FACTORIZATION APPROACH TO NONLINEAR FEEDBACK SYSTEMS

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Abstract

In its general algebraic framework, factorization theory has proven to be extremely useful in solving interesting control problems related to linear time-invariant systems that have transfer function representations. This work studies the extension of factorizations to nonlinear multiinput-multioutput maps.

The nonlinear maps considered are assumed to be causal (i.e., non-anticipatory) and are defined over input and output extended spaces; hence the setting is quite general and is suitable for analyzing unstable nonlinear maps. Due to the flexibility of choosing norms in input and output spaces, this input-output approach is suitable for generalized forms of bounded-input bounded-output stability analysis.

Factorization tools are applied to stability and robustness analysis of nonlinear

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additive feedback systems. These tools are also used to propose stabilizing feedback schemes.

Proper stable factorizations of linear time-invariant finite-dimensional systems and related key facts are reviewed for motivation; they lead to a compact self-contained formulation of stability and robustness properties.

Stabilizing feedback systems and existence of factorizations are studied based on a discussion of factorization tools for general linear maps. Using factorization tools, necessary and sufficient conditions are given for robust stability of the nominal linear unity-feedback system under nonlinear (possibly unstable) additive, feedback, pre-multiplicative and post-multiplicative plant perturbations.

Following a discussion for right-factorization tools for nonlinear causal maps, a stabilizing additive feedback configuration is proposed. Right-factorization and right-coprime factorization examples for some classes of nonlinear plants are explicitly worked out. After stating conditions on linear (not necessarily time-invariant) plants for parametrizing the set of all nonlinear stabilizing compensators in nonlinear unity-feedback systems, the parametrization of all stabilizing nonlinear compensators is obtained. Stability and robustness of nonlinear unity-feedback system and conditions for simultaneous stabilization are studied using factorization tools.

List of Symbols

$A, \ldots, Z$	matrix transfer functions
$\mathsf{A}, \ldots, \mathsf{Z}$	linear input-output maps
$\mathcal{A}, \ldots, \mathcal{Z}$	nonlinear input-output maps
$\mathrm{Adj}D$	adjoint matrix of $D \in \mathbb{R}(s)^{n \times n}$
$\mathrm{B}(r)$	$\{\ \Delta \in \mathbf{M}(\mathbb{R}_p(s)) \ \mid\ r \in \mathbf{R_U}\ ,\ \ \|\Delta(s)\| \leq \lvert r(s)\rvert \quad \forall\, s \in \partial\mathbf{U}\ \}$
$\mathrm{B}_{\mathbf{U}}(r)$	$\{\ \Delta \in \mathbf{M}(\mathbf{R_U}) \ \mid\ r \in \mathbf{R_U}\ ,\ \ \|\Delta(s)\| \leq \lvert r(s)\rvert \quad \forall\, s \in \partial\mathbf{U}\ \}$
$\mathbb{C}$	complex numbers
$\mathbb{C}^{o}_{-}$	open left-half plane, $\{\ s \in \mathbb{C} \ \mid\ \Re\mathrm{e}(s) < 0\ \}$
$\Delta\mathcal{P}$	nonlinear (possibly unstable) plant perturbation
$\partial\mathbf{U}$	boundary of $\mathbf{U} \subset \mathbb{C}$
$\det D$	determinant of $D \in \mathbb{R}(s)^{n \times n}$
$\mathbf{I}$	identity map
$\mathcal{I}_{\mathcal{P}}$	instabilities of $\mathcal{P}$ $\left\{\ e \in \Lambda^{\mathrm{i}} \ \mid\ \mathcal{P}e \in \Lambda^{\mathrm{o}}_{\mathrm{e}} \setminus \Lambda^{\mathrm{o}}\ \right\}$

$\Im\mathrm{m}(s)$	imaginary part of $s \in \mathbb{C}$, $\Im\mathrm{m}(\sigma + j\omega) = \omega$
Λ	normed vector space (Definition 3.2.1)
Λ_e	causal extension of the normed vector space Λ (Definition 3.2.1)
l.f.	left factorization
l.c.f.	left-coprime factorization
$\mathbf{M}(\mathbf{R_U})$	set of matrices with entries in $\mathbf{R_U}$
$\vert \cdot \vert$	norm on vectors
$\vert \cdot \vert_2$	euclidean norm on $\mathbb{R}^n$
$\Vert \cdot \Vert$	norm on function spaces, maps
$\Vert \cdot \Vert_{\mathbf{U}}$	norm on $\mathbf{M}(\mathbf{R_U})$
$P_{add}(\Delta)$	Figure 2.15 , $P \in \mathbf{M}(\mathbb{R}_p(s))$, $\Delta \in \mathbf{M}(\mathbf{R_U})$
$P_{feed}(\Delta)$	Figure 2.14 , $P \in \mathbf{M}(\mathbb{R}_p(s))$, $\Delta \in \mathbf{M}(\mathbf{R_U})$
$P_{post}(\Delta)$	Figure 2.13 , $P \in \mathbf{M}(\mathbb{R}_p(s))$, $\Delta \in \mathbf{M}(\mathbf{R_U})$
$P_{pre}(\Delta)$	Figure 2.12 , $P \in \mathbf{M}(\mathbb{R}_p(s))$, $\Delta \in \mathbf{M}(\mathbf{R_U})$
$\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{add}}$	i input j output additive nonlinear perturbation of linear plant P (Subsection 3.7.1)
$\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{feed}}$	i input j output feedback nonlinear perturbation of linear plant P (Subsection 3.7.2)

$\left(\mathsf{P}\,,\,\Delta\mathcal{P}\right)^{\mathbf{pre}}_{\mathbf{ij}}$	**i** input **j** output pre-multiplicative nonlinear perturbation of linear plant P (Subsection 3.7.3)
$\left(\mathsf{P}\,,\,\Delta\mathcal{P}\right)^{\mathbf{post}}_{\mathbf{ij}}$	**i** input **j** output post-multiplicative nonlinear perturbation of linear plant P (Subsection 3.7.4)
r.f.	right factorization
r.c.f.	right-coprime factorization
$\mathbb{R}$	real numbers
$\mathbb{R}^n$	vector space of ordered n-tuples in $\mathbb{R}$
$\mathbb{R}_+$	nonnegative real numbers, $\{\, x \in \mathbb{R} \mid x \geq 0 \,\}$
$\Re\mathrm{e}(s)$	real part of $s \in \mathbb{C}$, $\Re\mathrm{e}(\sigma + j\omega) = \sigma$
$\mathbb{R}(s)$	rational functions in s
$\mathbb{R}_p(s)$	proper rational functions in s with real coefficients, $\{\, h \in \mathbb{R}(s) \mid \lvert h(\infty)\rvert \neq \infty \,\}$
$\mathbb{R}_p(s)^{n_o \times n_i}$	$n_o \times n_i$ matrices with elements in $\mathbb{R}_p(s)$
$\mathbb{R}_{sp}(s)$	strictly proper rational functions in s with real coefficients, $\{\, h \in \mathbb{R}_p(s) \mid h(\infty) = 0 \,\}$
$\mathbf{R_U}$	proper rationals analytic in $\mathbf{U} \subset \mathbb{C}$
$\bar{s}$	complex conjugate of $s \in \mathbb{C}$
$\sigma(A)$	spectrum of A
$\sigma_{\max}(A)$	maximum singular-value of A

$S(P, C)$	unity-feedback system with linear time-invariant subsystems
$\Sigma(P, C)$	observer-controller feedback configuration with linear time-invariant subsystems (Figure 2.6)
$S(\mathsf{P}, \mathsf{C})$	unity-feedback system with linear subsystems
$\mathcal{S}(\mathsf{P}, \mathcal{C})$	unity-feedback system with linear plant and nonlinear compensator
$\mathcal{S}(\mathcal{P}, \mathcal{C})$	unity-feedback system with nonlinear subsystems
$\mathcal{\Sigma}(\mathcal{P}, \mathcal{Q})$	nonlinear feedback system with stable $\mathcal{Q}$ (Figure 4.6)
U	undesired pole location in $\mathbb{C}$, (Definition 2.2.1)
$\mathbb{Z}_+$	nonnegative integers, $\{0, 1, 2, \ldots\}$
$\xi_{\mathcal{P}}$	pseudo-state associated with an r.f. of $\mathcal{P}$

Contents

Chapter 1

Introduction

In the design of linear time-invariant multiinput-multioutput feedback systems, the parametrization of all stabilizing linear time-invariant compensators and the characterization of achievable performance (like achievable input-output maps, disturbance rejection, tracking, ...) by stabilizing compensators have been of great interest. For the lumped linear time-invariant, continuous-time and discrete-time cases, stabilizing compensators were obtained in [You.1,Per.1,Kuc.1]. Using a general algebraic formulation, [Des.4] generalizes these results to include the distributed cases, among others. Using an algebraic approach [Zam.2] considers stable plants, characterizes all stabilizing compensators and establishes the trade-off between input-output performance and robustness. For related work, see also [Ros.1,Doy.1,Des.5, Sae.1,Des.8] and the references therein. An excellent review of research in this area and related work can be found in [Vid.3] .

In its general algebraic framework, linear factorization theory has proven to be extremely useful in solving important control problems related to linear time-invariant systems that have transfer function representations. The multiinput-multioutput transfer functions that are considered can represent plants which are continuous-time or discrete-time, finite-dimensional or distributed, one-dimensional or multi-dimensional. Most plant models encountered in practice can be treated in this setting of a *commutative* algebra of linear maps. Using linear factorization theory for these systems, an abundance of results were obtained in the literature, where many control problems (like stabilization, parametrization of all stabilizing compensators, achievable input-output maps, robust stabilization, disturbance rejection, tracking, decoupling, decentralized control, ...) were solved.

There has been great interest in extending the existing linear factorization theory to linear time-varying and possibly to nonlinear maps. In [Fei.1] factorizations of linear input-output maps over Hilbert spaces are discussed. [Man.1] explicitly derives factorizations for a class of finite-dimensional linear time-varying plants which provide an extension to [Net.1,Kha.1] .

Along the same input-output approach, [Vid.1] introduces coprime factorizations (over Banach spaces) for nonlinear maps as a direct extension of the well-known Bezout-identity (see also [Des.6]) . [Ham.1,Ham.2,Ham.3,Ham.4,Ham.5] pose the stabilization problem of a time-invariant nonlinear discrete-time plant; using a set-theoretic approach, key points of linear factorization theory that are suitable for generalizations to this setting are emphasized. Finite-dimensional nonlinear systems with recursive descriptions are studied in detail. The structured extension of the Bezout-identity is used in feedback stabilization of such discrete-time nonlinear plants.

The purpose of this work is to study the extension of factorizations to nonlinear multiinput-multioutput maps. These factorizations will be used for analyzing nonlinear feedback interconnections and for proposing new stabilizing feedback configurations for nonlinear plants. The only assumption on the nonlinear maps that are considered is that they are causal and defined over input and output extended spaces. This standard input-output approach is quite general and is suitable for generalized forms of bounded-input bounded-output stability analysis.

The thesis is organized as follows:

Each chapter is a stepping stone to the next. Chapter 2 consists of an extensive review of proper stable factorizations (for finite-dimensional linear time-invariant systems); it also includes a different way of looking at the unity-feedback system and the observer-controller configuration (see Figures 2.3 and 2.6) and robustness results associated with the linear time-invariant unity-feedback system (Sections 2.5 – 2.7) . Key facts useful for our study are also stated (for a complete description see Section 2.1) .

Chapter 3 focuses on linear input-output maps; factorization tools for these maps are used in the analysis and the synthesis of linear feedback systems (for a complete description see Section 3.1) .

Chapter 4 studies the most general setting for factorizations of causal nonlinear input-output maps. Linearity and structure constraints are dropped. We study the tools of nonlinear factorization and apply them to stability and robustness analysis of nonlinear feedback systems; using these tools, stabilizing feedback configurations are proposed (see Section 4.1) .

The contribution of this work is its unified factorization approach to the analysis and synthesis of linear and nonlinear feedback systems. Examples are explicitly worked out to illustrate the conceptual tools.

Chapter 2

Proper Stable Factorizations

2.1 Introduction

In this chapter, all plants and compensators are multi-input multi-output linear time-invariant one-dimensional subsystems which are represented by continuous-time or discrete-time matrix transfer functions with proper rational entries; that is, all plants and compensators are realizable (with integrators or delay blocks) in a minimal state-space description (A, B, C, D). We discuss the stability and robust stability of two feedback interconnections of such subsystems:

i) the standard unity-feedback configuration $S(P, C)$ (see Figure 2.3),

ii) the observer-controller configuration $\Sigma(P, C)$ (see Figure 2.6), a special case of two-input one-output compensation.

The discussion is based on factorization theory [Vid.3]; for this reason, we give an extensive review of proper stable factorizations (a special case of the more general algebraic setting of factorization theory) and include the key facts that relate to our study.

The chapter is organized as follows:

Section 2.2 introduces the preliminary definitions and facts on proper stable factorizations. A proper, rational transfer function is called $\mathbf{R}_{\mathbf{U}}$-stable if its poles are *not* in the undesired region $\mathbf{U}$ (see Definition 2.2.1) . For the robustness analysis later on, the boundary $\partial\mathbf{U}$ of the undesired region $\mathbf{U}$ satisfies the matching condition in Definition 2.2.4 . Definition 2.2.6 introduces a norm on $\mathbf{R}_{\mathbf{U}}$-stable maps. This norm definition will be used also in robustness analysis. Definitions 2.2.7 – 2.2.10 introduce right and left factorizations for proper rational transfer functions. The properties of right and left factorizations are also stated.

Section 2.3 illustrates that two-input one-output compensation is the most general linear feedback compensation of a linear plant if all of the plant inputs and outputs are to be used. The set of all $\mathbf{R}_{\mathbf{U}}$-stabilizing two-input one-output compensators are given in Fact 2.3.3. A simple proof of this well-known fact points out the specific structure constraint on the compensator: all instabilities of the compensator must be due to one denominator map. All $\mathbf{R}_{\mathbf{U}}$-stabilizing compensators in the standard unity-feedback configuration follows from this fact (see Corollary 2.34) .

Any proper plant (with a minimal state-space description) can be stabilized by a two-input one-output compensator (e.g. a full-order observer-controller); this specific configuration is referred to as the observer-controller configuration $\Sigma(P, C)$ (see Figure 2.6) . The interesting point is that from an $\mathbf{R}_{\mathbf{U}}$-stable $\Sigma(P, C)$, one can derive an $\mathbf{R}_{\mathbf{U}}$-stabilizing compensator in the unity-feedback configuration; in the process, one Bezout-identity (see (2.11)) is sufficient. Using this observation together with all $\mathbf{R}_{\mathbf{U}}$-stabilizing compensators in $S(P, C)$ we give

i) a parametrization of all strictly proper compensators ,

ii) a simple proof of the fact that the parameter set is dense (see (2.15) and Fact 2.4.6) .

Section 2.5 considers robustness of the $\mathbf{R_U}$-stabilizing compensator in $S(P,C)$ under plant uncertainties. A standard way of characterizing unstructured plant uncertainties is to use a ball description: at each $s \in \partial\mathbf{U}$, the norm of the perturbation transfer function Δ (possibly unstable) is within a specified radius. Using this perturbation Δ , four cases of plant perturbation models are discussed in subsections 2.5.1 – 2.5.4 : pre-multiplicative, post-multiplicative, feedback and additive plant perturbations. In each subsection we state the necessary and sufficient condition for the nominal stabilizing compensator to stabilize the perturbed plant. Using the necessary and sufficient condition, we show that in a ball description, the uncertainty Δ can *not* be unstable. We show that given *any* radius map, there is always a destabilizing unstable Δ in the specified ball; hence the perturbations Δ in a ball must be $\mathbf{R_U}$-stable to get a uniform robustness condition.

Section 2.6 studies the special case that the perturbation Δ is $\mathbf{R_U}$-stable (which is proven to be necessary in Section 2.5) . Using this ball description of $\mathbf{R_U}$-stable perturbations, we state the necessary and sufficient conditions for robust stabilization for each of the four subcases in Section 2.5 .

2.2 Preliminaries

Definition 2.2.1 (undesired region U)

A nonempty closed set $\mathbf{U} \subset \mathbb{C}$ is called an *undesired region* iff

i) $s \in \mathbf{U} \Leftrightarrow \bar{s} \in \mathbf{U}$ and

ii) $\pm\infty \in \mathbf{U}$ and

iii) $\exists\, \alpha \in [0, \infty)$ such that $[\alpha, \infty] \subset \mathbf{U}$.

□

An undesired region $\mathbf{U}$ denotes the undesired closed-loop pole locations. The concept of undesired region $\mathbf{U}$ applies to transfer functions of both continuous-time and discrete-time systems. Consider the following examples of undesired regions:

i) For $\sigma \leq 0$,

$$\mathrm{R}_\sigma := \{\, s \in \mathbb{C} \mid \Re\mathrm{e}(s) \geq \sigma \,\} . \tag{2.1}$$

ii) For $\sigma \in (0, 1]$,

$$\mathrm{D}_\sigma := \{\, s \in \mathbb{C} \mid |s| \geq \sigma \,\} . \tag{2.2}$$

iii) For $\alpha \geq 0$,

$$\mathrm{K}_\alpha := \{\, s \in \mathbb{C} \mid \Re\mathrm{e}(s) \geq -\alpha |\Im\mathrm{m}(s)| \,\} . \tag{2.3}$$

iv) For $\alpha \geq 0$ and $\sigma \leq 0$, $\mathbf{U} = \mathrm{K}_\alpha \cap \mathrm{R}_\sigma$.

Definition 2.2.2 (the ring $\mathbf{R}_\mathbf{U}$)

For a given undesired region $\mathbf{U}$, the ring $\mathbf{R}_\mathbf{U} \subset \mathbb{R}_p(s)$ is defined as

$$\mathbf{R}_\mathbf{U} := \{\, u \in \mathbb{R}_p(s) \mid u \text{ is analytic in } \mathbf{U} \,\} .$$

Definition 2.2.3 ($\mathbf{R_U}$-stable)

A map $H \in \mathbb{R}_p(s)^{n_o \times n_i}$ is called $\mathbf{R_U}$-*stable* iff $H : \mathbf{R_U}^{n_i} \to \mathbf{R_U}^{n_o}$ (denoted by $H \in \mathbf{R_U}^{n_o \times n_i} \subset \mathrm{M}(\mathbf{R_U})$).

Definition 2.2.4 (matching condition)

An undesired region $\mathbf{U}$ is said to satisfy the *matching condition* iff for all $x_0 \in \mathbb{C}$, $s_0 \in \partial\mathbf{U}$, there exists an $h \in \mathbf{R_U}$ such that

i) $h(s_0) = x_0$ and

ii) $|h(s)| \leq |x_0| \quad \forall\, s \in \partial\mathbf{U}$.

□

Most undesired regions used in practice satisfy the matching condition; the following fact shows that the sets R_σ and D_σ do so.

Fact 2.2.5 (all-pass fit on $\partial\mathrm{R}_\sigma$ and $\partial\mathrm{D}_\sigma$)

i) For a given $x_0 \in \mathbb{C}$ and $s_0 \in \partial\mathrm{R}_\sigma$ (see (2.1)), there exists a parameter $\alpha \in (0, \infty)$ such that the map $h_\sigma : \mathbf{U} \to \mathbb{C}$ given by

$$h_\sigma(s) = |x_0| \left(\frac{s - \sigma - \alpha}{s - \sigma + \alpha} \right)^2$$

satisfies the matching condition in Definition 2.2.4.

ii) For a given $x_0 \in \mathbb{C}$ and $s_0 \in \partial\mathrm{D}_\sigma$ (see (2.2)), there exists an $n \geq 1$ and parameters $\alpha_i \in (\mathbb{C} \setminus \mathrm{D}_\sigma)$, $i = 1, \ldots, n$, such that the map $h_\sigma : \mathbf{U} \to \mathbb{C}$ given by

$$h_\sigma(s) = \frac{1}{\sigma} |x_0| \prod_{i=1}^{n} \frac{\bar{\alpha}_i s - \sigma^2}{s - \alpha_i}$$

satisfies the matching condition in Definition 2.2.4.

Definition 2.2.6 ($\|\cdot\|_{\mathbf{U}}$)

For any undesired region $\mathbf{U}$, the norm $\|\cdot\|_{\mathbf{U}} : \mathbf{M}(\mathbf{R_U}) \to \mathbb{R}_+$ is defined by

$$\|H\|_{\mathbf{U}} := \sup_{s \in \partial \mathbf{U}} \|H\| . \tag{2.4}$$

□

By definition, a map $H \in \mathbf{M}(\mathbf{R_U})$ has entries which are analytic in $\mathbf{U}$. For an analytic function h , $|h|^p$ is subharmonic for $p \in (0, \infty)$ [Rud.1]. Hence,

$$\|H(s_0)\| \leq \sup_{s \in \partial \mathbf{U}} \|H(s)\| = \|H\|_{\mathbf{U}} \qquad \forall\, s_0 \in \mathbf{U} . \tag{2.5}$$

When $\mathbf{U} = \mathrm{R}_0 = \mathbb{C}_+$, we have

$$\|H\|_{\mathbb{C}_+} = \sup_{s \in \partial \mathbb{C}_+} \|H(s)\| = \sup_{\omega \in \mathbf{R}} \|H(j\omega)\| = \|H\|_\infty .$$

Similarly when $\mathbf{U} = \mathrm{D}_1$, we have

$$\|H\|_{\mathrm{D}_1} = \sup_{s \in \partial \mathrm{D}_1} \|H(s)\| = \sup_{\theta \in [0, 2\pi]} \|H(e^{j\theta})\| = \|H\|_\infty .$$

Definition 2.2.7 (right factorization)

(N_p, D_p) is said to be a *right factorization (r.f.)* of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ iff

i) N_p , $D_p \in \mathbf{M}(\mathbf{R_U})$ and

ii) $D_p \in \mathbf{R_U}^{n_i \times n_i}$ has an inverse and

iii) $N_p D_p^{-1} = P$.

Definition 2.2.8 (left factorization)

$(\widetilde{D}_p, \widetilde{N}_p)$ is said to be a *left factorization (l.f.)* of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ iff

i) $\widetilde{N}_p$, $\widetilde{D}_p \in \mathbf{M}(\mathbf{R_U})$ and

ii) $\widetilde{D}_p \in \mathbf{R_U}^{n_o \times n_o}$ has an inverse and

iii) $\widetilde{D}_p^{-1}\widetilde{N}_p = P$.

□

In a right (left) factorization, the denominator map D ($\widetilde{D}$) is invertible, however the inverse D^{-1} ($\widetilde{D}^{-1}$) need not be proper.

Definition 2.2.9 (right-coprime factorization)

(N_p, D_p) is said to be a *right-coprime factorization (r.c.f.)* of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ iff

i) (N_p, D_p) is an r.f. of P <u>and</u>

ii) there exist $\widetilde{U}$, $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$ such that

$$\begin{bmatrix} \widetilde{U} & \widetilde{V} \end{bmatrix} \begin{bmatrix} N_p \\ D_p \end{bmatrix} = \mathrm{I} .$$

Definition 2.2.10 (left-coprime factorization)

$(\widetilde{D}_p, \widetilde{N}_p)$ is said to be a *left-coprime factorization (l.c.f.)* of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ iff

i) $(\widetilde{D}_p, \widetilde{N}_p)$ is an l.f. of P <u>and</u>

ii) there exist U , $V \in \mathbf{M}(\mathbf{R_U})$ such that

$$\begin{bmatrix} \widetilde{N}_p & \widetilde{D}_p \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \mathrm{I} .$$

□

An important property of the members of $\mathbf{M}(\mathbb{R}_p(s))$ is that they have both an r.c.f. and an l.c.f. description [Vid.3] . This greatly simplifies the analysis of arbitrary feedback interconnections of subsystems which have proper rational transfer function descriptions [Vid.3,Cal.2].

In a coprime factorization of a proper map, the denominator map always has a proper inverse. The following fact is proved for an l.c.f. of a proper map; the r.c.f. version follows similarly.

Fact 2.2.11

Let $(\widetilde{D}_p, \widetilde{N}_p)$ ((N_p, D_p)) be an l.c.f. (r.c.f.) of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$, as in Definition 2.2.10 (2.2.9) ; then $\widetilde{D}_p^{-1} \in \mathrm{M}(\mathbb{R}_p(s))$ ($D_p^{-1} \in \mathrm{M}(\mathbb{R}_p(s))$) .

Proof

Let U and V be $\mathbf{R}_{\mathrm{U}}$-stable maps satisfying condition ii) of Definition 2.2.10. Since $\widetilde{D}_p^{-1}$ exists, we have

$$\widetilde{D}_p^{-1} = \widetilde{D}_p^{-1}\widetilde{N}_p U + V = PU + V \quad ,$$

which is proper.

□

The following fact establishes a simple one point check at ∞ to determine the existence of a proper inverse.

Fact 2.2.12 (n&s condition for a proper inverse)

Let $D \in \mathbb{R}_p(s)^{n \times n}$; then D^{-1} exists and $D^{-1} \in \mathbb{R}_p(s)^{n \times n}$ if and only if $\det D(\infty) \neq 0$.

Proof

" if "

Let $\det D(\infty) \neq 0$. Since $\det D(\cdot) \neq 0$, $D^{-1} \in \mathbb{R}(s)^{n \times n}$ and is given by $D^{-1} = \mathrm{Adj}D / \det D$. Since $\mathrm{Adj}D$ and $1/\det D$ are proper, D^{-1} is proper.

" only if "

Let D and D^{-1} be proper; then $\det D(\infty)$ and $\det D^{-1}(\infty)$ are finite. Since $\det D(\infty) \det D^{-1}(\infty) = 1$, we have $\det D(\infty) \neq 0$.

□

A plant $P \in \mathrm{M}(\mathbb{R}_p(s))$ does *not* have a unique r.c.f. (l.c.f.); however, *all* r.c.f.s (l.c.f.s) of P can be obtained from a given r.c.f. (l.c.f.) of P .

Definition 2.2.13 ($\mathbf{R_U}$-unimodular)

$M \in \mathbf{R_U}^{n \times n}$ is said to be $\mathbf{R_U}$-*unimodular* iff $M^{-1} \in \mathbf{R_U}^{n \times n}$.

□

The following fact [Vid.3] shows that an r.c.f. (l.c.f.) of a given plant is unique up to unimodular factors.

Fact 2.2.14 (all r.c.f.s (l.c.f.s) are related by $\mathbf{R_U}$-unimodular maps)
Let (N_p, D_p) ($(\widetilde{D}_p, \widetilde{N}_p)$) be an r.c.f (l.c.f) of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$; then (N_1, D_1) ($(\widetilde{D}_1, \widetilde{N}_1)$) is an r.c.f. (l.c.f.) of $P \in \mathbf{M}(\mathbb{R}_p(s))$ if and only if there exists an $\mathbf{R_U}$-unimodular map $M \in \mathbf{R_U}^{n_i \times n_i}$ ($\widetilde{M} \in \mathbf{R_U}^{n_o \times n_o}$) such that

$$\begin{bmatrix} N_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} N_p \\ D_p \end{bmatrix} M \qquad (\begin{bmatrix} \widetilde{D}_1 & \widetilde{N}_1 \end{bmatrix} = \widetilde{M} \begin{bmatrix} \widetilde{D}_p & \widetilde{N}_p \end{bmatrix}) .$$

□

2.3 Feedback Interconnections

Consider an n_i input n_o output plant $P \in \mathbb{R}_p(s)^{n_o \times n_i}$. Since transfer function approach is an input-output approach, we assume that the plant description has no hidden modes in $\mathbf{U}$; that is, if the plant is input-output stabilized, all of the internal variables are guaranteed to be stabilized as well. Suppose that all of the plant inputs and outputs are to be used in a closed-loop compensation scheme. Since the measured output y_m and the control input v are the *only* signals that are available for computing the control signal y_c (see Figure 2.1), the most general feedback interconnection is the one that uses a two-input one-output nonlinear compensator $\mathcal{C}$ as in Figure 2.1. The signals d_i and d_o denote the input and output disturbances, respectively.

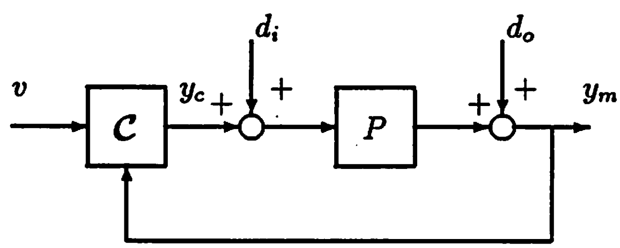

Figure 2.1:
Two-input one-output general compensation scheme

Definition 2.3.1 ($\mathbf{R_U}$-stable feedback system)

A *feedback system is* said to be $\mathbf{R_U}$*-stable* iff all of the closed-loop maps (mapping the closed-loop system inputs to the internal signals (i.e., input and output signals of each subsystem)) are $\mathbf{R_U}$-stable .

□

For example, the feedback system in Figure 2.1 is $\mathbf{R_U}$-stable if and only if the closed-loop map $(v\,,\, d_i\,,\, d_o) \mapsto (y_c\,,\, y_m)$ is $\mathbf{R_U}$-stable .

If the compensator $\mathcal{C}$ in Figure 2.1 is required to be proper, the *linearity* of the compensator implies

$$\mathcal{C}(v\,,\,y_m) = -C_1 y_m \;+\; C_2 v$$

for some proper maps C_1 and C_2 ; hence we obtain the additive feedback scheme shown in Figure 2.2 :

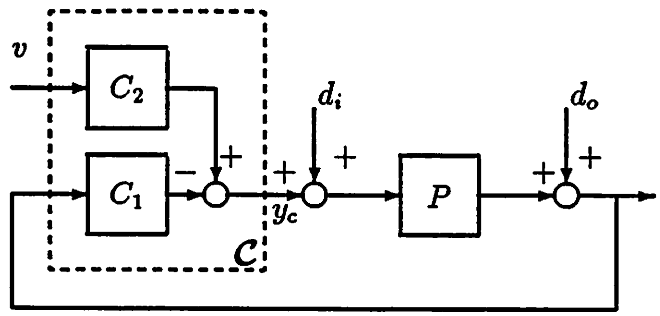

Figure 2.2:
Figure 2.1 with $\mathcal{C}$ *proper*

When C_2 in Figure 2.2 is set to zero, after re-labeling the inputs, we obtain the standard unity-feedback system $S(P, C)$ shown in Figure 2.3.

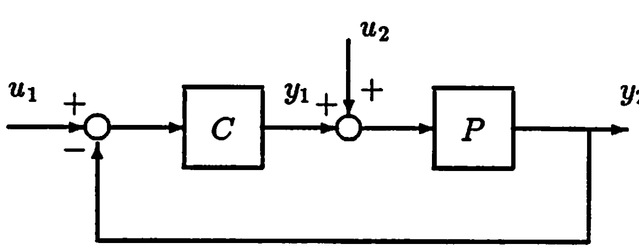

Figure 2.3:
The unity-feedback system $S(P, C)$

Definition 2.3.2 ($\mathbf{R_U}$-stabilizing compensator)

A map $C \in \mathrm{M}(\mathbb{R}_p(s))$ in a feedback system (say Figure 2.3) is said to be an

$\mathbf{R_U}$-*stabilizing compensator* iff the resulting feedback system is $\mathbf{R_U}$-stable .

□

It is well-known that if C in Figure 2.2 is required to be a $\mathbf{R_U}$-stabilizing compensator, C must satisfy an additional structure constraint; namely, C must have an l.c.f. of the form

$$\left(\widetilde{D}_c \, , \, \begin{bmatrix} \widetilde{N}_c & \widetilde{R} \end{bmatrix} \right)$$

where $(\widetilde{D}_c \, , \widetilde{N}_c)$ is an l.c.f. of C_1 and $\widetilde{R} \in \mathbf{M}(\mathbf{R_U})$ [Vid.3,Des.10,Net.2]. In other words, the instabilities in C_2 must be a subset of the instabilities of C_1 . To see the *necessity* of this structure constraint, consider the following siso example: let $\mathbf{U} = \mathbb{C}_+$, $p = 1$, $c_1 = \frac{1}{s}$ and $c_2 = \frac{1}{s-1}$; the closed-loop map $(0 \, , d_i \, , d_o) \mapsto y_c$ (see Figure 2.2) is $\mathbf{R_U}$-stable ; however the map $(v \, , 0 \, , 0) \mapsto y_c$ is *not*.

Fact 2.3.3 (all $\mathbf{R_U}$-stabilizing two-input one-output compensators)

Let (N_p, D_p) ($(\widetilde{D}_p, \widetilde{N}_p)$) be an r.c.f. (l.c.f.) of the plant $P \in \mathbb{R}_p(s)^{n_o \times n_i}$. Choose the $\mathbf{R_U}$-stable maps U , V , $\widetilde{U}$, $\widetilde{V}$ such that

$$\begin{bmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & V \\ D_p & -U \end{bmatrix} = \mathbf{I} \ . \tag{2.6}$$

Under these assumptions, the set of all two-input one-output $\mathbf{R_U}$-stabilizing compensators is given by

$$\Big\{ \, (\widetilde{V} - \widetilde{Q}\widetilde{N}_p)^{-1} [\, (\widetilde{U} + \widetilde{Q}\widetilde{D}_p) \quad \widetilde{R} \,] \quad | \quad \widetilde{R} \, , \widetilde{Q} \in \mathbf{M}(\mathbf{R_U}) \ \text{and} \ \det(\widetilde{V} - \widetilde{Q}\widetilde{N}_p)(\infty) \neq 0 \Big\} . \tag{2.7}$$

□

Note that equation (2.6) is of the form

$$A^{-1}A = \mathrm{I} \quad ,$$

where A and A^{-1} are $\mathbf{R_U}$-stable square matrices. Since

$$A^{-1}A = AA^{-1} = \mathrm{I} \quad ,$$

equation (2.6) holds if and only if

$$\begin{bmatrix} N_p & V \\ D_p & -U \end{bmatrix} \begin{bmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} = \mathrm{I} \quad . \tag{2.8}$$

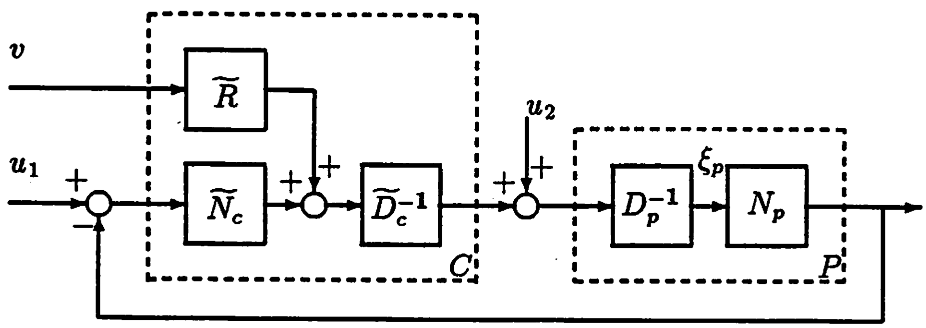

Figure 2.4:
A stable feedback system with two-input one-output compensator

($\widetilde{D}_c = \widetilde{V} - \widetilde{Q}\widetilde{N}_p$, $\widetilde{N}_c = \widetilde{U} + \widetilde{Q}\widetilde{D}_p$ for some $\widetilde{Q}$ satisfying (2.7))

Since (N_p, D_p) is an r.c.f. of P , the feedback system in Figure 2.4 is $\mathbf{R_U}$-stable if and only if the closed-loop pseudo-state map

$$(v\,,\,u_1\,,\,u_2) \mapsto \xi_p$$

is $\mathbf{R_U}$-stable (ξ_p denotes the pseudo-state of the r.c.f. description (N_p, D_p)) .

Proof of Fact 2.3.3

From Figure 2.2 and (2.6), direct calculation shows that any compensator in (2.7) is an $\mathbf{R_U}$-stabilizing compensator.

Conversely, if the feedback system in Figure 2.2 is $\mathbf{R_U}$-stable , (setting $v = 0$) C_1 has an l.c.f. $(\widetilde{D}_c, \widetilde{N}_c)$ such that $\widetilde{D}_c D_p + \widetilde{N}_c N_p = I$. Since the closed-loop map $v \mapsto \xi_p$ is $\mathbf{R_U}$-stable ,

$$\widetilde{D}_c C_2 \in \mathbf{M}(\mathbf{R_U}) \quad ,$$

hence the compensator C is of the form specified in (2.7).

□

For the special case where the map $\widetilde{R}$ in Figure 2.4 is set to zero, we obtain the $\mathbf{R_U}$-stable unity-feedback system $S(P,C)$ shown in Figure 2.5.

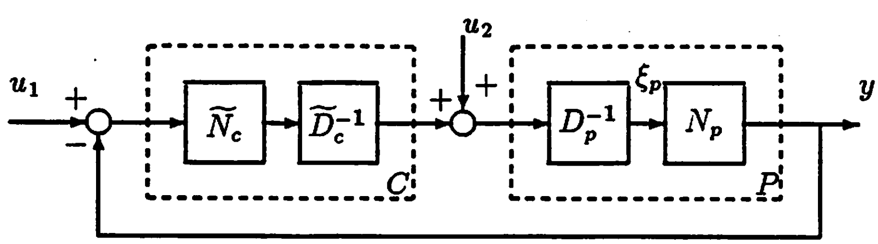

Figure 2.5:
A stable unity-feedback system $S(P,C)$

($\widetilde{D}_c = \widetilde{V} - \widetilde{Q}\widetilde{N}_p$, $\widetilde{N}_c = \widetilde{U} + \widetilde{Q}\widetilde{D}_p$ for some $\widetilde{Q}$ satisfying (2.9))

Setting $\widetilde{R}$ to zero in Fact 2.3.3 , we obtain the following corollary.

Corollary 2.3.4 (all $\mathbf{R_U}$-stabilizing compensators in $S(P,C)$)
Let (N_p, D_p) be an r.c.f. and $(\widetilde{D}_p, \widetilde{N}_p)$ be an l.c.f. of the plant $P \in \mathbb{R}_p(s)^{n_o \times n_i}$, satisfying equation (2.6). Then the set of all $\mathbf{R_U}$-stabilizing compensators in

$S(P,C)$ is given by

$$\left\{ (\widetilde{V} - \widetilde{Q}\,\widetilde{N}_p)^{-1}(\widetilde{U} + \widetilde{Q}\,\widetilde{D}_p) \quad | \quad \widetilde{Q} \in \mathbf{M}(\mathbf{R_U}) \text{ , } \det(\widetilde{V} - \widetilde{Q}\,\widetilde{N}_p)(\infty) \neq 0 \right\} ; \tag{2.9}$$

equivalently

$$\left\{ (U + D_pQ)(V - N_pQ)^{-1} \quad | \quad Q \in \mathbf{M}(\mathbf{R_U}) \text{ , } \det(V - N_pQ)(\infty) \neq 0 \right\}. \tag{2.10}$$

□

2.4 Proper stabilizing compensators in $S(P,C)$

Consider the following classical problem:

Problem 2.4.1

For a given proper plant $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ with an r.c.f. (N_p, D_p) satisfying

$$\widetilde{U} N_p + \widetilde{V} D_p = \mathrm{I} \tag{2.11}$$

for some $\widetilde{U}$, $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$, find a *proper* compensator C that stabilizes P in $S(P,C)$.

□

Note that we have only the (1,1)-entry of equation (2.6) at hand. There is no assumption that $\widetilde{V}$ is invertible or has a proper inverse. The standard way of answering Problem 2.4.1 is in two parts:

i) If P is strictly proper (i.e., P has a blocking zero at ∞ ; equivalently, $N_p(\infty) = 0$), from (2.11), we have $\det \widetilde{V}(\infty) \neq 0$; hence, by Fact 2.2.12 , we conclude that $\widetilde{V}^{-1}\widetilde{U}$ is a proper stabilizing compensator.

ii) If P is proper but not strictly proper (i.e., $P(\infty) \neq 0$), we also need an l.c.f. $(\widetilde{D}_p, \widetilde{N}_p)$ of P to pick a $\widetilde{Q}$ so that the determinant condition in (2.9) is satisfied.

In other words, if P is *not* strictly proper, then we need to bring in all the tools necessary to find the set of all proper stabilizing compensators (Corollary 2.3.4) in $S(P,C)$ to pick *only one*.

Consider the $\mathbf{R_U}$-stable observer-controller configuration $\Sigma(P,C)$ shown in Figure 2.6 (M is $\mathbf{R_U}$-unimodular).

