AN ELEMENTARY PROOF OF THE ROUTH-HURWITZ STABILITY CRITERION

by

J. J. Anagnost and C. A. Desoer

Memorandum No. UCB/ERL M89/74

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ABSTRACT

This paper presents an elementary proof of the well-known Routh-Hurwitz stability criterion. The novelty of the proof is that it requires only elementary geometric considerations in the complex plane. This feature makes it suitable for use in undergraduate control system courses.

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1. Introduction

The determination of stability of lumped parameter, linear, time invariant systems is one of the most fundamental problems in system theory. According to Gantmacher, [Gan. 1, p.172-173] this problem was first solved in essence by Hermite [Her. 1] in 1856, but remained unknown. In 1875, E. J. Routh also obtained conditions for stability of such systems [Rou. 1]. In 1895, A. Hurwitz, unaware of Routh's work, gave another solution based on Hermite's paper. The determinantal inequalities obtained by Hurwitz are known today as the Routh-Hurwitz conditions, taught in virtually every undergraduate course on control theory.

Unfortunately, Hurwitz's proof of the result is very complicated, involving algebraic manipulations. Indeed, the proof is so complicated that most elementary textbooks (for example, [Dorf 1], [Kuo 1]) choose not to prove it at all, but rather to state it as a fact.

In a recent paper, Mansour [Man. 1] proves the Routh-Hurwitz Theorem in a very simple manner using the Hermite-Bieler Theorem. Motivated by Mansour's proof, this paper presents a proof based on elementary geometric considerations in the complex plane. It thus provides a clear *geometric* insight into what makes the procedure work. It also slightly extends Mansour's work by providing a proof of the second part of the Routh-Hurwitz criterion: the number of sign changes in the first column of the Routh Table is the number of open right half-plane zeros.

The idea behind the proof of the theorem is simple. It will be shown that at each step the Routh procedure (i) eliminates precisely one zero of the characteristic polynomial (ii) preserves the position of the $j\omega$-axis zeros, and (iii) ensures that the remaining off $j\omega$-axis zeros do not cross the $j\omega$-axis. By observing the sign changes in the first column of the Routh table, it can be determined whether the eliminated zero is a zero in the open right half-plane or the open left half-plane. Thus, in n steps, precisely n zeros have been eliminated and the sign changes indicate the number of right half-plane zeros of the original polynomial.

2. Statement of Routh-Hurwitz Stability Criterion

Theorem 2.1 (Routh-Hurwitz) - Consider a nth order polynomial in s

$$p(s) = a_0 + a_1 s + \ldots a_{n-1} s^{n-1} + a_n s^n \quad \text{where } a_i \in \mathbb{R},\ i = 1, 2, \ldots n, \text{ and } a_n \neq 0.$$

Assume without loss of generality that n is even. Split the even polynomial p(s) into its even and odd part by $p(s) = h_{n/2+1}(s^2) + s g_{n/2}(s^2)$, where $h_{n/2+1}(s^2)$ is even and of degree n, and where $g_{n/2}(s^2)$ is even and of degree n-2. Consider the well-known Routh table, written in the following form:

$$
\begin{array}{r|llllll}
h_{n/2+1} & a_n & a_{n-2} & a_{n-4} & \cdots\cdots & a_2 & a_0 \\
g_{n/2} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots\cdots & a_1 & \\
h_{n/2} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots\cdots & b_1 & \\
g_{n/2-1} & c_{n-1} & c_{n-2} & c_{n-3} & \cdots\; c_2 & & \\
h_{n/2-1} & d_{n-1} & d_{n-2} & d_{n-4} & \cdots\; d_2 & & \\
 & \cdot & \cdot & \cdot & \cdot & & \\
 & \cdot & \cdot & \cdot & \cdot & & \\
g_2 & k_1 & k_2 & & & & \\
h_2 & l_1 & l_2 & & & & \\
g_1 & m_1 & & & & & \\
h_1 & n_1 & & & & &
\end{array}
$$

Table 1 - The Routh Table

where

$$b_{n-1} = a_{n-2} - \frac{a_n}{a_{n-1}} a_{n-3} \qquad b_{n-2} = a_{n-4} - \frac{a_n}{a_{n-1}} a_{n-5} \quad \cdots$$

$$c_{n-1} = a_{n-3} - \frac{a_{n-1}}{b_{n-1}} b_{n-2} \qquad c_{n-2} = a_{n-5} - \frac{a_{n-1}}{b_{n-1}} b_{n-3} \quad \cdots$$

etc.

