \begin{bmatrix} h_0(x) \\ L_{f_0+\epsilon f_1}h_0(x) \\ \vdots \\ L^{\gamma_0-1}_{f_0+\epsilon f_1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0} \\ \eta &:= \quad \eta(x) \qquad \in \mathbf{R}^{n-\gamma_0} \end{aligned} \tag{3.32}$$

and the development of Sections 3.1 and 3.2 can be repeated to yield the fast dynamics of equation (3.26) with the difference that $\alpha_0(\xi^{\epsilon}_1, \tilde{\xi}^{\epsilon}_2, \eta)$ is $L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0$.

4 Conclusion

In this paper, we have shown the effects of perturbation on the zero dynamics of both linear and nonlinear SISO systems. We have shown how regular perturbations in the state space descriptions of these systems can result in the appearance of singularly perturbed or fast zero dynamics. We have given explicit formulas for the locations in the complex plane that the zeros at ∞ migrate to under perturbation. For the most part, we have placed assumptions on the structure of the allowable perturbations so as to guarantee the appearance of fast time scale zero dynamics at one time scale alone. When these assumptions are not met, we conjecture that our qualitative results will be unaltered but that fast zero dynamics at multiple time scales will appear. Our theory bears a strong resemblance to the literature on high gain feedback and is in some sense to be thought of as a companion to that literature, since it reveals the *zero structure at* ∞ by the artifact of system perturbation.

We conclude by noting that the analysis presented in this paper can be extended albeit in much more subtle form to the MIMO case, and involving a multiplicity of time scales (see [HSK89]).

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