ON THE DESIGN OF FINITE DIMENSIONAL STABILIZING COMPENSATORS FOR INFINITE DIMENSIONAL FEEDBACK-SYSTEMS VIA SEMI-INFINITE OPTIMIZATION

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ABSTRACT

Recently, Polak and Wuu have presented a set of easily solvable, differentiable inequalities, which are related to the classical Nyquist stability criterion, and which constitute a necessary and sufficient condition of stability for finite dimensional systems. In this paper it is shown that a similar set of easily solvable inequalities can be used to design finite dimensional stabilizing compensators for a class of infinite dimensional feedback systems. Computational aspects of the new stability test are discussed.

1. INTRODUCTION

Control system design via semi-infinite optimization (see [Pol.3]) requires that all performance specifications be expressed as (semi-infinite) inequalities. For the finite dimensional case two approaches are possible. The first is to use Q–parametrization of the design, as in [Boy.1, Pol.5], which requires co-prime factorization of the plant transfer function. The second is to use a semi-infinite inequality, as in [Pol.2, Pol.6]. The main advantage of the first approach is that it leads to convex design problems; its main disadvantages are (i) the design problem convexity is achieved at the expense of fixing the location of the closed loop poles and using optimization only to manipulate the zeros of the closed loop transfer functions, and (ii) it does not permit the preselection of the controller order. The main advantages of the second approach are (i) it can be used in conjunction with broad classes of stability regions, and (ii) it permits the preselection of the controller order; its main disadvantage is that it does not result in convex design problems. On balance, we prefer the second approach.

The Q–parametrization approach can also be used for the the design of stabilizing compensators for infinite dimensional feedback systems [Net.1, Vid.1]. Since it generates an infinite dimensional stabilizing compensator, it has to be supplemented with approximation and order reduction techniques, and appears to be rather cumbersome. The results presented in [Cur.1, Sch.1], for the the design of finite dimensional compensators for infinite dimensional feedback systems, do not appear to be utilizable in a semi-infinite optimal design setting. In this paper, we develop an extension of the second approach described above, for the design of finite dimensional compensators for infinite dimensional feedback systems.

The first attempt to produce a frequency domain stability test for finite dimensional systems, which is compatible with the requirements of semi-infinite optimization, was presented in [Pol.1]. A significant improvement was presented in [Pol.2]. The necessary and sufficient stability criterion proposed in [Pol.2] is based on the following observation. Suppose that $\chi(s)$ is a characteristic polynomial. Then all the zeros of $\chi(s)$ are in $\overset{o}{\mathbb{C}}_- \triangleq \{s \in \mathbb{C},\ \mathrm{Re}(s) < 0\}$ if and only if there exists a polynomial $d(s)$, of the same degree as $\chi(s)$ and whose zeros are in $\overset{o}{\mathbb{C}}_-$, such that

$$\mathrm{Re}\,[\chi(j\omega) \,/\, d(j\omega)] > 0, \quad \forall \ \ \omega \in (-\infty\,,\,\infty)\,. \tag{1.1}$$

The proof of this result is simple. If all the zeros of $\chi(s)$ are in $\overset{o}{\mathbb{C}}_-$, then set $d(s) = \chi(s)$ and hence (1.1) holds. Alternatively, if (1.1) holds then the origin is not encircled by the locus of $\chi(j\omega)/d(j\omega)$ and hence the conclusion holds as for the Nyquist stability criterion. When used in design, the characteristic polynomial is also a differentiable function of compensator designable parameters $x \in \mathbb{R}^n$, and has the form $\chi(x,s)$; and the normalizing polynomial $d(s)$ is written in a factored form, such as $d(s,q) = \Pi_{j=1}^{n_p} (s^2 + a_j s + b_j)$, which makes it simple to ensure that the zeros of $d(s)$ are in $\overset{o}{\mathbb{C}}_-$ (q is a vector whose components are the a_j, b_j).

In this paper we extend the computational stability criterion presented in [Pol.2], to a form that can be used in the design of *finite dimensional* stabilizing compensators for a class of feedback systems with infinite dimensional plants, to be described in Sec. 2. Since in this case the characteristic function is not a polynomial, there is no simple way to define a normalizing polynomial (of finite degree) for a test of the form (1.1). Hence approximation theory has to be brought into play. Our new stability test guarantees not only input-output stability, but also internal stability of the feedback system. Furthermore, since the numerical implementation of the test does not depend on the use of a reduced plant model, the test does not lead to spill-over effects. Finally, because the compensator is parametrized in the state-space form, the order of the compensator can be assigned by the designer.

The new computational stability criterion will be presented in Sec. 3, and its numerical implementations in Sec. 4. A numerical design example involving a flexible beam will be given in Sec. 5.

2. PRELIMINARY RESULTS

Consider the feedback system $S(P,K)$, with n_i inputs and n_o outputs, shown in Fig. 1. We assume that the plant is described by a linear and time-invariant differential equation in a reflexive Banach space E:

$$\dot{x}_p(t) = A_p x_p(t) + B_p e_2(t); \quad y_2(t) = C_p x_p(t) + D_p e_2(t), \tag{2.1}$$

where $x_p(t) \in E$, $e_2(t) \in \mathbb{R}^{n_i}$, $y_2(t) \in \mathbb{R}^{n_o}$, for $t \geq 0$. The operator A_p from E to E, may be an unbounded operator with domain dense in E, which generates a strongly continuous (C_0) semigroup, $\{e^{A_p t}\}_{t \geq 0}$. The operators $C_p : E \to \mathbb{R}^{n_o}$ and $D_p : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ are assumed to be bounded. To allow boundary or point control action, we assume that the operator B_p can be unbounded in a certain sense. To clarify the class of operators B_p that are allowed, we need to extend the state space, as follows. First, we denote the adjoint operator of A_p by A_p^*, the dual space of E by E^* and the domain and the range of A_p by $D(A_p)$ and $R(A_p)$, respectively. As in [Cur.1], we then let

$$Z^* \triangleq D_{E^*}(A_p^*) \subset E^* \tag{2.2}$$

endowed with the graph norm of A_p^*. Then Z^* is a real, reflexive Banach space and the injection of Z^* into E^* is continuous and dense. Defining the *extended state space* Z be the dual space of Z^*, we obtain by duality that

$$E \subset Z, \tag{2.3}$$

with a continuous dense injection.

From now on, we will treat the state of the plant, x_p, as an element of the extended state space Z, and we will assume that $B_p : \mathbb{R}^{n_i} \to Z$, is bounded. Because E is dense in Z, C_p can be extended to a bounded operator from Z to $\mathbb{R}^{n_o}$. The operator A_p^* can be regarded as a bounded operator from Z^* into E^* and, by duality, A_p extends to a bounded operator from E to Z. Referring to [Cur.1, Sal.1], we see that this extension, regarded as an unbounded operator on Z, is the infinitesimal generator of the extended semigroup $\{e^{A_p t}\}_{t \geq 0} \in L(Z)$. The exponential growth rate of the semigroup $\{e^{A_p t}\}_{t \geq 0}$ is the same on the state spaces E and Z. Furthermore, the spectrum of A_p on the state space E coincides with the spectrum of A_p on Z[1].