Then p(s) is Hurwitz (i.e., p(s) has all its zeros in the open left half-plane) if and only if each element of the first column is nonzero, i.e. $a_n > 0$, $a_{n-1} > 0$, $b_{n-1} > 0, \ldots m_1 > 0$, $n_1 > 0$.

3. Preliminary Lemmas

We first start with a definition which makes precise the notion of net phase change.

Definition 3.1 Consider a polynomial p(s) and a continuous, oriented curve $C \subset \mathbb{C}$ which starts at $s_1 \in \mathbb{C}$ and ends at $s_2 \in \mathbb{C}$. Suppose $p(s) \neq 0$, for all $s \in C$. Let the curve be parameterized by the continuous function $\phi:[0, 1] \to C$. Since $p(s) \neq 0$ for all $s \in C$ this means that $\arg(p(s))$ is well-defined mod2π; so we choose $\arg(p(\phi(0)))$ arbitrarily and for all $r \in (0, 1]$, we choose $\arg(p(\phi(r)))$ such that $r \to \arg(p(\phi(r)))$ is continuous. Then we define the function

$$\underset{C}{\text{argnet}}\ (\ p(\bullet)\ ) := \arg(p(\phi(1))) - \arg(p(\phi(0))) = \arg(p(s_2)) - \arg(p(s_1))$$

Roughly speaking, $\underset{C}{\text{argnet}}\,(\,p(\bullet)\,)$ is simply the net phase change as $p(\bullet)$ traverses C. For example, in Figure 2, if the plotted solid locus is p(C), then $\underset{C}{\text{argnet}}\,(\,p(\bullet)\,) = 2\pi$.

The following lemma gives a relationship between the location of zeros of a polynomial and its net phase change.

Lemma 3.3 - Consider the polynomial $p(s) = a_0 + a_1 s + \ldots a_{n-1}s^{n-1} + a_n s^n$ where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots n$, with $a_n \neq 0$ and $a_0 \neq 0$ (so p(s) is of degree n, and $p(0) \neq 0$). Then p(s) has L zeros in the open left half-plane counting multiplicities, R zeros in the open right half-plane counting multiplicities and 2K zeros $\pm j\omega_i$ on the $j\omega$-axis with multiplicities m_i, $i=1, \ldots K$, (i.e., there are a total of $M = 2\sum_{i=1}^{K} m_i$ $j\omega$-axis zeros) if and only if

(i) $p^k(j\omega_i) = 0$ for $k=0, \ldots m_i-1$, $i=1, \ldots K$ but $p^{m_i}(j\omega_i) \neq 0$, $i=1, \ldots K$, and $p(j\omega) \neq 0$ for all $\omega \in \mathbb{R}^+ \setminus \{\omega_i\ i=1, \ldots K\}$.

(ii) $\underset{C}{\text{argnet}}\,(\,p(\bullet)\,) = \pi/2(\,L - R + M)$

where the oriented curve C is the $j\omega$-axis, except for indentations *on the right* at each $j\omega$-axis zero $j\omega_i$, i.e., it starts at 0 and ends at $+j\infty$.

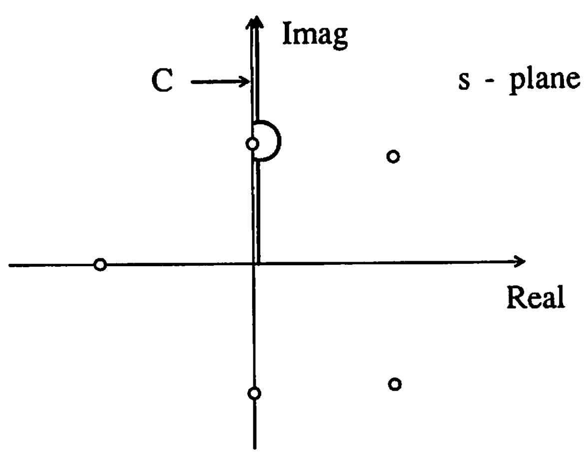

Figure 1 - Plot of the curve C

Proof of Lemma 3.3 - $\underline{\Rightarrow}$ Since p(s) is an nth degree polynomial, it has precisely n zeros. By assumption, precisely M are on the $j\omega$-axis, while the remaining zeros lie in the open right half-plane or open left half-plane. In addition, since each zero $j\omega_i$ has multiplicity