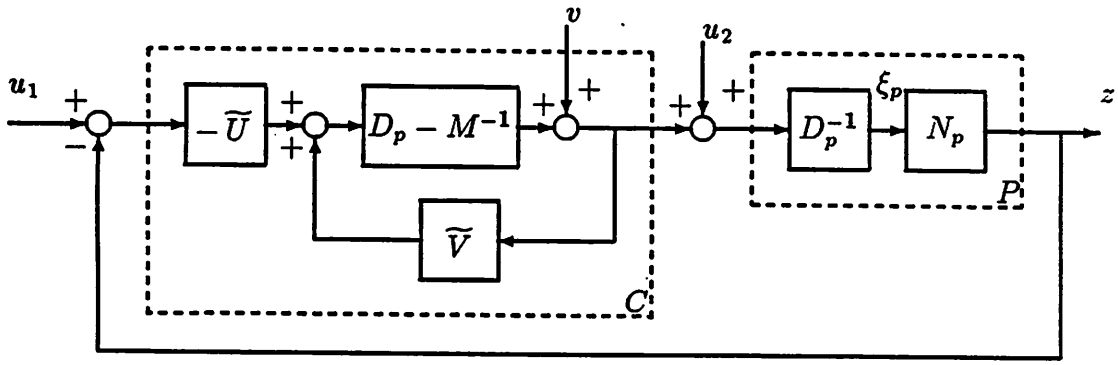

Figure 2.6:
Observer-controller configuration $\Sigma(P, C)$

Standard calculation shows that the feedback system $\Sigma(P, C)$ in Figure 2.6 is $\mathbf{R_U}$-stable for *all* $\mathbf{R_U}$-unimodular maps M and for *all* $\widetilde{U}$, $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$ satisfying the identity (2.11). In terms of stabilization, (2.11) is the crucial identity for both $S(P, C)$ and $\Sigma(P, C)$; however, the compensator in $\Sigma(P, C)$ is less restrictive in the sense that the map $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$ need *not* be invertible or have a proper inverse.

The feedback system $\Sigma(P, C)$ in Figure 2.6 is a special case of the general two-input one-output feedback system in Figure 2.4. To see this, redraw Figure 2.6 as Figure 2.7 in order to obtain the specific structure in Figure 2.4.

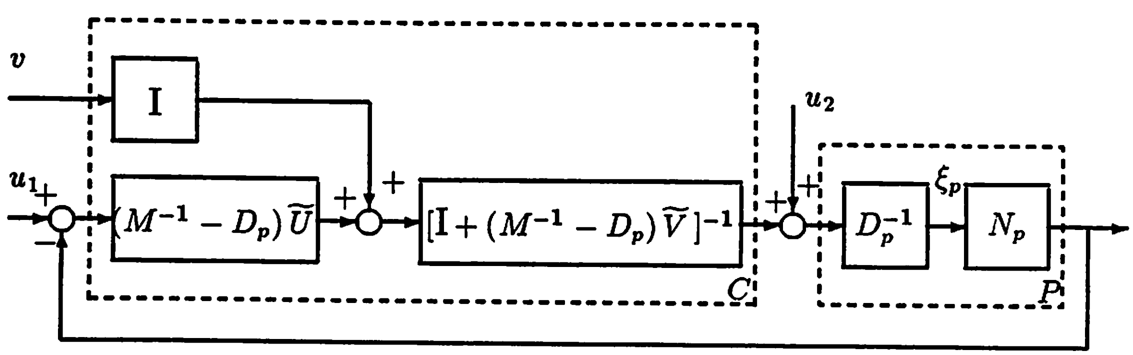

Figure 2.7:
Figure 2.6 redrawn as Figure 2.4

Set $v = 0$ in Figure 2.7 . Note that

$$\left(\left[\mathrm{I} + (M^{-1} - D_p)\widetilde{V} \right], (M^{-1} - D_p)\widetilde{U} \right)$$

is an l.c.f. of a proper C if and only if the map

$$\left[\mathrm{I} + (M^{-1} - D_p)\widetilde{V} \right]$$

has a proper inverse. An obvious way of satisfying this is by choosing

$$M^{-1} = D_p(\infty) \quad .$$

Hence the observer-controller configuration $\Sigma(P, C)$ allows us to answer Problem 2.4.1 in *one* step, using *only* the identity in (2.11).

Proposition 2.4.2 (deriving a proper C from (2.11) for $S(P, C)$)

Let $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ be given by an r.c.f. (N_p, D_p) , satisfying equation (2.11) ; then the pair

$$\left(\widetilde{D}_c, \widetilde{N}_c\right) := \left(\left[\mathrm{I} + (D_p(\infty) - D_p)\widetilde{V} \right], (D_p(\infty) - D_p)\widetilde{U} \right) \qquad (2.12)$$

is an l.c.f. of a <u>proper</u> compensator $C \in \mathbb{R}_p(s)^{n_i \times n_o}$ which stabilizes P in $S(P, C)$.

Proof

Let $\widetilde{D}_c$ and $\widetilde{N}_c$ be as in (2.12) . Note that $\widetilde{D}_c$, $\widetilde{N}_c \in \mathbf{M}(\mathbf{R_U})$ and $\widetilde{D}_c(\infty) = \mathrm{I}$. By Fact 2.2.12 , $\widetilde{D}_c^{-1} \in \mathbb{R}_p(s)^{n_i \times n_i}$. Since $D_p(\infty) \in \mathbf{M}(\mathbb{R})$ is $\mathbb{R}$-unimodular and

$$\widetilde{N}_c N_p + \widetilde{D}_c D_p = D_p(\infty) \quad ,$$

we conclude that $C = \widetilde{D}_c^{-1}\widetilde{N}_c$ is an $\mathbf{R_U}$-stabilizing compensator.

□

In the rest of this section, without loss of generality, we assume that the denominator maps D_p and $\widetilde{D}_p$ of r.c.f.s and l.c.f.s of P are normalized so that they satisfy

$$D_p(\infty) = \mathrm{I} \quad , \quad \widetilde{D}_p(\infty) = \mathrm{I} \quad .$$

If this is not the case, since $D_p(\infty)$ and $\widetilde{D}_p(\infty)$ are $\mathbf{R}_{\mathbf{U}}$-unimodular , modify the identity in (2.6) as follows:

$$\left\{\begin{bmatrix} D_p(\infty) & 0 \\ 0 & \widetilde{D}_p(\infty)^{-1} \end{bmatrix}\begin{bmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix}\right\}\left\{\begin{bmatrix} N_p & V \\ D_p & -U \end{bmatrix}\begin{bmatrix} D_p(\infty)^{-1} & 0 \\ 0 & \widetilde{D}_p(\infty) \end{bmatrix}\right\} = \mathrm{I}.$$

From a design point of view, it may cause concern that the compensator proposed in (2.12) may have more zeros than those of $\widetilde{U}$. There are many ways of constructing stabilizing compensators in $S(P, C)$ from Figure 2.7. It is possible to make the zeros of the compensator *identical* to those of $\widetilde{U}$, as the following proposition points out.

Proposition 2.4.3

Let $\widetilde{V}$, $D_p \in \mathbf{R}_{\mathbf{U}}{}^{n_i \times n_i}$ and $D_p(\infty) = \mathrm{I}$; then there exists an $\mathbf{R}_{\mathbf{U}}$-unimodular map $M \in \mathbf{R}_{\mathbf{U}}{}^{n_i \times n_i}$ such that

i) $(M^{-1} - D_p)$ is $\mathbf{R}_{\mathbf{U}}$-unimodular <u>and</u>

ii) $\left[\, \mathrm{I} + (M^{-1} - D_p)\widetilde{V} \,\right]$ has a proper inverse.

Proof

Choose $m_0 > 0$ such that

$$\| D_p \|_{\mathbf{U}} < 1 + m_0 \;\; ;$$

Let

$$M := \frac{1}{1+m}\mathrm{I} \quad , \; m \geq m_0 \; . \tag{2.13}$$

For any M in (2.13) , $(\mathrm{I} - MD_p)$ has an inverse and $\|(\mathrm{I} - MD_p)^{-1}\|_{\mathrm{U}}$ is bounded. Moreover

$$\det(\mathrm{I} - MD_p)(\infty) = (\frac{m}{1+m})^{n_i} \neq 0 \quad ;$$

hence $(\mathrm{I} - MD_p)$ is $\mathbf{R_U}$-unimodular . For the second condition, by Fact 2.2.12, it suffices to have

$$\det(\mathrm{I} + (M^{-1} - D_p)\widetilde{V})(\infty) = \det(\mathrm{I} + m\widetilde{V}(\infty)) = \prod_{j=1}^{n_i}(1 + m\lambda_j) \neq 0 \quad ,$$

where $\lambda_j \in \sigma(\widetilde{V}(\infty))$. Choosing $m \geq m_0$ such that $m \neq 1/|\lambda_j|$ for $\lambda_j \in (-\infty, 0)$ establishes the claim.

□

Note that the proposed compensator in Proposition 2.4.2 is a *strictly* proper compensator. Using the particular solution in Proposition 2.4.2, we can generate all solutions to (2.11); hence we get an equivalent characterization of the set of all proper stabilizing compensators in (2.9).

Corollary 2.4.4 (all proper stabilizing compensators in $S(P, C)$)

Let (N_p, D_p) be an r.c.f. of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ with $D_p(\infty) = \mathrm{I}$, satisfying equation (2.11) for some $\widetilde{U}$, $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$. Let $(\widetilde{D}_p, \widetilde{N}_p)$ be an l.c.f. of P , where $\widetilde{D}_p(\infty) = \mathrm{I}$. Under these assumptions, the set of all $\mathbf{R_U}$-stabilizing compensators in $S(P, C)$ is given by

$$\Big\{ \left[\mathrm{I} + (\mathrm{I} - D_p)\widetilde{V} - \widetilde{Q}\widetilde{N}_p\right]^{-1}\left[(\mathrm{I} - D_p)\widetilde{U} + \widetilde{Q}\widetilde{D}_p\right] \;\Big|\; \widetilde{Q} \in \mathbf{M}(\mathbf{R_U}) , \; \det\left(\mathrm{I} - \widetilde{Q}P\right)(\infty) \neq 0 \Big\} . \quad (2.14)$$

□

Note that the determinant condition in (2.14) can be expressed in terms of the plant P because,

$$\left[\mathrm{I} + (\mathrm{I} - D_p)\widetilde{V} - \widetilde{Q}\widetilde{N}_p\right](\infty) = \left(\mathrm{I} - \widetilde{Q}P\right)(\infty) \quad .$$

From (2.14) we also see that the set S_A of admissible parameters $\widetilde{Q}$'s (i.e., those that yield a proper compensator) is given by

$$S_A := \Big\{ \widetilde{Q}_{sp} + \widetilde{Q}_{\infty} \ \mid\ \widetilde{Q}_{sp} \in \mathbf{M}(\mathbf{R_U}) \cap \mathbb{R}_{sp}(s)^{n_i \times n_o} \ , \ \widetilde{Q}_{\infty} \in \mathbb{R}^{n_i \times n_o} \text{ and } \det(\mathrm{I} - \widetilde{Q}_{\infty} P(\infty)) \neq 0 \Big\} . \quad (2.15)$$

The characterization in (2.14) is due to a translation in the parameter $\widetilde{Q}$ in (2.9). To see this, let $D_p(\infty) = \mathrm{I}$, $\widetilde{D}_p(\infty) = \mathrm{I}$ and let equation (2.8) hold. From equation (2.6), we obtain

$$\begin{aligned} \mathrm{I} - D_p \widetilde{V} &= U \widetilde{N}_p \ , \\ D_p \widetilde{U} &= U \widetilde{D}_p \ ; \end{aligned}$$

substituting these in (2.14), we obtain

$$\Big\{ \big[\widetilde{V} - (\widetilde{Q} - U)\widetilde{N}_p \big]^{-1} \big[\widetilde{U} + (\widetilde{Q} - U)\widetilde{D}_p \big] \ \mid\ \widetilde{Q} \in \mathbf{M}(\mathbf{R_U}) \ , \ \det(\mathrm{I} - \widetilde{Q} P)(\infty) \neq 0 \Big\} . \quad (2.16)$$

Comparing (2.16) with (2.9), the translation U in the parameter $\widetilde{Q}$ is evident.

The description of the set of all proper stabilizing compensators in (2.14), allows us to obtain the *parametrization* of all *strictly* proper stabilizing compensators.

Corollary 2.4.5 (parametrization of all strictly proper compensators)

Let (N_p, D_p) be an r.c.f. of $P \in \mathbb{R}_p(s)^{n_o \times n_i}$ with $D_p(\infty) = \mathrm{I}$, satisfying (2.11), for some $\widetilde{U}$, $\widetilde{V} \in \mathbf{M}(\mathbf{R_U})$. Let $(\widetilde{D}_p, \widetilde{N}_p)$ be an l.c.f. of P , where $\widetilde{D}_p(\infty) = \mathrm{I}$. Under these assumptions, the set of all strictly proper $\mathbf{R_U}$-stabilizing compensators in $S(P, C)$ is given by

$$\left\{ \left[\mathrm{I} + (\mathrm{I} - D_p)\widetilde{V} - \widetilde{Q}\widetilde{N}_p \right]^{-1} \left[(\mathrm{I} - D_p)\widetilde{U} + \widetilde{Q}\widetilde{D}_p \right] \;\middle|\; \widetilde{Q} \in \mathbf{M}(\mathbf{R_U}) \cap \mathrm{I\!R}_{sp}(s)^{n_i \times n_o} \right\} . \quad (2.17)$$

□

The sets (2.17) and (2.15) imply that, if we insist on a proper but not strictly proper compensator, we need to determine a real matrix $\widetilde{Q}_\infty$ yielding a well-posed feedback interconnection with the DC-gain matrix of the plant ($P(\infty)$). An obvious way of satisfying this constraint is by choosing the maximum singular value $\sigma_{\max}$ of $\widetilde{Q}_\infty$ such that $\sigma_{\max} < 1/\|P(\infty)\|$.

From the characterization of $\mathrm{S_A}$ in (2.15), we conclude that $\mathrm{S_A}$ is an open dense subset of $\mathbf{R_U}^{n_i \times n_o}$ due to the following fact:

Fact 2.4.6

The set $\left\{ Q_\infty \in \mathrm{I\!R}^{n_i \times n_o} \;\middle|\; \det(\mathrm{I} - Q_\infty P(\infty)) \neq 0 \right\}$ is open and dense in $\mathrm{I\!R}^{n_i \times n_o}$.

Proof

If $P(\infty) = 0$, then there is no restriction on Q_∞ . Assume that $P(\infty) \neq 0$. Let the map $f : \mathrm{I\!R}^{n_i \times n_o} \rightarrow \mathrm{I\!R}$ be defined by

$$f(Q_\infty) := \det(\mathrm{I} - Q_\infty P(\infty)) .$$

The map f is continuous hence $f^{-1}[\mathrm{I\!R} \setminus \{0\}]$ is open in $\mathrm{I\!R}^{n_i \times n_o}$. Let Q^* be such that $f(Q^*) = 0$. Then $1 \in \sigma(Q^* P(\infty))$, where σ denotes the spectrum. Let

$$\delta_* := \begin{cases} 1 - \max\{\sigma(Q^* P(\infty)) \cap (0,1)\} & \text{if } \sigma(Q^* P(\infty)) \cap (0,1) \neq \emptyset \\ 2 & \text{otherwise} . \end{cases}$$

It suffices to show that given any $\epsilon > 0$, there exists a Q_∞ such that $\|Q_\infty - Q^*\| < \epsilon$ and $f(Q_\infty) \neq 0$. Fix $\epsilon > 0$. Choose $\epsilon' > 0$ such that

$$\epsilon' < \min\left\{ \delta_* \, , \, \frac{\epsilon}{\|Q^*\|} \right\} .$$

Let $Q_\infty := (1+\epsilon')Q^*$; then $||Q_\infty - Q^*|| < \epsilon$. We claim that

$$1 \notin \sigma(Q_\infty P(\infty)) = \sigma((1+\epsilon')Q^* P(\infty)) = (1+\epsilon')\sigma(Q^* P(\infty)) .$$

Since $0 < \epsilon' < \delta_*$, we have $(1+\epsilon')(1-\delta_*) < 1$; hence we conclude that $1 \notin \sigma(Q_\infty P(\infty))$, equivalently $f(Q_\infty) \neq 0$.

□

2.5 Robustness

Suppose that the nominal plant $P \in \mathbf{M}(\mathbb{R}_p(s))$ is stabilized by a compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$ in the unity-feedback configuration $S(P,C)$. From an input-output approach, one can model the uncertainties in the model of the plant in a number of ways by defining certain sets of admissible plant perturbations; for the specific uncertainty model in hand, one might determine, if possible, the necessary and sufficient conditions on the nominal compensator to guarantee that C stabilizes all possible plant models in the set of admissible plant perturbations.

If the set of admissible plant perturbations has finitely many plants, linear factorization theory gives the precise necessary and sufficient conditions for simultaneous stabilization of all of these plants [Vid.3]. If the set of admissible plant perturbations has infinitely many plants, certain ball descriptions may be used to define the plant perturbations.

In this section, we focus on a special class of plant perturbations. For a given undesired region $\mathbf{U}$ and a radius map $r \in \mathbf{R_U}$, let the set $\mathrm{B}(r)$ be defined as

$$\mathrm{B}(r) := \{\ \Delta \in \mathbf{M}(\mathbb{R}_p(s)) \ \mid\ \|\Delta(s)\| \leq |r(s)| \quad \forall s \in \partial\mathbf{U}\ \}\ . \tag{2.18}$$

Note that a perturbation $\Delta \in \mathrm{B}(r)$ is not required to be $\mathbf{R_U}$-stable ; all that is required is to have proper maps whose norms on $\partial\mathbf{U}$ (typically, the frequency response norms when $\mathbf{U} = \mathbb{C}_+$ or the complement of the open unit disk) are within the specified radius map r . Using the ball description in (2.18), we consider four cases of "unstructured" plant perturbations: pre- and post-multiplicative, feedback and additive perturbations. For each of the cases, we state the necessary and sufficient conditions for stability of the unity-feedback system with the perturbed plant; we show that the perturbation description in (2.18) must be further restricted to $\mathbf{R_U}$-stable maps for robustness results.

2.5.1 Pre-Multiplicative Perturbations

For a given undesired region $\mathbf{U}$, let the set of plant perturbations be given by

$$\left\{ P(\mathrm{I}+\Delta) \mid \Delta \in \mathrm{B}(r) \right\}, \tag{2.19}$$

where $\mathrm{B}(r)$ is defined in (2.18).

Lemma 2.5.1 (n&s condition for the stability of $S(P(\mathrm{I}+\Delta), C)$)
Let the maps P and $C \in \mathbf{M}(\mathbb{R}_p(s))$ be such that the feedback system $S(P, C)$ is $\mathbf{R}_\mathbf{U}$-stable. Choose the r.c.f.s (N_p, D_p) of P and (N_c, D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I},$$

for some l.c.f.s $(\widetilde{D}_p, \widetilde{N}_p)$, $(\widetilde{D}_c, \widetilde{N}_c)$ of P and C, respectively. Let (N_Δ, D_Δ) be an r.c.f. of $\Delta \in \mathbf{M}(\mathbb{R}_p(s))$. Consider the unity-feedback system $S(P(\mathrm{I}+\Delta), C)$ shown in Figure 2.8; note that only the input and the output of $P(\mathrm{I}+\Delta)$ are observed.

Under these assumptions, the feedback system $S(P(\mathrm{I}+\Delta), C)$ is $\mathbf{R}_\mathbf{U}$-stable (i.e., the closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is $\mathbf{R}_\mathbf{U}$-stable) if and only if the map

$$\widetilde{N}_p N_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p$$

is $\mathbf{R}_\mathbf{U}$-stable.

Proof

Let (N_Δ, D_Δ) be an r.c.f. of $\Delta \in \mathbf{M}(\mathbb{R}_p(s))$. By assumption, only the input and the output of $P(\mathrm{I}+\Delta)$ are observed. Hence, the unity feedback system $S(P(\mathrm{I}+\Delta), C)$ is $\mathbf{R}_\mathbf{U}$-stable if and only if the map $(u_1, u_2) \mapsto \xi_c$ (see Figure 2.8) is $\mathbf{R}_\mathbf{U}$-stable. Writing the summing node equations in Figure 2.8, we obtain

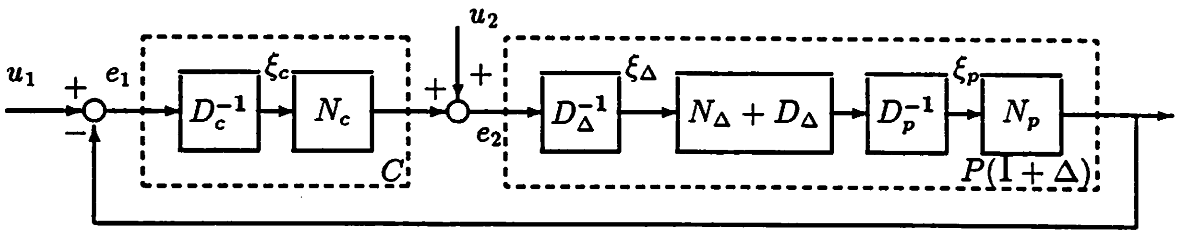

Figure 2.8:
Pre-multiplicative uncertainty model

$$\begin{bmatrix} D_c & N_p & 0 \\ -N_c & 0 & D_\Delta \\ 0 & -D_p & N_\Delta + D_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} . \tag{2.20}$$

Pre-multiplying both sides by the $\mathbf{R_U}$-unimodular matrices $\begin{bmatrix} I & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix}$, $\begin{bmatrix} \widetilde{D}_p & -\widetilde{N}_p & 0 \\ \widetilde{N}_c & \widetilde{D}_c & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & D_p & I \end{bmatrix}$ successively, we obtain

$$\begin{bmatrix} I & 0 & \widetilde{N}_p N_\Delta \\ 0 & I & -\widetilde{D}_c N_\Delta \\ 0 & 0 & D_\Delta + N_c \widetilde{N}_p N_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} \widetilde{D}_p u_1 - \widetilde{N}_p u_2 \\ \widetilde{N}_c u_1 + \widetilde{D}_c u_2 \\ D_p[\widetilde{N}_c u_1 + \widetilde{D}_c u_2] \end{bmatrix} . \tag{2.21}$$

Hence the map $(u_1, u_2) \mapsto \xi_c$ is given by

$$\xi_c = \widetilde{D}_p u_1 - \widetilde{N}_p u_2 - \widetilde{N}_p N_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p \begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{2.22}$$

Since the map $\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix}$ has a proper $\mathbf{R_U}$-stable right-inverse, we conclude that the map $(u_1, u_2) \mapsto \xi_c$ is $\mathbf{R_U}$-stable if and only if the map $\widetilde{N}_p N_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p$ is $\mathbf{R_U}$-stable .

□

We now prove that the ball description in (2.18) must be further restricted to guarantee that a compensator C stabilizes the class in (2.19) in the unity-feedback system.

Proposition 2.5.2 (admissible pre-multiplicative perturbations)

Let the plant $P \in \mathbf{M}(\mathbb{R}_p(s))$ be stabilized by the compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$

in the unity-feedback configuration $S(P,C)$; then given any undesired region $\mathbf{U}$ and any radius map $r \in \mathbf{R_U}$, there exists a $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S(P(\mathrm{I}+\Delta),C)$ is *not* $\mathbf{R_U}$-stable .

Proof

Let the radius map $r \in \mathbf{R_U}$ be given. We prove the claim by constructing a perturbation $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S(P(\mathrm{I}+\Delta),C)$ is *not* $\mathbf{R_U}$-stable . Using the notation of Lemma 2.5.1 , $S(P(\mathrm{I}+\Delta),C)$ is $\mathbf{R_U}$-stable if and only if the map

$$\widetilde{N}_p N_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p$$

is $\mathbf{R_U}$-stable . Hence, it is necessary that

$$N_c \widetilde{N}_p N_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p \;=\; [\mathrm{I} - D_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1}] D_p$$

is $\mathbf{R_U}$-stable . Then $S(P(\mathrm{I}+\Delta),C)$ is $\mathbf{R_U}$-stable *only if* the map

$$D_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p$$

is $\mathbf{R_U}$-stable . We now construct an r.c.f. (N_Δ, D_Δ) such that $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ and $D_\Delta (D_\Delta + N_c \widetilde{N}_p N_\Delta)^{-1} D_p$ is *not* $\mathbf{R_U}$-stable . Choose $s_0 \in \mathbb{R} \cap \mathbf{U}$ such that

$$s_0 > \max\Big\{ \text{ } (\mathbb{R} \cap \mathbf{U})\text{-blocking zeros of } N_c \widetilde{N}_p \text{ , } (\mathbb{R} \cap \mathbf{U})\text{-zeros of } r \text{ and } \det D_p \Big\},$$

and

$$\Big|\frac{1}{\mathrm{s} - \mathrm{s}_0}\Big| \leq 1 \quad \forall s \in \partial\mathbf{U} \text{ .}$$

Note that for all $\alpha \geq s_0$,

$$\Big|\frac{1}{\mathrm{s} - \alpha}\Big| \leq 1 \quad \forall s \in \partial\mathbf{U} \text{ .}$$

By the choice of s_0 , $N_c\widetilde{N}_p \neq 0$; hence there exist unitary matrices $A, B \in \mathbf{M}(\mathbb{R})$ and $\sigma_1 > 0$ such that

$$N_c\widetilde{N}_p(s_0) = A \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{n_i} \end{bmatrix} B \ .$$

Let $\delta := sgn(r(s_0))$, $\gamma \in \mathbb{R} \setminus \mathbf{U}$ and $\alpha \geq s_0$; define N_Δ and D_Δ as

$$\begin{aligned} N_\Delta(s) &:= \delta \frac{r(s)}{s+\gamma} \mathrm{I} \ , \\ D_\Delta(s) &:= \frac{s-\alpha}{s+\gamma} AB \ . \end{aligned}$$

Note that $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ by construction. We now determine a suitable $\alpha \geq s_0$. We have

$$(D_\Delta + N_c\widetilde{N}_p N_\Delta)(s_0) = \frac{1}{s_0+\gamma} A \begin{bmatrix} s_0 - \alpha + \delta\sigma_1 r(s_0) & & \\ & \ddots & \\ & & s_0 - \alpha + \delta\sigma_{n_i} r(s_0) \end{bmatrix} B \ .$$

Let

$$\alpha := s_0 + \delta\sigma_1 r(s_0) \ .$$

Clearly, $\alpha > s_0$ and $\det(D_\Delta + N_c\widetilde{N}_p N_\Delta)(s_0) = 0$. By construction,

$$\det D_p(s_0) \neq 0 \quad \text{and} \quad \det D_\Delta(s_0) \neq 0 \ ;$$

hence the map $D_\Delta(D_\Delta + N_c\widetilde{N}_p N_\Delta)^{-1} D_p$ has at least one pole at $s_0 \in \mathbf{U}$ and we conclude that the unity-feedback configuration $S(P(\mathrm{I}+\Delta), C)$ is *not* $\mathbf{R_U}$-stable .

□

2.5.2 Post-Multiplicative Perturbations

For a given undesired region $\mathbf{U}$, let the set of plant perturbations be given by

$$\left\{ (\mathrm{I}+\Delta)P) \mid \Delta \in \mathrm{B}(r) \right\}, \tag{2.23}$$

where $\mathrm{B}(r)$ is defined in (2.18) .

Lemma 2.5.3 (n&s condition for the stability of $S((\mathrm{I}+\Delta)P, C)$)
Let the maps P and $C \in \mathbf{M}(\mathbb{R}_p(s))$ be such that the feedback system $S(P, C)$ is $\mathbf{R_U}$-stable . Choose the r.c.f.s (N_p, D_p) of P and (N_c, D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I} \ ,$$

for some l.c.f.s $(\widetilde{D}_p, \widetilde{N}_p)$, $(\widetilde{D}_c, \widetilde{N}_c)$ of P and C , respectively. Let $(N_\Delta\,,\, D_\Delta)$ be an r.c.f. of $\Delta \in \mathbf{M}(\mathbb{R}_p(s))$. Consider the unity-feedback system $S((\mathrm{I}+\Delta)P, C)$ shown in Figure 2.9; note that only the input and the output of $(\mathrm{I}+\Delta)P$ are observed.

Under these assumptions, the feedback system $S((\mathrm{I}+\Delta)P, C)$ is $\mathbf{R_U}$-stable (i.e., the closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is $\mathbf{R_U}$-stable) if and only if the map

$$\widetilde{D}_p N_\Delta (D_\Delta + N_p \widetilde{N}_c N_\Delta)^{-1} N_p$$

is $\mathbf{R_U}$-stable .

Proof

Let (N_Δ, D_Δ) be an r.c.f. of $\Delta \in \mathbf{M}(\mathbb{R}_p(s))$. By assumption, only the input and the output of $(\mathrm{I}+\Delta)P$ are observed. Hence, the unity feedback system $S((\mathrm{I}+\Delta)P, C)$ is $\mathbf{R_U}$-stable if and only if the map $(u_1, u_2) \mapsto \xi_c$ (see Figure 2.9) is $\mathbf{R_U}$-stable . Writing the summing node equations in Figure 2.9, we obtain

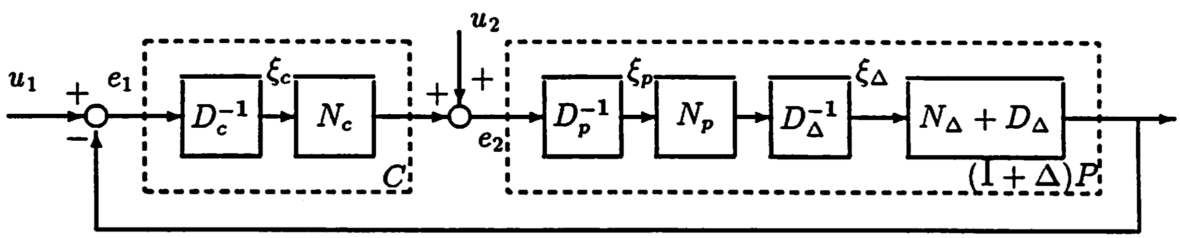

Figure 2.9:
Post-multiplicative uncertainty model

$$\begin{bmatrix} D_c & 0 & N_\Delta + D_\Delta \\ -N_c & D_p & 0 \\ 0 & -N_p & D_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} . \tag{2.24}$$

Operating on the left by $\begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, $\begin{bmatrix} \widetilde{D}_p & -\widetilde{N}_p & 0 \\ \widetilde{N}_c & \widetilde{D}_c & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & N_p & I \end{bmatrix}$ successively, we obtain

$$\begin{bmatrix} I & 0 & \widetilde{D}_p N_\Delta \\ 0 & I & \widetilde{N}_c N_\Delta \\ 0 & 0 & D_\Delta + N_p \widetilde{N}_c N_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} \widetilde{D}_p u_1 - \widetilde{N}_p u_2 \\ \widetilde{N}_c u_1 + \widetilde{D}_c u_2 \\ N_p[\widetilde{N}_c u_1 + \widetilde{D}_c u_2] \end{bmatrix} . \tag{2.25}$$

Hence the map $(u_1, u_2) \mapsto \xi_c$ is given by

$$\xi_c = \widetilde{D}_p u_1 - \widetilde{N}_p u_2 - \widetilde{D}_p N_\Delta (D_\Delta + N_p \widetilde{N}_c N_\Delta)^{-1} N_p \begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{2.26}$$

Since the map $\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix}$ has a proper $\mathbf{R_U}$-stable right-inverse, we conclude that the map $(u_1, u_2) \mapsto \xi_c$ is $\mathbf{R_U}$-stable if and only if the map $\widetilde{D}_p N_\Delta (D_\Delta + N_p \widetilde{N}_c N_\Delta)^{-1} N_p$ is $\mathbf{R_U}$-stable .

□

We now prove that the ball description in (2.18) must be further restricted to guarantee that a compensator C stabilizes the class in (2.23) in the unity-feedback system.

Proposition 2.5.4 (admissible post-multiplicative perturbations)

Let the plant $P \in \mathbf{M}(\mathbb{R}_p(s))$ be stabilized by the compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$

in the unity-feedback configuration $S(P,C)$; then given any undesired region $\mathbf{U}$ and any radius map $r \in \mathbf{R_U}$, there exists a $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S((\mathrm{I}+\Delta)P,C)$ is *not* $\mathbf{R_U}$-stable .

Proof

Let the radius map $r \in \mathbf{R_U}$ be given. We prove the claim by constructing a perturbation $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S((\mathrm{I}+\Delta)P,C)$ is *not* $\mathbf{R_U}$-stable . Using the notation of Lemma 2.5.3 , we choose a $\Delta \in \mathbf{M}(\mathrm{I\!R}_p(s))$ which has an r.c.f (N_Δ, D_Δ) such that $N_\Delta D_\Delta = D_\Delta N_\Delta$. When the numerator and denominator of Δ commute, the conclusion in Lemma 2.5.3 can be restated as : $S((I+\Delta)P,C)$ is $\mathbf{R_U}$-stable if and only if the map

$$\widetilde{D}_p(D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} N_\Delta N_p$$

is $\mathbf{R_U}$-stable . Hence, necessarily

$$\widetilde{D}_p(D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} N_\Delta N_p \widetilde{N}_c = \widetilde{D}_p[I - (D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} D_\Delta]$$

is $\mathbf{R_U}$-stable . Then $S((I+\Delta)P,C)$ is $\mathbf{R_U}$-stable *only if* the map

$$\widetilde{D}_p(D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} D_\Delta$$

is $\mathbf{R_U}$-stable . We now construct an r.c.f. (N_Δ, D_Δ) such that N_Δ and D_Δ commute and $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ and $\widetilde{D}_p(D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} D_\Delta$ is *not* $\mathbf{R_U}$-stable . Choose $s_0 \in \mathrm{I\!R} \cap \mathbf{U}$ such that

$$s_0 > \max\Big\{ \text{ } (\mathrm{I\!R} \cap \mathbf{U})\text{-blocking zeros of } N_p\widetilde{N}_c \text{ , } (\mathrm{I\!R} \cap \mathbf{U})\text{-zeros of } r \text{ and } \det \widetilde{D}_p \Big\},$$

and

$$\left|\frac{1}{s - s_0}\right| \leq 1 \quad \forall s \in \partial\mathbf{U} \text{ .}$$

Note that for all $\alpha \geq s_0$,

$$|\frac{1}{s-\alpha}| \leq 1 \quad \forall s \in \partial \mathbf{U} .$$

By the choice of s_0 , $N_p \widetilde{N}_c \neq 0$; hence there exist unitary matrices $A, B \in \mathbf{M}(\mathbb{R})$ and $\sigma_1 > 0$ such that

$$N_p \widetilde{N}_c(s_0) = A \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{n_o} \end{bmatrix} B .$$

Let $\delta := sgn(r(s_0))$, $\gamma \in \mathbb{R} \setminus \mathbf{U}$ and $\alpha \geq s_0$; define N_Δ and D_Δ as

$$\begin{aligned} N_\Delta(s) &:= \delta \frac{r(s)}{s+\gamma} I , \\ D_\Delta(s) &:= \frac{s-\alpha}{s+\gamma} AB . \end{aligned}$$

Note that $N_\Delta D_\Delta = D_\Delta N_\Delta$ and $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ by construction. Now we determine a suitable $\alpha \geq s_0$. We have

$$(D_\Delta + N_\Delta N_p \widetilde{N}_c)(s_0) = \frac{1}{s_0+\gamma} A \begin{bmatrix} s_0 - \alpha + \delta\sigma_1 r(s_0) & & \\ & \ddots & \\ & & s_0 - \alpha + \delta\sigma_{n_o} r(s_0) \end{bmatrix} B .$$

Let

$$\alpha := s_0 + \delta\sigma_1 r(s_0) .$$

Clearly, $\alpha > s_0$ and $\det(D_\Delta + N_\Delta N_p \widetilde{N}_c)(s_0) = 0$. By construction,

$$\det \widetilde{D}_p(s_0) \neq 0 \quad \text{and} \quad \det D_\Delta(s_0) \neq 0 \;;$$

hence the map $\widetilde{D}_p (D_\Delta + N_\Delta N_p \widetilde{N}_c)^{-1} D_\Delta$ has at least one pole at $s_0 \in \mathbf{U}$ and we conclude that the unity-feedback configuration $S((I+\Delta)P, C)$ is *not* $\mathbf{R_U}$-stable .

□

2.5.3 Feedback Perturbations

For a given undesired region $\mathbf{U}$, let the set of plant perturbations be given by

$$\left\{ P(\mathrm{I}+\Delta P)^{-1} \mid \Delta \in \mathrm{B}(r) \right\}. \tag{2.27}$$

where $\mathrm{B}(r)$ is defined in (2.18) .

Lemma 2.5.5 (n&s condition for the stability of $S(P(\mathrm{I}+\Delta P)^{-1},C)$)
Let the maps P and $C \in \mathrm{M}(\mathbb{R}_p(s))$ be such that the feedback system $S(P,C)$ is $\mathbf{R_U}$-stable . Choose the r.c.f.s (N_p,D_p) of P and (N_c,D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix}\begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I} \,,$$

for some l.c.f.s $(\widetilde{D}_p,\widetilde{N}_p)$, $(\widetilde{D}_c,\widetilde{N}_c)$ of P and C , respectively. Let (N_Δ,D_Δ) be an r.c.f. of $\Delta \in \mathrm{M}(\mathbb{R}_p(s))$. Consider the unity-feedback system $S(P(\mathrm{I} + \Delta P)^{-1},C)$ shown in Figure 2.10; note that only the input and the output of $P(\mathrm{I}+\Delta P)^{-1}$ are observed.

Under these assumptions, the feedback system $S(P(\mathrm{I}+\Delta P)^{-1},C)$ is $\mathbf{R_U}$-stable (i.e., the closed-loop map $(u_1,u_2)\mapsto(e_1,e_2)$ is $\mathbf{R_U}$-stable) if and only if the map

$$\widetilde{N}_p N_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} N_p$$

is $\mathbf{R_U}$-stable .

Proof

Let (N_Δ,D_Δ) be an r.c.f. of $\Delta \in \mathrm{M}(\mathbb{R}_p(s))$. By assumption, only the input and the output of $P(\mathrm{I}+\Delta P)^{-1}$ are observed. Hence, the unity feedback system $S(P(\mathrm{I}+\Delta P)^{-1},C)$ is $\mathbf{R_U}$-stable if and only if the map $(u_1,u_2)\mapsto\xi_c$ (see Figure 2.10) is $\mathbf{R_U}$-stable . Writing the summing node equations in Figure 2.10, we obtain

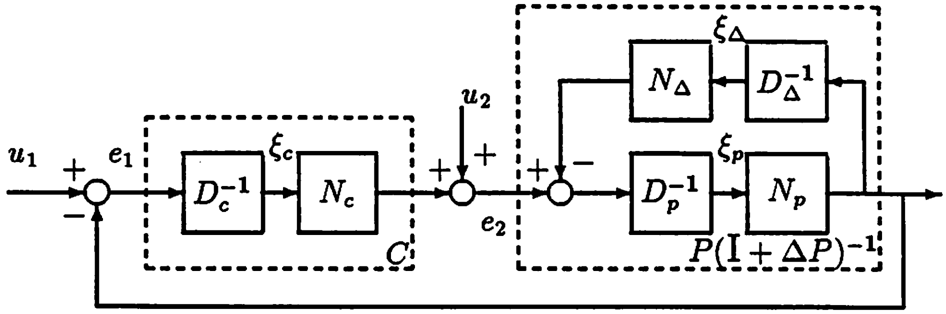

Figure 2.10:
The feedback system $S(P(\mathrm{I}+\Delta P)^{-1}, C)$

$$\begin{bmatrix} D_c & N_p & 0 \\ -N_c & D_p & N_\Delta \\ 0 & -N_p & D_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} . \tag{2.28}$$

Operating on the left by $\begin{bmatrix} \widetilde{D}_p & -\widetilde{N}_p & 0 \\ \widetilde{N}_c & \widetilde{D}_c & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix}$ and $\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & N_p & \mathrm{I} \end{bmatrix}$ successively, we obtain

$$\begin{bmatrix} \mathrm{I} & 0 & -\widetilde{N}_p N_\Delta \\ 0 & \mathrm{I} & \widetilde{D}_c N_\Delta \\ 0 & 0 & D_\Delta + N_p \widetilde{D}_c N_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} \widetilde{D}_p u_1 - \widetilde{N}_p u_2 \\ \widetilde{N}_c u_1 + \widetilde{D}_c u_2 \\ N_p[\widetilde{N}_c u_1 + \widetilde{D}_c u_2] \end{bmatrix} . \tag{2.29}$$

Hence the map $(u_1, u_2) \mapsto \xi_c$ is given by

$$\xi_c = \widetilde{D}_p u_1 - \widetilde{N}_p u_2 + \widetilde{N}_p N_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} N_p \begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{2.30}$$

Since the map $\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \end{bmatrix}$ has a proper $\mathbf{R}_\mathbf{U}$-stable right-inverse, we conclude that the map $(u_1, u_2) \mapsto \xi_c$ is $\mathbf{R}_\mathbf{U}$-stable if and only if the map $\widetilde{N}_p N_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} N_p$ is $\mathbf{R}_\mathbf{U}$-stable .