By the Hille-Yosida theorem [Paz.1], there exist $M \geq 1$ and $\gamma \in \mathbb{R}$ such that $\|e^{A_p t}\| \leq Me^{\gamma t}$, $\forall\ t \geq 0$. Let $\sigma(A_p)$ denote the *spectrum* of A_p and let $\rho(A_p)$ denote the *resolvent set* of A_p. We define the *transfer function* of the plant, $G_p(s)$, to be $C_p(sI - A_p)^{-1}B_p + D_p$, $\forall\ s \in \rho(A_p)$, which is analytic on $\rho(A_p)$ [Kat.1, Theorem III 6.7].

Definition 2.1: We will say that a function, $g: \mathbb{C} \to \mathbb{C}$, *converges at infinity* in a domain $D \subset \mathbb{C}$, if there exists a finite complex number, c, such that $\lim_{\rho \to \infty} \sup_{\substack{|s| \geq \rho \\ s \in D}} |g(s) - c| = 0$, and we shall write $c = \lim_{\substack{|s| \to \infty \\ s \in D}} g(s)$. We will say that a matrix function $G: \mathbb{C} \to \mathbb{C}^{m \times n}$ converges at infinity in a domain D if each of its elements converges at infinity in D. ■

It follows from [Jac.1] that there exists a $\gamma \in \mathbb{R}$ such that $\lim_{\substack{|s| \to \infty \\ \mathrm{Re}\, s > \gamma}} G_p(s) \to D_p$.

Definition 2.2: For any $\alpha \geq 0$, a semi-group $\{T(t)\}_{t \geq 0}$, defined on a Banach space, is said to be α*–stable* if there exist $M \in (0,\infty)$ and $\alpha_0 > \alpha$ such that

$$\|T(t)\| \leq Me^{-\alpha_0 t}, \quad \forall\ t \geq 0. \tag{2.4}$$

■

For any $\alpha \geq 0$, we define the *stability region* $D_{-\alpha} \triangleq \{s \in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}$, with compliment, in $\mathbb{C}$, $U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -\alpha\}$, whose boundary and interior will be denoted by $\partial U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = -\alpha\}$ and $U^o_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha\}$.

We assume that the plant in (2.1) is α*–stabilizable* and α*–detectable*, i.e., there exist bounded linear operators $K: Z \to \mathbb{R}^{n_i}$ and $F: \mathbb{R}^{n_o} \to Z$ such that $A_p - B_p K$ and $A_p - FC_p$ are the infinitesimal generators of α–stable C_0–semigroups. It can be shown that the plant is α–stabilizable and α–detectable if and only if there exists a decomposition of $Z = Z_- \oplus Z_+$, with Z_+ finite-dimensional, which induces a decomposition of the plant (2.1), of the form

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[1] Under the above assumptions, model (2.1) can represent a flexible beam with point actuators and sensors [Har.1].

$$\frac{d}{dt}\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} = \begin{bmatrix} A_{p-} & 0 \\ 0 & A_{p+} \end{bmatrix}\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} + \begin{bmatrix} B_{p-} \\ B_{p+} \end{bmatrix} u(t) \; ; \qquad y(t) = [C_{p-} \;\; C_{p+}]\begin{bmatrix} x_{p-}(t) \\ x_{p+}(t) \end{bmatrix} + D_p u(t), \tag{2.5}$$

such that $\sigma(A_{p+}) \subset U_{-\alpha}$, (A_{p+}, B_{p+}) is controllable, (A_{p+}, C_{p+}) is observable, and A_{p-} is the infinitesimal generator of an α–stable C_0–semigroup on Z_- [Nef.1, Jac.1][2]. We recall that a plant is α–stabilizable and α–detectable if and only if there exists a finite dimensional strictly proper compensator such that the feedback system is α–stable [Jac.1][3].

We assume the compensator to be *finite dimensional, linear, and time-invariant*, with state equations

$$\dot{x}_c(t) = A_c x_c(t) + B_c e_1(t); \quad y_1(t) = C_c x_c(t) + D_c e_1(t) \, , \tag{2.6}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, $e_1(t) \in \mathbb{R}^{n_o}$, $y_1(t) \in \mathbb{R}^{n_i}$ and A_c, B_c, C_c and D_c are matrices of appropriate dimension. The compensator transfer function is $G_c(s) = C_c(sI_{n_c} - A_c)^{-1}B_c + D_c$. The compensator is also assumed to be α–stabilizable and α–detectable. To ensure well-posedness of the feedback system, we assume that $\det(I_{n_i} + D_c D_p) \neq 0$.

We define the product space $H = Z \times \mathbb{R}^{n_c}$. Since $e_1 = u_1 - y_2$ and $e_2 = y_1 + u_2$, the state equations for the feedback system are

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = A \begin{bmatrix} x_p \\ x_c \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ; \qquad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = C \begin{bmatrix} x_p \\ x_c \end{bmatrix} + D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{2.7}$$

where

$$A = \begin{bmatrix} A_p - B_p D_c (I_{n_o} + D_p D_c)^{-1} C_p & B_p (I_{n_i} + D_c D_p)^{-1} C_c \\ -B_c (I_{n_o} + D_p D_c)^{-1} C_p & A_c - B_c (I_{n_o} + D_p D_c)^{-1} D_p C_c \end{bmatrix} \tag{2.8a}$$

$$B = \begin{bmatrix} B_p D_c (I_{n_o} + D_p D_c)^{-1} & B_p (I_{n_i} + D_c D_p)^{-1} \\ B_c (I_{n_o} + D_p D_c)^{-1} & -B_c (I_{n_o} + D_p D_c)^{-1} D_p \end{bmatrix} , \tag{2.8b}$$

[2] In [Nef.1, Jac.1] only 0-stability is considered. Our extension to α-stability is trivial.

[3] Although the state space of the plant is assumed to be a Hilbert space in [Jac.1], the results from [Jac.1] used in this section can be easily seen to remain true if we assume that the state space of the plant is a reflexive Banach space.

$$C = \begin{bmatrix} -(I_{n_o}+D_pD_c)^{-1}C_p & -(I_{n_o}+D_pD_c)^{-1}D_pC_c \\ -D_c(I_{n_o}+D_pD_c)^{-1}C_p & (I_{n_i}+D_cD_p)^{-1}C_c \end{bmatrix}; \quad D = \begin{bmatrix} (I_{n_o}+D_pD_c)^{-1} & -(I_{n_o}+D_pD_c)^{-1}D_p \\ D_c(I_{n_o}+D_pD_c)^{-1} & (I_{n_i}+D_cD_p)^{-1} \end{bmatrix}. \tag{2.8c}$$

The domain $D(A) = D(A_p) \times \mathbb{R}^{n_c} \subset H$. It follows from [Paz.1, p. 76], that because, with the exception of A_p, all the operators in the matrix A are bounded, and because diag(A_p,0) generates a C_0-semigroup, the operator A also generates a C_0–semigroup, $\{e^{At}\}_{t \geq 0}$.

Let $x = [x_p,x_c] \in H$. Then the formula $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ defines a *mild solution* of (2.7) [Paz.1]. We therefore define the *exponential stability* of the feedback system $S(P,K)$ in terms of the semigroup $\{e^{At}\}_{t \geq 0}$.