m_i, this implies $p^k(j\omega_i) = 0$ for k=0, ...m_i-1. Thus, (i) is proved. To prove (ii), note that each simple open right half-plane zero contributes $-\pi/2$ radians of phase to the net argument as s traverses C, while each simple open left half-plane zero contributes $\pi/2$ radians of phase. Due to the indentations on the right of the $j\omega$-axis zeros, each simple $j\omega$-axis zero pair contributes π radians of phase, etc. Thus,

$$\underset{C}{\text{argnet}}\,(\,p(\bullet)\,) = \pi/2(\,L - R + M)$$

This proves (ii).

$\Leftarrow$ By assumption p(s) has precisely M/2 pairs of $j\omega$-axis zeros counting multiplicities, so it can be factored as

$$p(s) = \prod_{i=1}^{K} (s^2 + \omega_i^2)^{m_i} \prod_{i=1}^{n-M} (s - s_{zi})$$

where $\{s_{zi},\ i=1, \ldots\ n\text{-}M\}$ denotes the remaining zeros of p(s). If we make the curve indented to the right of the $j\omega$-axis, we can define $\underset{C}{\text{argnet}}\,(\,p(\bullet)\,)$. By computation its value is

$$\underset{C}{\text{argnet}}\,(\,p(\bullet)\,) = \pi\, M/2 + \underset{C}{\text{argnet}}\left(\prod_{i=1}^{n-M} (s - s_{zi})\right)$$

Now, if the number of open right half-plane zeros does <u>not</u> equal R, then the $\underset{C}{\text{argnet}}\,(\,p(\bullet)\,) \neq \pi/2(\,L - R + M)$, which leads to a contradiction. ■

The following result is the main result of the section which characterizes the effect of one step of the Routh-Hurwitz procedure.

Lemma 3.4 - Let $p(s) = a_0 + a_1 s + \ldots a_{n-1}s^{n-1} + a_n s^n$ where $a_i \in \mathbb{R}$, i= 1, 2, ...n with $a_n \neq 0$ and $a_0 \neq 0$. Assume n is even. Let $h(s^2)$ and $sg(s^2)$ be the even and odd parts of p(s), respectively, i.e.,

$$h(s^2) := a_0 + a_2 s^2 + \ldots + a_{n-2}s^{n-2} + a_n s^n$$
$$sg(s^2) := a_1 s + a_3 s^3 + \ldots + a_{n-1}s^{n-1}.$$

Assume $a_{n-1} \neq 0$. Suppose that p(s) has L zeros in the open left half-plane counting multiplicities, M $j\omega$-axis zeros counting multiplicities, and R (=n-L-M) zeros in the open right half-plane counting multiplicities. Define

$$N(s, \lambda) := p(s) + \lambda s^2 g(s^2)$$

$$= h(s^2) + \lambda s^2 g(s^2) + s g(s^2).$$

Then,

(i) $j\omega_i$ is a $j\omega$-axis zero of $p(s)$ with multiplicity m_i if and only if $j\omega_i$ is a $j\omega$-axis zero of $N(s, \lambda)$ with multiplicity m_i for arbitrary $\lambda \in \mathbb{R}$;

(ii) Given any closed, bounded interval $I \subset \mathbb{R}$, there exists a curve C as in Figure 1 such that $N(s, \lambda) \neq 0$ for all $s \in C$, and for *all* $\lambda \in I$. Thus $\underset{C}{\text{argnet}} N(\bullet, \lambda)$ is well-defined for $\lambda \in I$.