□

We now prove that the ball description in (2.18) must be further restricted to guarantee that a compensator C stabilizes the class in (2.27) in the unity-feedback system.

Proposition 2.5.6 (admissible feedback perturbations)

Let the plant $P \in \mathbf{M}(\mathbb{R}_p(s))$ be stabilized by the compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$

in the unity-feedback configuration $S(P,C)$; then given any undesired region $\mathbf{U}$ and any radius map $r \in \mathbf{R_U}$, there exists a $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is *not* $\mathbf{R_U}$-stable .

Proof

Let the radius map $r \in \mathbf{R_U}$ be given. We prove the claim by constructing a perturbation $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is *not* $\mathbf{R_U}$-stable . Using the notation of Lemma 2.5.5 , $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is $\mathbf{R_U}$-stable if and only if the map

$$\widetilde{N}_p N_\Delta (D_\Delta + D_c \widetilde{N}_p N_\Delta)^{-1} N_p$$

is $\mathbf{R_U}$-stable . Hence, necessarily

$$D_c \widetilde{N}_p N_\Delta (D_\Delta + D_c \widetilde{N}_p N_\Delta)^{-1} N_p \;=\; [\mathrm{I} - D_\Delta (D_\Delta + D_c \widetilde{N}_p N_\Delta)^{-1}] N_p$$

is $\mathbf{R_U}$-stable . Then $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is $\mathbf{R_U}$-stable *only if* the map

$$D_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} N_p$$

is $\mathbf{R_U}$-stable . Hence, necessarily

$$D_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} N_p \widetilde{D}_c N_\Delta \;=\; D_\Delta [\mathrm{I} - (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} D_\Delta]$$

is $\mathbf{R_U}$-stable . Then $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is $\mathbf{R_U}$-stable *only if* the map

$$D_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} D_\Delta$$

is $\mathbf{R_U}$-stable . We now construct an r.c.f. (N_Δ, D_Δ) such that $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ and $D_\Delta (D_\Delta + N_p \widetilde{D}_c N_\Delta)^{-1} D_\Delta$ is *not* $\mathbf{R_U}$-stable . Choose $s_0 \in \mathbb{R} \cap \mathbf{U}$ such that

$$s_0 \;>\; \max\Big\{ \text{ } (\mathbb{R} \cap \mathbf{U})\text{-blocking zeros of } N_p \widetilde{D}_c \text{ , } \text{ and } (\mathbb{R} \cap \mathbf{U})\text{-zeros of } r \Big\} \text{ ,}$$

and

$$|\frac{1}{s-s_0}| \leq 1 \quad \forall s \in \partial \mathbf{U} \; .$$

Note that for all $\alpha \geq s_0$,

$$|\frac{1}{s-\alpha}| \leq 1 \quad \forall s \in \partial \mathbf{U} \; .$$

By the choice of s_0 , $N_p\widetilde{D}_c \neq 0$; hence there exist unitary matrices $A, B \in \mathrm{M}(\mathbb{R})$ and $\sigma_1 > 0$ such that

$$N_p\widetilde{D}_c(s_0) = A \begin{bmatrix} \sigma_1 & & & \vdots & 0 & \dots & 0 \\ & \sigma_2 & & \vdots & 0 & \dots & 0 \\ & & \ddots & \vdots & & \vdots & \\ & & & \sigma_{n_o} \; \vdots & 0 & \dots & 0 \end{bmatrix} B \; .$$

For the above singular-value decomposition, we assumed that the plant P has less outputs than inputs. The other cases follow similarly.

Let $\delta := sgn(r(s_0))$, $\gamma \in \mathbb{R} \setminus \mathbf{U}$ and $\alpha \geq s_0$; define N_Δ and D_Δ as

$$\begin{aligned} N_\Delta(s) &:= B^{-1} \begin{bmatrix} \mathrm{I}_{n_o} \\ \dots \\ 0 \end{bmatrix} \delta \frac{r(s)}{s+\gamma} \; , \\ D_\Delta(s) &:= \frac{s-\alpha}{s+\gamma} A \; . \end{aligned}$$

Note that $N_\Delta D_\Delta^{-1} \in \mathrm{B}(r)$ by construction. Now we determine a suitable $\alpha \geq s_0$. We have

$$(D_\Delta + N_p\widetilde{D}_c N_\Delta)(s_0) = \frac{1}{s_0+\gamma} A \begin{bmatrix} s_0 - \alpha + \delta\sigma_1 r(s_0) & & \\ & \ddots & \\ & & s_0 - \alpha + \delta\sigma_{n_o} r(s_0) \end{bmatrix} .$$

Let

$$\alpha := s_0 + \delta\sigma_1 r(s_0) \; .$$

Clearly, $\alpha > s_0$ and $\det(D_\Delta + N_p\widetilde{D}_c N_\Delta)(s_0) = 0$. By construction,

$$\det D_\Delta(s_0) \neq 0 \quad ;$$

hence the map $D_\Delta(D_\Delta + N_p\widetilde{D}_c N_\Delta)^{-1}D_\Delta$ has at least one pole at $s_0 \in \mathbf{U}$ and we conclude that the unity-feedback configuration $S(P(\mathrm{I}+\Delta P)^{-1}, C)$ is *not* $\mathbf{R_U}$-stable .

□

2.5.4 Additive Perturbations

For a given undesired region $\mathbf{U}$, let the set of plant perturbations be given by

$$\{ P + \Delta \mid \Delta \in \mathrm{B}(r) \} . \tag{2.31}$$

If one is interested in determining necessary and sufficient conditions on a fixed compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$ to stabilize P in the unity-feedback system $S(P, C)$ for *all* P in (2.31) , the description in (2.31) must be *necessarily* modified. The following lemma establishes that the *convexity* of $\mathrm{B}(r)$ in (2.18) implies that the additive perturbation class in (2.31) *must* be further restricted as follows:

$$\{ P + \Delta \mid \Delta \in \mathbf{M}(\mathbf{R_U}) \cap \mathrm{B}(r) \} , \tag{2.32}$$

i.e. Δ must also be $\mathbf{R_U}$-stable ($\Delta \in \mathbf{M}(\mathbf{R_U})$).

Lemma 2.5.7 (admissible additive perturbations)
Let $\mathbf{U}$ be an undesired region, $P \in \mathbf{M}(\mathbb{R}_p(s))$ denote the plant and $C \in \mathbf{M}(\mathbb{R}_p(s))$ denote the compensator. For a given radius map $r \in \mathbf{R_U}$ let $\Delta \in \mathrm{B}(r)$. Under these assumptions, if the unity-feedback system $S(P + \lambda\Delta, C)$ is $\mathbf{R_U}$-stable for all $\lambda \in [0, 1]$, then $\Delta \in \mathrm{B}(r) \cap \mathbf{M}(\mathbf{R_U})$.

Proof

By assumption ($\lambda = 0$) , the unity-feedback system $S(P, C)$ is $\mathbf{R_U}$-stable . Hence, for any l.c.f. $(\widetilde{D}_p, \widetilde{N}_p)$ of P , there is an r.c.f. (N_c, D_c) of C such that

$$\widetilde{D}_p D_c + \widetilde{N}_p N_c = \mathrm{I} .$$

For the *same* r.c.f. (N_c, D_c) of C , for any $\Delta \in \mathrm{B}(r)$, there exists an l.c.f. $(\widetilde{D}_{P+\Delta}, \widetilde{N}_{P+\Delta})$ of $(P + \Delta)$ such that

$$\widetilde{D}_{P+\Delta} D_c + \widetilde{N}_{P+\Delta} N_c = \mathrm{I} .$$

For any $\Delta \in \mathrm{B}(r)$, we have

$$[\mathrm{I} + (P+\Delta)C] = \widetilde{D}_{P+\Delta}^{-1} D_c^{-1} \quad .$$

Since the radius map r is analytic on $\partial\mathbf{U}$ by assumption, any $\Delta \in \mathrm{B}(r)$ has no $\partial\mathbf{U}$-poles; hence for all $\Delta \in \mathrm{B}(r)$, the $\partial\mathbf{U}$-poles of P and $(P+\Delta)$ are *identical*. Let k_p and k_c denote the number of $(\mathbf{U} \setminus \partial\mathbf{U})$-poles of P and C , respectively (i.e. k_p and k_c are the number of zeros of $\det \widetilde{D}_p$ and $\det D_c$ in $(\mathbf{U} \setminus \partial\mathbf{U})$, counting multiplicities) .

Let Γ be a closed oriented path such that

$$\partial\mathbf{U} \subset \Gamma \subset \mathbf{U} \quad ,$$

(the orientation of Γ is such that $\mathbf{U}$ stays on the right) with indentations *into* $\mathbf{U}$ at the $\partial\mathbf{U}$-poles of P and C . Since $S(P,C)$ is $\mathbf{R}_{\mathbf{U}}$-stable and PC is analytic on Γ , the *closed* curve $\det(\mathrm{I}+PC)(\Gamma)$ encircles the origin counter-clockwise $k_p + k_c$ times.

Fix $\Delta \in \mathrm{B}(r)$. For a contradiction, suppose that there exists a $\lambda_0 \in [0\,,\,1]$ for which the number of $(\mathbf{U} \setminus \partial\mathbf{U})$-zeros of $\det \widetilde{D}_{P+\lambda_0\Delta}$ is $k_p \pm k$, where $k \neq 0$. Then by the stability of $S(P+\lambda_0\Delta, C)$, the closed curve

$$\det(\mathrm{I} + PC + \lambda_0 \Delta C)(\Gamma)$$

encircles the origin counter-clockwise $k_p + k_c \pm k$ times. Now consider the homotopy

$$\det(\mathrm{I} + PC + \mu\lambda_0\Delta C)(\Gamma) \quad , \quad \mu \in [0\,,\,1] \quad ,$$

which continuously (in μ) deforms the closed curve at $\mu = 0$ to the closed curve at $\mu = 1$. Since $k \neq 0$, there exist $\mu_0 \in (0,1)$ and a point $\gamma_0 \in \Gamma \subset \mathbf{U}$ such that

$$\det(\mathrm{I} + PC + \mu_0\lambda_0\Delta C)(\gamma_0) = 0 \quad .$$

Since $\mu_0\lambda_0 \in [0\,,1]$ and since both $\det \widetilde{D}_{P+\mu_0\lambda_0\Delta}$ and $\det D_c$ are analytic on Γ , this contradicts the assumption that

$$\det \widetilde{D}_{P+\mu_0\lambda_0\Delta} \det(\mathrm{I} + PC + \mu_0\lambda_0\Delta C) \det D_c \;=\; 1 \;\;.$$

Hence we conclude that the unity-feedback system $S(P+\lambda\Delta, C)$ is $\mathbf{R_U}$-stable for all $\lambda \in [0\,,1]$ only if $k=0$ for all $\lambda \in [0\,,1]$.

We complete the proof by showing that $k=0$ for all $\lambda \in [0\,,1]$ if and only if $\Delta \in \mathbf{M}(\mathbf{R_U})$. If Δ is $\mathbf{R_U}$-stable , for all $\lambda \in [0\,,1]$, the $\mathbf{U}$-poles of P and $(P+\lambda\Delta)$ are *identical* with the same multiplicities. Now for a contradiction, suppose that $k=0$ for all $\lambda\Delta$, $\lambda \in [0\,,1]$; suppose also that $\Delta \notin \mathbf{M}(\mathbf{R_U})$. Since $k = 0$, Δ cancels at least one $(\mathbf{U} \setminus \partial\mathbf{U})$-pole of P and introduces another. Consequently, for any $\lambda \in (0\,,1)$, $P + \lambda\Delta$ has at least one more $(\mathbf{U} \setminus \partial\mathbf{U})$-pole than P . This contradicts the fact that $k=0$.

□

We now prove Lemma 2.5.7 in a way similar to the previous sections. The following lemma (see also [Hua.2]) establishes the necessary and sufficient condition for the perturbed system to be stable.

Lemma 2.5.8 (n&s condition for the stability of $S(P+\Delta, C)$)

Let the maps P and $C \in \mathbf{M}(\mathrm{IR}_p(s))$ be such that the feedback system $S(P,C)$ is $\mathbf{R_U}$-stable . Choose the r.c.f.s (N_p, D_p) of P and (N_c, D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I} \;,$$

for some l.c.f.s $(\widetilde{D}_p, \widetilde{N}_p)$, $(\widetilde{D}_c, \widetilde{N}_c)$ of P and C , respectively. Let (N_Δ, D_Δ) be an r.c.f. of $\Delta \in \mathbf{M}(\mathrm{IR}_p(s))$. Consider the feedback system $S(P+\Delta, C)$ shown in Figure 2.11 ; note that only the input and the output of $(P+\Delta)$ are observed.

Under these assumptions, the feedback system $S(P+\Delta, C)$ is $\mathbf{R}_{\mathbf{U}}$-stable (i.e. the closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is $\mathbf{R}_{\mathbf{U}}$-stable) if and only if the map

$$\widetilde{D}_p N_\Delta (D_\Delta + D_p \widetilde{N}_c N_\Delta)^{-1} D_p$$

is $\mathbf{R}_{\mathbf{U}}$-stable .

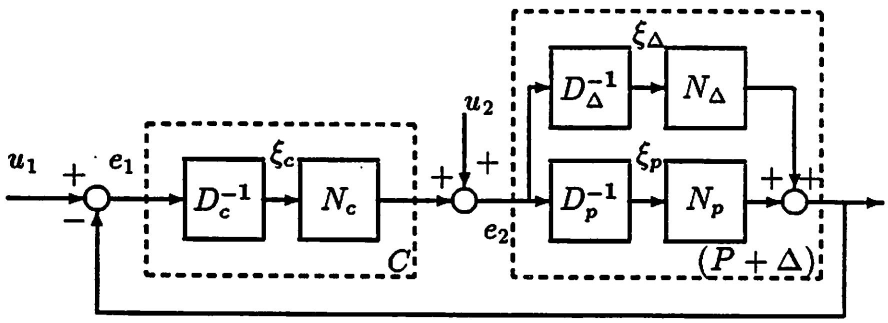

Figure 2.11:
The feedback system $S(P+\Delta, C)$

Proof

Let (N_Δ, D_Δ) be an r.c.f. of $\Delta \in \mathbf{M}(\mathbb{R}_p(s))$. By assumption, only the input and the output of $(P+\Delta)$ are observed. Hence, the unity feedback system $S(P+\Delta, C)$ is $\mathbf{R}_{\mathbf{U}}$-stable if and only if the map $(u_1, u_2) \mapsto \xi_c$ (see Figure 2.11) is $\mathbf{R}_{\mathbf{U}}$-stable . Writing the summing node equations in Figure 2.11 , we obtain

$$\begin{bmatrix} D_c & N_p & N_\Delta \\ -N_c & D_p & 0 \\ 0 & -D_p & D_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} . \qquad (2.33)$$

Operating on the left by $\begin{bmatrix} \widetilde{D}_p & -\widetilde{N}_p & 0 \\ \widetilde{N}_c & \widetilde{D}_c & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix}$ and $\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & D_p & \mathrm{I} \end{bmatrix}$ successively, we obtain

$$\begin{bmatrix} \mathrm{I} & 0 & \widetilde{D}_p N_\Delta \\ 0 & \mathrm{I} & \widetilde{N}_c N_\Delta \\ 0 & 0 & D_\Delta + D_p \widetilde{N}_c N_\Delta \end{bmatrix} \begin{bmatrix} \xi_c \\ \xi_p \\ \xi_\Delta \end{bmatrix} = \begin{bmatrix} \widetilde{D}_p u_1 - \widetilde{N}_p u_2 \\ \widetilde{N}_c u_1 + \widetilde{D}_c u_2 \\ D_p[\widetilde{N}_c u_1 + \widetilde{D}_c u_2] \end{bmatrix} . \qquad (2.34)$$

Hence the map $(u_1, u_2) \mapsto \xi_c$ is given by

$$\xi_c = \widetilde{D}_p u_1 - \widetilde{N}_p u_2 - \widetilde{D}_p N_\Delta (D_\Delta + D_p \widetilde{N}_c N_\Delta)^{-1} D_p \big[\widetilde{N}_c \;\; \widetilde{D}_c\big] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \qquad (2.35)$$

Since the map $\begin{bmatrix}\widetilde{N}_c & \widetilde{D}_c\end{bmatrix}$ has a proper $\mathbf{R_U}$-stable right-inverse, we conclude that the map $(u_1, u_2) \mapsto \xi_c$ is $\mathbf{R_U}$-stable if and only if the map $\widetilde{D}_p N_\Delta (D_\Delta + D_p \widetilde{N}_c N_\Delta)^{-1} D_p$ is $\mathbf{R_U}$-stable .

□

Proposition 2.5.9 (admissible additive perturbations)

Let the plant $P \in \mathbf{M}(\mathbb{R}_p(s))$ be $\mathbf{R_U}$-stabilized by the compensator $C \in \mathbf{M}(\mathbb{R}_p(s))$ in the unity-feedback configuration $S(P, C)$; then given any undesired region $\mathbf{U}$ and any radius map $r \in \mathbf{R_U}$, there exists a $\Delta \in \mathrm{B}(r) \setminus \mathbf{M}(\mathbf{R_U})$ such that the unity-feedback configuration $S(P + \Delta, C)$ is *not* $\mathbf{R_U}$-stable .

Proof

The proof is similar to the proof of Proposition 2.5.6 . The perturbation Δ can be constructed so that the map

$$D_\Delta \left(D_\Delta + N_c \widetilde{D}_p N_\Delta\right)^{-1} D_p$$

is *not* $\mathbf{R_U}$-stable.

□

2.6 Robustness under Stable Uncertainties

In the previous section, four classes of plant perturbations were considered (pre- and post-multiplicative, feedback, additive); for each of these cases, robust stabilization by a fixed compensator in the unity-feedback system necessarily required the uncertainty Δ to be $\mathbf{R_U}$-stable .

In this section, we briefly go over the simplifications due to the change of uncertainty description to one that is $\mathbf{R_U}$-stable .

Definition 2.6.1 (ball of uncertainty $\mathrm{B}_{\mathbf{U}}(r)$)

For an undesired region $\mathbf{U}$ and a radius map $r \in \mathbf{R_U}$ (hence no poles on $\partial\mathbf{U}$), the *ball of uncertainty* $\mathrm{B}_{\mathbf{U}}(r)$ is defined as

$$\mathrm{B}_{\mathbf{U}}(r) := \{ \Delta \in \mathbf{M}(\mathbf{R_U}) \mid \|\Delta(s)\| \leq |r(s)| \;\; \forall s \in \partial\mathbf{U} \} . \tag{2.36}$$

□

Note that unless $|r(s)| = k$ for all $s \in \partial\mathbf{U}$,

$$\mathrm{B}_{\mathbf{U}}(r) \not\equiv \{ \Delta \in \mathbf{M}(\mathbf{R_U}) \mid \|\Delta\|_{\mathbf{U}} \leq \|r\|_{\mathbf{U}} \} .$$

For a given $P \in \mathbf{M}(\mathbb{R}_p(s))$ and $\Delta \in \mathbf{M}(\mathbf{R_U})$, the four perturbation classes $P_{pre}(\Delta)$, $P_{post}(\Delta)$, $P_{feed}(\Delta)$ and $P_{add}(\Delta)$ and are shown in Figures 2.12 , 2.13 , 2.14 and 2.15, respectively.

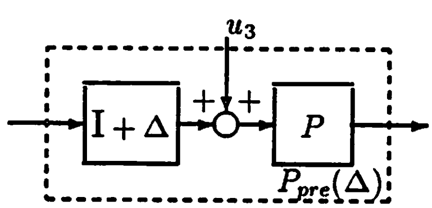

Figure 2.12:
The perturbation class $P_{pre}(\Delta)$

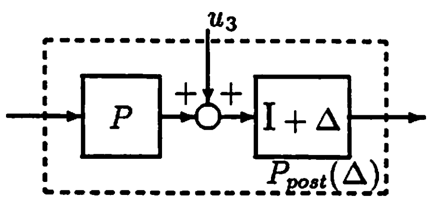

Figure 2.13:
The perturbation class $P_{post}(\Delta)$

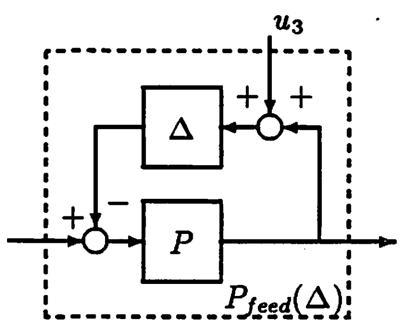

Figure 2.14:
The perturbation class $P_{feed}(\Delta)$

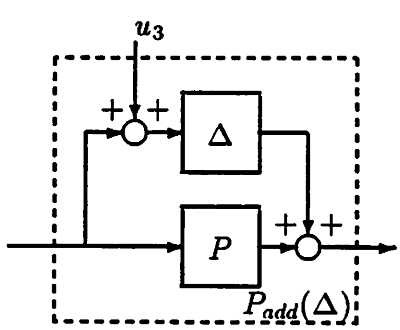

Figure 2.15:
The perturbation class $P_{add}(\Delta)$

Lemma 2.6.2 (n&s condition for stability with stable Δ)

Let the maps P and $C \in \mathbf{M}(\mathbb{R}_p(s))$ be such that the feedback system $S(P,C)$ is $\mathbf{R_U}$-stable . Choose the r.c.f.s (N_p, D_p) of P and (N_c, D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I} \ ,$$

for some l.c.f.s $(\widetilde{D}_p, \widetilde{N}_p)$, $(\widetilde{D}_c, \widetilde{N}_c)$ of P and C , respectively. Let $\Delta \in \mathbf{M}(\mathbf{R_U})$ be given; under these assumptions,

i) The feedback system $S(P_{pre}(\Delta), C)$ is $\mathbf{R_U}$-stable if and only if

$$(\mathrm{I} + N_c \widetilde{N}_p \Delta) \tag{2.37}$$

is $\mathbf{R_U}$-unimodular .

ii) The feedback system $S(P_{post}(\Delta), C)$ is $\mathbf{R_U}$-stable if and only if

$$(\mathrm{I} + N_p \widetilde{N}_c \Delta) \tag{2.38}$$

is $\mathbf{R_U}$-unimodular .

iii) The feedback system $S(P_{feed}(\Delta), C)$ is $\mathbf{R_U}$-stable if and only if

$$(\mathrm{I} + N_p \widetilde{D}_c \Delta) \tag{2.39}$$

is $\mathbf{R_U}$-unimodular .

iv) The feedback system $S(P_{add}(\Delta), C)$ is $\mathbf{R_U}$-stable if and only if

$$(\mathrm{I} + N_c \widetilde{D}_p \Delta) \tag{2.40}$$

is $\mathbf{R_U}$-unimodular .

Lemma 2.6.3 (small gain)

For a given $H \in \mathbf{R_U}^{n \times n}$ with $||H||_{\mathrm{U}} < 1$, the map $(\mathrm{I}+H)$ is $\mathbf{R_U}$-unimodular .

Comment 2.6.4

The lemma follows by the contraction mapping theorem; H is the transfer function description of the linear map H with the Lipschitz constant $\| H \|_{\mathbf{U}}$. The following proof makes use of the transfer function description of the linear map H .

Proof of Lemma 2.6.3

Let $H \in \mathbf{R_U}^{n \times n}$; then $(\mathrm{I} + H) \in \mathbf{R_U}^{n \times n}$. The map $(\mathrm{I} + H)$ is $\mathbf{R_U}$-unimodular if and only if $\mathrm{rank}(\mathrm{I} + H(s)) = n$ for all $s \in \mathbf{U}$. Let $\|H\|_{\mathbf{U}} < 1$ and for the sake of contradiction suppose that there exists an $s_0 \in \mathbf{U}$ such that $\mathrm{rank}(\mathrm{I} + H(s_0)) < n$. Then there exists a *nonzero* $x \in \mathbb{C}^n$ such that $(\mathrm{I} + H(s_0))x = 0$. Hence

$$|x| = |H(s_0)x| \le \| H(s_0) \| \, |x| \le \| H \|_{\mathbf{U}} \, |x| < |x| \quad ,$$

which contradicts the fact that $|x| \neq 0$. Hence we conclude that the map $(\mathrm{I} + H)$ is $\mathbf{R_U}$-unimodular .

□

Proposition 2.6.5 (robust stability under a ball of uncertainty)

Let $\mathbf{U}$ be an undesired region satisfying the matching condition in Definition 2.2.4 and let the map $H \in \mathbf{M}(\mathbf{R_U})$. Let $r \in \mathbf{R_U}$ be a radius map.

Under these assumptions the map $(\mathrm{I} + \Delta H)$ is $\mathbf{R_U}$-unimodular $\forall \, \Delta \in \mathrm{B_U}(r)$ if and only if

$$|r(s)| \, \| H(s) \| < 1 \quad \forall \, s \in \partial \mathbf{U} \quad .$$

Proof

" if "

Let the perturbation $\Delta \in \mathrm{B}_{\mathrm{U}}(r)$. Since Δ and $H \in \mathrm{M}(\mathbf{R}_{\mathrm{U}})$ by assumption, $\Delta H \in \mathrm{M}(\mathbf{R}_{\mathrm{U}})$ and

$$\|\Delta H(s)\| \le \|\Delta(s)\|\,\|H(s)\| \le |r(s)|\,\|H(s)\| < 1 \quad , \forall\, s \in \partial\mathrm{U} .$$

Hence $\|\Delta H\|_{\mathrm{U}} < 1$ for all $\Delta \in \mathrm{B}_{\mathrm{U}}(r)$. By Lemma 2.6.3 , the map $(\mathrm{I}+\Delta H)$ is $\mathbf{R}_{\mathrm{U}}$-unimodular for all $\Delta \in \mathrm{B}_{\mathrm{U}}(r)$.

" only if "

To prove the contrapositive, suppose that there exists an $s_0 \in \partial\mathrm{U}$ such that $|r(s_0)|\,\|H(s_0)\| \ge 1$. We show that there exists a $\Delta_0 \in \mathrm{B}_{\mathrm{U}}(r)$ such that the map $(\mathrm{I}+\Delta_0 H)$ is *not* $\mathbf{R}_{\mathrm{U}}$-unimodular .

Without loss of generality, we consider the case where the map $H \in \mathbf{R}_{\mathrm{U}}{}^{n\times n}$ for some $n \ge 1$ (if the map H is *not* square, then the singular-value decomposition obtained below will be augmented by a band of zero rows or zero columns and the rest of the proof will still hold after taking care of dimensions).

Let A and $B \in \mathrm{M}(\mathbb{C})$ be the unitary matrices defined by the singular-value decomposition of $H(s_0)$, where

$$H(s_0) =: A \begin{bmatrix} \|H(s_0)\| & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} B \ . \tag{2.41}$$

Let $K_0 \in \mathrm{M}(\mathbb{C})$ be defined as

$$K_0 := B^{-1} \begin{bmatrix} \frac{-1}{r(s_0)\|H(s_0)\|} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} A^{-1} \ . \tag{2.42}$$

Let x_0 , $y_0 \in \mathbb{C}^n$ be such that

$$K_0 =: x_0 y_0^T \ .$$

Since $\mathbf{U}$ satisfies the matching condition by assumption, there exist x , $y \in \mathbf{R_U}^n$ such that

i) $x(s_0) = x_0$, $y(s_0) = y_0$ and

ii) $|x(s)| \leq |x_0|$, $|y(s)| \leq |y_0|$ for all $s \in \partial\mathbf{U}$.

Let $K := xy^T \in \mathbf{M}(\mathbf{R_U})$. Note that by construction $K(s_0) = K_0$ and

$$||K(s)|| = ||x(s)y^T(s)|| = |x(s)|\,|y(s)| \leq |x_0|\,|y_0| = ||x_0 y_0^T|| = ||K_0|| \ , \quad \forall s \in \partial\mathbf{U} \ .$$

Let the perturbation Δ_0 be defined by

$$\Delta_0(s) := r(s)K(s) \quad \forall s \in \mathbb{C} \ .$$

Clearly, $\Delta_0 \in \mathbf{M}(\mathbf{R_U})$; moreover $\Delta_0 \in \mathrm{B_U}(r)$ since

$$\begin{aligned} ||\Delta_0(s)|| &= |r(s)|\,||K(s)|| \\ &\leq |r(s)|\,||K_0|| \\ &\leq \frac{|r(s)|}{|r(s_0)|\,||H(s_0)||} \\ &\leq |r(s)| \ , \qquad\qquad \forall\, s \in \partial\mathbf{U} \ . \end{aligned}$$

By calculation from (2.41) and (2.42) , we have

$$\det(\mathbf{I} + \Delta_0(s_0)H(s_0)) = 0 \quad ;$$

hence the map $(\mathbf{I} + \Delta_0 H)$ is *not* $\mathbf{R_U}$-unimodular .

□

From Lemma 2.6.2 and Proposition 2.6.5 , we obtain the following corollary.

Corollary 2.6.6 (n&s conditions for robust stability when $\Delta \in \mathrm{B_U}(r)$)
Let $\mathbf{U}$ be an undesired region satisfying the matching condition in Definition 2.2.4 .

Let the maps P and $C \in \mathbf{M}(\mathbb{R}_p(s))$ be such that the feedback system $S(P,C)$ is $\mathbf{R_U}$-stable . Choose the r.c.f.s (N_p, D_p) of P and (N_c, D_c) of C such that

$$\begin{bmatrix} \widetilde{N}_c & \widetilde{D}_c \\ \widetilde{D}_p & -\widetilde{N}_p \end{bmatrix} \begin{bmatrix} N_p & D_c \\ D_p & -N_c \end{bmatrix} = \mathrm{I} ,$$

for some l.c.f.s $(\widetilde{D}_p, \widetilde{N}_p)$, $(\widetilde{D}_c, \widetilde{N}_c)$ of P and C , respectively. Let $r \in \mathbf{R_U}$ be a radius map.

Under these assumptions, the feedback system

i) $S(P_{pre}(\Delta), C)$ is $\mathbf{R_U}$-stable for all $\Delta \in \mathrm{B_U}(r)$ if and only if

$$|r(s)|\,||N_c\widetilde{N}_p(s)|| < 1 , \quad \forall s \in \partial\mathbf{U} . \tag{2.43}$$

ii) $S(P_{post}(\Delta), C)$ is $\mathbf{R_U}$-stable for all $\Delta \in \mathrm{B_U}(r)$ if and only if

$$|r(s)|\,||N_p\widetilde{N}_c(s)|| < 1 , \quad \forall s \in \partial\mathbf{U} . \tag{2.44}$$

iii) $S(P_{feed}(\Delta), C)$ is $\mathbf{R_U}$-stable for all $\Delta \in \mathrm{B_U}(r)$ if and only if

$$|r(s)|\,||N_p\widetilde{D}_c(s)|| < 1 , \quad \forall s \in \partial\mathbf{U} . \tag{2.45}$$

iv) $S(P_{add}(\Delta), C)$ is $\mathbf{R_U}$-stable for all $\Delta \in \mathrm{B_U}(r)$ if and only if

$$|r(s)|\,||N_c\widetilde{D}_p(s)|| < 1 , \quad \forall s \in \partial\mathbf{U} . \tag{2.46}$$

□

Chapter 3

Factorizations of Linear Maps

3.1 Introduction

In this chapter, all plants and compensators are represented by *linear* maps (*not necessarily time-invariant nor finite-dimensional*) . Clearly this general setting includes the contents of Chapter 2 as a special case of factorizations of linear maps. However this should not give the impression that the results of Chapter 2 will be simply repeated as generalizations. Chapter 3 will still have an input-output approach; however a transfer function description will no longer be available for general linear maps (although transfer-function-like representations of certain time-varying finite-dimensional state-space representations are available in the literature, the manipulations in this *noncommutative algebra* definitely requires caution and even so, it does not provide the insight that its time-invariant counterpart does) . For this reason, Chapter 3 studies the key points of factorizations of transfer functions (which is studied only for the proper finite-dimensional case in Chapter 2 due to its simplicity) and extracts those which will be the basis of Chapter 4 . In other words, Chapter 3 is a stepping stone to Chapter 4 .

In Chapter 3 , each subsystem is considered as a black-box whose input-output pairs are *uniquely* determined by a causal linear map. Clearly the set of causal linear maps (with the composition and addition operations) have certain properties that are useful in derivations: **i)** linear maps over product spaces have a *structured* form; i.e. $\mathsf{A} : \Lambda_{1e} \times \Lambda_{2e} \mapsto \Lambda_{3e} \times \Lambda_{4e}$ can be expressed as $\mathsf{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{bmatrix}$ for some causal linear maps A_{11} , $\ldots$, A_{22} over appropriate input and output spaces, **ii)** the set of linear maps is left- and right-distributive (composition over addition) . These specific tools are used in the manipulations in Chapter 3 . For this reason, a justification of "left-factorizations" can still be made for general linear maps. When the algebra of input-output maps is restricted to be left-distributive only (Chapter 4) , certain constraints on the manipulations will be readily apparent.

This chapter is organized as follows:

Section 3.2 introduces the preliminary definitions on causal *nonlinear* input-output maps; this section serves as the tool-box for the rest of the thesis. In order to cut down on repetitions, the standard definitions on causal input-output maps over extended spaces are introduced once and for all in the general *nonlinear* context (see Definition 3.2.1) . As pointed out in the List of Symbols, sans serif style $(\mathsf{A}, \ldots, \mathsf{Z})$ is used for causal *linear* maps; calligraphic style $(\mathcal{A}, \ldots, \mathcal{Z})$ is used for causal *nonlinear* maps. Hence the set of results for each style, say Ω_P , Ω_{P} and $\Omega_{\mathcal{P}}$, can be ordered as follows:

$$\Omega_P \subset \Omega_{\mathsf{P}} \subset \Omega_{\mathcal{P}} \ .$$

Note that Chapter 3 deals with the *first* inclusion. Standard bounded-input bounded-output stability definitions are introduced in Definitions 3.2.3 – 3.2.6 .

Section 3.3 starts with right and left factorizations for linear maps (see Definitions 3.3.1 – 3.3.4) . The properties of these factorizations are stated and established.

In Section 3.4 , we study the general unity-feedback system $S(\mathsf{P},\mathsf{C})$ (see Figure 3.1) using the tools in the previous section. Lemma 3.4.6 shows that in a stable $S(\mathsf{P},\mathsf{C})$, the plant <u>and</u> the compensator both have right <u>and</u> left factorizations. Furthermore, the properties of the pseudo-state maps (mapping the closed-loop system inputs to the pseudo-state of the right factorizations) in $S(\mathsf{P},\mathsf{C})$ are investigated. We show the necessary and sufficient conditions on stabilizing linear compensators provided that the plant has coprime factorizations (Propositions 3.4.9 – 3.4.10) . If the plant has both right and left coprime factorizations, all causal stabilizing linear compensators are parametrized in Fact 3.4.12 .

Although the class of linear maps in Chapter 2 has both right and left coprime factorizations, this is *not* the case for general linear maps. Note that in general, the composition of linear single-input single-output maps is not commutative. Recall that in Section 3.4, the existence of coprime factorizations is crucial for the results stated. For this reason, Section 3.5 studies the conditions under which coprime factorizations exist for linear maps. First we show that in a stable $S(\mathsf{P},\mathsf{C})$, the plant and the compensator have right-coprime factorizations *if and only if* they both have left-coprime factorizations (Proposition 3.5.1) . This parallel existence properties of right an left coprime factorizations is exclusively due to *linearity* . Propositions 3.5.3 – 3.5.4 show that in a stable $S(\mathsf{P},\mathsf{C})$, the plant has right (left) coprime factorization *if and only if* the compensator has left (right) coprime factorization. These properties simplify the search for coprime factorizations of linear maps. The crux of this section is Theorem 3.5.6 ; it states the necessary and sufficient conditions to obtain coprime factorizations of the subsystems in a diagonal linear map with coprime factorizations. In a stable $S(\mathsf{P},\mathsf{C})$, the diagonal map

$\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$ has coprime factorizations (Fact 3.5.5) .

Section 3.6 briefly introduces the *nonlinear* unity-feedback system $\mathcal{S}(\mathcal{P}, \mathcal{C})$ and the related definitions.

Section 3.7 consists of a general robustness analysis of the *linear* compensator in a stable $S(\mathsf{P}, \mathsf{C})$ under *nonlinear* plant perturbations. Using coprime factorizations of linear maps and the results in the previous sections, we study four cases in subsections 3.7.1 – 3.7.4 : additive, feedback, pre-multiplicative and post-multiplicative *nonlinear* plant perturbations. In each of these cases, the perturbed plant model has four subcases resulting from the number of inputs and outputs considered. For each subcase we state the necessary and sufficient condition for the nominal compensator to simultaneously stabilize the perturbed plant. Comment 3.7.2 gives an intuitive explanation on the form of the necessary and sufficient conditions in Theorem 3.7.1 (the additive case) . The rest of the subsections state the results in detail; however the proofs are omitted since they are in essence identical to the proof of Theorem 3.7.1 . The corollaries at the end of each subsection characterize the set of all *nonlinear* perturbations for which the associated perturbed system remains stable.

This section is important for *design* purposes: the results allow the designer to check whether the expected perturbations will destabilize the design; furthermore the conditions for robust stabilization are necessary and sufficient.

3.2 Preliminaries

The following definitions are introduced to build the framework of the input-output approach used in the rest of the thesis. Note that all sans serif letters $\mathsf{A}, \ldots, \mathsf{Z}$ denote *linear* maps; all calligraphic letters $\mathcal{A}, \ldots, \mathcal{Z}$ denote *nonlinear* maps.

Definition 3.2.1 (Π_T and extended space Λ_e)

Let $\mathcal{T} \subset \mathbb{R}$ and let $\mathcal{V}$ be a normed vector space. Let

$$\zeta := \{u \mid u : \mathcal{T} \rightarrow \mathcal{V}\}$$

be the vector space of $\mathcal{V}$-valued functions on $\mathcal{T}$.

(For example: $\zeta := \{u \mid u : \mathbb{Z}_+ \rightarrow \mathbb{R}^n\}$)

For $\mathrm{T} \in \mathcal{T}$, the projection map $\Pi_\mathrm{T} : \zeta \rightarrow \zeta$ is defined by

$$\Pi_\mathrm{T} u(t) := \begin{cases} u(t) & t \leq \mathrm{T}, t \in \mathcal{T} \\ \theta_\zeta & t > \mathrm{T}, t \in \mathcal{T} \end{cases} ,$$

where θ_ζ is the zero element in ζ .

Let $\Lambda \subset \zeta$ be a normed vector space which is closed under the family of projection maps $\{\Pi_\mathrm{T}\}_{\mathrm{T}\in\mathcal{T}}$. For a given $u \in \Lambda$, let the norm

$$\|\Pi_{(\cdot)} u\| : \mathcal{T} \rightarrow \mathbb{R}_+$$

be a nondecreasing function.

(For example: $\Lambda := \left\{u : \mathbb{R}_+ \rightarrow \mathbb{R}^n \mid \sup_{t\in\mathbb{R}_+} \|u(t)\| < \infty\right\}$)

The *extended space* Λ_e is defined by

$$\Lambda_\mathrm{e} := \left\{u \in \zeta \mid \forall\, \mathrm{T} \in \mathcal{T} , \Pi_\mathrm{T} u \in \Lambda\right\} .$$

□

Note that the causal extension Λ_e of Λ is introduced to take "unbounded" signals into account. The norm $\|\cdot\|$ is defined for the members of Λ only.

If Λ_e was *not* introduced, then the input-output approach could represent only "stable" maps.

Definition 3.2.2 (causal map)

A nonlinear map $\mathcal{F} : \Lambda_e \to \Lambda_e$ is said to be causal iff

$$\Pi_T \mathcal{F} = \Pi_T \mathcal{F} \Pi_T \quad \forall\, T \in \mathcal{T} .$$

□

Typically all realizable models are causal. At any time instant T we have access to the input signals up to T, hence the map should not require "future" values of the input to determine the output at the time instant T. For example, the inverse of the delay operator

$$\mathsf{D} : e \mapsto y \begin{cases} y(k) = e(k-1) & k \in \mathbb{Z}_+ \\ y(0) = 0 \end{cases}$$

is <u>not</u> causal. A convolution system H, where

$$\mathsf{H}u(t) = \int_{-\infty}^{t} h(t\,,\,\tau)u(\tau)d\tau$$

is causal.

We define two function spaces closely related to Λ_e (the superscripts i and o refer to "input" and "output", respectively) : Let Λ_e^i and Λ_e^o be extended function spaces analogous to Λ_e except that their members take values in the normed vector spaces $\mathcal{V}^i$ and $\mathcal{V}^o$, respectively; the associated projections Π_T are redefined accordingly.

Definition 3.2.3 (Λ-stable)

A *causal nonlinear map* $\mathcal{H} : \Lambda_e^i \to \Lambda_e^o$ is said to be Λ-*stable* (see also [Des.9]) iff there exists a continuous nondecreasing function $\phi_{\mathcal{H}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$||\mathcal{H}u|| \leq \phi_{\mathcal{H}}(||u||) \quad \forall u \in \Lambda^{\mathrm{i}} .$$

□

For the sake of brevity, the composition $\mathcal{H}_1 \circ \mathcal{H}_2$ of two causal maps will be denoted by $\mathcal{H}_1\mathcal{H}_2$. For an input u the output $\mathcal{H}(u)$ will be denoted by $\mathcal{H}u$.