Definition 2.3: For any $\alpha \geq 0$, the feedback system S(P , K) is α–stable if the semi-group $\{e^{At}\}_{t \geq 0}$ is α–stable. ■

It was shown in [Jac.1] that, under the above assumptions, the feedback system $S(P,K)$ is also α–stabilizable and α–detectable. From the decomposition property in (2.5) for α–stabilizable and α–detectable systems, we can easily deduce the following relationship between α–stability of the feedback system and the spectrum of A, first established in [Jac.1]:

Proposition 2.1: If the above assumptions hold, the feedback system is α–stable if and only if $U_{-\alpha}$ is contained in $\rho(A)$. ■

3. A COMPUTATIONAL STABILITY CRITERION

We define the characteristic function $\chi: \mathbb{C} \to \mathbb{C}$, of the feedback system $S(P,K)$, by

$$\chi(s) \triangleq \det(sI_{n_+} - A_{p+})\det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s)) , \tag{3.1}$$

where A_{p+} is defined as in (2.5) and n_+ is the dimension of A_{p+}.

To establish the next result, we have to apply the following Weinstein-Aronszajn formula ([Kat.1, p. 247]).

The W-A Formula: Let F be a closed operator in the Banach space X. Let Q be a bounded operator in X and suppose that $R \triangleq R(Q)$ is finite-dimensional. Let $y: \mathbb{C} \to \mathbb{C}$, defined by $y(s) = \det(I_R + (Q(F - sI)^{-1})|_R)$, be the associated *W-A determinant*, with I_R the identity operator in R

and $(Q(F-sI)^{-1})|_R$ the restriction of $Q(F-sI)^{-1}$ to R. If Δ is a domain of the complex plane consisting of points of $\rho(F)$ and of isolated eigenvalues of F with finite multiplicities, then $y(s)$ is meromorphic in Δ. Next, we define the *multiplicity function* $\nu(s;y)$ of $y(s)$ by

$$\nu(s;y) = \begin{cases} k & \text{if } s \text{ is a zero of } \phi \text{ of order } k \\ -k & \text{if } s \text{ is a pole of } \phi \text{ of order } k \\ 0 & \text{for other } s \in \Delta \end{cases}, \tag{3.2a}$$

and, for any closed operator $G: X \to X$, we define the *multiplicity function* $\tilde{\nu}(s;G)$ by

$$\tilde{\nu}(s;G) = \begin{cases} 0 & \text{if } s \in \rho(G) \\ \dim(P) & \text{if } s \text{ is an isolated point of } \sigma(G) \\ +\infty & \text{in all other cases} \end{cases}, \tag{3.2b}$$

where P is the projection associated with an isolated point of $\sigma(G)$ (see [Kat.1, p.180]). Then, the W-A formula relates the multiplicity function of the operator $F+Q$ to those of F and $y(s)$, as follows:

$$\tilde{\nu}(s;F+Q) = \tilde{\nu}(s;F) + \nu(s;y),\ \forall\ s \in \Delta\ , \qquad \blacksquare \tag{3.2c}$$

Next, for any function $f: \mathbb{C} \to \mathbb{C}$, we define $Z(f(s)) \triangleq \{s \in \mathbb{C} \mid f(s) = 0\}$ to be its set of zeros.

Theorem 3.1: The system $S(P,K)$ is α–stable if and only if $Z(\chi(s)) \subset D_{-\alpha}$.

Proof: We begin by decomposing the operator A (in (2.8a)) into the form $A = F + Q$, as shown below, with the plant decomposed as in (2.5) and λ_c such that $\mathrm{Re}(\lambda_c) < -\alpha$,

$$F = \begin{bmatrix} A_{p-} & 0 & 0 \\ 0 & \lambda_c I_{n_+} & 0 \\ 0 & 0 & \lambda_c I_{n_c} \end{bmatrix}; \quad Q = \begin{bmatrix} -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-} & -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+} & B_{p-}(I_{n_i}+D_cD_p)^{-1}C_c \\ -B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-} & A_{p+}-B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}-\lambda_c I_{n+} & B_{p+}(I_{n_i}+D_cD_p)^{-1}C_c \\ -B_c(I_{n_o}+D_pD_c)^{-1}C_{p-} & -B_c(I_{n_o}+D_pD_c)^{-1}C_{p+} & A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_c I_{n_c} \end{bmatrix}. \tag{3.3}$$

It is easy to see that F generates the C_0-semigroup $\{e^{Ft}\}_{t\geq 0}$, where $e^{Ft} = diag(e^{A_{p-}t}, e^{\lambda_c t}I_{n+}, e^{\lambda_c t}I_{n_c})$, and that $(F-sI)$ is invertible for $s \in U_{-\alpha}$; $Q = A - F$ is a bounded operator and $R(Q)$ is finite dimensional. Consider $s \in U_{-\alpha} \subset \rho(A_{p-})$. Since $(F-sI)^{-1}$ exists and is bounded, we can define $V(s)$ by

$$V(s) = Q(F-sI)^{-1}$$

$$= \begin{bmatrix} -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1} & -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & B_{p-}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1} & (A_{p+}-B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}-\lambda_cI_{n+})(\lambda_c-s)^{-1} & B_{p+}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1} & -B_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & (A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_cI_{n_c})(\lambda_c-s)^{-1} \end{bmatrix} . \tag{3.4}$$

Let $B_0 \triangleq R(Q) = R(B_p) \times \mathbb{R}^{n_c} = R(B_{p-}) \times R(B_{p+}) \times \mathbb{R}^{n_c}$ and let $V_{B_0}(s)$ denote the restriction of $V(s)$ to B_0. Then $\det(I + V(s)) \triangleq \det(I_{B_0} + V_{B_0}(s))$ is well defined [Kat.1]. We will show that $\det(I_{B_0} + V_{B_0}) = \chi(s)$ and then apply the W-A formula.

Let $b_j \triangleq B_{p-}e_j$, $j = 1,2,\ldots,n_i$, where $\{e_j\}_{j=1}^{n_i}$ is the standard unit basis in $\mathbb{R}^{n_i}$. Suppose, without loss of generality, that $\bar{n} \le n_i$ is a positive integer such that $\{b_j\}_{j=1}^{\bar{n}}$ is the largest linearly independent subset of $\{b_j\}_{j=1}^{n_i}$. Under the basis $\{b_j\}_{j=1}^{\bar{n}}$ as a basis for $R(B_{p-})$, the linear operator B_{p-} assumes the matrix form $B_{p-} = (I_{\bar{n}\times\bar{n}} \mid \tilde{B}_{p-}) \in \mathbb{R}^{\bar{n}\times n_i}$, where the i-th column of $\tilde{B}_{p-}$ is obtained by expressing $b_{\bar{n}+i}$ in terms of the basis $\{b_j\}_{j=1}^{\bar{n}}$. Let $\bar{B} \triangleq (b_1,b_2,\ldots,b_{\bar{n}})$. Then it is easy to show that