Choose an interval $I = [-|a_n/a_{n-1}|, |a_n/a_{n-1}|]$. Choose the curve C so that $\underset{C}{\text{argnet}} N(\bullet, \lambda)$ is well-defined for all $\lambda \in I$. (This can be done by part (ii).) Then,

(iii) $\left|\underset{C}{\text{argnet}} N(\bullet,\lambda) - \underset{C}{\text{argnet}}(p(\bullet))\right| \leq \pi$, for all $\lambda \in I$;

(iv) $(\underset{C}{\text{argnet}} N(\bullet,-a_n/a_{n-1}) - \underset{C}{\text{argnet}}(p(\bullet)))\text{sign}(a_n/a_{n-1}) = \pi/2$;

(v) If, in addition, $a_n/a_{n-1} > 0$, then $N(s, -a_n/a_{n-1})$ has L - 1 zeros in the open left half-plane, M zeros on the $j\omega$-axis, and R zeros in the open right half-plane, in each case counting multiplicities. If on the other hand $a_n/a_{n-1} < 0$, then $N(s, -a_n/a_{n-1})$ has L zeros in the open left half-plane, M zeros on the $j\omega$-axis, and R - 1 zeros in the open right half-plane, in each case counting multiplicities.

Proof of Lemma 3.4 -

Proof of (i) - $\Leftarrow$ Take $\lambda \in \mathbb{R}$, arbitrary; $j\omega_i$ is a $j\omega$-axis zero of $N(s, \lambda)$ with multiplicity m_i means $\frac{d^k}{ds^k} N(j\omega_i, \lambda) = 0$ for $k=0, \ldots m_i-1$. Equivalently, $h^k(-\omega_i^2) + \lambda(s^2g(s^2))^k|_{j\omega_i} + (sg(s^2))^k|_{j\omega_i} = 0$ for $k=0, \ldots m_i-1$. Since both the real and imaginary parts of $N^k(j\omega_i, \lambda)$ must be zero, this means that $h^k(-\omega_i^2) + \lambda(s^2g(s^2))^k|_{j\omega_i} = 0$ and $(sg(s^2))^k|_{j\omega_i} = 0$ for $k=0, \ldots m_i-1$. The latter expression implies that $g^k(-\omega_i^2) = 0$ for $k=0, \ldots m_i-1$. This in turn implies $h^k(-\omega_i^2) = 0$ and $g^k(-\omega_i^2) = 0$ for $k=0, \ldots m_i-1$, which implies $p^k(j\omega_i) = h^k(-\omega_i^2) + (sg(s^2))^k|_{j\omega_i} = 0$ for $k=0, \ldots m_i-1$.

$\Rightarrow$ $j\omega_i$ is a $j\omega$-axis zero of p(s) with multiplicity m_i means $p^k(j\omega_i) = 0$ for k=0, ...m_i-1. Equating the real and imaginary parts of $p^k(j\omega_i) = 0$, we have $h^k(-\omega_i^2) = 0$ and $g^k(-\omega_i^2) = 0$ for k=0, ... m_i-1. This in turn implies $h^k(-\omega_i^2) + \lambda(s^2g(s^2))^k|_{j\omega_i} = 0$ for any $\lambda \in \mathbb{R}$ and for k=0, ...m_i-1. Finally, this implies that for arbitrary $\lambda \in \mathbb{R}$, $N^k(j\omega_i,\lambda) = h^k(-\omega_i^2) + \lambda(s^2g(s^2))^k|_{j\omega_i} + (sg(s^2))^k|_{j\omega_i} = 0$ for k=0, ... m_i-1. This proves (i).

Proof of (ii) - This statement merely asserts the existence of a curve C which insures $\underset{C}{\text{argnet}}$ (N(•, λ)) is well-defined for all λ in the closed, bounded interval I. Since the details are not relevant to the rest of the proof, the details are left to the Appendix.

Proof of (iii) - For simplicity, first assume that p(s) has no $j\omega$-axis zeros. For this case we take the curve C to be the positive $j\omega$-axis.