In the *linear* case, the output bounding function can be taken as linear.

Definition 3.2.4 (finite-gain-stable)

A *causal linear map* $\mathsf{H} : \Lambda_{\mathrm{e}}^{\mathrm{i}} \rightarrow \Lambda_{\mathrm{e}}^{\mathrm{o}}$ is said to be *finite-gain-stable* [Des.1] iff there exists $k > 0$ such that

$$||\mathsf{H}u|| \leq k||u|| \quad \forall u \in \Lambda^{\mathrm{i}} .$$

□

Note that the "gain" of a finite-gain-stable linear map H can be defined as

$$\sup_{u \in \Lambda^{\mathrm{i}},\, u \neq 0} \frac{||\mathsf{H}u||}{||u||} =: k \ .$$

Definition 3.2.5 (Λ-unimodular)

A causal nonlinear map $\mathcal{M} : \Lambda_{\mathrm{e}} \rightarrow \Lambda_{\mathrm{e}}$ is called Λ-*unimodular* iff

i) $\mathcal{M}$ is Λ-stable, bijective <u>and</u>

ii) $\mathcal{M}^{-1} : \Lambda_{\mathrm{e}} \rightarrow \Lambda_{\mathrm{e}}$ is Λ-stable.

□

For the linear case, we define finite-gain-unimodular maps.

Definition 3.2.6 (finite-gain-unimodular)

A causal linear map $\mathsf{M} : \Lambda_{\mathrm{e}} \rightarrow \Lambda_{\mathrm{e}}$ is called *finite-gain-unimodular* iff

i) M is finite-gain-stable, bijective <u>and</u>

ii) $M^{-1} : \Lambda_e \rightarrow \Lambda_e$ is finite-gain-stable.

□

Note that the sum and composition of Λ-stable (finite-gain-stable) maps are Λ-stable (finite-gain-stable) ; the composition of Λ-unimodular (finite-gain- unimodular) maps are Λ-unimodular (finite-gain-unimodular) .

In this chapter, we consider factorizations of *linear* maps only. Since the previous chapter was specifically focused on factorizations in $\mathbf{R}_{\mathbf{U}}$, due care must be taken in the generalization of coprimeness definitions.

The following definitions are introduced using the same names as in the previous chapter because they are direct generalizations. However, the existence of such factorizations are not as straightforward as in $\mathbf{R}_{\mathbf{U}}$.

3.3 Factorizations of linear maps

Definition 3.3.1 (right-factorization)

(N,D) is said to be a *right-factorization (r.f.)* of a causal linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ iff

i) the linear map $\mathsf{N} : \Lambda_e^i \to \Lambda_e^o$ is finite-gain-stable <u>and</u>

ii) the linear map $\mathsf{D} : \Lambda_e^i \to \Lambda_e^i$ is finite-gain-stable, bijective and has a causal inverse <u>and</u>

iii) $\mathsf{N}\mathsf{D}^{-1} = \mathsf{P}$.

Definition 3.3.2 (left-factorization)

$(\widetilde{\mathsf{D}},\widetilde{\mathsf{N}})$ is said to be a *left-factorization (l.f.)* of a causal linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ iff

i) the linear map $\widetilde{\mathsf{N}} : \Lambda_e^i \to \Lambda_e^o$ is finite-gain-stable <u>and</u>

ii) the linear map $\widetilde{\mathsf{D}} : \Lambda_e^o \to \Lambda_e^o$ is finite-gain-stable, bijective and has a causal inverse <u>and</u>

iii) $\widetilde{\mathsf{D}}^{-1}\widetilde{\mathsf{N}} = \mathsf{P}$.

Definition 3.3.3 (right-coprime factorization)

(N,D) is said to be a *right-coprime factorization (r.c.f.)* of the causal linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ iff

i) (N,D) is an r.f. of P <u>and</u>

ii) there exist linear finite-gain-stable maps $\widetilde{\mathsf{U}} : \Lambda_e^o \to \Lambda_e^i$, $\widetilde{\mathsf{V}} : \Lambda_e^i \to \Lambda_e^i$ such that

$$\begin{bmatrix} \widetilde{\mathsf{U}} & \widetilde{\mathsf{V}} \end{bmatrix} \begin{bmatrix} \mathsf{N} \\ \mathsf{D} \end{bmatrix} = I\,, \tag{3.1}$$

where I denotes the identity map on Λ_e^i .

□

Note that the identity in (3.1) resembles the Bezout-identity [Vid.3] in Definition 2.2.9 **ii)** . The finite-gain-stable maps (3.1) may <u>not</u> have transfer function representations. The matrix notation in (3.1) should be interpreted as: $\widetilde{U}$ <u>composed</u> with N <u>plus</u> $\widetilde{V}$ <u>composed</u> with D <u>equals</u> the identity map. For this reason, identities as in (3.1) will be often referred to as "Bezout-like" identities.

Definition 3.3.4 (left-coprime factorization)

$(\widetilde{D},\widetilde{N})$ is said to be a *right-coprime factorization (l.c.f.)* of the causal linear map $P : \Lambda_e^i \to \Lambda_e^o$ iff

i) $(\widetilde{D},\widetilde{N})$ is an l.f. of P <u>and</u>

ii) there exist linear finite-gain-stable maps $U : \Lambda_e^o \to \Lambda_e^i$, $V : \Lambda_e^o \to \Lambda_e^o$ such that

$$\begin{bmatrix} \widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = I ,$$

where I denotes the identity map on Λ_e^o.

Fact 3.3.5 (all r.c.f.s are related by finite-gain-unimodular maps)

Let (N, D) be an r.c.f. of the linear map $P : \Lambda_e^i \to \Lambda_e^o$; then (N_1, D_1) is an r.c.f. of P if and only if there exists a linear finite-gain-unimodular map $M : \Lambda_e^i \to \Lambda_e^i$ such that

$$\begin{bmatrix} N_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} M .$$

Proof

" if "

Since (N, D) is an r.c.f. of P and the linear map M is finite-gain-unimodular, (N_1, D_1) is also an r.f. of P . By assumption there exist causal linear finite-gain-stable maps $\widetilde{U}$ and $\widetilde{V}$ such that (3.1) holds. Substituting $N = N_1 M^{-1}$ and $D = D_1 M^{-1}$, we obtain

$$\left[\widetilde{U} N_1 + \widetilde{V} D_1\right] M^{-1} = I .$$

Composing on the left by M^{-1} and on the right by M and using the linearity of M^{-1} , we obtain

$$\left(M^{-1}\widetilde{U}\right) N_1 + \left(M^{-1}\widetilde{V}\right) D_1 = I ;$$

hence (N_1, D_1) is an r.c.f. of P .

" only if "

By assumption there exist linear finite-gain-stable maps $\widetilde{U}$, $\widetilde{V}$, $\widetilde{U}_1$ and $\widetilde{V}_1$ such that

$$\widetilde{U} N + \widetilde{V} D \;=\; I \;, \tag{3.2}$$

$$\widetilde{U}_1 N_1 + \widetilde{V}_1 D_1 \;=\; I \;. \tag{3.3}$$

Let $M := D^{-1} D_1$. Composing both sides of (3.2) by $D^{-1} D_1$ and using $N D^{-1} = N_1 D_1^{-1}$, we obtain

$$M = \widetilde{U} N_1 + \widetilde{V} D_1 \;\; ; \tag{3.4}$$

hence the linear map M is finite-gain-stable. By the definition of M , the map M is bijective and has a causal inverse.

Composing both sides of (3.3) by $M^{-1} = D_1^{-1} D$ and using $N D^{-1} = N_1 D_1^{-1}$, we obtain

$$M^{-1} = \widetilde{U}_1 N + \widetilde{V}_1 D \;\; ; \tag{3.5}$$

hence M^{-1} is finite-gain-stable. From (3.4) and (3.5) , we conclude that the finite-gain-stable map M is in fact finite-gain-unimodular . Furthermore

$$\mathsf{D}_1 = \mathsf{D}\,\mathsf{M} \quad \text{and} \quad \mathsf{N}_1 = \mathsf{N}\mathsf{M} \quad .$$

□

Fact 3.3.6 (all l.c.f.s are related by finite-gain-unimodular maps)
Let $(\widetilde{\mathsf{D}}, \widetilde{\mathsf{N}})$ be an l.c.f. of the causal linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$; then $(\widetilde{\mathsf{D}}_1, \widetilde{\mathsf{N}}_1)$ is an l.c.f. of P if and only if there exists a linear finite-gain-unimodular map $\widetilde{\mathsf{M}} : \Lambda_e^o \to \Lambda_e^o$ such that

$$\begin{bmatrix} \widetilde{\mathsf{D}}_1 & \widetilde{\mathsf{N}}_1 \end{bmatrix} = \widetilde{\mathsf{M}} \begin{bmatrix} \widetilde{\mathsf{D}} & \widetilde{\mathsf{N}} \end{bmatrix} .$$

□

3.4 Linear unity-feedback system

Definition 3.4.1 (linear unity-feedback system $S(\mathsf{P},\mathsf{C})$)

Let $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ be causal *linear* maps (not necessarily time-invariant). The unity-feedback system $S(\mathsf{P},\mathsf{C})$ is shown in Figure 3.1.

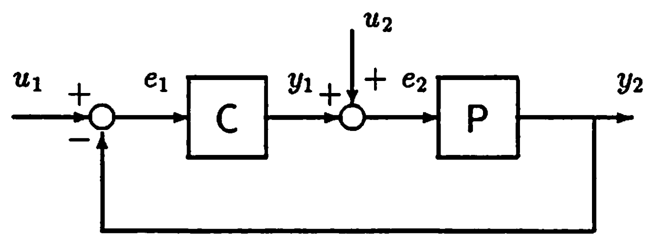

Figure 3.1:
Linear unity-feedback system $S(\mathsf{P},\mathsf{C})$

In Figure 3.1 , u_1 and u_2 denote the exogenous inputs; the outputs of C and P are denoted by y_1 and y_2 , respectively. From the summing node equations in Figure 3.1 , the pair $(e_1\,,\,e_2)$ is determined by

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} . \tag{3.6}$$

Definition 3.4.2 (well-posed $S(\mathsf{P},\mathsf{C})$)

The unity-feedback system $S(\mathsf{P},\mathsf{C})$ (see Figure 3.1) , where P and C are linear, is said to be *well-posed* iff there exists a causal map $(u_1\,,\,u_2) \mapsto (e_1\,,\,e_2)$.

Fact 3.4.3

For the linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ in Figure 3.1 , the following statements are equivalent:

i) The linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ is well-posed.

ii) There exists a causal map $(u_1\,,\,u_2) \mapsto (y_1\,,\,y_2)$.

iii) The linear map $(\mathrm{I} + \mathsf{PC})$ has a causal inverse.

iv) The linear map $(\mathrm{I} + \mathsf{CP})$ has a causal inverse .

Proof

i) $\Leftrightarrow$ ii)

Follows by (3.6).

i) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv)

Writing the summing node equations in Figure 3.1 in terms of u_1 , u_2 , e_1 and e_2 , we obtain

$$\begin{bmatrix} \mathrm{I} & \mathsf{P} \\ -\mathsf{C} & \mathrm{I} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad ;$$

hence the closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is given by

$$\begin{bmatrix} \mathrm{I} & \mathsf{P} \\ -\mathsf{C} & \mathrm{I} \end{bmatrix}^{-1} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} . \tag{3.7}$$

By calculation and using linearity, (3.7) is equivalent to

$$\begin{aligned} \begin{bmatrix} \mathrm{I} & \mathsf{P} \\ -\mathsf{C} & \mathrm{I} \end{bmatrix}^{-1} &= \begin{bmatrix} (\mathrm{I}+\mathsf{PC})^{-1} & -\mathsf{P}(\mathrm{I}+\mathsf{CP})^{-1} \\ \mathsf{C}(\mathrm{I}+\mathsf{PC})^{-1} & (\mathrm{I}+\mathsf{CP})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (\mathrm{I}+\mathsf{PC})^{-1} & -(\mathrm{I}+\mathsf{PC})^{-1}\mathsf{P} \\ (\mathrm{I}+\mathsf{CP})^{-1}\mathsf{C} & (\mathrm{I}+\mathsf{CP})^{-1} \end{bmatrix} . \end{aligned} \tag{3.8}$$

□

Definition 3.4.4 (finite-gain-stable $S(\mathsf{P},\mathsf{C})$)

A well-posed linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ is said to be *finite-gain-stable* iff the causal linear closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is finite-gain-stable.

Fact 3.4.5

For the well-posed linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ in Figure 3.1 , the following statements are equivalent:

i) The unity-feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable.

ii) The causal linear map $(u_1, u_2) \mapsto (y_1, y_2)$ is finite-gain-stable .

iii) The causal linear map

$$\begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i$$

in (3.7) is finite-gain-stable.

Proof

Follows by (3.6) and (3.7).

□

Lemma 3.4.6 (a necessary condition on P and C in stable $\mathcal{S}(P, C)$) .

Let the causal linear maps $P : \Lambda_e^i \to \Lambda_e^o$ and $C : \Lambda_e^o \to \Lambda_e^i$ be such that the unity-feedback system $\mathcal{S}(P, C)$ is finite-gain-stable; then the linear maps P <u>and</u> C have right- <u>and</u> left-factorizations.

Proof

We prove the result for C only; the case for P follows similarly.

By assumption the unity-feedback system $\mathcal{S}(P, C)$ is finite-gain-stable; hence the linear maps

$$\begin{aligned} \widetilde{D}_C &:= (I + CP)^{-1} , \\ \widetilde{N}_C &:= (I + CP)^{-1} C , \\ N_C &:= C(I + PC)^{-1} , \\ D_C &:= (I + PC)^{-1} \end{aligned}$$

are finite-gain-stable (see (3.8)). Since $\widetilde{D}_C^{-1}$ and D_C^{-1} are causal and

$$N_C D_C^{-1} = C = \widetilde{D}_C^{-1} \widetilde{N}_C ,$$

we conclude that (N_C, D_C) is an r.f. of C and $(\widetilde{D}_C, \widetilde{N}_C)$ is an l.f. of C .

□

Lemma 3.4.7 (n&s condition for finite-gain-stable $\mathcal{S}(\mathsf{P},\mathsf{C})$)

Let the linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ be given by an r.c.f. $(\mathsf{N_P},\mathsf{D_P})$ (hence there exist linear finite-gain-stable maps $\widetilde{\mathsf{U}}$, $\widetilde{\mathsf{V}}$ such that

$$\widetilde{\mathsf{U}}\,\mathsf{N_P} + \widetilde{\mathsf{V}}\,\mathsf{D_P} = \mathrm{I} \quad); \tag{3.9}$$

then the linear unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable if and only if the map

$$(u_1,u_2) \mapsto \xi_{\mathsf{P}}$$

from closed-loop inputs into the plant pseudo-state (see Figure 3.2) is finite-gain-stable.

Proof

By Definition 3.4.4 , the unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable if and only if the closed-loop map $(u_1\,,\,u_2) \mapsto (e_1\,,\,e_2)$ is finite-gain-stable. By assumption, $(\mathsf{N_P},\mathsf{D_P})$ is an r.c.f. of P ; hence from the summing node equations in Figure 3.2 and (3.9) , we obtain

$$\xi_{\mathsf{P}} \;=\; \widetilde{\mathsf{U}}\,(u_1 \;-\; e_1) \;+\; \widetilde{\mathsf{V}}\,e_2 \quad . \tag{3.10}$$

Since the linear map $\widetilde{\mathsf{U}}$ is finite-gain-stable, from (3.10) we conclude that the map $(u_1\,,\,u_2) \mapsto (e_1\,,\,e_2)$ is finite-gain-stable if and only if the map $(u_1,u_2) \mapsto \xi_{\mathsf{P}}$ is finite-gain-stable.

□

Corollary 3.4.8 (the pseudo-state map $(u_1,u_2) \mapsto (\xi_{\mathsf{C}},\xi_{\mathsf{P}})$)

Let the linear maps $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ have r.c.f.s $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively; then the unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable if and only if the causal linear map M , defined by

$$\mathsf{M} := \begin{bmatrix} \mathsf{D_C} & \mathsf{N_P} \\ -\mathsf{N_C} & \mathsf{D_P} \end{bmatrix} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i \quad , \tag{3.11}$$

is finite-gain-unimodular.

Proof

Writing the summing node equations in Figure 3.2 in terms of the pseudo-states ξ_P and ξ_C, we obtain

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathsf{M} \begin{bmatrix} \xi_C \\ \xi_P \end{bmatrix} ,$$

where the map M is as in (3.11). The proof follows by Lemma 3.4.7 .

□

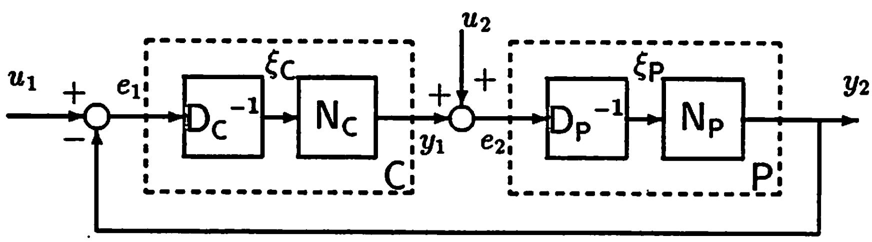

Figure 3.2:
Linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ with individual r.c.f.s

Proposition 3.4.9 (n&s condition on a stabilizing compensator)
Let the linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ have an l.c.f. $(\widetilde{D}_P, \widetilde{N}_P)$; then the unity-feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable if and only if C has an r.c.f. (N_C, D_C) such that (3.12) below holds:

$$\widetilde{N}_P N_C + \widetilde{D}_P D_C = I \tag{3.12}$$

Proof
" **if** "

By assumption (N_C, D_C) is an r.c.f of C and $(\widetilde{D}_P, \widetilde{N}_P)$ is an l.c.f. of P . Writing the summing node equations from Figure 3.2 in terms of the pseudo-state ξ_C, we obtain

$$\left(\widetilde{N}_P N_C + \widetilde{D}_P D_C\right)\xi_C = \begin{bmatrix} \widetilde{D}_P & -\widetilde{N}_P \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ;$$

substituting (3.12) we see that the map $(u_1, u_2) \mapsto \xi_C$ is finite-gain-stable. By Lemma 3.4.7 , we conclude that the feedback system $S(\mathsf{P}, \mathsf{C})$ is finite-gain-stable.

" only if "

By assumption, $(\widetilde{\mathsf{D}}_\mathsf{P}, \widetilde{\mathsf{N}}_\mathsf{P})$ is an l.c.f. of P and $S(\mathsf{P}, \mathsf{C})$ is finite-gain-stable. By Lemma 3.4.6 , C has an r.f. ; call it $(\mathsf{N}_1, \mathsf{D}_1)$. Writing the summing node equations from Figure 3.2 using the r.f. $(\mathsf{N}_1, \mathsf{D}_1)$ of C , we obtain

$$\left(\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_1 + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_1\right)\xi_C = \left[\widetilde{\mathsf{D}}_\mathsf{P} \quad -\widetilde{\mathsf{N}}_\mathsf{P}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{3.13}$$

By assumption, there exist finite-gain-stable maps U and V such that $\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{U} + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{V} = I$. Set

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathsf{V} \\ -\mathsf{U} \end{bmatrix}\eta .$$

The closed-loop map

$$\eta \mapsto (e_1, y_1) = \left(\mathsf{D}_1\xi_C , \mathsf{N}_1\xi_C\right)$$

is finite-gain-stable. From (3.13) , the finite-gain-stable maps $\mathsf{N}_C : \eta \mapsto y_1$ and $\mathsf{D}_C : \eta \mapsto e_1$ are given by

$$\begin{aligned} \mathsf{N}_C &= \mathsf{N}_1\left(\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_1 + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_1\right)^{-1} , \\ \mathsf{D}_C &= \mathsf{D}_1\left(\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_1 + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_1\right)^{-1} . \end{aligned}$$

Moreover D_C^{-1} is causal. Hence $(\mathsf{N}_C, \mathsf{D}_C)$ is an r.c.f. of C and

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_C + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_C = I .$$

□

A result similar to Proposition 3.4.9 can be obtained by requiring only that the plant P has an r.c.f.

Proposition 3.4.10 (n&s condition on a stabilizing compensator)

Let the linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ have an r.c.f. $(\mathsf{N}_\mathsf{P}, \mathsf{D}_\mathsf{P})$; then the unity-feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable if and only if C has an l.c.f. $(\widetilde{\mathsf{D}}_\mathsf{C},\widetilde{\mathsf{N}}_\mathsf{C})$ such that (3.14) below holds:

$$\widetilde{\mathsf{N}}_\mathsf{C}\mathsf{N}_\mathsf{P}+\widetilde{\mathsf{D}}_\mathsf{C}\mathsf{D}_\mathsf{P}=\mathrm{I} \tag{3.14}$$

Lemma 3.4.11 (all solutions $(\mathsf{N}_\mathsf{C},\mathsf{D}_\mathsf{C})$ of (3.12))

Let the linear map $\mathsf{P}:\Lambda_e^i\to\Lambda_e^o$ have an r.c.f. $(\mathsf{N}_\mathsf{P},\mathsf{D}_\mathsf{P})$ and an l.c.f. $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$. Choose the linear finite-gain-stable maps U, $\widetilde{\mathsf{U}}$, V and $\widetilde{\mathsf{V}}$ such that

$$\widetilde{\mathsf{U}}\mathsf{N}_\mathsf{P}+\widetilde{\mathsf{V}}\mathsf{D}_\mathsf{P} = \mathrm{I} \tag{3.15}$$

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{U}+\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{V} = \mathrm{I}\ . \tag{3.16}$$

Under these assumptions, the set of <u>all</u> linear finite-gain-stable solutions N_C, D_C of equation (3.12), namely

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_\mathsf{C}=\mathrm{I}\ ,$$

is given by

$$\left\{\begin{bmatrix}\mathsf{N}_\mathsf{C}\\ \mathsf{D}_\mathsf{C}\end{bmatrix}=\begin{bmatrix}\mathsf{U}+\mathsf{D}_\mathsf{P}\mathsf{Q}\\ \mathsf{V}-\mathsf{N}_\mathsf{P}\mathsf{Q}\end{bmatrix}\ \middle|\ \mathsf{Q}:\Lambda_e^o\to\Lambda_e^i \text{ is linear and finite-gain-stable}\right\}\ . \tag{3.17}$$

Proof

Any pair N_C, D_C in (3.17) is a solution of (3.12) : using the linearity of $\widetilde{\mathsf{N}}_\mathsf{P}$, $\widetilde{\mathsf{D}}_\mathsf{P}$ and (3.16), we obtain

$$\begin{aligned}\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_\mathsf{C} &= \widetilde{\mathsf{N}}_\mathsf{P}\left(\mathsf{U}+\mathsf{D}_\mathsf{P}\mathsf{Q}\right)+\widetilde{\mathsf{D}}_\mathsf{P}\left(\mathsf{V}-\mathsf{N}_\mathsf{P}\mathsf{Q}\right)\\ &= \mathrm{I}+\left(\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{D}_\mathsf{P}-\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{N}_\mathsf{P}\right)\mathsf{Q}\\ &= \mathrm{I}\ .\end{aligned}$$

Conversely, let N_C and D_C be a solution of (3.12); then

$$\widetilde{N}_P U + \widetilde{D}_P V = \widetilde{N}_P N_C + \widetilde{D}_P D_C = I \ .$$

Composing on the left by $\widetilde{D}_P^{-1}$, and substituting $N_P D_P^{-1} = \widetilde{D}_P^{-1} \widetilde{N}_P$, we obtain

$$V - D_C = N_P D_P^{-1} \left(N_C - U \right) \ . \tag{3.18}$$

Let

$$Q := D_P^{-1} \left(N_C - U \right) \ . \tag{3.19}$$

From (3.15), (3.18) and (3.19), we obtain

$$\begin{aligned} Q &= \left(\widetilde{U} N_P + \widetilde{V} D_P \right) Q \\ &= \widetilde{U} \left(V - D_C \right) + \widetilde{V} \left(N_C - U \right) \ , \end{aligned}$$

hence the linear map Q in (3.19) is finite-gain-stable. From (3.19) we also have $N_C = U + D_P Q$. From (3.18) we have $D_C = V - N_P Q$.

□

Fact 3.4.12 (parametrization of all stabilizing compensators C)

Let the linear map $P : \Lambda_e^i \to \Lambda_e^o$ have an r.c.f. (N_P, D_P) and an l.c.f. $(\widetilde{D}_P, \widetilde{N}_P)$. Choose the linear finite-gain-stable maps U , $\widetilde{U}$, V and $\widetilde{V}$ such that

$$\widetilde{U} N_P + \widetilde{V} D_P = I \tag{3.20}$$

$$\widetilde{N}_P U + \widetilde{D}_P V = I \ . \tag{3.21}$$

Under these assumptions, the set of *all* linear compensators $C : \Lambda_e^o \to \Lambda_e^i$ such that the linear unity-feedback system $S(P, C)$ is finite-gain-stable is given by

$$\left\{ \mathsf{C} = \left(\mathsf{U} + \mathsf{D_P Q}\right)\left(\mathsf{V} - \mathsf{N_P Q}\right)^{-1} \quad \middle| \quad \mathsf{Q} : \Lambda_e^o \to \Lambda_e^i \text{ is linear finite-gain-stable } \underline{\text{and}} \ \left(\mathsf{V} - \mathsf{N_P Q}\right)^{-1} \text{ is causal} \right\} . \tag{3.22}$$

Moreover, the map $\mathsf{Q} \mapsto \mathsf{C}$ defined in (3.22) is a bijection onto finite-gain-stabilizing compensators.

Proof

By Lemma 3.4.11 , the $\mathsf{Q} \mapsto \mathsf{C}$ map in (3.22) gives <u>all</u> compensators C such that $\mathcal{S}(\mathsf{P}, \mathsf{C})$ is finite-gain-stable; hence it is surjective. Let a member C in (3.22) be described by two parameters Q_1 and Q_2 . We show that

$$\mathsf{C} = \left(\mathsf{U} + \mathsf{D_P Q_1}\right)\left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} \tag{3.23}$$

$$= \left(\mathsf{U} + \mathsf{D_P Q_2}\right)\left(\mathsf{V} - \mathsf{N_P Q_2}\right)^{-1} \tag{3.24}$$

implies that

$$\mathsf{Q}_1 = \mathsf{Q}_2 \ .$$

From (3.21) , (3.23) and $\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{N}_\mathsf{P} = \widetilde{\mathsf{N}}_\mathsf{P}\mathsf{D}_\mathsf{P}$, we obtain

$$\begin{aligned}
\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{C} &= \widetilde{\mathsf{N}}_\mathsf{P}\left(\mathsf{U} + \mathsf{D_P Q_1}\right)\left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} \\
&= \left(\mathrm{I} - \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{V} + \widetilde{\mathsf{D}}_\mathsf{P}\mathsf{N_P Q_1}\right)\left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} \\
&= \left[\mathrm{I} - \widetilde{\mathsf{D}}_\mathsf{P}\left(\mathsf{V} - \mathsf{N_P Q_1}\right)\right]\left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} \\
&= \left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} - \widetilde{\mathsf{D}}_\mathsf{P} .
\end{aligned} \tag{3.25}$$

Similarly, from (3.21) and (3.24) , we obtain

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{C} = \left(\mathsf{V} - \mathsf{N_P Q_2}\right)^{-1} - \widetilde{\mathsf{D}}_\mathsf{P} \ . \tag{3.26}$$

From (3.25) and (3.26) , we obtain

$$\left(\mathsf{V} - \mathsf{N_P Q_1}\right)^{-1} = \left(\mathsf{V} - \mathsf{N_P Q_2}\right)^{-1} \ ;$$

substituting in (3.23-3.24) , we have

$$U + D_P Q_1 = U + D_P Q_2 \ .$$

Since $D_P{}^{-1}$ exists, we conclude that $Q_1 = Q_2$. Thus the map $Q \mapsto C$ is injective.

□

3.5 Existence of r.c.f.s and l.c.f.s

In this section we study the conditions under which coprime factorizations exist for linear maps.

Proposition 3.5.1 (r.c.f. $\Longleftrightarrow$ l.c.f.)

Let the linear unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ be finite-gain-stable; then the linear maps P and C have r.c.f.s if and only if the linear maps P and C have l.c.f.s .

Proof

We prove the "only if" since the other direction follows similarly.

By assumption, the linear maps P and C have r.c.f.s; call them $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively . Since the linear feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable, the linear map M defined in (3.11) is finite-gain-unimodular by Corollary 3.4.8 . Partition the finite-gain-stable inverse map M^{-1} as

$$\mathsf{M}^{-1} := \begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P} \\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} . \tag{3.27}$$

We claim that $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C},\widetilde{\mathsf{N}}_\mathsf{C})$ defined in (3.27) are l.c.f.s for P and C , respectively. From (3.11) ,

$$\mathsf{M} = \begin{bmatrix} I & \mathsf{P} \\ -\mathsf{C} & I \end{bmatrix} \begin{bmatrix} \mathsf{D_C} & 0 \\ 0 & \mathsf{D_P} \end{bmatrix} ;$$

taking the inverse and using (3.8) ,

$$\begin{aligned} \mathsf{M}^{-1} &= \begin{bmatrix} \mathsf{D_C}^{-1} & 0 \\ 0 & \mathsf{D_P}^{-1} \end{bmatrix} \begin{bmatrix} I & \mathsf{P} \\ -\mathsf{C} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathsf{D_C}^{-1} & 0 \\ 0 & \mathsf{D_P}^{-1} \end{bmatrix} \begin{bmatrix} (I+\mathsf{PC})^{-1} & -\mathsf{P}(I+\mathsf{CP})^{-1} \\ \mathsf{C}(I+\mathsf{PC})^{-1} & (I+\mathsf{CP})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{D_C}^{-1}(I+\mathsf{PC})^{-1} & -\mathsf{D_C}^{-1}\mathsf{P}(I+\mathsf{CP})^{-1} \\ \mathsf{D_P}^{-1}\mathsf{C}(I+\mathsf{PC})^{-1} & \mathsf{D_P}^{-1}(I+\mathsf{CP})^{-1} \end{bmatrix} . \end{aligned} \tag{3.28}$$

Setting (3.27) equal to (3.28) , we see that the maps $\widetilde{D}_P$ and $\widetilde{D}_C$ have causal inverses; moreover

$$\begin{aligned}\widetilde{D}_P^{-1}\widetilde{N}_P &= (I+PC)D_C D_C^{-1} P(I+CP)^{-1} = P\\ \widetilde{D}_C^{-1}\widetilde{N}_C &= (I+CP)D_P D_P^{-1} C(I+PC)^{-1} = C\ .\end{aligned}$$

Hence $(\widetilde{D}_P, \widetilde{N}_P)$ and $(\widetilde{D}_C, \widetilde{N}_C)$ are r.f.s of P and C , respectively. Using the (1,1)- and (2,2)-entries of the identity $M^{-1}M = I$, we conclude that $(\widetilde{D}_P, \widetilde{N}_P)$ is an l.c.f. of P and $(\widetilde{D}_C, \widetilde{N}_C)$ is an l.c.f. of C .

□

Comment 3.5.2

The proof of Proposition 3.5.1 is by construction; hence (3.28) shows how to obtain the l.c.f.s of P and C from their r.c.f.s , provided that the linear unity-feedback system $S(P, C)$ is finite-gain-stable.

□

The following proposition is similar to Proposition 3.4.9 ; however the emphasis is on the existence of coprime factorizations. In a finite-gain-stable $S(P, C)$, the existence of an r.c.f. (l.c.f.) of one of the blocks implies the existence of an l.c.f. (r.c.f.) of the other.

Proposition 3.5.3 (P has an l.c.f $\Longleftrightarrow$ C has an r.c.f.)

Let the linear maps $P : \Lambda_e^i \rightarrow \Lambda_e^o$ and $C : \Lambda_e^o \rightarrow \Lambda_e^i$ be such that the linear unity-feedback system $S(P, C)$ is finite-gain-stable. Under this assumption, the map P has an l.c.f. if and only if C has an r.c.f.

Proof

" only if "

Identical to the proof of the **"only if"** part in Proposition 3.4.9.

" if "

By assumption, the linear feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable. By Lemma 3.4.6 , the plant P has an l.f. ; call it $(\widetilde{\mathsf{D}}_1,\widetilde{\mathsf{N}}_1)$. By assumption C has an r.c.f.; call it $(\mathsf{N}_\mathsf{C},\mathsf{D}_\mathsf{C})$. Writing the summing node equations in Figure 3.2 using the r.c.f. $(\mathsf{N}_\mathsf{C},\mathsf{D}_\mathsf{C})$ of C , we obtain

$$\left(\widetilde{\mathsf{N}}_1\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_1\mathsf{D}_\mathsf{C}\right)\xi_\mathsf{C} = \begin{bmatrix}\widetilde{\mathsf{D}}_1 & -\widetilde{\mathsf{N}}_1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} .$$

Since $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable, by Lemma 3.4.7 the closed-loop map

$$(u_1,u_2)\mapsto\xi_\mathsf{C}$$

is finite-gain-stable. Hence the linear maps $\widetilde{\mathsf{D}}_\mathsf{P}$ and $\widetilde{\mathsf{N}}_\mathsf{P}$ defined as

$$\widetilde{\mathsf{D}}_\mathsf{P} := \left(\widetilde{\mathsf{N}}_1\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_1\mathsf{D}_\mathsf{C}\right)^{-1}\widetilde{\mathsf{D}}_1 \qquad (3.29)$$

$$\widetilde{\mathsf{N}}_\mathsf{P} := \left(\widetilde{\mathsf{N}}_1\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_1\mathsf{D}_\mathsf{C}\right)^{-1}\widetilde{\mathsf{N}}_1 \qquad (3.30)$$

are finite-gain-stable; moreover $\widetilde{\mathsf{D}}_\mathsf{P}^{-1}$ is causal. From (3.29) and (3.30) ,

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathsf{N}_\mathsf{C}+\widetilde{\mathsf{D}}_\mathsf{P}\mathsf{D}_\mathsf{C} = I \ ;$$

hence $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$ is an l.c.f. of P .

□

The following proposition is stated without proof; the proof is similar to that of Proposition 3.5.3 .

Proposition 3.5.4 (P has an r.c.f $\Longleftrightarrow$ C has an l.c.f.)

Let the linear maps $\mathsf{P}:\Lambda_e^i\to\Lambda_e^o$ and $\mathsf{C}:\Lambda_e^o\to\Lambda_e^i$ be such that the linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable. Under this assumption, the linear map P has an r.c.f. if and only if the linear map C has an l.c.f.

□

Consider the linear (not necessarily time-invariant) feedback system $S(\mathsf{P},\mathsf{C})$ shown in Figure 3.1 . If $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable, then the linear map

$$\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix} : \begin{bmatrix} e_2 \\ e_1 \end{bmatrix} \mapsto \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}$$

has both an r.c.f. (N,D) and an l.c.f. $(\widetilde{\mathsf{D}},\widetilde{\mathsf{N}})$. This result was proven in [Vid.2] for the case where P has elements in the quotient field of an entire ring. However, the conditions for existence of individual r.c.f. and l.c.f. of P and C was left as an open question.

To show that the stability of the closed-loop does <u>not</u> imply that P and C individually have coprime factorizations, a special non-unique factorization domain was constructed in [Ana.2] ; scalar p and c in the quotient field of this particular ring have no stable coprime factorizations although $\begin{bmatrix} \mathsf{p} & 0 \\ 0 & \mathsf{c} \end{bmatrix}$ has an r.c.f.

We consider this problem from a general input-output approach, where the multiinput-multioutput subsystems P and C are represented by linear (not necessarily time-invariant) maps defined over extended spaces. We obtain r.c.f.s and l.c.f.s of $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$ when the feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable . The main result is Theorem 3.5.6 , which states that: given coprime factorizations of $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$, individual coprime factorizations for P and C exist if and only if an r.c.f. of P has a lower block-triangular "denominator" D . Note that Theorem 3.5.6 answers the question posed in [Vid.2] ; the example constructed in [Ana.2] is only one case where the conditions of Theorem 3.5.6 fail. In the linear time-invariant finite-dimensional case where P and C have descriptions with rational function entries, the necessary and sufficient conditions in Theorem 3.5.6 are automatically satisfied due to the existence of triangular (Hermite) forms [Vid.3] .

Fact 3.5.5 ($\mathcal{S}(\mathsf{P},\mathsf{C})$ stable $\Rightarrow \begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$ has r.c.f. and l.c.f.)
Let the linear maps $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ be such that the linear unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable; then the map

$$\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix} : \Lambda_e^i \times \Lambda_e^o \to \Lambda_e^o \times \Lambda_e^i$$

has both an r.c.f. and an l.c.f.

Proof

Let the maps J and T be defined as follows :

$$\mathsf{J} : \Lambda_e^i \times \Lambda_e^o \to \Lambda_e^o \times \Lambda_e^i \quad , \quad \mathsf{J} := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad ,$$

$$\mathsf{T} : \Lambda_e^i \times \Lambda_e^o \to \Lambda_e^o \times \Lambda_e^i \quad , \quad \mathsf{T} := \begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix} \quad .$$

Note that

$$\left(I + \mathsf{TJ}\right)^{-1} = \begin{bmatrix} I & \mathsf{P} \\ -\mathsf{C} & I \end{bmatrix}^{-1} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} .$$

Since $\mathsf{J}^T\mathsf{J} = I$, J is finite-gain-unimodular. By assumption, the linear unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ is finite-gain-stable; hence the linear maps

$$\mathsf{N} := \mathsf{T}\left(I + \mathsf{JT}\right)^{-1} , \tag{3.31}$$

$$\mathsf{D} := \left(I + \mathsf{JT}\right)^{-1} \tag{3.32}$$

are finite-gain-stable. From (3.31) and (3.32) , $\mathsf{T} = \mathsf{N}\mathsf{D}^{-1}$ and $\mathsf{JN} + \mathsf{D} = I$; hence (N,D) is an r.c.f. of T . Let

$$\widetilde{\mathsf{N}} := \mathsf{N} \tag{3.33}$$

$$\widetilde{\mathsf{D}} := \mathsf{J}^{-1}\mathsf{D}\mathsf{J} . \tag{3.34}$$

From (3.33) and (3.34) , $\widetilde{\mathsf{D}}^{-1}\widetilde{\mathsf{N}} = \mathsf{T}$ and $\widetilde{\mathsf{N}}\mathsf{J} + \widetilde{\mathsf{D}} = I$; hence we conclude that $(\widetilde{\mathsf{D}},\widetilde{\mathsf{N}})$ is an l.c.f. of T .

□

Theorem 3.5.6 (coprime factorizations of P and C from $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$)

Let $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix} : \Lambda_e^i \times \Lambda_e^o \to \Lambda_e^o \times \Lambda_e^i$, where P and C are linear maps.

i) Let (N,D) be an r.c.f. of $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$; then P <u>and</u> C have r.c.f.s $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively, if and only if there exists a finite-gain-unimodular map R such that

$$\mathsf{DR} = \begin{bmatrix} \mathsf{D_P} & 0 \\ \mathsf{X_1} & \mathsf{D_C} \end{bmatrix} , \tag{3.35}$$

for some linear finite-gain-stable map $\mathsf{X}_1 : \Lambda_e^i \to \Lambda_e^o$ and where

$$\mathsf{NR} =: \begin{bmatrix} \mathsf{N_P} & \mathsf{X_2} \\ \mathsf{X_3} & \mathsf{N_C} \end{bmatrix} . \tag{3.36}$$

ii) Let $(\widetilde{\mathsf{D}},\widetilde{\mathsf{N}})$ be an l.c.f. of $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$; then P <u>and</u> C have l.c.f.s $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C},\widetilde{\mathsf{N}}_\mathsf{C})$, respectively if and only if there exists a finite-gain-unimodular map L such that

$$\mathsf{L}\widetilde{\mathsf{D}} = \begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & \tilde{\mathsf{X}}_1 \\ 0 & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} , \tag{3.37}$$

for some finite-gain-stable map $\tilde{\mathsf{X}}_1 : \Lambda_e^i \to \Lambda_e^o$ and where

$$\mathsf{L}\widetilde{\mathsf{N}} =: \begin{bmatrix} \widetilde{\mathsf{N}}_\mathsf{P} & \tilde{\mathsf{X}}_2 \\ \tilde{\mathsf{X}}_3 & \widetilde{\mathsf{N}}_\mathsf{C} \end{bmatrix} . \tag{3.38}$$

Comments 3.5.7

i) Equation (3.35) is a structure test on the "denominator" map : $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$ *must* have an r.c.f. (N,D) , where D is of the specific *lower block-triangular* form. In order to find the individual r.c.f.s of the subsystems from the given r.c.f. (N,D) of $\begin{bmatrix} \mathsf{P} & 0 \\ 0 & \mathsf{C} \end{bmatrix}$, we only need to determine $\mathsf{D_P}$ and $\mathsf{D_C}$; calculating $\mathsf{X_1}$ is not needed. Similar comments apply for the *upper block-triangular* form in (3.37) .