$$V_{B_0} = \begin{bmatrix} -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1}\bar{B} & -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & B_{p-}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1}\bar{B} & (A_{p+}-B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}-\lambda_cI_{n+})(\lambda_c-s)^{-1} & B_{p+}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_c(I_{n_o}+D_pD_c)^{-1}C_{p-}(A_{p-}-sI)^{-1}\bar{B} & -B_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & (A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_cI_{n_c})(\lambda_c-s)^{-1} \end{bmatrix} , \tag{3.5a}$$

$$= \begin{bmatrix} -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}M(s) & -B_{p-}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & B_{p-}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}M(s) & (A_{p+}-B_{p+}D_c(I_{n_o}+D_pD_c)^{-1}C_{p+}-\lambda_cI_{n+})(\lambda_c-s)^{-1} & B_{p+}(I_{n_i}+D_cD_p)^{-1}C_c(\lambda_c-s)^{-1} \\ -B_c(I_{n_o}+D_pD_c)^{-1}M(s) & -B_c(I_{n_o}+D_pD_c)^{-1}C_{p+}(\lambda_c-s)^{-1} & (A_c-B_c(I_{n_o}+D_pD_c)^{-1}D_pC_c-\lambda_cI_{n_c})(\lambda_c-s)^{-1} \end{bmatrix} , \tag{3.5b}$$

where $M(s) \triangleq [r_1(s),r_2(s),\ldots,r_{\bar{n}}(s)] \in \mathbb{C}^{n_o\times\bar{n}}$ with $r_i(s) \triangleq C_{p-}(A_{p-} - sI)^{-1}b_i$, $1 \le i \le \bar{n}$. Because each element in (3.5b) is in matrix form, it is straightforward to show that

$$\det(I_{B_0} + V_{B_0}(s)) = \det(sI_{n+} - A_{p+})\det(sI_{n_c} - A_c)\det(I_{n_i} + G_c(s)G_p(s)) = \chi(s) . \tag{3.6}$$

Now we make use of the W-A formula. Let F and Q be defined as in (3.3), so that $A = F + Q$. Since $U_{-\alpha} \subset \rho(F)$, we can choose $\Delta = U_{-\alpha}$. Then $\chi(s) = \det(I_R + (Q(F - sI)^{-1})|_R)$, for $s \in U_{-\alpha}$ and hence applying the W-A formula, we obtain that $\tilde{v}(s;A) = \tilde{v}(s;F) + v(s;\chi)$ for all $s \in U_{-\alpha}$. Since $U_{-\alpha} \subset \rho(F)$, it follows that $\tilde{v}(s;F) = 0$ for all $s \in U_{-\alpha}$, and hence $\tilde{v}(s;A) = v(s;\chi)$ for all $s \in U_{-\alpha}$, which implies that (i) the operator A has only finitely many eigenvalues in $U_{-\alpha}$ and (ii)

$$U_{-\alpha} \cap \sigma(A) = U_{-\alpha} \cap Z(\chi(s)) \,. \tag{3.7}$$

Now suppose that the system $S(P,K)$ is α–stable. Then it follows from Proposition 2.1 that $U_{-\alpha} \subset \rho(A)$, which is equivalent to saying that $U_{-\alpha} \cap \sigma(A)$ is the empty set. Hence it follows from (3.7) that $U_{-\alpha} \cap Z(\chi(s))$ is the empty set, which implies that $Z(\chi(s)) \subset D_{-\alpha}$.

Next, suppose that $Z(\chi(s)) \subset D_{-\alpha}$. Then we must have that $U_{-\alpha} \cap Z(\chi(s))$ is empty. It now follows from (3.7) and Proposition 2.1 that $S(P,K)$ is α–stable, which completes our proof. ■

Next we introduce an approximation result.

Proposition 3.1: Given $\alpha \geq 0$, any function $f\colon \mathbb{C} \to \mathbb{C}$ which is analytic in $U^o_{-\alpha}$, continuous on $\partial U_{-\alpha}$, and converges at infinity in $U_{-\alpha}$, can be approximated uniformly by a rational function which is also analytic on the same domain.

Proof : Let $f(s)$ be an analytic function on $U_{-\alpha}$. Define the bilinear transformation

$$z \triangleq \frac{s-p+\alpha}{s+p+\alpha} \; ; \; s \triangleq -\alpha + p\frac{1+z}{1-z} \,, \tag{3.8}$$

and let $g(z) = f(-\alpha + p(1+z)/(1-z))$. Since $U_{-\alpha}$ is mapped onto the unit disc, and $f(s)$ is analytic in $U_{-\alpha}$ and continuous on ∂U and converges at infinity, $g(z)$ is analytic in the open unit disc and continuous on the unit circle. By Mergelyan's Theorem [Rud.1], $g(z)$ can be uniformly approximated arbitrarily closely on the unit circle by a polynomial in z. Since the transformation (3.8) is H_∞-norm preserving, the desired result follows. ■

Theorem 3.2: Let n_+ and n_c be the dimensions of the matrices A_{p+} in (2.5) and A_c in (2.6), respectively. $Z(\chi(s)) \subset D_{-\alpha}$ if and only if there exists an integer $N_n > 0$, and polynomials $d_0(s)$ and $n_0(s)$, of degree $N_d = N_n + n_s$ and N_n, respectively, with $n_s = n_c + n_+$, such that

$$(i)\;\; Z(d_0(s)) \subset D_{-\alpha}\,, \quad Z(n_0(s)) \subset D_{-\alpha}\,; \qquad (ii)\;\; \mathrm{Re}\left[\frac{\chi(s)n_0(s)}{d_0(s)}\right] > 0 \;\; \forall\; s \in \partial U_{-\alpha}\,. \tag{3.9}$$

Proof: (i) Suppose that (3.9) holds. Since A_{p-} is α–stable, there exists $\varepsilon > 0$ such that $U_{-(\alpha+\varepsilon)}$ is a subset of $\rho(A_{p-})$, and $(sI - A_{p-})^{-1}$ is analytical on $U_{-(\alpha+\varepsilon)}$. From (3.6), (3.5b), we observe that $\chi(s)$ is an analytic function over $U_{-(\alpha+\varepsilon)}$. Then it follows from the Argument Principle [Chu.1] that $Z(\chi(s)) \subset D_{-\alpha}$.

(ii) Suppose that $Z(\chi(s)) \subset D_{-\alpha}$. We first apply the approximation result given in Proposition 3.1 to the function $\chi(s)/(s+\beta)^{n_s}$, where $\beta \geq \alpha$. Clearly, there exists some real number $\gamma_0 > -\alpha$ such that $\lim_{\substack{|s| \to \infty \\ \mathrm{Re}\, s \geq \gamma_0}} G_p(s) \to D_p$ and $\lim_{\substack{|s| \to \infty \\ \mathrm{Re}\, s \geq \gamma_0}} G_c(s) \to D_c$. Because the degree of $\det(sI_{n_+} - A_{p+})\det(sI_{n_c} - A_c)$ is n_s, we have that

$$\lim_{\substack{|s|\to\infty \\ s \in U_{\gamma_0}}} \left| \frac{\chi(s)}{(s+\beta)^{n_s}} \right| = \lim_{\substack{|s|\to\infty \\ s \in U_{\gamma_0}}} \left| \frac{\det(sI_{n_+} - A_{p+})\det(sI_{n_c} - A_c)}{(s+\beta)^{n_s}} \right| |\det(I_{n_i} + G_c(s)G_p(s))| = |\det(I_{n_i} + D_cD_p)|.$$