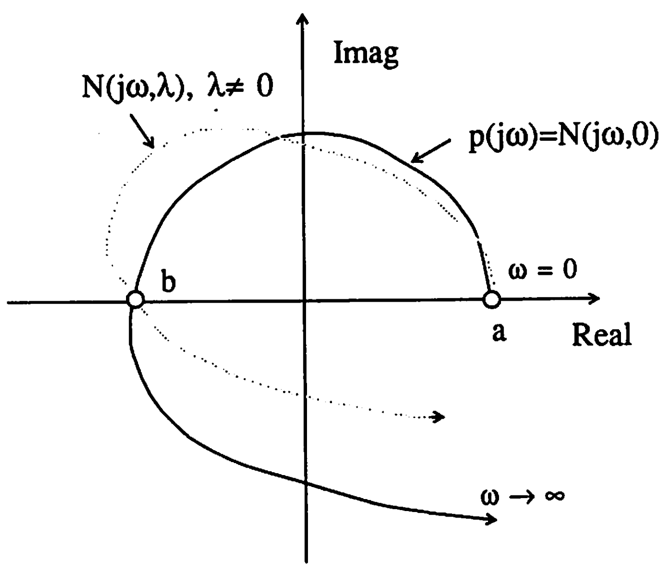

Figure 2 - Graph of $\omega \rightarrow N(j\omega,\lambda)$, $0 \leq \omega < \infty$

Since $s^2g(s^2)$ is an even function of s, $-\lambda\omega^2g(-\omega^2)$ only contributes to the *real* part of $N(j\omega,\lambda)$. In particular, points a and b in Figure 2 above are fixed points: indeed, let ω_1 be such that $p(j\omega_1) = b$. Then $g(-\omega_1^2) = 0$, which means that $N(j\omega_1, \lambda) = b$, for all $\lambda \in$ I. Next, order the zeros of $\omega^2g(-\omega^2)$ by $0 = \omega_0 < \omega_1 \leq \omega_2 \leq ...\omega_k < \infty$.

First consider $\omega \in (0, \omega_1)$. Since there are no zeros of $\omega g(-\omega^2)$ in this interval, this implies that sign(Im($N(j\omega, \lambda)$)) is a *constant* on $(0, \omega_1) \times I$. By part (ii) above, $N(j\omega, \lambda) \neq 0$ for all $\omega \in \mathbb{R}^+$ and for all $\lambda \in I$, so $\underset{[0,\infty]}{\text{argnet}} (N(\bullet, \lambda))$ is well-defined. In particular, $\underset{[0,\omega_1]}{\text{argnet}}(N(\bullet, \lambda))$ is well-defined and continuous which implies that $\underset{[0,\omega_1]}{\text{argnet}}(N(\bullet, \lambda)) = \arg(N(j\omega_1,\lambda)) - \arg(N(j0, \lambda))$. But $\arg(N(j\omega_1, \lambda)) = \arg(p(j\omega_1))$, and $\arg(N(j0, \lambda)) = \arg(p(j0))$ since 0 and ω_1 correspond to the fixed points a and b in Figure 2. Thus, the net argument change between 0 and ω_1 is *independent* of λ.

The same reasoning applies for $\omega \in (\omega_1, \omega_2)$, for $\omega \in (\omega_2, \omega_3)$, etc., up to $\omega \in (\omega_{k-1}, \omega_k)$. Hence,

$$\underset{[0,\omega_k]}{\text{argnet}} (N(\bullet, \lambda)) = \underset{[0,\omega_k]}{\text{argnet}} (p(\bullet))$$

Thus, the only difference in argument occurs for $\omega \in (\omega_k, \infty)$. Since there are no zeros of $\omega g(-\omega^2)$ in this interval, this again implies that sign(Im($N(j\omega, \lambda)$)) is constant (See Figure 2). This in turn implies that $\left|\underset{[\omega_k,\infty]}{\text{argnet}} N(\bullet,\lambda)\right| \leq \pi$. Since $\underset{[0,\omega_k]}{\text{argnet}} (N(\bullet, \lambda)) = \underset{[0,\omega_k]}{\text{argnet}} (p(\bullet))$, we then have

$$\left|\underset{[0,\infty]}{\text{argnet}} N(\bullet,\lambda) - \underset{[0,\infty]}{\text{argnet}}(p(\bullet))\right| \leq \pi$$

for all $\lambda \in I$, which proves (iii) for the case where p(s) has no $j\omega$-axis zeros.