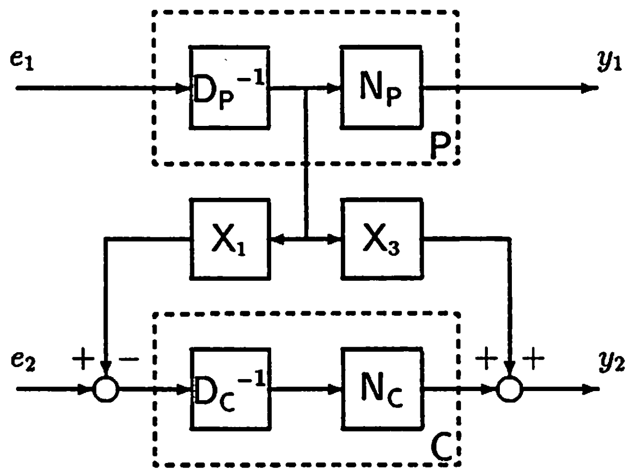

Figure 3.3:
A representation of linear $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$

ii) Suppose that the block-diagonal matrix of Theorem 3.5.6 would involve n subsystems where the map

$$\begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_n \end{bmatrix}$$

has an r.c.f. or an l.c.f. ; then the theorem still holds.

iii) If condition **i)** of Theorem 3.5.6 holds, then the map $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ has the structure in Figure 3.3 . As we shall see below with (3.42) , $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ is in fact *decoupled* into two subsystems P and C . In other words, the blocks X_1 and X_3 in Figure 3.3 can be removed for a simpler r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$.

iv) By Fact 3.5.5 , the linear map $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ in a finite-gain-stable unity-feedback system $\mathcal{S}(P, C)$ has an r.c.f. (l.c.f.) ; the individual subsystems also have r.c.f.s (l.c.f.s) *if and only if* the condition stated in Theorem 3.5.6 is satisfied.

Proof of Theorem 3.5.6

We only prove part **(i)** ; the proof of **(ii)** follows similarly.

" if "

By assumption, (N, D) is an r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ and R in (3.35) is finite-gain-unimodular. By Fact 3.3.5 , (NR, DR) is an r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$. From (3.35) , $(DR)^{-1}$ is given by

$$(DR)^{-1} = \begin{bmatrix} D_P^{-1} & 0 \\ -D_C^{-1}X_1D_P^{-1} & D_C^{-1} \end{bmatrix} . \tag{3.39}$$

Note that D_P^{-1} and D_C^{-1} are causal. Substituting (3.36) and (3.39) in

$$\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix} = (NR)(DR)^{-1}$$

we obtain $X_2D_C^{-1} = 0$; hence

$$X_2 = 0 .$$

Therefore

$$P = N_PD_P^{-1} , \tag{3.40}$$

$$C = N_CD_C^{-1} , \tag{3.41}$$

$$X_3 = N_CD_C^{-1}X_1 . \tag{3.42}$$

At this point, (3.40) and (3.41) imply that (N_P, D_P) and (N_C, D_C) are <u>r.f.</u>'s of P and C , respectively.

Since (NR, DR) is an r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$, there exist linear finite-gain-stable maps

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} N_P & 0 \\ X_3 & N_C \end{bmatrix} + \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\begin{bmatrix} D_P & 0 \\ X_1 & D_C \end{bmatrix} = I . \tag{3.43}$$

From (3.43) , we obtain

$$U_{22}N_C + V_{22}D_C = I ; \tag{3.44}$$

furthermore, by (3.43)

$$U_{12}N_C + V_{12}D_C = 0 \ ,$$

$$U_{11}N_P + V_{11}D_P + U_{12}X_3 + V_{12}X_1 = I \ . \tag{3.45}$$

From (3.42) , equation (3.45) becomes

$$U_{11}N_P + V_{11}D_P = I \ . \tag{3.46}$$

From (3.44) and (3.46) we conclude that (N_P, D_P) and (N_C, D_C) are in fact r.c.f.'s of P and C , respectively.

" only if "

Let (N_P, D_P) and (N_C, D_C) be r.c.f.s of P and C , respectively. Let

$$N_1 := \begin{bmatrix} N_P & 0 \\ 0 & D_P \end{bmatrix}$$

and

$$D_1 := \begin{bmatrix} D_P & 0 \\ 0 & D_C \end{bmatrix} .$$

Clearly (N_1, D_1) is an r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$. By assumption, (N, D) is also an r.c.f. of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$. By Fact 3.3.5 , there exists a linear finite-gain-unimodular map R such that

$$DR = \begin{bmatrix} D_P & 0 \\ 0 & D_C \end{bmatrix}$$

and $NR = N_1$. Therefore the r.c.f. (N, D) of $\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ is related to r.c.f.s (N_P, D_P) of P and (N_C, D_C) of C so that conditions (3.35) and (3.36) hold .

□

3.6 Nonlinear unity-feedback system

The results in this section are derived in parallel to the ones in Section 3.4 . A direct comparison of what can be done in the nonlinear case with what has been done in the linear case proves useful.

Definition 3.6.1 (<u>nonlinear</u> unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$)

Let $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$ be causal *nonlinear* maps. The unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is shown in Figure 3.4.

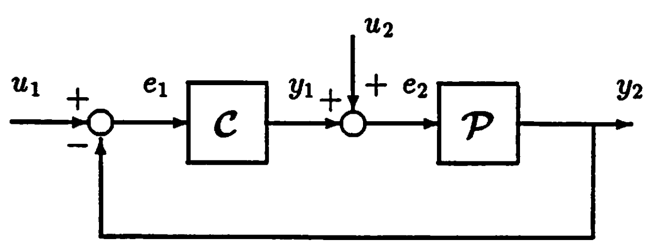

Figure 3.4:
Nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$

Definition 3.6.2 (well-posed $\mathcal{S}(\mathcal{P},\mathcal{C})$)

The unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ where $\mathcal{P}$ and $\mathcal{C}$ are not necessarily linear, is said to be *well-posed* iff there exists a causal map

$$(u_1\,,\,u_2) \mapsto (e_1\,,\,e_2) \quad .$$

Fact 3.6.3

For the unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ in Figure 3.4 , the following three statements are equivalent:

i) The unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is well-posed.

ii) There exists a causal map $(u_1\,,\,u_2) \mapsto (y_1\,,\,y_2)$.

iii) The map

$$\begin{bmatrix} \mathrm{I} & \mathcal{P} \\ -\mathcal{C} & \mathrm{I} \end{bmatrix} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i \tag{3.47}$$

has a causal inverse.

Proof

Follows by (3.6) and the summing node equations in Figure 3.4 :

$$\begin{bmatrix} \mathrm{I} & \mathcal{P} \\ -\mathcal{C} & \mathrm{I} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{3.48}$$

□

Definition 3.6.4 (Λ-stable $\mathcal{S}(\mathcal{P},\mathcal{C})$)

A well-posed unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is said to be Λ-*stable* iff the causal closed-loop map $(u_1, u_2) \mapsto (e_1, e_2)$ is Λ-stable.

Fact 3.6.5

For the well-posed unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ in Figure 3.4 , the following three statements are equivalent:

i) The unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable.

ii) The causal map $(u_1, u_2) \mapsto (y_1, y_2)$ is Λ-stable.

iii) The causal map $\begin{bmatrix} \mathrm{I} & \mathcal{P} \\ -\mathcal{C} & \mathrm{I} \end{bmatrix}^{-1} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i$ is Λ-stable.

Proof

Follows by (3.6) and (3.48).

□

3.7 Stable linear unity-feedback system and necessary and sufficient conditions for stability under nonlinear plant perturbations

Robust stability of feedback systems under *unstructured* perturbations of the plant model has been studied extensively. In the nonlinear case, the small gain theorem [Zam.1,Des.1] gives a *sufficient* condition for robust stability of a stable system under nonlinear *stable* additive perturbations. *Sufficient* robust stability conditions were also obtained in [Åst.1,Cru.1,Des.3,Fra.1,Owe.1,Pos.1,San.1] . In the linear time-invariant case, *necessary and sufficient* conditions for robust stability for a certain class of possibly *unstable* plant perturbations have been obtained in [Doy.1, and references therein] [Chen1] ; for a *general* class of possibly unstable perturbations, the factorization approach yields *necessary and sufficient* conditions for robust stability of the feedback system under *fractional* perturbations of the subsystems [Chen2] . Furthermore, necessary and sufficient conditions for the *existence* of a controller for plants with additive or multiplicative uncertainty are given in [Vid.4] .

For linear time-invariant stable unity-feedback systems with *nonlinear additive* plant perturbations, necessary and sufficient conditions have been obtained in two cases:

i) The additive perturbation has an independent input; hence unmodelled dynamics, which is not coupled to the nominal plant inputs, can be taken into account [Bha.1] .

ii) The perturbed plant is considered as a one-input one-output plant [Hua.1] (see also [Hua.2] for the linear time-invariant additive perturbation case) .

In this section we consider a linear (not necessarily time-invariant) finite-gain-stable unity-feedback system $S(\mathsf{P}, \mathsf{C})$, where the plant and the compensator have r.c.f.s ; we study four cases of *nonlinear* plant perturbations (*additive* , *feedback* , *pre-* and *post-multiplicative*). The plant perturbation $\Delta\mathcal{P}$ is *not* required to be Λ-stable . Using the factorization approach we obtain *necessary* and *sufficient* conditions for all cases in terms of two pairs of nonlinear pseudo-state maps. Simple physical considerations explain the form of these necessary and sufficient conditions.

3.7.1 Additive Perturbation Case

Let $\Delta\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be a causal nonlinear map . For $\mathbf{i},\mathbf{j} = 1,2$, the nonlinear perturbed model of the linear plant P is denoted by $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{add}}$. The one-input one-output plant

$$\mathsf{P} : e_2 \mapsto y_2$$

is perturbed to an i-input j-output $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{add}}$, where (see Figure 3.5)

i) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{11}}^{\mathbf{add}} : (e_2, 0) \mapsto z_2$,

ii) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{12}}^{\mathbf{add}} : (e_2, 0) \mapsto (z_2, y_3)$,

iii) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{21}}^{\mathbf{add}} : (e_2, u_3) \mapsto z_2$,

iv) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{add}} : (e_2, u_3) \mapsto (z_2, y_3)$.

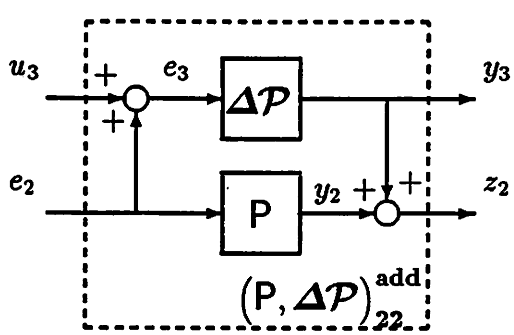

Figure 3.5:
$\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{add}}$: 2-input 2-ouput additive perturbation of P

For $\mathbf{i},\mathbf{j} = 1,2$, the nonlinear unity-feedback system $\mathcal{S}(\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{add}},\mathsf{C})$ is shown in Figure 3.6 : the input-output pair

$$(e_2, z_2)$$

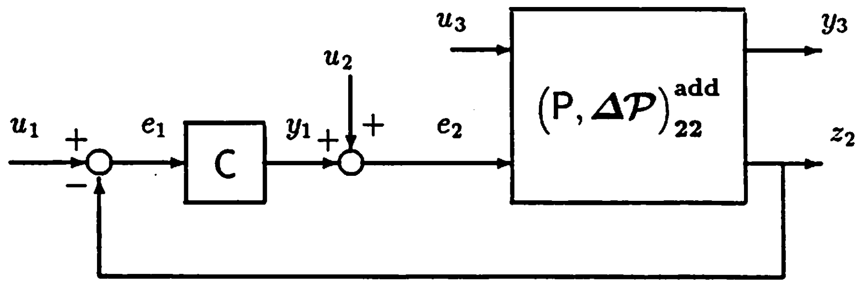

Figure 3.6:
Nonlinear unity-feedback system $\mathcal{S}((\mathsf{P}, \Delta\mathcal{P})_{22}^{\text{add}}, \mathsf{C})$

is used in feedback compensation; that is, u_3 is an *exogenous* input and y_3 is an observed output which is not used in feedback (see Figure 3.6) .

Note that for $\mathbf{i}, \mathbf{j} = 1, 2$, the (i+1)-input feedback system

$$\mathcal{S}((\mathsf{P}, \Delta\mathcal{P})_{\mathbf{ij}}^{\text{add}}, \mathsf{C})$$

is Λ-stable iff the (j+1) outputs (i.e. j outputs of $(\mathsf{P}, \Delta\mathcal{P})_{\mathbf{ij}}^{\text{add}}$ and the output y_1) are determined by Λ-stable maps of the (i+1)-inputs.

Theorem 3.7.1 (n&s condition for robust stability)

Let the linear unity-feedback system $S(\mathsf{P}, \mathsf{C})$ be finite-gain-stable , where $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ are causal linear maps with r.c.f.s $(\mathsf{N_P}, \mathsf{D_P})$ and $(\mathsf{N_C}, \mathsf{D_C})$, respectively. (Hence by Proposition 3.5.1 , P and C have l.c.f.s $(\widetilde{\mathsf{D}}_\mathsf{P}, \widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C}, \widetilde{\mathsf{N}}_\mathsf{C})$, respectively , defined as

$$\begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P} \\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} := \begin{bmatrix} \mathsf{D_C} & \mathsf{N_P} \\ -\mathsf{N_C} & \mathsf{D_P} \end{bmatrix}^{-1} \quad)$$

Under these assumptions, for *any* causal nonlinear map

$$\Delta\mathcal{P} : \Lambda_e^i \to \Lambda_e^o \ ,$$

i) the well-posed $\mathcal{S}((\mathsf{P}, \Delta\mathcal{P})_{11}^{\text{add}}, \mathsf{C})$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{D}}_\mathsf{P} \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1}\mathsf{D_P} \tag{3.49}$$

is Λ-stable;

ii) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{12}^{\mathrm{add}},\mathsf{C})$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(I+\mathsf{N_C}\widetilde{\mathsf{D}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1}\mathsf{D_P} \tag{3.50}$$

is Λ-stable;

iii) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{21}^{\mathrm{add}},\mathsf{C})$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{D}}_{\mathsf{P}}\Delta\mathcal{P}\left(I+\mathsf{N_C}\widetilde{\mathsf{D}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \tag{3.51}$$

is Λ-stable;

iv) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{22}^{\mathrm{add}},\mathsf{C})$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(I+\mathsf{N_C}\widetilde{\mathsf{D}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \tag{3.52}$$

is Λ-stable.

Comments 3.7.2

We offer the following explanation on the forms of the necessary and sufficient conditions for $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{ij}^{\mathrm{add}},\mathsf{C})$ to be Λ-stable, for $\mathbf{i},\mathbf{j}=\mathbf{1},\mathbf{2}$.

i) The effect of *not* observing y_3 :

Since y_3 is not observed, we consider the stability of the map

$$(u_1,u_2,u_3)\mapsto(y_1,z_2) \quad .$$

By assumption, the linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable and the linear maps P and C have r.c.f.s ; hence by Proposition 3.5.1 , the map P has an l.c.f. $(\widetilde{\mathsf{D}}_{\mathsf{P}},\widetilde{\mathsf{N}}_{\mathsf{P}})$. Using this l.c.f. of P , we redraw the feedback system $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{21}^{\mathrm{add}},\mathsf{C})$ as in Figure 3.7 ; note that we use the linearity of $\widetilde{\mathsf{D}}_{\mathsf{P}}$.

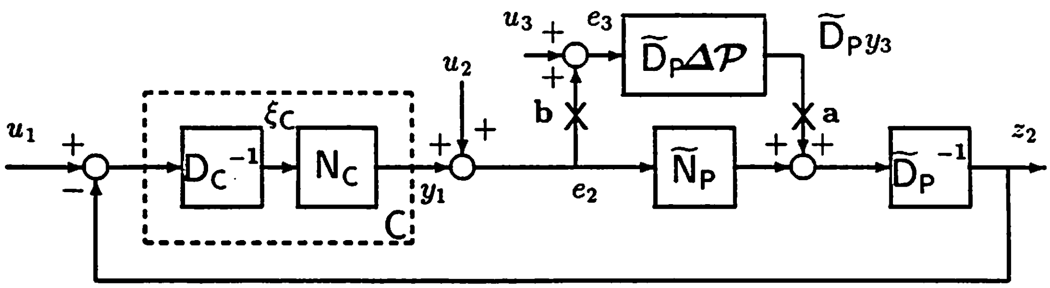

Figure 3.7:
The unity-feedback system $\mathcal{S}\left(\left(\widetilde{D}_P^{-1}\widetilde{N}_P, \Delta\mathcal{P}\right)_{21}^{\text{add}}, N_C D_C^{-1}\right)$

Now view Figure 3.7 as a feedback system consisting of the nonlinear, possibly unstable, subsystem $\widetilde{D}_P\Delta\mathcal{P}$ closed in a feedback-loop by the finite-gain-stable subsytem whose input is at **a** and output at **b** ; note that

$$\begin{aligned}
\mathbf{b} &= u_2 + C\left(u_1 - \widetilde{D}_P^{-1}\left(\mathbf{a} + \widetilde{N}_P\mathbf{b}\right)\right) \\
&= -\left(I + CP\right)^{-1} C\widetilde{D}_P^{-1}\mathbf{a} + \left(I + CP\right)^{-1} C u_1 + \left(I + CP\right)^{-1} u_2 \\
&= -N_C\mathbf{a} + D_P\widetilde{N}_C u_1 + D_P\widetilde{D}_C u_2 \ .
\end{aligned}$$

The resulting closed-loop system is Λ-stable if and only if

$$\left(\widetilde{D}_P\Delta\mathcal{P}\right)\left(I + N_C\left(\widetilde{D}_P\Delta\mathcal{P}\right)\right)^{-1}$$

is Λ-stable [Des.2] .

In conclusion, whenever we fail to observe y_3 , the necessary and sufficient condition for Λ-stability has $\widetilde{D}_P$ as an additional <u>left</u> factor.

ii) The effect of setting $u_3 = 0$:

By linearity and finite-gain-stability of $S(P, C)$, the map

$$y_3 \mapsto e_2$$

(see Figure 3.8) is given by

$$e_2 = -\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P} y_3 + \mathsf{D_P}\left(\widetilde{\mathsf{N}}_\mathsf{C} u_1 + \widetilde{\mathsf{D}}_\mathsf{C} u_2\right) .$$

Now consider the system in Figure 3.8 as a feedback system consisting of the subsystem $\Delta\mathcal{P}$ in a closed-loop with the finite-gain-stable subsystem whose input is y_3 and output is e_2 . Whenever $u_3 = 0$, the input e_2 is in the range of $\mathsf{D_P}$, hence the necessary and sufficient condition for Λ-stability has $\mathsf{D_P}$ as a *right* factor.

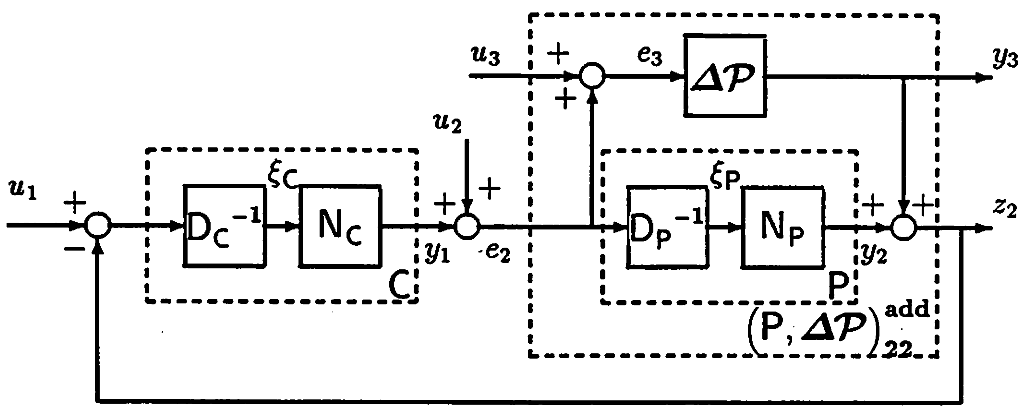

Figure 3.8:
The feedback system $\mathcal{S}\left(\left(\mathsf{N_P}\mathsf{D_P}^{-1},\Delta\mathcal{P}\right)^{\mathrm{add}}_{22},\mathsf{N_C}\mathsf{D_C}^{-1}\right)$

Proof of Theorem 3.7.1

By assumption, the maps P and C have r.c.f.s $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively; moreover the feedback system $S(\mathsf{P},\mathsf{C})$ is finite-gain-stable . By Corollary 3.4.8 , the finite-gain-unimodular map M defined in (3.11) ,

$$\mathsf{M} := \begin{bmatrix} \mathsf{D_C} & \mathsf{N_P} \\ -\mathsf{N_C} & \mathsf{D_P} \end{bmatrix}$$

has a finite-gain-stable inverse M^{-1} as defined in (3.27)

$$\mathsf{M}^{-1} := \begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P} \\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} .$$

Writing the summing node equations in Figure 3.8 in terms of ξ_C , ξ_P and e_3 , we obtain

$$\mathsf{M}\begin{bmatrix} \xi_C \\ \xi_P \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \Delta\mathcal{P} \\ 0 \end{bmatrix} e_3 \tag{3.53}$$

$$e_3 = u_2 + u_3 + \mathsf{N}_C \xi_C \ . \tag{3.54}$$

Composing both sides of (3.53) by the *linear* map M^{-1} in (3.27) , we obtain

$$\begin{bmatrix} \xi_C \\ \xi_P \end{bmatrix} = \mathsf{M}^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \widetilde{\mathsf{D}}_P \\ \widetilde{\mathsf{N}}_C \end{bmatrix} \Delta\mathcal{P}\, e_3 \ . \tag{3.55}$$

Substituting ξ_C determined by (3.55) in (3.54) and using the identity

$$\mathsf{M}\,\mathsf{M}^{-1} = \mathrm{I} \ ,$$

we obtain

$$e_3 = \left(\mathrm{I} + \mathsf{N}_C\widetilde{\mathsf{D}}_P\Delta\mathcal{P}\right)^{-1}\left[\mathsf{D}_P\left[\widetilde{\mathsf{N}}_C \quad \widetilde{\mathsf{D}}_C\right] \quad \mathrm{I}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \tag{3.56}$$

Substituting (3.56) in (3.55) , the closed-loop pseudo-state map

$$(u_1, u_2, u_3) \mapsto (\xi_C, \xi_P)$$

is given by

$$\begin{aligned} \begin{bmatrix} \xi_C \\ \xi_P \end{bmatrix} &= \mathsf{M}^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &- \begin{bmatrix} \widetilde{\mathsf{D}}_P \\ \widetilde{\mathsf{N}}_C \end{bmatrix} \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_C\widetilde{\mathsf{D}}_P\Delta\mathcal{P}\right)^{-1}\left[\mathsf{D}_P\left[\widetilde{\mathsf{N}}_C \quad \widetilde{\mathsf{D}}_C\right] \quad \mathrm{I}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned} \tag{3.57}$$

We now state the necessary and sufficient conditions for the four cases in terms of the pseudo-state maps given by (3.57) .

i) The well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\text{add}}_{11},\mathsf{C}\big)$ is Λ-stable if and only if

$$(u_1,u_2,0)\mapsto\xi_{\mathsf{C}}$$

is Λ-stable (recall Lemma 3.4.7) . Since the finite-gain-stable map

$$\begin{bmatrix}\widetilde{\mathsf{N}}_{\mathsf{C}} & \widetilde{\mathsf{D}}_{\mathsf{C}}\end{bmatrix}$$

has a finite-gain-stable right-inverse (follows by the identity $\mathsf{M}^{-1}\mathsf{M}=I$) , condition (3.49) follows from (3.57) .

ii) The well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\text{add}}_{12},\mathsf{C}\big)$ is Λ-stable if and only if

$$(u_1,u_2,0)\mapsto(\xi_{\mathsf{C}},\xi_{\mathsf{P}})$$

is Λ-stable (the sufficiency follows from Figure 3.8 and the finite-gain-stability of the maps N_{P} , D_{P} , N_{C} and D_{C} . The necessity follows by the fact that $(\mathsf{N}_{\mathsf{C}},\mathsf{D}_{\mathsf{C}})$ and $(\mathsf{N}_{\mathsf{P}},\mathsf{D}_{\mathsf{P}})$ are r.c.f.s of C and P , respectively. Using similar reasoning as in case **i)** and noting that the finite-gain-stable map $\begin{bmatrix}\widetilde{\mathsf{D}}_{\mathsf{P}}\\ \widetilde{\mathsf{N}}_{\mathsf{C}}\end{bmatrix}$ has a finite-gain-stable left-inverse (follows by the identity $\mathsf{M}\mathsf{M}^{-1}=I$) , we obtain condition (3.50) from (3.57) .

Using similar reasoning, we obtain the proofs of the following: since now $u_3\neq 0$, the identity map in the last term of (3.57) plays a crucial role.

iii) The well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\text{add}}_{21},\mathsf{C}\big)$ is Λ-stable if and only if

$$(u_1,u_2,u_3)\mapsto\xi_{\mathsf{C}}$$

is Λ-stable if and only if the map in (3.51) is Λ-stable.

iv) The well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\text{add}}_{22},\mathsf{C}\big)$ is Λ-stable if and only if

$$(u_1, u_2, u_3) \mapsto (\xi_C, \xi_P)$$

is Λ-stable if and only if the map in (3.52) is Λ-stable.

□

The necessary and sufficient conditions in Theorem 3.7.1 can be also stated in terms of "admissible" perturbations $\Delta\mathcal{P}$; i.e. those nonlinear (possibly unstable) perturbations $\Delta\mathcal{P}$ such that the perturbed plant is still stabilized by the nominal compensator.

Corollary 3.7.3 (parametrization of admissible perturbations)

Let the assumptions of Theorem 3.7.1 hold. Under these assumptions,

i) the well-posed feedback system $S(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{11}^{\mathbf{add}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \widetilde{\mathsf{D}}_{\mathsf{P}}^{-1}\mathcal{Q}\left(\mathsf{D}_{\mathsf{P}} - \mathsf{N}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.58}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

ii) the well-posed feedback system $S(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{12}^{\mathbf{add}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \widetilde{\mathsf{N}}_{\mathsf{C}}\mathcal{Q}\right)^{-1}\mathsf{D}_{\mathsf{P}}^{-1} \tag{3.59}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iii) the well-posed feedback system $S(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{21}^{\mathbf{add}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \widetilde{\mathsf{D}}_{\mathsf{P}}^{-1}\mathcal{Q}\left(\mathrm{I} - \mathsf{N}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.60}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iv) the well-posed feedback system $S((\mathsf{P},\Delta\mathcal{P})^{\text{add}}_{22},C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I}-\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\mathcal{Q}\right)^{-1} \tag{3.61}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$.

Proof

The proof follows by establishing that conditions **i)** – **iv)** in Theorem 3.7.1 are equivalent to the conditions **i)** – **iv)** in Corollary 3.7.3 , respectively.

We first show the equivalence in (3.62) :

$$\mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} \quad \Leftrightarrow \quad \Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\mathcal{Q}\right)^{-1} . \tag{3.62}$$

Composing both sides of the left equation in (3.62) by $\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}$, we obtain

$$\begin{aligned} \mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\mathcal{Q} &= \mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} \\ &= \left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}-\mathrm{I}\right)\left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} \\ &= \mathrm{I} - \left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} . \end{aligned} \tag{3.63}$$

From (3.63), we obtain

$$\left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} = \left(\mathrm{I}-\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\mathcal{Q}\right) . \tag{3.64}$$

Substituting (3.64) in the left equation of (3.62), the right equality in (3.62) follows. The proof of the converse direction of (3.62) is identical.

From (3.62) , we conclude that condition **iv)** in Corollary 3.7.3 is equivalent to condition **iv)** in Theorem 3.7.1 .

Substituting $\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}$ for $\mathcal{Q}$ in (3.62) , we obtain

$$\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N_C}\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1} \quad \Leftrightarrow \quad \Delta\mathcal{P} = \widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\left(\mathrm{I}-\mathsf{N_C}\mathcal{Q}\right)^{-1} ;$$

hence conditions **iii)** in Theorem 3.7.1 and Corollary 3.7.3 are equivalent.

Substituting $\mathcal{Q}D_P^{-1}$ for $\mathcal{Q}$ in (3.62), we obtain

$$\mathcal{Q}D_P^{-1} = \Delta\mathcal{P}\left(I + N_C\widetilde{D}_P\Delta\mathcal{P}\right)^{-1}$$

$$\Updownarrow$$

$$\begin{aligned}\Delta\mathcal{P} &= \mathcal{Q}D_P^{-1}\left(I - N_C\widetilde{D}_P\mathcal{Q}D_P^{-1}\right)^{-1} \\ &= \mathcal{Q}\left(I - \widetilde{N}_C\mathcal{Q}\right)^{-1}D_P^{-1}\end{aligned}$$

(recall that $N_C\widetilde{D}_P = D_P\widetilde{N}_C$ by (3.11) and (3.27)); hence conditions **ii)** in Theorem 3.7.1 and Corollary 3.7.3 are equivalent.

Substituting $\widetilde{D}_P^{-1}\mathcal{Q}D_P^{-1}$ for $\mathcal{Q}$ in (3.62), we obtain

$$\widetilde{D}_P^{-1}\mathcal{Q}D_P^{-1} = \Delta\mathcal{P}\left(I + N_C\widetilde{D}_P\Delta\mathcal{P}\right)^{-1}$$

$$\Updownarrow$$

$$\begin{aligned}\Delta\mathcal{P} &= \widetilde{D}_P^{-1}\mathcal{Q}D_P^{-1}\left(I - N_C\mathcal{Q}D_P^{-1}\right)^{-1} \\ &= \widetilde{D}_P^{-1}\mathcal{Q}\left(D_P - N_C\mathcal{Q}\right)^{-1};\end{aligned}$$

hence conditions **i)** in Theorem 3.7.1 and Corollary 3.7.3 are equivalent.

□

3.7.2 Feedback Perturbation Case

Let $\Delta\mathcal{P} : \Lambda_e^o \to \Lambda_e^i$ be a causal <u>nonlinear</u> map . For $\mathbf{i},\mathbf{j} = 1,2$ the nonlinear perturbed model of the <u>linear</u> plant P is denoted by $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{feed}}$. The one-input one-output plant $\mathsf{P} : e_2 \mapsto y_2$ is perturbed to an **i**-input **j**-output $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{feed}}$, where (see Figure 3.9)

i) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{11}}^{\mathbf{feed}} : (e_2,0) \mapsto y_2$,

ii) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{12}}^{\mathbf{feed}} : (e_2,0) \mapsto (y_2,y_3)$,

iii) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{21}}^{\mathbf{feed}} : (e_2,u_3) \mapsto y_2$,

iv) $\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{feed}} : (e_2,u_3) \mapsto (y_2,y_3)$.

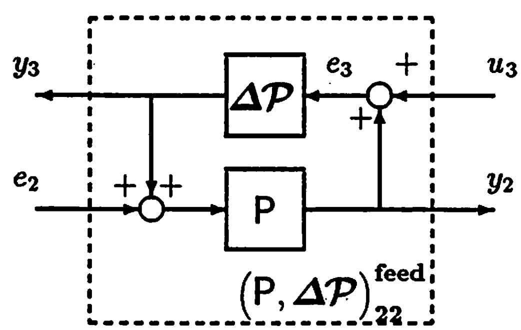

Figure 3.9:
$\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{feed}}$: 2-input 2-ouput feedback perturbation of P

For $\mathbf{i},\mathbf{j} = 1,2$, the nonlinear unity-feedback system $\mathcal{S}(\left(\mathsf{P},\Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{feed}},\mathsf{C})$ is shown in Figure 3.10 : the input-output pair

$$(e_2,y_2)$$

is used in feedback compensation; that is, u_3 is an *exogenous* input and y_3 is an observed output which is <u>not</u> used in feedback (see Figure 3.10) .

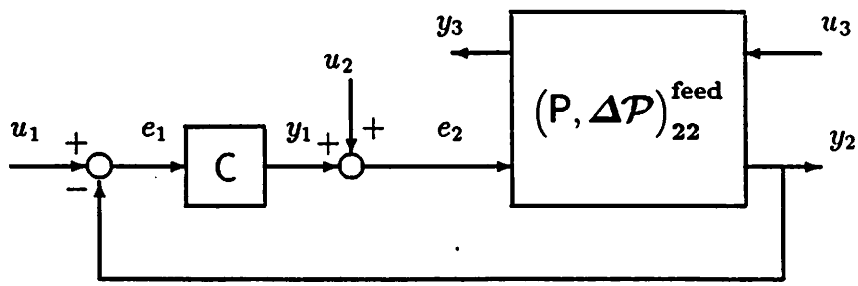

Figure 3.10:
Nonlinear unity-feedback system $\mathcal{S}\left((\mathsf{P}, \Delta\mathcal{P})_{22}^{\text{feed}}, \mathsf{C}\right)$

Similar to the additive perturbation case, for $\mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, the $(\mathbf{i}+\mathbf{1})$-input feedback system $\mathcal{S}\left((\mathsf{P}, \Delta\mathcal{P})_{\mathbf{ij}}^{\text{feed}}, \mathsf{C}\right)$ is Λ-stable iff the $(\mathbf{j}+\mathbf{1})$ outputs (i.e. $\mathbf{j}$ outputs of $(\mathsf{P}, \Delta\mathcal{P})_{\mathbf{ij}}^{\text{feed}}$ and the output y_1) are determined by Λ-stable maps of the $(\mathbf{i}+\mathbf{1})$ inputs.

We state the following theorem without proof since the proof is identical to that of Theorem 3.7.1 .

Theorem 3.7.4 (n&s condition for robust stability)

Let the linear unity-feedback system $S(\mathsf{P}, \mathsf{C})$ be finite-gain-stable , where $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ are causal <u>linear</u> maps with r.c.f.s $(\mathsf{N_P}, \mathsf{D_P})$ and $(\mathsf{N_C}, \mathsf{D_C})$, respectively. (Hence by Proposition 3.5.1 , P and C have l.c.f.s $(\widetilde{\mathsf{D}}_\mathsf{P}, \widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C}, \widetilde{\mathsf{N}}_\mathsf{C})$, respectively , defined as

$$\begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P} \\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} := \begin{bmatrix} \mathsf{D_C} & \mathsf{N_P} \\ -\mathsf{N_C} & \mathsf{D_P} \end{bmatrix}^{-1} \quad)$$

Under these assumptions, for *any* causal <u>nonlinear</u> map

$$\Delta\mathcal{P} : \Lambda_e^o \to \Lambda_e^i \ ,$$

i) the well-posed $\mathcal{S}\left((\mathsf{P}, \Delta\mathcal{P})_{11}^{\text{feed}}, \mathsf{C}\right)$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{N}}_\mathsf{P} \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_P}\widetilde{\mathsf{D}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N_P} \tag{3.65}$$

is Λ-stable;

ii) the well-posed $\mathcal{S}\left(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{12}^{\text{feed}}, \mathsf{C}\right)$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1}\mathsf{N}_{\mathsf{P}} \tag{3.66}$$

is Λ-stable;

iii) the well-posed $\mathcal{S}\left(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{21}^{\text{feed}}, \mathsf{C}\right)$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1} \tag{3.67}$$

is Λ-stable;

iv) the well-posed $\mathcal{S}\left(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{22}^{\text{feed}}, \mathsf{C}\right)$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1} \tag{3.68}$$

is Λ-stable.

□

The necessary and sufficient conditions ii) – iv) in Theorem 3.7.4 can be also stated in terms of "admissible" perturbations $\Delta\mathcal{P}$; i.e. those nonlinear (possibly unstable) perturbations $\Delta\mathcal{P}$ such that the perturbed plant is still stabilized by the nominal compensator.

Corollary 3.7.5 (representation of admissible perturbations)

Let the assumptions of Theorem 3.7.4 hold. Under these assumptions,

i) the well-posed feedback system $S\left(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{12}^{\text{feed}}, C\right)$ is Λ-stable if and only if

$$\Delta\mathcal{P}\mathsf{N}_{\mathsf{P}} = \mathcal{Q}\left(\mathrm{I} - \widetilde{\mathsf{D}}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.69}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \rightarrow \Lambda_e^o$;

ii) the well-posed feedback system $S\left(\left(\mathsf{P},\Delta\mathcal{P}\right)_{21}^{\mathbf{feed}},C\right)$ is Λ-stable if and only if

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I}-\mathsf{D}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.70}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iii) the well-posed feedback system $S\left(\left(\mathsf{P},\Delta\mathcal{P}\right)_{22}^{\mathbf{feed}},C\right)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I}-\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.71}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$.

Proof

The proof follows by establishing that conditions **ii)** – **iv)** in Theorem 3.7.4 are equivalent to conditions **i)** – **iii)** in Corollary 3.7.5 , respectively.

Substituting $\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}$ for $\mathsf{N}_{\mathsf{C}}\widetilde{\mathsf{D}}_{\mathsf{P}}$ in (3.62) (see the proof of Corollary 3.7.3) , we obtain

$$\mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1} \;\Leftrightarrow\; \Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I}-\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\mathcal{Q}\right)^{-1} . \tag{3.72}$$

From (3.72) , we conclude that condition **iii)** in Corollary 3.7.5 is equivalent to condition **iv)** in Theorem 3.7.4 .

Since $\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}} = \mathsf{D}_{\mathsf{C}}\widetilde{\mathsf{N}}_{\mathsf{P}}$ by (3.11) and (3.27) , we can write (3.67) as

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1} = \left(\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)\left(\mathrm{I}+\mathsf{D}_{\mathsf{C}}\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)\right)^{-1} . \tag{3.73}$$

Substituting $\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}$ for $\Delta\mathcal{P}$ and D_{C} for $\mathsf{N}_{\mathsf{C}}\widetilde{\mathsf{D}}_{\mathsf{P}}$ in (3.62) and using (3.73) , we obtain

$$\mathcal{Q} = \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{P}}\widetilde{\mathsf{D}}_{\mathsf{C}}\Delta\mathcal{P}\right)^{-1} \;\Leftrightarrow\; \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I}-\mathsf{D}_{\mathsf{C}}\mathcal{Q}\right)^{-1} ;$$

hence conditions **iii)** in Theorem 3.7.4 and **ii)** in Corollary 3.7.5 are equivalent.

Since the map $\mathsf{N_P}$ is linear, (3.66) can be written as

$$\Delta\mathcal{P}\left(I+\mathsf{N_P}\widetilde{\mathsf{D}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N_P} = \Delta\mathcal{P}\mathsf{N_P}\left(I+\widetilde{\mathsf{D}}_\mathsf{C}\Delta\mathcal{P}\mathsf{N_P}\right)^{-1} .$$

By a similar reasoning as the one above, the equivalence of conditions **ii)** in Theorem 3.7.4 and **i)** in Corollary 3.7.5 follows since

$$\mathcal{Q} = \Delta\mathcal{P}\left(I+\mathsf{N_P}\widetilde{\mathsf{D}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N_P} \quad\Leftrightarrow\quad \Delta\mathcal{P}\mathsf{N_P} = \mathcal{Q}\left(I-\widetilde{\mathsf{D}}_\mathsf{C}\mathcal{Q}\right)^{-1} .$$

□

3.7.3 Pre-Multiplicative Perturbation Case

Let $\Delta\mathcal{P} : \Lambda_e^i \to \Lambda_e^i$ be a causal <u>nonlinear</u> map . For $\mathbf{i}, \mathbf{j} = 1, 2$ the nonlinear perturbed model of the <u>linear</u> plant P is denoted by $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{pre}}$. The one-input one-output plant $\mathsf{P} : e_2 \mapsto y_2$ is perturbed to an i-input j-output $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{pre}}$, where (see Figure 3.11)

i) $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{11}}^{\mathbf{pre}} : (e_3, 0) \mapsto y_2$,

ii) $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{12}}^{\mathbf{pre}} : (e_3, 0) \mapsto (y_2, y_3)$,

iii) $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{21}}^{\mathbf{pre}} : (e_3, u_3) \mapsto y_2$,

iv) $\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{pre}} : (e_3, u_3) \mapsto (y_2, y_3)$.