(3.10a)

Since $\chi(s)$ is analytic on $U_{-(\alpha+\varepsilon)}$ for some $\varepsilon > 0$, it is uniquely determined by its values over U_{γ_0} [Chu.1, p.286]. Hence[4]

$$\lim_{\substack{|s|\to\infty \\ s \in U_{-\alpha}}} \left| \frac{\chi(s)}{(s+\beta)^{n_s}} \right| = |\det(I_{n_i} + D_cD_p)| \neq 0 . \qquad (3.10b)$$

Note that $|\det(I_{n_i} + D_cD_p)|$ is not equal to zero because of the assumption of well-posedness of the feedback system. Therefore it follows from Proposition 3.1, that for any $\delta > 0$, we can find a rational function $d(s)/n(s)$, such that all the zeros of $n(s) \subset D_{-\alpha}$, and

$$\|\chi(s)/(s+\beta)^{n_s} - d(s)/n(s)\| \triangleq \sup_{s \in \partial U_{-\alpha}} |\chi(s)/(s+\beta)^{n_s} - d(s)/n(s)| < \delta . \qquad (3.11)$$

Since $Z(\chi(s)) \subset D_{-\alpha}$ and for $s \in U_{-\alpha}$, $|\chi(s)| \to \infty$ as $|s| \to \infty$. It is easy to show that $\inf_{s \in U_{-\alpha}} |\chi(s)| = c_0 > 0$. Because of (3.10b), for any given $\eta > 0$ sufficiently small, there exists r_η such that $|\chi(s)/(s+\beta)^{n_s}| > |\det(I_{n_i} + D_cD_p)| - \eta$, for all $s \in U_{-\alpha}$ and $|s| \geq r_\eta$. Next we show that if $\delta < \min\{|\det(I_{n_i} + D_cD_p)| - \eta \, , \, c_0/(r_\eta+\beta)^{n_s}\}$, then $Z(d(s)) \subset D_{-\alpha}$. If not, then there exists $s_0 \in U_{-\alpha}$ such that $d(s_0) = 0$. Now, by (3.11), $|\chi(s_0)/(s_0+\beta)^{n_s} - d(s_0)/n(s_0)| = |\chi(s_0)/(s_0+\beta)^{n_s}| < \delta$. If $|s_0| > r_\eta$, we obtain a contradiction of $|\chi(s_0)/(s_0+\beta)^{n_s}| > |\det(I_{n_i} + D_cD_p)| - \eta > \delta$, while if $|s_0| \leq r_\eta$, we obtain a contradiction of $|\chi(s_0)/(s_0+\beta)^{n_s}| > c_0/(r_\eta+\beta)^{n_s} > \delta$.

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4 The following is a sketch of the proof for (3.10b). Consider the function $f(s): U_{\gamma_0} \to \mathbb{C}$ such that $f(s) = \chi(s)$ for $s \in U_{\gamma_0}$. By using the transformation defined in (3.8), we transform U_{γ_0} in the s plane unto a subset of unit disc in the z plane, which includes the point $z = 1$. Then there exists a unique analytic extension of the function $g(z) = f(-\alpha + p\frac{1+z}{1-z})$ to the unit disk, which is $h(z) = \chi(-\alpha + p\frac{1+z}{1-z})$, $|z| \leq 1$. Therefore (3.10b) is just the consequence of $h(1) = g(1)$.

From (3.10b) and the fact that $\inf_{s\in U_{-\alpha}} |\chi(s)| = c_0 > 0$, it is easy to show that $\inf_{s\in\partial U_{-\alpha}} |\chi(s)/(s+\beta)^{n_s}| = l_0 \neq 0$. From (3.11), if $\delta < l_0/2$, then for $s\in\partial U_{-\varepsilon}$, $|d(s)/n(s)| > |\chi(s)/(s+\beta)^{n_s}| - \delta > |\chi(s)/(s+\beta)^{n_s}|/2$. Therefore if δ is chosen less than $\min\{l_0/2, |\det(I_{n_t} + D_cD_p)| - \eta, c_0/(r_\eta + \beta)^{n_s}\}$, from (3.11), we obtain that

$$|\chi(s)/(s+\beta)^{n_s} - d(s)/n(s)| \,/\, |d(s)/n(s)| < \delta \,/\, |d(s)/n(s)| < 2\delta \,/\, |\chi(s)/(s+\beta)^{n_s}| < 1 \;,\quad s\in\partial U_{-\alpha}\,. \tag{3.12}$$

It follows from the above that for all $s\in\partial U_{-\alpha}$, $|[\chi(s)n(s)/(s+\beta)^{n_s}d(s)] - 1| < 1$, and hence that

$$\mathrm{Re}[\,\chi(s)\frac{n(s)}{(s+\beta)^{n_s}d(s)}\,] > 0\;,\;\; \forall\; s\in\partial U_{-\alpha}\;, \tag{3.13}$$

Let $n_0(s) = n(s)$ and $d_0(s) = (s+\beta)^{n_s}d(s)$. This completes our proof. ■

4. NUMERICAL IMPLEMENTATION OF THE STABILITY CRITERION

In practice, the test (3.9) can only be used as a sufficient condition, because one must choose in advance the degree N_d of the polynomial $d_0(s)$. We shall now sketch out some of the numerical aspects of using the test (3.9) in the design of a stabilizing compensators. First, the order n_c of the compensators (2.6) must be selected and the elements of the matrices in (2.6) must be made continuously differentiable in the design parameter vector p_c. Second, the polynomials $d_0(s)$ and $n_0(s)$ must be parametrized. In [Pol.2] we find a computationally efficient parametrization for $d_0(s)$ and $n_0(s)$ which is based on the following observation. When $a,b\in\mathbb{R}$, $Z[(s+\alpha)+a)] \subset D_{-\alpha}$ if and only if $a > 0$, and $Z[(s+\alpha)^2 + a(s+\alpha) + b] \subset D_{-\alpha}$ if and only if $a > 0$, $b > 0$. Hence, when the degree of $d_0(s)$ is odd, we set $d_0(s,q_d) \triangleq ((s+\alpha)+a_0)\prod_{i=1}^{m}((s+\alpha)^2 + a_i(s+\alpha) + b_i)$, where $q_d \triangleq (a_0,a_1,a_2,\cdots,a_m,b_1,b_2,\cdots,b_m)^T\in\mathbb{R}^{2m+1}$ and $N_d = 2m+1$. When N_d is even, the linear term is omitted. The polynomial $n_0(s)$, which is of degree $N_n = N_d - n_s$ can be parametrized similarly, with corresponding parameter vector q_n. As a result, the system of inequalities (3.9) becomes

$$q_d^i - \varepsilon \geq 0,\quad \text{for } i = 1,2,\cdots,N_d\;;\quad q_n^i - \varepsilon \geq 0,\quad \text{for } i = 1,2,\cdots,N_n, \tag{4.1a}$$

$$\mathrm{Re}\left(\frac{\chi(-\alpha+j\omega,p_c)n_0(-\alpha+j\omega,q_n)}{d_0(-\alpha+j\omega,q_d)}\right) - \varepsilon \geq 0,\quad \forall\;\omega\in[0,\infty)\;, \tag{4.1b}$$

where q_d^i, q_n^i are the components of q_d, q_n, and ε is a small positive number.