Proof of (iv) - Note by the definition of I that $-a_n/a_{n-1} \in I$. Order the zeros of $\omega g(-\omega^2)$ as before, and use arguments identical to that of part (iii) to obtain

$$\underset{[0,\infty]}{\text{argnet}}(p(\bullet)) - \underset{[0,\infty]}{\text{argnet}}(N(\bullet,-a_n/a_{n-1})) = \underset{[\omega_k,\infty]}{\text{argnet}}(p(\bullet)) - \underset{[\omega_k,\infty]}{\text{argnet}}(N(\bullet,-a_n/a_{n-1}))$$

Since ω_k is a fixed point (i.e., independent of λ), we then obtain

$$= \underset{\omega \to \infty}{\arg}(p(j\omega)) - \underset{\omega \to \infty}{\arg}(N(j\omega,-a_n/a_{n-1}))$$

Since we are taking the limit as $\omega \to \infty$, we only need to consider the leading term of each polynomial. Performing this operation, and using properties of arg, we obtain in succession

$$= \arg_{\omega \to \infty}(a_n(j\omega)^n) \ - \ \arg_{\omega \to \infty}(a_{n-1}(j\omega)^{n-1})$$

$$= \arg_{\omega \to \infty} [a_n(j\omega)^n/(a_{n-1}(j\omega)^{n-1}]$$

$$= \arg_{\omega \to \infty} [a_n j\omega/a_{n-1}]$$

If $a_n/a_{n-1}>0$, then the net argument difference is $\pi/2$, and if $a_n/a_{n-1} < 0$, then the net argument difference is $-\pi/2$. This proves (iv) for the case where p(s) has no $j\omega$-axis zeros.

If p(s) has $j\omega$-axis zeros, then part (i) shows that $N(s, \lambda)$ has the same $j\omega$-axis zeros with the same multiplicities. This means that the only difference in argument can come from the non-$j\omega$-axis zeros. If we extract the $j\omega$-axis zeros by $p_1(s) = p(s)/ \prod_{i=1}^{M/2} (s^2 + \omega_i^2)$, then $p_1(s)$ has no $j\omega$-axis zeros, so we can apply the arguments above. For example, to prove (iii) we know from above that

$$\left| \underset{[0,\infty]}{\operatorname{argnet}} N_1(\bullet,\lambda) \ - \ \underset{[0,\infty]}{\operatorname{argnet}}(p_1(\bullet)) \right| \leq \pi$$

where $N_1(s, \lambda) = N_1(s, \lambda)/ \prod_{i=1}^{M/2} (s^2 + \omega_i^2)$ for all $\lambda \in I$. This then implies

$$\left| \underset{C}{\operatorname{argnet}} N(\bullet,\lambda) \ - \ \underset{C}{\operatorname{argnet}}(p(\bullet)) \right| \leq \pi$$

which proves (iii). Statement (iv) is proved similarily.

Proof of (v) - The net argument difference between $N(s, -a_n/a_{n-1})$ and p(s) as s traverses C is $\text{sign}(a_n/a_{n-1})\pi/2$, by applying part (iv) above. Applying the converse of Lemma 3.3 shows that $N(s, -a_n/a_{n-1})$ and p(s) have the same number of zeros on the $j\omega$-axis, and a difference of a most one in the number of open right half-plane or open left half-plane zeros, depending on the sign of a_n/a_{n-1}. This proves (v). ■

4. Proof of Theorem 2.1 (Routh-Hurwitz)

Let us first emphasize some notation. As before, assume n is even.

Let $h_{n/2-w}(s^2)$ be the even polynomial of degree n-2w whose coefficients lie in row 2w+3. (See Table 1). Let $g_{n/2-w}(s^2)$ be the even polynomial of degree n-2w whose coefficients lie in row 2w+2. (Again see Table 1).

To construct the Routh Table, perform the calculations indicated in section 1. This corresponds at each step to finding a $\lambda_{n/2-w} \in \mathbb{R}$, or a $\mu_{n/2-w} \in \mathbb{R}$ such that

$$h_{n/2-w}(s^2) = h_{n/2-w+1}(s^2) + \lambda_{n/2-w}s^2 g_{n/2-w}(s^2)$$
$$g_{n/2-w}(s^2) = g_{n/2-w+1}(s^2) + \mu_{n/2-w}h_{n/2-w+1}(s^2)$$

where the leading term of $h_{n/2-w}(s^2)$ and $g_{n/2-w}(s^2)$, respectively, is cancelled. If this procedure cannot be performed (i.e., the leading term of $g_{n/2-w}(s^2)$ or $h_{n/2-w+1}(s^2)$ is zero), then a zero is in the first column of the Routh Table. The standard procedure given in elementary textbooks is to replace the zero by $\varepsilon > 0$, and proceed. See section 5 for some of the implications of this.