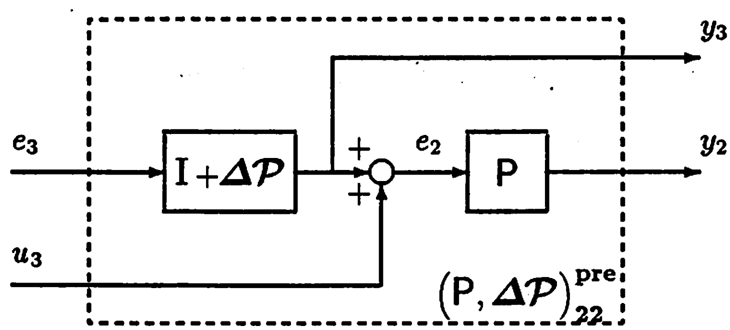

Figure 3.11:
$\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{22}}^{\mathbf{pre}}$: 2-input 2-ouput pre-multiplicative perturbation of P

For $\mathbf{i}, \mathbf{j} = 1, 2$, the nonlinear unity-feedback system $\mathcal{S}(\left(\mathsf{P}, \Delta\mathcal{P}\right)_{\mathbf{ij}}^{\mathbf{pre}}, \mathsf{C})$ is shown in Figure 3.12 : the input-output pair

$$(e_3, y_2)$$

is used in feedback compensation; that is, u_3 is an *exogenous* input and y_3 is an observed output which is <u>not</u> used in feedback (see Figure 3.12) .

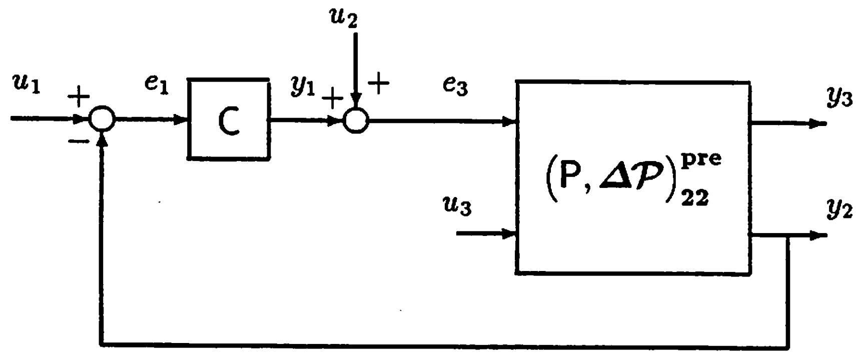

Figure 3.12:
Nonlinear unity-feedback system $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{22}^{\mathbf{pre}},\mathsf{C})$

Similar to the additive perturbation case, for $\mathbf{i}\,,\mathbf{j}=\mathbf{1},\mathbf{2}$, the $\mathbf{(i+1)}$-input feedback system $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{\mathbf{ij}}^{\mathbf{pre}},\mathsf{C})$ is Λ-stable iff the $\mathbf{(j+1)}$ outputs (i.e. $\mathbf{j}$ outputs of $(\mathsf{P},\Delta\mathcal{P})_{\mathbf{ij}}^{\mathbf{pre}}$ and the output y_1) are determined by Λ-stable maps of the $\mathbf{(i+1)}$ inputs.

We state the following theorem without proof since its proof is similar to that of Theorem 3.7.1 .

Theorem 3.7.6 (n&s condition for robust stability)

Let the linear unity-feedback system $\mathcal{S}(\mathsf{P},\mathsf{C})$ be finite-gain-stable , where $\mathsf{P}:\Lambda_e^i\to\Lambda_e^o$ and $\mathsf{C}:\Lambda_e^o\to\Lambda_e^i$ are causal linear maps with r.c.f.s $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively. (Hence by Proposition 3.5.1 , P and C have l.c.f.s $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C},\widetilde{\mathsf{N}}_\mathsf{C})$, respectively , defined as

$$\begin{bmatrix}\widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P}\\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C}\end{bmatrix}:=\begin{bmatrix}\mathsf{D}_\mathsf{C} & \mathsf{N}_\mathsf{P}\\ -\mathsf{N}_\mathsf{C} & \mathsf{D}_\mathsf{P}\end{bmatrix}^{-1}\ )$$

Under this assumption, for *any* causal nonlinear map

$$\Delta\mathcal{P}:\Lambda_e^i\to\Lambda_e^i\ ,$$

i) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{11}^{\mathbf{pre}},\mathsf{C})$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{N}}_\mathsf{P}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_\mathsf{C}\widetilde{\mathsf{N}}_\mathsf{P}\Delta\mathcal{P}\right)^{-1}\mathsf{D}_\mathsf{P} \qquad (3.74)$$

is Λ-stable;

ii) the well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\mathrm{pre}}_{12},\mathsf{C}\big)$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{C}}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1}\mathsf{D}_{\mathsf{P}} \tag{3.75}$$

is Λ-stable;

iii) the well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\mathrm{pre}}_{21},\mathsf{C}\big)$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{C}}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \tag{3.76}$$

is Λ-stable;

iv) the well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})^{\mathrm{pre}}_{22},\mathsf{C}\big)$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N}_{\mathsf{C}}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \tag{3.77}$$

is Λ-stable.

□

The necessary and sufficient conditions **i) – iv)** in Theorem 3.7.6 can be also stated in terms of "admissible" perturbations $\Delta\mathcal{P}$; i.e. those nonlinear (possibly unstable) perturbations $\Delta\mathcal{P}$ such that the perturbed plant is still stabilized by the nominal compensator.

Corollary 3.7.7 (representation of admissible perturbations)

Let the assumptions of Theorem 3.7.6 hold. Under these assumptions,

i) the well-posed feedback system $S\big((\mathsf{P},\Delta\mathcal{P})^{\mathrm{pre}}_{11},C\big)$ is Λ-stable if and only if

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathsf{D}_{\mathsf{P}}-\mathsf{N}_{\mathsf{C}}\mathcal{Q}\right)^{-1} \tag{3.78}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$;

ii) the well-posed feedback system $S((\mathsf{P},\Delta\mathcal{P})_{12}^{\text{pre}},C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathsf{D_P} - \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{Q}\right)^{-1} \tag{3.79}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iii) the well-posed feedback system $S((\mathsf{P},\Delta\mathcal{P})_{21}^{\text{pre}},C)$ is Λ-stable if and only if

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N_C}\mathcal{Q}\right)^{-1} \tag{3.80}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iv) the well-posed feedback system $S((\mathsf{P},\Delta\mathcal{P})_{22}^{\text{pre}},C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{Q}\right)^{-1} \tag{3.81}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$.

Proof

The proof follows from the following equivalences (see the proofs of Corollaries 3.7.3 and 3.7.5 for similar manipulations) :

$$\mathcal{Q} = \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1}\mathsf{D_P} \Leftrightarrow \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathsf{D_P} - \mathsf{N_C}\mathcal{Q}\right)^{-1}$$

$$\mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1}\mathsf{D_P} \Leftrightarrow \Delta\mathcal{P} = \mathcal{Q}\left(\mathsf{D_P} - \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{Q}\right)^{-1}$$

$$\mathcal{Q} = \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \Leftrightarrow \widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N_C}\mathcal{Q}\right)^{-1}$$

$$\mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\Delta\mathcal{P}\right)^{-1} \Leftrightarrow \Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N_C}\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{Q}\right)^{-1}$$

□

3.7.4 Post-Multiplicative Perturbation Case

Let $\mathcal{\Delta P} : \Lambda_e^o \to \Lambda_e^o$ be a causal <u>nonlinear</u> map . For $\mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, the nonlinear perturbed model of the <u>linear</u> plant P is denoted by $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{ij}}^{\mathbf{post}}$. The one-input one-output plant $\mathsf{P} : e_2 \mapsto y_2$ is perturbed to an **i**-input **j**-output $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{ij}}^{\mathbf{post}}$, where (see Figure 3.13)

i) $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{11}}^{\mathbf{post}} : (e_2, 0) \mapsto y_3$,

ii) $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{12}}^{\mathbf{post}} : (e_2, 0) \mapsto (y_2, y_3)$,

iii) $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{21}}^{\mathbf{post}} : (e_2, u_3) \mapsto y_2$,

iv) $\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{22}}^{\mathbf{post}} : (e_2, u_3) \mapsto (y_2, y_3)$.

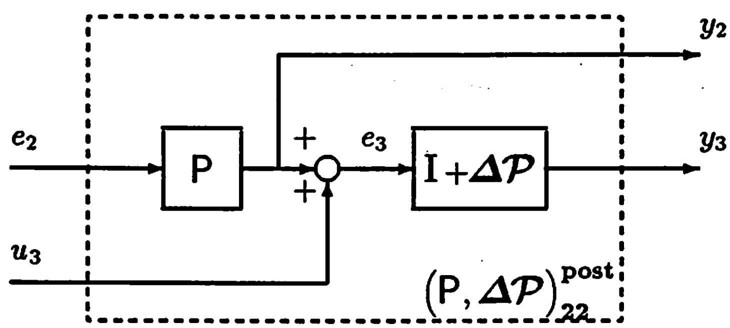

Figure 3.13:
$\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{22}}^{\mathbf{post}}$: 2-input 2-ouput post-multiplicative perturbation of P

For $\mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, the nonlinear unity-feedback system $\mathcal{S}(\left(\mathsf{P}, \mathcal{\Delta P}\right)_{\mathbf{ij}}^{\mathbf{post}}, \mathsf{C})$ is shown in Figure 3.14 : the input-output pair

$$(e_2, y_3)$$

is used in feedback compensation; that is, u_3 is an *exogenous* input and y_2 is an observed output which is <u>not</u> used in feedback (see Figure 3.14) .

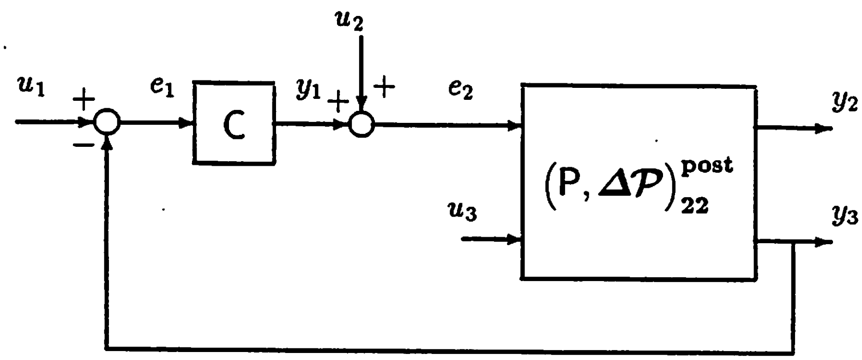

Figure 3.14:
Nonlinear unity-feedback system $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})_{22}^{\text{post}},\mathsf{C}\big)$

Similar to the additive perturbation case, for $\mathbf{i}\ ,\ \mathbf{j} = \mathbf{1},\mathbf{2}$, the (i+1)-input feedback system $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})_{\mathbf{ij}}^{\text{post}},\mathsf{C}\big)$ is Λ-stable iff the (j+1) outputs (i.e. j outputs of $(\mathsf{P},\Delta\mathcal{P})_{\mathbf{ij}}^{\text{post}}$ and the output y_1) are determined by Λ-stable maps of the (i+1) inputs.

We state the following theorem without proof since its proof is simialr to the proof of Theorem 3.7.1 .

Theorem 3.7.8 (n&s condition for robust stability)

Let the linear unity-feedback system $S(\mathsf{P},\mathsf{C})$ be finite-gain-stable , where $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathsf{C} : \Lambda_e^o \to \Lambda_e^i$ are causal <u>linear</u> maps with r.c.f.s $(\mathsf{N_P},\mathsf{D_P})$ and $(\mathsf{N_C},\mathsf{D_C})$, respectively. (Hence by Proposition 3.5.1 , P and C have l.c.f.s $(\widetilde{\mathsf{D}}_\mathsf{P},\widetilde{\mathsf{N}}_\mathsf{P})$ and $(\widetilde{\mathsf{D}}_\mathsf{C},\widetilde{\mathsf{N}}_\mathsf{C})$, respectively , defined as

$$\begin{bmatrix} \widetilde{\mathsf{D}}_\mathsf{P} & -\widetilde{\mathsf{N}}_\mathsf{P} \\ \widetilde{\mathsf{N}}_\mathsf{C} & \widetilde{\mathsf{D}}_\mathsf{C} \end{bmatrix} := \begin{bmatrix} \mathsf{D_C} & \mathsf{N_P} \\ -\mathsf{N_C} & \mathsf{D_P} \end{bmatrix}^{-1} \quad)$$

Under this assumption, for *any* causal <u>nonlinear</u> map

$$\Delta\mathcal{P} : \Lambda_e^o \to \Lambda_e^o \ ,$$

i) the well-posed $\mathcal{S}\big((\mathsf{P},\Delta\mathcal{P})_{11}^{\text{post}},\mathsf{C}\big)$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\big(\mathrm{I}+\mathsf{N_P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\big)^{-1}\mathsf{N_P} \tag{3.82}$$

is Λ-stable;

ii) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{12}^{\text{post}},\mathsf{C})$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N_P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N_P} \tag{3.83}$$

is Λ-stable;

iii) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{21}^{\text{post}},\mathsf{C})$ is Λ-stable if and only if the map

$$\widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N_P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1} \tag{3.84}$$

is Λ-stable;

iv) the well-posed $\mathcal{S}((\mathsf{P},\Delta\mathcal{P})_{22}^{\text{post}},\mathsf{C})$ is Λ-stable if and only if the map

$$\Delta\mathcal{P}\left(\mathrm{I}+\mathsf{N_P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1} \tag{3.85}$$

is Λ-stable.

□

The necessary and sufficient conditions **i) – iv)** in Theorem 3.7.8 can be also stated in terms of "admissible" perturbations $\Delta\mathcal{P}$; i.e. those nonlinear (possibly unstable) perturbations $\Delta\mathcal{P}$ such that the perturbed plant is still stabilized by the nominal compensator.

Corollary 3.7.9 (representation of admissible perturbations)

Let the assumptions of Theorem 3.7.8 hold. Under these assumptions,

i) the well-posed feedback system $S((\mathsf{P},\Delta\mathcal{P})_{11}^{\text{post}},C)$ is Λ-stable if and only if

$$\Delta\mathcal{P}\mathsf{N_P} = \widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\left(\mathrm{I}-\widetilde{\mathsf{N}}_\mathsf{C}\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\right)^{-1} \tag{3.86}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

ii) the well-posed feedback system $S((\mathsf{P}, \Delta\mathcal{P})_{12}^{\text{post}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P}\mathsf{N}_\mathsf{P} = \mathcal{Q}\left(\mathrm{I} - \widetilde{\mathsf{N}}_\mathsf{C}\mathcal{Q}\right)^{-1} \tag{3.87}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iii) the well-posed feedback system $S((\mathsf{P}, \Delta\mathcal{P})_{21}^{\text{post}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\left(\mathrm{I} - \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\right)^{-1} \tag{3.88}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$;

iv) the well-posed feedback system $S((\mathsf{P}, \Delta\mathcal{P})_{22}^{\text{post}}, C)$ is Λ-stable if and only if

$$\Delta\mathcal{P} = \mathcal{Q}\left(\mathrm{I} - \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\mathcal{Q}\right)^{-1} \tag{3.89}$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^i \to \Lambda_e^o$.

Proof

The proof follows from the following equivalences (see the proofs of Corollaries 3.7.3 and 3.7.5 for similar manipulations) :

$$\left\{\begin{array}{c} \mathcal{Q} = \widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N}_\mathsf{P} \\ \Updownarrow \\ \Delta\mathcal{P}\mathsf{N}_\mathsf{P} = \widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\left(\mathrm{I} - \widetilde{\mathsf{N}}_\mathsf{C}\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\right)^{-1} \end{array}\right\}$$

$$\left\{\begin{array}{c} \mathcal{Q} = \Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1}\mathsf{N}_\mathsf{P} \\ \Updownarrow \\ \Delta\mathcal{P}\mathsf{N}_\mathsf{P} = \mathcal{Q}\left(\mathrm{I} - \widetilde{\mathsf{N}}_\mathsf{C}\mathcal{Q}\right)^{-1} \end{array}\right\}$$

$$\left\{\begin{array}{c} \mathcal{Q} = \widetilde{\mathsf{D}}_\mathsf{P}\Delta\mathcal{P}\left(\mathrm{I} + \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\Delta\mathcal{P}\right)^{-1} \\ \Updownarrow \\ \Delta\mathcal{P} = \widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\left(\mathrm{I} - \mathsf{N}_\mathsf{P}\widetilde{\mathsf{N}}_\mathsf{C}\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\mathcal{Q}\right)^{-1} \end{array}\right\}$$

$$\left\{\begin{array}{c} \mathcal{Q} = \Delta\mathcal{P}\left(I+N_P\widetilde{N}_C\Delta\mathcal{P}\right)^{-1} \\ \Updownarrow \\ \Delta\mathcal{P} = \mathcal{Q}\left(I-N_P\widetilde{N}_C\mathcal{Q}\right)^{-1} \end{array}\right\}$$

□

Chapter 4

Factorizations of Nonlinear Maps

4.1 Introduction

In this chapter, we study coprime factorizations for causal *nonlinear* maps; we use these in stability and robustness analysis of nonlinear feedback systems and in synthesis of stabilizing feedback configurations.

All causal nonlinear maps (denoted by calligraphic style letters) are defined over input and output extended spaces (see Section 3.2) . For a causal nonlinear map $\mathcal{P}$, where $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$, $\mathcal{P} : e \mapsto y$, the output $y := \mathcal{P}e \in \Lambda_e^o$ is <u>uniquely</u> determined for all inputs $e \in \Lambda_e^i$. In the case that the nonlinear causal map $\mathcal{P}$ has a state-space description, we assume that the initial conditions are fixed once and for all.

The chapter is organized as follows:

Section 4.2 emphasizes a fact about causal nonlinear plants: unlike the special case of finite-dimensional linear time-invariant maps, not all nonliner plants are stabilizable (in *any* configuration) (see Fact 4.2.3) . Right factorizations of causal nonlinear maps are defined in Definition 4.2.4 ; Fact 4.2.5 justifies this definition by establishing that stabilizability implies the existence of right factorizations. In other words, causal nonlinear maps that are stabilizable by feedback through a summing node, have a right factorization. Although left factorization definitions for nonlinear maps are available in the literature, a similar result (see Fact 4.2.5) is not available to justify the existence of left factorizations. It is shown that in a right factorization, the "denominator" map includes the instabilities of the plant.

Section 4.3 is a self-contained example; we explicitly derive a right-factorization for a class of time-varying nonlinear causal plants. Proposition 4.3.2 gives a constructive proof.

In Section 4.4 , we define right-coprime factorizations for nonlinear causal maps (Definition 4.4.1) . (Note that coprime factorizations may not exist even for certain linear maps.) The existence of an unstructured two-input one-output causal stable "pseudo-state" observer is the key point. If a plant has a right-coprime factorization, the pseudo-state can be reconstructed from (noiseless) input and output measurements. Fact 4.4.2 shows that the denominator map in a right-*coprime* factorization *completely* characterizes the instabilities of the plant. Fact 4.4.3 proves that all right-coprime factorizations of a plant are related by unimodular maps; hence once we find one, we have found them all. The section ends with a more restricted stability definition (incremental Λ-stability, Definition 4.4.4), which is extremely useful in manipulations of summing nodes in analyzing nonlinear feedback interconnections.

In Section 4.5, for the class of nonlinear causal plants which have right-coprime factorizations, we study the stabilizing two-input one-output feedback configuration $\Sigma(\mathcal{P}, \mathcal{Q})$ (see Figure 4.6) . Proposition 4.5.1 shows that the existence of a right-coprime factorization of the nonlinear plant is a necessary condition for stabilizability in $\Sigma(\mathcal{P}, \mathcal{Q})$. Proposition 4.5.3 establishes the converse: if $\mathcal{P}$

has an incrementally Λ-stable right-coprime factorization, it can be stabilized in $\Sigma(\mathcal{P}, \mathcal{Q})$.

Section 4.6 is another self-contained example; for a class of nonlinear causal plants, we explicitly derive a right-coprime factorization (Proposition 4.6.2) . We show that the specific right-coprime factorization is in fact incrementally Λ-stable; hence using the result in the previous section, we propose a stabilizing feedback configuration with a free parameter (Corollary 4.6.4) .

Section 4.7 studies the nonlinear unity-feedback system, where one of the subsystems is *linear*. In the case that the plant is linear, we parametrize the set of all nonlinear causal stabilizing compensators (Theorem 4.7.4) .

In Section 4.8 we study stability and robustness of the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P}, \mathcal{C})$ from a factorization point of view. Theorem 4.8.1 states the necessary and sufficient condition for stability of the nonlinear unity-feedback system when one of the subsystems has a right-coprime factorization. Theorem 4.8.4 shows that if the nonlinear plant is incrementally stable, all stabilizing compensators have a specific right-coprime factorization. Theorem 4.8.7 is a robustness result: for a *family* of incrementally stable plants, a necessary and sufficient condition on the compensator to simultaneously stabilize this family is stated. Theorem 4.8.9 considers robust stability of a nominal nonlinear unity-feedback system under nonlinear (possibly unstable) plant perturbations.

4.2 Right-factorizations for nonlinear maps

Consider a nonlinear causal plant

$$\mathcal{P} : \Lambda_e^i \to \Lambda_e^o \quad , \quad \mathcal{P} : e \mapsto y \ .$$

If we model the disturbances at the plant input (d_i) and the plant output (d_o) *additively* , then the most general feedback system will be as shown in Figure 4.1 : the control input is denoted by v and the causal map $\mathcal{C}$ denotes the compensator.

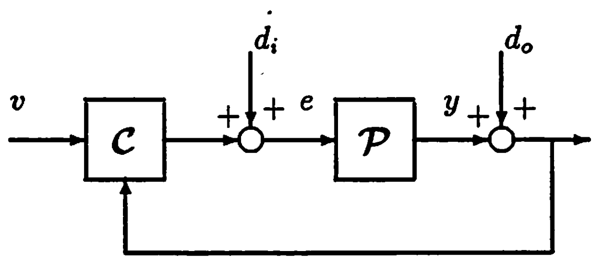

Figure 4.1:
The general feedback system

Definition 4.2.1 (well-posed feedback system)

A feedback system is said to be *well-posed* iff there exists a causal closed-loop map mapping the closed-loop system inputs to the internal signals.

□

If a feedback system is well-posed, for a given closed-loop system input the internal signals exist and are *uniquely* determined by maps (i.e., they are not set-valued).

Definition 4.2.2 (Λ-stable feedback system)

A well-posed *feedback system* is said to be Λ-*stable* iff all of the closed-loop maps (mapping the closed-loop system inputs to the internal signals) are Λ-stable.

□

Note that the feedback system in Figure 4.1 is Λ-stable if and only if the closed-loop map

$$(v, d_i, d_o) \mapsto (e, y)$$

is Λ-stable.

In any Λ-stable feedback system, the plant $\mathcal{P}$ is constrained to operate on a *subset* of its input-output pairs: those special bounded inputs e for which the output y is also bounded. This implies an obvious requirement on the classes of maps that can be stabilized in a feedback system.

Fact 4.2.3 (not all causal plants are stabilizable)

A causal plant $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ can be stabilized in a feedback system <u>only if</u> there exists at least one $e \in \Lambda^i$ such that $\mathcal{P}e \in \Lambda^o$.

□

Consider the following state-space description:

$$\mathcal{P} : e \mapsto y \begin{cases} x(k+1) &= 2x(k) + x(k)[e(k)]^2 \quad k \in \mathbb{Z}_+ \\ x(0) &= 1 \\ y(k) &= x(k) \quad . \end{cases} \tag{4.1}$$

The causal plant $\mathcal{P}$ described in (4.1) can <u>not</u> be stabilized in any feedback configuration since there does <u>not</u> exist a bounded sequence e such that the sequence $\mathcal{P}e$ is bounded.

Definition 4.2.4 (right-factorization of a nonlinear map)

$(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ is said to be a *right-factorization (r.f.)* of a causal map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ iff

i) $\mathcal{N}_{\mathcal{P}} : \Lambda_e^i \to \Lambda_e^o$ is Λ-stable <u>and</u>

ii) $\mathcal{D}_{\mathcal{P}} : \Lambda_e^i \to \Lambda_e^i$ is Λ-stable, bijective and has a causal inverse and

iii) $\mathcal{N}_{\mathcal{P}} \mathcal{D}_{\mathcal{P}}^{-1} = \mathcal{P}$.

□

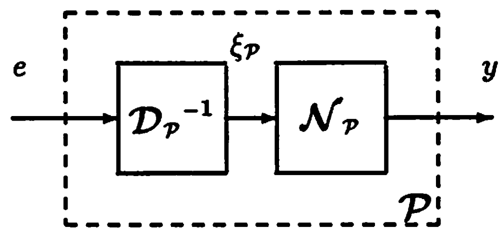

Figure 4.2:
$\mathcal{P}$ with an r.f. $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$

At a first glance, Definition 4.2.4 looks like just another definition extending Definition 3.3.1 to nonlinear maps. The crucial point is that the existence of r.f.'s is a necessary condition for stabilizability of systems with an additive exogenous input d_i at the input of $\mathcal{P}$ (see Figure 4.1) .

Fact 4.2.5 (stabilizability ⇒ r.f.)

Let the plant $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be such that there exists a causal map $\mathcal{C} : \Lambda_e \times \Lambda_e^o \to \Lambda_e^i$, where the well-posed general feedback system shown in Figure 4.1 is Λ-stable. Under these assumptions $\mathcal{P}$ has an r.f.

Comment 4.2.6

Note that Lemma 3.4.6 is a special case of Fact 4.2.5 for the case

$$\mathcal{C} = \begin{bmatrix} 0 & \mathsf{C} \end{bmatrix} ,$$

for some linear map C .

Proof of Fact 4.2.5

By assumption, the feedback system in Figure 4.1 is Λ-stable; hence the closed-loop map $(v, d_i, d_o) \mapsto (e, y)$ is Λ-stable (see Figure 4.1). Choose the specific inputs v and d_o as

$$v \;:=\; v^* \in \Lambda \quad \text{and}$$
$$d_o \;:=\; d_o^* \in \Lambda^{\mathrm{o}} \quad .$$

Then by well-posedness, there is a causal closed-loop map $\mathcal{D}_{\mathcal{P}}$

$$\mathcal{D}_{\mathcal{P}} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{i}}_{\mathrm{e}} \quad , \quad \mathcal{D}_{\mathcal{P}} : d_i \mapsto e \quad . \tag{4.2}$$

Since the feedback system is Λ-stable, the causal map $\mathcal{D}_{\mathcal{P}}$ in (4.2) is Λ-stable. Since

$$d_i = e - \mathcal{C}(v^*, \mathcal{P}e + d_o^*) \quad ,$$

$\mathcal{D}_{\mathcal{P}}$ is bijective and has a causal inverse.

Since the feedback system is Λ-stable, the causal map $\mathcal{N}_{\mathcal{P}}$ defined as

$$\mathcal{N}_{\mathcal{P}} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}} \quad , \quad \mathcal{N}_{\mathcal{P}} : d_i \mapsto y \quad , \tag{4.3}$$

is also Λ-stable. Moreover

$$\mathcal{P}\mathcal{D}_{\mathcal{P}} = \mathcal{N}_{\mathcal{P}} \quad ;$$

hence we conclude that $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ defined in (4.2-4.3) is an r.f. of $\mathcal{P}$.

□

For a causal map $\mathcal{P} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$, define the subset of its inputs $\mathcal{I}_{\mathcal{P}}$ by

$$\mathcal{I}_{\mathcal{P}} := \left\{ e \in \Lambda^{\mathrm{i}} \;\middle|\; \mathcal{P}e \in \Lambda^{\mathrm{o}}_{\mathrm{e}} \setminus \Lambda^{\mathrm{o}} \right\} \quad ,$$

i.e., $\mathcal{I}_{\mathcal{P}}$ is the set of all <u>bounded inputs</u> of $\mathcal{P}$ that produce <u>unbounded outputs</u> . The set $\mathcal{I}_{\mathcal{P}}$ characterizes the "instabilites" of the map $\mathcal{P}$.

For a Λ-stable $\mathcal{P}$, the set $\mathcal{I}_{\mathcal{P}}$ is the empty set; there is no bounded input that one must avoid.

Note that if $\mathcal{I}_{\mathcal{P}} = \Lambda^{i}$, then it is hopeless; $\mathcal{P}$ can <u>not</u> be stabilized (recall Fact 4.2.3) .

Fact 4.2.7 ($\mathcal{I}_{\mathcal{P}} \subset \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}}$)
Let $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ be an r.f. of $\mathcal{P} : \Lambda_{e}^{i} \to \Lambda_{e}^{o}$; then

$$\mathcal{I}_{\mathcal{P}} \subset \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}} \ .$$

Proof

By definition,

$$e \in \mathcal{I}_{\mathcal{P}} \ \Leftrightarrow \ (e, \mathcal{P}e) \ = \ (e, \mathcal{N}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}{}^{-1}e) \in \Lambda^{i} \times \left(\Lambda_{e}^{o} \setminus \Lambda^{o}\right) \ .$$

Since the causal map $\mathcal{N}_{\mathcal{P}}$ is Λ-stable (see Figure 4.2) , the last inclusion implies

$$(e, \mathcal{D}_{\mathcal{P}}{}^{-1}e) \in \Lambda^{i} \times \left(\Lambda_{e}^{i} \setminus \Lambda^{i}\right) \ .$$

□

When $\mathcal{P}$ has an r.f. , by Fact 4.2.7 , all of the "instabilities" of $\mathcal{P}$ are contained in the "instabilities" of $\mathcal{D}_{\mathcal{P}}{}^{-1}$. Hence one might think that pre-compensation by Λ-stable $\mathcal{D}_{\mathcal{P}}$ achieves input-output stabilization (see Figure 4.3).

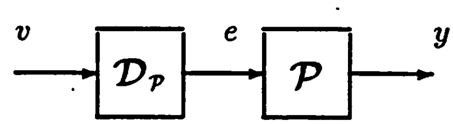

Figure 4.3:
Undesired pre-compensation scheme

Clearly, the input-output map in Figure 4.3

$$v \mapsto (e, y) \ = \ (\mathcal{D}_{\mathcal{P}}v, \mathcal{N}_{\mathcal{P}}v)$$

is Λ-stable. However, this "exact cancellation" is undesirable for at least two reasons:

i) the denominator map $\mathcal{D}_{\mathcal{P}}$ may not be known exactly,

ii) even if $\mathcal{D}_{\mathcal{P}}$ is known exactly, the plant input may be subject to an input disturbance d_i . If $e^* \in \mathcal{I}_{\mathcal{P}}$, for a *fixed* input $v^* \in \Lambda^{\mathrm{i}}$, a disturbance of the form

$$d_i^* := e^* - \mathcal{D}_{\mathcal{P}} v^*$$

will result in an unbounded output y (see Figure 4.4).

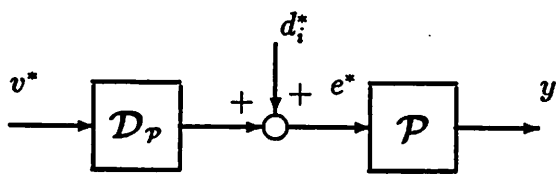

Figure 4.4:
Destabilizing input disturbance

Clearly, these problems arise due to the cascade structure: the compensator has no access to the internal signals in the later stages. The standard way of avoiding this problem is feeding back the internal signals as in the general "feedback" scheme in Figure 4.1.

Note that even if the plant is Λ-stable, (i.e., $\mathcal{I}_{\mathcal{P}} = \emptyset$) open-loop compensation may not be desirable since any disturbance at the plant output and/or any plant perturbation will not be attenuated: the purpose of feedback is to use the loop dynamics to compensate for exogenous output disturbances or perturbations in the plant.

4.3 Right-factorization of a Class of Time-varying Nonlinear Plants

Notation

We choose the ∞-norm for vectors in $\mathbb{R}^n$ and the corresponding induced-norm for matrices; we denote them by $|\cdot|$ and $\|\cdot\|$, respectively.. For vector-valued functions

$$x : \mathbb{R}_+ \to \mathbb{R}^n \quad ,$$

we write

$$\|x\| := \sup_{t\in[0,\infty)} |x(t)| \quad .$$

We set

$$\Lambda := L_\infty^n = \left\{ x : \mathbb{R}_+ \to \mathbb{R}^n \;\middle|\; \|x\| = \sup_{t\geq 0} |x(t)| < \infty \right\}$$

and

$$\Lambda_{\mathrm{e}} := L_{\infty\mathrm{e}}^n = \left\{ x : \mathbb{R}_+ \to \mathbb{R}^n \;\middle|\; \forall\, \mathrm{T} \in \mathbb{R}_+ \, , \sup_{t\in[0,\mathrm{T}]} |x(t)| < \infty \right\} \quad .$$

For $n = n_i$ and $n = n_o$, we write Λ^{i} , $\Lambda^{\mathrm{i}}_{\mathrm{e}}$ and Λ^{o} , $\Lambda^{\mathrm{o}}_{\mathrm{e}}$, respectively.

Description of the Class of Nonlinear Plants

Consider a causal nonlinear time-varying plant whose input-output map

$$\mathcal{P} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$$

is specified by the following state-space description:

$$\mathcal{P} : u \mapsto y \left\{ \begin{array}{rcl} \dot{x} & = & A(t)x + f(t,x) + B(t)u \\ y & = & h(t,x,u) \\ x(0) & = & 0 \quad , \end{array} \right. \tag{4.4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_i}$ and $y(t) \in \mathbb{R}^{n_o}$, $\forall t \in \mathbb{R}_+$.

On the functions $A(\cdot)$, $B(\cdot)$, $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ we impose the following assumptions:

Assumptions 4.3.1

I. For the specified initial condition and for all inputs $u \in \Lambda_e^i$, the differential equation in (4.4) has a unique solution. (Consequently, $\mathcal{P} : u \mapsto y$ is a map.)

II. The nonlinearity f is bounded on $\mathbb{R}_+ \times \mathbb{R}^n$; that is, there exists $m > 0$ such that

$$\sup_{t \in [0,\infty), x \in \mathbb{R}^n} |f(t,x)| \leq m \quad .$$

III. For any Λ-stable map

$$\mathcal{H}_x : \Lambda_e^i \to \Lambda_e \quad , \quad \mathcal{H}_x : u \mapsto x \quad ,$$

the causal map $\mathcal{H}_y$ defined by

$$\mathcal{H}_y : u \mapsto y \quad \left\{ \quad y(t) = h(t, \left(\mathcal{H}_x u\right)(t), u(t)) \right.$$

is Λ-stable, where

$$h : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_o} \quad .$$

That is, if the state-map is stabilized the input-output map is also stabilized.

IV. The pair $(A(\cdot), B(\cdot))$ is uniformly completely controllable; equivalently there exist $\delta > 0$, $w_{\max} \geq w_{\min} > 0$, such that for all $t \in \mathbb{R}_+$,

$$w_{\min} \mathrm{I} \leq W(t, t+\delta) \leq w_{\max} \mathrm{I} \quad , \tag{4.5}$$

where $W(t,t+\delta)$ is the controllability Gramian [Bro.1]

$$W(t,t+\delta) := \int_t^{t+\delta} \Phi(t,\tau) B(\tau) B^T(\tau) \Phi^T(t,\tau) d\tau \tag{4.6}$$

and $\Phi(\cdot,\cdot)$ is the state-transition map for the linear differential equation

$$\dot{x} = A(t)x \quad .$$

V. The map $B(\cdot)$ is bounded on $\mathbb{R}_+$; that is there exits $b > 0$ such that

$$\sup_{t \in [0,\infty)} \|B(t)\| \leq b \quad .$$

□

We now construct a right-factorization of $\mathcal{P}$.

Proposition 4.3.2 (an r.f. of $\mathcal{P}$ defined in (4.4))

Let the plant $\mathcal{P}$ be described by (4.4) and satisfy Assumptions 4.3.1 **I - V** ; then $\mathcal{P}$ has an r.f.

Proof

The proof is in two steps:

i) Using Assumption 4.3.1 **I** , we obtain a causal bijective map $\mathcal{D}_{\mathcal{P}} : \Lambda_e^i \to \Lambda_e^i$, which has a causal inverse $\mathcal{D}_{\mathcal{P}}^{-1}$ and a causal map $\mathcal{N}_{\mathcal{P}} : \Lambda_e^i \to \Lambda_e^o$ such that $\mathcal{P} = \mathcal{N}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1}$.

ii) Using Assumptions 4.3.1 **II - V** , we show that both $\mathcal{N}_{\mathcal{P}}$ and $\mathcal{D}_{\mathcal{P}}$ are Λ-stable maps.

Step 1

Define the causal map $\mathcal{D}_{\mathcal{P}} : \Lambda_e^i \to \Lambda_e^i$ as

$$\mathcal{D}_{\mathcal{P}} : \xi_1 \mapsto u_1 \left\{ \begin{array}{rcl} \dot{x}_1 & = & (A+BK)(t)x_1 + f(t,x_1) + B(t)\xi_1 \\ u_1 & = & K(t)x_1 + \xi_1 \\ x_1(0) & = & 0 \quad , \end{array} \right. \tag{4.7}$$

for some piecewise continuous

$$K : \mathbb{R}_+ \to \mathbb{R}^{n_i \times n} \quad .$$

Now the map $\mathcal{D}_{\mathcal{P}}$ in (4.7) has a causal inverse $\mathcal{D}_{\mathcal{P}}^{-1} : \Lambda_e^i \to \Lambda_e^i$; indeed $\mathcal{D}_{\mathcal{P}}^{-1}$ is given by

$$\mathcal{D}_{\mathcal{P}}^{-1} : u_2 \mapsto \xi_2 \left\{ \begin{array}{rcl} \dot{x}_2 &=& A(t)x_2 + f(t,x_2) + B(t)u_2 \\ \xi_2 &=& -K(t)x_2 + u_2 \\ x_2(0) &=& 0 \quad . \end{array} \right. \tag{4.8}$$

We show that $\mathcal{D}_{\mathcal{P}}$ is bijective by verifying

$$\mathcal{D}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1} = \mathcal{D}_{\mathcal{P}}^{-1}\mathcal{D}_{\mathcal{P}} = \mathrm{I} \ .$$

Consider the map $\mathcal{D}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1} : u_2 \mapsto u_1$; note that the interconnection constraint is

$$\xi_1 = \xi_2 = -K(t)x_2 + u_2 \ ,$$

moreover, from (4.7) and (4.8), we obtain

$$\mathcal{D}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1} : u_2 \mapsto u_1 \left\{ \begin{array}{rcl} \dot{x}_1 &=& (A+BK)(t)x_1 + f(t,x_1) + B(t)u_2 - (BK)(t)x_2 \\ \dot{x}_2 &=& A(t)x_2 + f(t,x_2) + B(t)u_2 \\ u_1 &=& K(t)(x_1 - x_2) + u_2 \\ x_1(0) &=& x_2(0) = 0 \quad . \end{array} \right. \tag{4.9}$$

For any input u_2 , using Assumption 4.3.1 I it is easy to check that

$$\begin{bmatrix} x_1(t) = x_2(t) \\ x_2(t) \end{bmatrix}$$

is *the* solution of the system of differential equations in (4.9) under the specified initial conditions. Hence from (4.9), we obtain $u_1 = u_2$ and we conclude that $\mathcal{D}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1} = \mathrm{I}$ on Λ_e^i .

Similarly, consider the map $\mathcal{D}_{\mathcal{P}}^{-1}\mathcal{D}_{\mathcal{P}} : \xi_1 \mapsto \xi_2$; the interconnection constraint is

$$u_1 = u_2 = K(t)x_1 + \xi_1 \quad .$$

From (4.7) and (4.8), we obtain

$$\mathcal{D}_{\mathcal{P}}^{-1}\mathcal{D}_{\mathcal{P}} : \xi_1 \mapsto \xi_2 \left\{ \begin{array}{rcl} \dot{x}_1 &=& (A+BK)(t)x_1 + f(t,x_1) + B(t)\xi_1 \\ \dot{x}_2 &=& A(t)x_2 + f(t,x_2) + (BK)(t)x_1 + B(t)\xi_1 \\ \xi_2 &=& K(t)(x_1 - x_2) + \xi_1 \\ x_1(0) &=& x_2(0) = 0 \quad . \end{array} \right. \tag{4.10}$$

For any input ξ_1 ,

$$\begin{bmatrix} x_1(t) \\ x_2(t) = x_1(t) \end{bmatrix}$$

is *the* solution of the system of differential equations in (4.10) under the specified initial condition. Hence, from (4.10), we obtain $\xi_2 = \xi_1$; so $\mathcal{D}_\mathcal{P}{}^{-1}\mathcal{D}_\mathcal{P} = \mathrm{I}$ on $\Lambda^{\mathrm{i}}_{\mathrm{e}}$. We conclude that $\mathcal{D}_\mathcal{P}$ is bijective and the map $\mathcal{D}_\mathcal{P}{}^{-1}$ defined in (4.8) is the causal inverse of $\mathcal{D}_\mathcal{P}$.