The design of a stabilizing compensators by means of semi-infinite optimization requires the evaluation of $\chi(-\alpha + j\omega, p_c)$ and its partial derivatives with respect to p_c^i for many values of ω. In [Pol.2] we find an efficient method for evaluating transfer functions, characteristic polynomials, and their derivatives for finite dimensional systems. This method can be used for evaluating the finite dimensional parts in (3.1). Thus, it remains to discuss the evaluation of the transfer function $G_p(s)$. A truly efficient method for evaluating $G_p(-\alpha + j\omega)$ for many values of ω remains to be developed. In our work, we have either been able to obtain a closed-form formula for $G_p(-\alpha + j\omega)$, as for the example below, or else we compute it by solving a two point boundary value problem, using shooting methods [Kel.1, Pol.4]. For example, consider the planar bending motion of a flexible beam, shown in Fig. 2. One end of the beam is fixed; a particle with mass M is attached to the other end. The x-axis is the undeformed-beam centroidal line; the y-axis is the cross section principal axis. The associated control system is required to damp out vibrations. Assuming that the beam is of unit length, its bending motion can be described by the partial differential equation

$$m\frac{\partial^2 w(t,x)}{\partial t^2} + cI\frac{\partial^5 w(t,x)}{\partial x^4 \partial t} + EI\frac{\partial^4 w(t,x)}{\partial x^4} = \sum_{j=1}^{n_i} f^j(t)\zeta^j(x,x^j), \;\; t \geq 0, \;\; 0 \leq x \leq 1 \;, \tag{4.2a}$$

with boundary conditions

$$w(t,0) = 0 \;, \; \frac{\partial w}{\partial x}(t,0) = 0, \tag{4.2b}$$

$$J\frac{\partial^3 w}{\partial x \partial t^2}(t,1) + cI\frac{\partial^3 w}{\partial x^2 \partial t}(t,1) + EI\frac{\partial^2 w}{\partial x^2}(t,1) = 0 \;, \quad M\frac{\partial^2 w}{\partial t^2}(t,1) - cI\frac{\partial^4 w}{\partial x^3 \partial t}(t,1) - EI\frac{\partial^3 w}{\partial x^3}(t,1) = 0, \tag{4.2c}$$

where $w(t,x)$ is the vibration along the y-axis, $f^j(t)$ is a control force, and $\zeta^j(x,x^j)$ is the influence function of the j–th actuator, centered at $x = x^j$; m is the distributed mass per unit length of the beam, c is the material viscous damping coefficient, E is Young's modulus, M is the end mass, I is the beam sectional moment of inertia with respect to y-axis, J is the inertia of the end mass in the direction of y-axis, and n_i is the number of actuators. The output sensors can be assumed to be modeled by

$$y^i(t) = \int_0^1 \kappa^i(\nu, z^i) w(t,\nu) d\nu \;, \;\; t \geq 0 \;, \;\; 1 \leq i \leq n_o, \tag{4.3}$$

where n_o is the number of the sensors, and $\kappa^i(\nu, z^i)$ is the distribution function of the i-th sensor centered at $x = z^i$. The distribution functions ζ in (4.2a) and κ in (4.3) can be delta functions, since it is shown in [Har.1] that the resulting operators B and C in the state form (2.1) satisfy the model assumptions in the space Z. In [Hua.1], it is shown that the resulting infinitesimal generator A_p generates an analytic semigroup. As a result, it is easy to show that a decomposition of the form (2.5) is possible for the plant, (4.2a-c), (4.3), as long as the stability margin $\alpha < E/c$ [Gib.1].

Taking the Laplace transforms of the equations (4.2a-c) and (4.3) with respect to time, we obtain, for each value of $s = -\alpha + j\omega$, the *ordinary differential equation*

$$(cIs + EI)\frac{d^4W(s,x)}{dx^4} + ms^2W(s,x) = \sum_{j=1}^{n} F^j(s)\zeta^j(x,x^j), \quad 0 \le x \le 1 , \tag{4.4a}$$

with boundary conditions

$$W(s,0) = 0 , \quad \frac{dW}{dx}(s,0) = 0 , \tag{4.4b}$$

$$(cIs + EI)\frac{d^2W}{dx^2}(s,1) + Js^2\frac{dW}{dx}(s,1) = 0 , \qquad (cIs + EI)\frac{d^3W}{dx^3}(s,1) - Ms^2W(s,1) = 0 . \tag{4.4c}$$

The Laplace transforms of the output components are given by

$$Y^i(s) = \int_0^1 \kappa^i(\nu, z^i)W(s,\nu)d\nu , \quad 1 \le i \le n_o, \tag{4.5}$$

where $W(s,x)$, $F^j(s)$ and $Y^i(s)$ are the Laplace transforms of $w(t,x)$, $f^j(t)$ and $y^i(t)$, respectively. It follows that the (i,j)-th element of $G_p(s)$ can be obtained by setting $F^j(s) = 1$ and $F^k(s) = 0$ for all other k and then solving (4.4a) - (4.4d) and (4.5).

The above boundary-value problem can be solved by shooting methods, using high precision at critical frequencies, and low precision otherwise. This is preferable to the use of a fixed modal truncation approach which may lead to serious "spill-over" effects.

5. A NUMERICAL EXAMPLE

We shall now describe our numerical experience in designing a fourth order compensator for a single-input single-output feedback system with the plant described by (4.2a-c), (4.3). We assumed that

$m = 2$, $cI = 0.01$, $EI = 1$, $M = 5$, $J = 0.5$, that the required stability margin $\alpha = 0.2$, and that the colocated point force actuator and point displacement sensor are located at $x = 1$.

To obtain an initial compensator design and to provide a testbed for the study of truncation effects, we carried out a modal expansion of the plant dynamics to obtain the first eight modes: $-0.0023 \pm 0.6716i$, $-0.0447 \pm 2.9890i$, $-1.3718 \pm 16.5069i$, $-9.7845 \pm 43.1411i$. In the corresponding truncated state space plant model, the matrix A_p has the form $A_p = diag(A_{11}, A_{22}, A_{33}, A_{44})$, where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -0.451053 & -0.004511 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0 & 1 \\ -8.936154 & -0.089362 \end{bmatrix},$$
$$A_{33} = \begin{bmatrix} 0 & 1 \\ -274.359603 & -2.743596 \end{bmatrix} \quad A_{44} = \begin{bmatrix} 0 & 1 \\ -1956.894214 & -19.568942 \end{bmatrix}. \tag{5.1}$$

$B_p = (0, -0.272993, 0, -0.112681, 0, 0.073277, 0, -0.047885)^T$, $C_p = (-0.545986, 0, -0.225362, 0\ \ 0.146553\ , 0\ , -0.095770\ , 0)$, and $D_p = 0$. We chose to design the compensator in transfer function form: $G_c(p_c, s) = c_0(c_1 s^2 + c_2 s + 1)(c_3 s^2 + c_4 s + 1)/(d_1 s^2 + d_2 s + 1)(d_3 s^2 + d_4 s + 1)$, which results in $p_c = (c_o, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4)^T$. We set $n_0(s) = 1$ and $d_0(s, q_d) = \prod_{i=1}^{4} ((s + \alpha) + a_i(s + \alpha) + b_i)$, so that $q_d \triangleq (a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)^T$. We set $\varepsilon = 0$ in (4.1a,b).