Proof of Theorem 2.1 (Routh-Hurwitz)

$\Rightarrow$ If p(s) is Hurwitz, then each of its zeros are in the open left half-plane. Consider the first step of the Routh procedure. By Lemma 3.4, part (v), $N(s, -a_n/a_{n-1}) = h_{n/2+1}(s^2) - a_n/a_{n-1}s^2 g_{n/2}(s^2) + sg_{n/2}(s^2)$ has the same number of zeros in the open left half-plane as p(s) except for the eliminated zero. Since all the zeros of p(s) are in the open left half-plane, the eliminated zero must also be in the left half-plane. Thus, by the definition of $h_{n/2}(s^2)$ and Lemma 3.4, $sg_{n/2}(s^2) + h_{n/2}(s^2)$ has precisely n-1 zeros in the open left half-plane, and a_{n-1}/a_n is positive. By exactly the same reasoning in the next step we have that $h_{n/2}(s^2) + sg_{n/2-1}(s^2)$ has precisely n-2 zeros in the open left half-plane, and b_{n-1}/a_{n-1} is positive. After n steps, all zeros have been eliminated and each element in the first column is positive.

$\Leftarrow$ If each element in the first column is positive then Lemma 3.4, part (v), shows that precisely n zeros in the open left half-plane have been eliminated. Thus p(s) is Hurwitz. ■

5. The Second Part of the Routh-Hurwitz Theorem

Based on Lemma 3.4, we have the second part of the Routh-Hurwitz criterion.

Theorem 5.1 - Consider a nth order polynomial in s

$$p(s) = a_0 + a_1 s + \dots a_{n-1}s^{n-1} + a_n s^n \quad \text{where } a_i \in \mathbb{R},\ i = 1, 2, \dots n.$$

As before, assume for simplicity that n is even. Suppose when calculating the Routh Table that no element in the first column is zero. Then the number of sign changes in the first column of the Routh Table is the number of *open* right half-plane zeros of p(s).

Proof of Theorem 5.1 - At each step the algorithm (i) eliminates precisely one zero of p(s), (ii) preserves the position of the $j\omega$-axis zeros, and (iii) ensures that the remaining off $j\omega$-axis zeros do not cross the $j\omega$-axis. By Lemma 3.4 part (v), the eliminated zero is in the open left half-plane if the ratio of the associated coefficients is positive, whereas the eliminated zero is the open right half-plane if the ratio of the associated coefficients is negative. Thus the number of sign changes in the first column indicates the number of open right half-plane zeros of p(s) that were eliminated. ■

Remark 5.2 - If a zero does appear in the first column during the Routh procedure, care must be exercised in ascertaining the zero positions of the original polynomial. By adding an $\varepsilon > 0$ to a column, the position of the zeros are being perturbed (since the zeros of a polynomial are continuous functions of their coefficients provided a_n remains bounded away from zero). Attempting to deduce properties of the zeros of the original polynomial based on the properties of the perturbed polynomial can often lead to erroneous conclusions as the following examples show.

Example - Let $p(s) = s^2 + 1$. The Routh table for this example is

$$\begin{array}{|lll} 1 & 1 & 0 \\ \varepsilon & 0 & \\ 1 & & \end{array}$$

where $\varepsilon > 0$. Since there are no sign changes, the "Theorem" states "there are no zeros in open right half-plane", which is true. However, by choosing $p(s) = -s^2 - 1$ the corresponding Routh table is then

$$\begin{array}{|cc} -1 & -1 \\ \varepsilon & 0 \\ -1 & \end{array}$$

where $\varepsilon > 0$. This leads to the "conclusion" that there are two zeros in the open right half-plane. Note that adding the $\varepsilon > 0$ merely pushes the $j\omega$-axis zeros of the original polynomial off the $j\omega$-axis. Much more insidious examples can be constructed that make it very difficult to tell the position of the zeros of the original polynomial. (See, for example, [Gan. 1, p.184, Example 4].) However, we do have the following proposition.