Now for the same $K(\cdot)$ in (4.7), define

$$\mathcal{N}_\mathcal{P} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$$

as follows:

$$\mathcal{N}_\mathcal{P} : \xi \mapsto y \left\{ \begin{array}{rcl} \dot{x}_3 & = & (A+BK)(t)x_3 + f(t,x_3) + B(t)\xi \\ y & = & h(t\,,\, x_3\,,\, K(t)x_3 + \xi) \\ x_3(0) & = & 0 \quad . \end{array} \right. \tag{4.11}$$

From (4.8) and (4.11), we obtain

$$\mathcal{N}_\mathcal{P}\mathcal{D}_\mathcal{P}{}^{-1} : u \mapsto y \left\{ \begin{array}{rcl} \dot{x}_2 & = & A(t)x_2 + f(t,x_2) + B(t)u \\ \dot{x}_3 & = & (A+BK)(t)x_3 + f(t,x_3) + B(t)u - (BK)(t)x_2 \\ y & = & h(t\,,\, x_3\,,\, K(t)(x_3\, -\, x_2) + u) \\ x_2(0) & = & x_3(0) = 0 \quad . \end{array} \right. \tag{4.12}$$

For any input u , by Assumption 4.3.1 **I** ,

$$(x_2(t)\,,\; x_3(t) = x_2(t))$$

is *the* solution of the system of differential equations in (4.12) under the specified initial conditions. Hence (4.12) is an equivalent description of $\mathcal{P}$ as $\mathcal{N}_\mathcal{P}\mathcal{D}_\mathcal{P}{}^{-1}$.

Step 2

We use a technique due to [Che.1] to show that there exists a

$$K : \mathbb{R}_+ \to \mathbb{R}^{n_i \times n}$$

such that the causal map

$$\mathcal{H}_x \,:\, \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$$

defined by

$$\mathcal{H}_x : \xi \mapsto x \begin{cases} \dot{x} &= (A+BK)(t)x + f(t,x) + B(t)\xi \\ x(0) &= 0 \quad , \end{cases} \tag{4.13}$$

is Λ-stable: Let

$$W_1(t,t+\delta) := \int_t^{t+\delta} e^{(t-\tau)}\Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)\mathrm{d}\tau \quad . \tag{4.14}$$

Using (4.5) , (4.6) and (4.14) , for all $t \in \mathbb{R}_+$,

$$e^{-\delta}w_{\min}\mathrm{I} \le W_1(t,t+\delta) \le w_{\max}\mathrm{I} \quad ;$$

hence, for all $t \in \mathbb{R}_+$,

$$w_{\max}^{-1}\mathrm{I} \le W_1^{-1}(t,t+\delta) \le e^{\delta}w_{\min}^{-1}\mathrm{I} \quad . \tag{4.15}$$

Note that

$$\begin{aligned} \frac{d}{dt}W_1(t,t+\delta) &= e^{-\delta}\Phi(t,t+\delta)B(t+\delta)B^T(t+\delta)\Phi^T(t,t+\delta) \\ &- B(t)B^T(t) + W_1(t,t+\delta) \\ &+ A(t)W_1(t,t+\delta) + W_1(t,t+\delta)A^T(t) \quad . \end{aligned} \tag{4.16}$$

For all $t \in \mathbb{R}_+$, let $K(\cdot)$ be defined as

$$K(t) := -B^T(t)W_1^{-1}(t,t+\delta) \quad . \tag{4.17}$$

So the map $K : \mathbb{R}_+ \to \mathbb{R}^{n_i \times n}$ is bounded on $\mathbb{R}_+$. Let

$$\mathcal{V} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$$

be a Lyapunov function candidate, where

$$\mathcal{V}(t\,,\,x) := x^T(t)W_1^{-1}(t\,,\,t+\delta)x(t) \quad . \tag{4.18}$$

Differentiating (4.18) along the solution of (4.13) , we obtain for all $(t, x(t)) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\left.\frac{d}{dt}\mathcal{V}(t, x(t))\right|_{(4.13)} = 2\dot{x}^T(t)W_1^{-1}(t,t+\delta)x(t) - x^T(t)W_1^{-1}(t,t+\delta)\tfrac{d}{dt}\{W_1\}(t,t+\delta)W_1^{-1}(t,t+\delta)x(t) \quad (4.19)$$

Substituting (4.17) and (4.16) in (4.19) , we obtain

$$\begin{aligned}\left.\frac{d}{dt}\mathcal{V}(t, x(t))\right|_{(4.13)} &= 2x^T(t)A^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &- 2x^T(t)W_1^{-1}(t,t+\delta)B(t)B^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &+ 2f^T(t, x(t))W_1^{-1}(t,t+\delta)x(t) \\ &+ 2\xi^T(t)B^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &- e^{-\delta}x^T(t)W_1^{-1}(t,t+\delta)\Phi(t,t+\delta)B(t+\delta) \\ &\qquad \bullet B^T(t+\delta)\Phi^T(t,t+\delta)W_1^{-1}(t,t+\delta)x(t) \\ &+ x^T(t)W_1^{-1}(t,t+\delta)B(t)B^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &- x^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &- 2x^T(t)A^T(t)W_1^{-1}(t,t+\delta)x(t) \quad . \end{aligned} \quad (4.20)$$

Performing the appropriate cancellations and neglecting some of the nonpositive terms in (4.20) , we obtain

$$\begin{aligned}\left.\frac{d}{dt}\mathcal{V}(t, x(t))\right|_{(4.13)} &\leq -x^T(t)W_1^{-1}(t,t+\delta)x(t) \\ &\quad + 2f^T(t, x(t))W_1^{-1}(t,t+\delta)x(t) \\ &\quad + 2\xi^T(t)B^T(t)W_1^{-1}(t,t+\delta)x(t) \quad . \end{aligned} \quad (4.21)$$

By Assumptions 4.3.1 II , IV , V and (4.15) , (4.21) , we obtain

$$\left.\frac{d}{dt}\mathcal{V}(t, x(t))\right|_{(4.13)} \leq -w_{\max}^{-1}|x(t)|_2\Big(|x(t)|_2 - 2e^{\delta}w_{\max}w_{\min}^{-1}\sqrt{n}(m + b||\xi||)\Big) \quad . \quad (4.22)$$

Let the map $\phi_{\mathcal{H}_x} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$\phi_{\mathcal{H}_x}(\|\xi\|) := 2e^{\delta} w_{\max} w_{\min}^{-1} \sqrt{n}(m + b\|\xi\|) \ .$$

Note that the map $\phi_{\mathcal{H}_x}$ is continuous and nondecreasing. From the inequality in (4.22) we conclude that

$$\forall\, t \in \mathbb{R}_+ \ , \ \ |x(t)|_2 \le \phi_{\mathcal{H}_x}(\|\xi\|) \ . \tag{4.23}$$

Since all norms are equivalent in $\mathbb{R}^n$, from (4.23), we conclude that the map $\mathcal{H}_x$ in (4.13) is Λ-stable.

For the choice of K in (4.17), by Assumption 4.3.1 **V** and (4.15), there exists $\alpha > 0$ such that

$$\sup_{t \in [0,\infty)} \|K(t)\| \le \alpha \ . \tag{4.24}$$

Hence, by Assumption 4.3.1 **III**, (4.24) and the Λ-stability of $\mathcal{H}_x$, the causal map $\mathcal{N}_{\mathcal{P}}$ in (4.11) is Λ-stable. By (4.24) and the Λ-stability of $\mathcal{H}_x$, the causal map $\mathcal{D}_{\mathcal{P}}$ in (4.7) is also Λ-stable. Hence $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ is an r.f. of $\mathcal{P}$.

□

4.4 Right-coprime factorizations

For a given causal plant $\mathcal{P}$ with an r.f. $(\mathcal{N}_\mathcal{P}, \mathcal{D}_\mathcal{P})$, Fact 4.2.7 states that the instabilities of $\mathcal{P}$ are contained in the instabilities of ${\mathcal{D}_\mathcal{P}}^{-1}$. The converse is *not* true in general. However, for a special class of right-factorizations the instabilities of ${\mathcal{D}_\mathcal{P}}^{-1}$ are identical to those of the plant.

Definition 4.4.1 (right-coprime factorization)

$(\mathcal{N}_\mathcal{P}, \mathcal{D}_\mathcal{P})$ is said to be a *right-coprime factorization (r.c.f.)* of the causal map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ iff

i) $(\mathcal{N}_\mathcal{P}, \mathcal{D}_\mathcal{P})$ is an r.f. of $\mathcal{P}$ (see Definition 4.2.4) and

ii) $\mathcal{F}_\mathcal{P} := \begin{bmatrix} \mathcal{N}_\mathcal{P} \\ \mathcal{D}_\mathcal{P} \end{bmatrix} : \Lambda_e^i \to \Lambda_e^o \times \Lambda_e^i$ has a causal Λ-stable left-inverse $\mathcal{F}_\mathcal{P}^\mathcal{I}$, i.e.,

$$\mathcal{F}_\mathcal{P}^\mathcal{I} \begin{bmatrix} \mathcal{N}_\mathcal{P} \\ \mathcal{D}_\mathcal{P} \end{bmatrix} = \mathrm{I} \ , \qquad (4.25)$$

where I is the identity map on Λ_e^o .

□

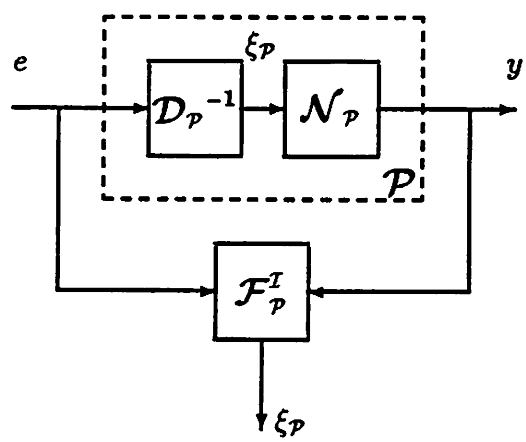

Figure 4.5:
Pseudo-state observer map $\mathcal{F}_\mathcal{P}^\mathcal{I}$

Definition 4.4.1 generalizes the right-coprime factorization definition in [Vid.1, Des.13,Des.15] by relaxing the constraint on the Λ-stable inverse of the map $\mathcal{F}_\mathcal{P}$

to be of the form $\begin{bmatrix}\mathcal{U} & \mathcal{V}\end{bmatrix}$, ($\begin{bmatrix}\mathcal{U} & \mathcal{V}\end{bmatrix}$ is described by

$$\begin{bmatrix}\mathcal{U} & \mathcal{V}\end{bmatrix}\begin{bmatrix} y \\ e \end{bmatrix} = \mathcal{U}y + \mathcal{V}e \ ,$$

a slight abuse of matrix notation). Note that the map $\mathcal{F}_{\mathcal{P}} = \begin{bmatrix}\mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}}\end{bmatrix}$ is injective since the "denominator" map $\mathcal{D}_{\mathcal{P}}$ is bijective; hence the map $\mathcal{F}_{\mathcal{P}}$ *always* has a causal left-inverse, for example $\begin{bmatrix}0 & \mathcal{D}_{\mathcal{P}}^{-1}\end{bmatrix}$: the point is that the map $\mathcal{F}_{\mathcal{P}}$ has a causal left-inverse that is Λ-stable.

A Λ-stable left-inverse $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ defines a pseudo-state observer:

$$\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}\begin{bmatrix} y \\ e \end{bmatrix} = \xi_{\mathcal{P}} \quad ;$$

it allows the pseudo-state $\xi_{\mathcal{P}}$ to be reconstructed from the (noiseless) input-ouput measurements e and y (see Figure 4.5) .

Fact 4.4.2 ($\mathcal{I}_{\mathcal{P}} = \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}}$)
Let $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ be an r.c.f. of $\mathcal{P} : \Lambda_e^i \rightarrow \Lambda_e^o$; then

$$\mathcal{I}_{\mathcal{P}} = \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}} \ .$$

Proof

$\mathcal{I}_{\mathcal{P}} \subset \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}}$ follows by Fact 4.2.7 . We show the reverse inclusion by contradiction. Suppose that for some $e \in \Lambda^i$,

$$e \in \mathcal{I}_{\mathcal{D}_{\mathcal{P}}^{-1}} \ \underline{\text{and}} \ e \notin \mathcal{I}_{\mathcal{P}} \ . \tag{4.26}$$

Then we have

$$\xi_{\mathcal{P}} := \mathcal{D}_{\mathcal{P}}^{-1}e \in \Lambda_e^i \setminus \Lambda^i \ . \tag{4.27}$$

By assumption, the identity in (4.25) holds. Since by (4.26) , $(e, \mathcal{P}e) \in \Lambda^i \times \Lambda^o$, we have

$$\xi_{\mathcal{P}} = \mathcal{F}_{\mathcal{P}}^{\mathcal{I}}\begin{bmatrix} \mathcal{P}e \\ e \end{bmatrix} \in \Lambda^i$$

which is a contradiction to (4.27) .

□

Fact 4.4.3 (all r.c.f.s are related by Λ-unimodular maps)

Let $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ be an r.c.f. of $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$; then $(\widehat{\mathcal{N}_{\mathcal{P}}}\ \widehat{\mathcal{D}_{\mathcal{P}}})$ is an r.c.f. of $\mathcal{P}$ if and only if there exists a Λ-unimodular map $\mathcal{M} : \Lambda_e^i \to \Lambda_e^i$ such that

$$\begin{bmatrix} \widehat{\mathcal{N}_{\mathcal{P}}} \\ \widehat{\mathcal{D}_{\mathcal{P}}} \end{bmatrix} = \begin{bmatrix} \mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}} \end{bmatrix} \mathcal{M} \ . \tag{4.28}$$

Proof

" if "

By assumption, $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ is an r.c.f. of $\mathcal{P}$ and $\mathcal{M}$ is a Λ-unimodular map such that $\widehat{\mathcal{N}_{\mathcal{P}}} = \mathcal{N}_{\mathcal{P}}\mathcal{M}$ and $\widehat{\mathcal{D}_{\mathcal{P}}} = \mathcal{D}_{\mathcal{P}}\mathcal{M}$. Then $(\widehat{\mathcal{N}_{\mathcal{P}}}\ \widehat{\mathcal{D}_{\mathcal{P}}})$ is an r.f. of $\mathcal{P}$. Let

$$\widehat{\mathcal{F}_{\mathcal{P}}} := \begin{bmatrix} \widehat{\mathcal{N}_{\mathcal{P}}} \\ \widehat{\mathcal{D}_{\mathcal{P}}} \end{bmatrix}$$

and

$$\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}} := \mathcal{M}^{-1}\mathcal{F}_{\mathcal{P}}^{\mathcal{I}} \ , \tag{4.29}$$

where the map $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ is as in (4.25) . The map $\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}}$ in (4.29) is causal and Λ-stable. Since $\widehat{\mathcal{F}_{\mathcal{P}}} = \mathcal{F}_{\mathcal{P}}\mathcal{M}$, we have

$$\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}}\widehat{\mathcal{F}_{\mathcal{P}}} = \mathcal{M}^{-1}\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}}\mathcal{F}_{\mathcal{P}}\mathcal{M} = \mathrm{I} \ .$$

Hence $\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}}$ is a causal Λ-stable left-inverse of $\widehat{\mathcal{F}_{\mathcal{P}}}$ and by Definition 4.4.1 , $(\widehat{\mathcal{N}_{\mathcal{P}}}\ \widehat{\mathcal{D}_{\mathcal{P}}})$ is an r.c.f. of $\mathcal{P}$.

" only if "

By assumption $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ and $(\widehat{\mathcal{N}_{\mathcal{P}}}\ \widehat{\mathcal{D}_{\mathcal{P}}})$ are two r.c.f.'s of $\mathcal{P}$. Let $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ and $\widehat{\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}}$ be the corresponding causal Λ-stable left-inverses. Let

$$\mathcal{M} := \mathcal{D}_{\mathcal{P}}{}^{-1}\widehat{\mathcal{D}_{\mathcal{P}}} \ .$$

Since $\mathcal{N}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1} = \widetilde{\mathcal{N}}_{\mathcal{P}}\widetilde{\mathcal{D}}_{\mathcal{P}}^{-1}$, using the map $\mathcal{M}$ above, (4.28) holds by calculation. Moreover

$$\mathcal{M} = \mathcal{F}_{\mathcal{P}}^{\mathcal{I}}\widehat{\mathcal{F}}_{\mathcal{P}} \text{ and } \mathcal{M}^{-1} = \widehat{\mathcal{F}}_{\mathcal{P}}^{\mathcal{I}}\mathcal{F}_{\mathcal{P}}$$

are Λ-stable.

□

We conclude this section by defining a special class of Λ-stable maps. They are a generalization of Λ-stable *linear* maps.

Definition 4.4.4 (incrementally Λ-stable maps)

A *causal map* $\mathcal{H} : \Lambda_e^i \rightarrow \Lambda_e^o$ is said to be *incrementally Λ-stable* [Des.9] iff

i) the map $\mathcal{H}$ is Λ-stable (see Definition 3.2.3) <u>and</u>

ii) there exists a continuous nondecreasing function $\widetilde{\phi_{\mathcal{H}}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, $\forall\, u \in \Lambda_e^i$,

$$\|\mathcal{H}(u + \Delta u) - \mathcal{H}u\| \leq \widetilde{\phi_{\mathcal{H}}}(\|\Delta u\|) \qquad \forall\, \Delta u \in \Lambda^i .$$

□

In other words, a Λ-stable map is incrementally Λ-stable if "bounded" deviations in the input result in "bounded" deviations at the output; the bound on the output deviation is <u>independent</u> of the nominal input signal u .

In particular, any linear Λ-stable map is also incrementally Λ-stable.

4.5 The feedback system $\Sigma(\mathcal{P}, \mathcal{Q})$

Consider the well-posed feedback system $\Sigma(\mathcal{P}, \mathcal{Q})$ shown in Figure 4.6 :

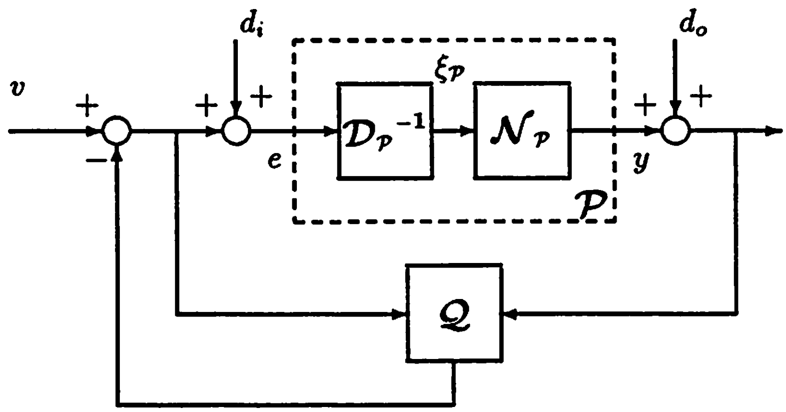

Figure 4.6:
The feedback system $\Sigma(\mathcal{P}, \mathcal{Q})$

The causal map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ denotes the plant and the causal map $\mathcal{Q}$ denotes a nonlinear Λ-*stable* two-input one-output compensator.

By Definition 4.2.2 , the feedback system $\Sigma(\mathcal{P}, \mathcal{Q})$ is Λ-stable iff the closed-loop map

$$(v, d_i, d_o) \mapsto (e, y)$$

is Λ-stable.

We now show that any plant $\mathcal{P}$ stabilized in $\Sigma(\mathcal{P}, \mathcal{Q})$ necessarily has an *r.c.f.*

Proposition 4.5.1 ($\Sigma(\mathcal{P}, \mathcal{Q})$ Λ-stable $\Rightarrow$ $\mathcal{P}$ has an r.c.f)
Let $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be a causal plant and $\mathcal{Q} : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^i$ be Λ-stable such that the feedback system $\Sigma(\mathcal{P}, \mathcal{Q})$ is Λ-stable; then $\mathcal{P}$ has an r.c.f.

Comment 4.5.2

By Fact 4.2.5 , the causal map $\mathcal{P}$ has an r.f. since the well-posed feedback system

$\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is Λ-stable. The fact that there is an r.c.f. follows by the fact that the causal compensator in $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is Λ-stable.

Proof of Proposition 4.5.1

Set the input and output disturbances d_i and d_o equal to zero. By assumption the well-posed feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is Λ-stable; hence the causal map

$$\mathcal{H}_{ev} : \Lambda_e^i \to \Lambda_e^i \ , \ \mathcal{H}_{ev} : v \mapsto e$$

is Λ-stable. By Fact 4.2.5 , $(\mathcal{P}\mathcal{H}_{ev}, \mathcal{H}_{ev})$ is an r.f. of $\mathcal{P}$. Writing the summing node equation at v , we obtain

$$\left\{ \begin{bmatrix} 0 & I \end{bmatrix} + \mathcal{Q} \right\} \begin{bmatrix} \mathcal{P}\mathcal{H}_{ev} \\ \mathcal{H}_{ev} \end{bmatrix} = I \ ;$$

hence $(\mathcal{P}\mathcal{H}_{ev}, \mathcal{H}_{ev})$ is in fact an r.c.f. of $\mathcal{P}$.

□

We now try to answer the converse: can any causal map $\mathcal{P}$ with an r.c.f. be stabilized in the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$?

Proposition 4.5.3 (incrementally Λ-stable r.c.f $\Rightarrow$ stabilization)

Let the map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ have an r.c.f. $(\mathcal{N}_\mathcal{P}, \mathcal{D}_\mathcal{P})$, where the maps $\mathcal{N}_\mathcal{P}$, $\mathcal{D}_\mathcal{P}$ and the chosen left-inverse $\mathcal{F}_\mathcal{P}^\mathcal{I}$ are incrementally Λ-stable . For a given *incrementally Λ-stable* Λ-unimodular map $\mathcal{M} : \Lambda_e^i \to \Lambda_e^i$ let the Λ-stable map $\mathcal{Q}$ be defined as

$$\mathcal{Q} := \left(\mathcal{M} - \mathcal{D}_\mathcal{P}\right)\mathcal{F}_\mathcal{P}^\mathcal{I} \ , \tag{4.30}$$

where the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is assumed to be well-posed. Under these assumptions the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is Λ-stable.

Proof

Writing the summing node equations in Figure 4.6 in terms of the plant pseudo-state $\xi_{\mathcal{P}}$, we obtain

$$\mathcal{D}_{\mathcal{P}}\xi_{\mathcal{P}} - d_i = v - \big(\mathcal{M} - \mathcal{D}_{\mathcal{P}}\big)\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}\begin{bmatrix} \mathcal{N}_{\mathcal{P}}\xi_{\mathcal{P}} + d_o \\ \mathcal{D}_{\mathcal{P}}\xi_{\mathcal{P}} - d_i \end{bmatrix} . \tag{4.31}$$

It suffices to show that the causal map

$$(v , d_i , d_o) \mapsto \xi_{\mathcal{P}}$$

determined by (4.31) , is Λ-stable. Let

$$\zeta := \mathcal{F}^{\mathcal{I}}_{\mathcal{P}}\begin{bmatrix} \mathcal{N}_{\mathcal{P}}\xi_{\mathcal{P}} + d_o \\ \mathcal{D}_{\mathcal{P}}\xi_{\mathcal{P}} - d_i \end{bmatrix} - \xi_{\mathcal{P}} . \tag{4.32}$$

Since $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}$ is incrementally Λ-stable, using the identity in (4.25) , we obtain

$$||\zeta|| \leq \tilde{\phi}_{\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}}(||d_i|| + ||d_o||) \ ; \tag{4.33}$$

hence the causal map $(v , d_i , d_o) \mapsto \zeta$ is Λ-stable. After substituting (4.32) in (4.31) , adding $\mathcal{M}\xi_{\mathcal{P}}$ to both sides of (4.31) and rearranging (4.31) , we obtain

$$\mathcal{M}\xi_{\mathcal{P}} = v + d_i + \Big\{\mathcal{D}_{\mathcal{P}}(\xi_{\mathcal{P}} + \zeta) - \mathcal{D}_{\mathcal{P}}\xi_{\mathcal{P}}\Big\} + \Big\{\mathcal{M}\xi_{\mathcal{P}} - \mathcal{M}(\xi_{\mathcal{P}} + \zeta)\Big\} .$$

Since the maps $\mathcal{M}$ and $\mathcal{D}_{\mathcal{P}}$ are incrementally Λ-stable, we obtain

$$||\mathcal{M}\xi_{\mathcal{P}}|| \leq ||v|| + ||d_i|| + \tilde{\phi}_{\mathcal{D}_{\mathcal{P}}}(||\zeta||) + \tilde{\phi}_{\mathcal{M}}(||\zeta||) \ ;$$

together with (4.33) , we have shown that the map

$$(v , d_i , d_o) \mapsto \mathcal{M}\xi_{\mathcal{P}}$$

is Λ-stable. Since the map $\mathcal{M}$ is Λ-unimodular, we conclude that

$$(v , d_i , d_o) \mapsto \xi_{\mathcal{P}}$$

is Λ-stable and consequently, $\Sigma(\mathcal{P} , \mathcal{Q})$ is Λ-stable.

□

4.6 A Class of Nonlinear Plants with Incrementally Λ-stable r.c.f.

In this section, let the input and output spaces Λ_e^i and Λ_e^o be defined as $L_{\infty e}^{n_i}[0, \infty)$ and $L_{\infty e}^{n_o}[0, \infty)$, respectively (see Section 4.3).

We assume that for all of the differential equation representations below, for any input, the state and the output are uniquely determined on $[0, \infty)$.

Description of the Class

Consider a nonlinear plant whose input-output map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ is specified by the following state-space description:

$$\mathcal{P} : e \mapsto y \left\{ \begin{array}{rcl} \dot{x} & = & Ax + f(t,x) + Be \\ y & = & Cx + h(t,x) + Ee \\ x(0) & = & 0 \ . \end{array} \right. \tag{4.34}$$

□

We impose the following assumptions: on the plant description in (4.34) :

Assumptions 4.6.1

I. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_i}$, $C \in \mathbb{R}^{n_o \times n}$, $E \in \mathbb{R}^{n_o \times n_i}$ and (A, B, C, E) is minimal .

II. The maps $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n_o}$ are piecewise continuous in t and globally uniformly Lipschitz continuous in x ; moreover there exist m_f and $m_h > 0$ such that

$$\sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^n} |f(t, x)| =: m_f \quad ,$$

$$\sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^n} |h(t, x)| =: m_h \quad .$$

Proposition 4.6.2 (an r.c.f. of $\mathcal{P}$ defined in (4.34))

Let the causal plant $\mathcal{P} : \Lambda^{\mathrm{i}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}}$ be described by (4.34) and satisfy Assumptions 4.6.1 **I**, **II** . Let the real matrices K and L be chosen such that $\sigma(A+BK) \cup \sigma(A+LC) \subset \mathbb{C}^{\circ}_{-}$. Let the maps $\mathcal{N}_{\mathcal{P}}$, $\mathcal{D}_{\mathcal{P}}$ and $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}$ be given by (4.35 - 4.37) , below:

$$\mathcal{N}_{\mathcal{P}} : \xi \mapsto y \quad \left\{ \begin{array}{rcl} \dot{x}_N &=& (A+BK)x_N + f(t,x_N) + B\xi \\ y &=& (C+EK)x_N + h(t,x_N) + E\xi \\ x_N(0) &=& 0 \quad , \end{array} \right. \tag{4.35}$$

$$\mathcal{D}_{\mathcal{P}} : \xi \mapsto e \quad \left\{ \begin{array}{rcl} \dot{x}_D &=& (A+BK)x_D + f(t,x_D) + B\xi \\ e &=& Kx_D + \xi \\ x_D(0) &=& 0 \quad , \end{array} \right. \tag{4.36}$$

$$\mathcal{F}^{\mathcal{I}}_{\mathcal{P}} : (e\,,\, y) \mapsto \hat{\xi} \quad \left\{ \begin{array}{rcl} \dot{x}_1 &=& (A+LC)x_1 + f(t,x_1) + Lh(t,x_1) \\ && \qquad\qquad -Ly + (B+LE)e \\ \hat{\xi} &=& -Kx_1 + e \\ x_1(0) &=& 0 \quad . \end{array} \right. \tag{4.37}$$

Under these assumptions,

i) the maps $\mathcal{N}_{\mathcal{P}}$, $\mathcal{D}_{\mathcal{P}}$ and $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}$ are incrementally Λ-stable, and

ii) $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ is an r.f. of $\mathcal{P}$ in (4.34) , and

$$\mathcal{F}^{\mathcal{I}}_{\mathcal{P}} \begin{bmatrix} \mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}} \end{bmatrix} = \mathrm{I} \ . \tag{4.25}$$

Comment 4.6.3

Proposition 4.6.2 shows the existence of an r.c.f. of $\mathcal{P}$ by calculating the three maps in (4.35 - 4.37). There are two cases which do not require any calculation :

i) If $\sigma(A) \subset \mathbb{C}^{\circ}_{-}$, then the map $\mathcal{P}$ is Λ-stable; $(\mathcal{P}\,,\mathrm{I})$ is an r.c.f. of $\mathcal{P}$ and the corresponding $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}$ is $\begin{bmatrix} 0 & \mathrm{I} \end{bmatrix}$.

ii) If the triple $(A\,,\,B\,,\,C)$ is such that there exists a real matrix K such that $\sigma(A+BKC) \subset \mathbb{C}^{\circ}_{-}$, then $\mathcal{P}$ has an r.c.f. (set $\mathcal{Q} = -K[\mathrm{I} \ \ 0]$

in Proposition 4.5.1); note that not all minimal triples (A, B, C) can be stabilized by stable dynamic output-feedback [You.1] .

Proof of Proposition 4.6.2

By Assumption 4.6.1 I , (A, B) is a controllable pair; hence there exists a $K \in \mathbb{R}^{n_i \times n}$ such that $\sigma(A+BK) \subset \mathbb{C}_-^o$. Let the numerator map $\mathcal{N}_\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ and the denominator map $\mathcal{D}_\mathcal{P} : \Lambda_e^i \to \Lambda_e^i$ be given by (4.35) and (4.36) , respectively. The map $\mathcal{D}_\mathcal{P}$ in (4.36) is bijective and has a causal inverse (recall Proof of Proposition 4.3.2) . Cascading $\mathcal{D}_\mathcal{P}^{-1}$ and $\mathcal{N}_\mathcal{P}$, we obtain (note that $\xi = e - Kx_D$ by (4.36))

$$\mathcal{N}_\mathcal{P}\mathcal{D}_\mathcal{P}^{-1} : e \mapsto y \begin{cases} \dot{x}_N &= (A+BK)x_N + f(t,x_N) + Be - BKx_D \\ \dot{x}_D &= Ax_D + f(t,x_D) + Be \\ y &= (C+EK)x_N + h(t,x_N) + Ee - EKx_D \\ x_N(0) &= x_D = 0 \quad . \end{cases} \tag{4.38}$$

Let $x_N - x_D =: z$; then the system of differential equations in (4.38) is equivalent to

$$\begin{cases} \dot{z} &= (A+BK)z + f(t,x_D + z) - f(t,x_D) \\ \dot{x}_D &= Ax_D + f(t,x_D) + Be \\ z(0) &= x_D = 0 \quad . \end{cases} \tag{4.39}$$

By assumption, the system of differential equations in (4.39) has a unique solution on $[0, \infty)$ for all inputs in Λ_e^i ; it is easy to see that $z(t) = 0$ for all $t \geq 0$ is the unique solution of (4.39) . Hence (4.35) and (4.39) describe the same map $\mathcal{P}$. Since $\sigma(A+BK) \subset \mathbb{C}_-^o$ and $f(\cdot, \cdot)$ is bounded, the maps $\mathcal{N}_\mathcal{P}$ and $\mathcal{D}_\mathcal{P}$ are Λ-stable; in fact, they are incrementally Λ-stable. To see this, let $\Delta\xi \in \Lambda^i$ and let (x_N, y) and $(\hat{x}_N, \hat{y})$ be the solution pairs of (4.36) corresponding to the inputs ξ and $(\xi + \Delta\xi)$, respectively. Then we obtain

$$x_N(t) = \int_0^t e^{(A+BK)(t-\tau)}(f(\tau, x_N(\tau)) + B\xi(\tau))d\tau \ , \tag{4.40}$$

$$\hat{x}_N(t) = \int_0^t e^{(A+BK)(t-\tau)}(f(\tau, x_N(\tau)) + B\xi(\tau) + B\Delta\xi(\tau))d\tau \ . \tag{4.41}$$

Since $\sigma(A+BK) \subset \mathbb{C}_-^o$, there exist $\alpha \geq 1$ and $\lambda > 0$ such that

$$\forall\, t \geq 0 \;\; , \;\; \|\, e^{(A+BK)t} \,\| \leq \alpha\, e^{-\lambda t} \;\; ;$$

from (4.40) and (4.41) , $[\,x_N(t) \,-\, \hat{x}_N(t)\,]$ satisfies

$$\begin{aligned} |\,x_N(t) \,-\, \hat{x}_N(t)\,| \;&\leq\; (2m_f \,+\, \|\,B\,\|\,\|\,\Delta\xi\,\|)\alpha \int_0^t e^{-\lambda(t-\tau)} d\tau \\ &\leq\; (2m_f \,+\, \|\,B\,\|\,\|\,\Delta\xi\,\|)\frac{\alpha}{\lambda} \;\; . \end{aligned} \tag{4.42}$$

From (4.42), we conclude that there exists a function $\tilde{\phi}_{\mathcal{N}_{\mathcal{P}}}$ such that

$$\|\, y \,-\, \hat{y}\,\| \leq \tilde{\phi}_{\mathcal{N}_{\mathcal{P}}}(\,\|\,\Delta\xi\,\|\,) \;\; .$$

A similar argument shows that the map $\mathcal{D}_{\mathcal{P}}$ is also incrementally Λ-stable.

By assumption, the pair $(C\,,\,A)$ is observable; hence there exists an L such that $\sigma(A+LC) \subset \mathbb{C}_-^o$. Let the pseudo-state observer map candidate $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ be given by (4.38) ; clearly, the map $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ is causal Λ-stable; in fact, it is *incrementally* Λ-stable. Cascading $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ with $\begin{bmatrix} \mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}} \end{bmatrix}$, we obtain

$$\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}\begin{bmatrix} \mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}} \end{bmatrix} : \xi \mapsto \hat{\xi} \; \left\{ \begin{aligned} \dot{x}_1 \;&=\; (A+LC)x_1 + f(t,x_1) + Lh(t,x_1) \\ &\qquad\qquad -Lh(t,x_N) - L(C+EK)x_N \\ &\qquad\qquad -LE\xi + (B+LE)(Kx_D+\xi) \\ \dot{x}_N \;&=\; (A+BK)x_N + f(t,x_N) + B\xi \\ \dot{x}_D \;&=\; (A+BK)x_D + f(t,x_D) + B\xi \\ \hat{\xi} \;&=\; -Kx_1 + Kx_D + \xi \\ x_1(0) \;&=\; x_N(0) = x_D(0) = 0 \quad . \end{aligned} \right. \tag{4.43}$$

By assumption, the system of differential equations in (4.43) has a unique solution; hence

$$\forall\, t \geq 0 \;\; , \;\; x_1(t) \,=\, x_N(t) \;\; ,$$

and the identity (4.25) is satisfied.

□

Corollary 4.6.4 (a stabilizing feedback system for $\mathcal{P}$ in (4.34))

Let the map $\mathcal{P}$ satisfy the assumptions in Proposition 4.6.2 . Let $\mathcal{M}$ be an incrementally Λ-stable Λ-unimodular map . Let $\mathcal{Q} := \left(\mathcal{M} - \mathcal{D}_{\mathcal{P}}\right)\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$, where the maps $\mathcal{D}_{\mathcal{P}}$ and $\mathcal{F}_{\mathcal{P}}^{\mathcal{I}}$ are given by (4.36) and (4.37) , respectively. Assume that the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is well-posed. Under these assumptions, the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ is Λ-stable (see Figure 4.6).

Proof

Follows by Proposition 4.6.2 and Proposition 4.5.3 .

□

4.7 Nonlinear stable unity-feedback systems with one linear subsystem

The problem of characterizing all linear time-invariant compensators which stabilize a linear time-invariant plant in the unity-feedback configuration has been solved using tools of algebraic control theory; the characterization is obtained by finding solutions of certain Bezout identities [You.1,Cal.1,Des.4,Vid.2,Vid.3] . A generalization of this approach to linear input-output maps can be found in [Fei.1] ; see [Man.1] for the time-varying continuous-time case. In [Kha.2] , the set of all stabilizing discrete-time possibly nonlinear time-varying compensators for a discrete-time linear time-invariant plant is obtained using periodic compensators and two-step compensation schemes. In [Des.6,Des.9] , the set of all stabilizing compensators for an *incrementally* stable nonlinear plant (e.g. *stable* linear plant) is obtained. Using left and right factorizations of a class of causal nonlinear discrete-time plants, a complete parametrization of the set of all stable solutions $\mathcal{U}$, $\mathcal{V}$ of the equation $\mathcal{U}\mathcal{N}+\mathcal{V}\mathcal{D}=\mathcal{M}$ is given in [Ham.5] .

In this section, we consider the nonlinear unity-feedback configuration where one of the two subsystems (either the plant or the compensator) is specified by a *linear* (not necessarily time-invariant) map. Since the plant and the compensator appear symmetrically in the stability analysis of the unity-feedback system, we choose to derive the results for a fixed linear plant. Assuming that the linear plant has a "generalized" left-coprime factorization, we show that all nonlinear stabilizing compensators have right-coprime factorizations which satisfy a Bezout-like identity. In the case where the linear plant also has a right-coprime factorization, we obtain the set of all solutions satisfying the identity; in fact, we obtain a parametrization of the set of all nonlinear stabilizing compensators. Interchanging the roles of the plant and the compensator, this result gives the set of all nonlinear plant perturbations which maintain feedback-system stability for a given linear compensator.

Results

Consider the nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ shown in Figure 4.7: the linear plant is given by a causal linear (not necessarily time-invariant) map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and the possibly nonlinear compensator is given by a causal map $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$. We assume that the the linear map P satisfies Assumption 4.7.1 :

Assumption 4.7.1

The causal linear map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ has the following properties :

i) The map P has an l.f. : that is, there exist causal linear finite-gain-stable maps $\widetilde{\mathsf{N}}_\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ and $\widetilde{\mathsf{D}}_\mathsf{P} : \Lambda_e^o \to \Lambda_e^o$, where $\widetilde{\mathsf{D}}_\mathsf{P}$ is bijective, has a causal inverse and $\widetilde{\mathsf{D}}_\mathsf{P}^{-1}\widetilde{\mathsf{N}}_\mathsf{P} = \mathsf{P}$.

ii) There exist causal Λ-stable (not necessarily linear) maps $\mathcal{U} : \Lambda_e^o \to \Lambda_e^i$ and $\mathcal{V} : \Lambda_e^o \to \Lambda_e^o$ such that

$$\widetilde{\mathsf{N}}_\mathsf{P}\mathcal{U} + \widetilde{\mathsf{D}}_\mathsf{P}\mathcal{V} = I \quad . \tag{4.44}$$

□

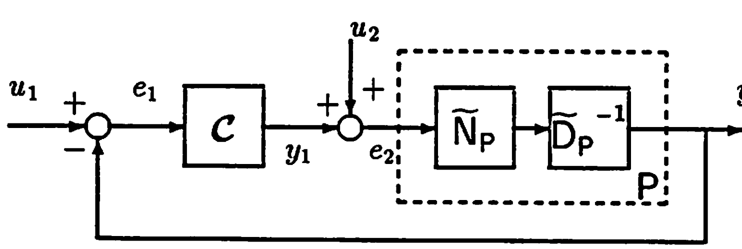

Figure 4.7:
Nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$

Theorem 4.7.2 (Λ-stable $\mathcal{S}(\mathsf{P},\mathcal{C})$)

Consider the nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ in Figure 4.7 , where the causal <u>linear</u> map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ satisfies Assumption 4.7.1 and $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$. Under these assumptions, the nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ is well-posed and Λ-stableif and only if the map $\mathcal{C}$ has an r.c.f. $(\mathcal{N}_{\mathcal{C}} , \mathcal{D}_{\mathcal{C}})$ such that

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}} = I \ . \tag{4.45}$$

Proof

" if "

Let $(\mathcal{N}_{\mathcal{C}} , \mathcal{D}_{\mathcal{C}})$ be an r.c.f. of $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$, satisfying (4.45). From the summing node equations in Figure 4.7 , we obtain

$$e_1 = \mathcal{D}_{\mathcal{C}}\xi_{\mathcal{C}} = u_1 - y_2 \tag{4.46}$$

$$\widetilde{\mathsf{D}}_{\mathsf{P}}y_2 = \widetilde{\mathsf{N}}_{\mathsf{P}}(u_2 + y_1) = \widetilde{\mathsf{N}}_{\mathsf{P}}\left(u_2 + \mathcal{N}_{\mathcal{C}}\xi_{\mathcal{C}}\right) \ , \tag{4.47}$$

where $\xi_{\mathcal{C}}$ denotes the pseudo-state of the compensator $\mathcal{C}$. Using the *linearity* of $\widetilde{\mathsf{N}}_{\mathsf{P}}$ and $\widetilde{\mathsf{D}}_{\mathsf{P}}$ in (4.46-4.47) and using assumption (4.45) ,

$$\xi_{\mathcal{C}} = \begin{bmatrix} \widetilde{\mathsf{D}}_{\mathsf{P}} & -\widetilde{\mathsf{N}}_{\mathsf{P}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{4.48}$$

By (4.48) , the map

$$(u_1 , u_2) \mapsto \xi_{\mathcal{C}}$$

is Λ-stable. Since the maps $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{D}_{\mathcal{C}}$ are Λ-stable, the map

$$(u_1 , u_2) \mapsto (e_1 , y_1) = (\mathcal{D}_{\mathcal{C}}\xi_{\mathcal{C}} , \mathcal{N}_{\mathcal{C}}\xi_{\mathcal{C}})$$

is Λ-stable; hence the feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ is well-posed and Λ-stable.