Using pole assignment on the fourth order truncated model, we obtained the initial compensator transfer function: $G_c(p_c, s) = \dfrac{21044.8s^3 + 96356.5s^2 + 88286.1s + 858018}{s^4 + 2.94613s^3 + 177.301s^2 - 3333.83s - 7930.13}$, which stabilizes the truncated model. However, it fails to stabilize the truncated plant of order 6 and 8, as well as the full precision model.

Using this compensator as the starting point for our semi-infinite optimization algorithm, we obtained in two iterations of a semi-infinite minimax algorithm the following transfer function of the stabilizing compensator for our controlled flexible structure: $G_c(p_c, s) = \dfrac{-12.5806s^4 + 20658.8s^3 + 94255.7s^2 + 87402.1s + 841483}{s^4 + 2.12762s^3 + 171.79s^2 - 3262.91s - 7774.42}$. The critical frequency interval for the evaluation of $\chi(p_c, s)$ was [0.1 , 200] and the number of sampling points used was 50; 500 points were used to produce the plots in Figures 3 and 4. The plot corresponding to (4.1b) for the initial value of the compensator is shown in Fig. 3 and for the final value in Fig. 4.

It is interesting to observe that when this stabilizing compensator is used with the plant model truncated to order 4, the resulting feedback system has two unstable poles. However, when this stabilizing compensator is used with the plant model truncated to order 6 or 8, the resulting feedback system is stable.

6. CONCLUSION

We have developed a new characteristic function for a class of feedback systems with infinite-dimensional plants, and have used it to construct a necessary and sufficient computational stability criterion. We expect that our stability criterion will be useful in the design of finite dimensional compensators for a large class of infinite dimensional systems, such as flexible structures with point actuators and sensors, subject to specified stability margins. Our design example shows that our stability inequalities are well conditioned with respect to semi-infinite optimization algorithms. Although we did not do so in our design example, it should be clear that our stability inequalities can be combined with other performance inequalities, ensuring robustness, disturbance rejection, and satisfactory transient responses, into a tractable, semi-infinite optimal design problem.

There remains a certain amount of numerical analysis type work to be done in developing efficient techniques for the repeated evaluation of frequency responses of distributed parameter systems, and for the computation of their unstable poles. Furthermore, because of local minima effects, the successful use of our stability criterion may be predicated on a good initial design of a stabilizing compensators and normalizing polynomials.

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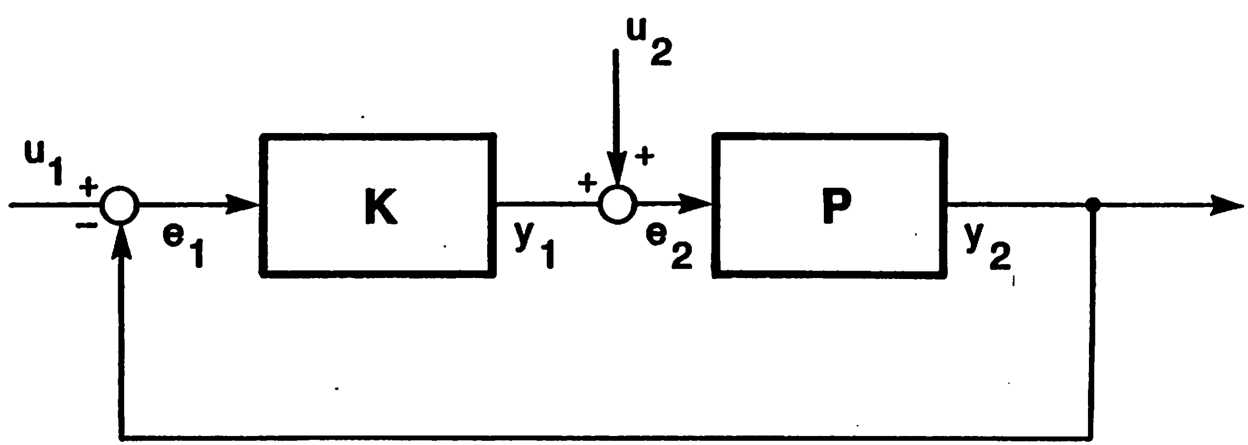

Figure 1: The feedback system S(P,K).

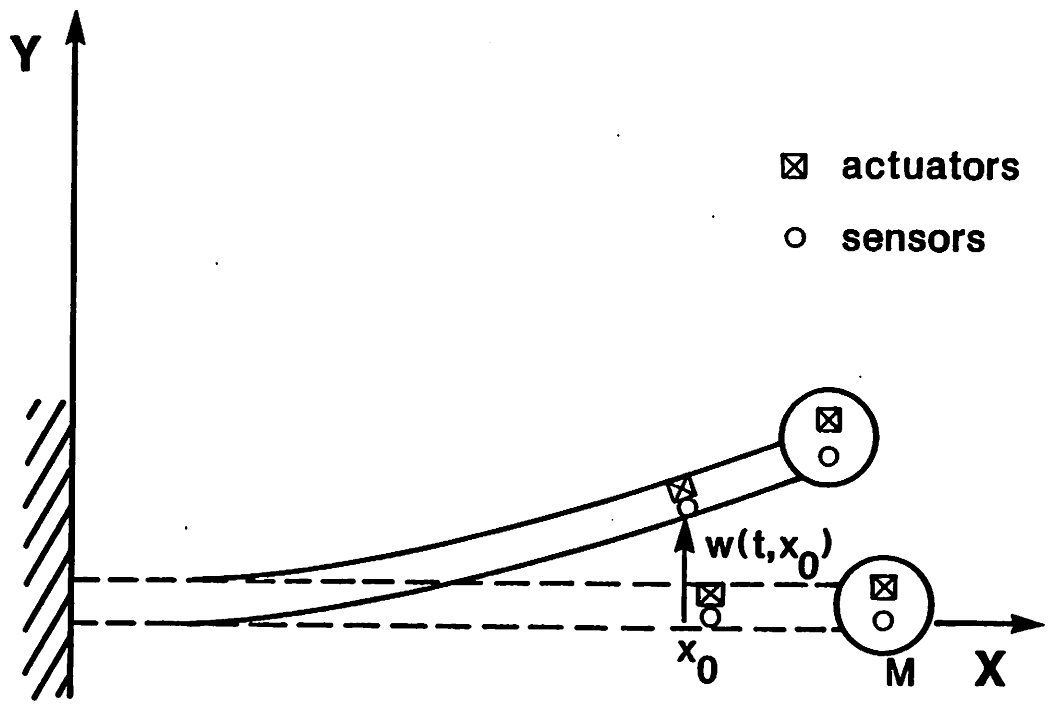

Figure 2: Planar bending motion of a flexible beam.

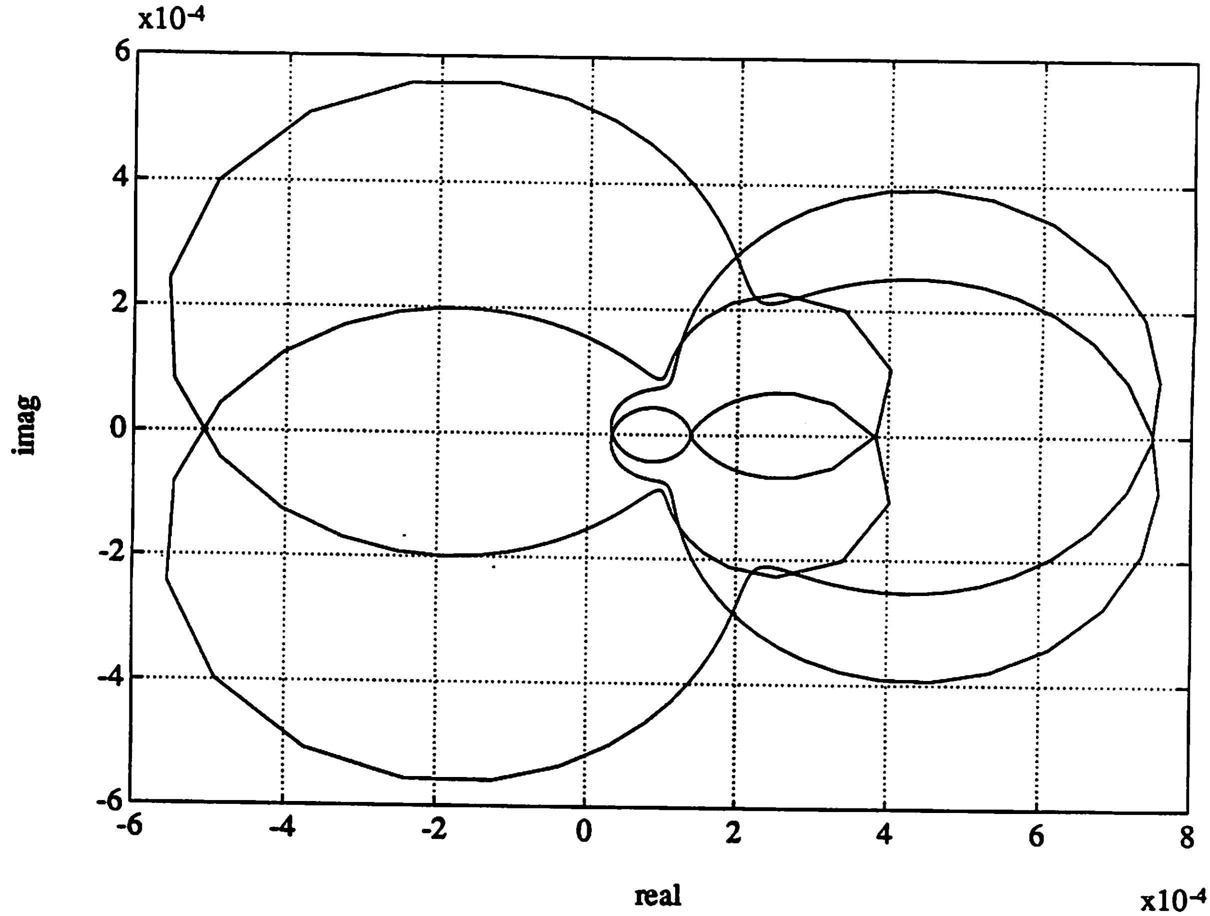

Figure 3: Modified Nyquist diagram (initial design).

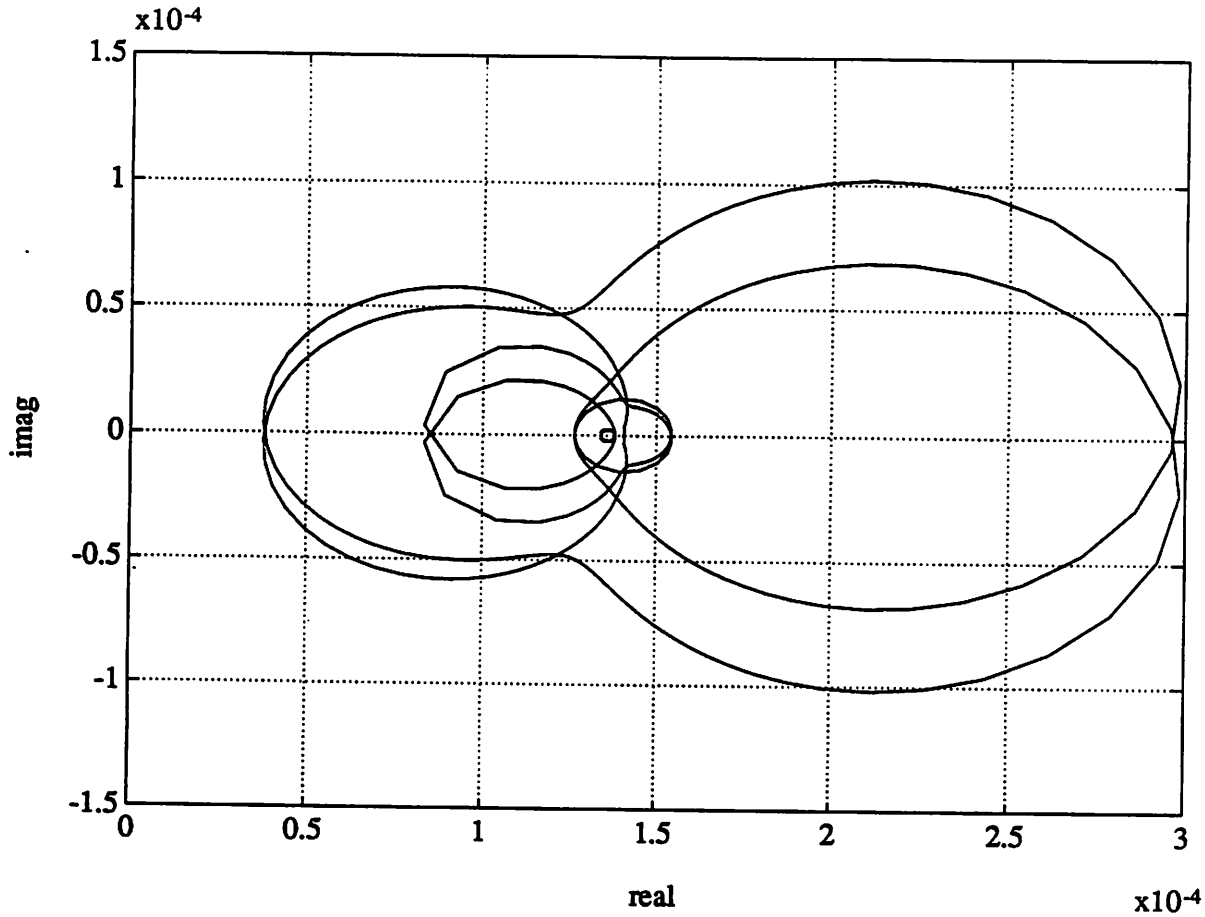

Figure 4: Modified Nyquist diagram for the stabilized system.