" **only if** "

By well-posedness and Λ-stability of $\mathcal{S}(\mathsf{P},\mathcal{C})$, the map $\mathcal{C}$ has an r.f. $(\mathcal{N}_{\mathcal{C}}\,,\mathcal{D}_{\mathcal{C}})$; namely

$$\mathcal{N}_{\mathcal{C}} \;=\; \mathcal{C}\left(\mathrm{I}+\mathsf{P}\mathcal{C}\right)^{-1} \;,$$
$$\mathcal{D}_{\mathcal{C}} \;=\; \left(\mathrm{I}+\mathsf{P}\mathcal{C}\right)^{-1} \;.$$

Using this r.f. of $\mathcal{C}$ in the summing node equations (4.46-4.47) of the feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ and using the *linearity* of $\widetilde{\mathsf{N}}_{\mathsf{P}}$ and $\widetilde{\mathsf{D}}_{\mathsf{P}}$,

$$\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)\xi_{\mathcal{C}} \;=\; \begin{bmatrix} \widetilde{\mathsf{D}}_{\mathsf{P}} & -\widetilde{\mathsf{N}}_{\mathsf{P}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \;. \tag{4.49}$$

By well-posedness of $\mathcal{S}(\mathsf{P},\mathcal{C})$ and the existence of the causal inverse ${\mathcal{D}_{\mathcal{C}}}^{-1}$, there exists a *causal* map $(u_1\,, u_2) \mapsto \xi_{\mathcal{C}}$, which need not be Λ-stable even if the feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ is. Choose the inputs as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \;:=\; \begin{bmatrix} \mathcal{V} \\ -\mathcal{U} \end{bmatrix} \nu \;, \tag{4.50}$$

where $\nu \in \Lambda^{\mathrm{o}}$. Substituting (4.50) in (4.49) ,

$$\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)\xi_{\mathcal{C}} \;=\; \nu \;. \tag{4.51}$$

Equation (4.51) determines a causal map

$$\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)^{-1} \;:\; \Lambda^{\mathrm{o}}_{\mathrm{e}} \to \Lambda^{\mathrm{o}}_{\mathrm{e}} \;,\quad \left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)^{-1} \;:\; \nu \mapsto \xi_{\mathcal{C}} \;.$$

Hence the causal Λ-stable maps $\nu \mapsto e_1$ and $\nu \mapsto y_1$ are given by

$$\mathcal{D}_{\mathcal{C}}\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)^{-1} \;:\; \nu \mapsto e_1 \;, \tag{4.52}$$
$$\mathcal{N}_{\mathcal{C}}\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}}\right)^{-1} \;:\; \nu \mapsto y_1 \;. \tag{4.53}$$

From (4.52-4.53) and

$$y_1 \;=\; \mathcal{C}e_1 \;,$$

we conclude that

$$\left(\mathcal{N}_c\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_c + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_c\right)^{-1} ,\ \mathcal{D}_c\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_c + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_c\right)^{-1} \right)$$

is an r.f. of $\mathcal{C}$; furthermore, since

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\left[\mathcal{N}_c\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_c + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_c\right)^{-1}\right] + \widetilde{\mathsf{D}}_{\mathsf{P}}\left[\mathcal{D}_c\left(\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_c + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_c\right)^{-1}\right] = I \ ,$$

it is an r.c.f. of $\mathcal{C}$.

□

We now prove an algebraic lemma which characterizes the set of <u>all</u> solutions of (4.45) .

Lemma 4.7.3 (all solutions of (4.45))

Let the causal *linear* map $\mathsf{P} : \Lambda_e^i \to \Lambda_e^o$ satisfy Assumption 4.7.1 . Suppose also that $(\mathcal{N}_\mathcal{P}, \mathcal{D}_\mathcal{P})$ is an r.c.f. of P (note that the maps $\mathcal{N}_\mathcal{P}$ and $\mathcal{D}_\mathcal{P}$ need <u>not</u> be linear), where the conditions in Definition 4.4.1 hold. Under these assumptions the set of all causal Λ-stable solutions of (4.45) is given by

$$\left\{ \begin{bmatrix} \mathcal{N}_c \\ \mathcal{D}_c \end{bmatrix} : \Lambda_e^o \to \Lambda_e^i \times \Lambda_e^o \quad \middle| \quad \begin{bmatrix} \mathcal{N}_c \\ \mathcal{D}_c \end{bmatrix} = \begin{bmatrix} \mathcal{U} + \mathcal{D}_\mathcal{P}\mathcal{Q} \\ \mathcal{V} - \mathcal{N}_\mathcal{P}\mathcal{Q} \end{bmatrix} ,\right. \tag{4.54}$$

$$\left. \mathcal{Q} : \Lambda_e^o \to \Lambda_e^i \text{ is } \Lambda\text{-stable} \right\} .$$

Proof

We first show that any pair of Λ-stable maps specified by (4.54) is a solution of equation (4.45) . Substituting $\mathcal{N}_c$ and $\mathcal{D}_c$ given by (4.54) in (4.45) and using the *linearity* of $\widetilde{\mathsf{N}}_{\mathsf{P}}$ and $\widetilde{\mathsf{D}}_{\mathsf{P}}$, we obtain

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_c + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_c = \widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{U} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{V} + \left(\widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{N}_\mathcal{P} - \widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{D}_\mathcal{P}\right)\mathcal{Q} \ . \tag{4.55}$$

Substituting

$$\widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{N}_{\mathcal{P}} = \widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{D}_{\mathcal{P}} \tag{4.56}$$

in (4.55) and using (4.44) , we see that (4.45) is satisfied.

Now consider any pair of causal Λ-stable maps $\mathcal{N}_{\mathcal{C}}$, $\mathcal{D}_{\mathcal{C}}$ satisfying equation (4.45) . Then from (4.44) ,

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{N}_{\mathcal{C}} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{D}_{\mathcal{C}} = \mathrm{I} = \widetilde{\mathsf{N}}_{\mathsf{P}}\mathcal{U} + \widetilde{\mathsf{D}}_{\mathsf{P}}\mathcal{V} \ . \tag{4.57}$$

Using the *linearity* of $\widetilde{\mathsf{N}}_{\mathsf{P}}$ and $\widetilde{\mathsf{D}}_{\mathsf{P}}$, and substituting $\widetilde{\mathsf{D}}_{\mathsf{P}}^{-1}\widetilde{\mathsf{N}}_{\mathsf{P}} = \mathcal{N}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1}$ in (4.57) ,

$$\mathcal{N}_{\mathcal{P}}\mathcal{D}_{\mathcal{P}}^{-1}\left(\mathcal{N}_{\mathcal{C}} - \mathcal{U}\right) = \mathcal{V} - \mathcal{D}_{\mathcal{C}} \ . \tag{4.58}$$

Let the parameter $\mathcal{Q}$ be defined by

$$\mathcal{Q} := \mathcal{D}_{\mathcal{P}}^{-1}\left(\mathcal{N}_{\mathcal{C}} - \mathcal{U}\right) \ . \tag{4.59}$$

Clearly, the map in (4.59) is causal. By assumption, $(\mathcal{N}_{\mathcal{P}}, \mathcal{D}_{\mathcal{P}})$ is an r.c.f. of P : hence there exists a causal Λ-stable map $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}}$ such that the identity (4.25) holds. From (4.25) , (4.58) and (4.59) ,

$$\begin{aligned} \mathcal{Q} &= \mathcal{F}^{\mathcal{I}}_{\mathcal{P}}\begin{bmatrix} \mathcal{N}_{\mathcal{P}} \\ \mathcal{D}_{\mathcal{P}} \end{bmatrix}\mathcal{Q} \\ &= \mathcal{F}^{\mathcal{I}}_{\mathcal{P}}\begin{bmatrix} \mathcal{N}_{\mathcal{P}}\mathcal{Q} \\ \mathcal{D}_{\mathcal{P}}\mathcal{Q} \end{bmatrix} \\ &= \mathcal{F}^{\mathcal{I}}_{\mathcal{P}}\begin{bmatrix} \mathcal{V} - \mathcal{D}_{\mathcal{C}} \\ \mathcal{N}_{\mathcal{C}} - \mathcal{U} \end{bmatrix} \end{aligned} \tag{4.60}$$

From (4.60) , we conclude that the map $\mathcal{Q}$ is Λ-stable. From (4.58) and (4.59) , (4.54) follows.

□

Theorem 4.7.4 (parametrization of all stabilizing compensators)

Let the causal <u>linear</u> map $\mathsf{P} : \Lambda_e^i \rightarrow \Lambda_e^o$ satisfy all of the assumptions in

Lemma 4.7.3 . Under these assumptions, <u>the set of all compensators</u> $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$ which Λ-stabilize the nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$ is given by

$$\Big\{ \mathcal{C} = \big(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}\big)\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}\big)^{-1} \quad \Big| \quad \mathcal{Q} : \Lambda_e^o \to \Lambda_e^i \text{ is } \Lambda\text{-stable and } \big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}\big)^{-1} \text{ is causal} \Big\} . \tag{4.61}$$

Moreover, the map $\mathcal{Q} \mapsto \mathcal{C}$ in (4.61) is bijective.

Proof

From Theorem 4.7.2 and Lemma 4.7.3 , we conclude that the map in (4.61) is <u>onto</u> the set of all causal compensators which Λ-stabilize the nonlinear unity-feedback system $\mathcal{S}(\mathsf{P},\mathcal{C})$. By inspection, the map $\mathcal{Q} \mapsto \mathcal{C}$ in (4.61) is surjective.

We need to show that the map $\mathcal{Q} \mapsto \mathcal{C}$ in (4.61) is injective: it suffices to show that

$$\big(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}_1\big)\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_1\big)^{-1} = \big(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}_2\big)\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_2\big)^{-1} \tag{4.62}$$

implies that

$$\mathcal{Q}_1 = \mathcal{Q}_2 .$$

First note that by the *linearity* of $\widetilde{\mathsf{N}}_{\mathsf{P}}$ and $\widetilde{\mathsf{D}}_{\mathsf{P}}$, (4.44) and (4.56) imply that

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\big(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}_1\big) + \widetilde{\mathsf{D}}_{\mathsf{P}}\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_1\big) = \mathrm{I} , \tag{4.63}$$

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\big(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}_2\big) + \widetilde{\mathsf{D}}_{\mathsf{P}}\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_2\big) = \mathrm{I} . \tag{4.64}$$

Composing (4.63) on the right with the nonlinear map

$$\big(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_1\big)^{-1}$$

and using (4.62) ,

$$\widetilde{\mathsf{N}}_{\mathsf{P}}\left(\mathcal{U} + \mathcal{D}_{\mathcal{P}}\mathcal{Q}_2\right)\left(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_2\right)^{-1} + \widetilde{\mathsf{D}}_{\mathsf{P}} = \left(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_1\right)^{-1} . \tag{4.65}$$

Composing equation (4.65) on the right with the nonlinear map

$$\left(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_2\right)$$

and using (4.64) , we obtain

$$\left(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_1\right)^{-1}\left(\mathcal{V} - \mathcal{N}_{\mathcal{P}}\mathcal{Q}_2\right) = \mathrm{I} . \tag{4.66}$$

Substituting (4.66) in (4.62) , we obtain

$$\mathcal{D}_{\mathcal{P}}\mathcal{Q}_1 = \mathcal{D}_{\mathcal{P}}\mathcal{Q}_2 \; ;$$

since $\mathcal{D}_{\mathcal{P}}$ is bijective, the claim follows.

□

4.8 Nonlinear unity-feedback system

The following theorem gives a necessary and sufficient condition for the stability of a well-posed nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$, provided that either $\mathcal{P}$ or $\mathcal{C}$ have r.c.f.'s. Since the roles of $\mathcal{P}$ and $\mathcal{C}$ can be interchanged, we state only the case where $\mathcal{C}$ has an r.c.f.

Theorem 4.8.1 (n&s condition for Λ-stable $\mathcal{S}(\mathcal{P},\mathcal{C})$)
Let $(\mathcal{N}_C \, , \, \mathcal{D}_C \,)$ be an r.c.f. of the causal nonlinear map $\mathcal{C} \, : \, \Lambda_e^o \to \Lambda_e^i$. Let $\mathcal{P} \, : \, \Lambda_e^i \to \Lambda_e^o$ be a causal nonlinear plant. Under these assumptions, the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable if and only if the causal pseudo-state map

$$(u_1 \, , \, u_2) \mapsto \xi_C$$

is Λ-stable (see Figure 4.8).

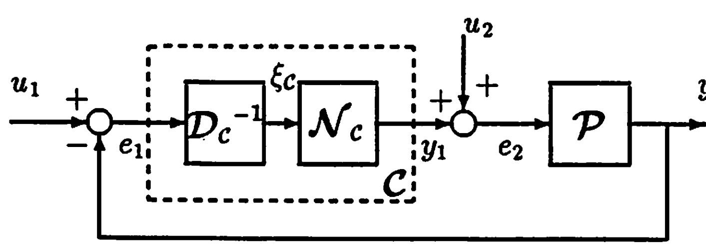

Figure 4.8:
Nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$

Proof

" if "

By assumption, the causal map $(u_1 \, , \, u_2) \mapsto \xi_C$ is Λ-stable . Since $\mathcal{N}_C$ and $\mathcal{D}_C$ are Λ-stable maps, the closed-loop map $(u_1 \, , \, u_2) \mapsto (e_1 \, , \, e_2)$ is given by

$$\begin{aligned} e_1 &= \mathcal{D}_C \xi_C \\ e_2 &= u_2 + \mathcal{N}_C \xi_C \ , \end{aligned} \tag{4.67}$$

and is also Λ-stable . Hence the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable .

" only if "

By assumption, $(\mathcal{N}_C\ ,\ \mathcal{D}_C\ )$ is an r.c.f. of $\mathcal{C}$; hence there exists a causal Λ-stable map $\mathcal{F}_C^{\mathcal{I}}$ such that

$$\mathcal{F}_C^{\mathcal{I}}\begin{bmatrix} \mathcal{N}_C \\ \mathcal{D}_C \end{bmatrix} = \mathrm{I}\ . \tag{4.68}$$

Moreover, by assumption, the closed-loop map $(u_1\ ,\ u_2) \mapsto (e_1\ ,\ e_2)$ is Λ-stable. Then by (4.67) and (4.68) ,

$$\xi_C = \mathcal{F}_C^{\mathcal{I}}\begin{bmatrix} e_1 \\ e_2 - u_2 \end{bmatrix}\ . \tag{4.69}$$

By definition of r.c.f., the causal map $\mathcal{F}_C^{\mathcal{I}}$ is Λ-stable; hence from (4.69), we conclude that the map $(u_1\ ,\ u_2) \mapsto \xi_C$ is Λ-stable.

□

Comment 4.8.2

The idea in Theorem 4.8.1 can be generalized to well-posed feedback systems other than $\mathcal{S}(\mathcal{P},\mathcal{C})$:

In any well-posed Λ-stable feedback system, if a subsystem (say $\mathcal{C}$) has an r.c.f., then the closed-loop pseudo-state map (mapping the closed-loop system inputs to the pseudo-state ξ_C) is Λ-stable.

If a subsystem (say $\mathcal{C}$) has an r.c.f., the Λ-stability of the closed-loop pseudo-state map may or may not guarantee Λ-stability of the overall feedback system; it holds for $\mathcal{S}(\mathcal{P},\mathcal{C})$ but it may also hold for other well-posed systems (for example the feedback system $\Sigma(\mathcal{P}\ ,\ \mathcal{Q})$ in Figure 4.6).

Lemma 4.8.3

Let the nonlinear maps $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ and $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$ be Λ-*stable* . If the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable, then the maps $\left(\mathrm{I} + \mathcal{P}\mathcal{C}\right)$ and $\left(\mathrm{I} + \mathcal{C}\mathcal{P}\right)$ are Λ-unimodular.

Proof

We only show that the map $\left(\mathrm{I} + \mathcal{P}\mathcal{C}\right)$ is Λ-unimodular. Proof of the other is similar.

Writing the summing node equations in Figure 4.8 and setting $u_2 = 0$,

$$\left(\mathrm{I} + \mathcal{P}\mathcal{C}\right)e_1 = u_1 \ .$$

By assumption, the closed-loop map $u_1 \mapsto e_1$ is causal Λ-stable. Hence the claim follows.

□

Theorem 4.8.4 (parametrization of all compensators)

Let the nonlinear map $\mathcal{P} : \Lambda_e^i \rightarrow \Lambda_e^o$ be *incrementally* Λ*-stable* . Then the well-posed nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable if and only if the map $\mathcal{C} : \Lambda_e^o \rightarrow \Lambda_e^i$ has an r.c.f.

$$\left(\mathcal{Q}, \mathrm{I} - \mathcal{P}\mathcal{Q}\right)$$

for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^o \rightarrow \Lambda_e^i$.

Comment 4.8.5

Theorem 4.8.4 gives a parametrization of <u>all</u> stabilizing nonlinear compensators $\mathcal{C}$, provided that the nonlinear plant $\mathcal{P}$ is incrementally Λ-stable. This theorem [Des.9] extends the linear Q-parametrization result [Zam.2] . Note that Theorem 4.8.4 motivates an r.c.f. approach.

Proof of Theorem 4.8.4

" only if "

By assumption, the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is well-posed and Λ-stable. Hence, with $u_2 = 0$, the closed-loop map $u_1 \mapsto y_1$, namely

$$\mathcal{Q} := \mathcal{C}\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)^{-1} \;:\; \Lambda_e^o \to \Lambda_e^i$$

is causal and Λ-stable. By calculation,

$$\big(\mathcal{Q}, \mathrm{I} - \mathcal{P}\mathcal{Q}\big)$$

is an r.f. of $\mathcal{C}$. Furthermore,

$$\begin{bmatrix} \mathcal{P} & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathcal{Q} \\ \mathrm{I} - \mathcal{P}\mathcal{Q} \end{bmatrix} = \mathrm{I} \;.$$

Since the map $\mathcal{P}$ is Λ-stable, we conclude that $\big(\mathcal{Q}, \mathrm{I} - \mathcal{P}\mathcal{Q}\big)$ is r.c.f. of $\mathcal{C}$.

" if "

By assumption $\big(\mathcal{Q}, \mathrm{I} - \mathcal{P}\mathcal{Q}\big)$ is an r.c.f. of $\mathcal{C}$ for some causal Λ-stable map $\mathcal{Q}$. It suffices to show that the closed-loop map $(u_1, u_2) \mapsto \xi_c$ is Λ-stable. Writing the summing node equations in Figure 4.8 for $\mathcal{C} = \mathcal{Q}\big(\mathcal{I} - \mathcal{P}\mathcal{Q}\big)^{-1}$,

$$\big(\mathrm{I} - \mathcal{P}\mathcal{Q}\big)\xi_c = u_1 - \mathcal{P}\big(u_2 + \mathcal{Q}\xi_c\big) \;. \tag{4.70}$$

By assumption, the feedback system $\mathcal{S}(\mathcal{P}, \mathcal{Q}(\mathrm{I} - \mathcal{P}\mathcal{Q})^{-1})$ is well-posed; hence (4.70) determines a causal pseudo-state map. From (4.70) , using the incremental Λ-stability of $\mathcal{P}$, we obtain

$\forall\, (u_1, u_2) \in \Lambda^o \times \Lambda^i$, $\forall\, \mathrm{T} \in \mathcal{T}$

$$\begin{aligned} \|\Pi_{\mathrm{T}}\xi_c\| &\le \|u_1\| + \|\mathcal{P}\mathcal{Q}\xi_c - \mathcal{P}\big(\mathcal{Q}\xi_c + u_2\big)\| \\ &\le \|u_1\| + \widetilde{\phi_{\mathcal{P}}}(\|u_2\|) \;. \end{aligned}$$

Hence the pseudo-state map in (4.70) is Λ-stable.

□

Using the incremental Λ-stability argument, we show that the unimodularity condition in Lemma 4.8.3 is also a sufficient condition.

Theorem 4.8.6

Let the nonlinear map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be incrementally Λ-*stable* and let the nonlinear map $\mathcal{C} : \Lambda_e^o \to \Lambda_e^i$ be Λ-*stable* . Then the well-posed nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ is Λ-stable if and only if the map $\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)$ is Λ-unimodular.

Proof

" only if "

Follows by Lemma 4.8.3 .

" if "

By assumption, the map $\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)$ is Λ-unimodular. Since the map $\mathcal{C}$ is Λ-stable, $(\mathcal{C}\,, \mathrm{I})$ is an r.c.f. of $\mathcal{C}$; hence $e_1 := \xi_c$ (see Figure 4.8). By Theorem 4.8.1 , it suffices to show that the causal map $(u_1\,, u_2) \mapsto e_1$ is Λ-stable. Writing the summing node equations in Figure 4.8 , we obtain

$$e_1 \;=\; u_1 - \mathcal{P}\big(\mathcal{C}e_1 + u_2\big) \quad ,$$

adding $\mathcal{P}\mathcal{C}e_1$ to both sides,

$$\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)e_1 \;=\; u_1 + \mathcal{P}\mathcal{C}e_1 - \mathcal{P}\big(\mathcal{C}e_1 + u_2\big) \quad .$$

By the Λ-unimodularity of $\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)$, $e_1 \in \Lambda^o$ if and only if $\big(\mathrm{I} + \mathcal{P}\mathcal{C}\big)e_1 \in \Lambda^o$. Using the incremental Λ-stability of $\mathcal{P}$,

$\forall\,(u_1\,,\,u_2) \in \Lambda^{\mathrm{o}} \times \Lambda^{\mathrm{i}}\ ,\ \forall\,\mathrm{T} \in \mathcal{T}$

$$\|\,\Pi_{\mathrm{T}}\big(\mathrm{I}+\mathcal{P}\mathcal{C}\big)e_1\,\| \ \le\ \|\,u_1\,\| \ +\ \|\,\mathcal{P}\mathcal{C}e_1 - \mathcal{P}\big(\mathcal{C}e_1 + u_2\big)\,\|$$
$$\le\ \|\,u_1\,\| \ +\ \widetilde{\phi_{\mathcal{P}}}(\|\,u_2\,\|)\ .$$

Hence we conclude that the closed-loop map $(u_1\,,\,u_2) \mapsto \big(\mathrm{I}+\mathcal{P}\mathcal{C}\big)e_1$ is Λ-stable.

□

Theorem 4.8.7

Let the nonlinear map

$$\mathcal{P}\,:\,\Lambda_{\mathrm{e}} \times \Lambda_{\mathrm{e}}^{\mathrm{i}} \to \Lambda_{\mathrm{e}}^{\mathrm{o}} \quad , \quad \mathcal{P}\,:\,(v\,,\,e_1) \mapsto y_1$$

be causal and *incrementally* Λ-stable. For some fixed causal nonlinear map $\mathcal{C}$: $\Lambda_{\mathrm{e}}^{\mathrm{o}} \to \Lambda_{\mathrm{e}}^{\mathrm{i}}$, let the nonlinear unity-feedback system $\mathcal{S}(\,\mathcal{P}(\nu\,,\,\cdot\,)\,,\,\mathcal{C})$ be well-posed for all $\nu \in \Lambda$. Under these assumptions, $\mathcal{S}(\,\mathcal{P}(\nu_0\,,\,\cdot\,)\,,\,\mathcal{C})$ is Λ-stable for some $\nu_0 \in \Lambda$ if and only if $\mathcal{S}(\,\mathcal{P}(\nu\,,\,\cdot\,)\,,\,\mathcal{C})$ is Λ-stable for all $\nu \in \Lambda$.

Comments 4.8.8

From Theorem 4.8.7 , we have the following interpretations:

i) If we have a family of incrementally Λ-stable plants

$$\big\{\,\mathcal{P}(\,\nu\,,\,\cdot\,)\,\big\}_{\nu\,\in\,\Lambda}$$

and if one member of this family is stabilized by some $\mathcal{C}$, then the whole family is stabilized by that $\mathcal{C}$.

ii) The one-input one-output plant $\mathcal{P}(\,\nu\,,\,\cdot\,)$ can be considered as an input-output description for a fixed parameter ν or for a fixed bounded auxiliary input ν . In any case, the map $\mathcal{P}(\,\nu\,,\,\cdot\,)$ is assumed to be a complete description of the plant for any $\nu \in \Lambda$ [Bha.2] .

Proof of Theorem 4.8.7

We only prove the necessity; the sufficiency proof is obvious.

By assumption, there exists a $\nu_0 \in \Lambda$ such that the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P}(\nu_0, \cdot), \mathcal{C})$ is Λ-stable. By Theorem 4.8.4 , there exists a causal Λ-stable map $\mathcal{Q} : \Lambda^{\mathrm{o}}_{\mathrm{e}} \to \Lambda^{\mathrm{i}}_{\mathrm{e}}$ such that

$$\big(\mathcal{Q} \, , \, \mathrm{I} - \mathcal{P}(\nu_0 \, , \, \mathcal{Q}(\cdot)) \big)$$

is an r.c.f. of $\mathcal{C}$. For any $\nu \in \Lambda$, consider the unity-feedback system $\mathcal{S}(\mathcal{P}(\nu, \mathcal{Q}(\cdot)) \, , \, \mathcal{Q}\big(\mathrm{I} - \mathcal{P}(\nu_0, \mathcal{Q}(\cdot))\big)^{-1})$ in Figure 4.9 .

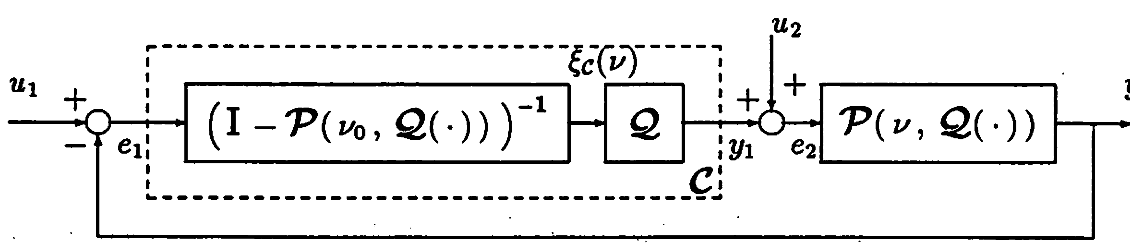

Figure 4.9:
$\mathcal{S}(\mathcal{P}(\nu, \mathcal{Q}(\cdot)) \, , \, \mathcal{Q}\big(\mathrm{I} - \mathcal{P}(\nu_0, \mathcal{Q}(\cdot))\big)^{-1})$

By Theorem 4.8.1 , it suffices to show that the causal pseudo-state map

$$(u_1 \, , \, u_2) \mapsto \xi_C(\nu)$$

associated with $\mathcal{S}(\mathcal{P}(\nu, \mathcal{Q}(\cdot))\cdot, \mathcal{Q}\big(\mathrm{I} - \mathcal{P}(\nu_0, \mathcal{Q}(\cdot))\big)^{-1})$ is Λ-stable for all $\nu \in \Lambda$. The summing node equations in Figure 4.9 give

$$\xi_C(\nu) \;=\; u_1 + \mathcal{P}(\nu_0 \, , \, \mathcal{Q}\xi_C(\nu)) \; - \; \mathcal{P}(\nu \, , \, \mathcal{Q}\xi_C(\nu) + u_2) \quad . \tag{4.71}$$

Since the map $\mathcal{P}(\cdot, \cdot)$ is incrementally Λ-stable, (4.71) gives

$\forall \, (u_1 \, , u_2) \in \Lambda^{\mathrm{o}} \times \Lambda^{\mathrm{i}}$, $\forall \, \nu \in \Lambda$, $\forall \, \mathrm{T} \in \mathcal{T}$,

$$||\Pi_T \xi_{\mathcal{C}}(\nu)|| \;\leq\; ||u_1|| \;+\; \widetilde{\phi_{\mathcal{P}}}(||\nu - \nu_0|| + ||u_2||) \;.$$

Hence the closed-loop pseudo-state map $(u_1\,,\,u_2) \mapsto \xi_{\mathcal{C}}(\nu)$ is Λ-stable for all $\nu \in \Lambda$.

□

Consider the case

$$\mathcal{P}(\nu\,,\cdot) \;:=\; \widehat{\mathcal{H}}_{y_2}\Big|_{u_2=\nu} \;:\; (u_1,\nu) \mapsto y_2 \quad ,$$

where the map $\widehat{\mathcal{H}}_{y_2}$ is the restricted input-output map of an incrementally Λ-stable $\mathcal{S}(\widehat{\mathcal{P}},\widehat{\mathcal{C}})$ for some causal maps $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{C}}$; then the two-step stabilization results in [Ana.1] and [Des.7] become special cases of Theorem 4.8.7 .

The following theorem [Des.9] establishes a necessary and sufficient condition for simultaneous stabilization of two plants which need not be members of the family of incrementally Λ-stable maps in Theorem 4.8.7 . Our use of the factorization approach greatly simplifies the proof.

Theorem 4.8.9

Let the nonlinear map $\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be causal and *incrementally* Λ-stable. Let the nonlinear unity-feedback system $\mathcal{S}(\mathcal{P},\mathcal{C})$ be well-posed and Λ-stable (hence by Theorem 4.8.4 , $\mathcal{C}$ has an r.c.f $\big(\mathcal{Q}\,,\,\mathrm{I} - \mathcal{P}\mathcal{Q}\big)$ for some causal Λ-stable map $\mathcal{Q} : \Lambda_e^o \to \Lambda_e^i$). Let the perturbation $\Delta\mathcal{P} : \Lambda_e^i \to \Lambda_e^o$ be a causal nonlinear map such that the nonlinear unity-feedback systems $\mathcal{S}(\mathcal{P}+\Delta\mathcal{P},\mathcal{C})$ and $\mathcal{S}(\Delta\mathcal{P},\mathcal{Q})$ are well-posed. Under these assumptions, $\mathcal{S}(\mathcal{P}+\Delta\mathcal{P},\mathcal{C})$ is Λ-stable if and only if $\mathcal{S}(\Delta\mathcal{P},\mathcal{Q})$ is Λ-stable.

Comment 4.8.10

The nonlinear perturbed plant $\mathcal{P}+\Delta\mathcal{P}$ need <u>not</u> be Λ-stable. The perturbation

$\mathcal{\Delta P}$ is only subject to the condition that $\mathcal{S}(\mathcal{\Delta P}, \mathcal{Q})$ is Λ-stable in order to have $\mathcal{S}(\mathcal{P} + \mathcal{\Delta P}, \mathcal{C})$ Λ-stable.

Proof of Theorem 4.8.9

Consider Figures 4.10 and 4.11 : Let the pseudo-state maps $\mathcal{H}_\xi$ and $\widehat{\mathcal{H}}_{\hat{\xi}}$

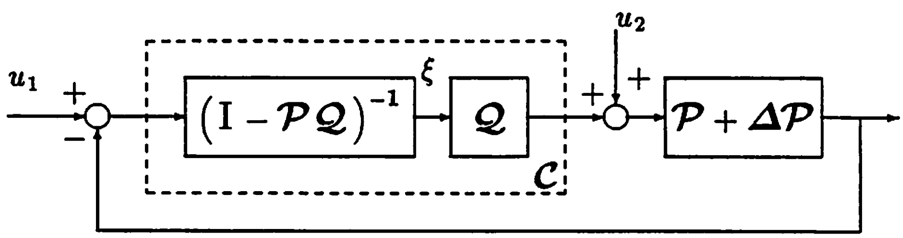

Figure 4.10:
$\mathcal{S}(\mathcal{P} + \mathcal{\Delta P}, \mathcal{C})$

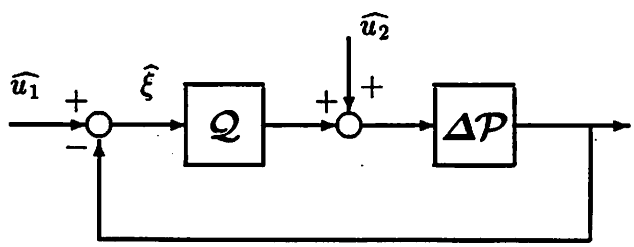

Figure 4.11:
$\mathcal{S}(\mathcal{\Delta P}, \mathcal{Q})$

be defined as

$$\mathcal{H}_\xi \;:\; (u_1\,,\, u_2) \mapsto \xi$$

$$\widehat{\mathcal{H}}_{\hat{\xi}} \;:\; (\widehat{u_1}\,,\, \widehat{u_2}) \mapsto \hat{\xi}\ .$$

By Theorem 4.8.1 , it suffices to show that the map $\mathcal{H}_\xi$ is Λ-stable if and only if $\widehat{\mathcal{H}}_{\hat{\xi}}$ is Λ-stable.

" if "

By assumption, the map $\widehat{\mathcal{H}}_{\hat{\xi}}$ is Λ-stable. Writing the summing node equations from Figure 4.11 , $\widehat{\mathcal{H}}_{\hat{\xi}}$ is given by

$$\hat{\xi} = \widehat{u_1} - \Delta\mathcal{P}(\mathcal{Q}\hat{\xi} + \widehat{u_2}) \quad . \tag{4.72}$$

By assumption, $\mathcal{H}_{\xi}$ is a causal map. Writing the summing node equations from Figure 4.10 , $\mathcal{H}_{\xi}$ is determined by

$$\xi = \mathcal{R}(u_1 , u_2) - \Delta\mathcal{P}(\mathcal{Q}\xi + u_2) \quad , \tag{4.73}$$

where the causal map $\mathcal{R}$ is defined by

$$\mathcal{R}(u_1 , u_2) := u_1 + \mathcal{P}\mathcal{Q}\mathcal{H}_{\xi}(u_1 , u_2) - \mathcal{P}\Big(\mathcal{Q}\mathcal{H}_{\xi}(u_1 , u_2) + u_2\Big) \quad . \tag{4.74}$$

Note that the map $\mathcal{R}$ in (4.74) is Λ-stable since by incremental Λ-stability of $\mathcal{P}$,

$\forall\, (u_1 , u_2) \in \Lambda^{\mathrm{o}} \times \Lambda^{\mathrm{i}}$, $\forall\, \mathrm{T} \in \mathcal{T}$,

$$\| \Pi_{\mathrm{T}} \mathcal{R}(u_1 , u_2) \| \le \| u_1 \| + \widetilde{\phi_{\mathcal{P}}}(\| u_2 \|) \quad .$$

Since (4.72) and (4.73) have the same form, we conclude that

$$\mathcal{H}_{\xi}(u_1 , u_2) := \widehat{\mathcal{H}}_{\hat{\xi}}\Big(\mathcal{R}(u_1 , u_2) , u_2\Big) \quad ;$$

hence, $\mathcal{H}_{\xi}$ is Λ-stable since both $\widehat{\mathcal{H}}_{\hat{\xi}}(\cdot\, , \cdot)$ and $\mathcal{R}(\cdot\, , \cdot)$ are Λ-stable.

" only if "

By assumption, the map $\mathcal{H}_{\xi}$ is Λ-stable. Writing the summing node equations from Figure 4.10 , $\mathcal{H}_{\xi}$ is found by solving the solution of

$$(\mathrm{I} - \mathcal{P}\mathcal{Q})\xi = u_1 - (\mathcal{P} + \Delta\mathcal{P})(\mathcal{Q}\xi + u_2) \tag{4.75}$$

for ξ . By assumption, $\widehat{\mathcal{H}}_{\hat{\xi}}$ is a causal map. Writing the summing node equations from Figure 4.11 , $\widehat{\mathcal{H}}_{\hat{\xi}}$ is determined by

$$(\mathrm{I} - \mathcal{P}\mathcal{Q})\hat{\xi} = \widehat{\mathcal{R}}(\widehat{u_1}, \widehat{u_2}) - (\mathcal{P} + \Delta\mathcal{P})(\mathcal{Q}\hat{\xi} + \widehat{u_2}) \ , \tag{4.76}$$

where the causal map $\widehat{\mathcal{R}}$ is defined by

$$\widehat{\mathcal{R}}(\widehat{u_1}, \widehat{u_2}) := \widehat{u_1} + \mathcal{P}\Big(\mathcal{Q}\widehat{\mathcal{H}}_{\hat{\xi}}(\widehat{u_1}, \widehat{u_2}) + \widehat{u_2}\Big) - \mathcal{P}\mathcal{Q}\widehat{\mathcal{H}}_{\hat{\xi}}(\widehat{u_1}, \widehat{u_2}) \ . \tag{4.77}$$

Note that the map $\widehat{\mathcal{R}}$ in (4.77) is Λ-stable since by incremental Λ-stability of $\mathcal{P}$,

$\forall\, (u_1, u_2) \in \Lambda^{\mathrm{o}} \times \Lambda^{\mathrm{i}}$, $\forall\, \mathrm{T} \in \mathcal{T}$,

$$\| \Pi_{\mathrm{T}} \widehat{\mathcal{R}}(\widehat{u_1}, \widehat{u_2}) \| \ \leq \ \| \widehat{u_1} \| + \widetilde{\phi_{\mathcal{P}}}(\| \widehat{u_2} \|) \ .$$

Comparing (4.75) and (4.76) , we conclude that

$$\widehat{\mathcal{H}}_{\hat{\xi}}(\widehat{u_1}, \widehat{u_2}) := \mathcal{H}_{\xi}\Big(\widehat{\mathcal{R}}(\widehat{u_1}, \widehat{u_2}), \widehat{u_2}\Big) \ ;$$

hence $\widehat{\mathcal{H}}_{\hat{\xi}}$ is Λ-stable (because both $\mathcal{H}_{\xi}(\cdot, \cdot)$ and $\widehat{\mathcal{R}}(\cdot, \cdot)$ are Λ-stable).

□

Chapter 5

Conclusion

The main focus of this work is on three items:

i) stable additive feedback systems,

ii) right factorizations,

iii) right-coprime factorizations.

Items i) – iii) are studied for three classes of causal input-output maps:

1. linear time-invariant finite-dimensional maps (Chapter 2) ,
2. linear maps (Chapter 3) ,
3. nonlinear maps (Chapter 4) .

The three classes are definitely nested in one another; however the point is that we investigated each one of them using all of the properties available for that class. This approach led to some interesting observations. The implications

$$\text{iii)} \Rightarrow \text{i)} \Rightarrow \text{ii)}$$

always hold. The implication

$$\text{ii)} \Rightarrow \text{iii)}$$

requires attention: In the setting of Chapter 2 , we have

$$\text{ii)} \Leftrightarrow \text{iii)}$$

due to the ring properties of proper rational transfer functions in one complex variable. In the setting of Chapter 3 , the implication

$$\text{ii)} \Rightarrow \text{iii)}$$

no longer exists. There are linear systems with right-factorizations that do not admit right-coprime factorizations [Fei.2] . Hence for general nonlinear maps, the conditions under which the implication

$$\text{ii)} \Rightarrow \text{iii)}$$

holds will be extremely useful. The answer is expected to be through case studies: such an attempt has already been made in Sections 3.5 , 4.3 and 4.6 .

The definitions of left-factorizations can be introduced for *linear* maps with no extra work. As proved in Section 3.4 , in a stable linear unity-feedback system the plant *and* the compensator have both right- *and* left-factorizations. In other words, linearity allows a left factorization tool to be developed in parallel with right factorization tools.

The implication

$$\text{i)} \Rightarrow \text{ii)}$$

(see also [Ham.2]) is a very important one justifying right factorization tools. A similar justification for left factorizations of nonlinear causal maps is not yet known.

Factorization tools bring a better understanding of robustness analysis of stable unity-feedback systems (Sections 2.5 , 2.6 , 3.7 , 4.8) .

In Section 4.6 we worked out right-coprime factorizations for a class of nonlinear plants and found a stabilizing feedback configuration, which has a free parameter that can be assigned; in other words, a class of stabilizing compensators was proposed. The explicit calculations are readily implementable.

Finding right-coprime factorizations for a given plant is an answer to stabilization; investigating special cases will be extremely useful.

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