Proposition 5.3 - Suppose that during the construction of the Routh table that a zero in the first column is encountered. Then

(i) If there are one or more nonzero elements in the same row, then p(s) has a least one zero in the *open* right half-plane.

(ii) If the row is zero, then (a) p(s) has at least one pair of $j\omega$-axis zeros, or (b) p(s) contains a factor of the form $(s + \alpha_0)(s - \alpha_0)$ for some $\alpha_0 \in \mathbb{R}$, or (c) p(s) contains a factor of the form $(s + \alpha_0 + \beta_0 j)(s + \alpha_0 - \beta_0 j)(s - \alpha_0 + \beta_0 j)(s - \alpha_0 - \beta_0 j)$ for some $\alpha_0, \beta_0 \in \mathbb{R}$.

Proof of Proposition 5.3 - *Proof of (i).* Since there is a zero in the first column in the Routh table, Lemma 3.4 shows that p(s) has at least one zero in the closed right half-plane. Without loss of generality, assume that the zero is the second element of the first column, i.e. $sg(s^2)$ has a leading coefficient of n-3. Suppose that the only zeros of p(s) in the closed right half-plane are $j\omega$-axis zeros, say M counting multiplicities. Then extract the $j\omega$-axis zero pairs from p(s) by $p_1(s) = p(s)/\prod_{i=1}^{M/2}(s^2 + \omega_i^2)$. Equivalently, $p_1(s) = h(s^2)/\prod_{i=1}^{M/2}(s^2 + \omega_i^2) + sg(s^2)/\prod_{i=1}^{M/2}(s^2 + \omega_i^2)$. By assumption, $p_1(s)$ is Hurwitz and thus has every coefficient positive. However, $h(s^2)/\Pi(s^2 + \omega_i^2)$ is of order n-M, while $sg(s^2)/\Pi(s^2 + \omega_i^2)$ is of order n - M - 3. Thus, $p_1(s)$ has its n - M - 1 coefficient equal to 0, which contradicts the fact that $p_1(s)$ is Hurwitz. This proves (i).

Proof of (ii) - Encountering a zero row during construction of the Routh table means that at some step

$$h_{n/2-w+1}(s^2) = -\lambda_{n/2-w}s^2 g_{n/2-w}(s^2)$$
$$g_{n/2-w+1}(s^2) = -\mu_{n/2-w}h_{n/2-w+1}(s^2)$$

Hence, the polynomial $h_{n/2-w+1}(s^2) + sg_{n/2-w}(s^2)$ or $sg_{n/2-w+1}(s^2) + h_{n/2-w+1}(s^2)$ equals $(1 - \lambda_{n/2-w}s)sg_{n/2-w}(s^2)$ or $(1 - \mu_{n/2-w+1}s)h_{n/2-w+1}(s^2)$, respectively. Thus, since $g_{n/2-w}(s^2)$ and $h_{n/2-w+1}(s^2)$ are even and real, this means that the zeros are of the type stated in the Proposition. Working our way back up the Routh table, note that p(s) can be written as linear combinations of $g_{n/2-w}(s^2)$ and $h_{n/2-w+1}(s^2)$, or $g_{n/2-w+1}(s^2)$ and $h_{n/2-w+1}(s^2)$. Thus, p(s) also has the stated property. ■

Appendix - Proof of Lemma 3.4 (ii)

The goal is to find a curve C as in Figure 1 with a sufficiently small indentation about each $j\omega$-axis zero so that $N(s, \lambda) \neq 0$ for any $\lambda \in I$ and any $s \in C$.

Take any bounded interval $I \subset \mathbb{R}$. From part (i) of Lemma 3.4, $N(s, \lambda)$ has the same $j\omega$-axis zeros with the same multiplicities as p(s), say a total of M. Therefore, only n-M zeros of $N(s, \lambda)$ depend on λ. Let $\{z_i(\lambda), i=1, ...n - M\}$ denote these zeros of $N(s, \lambda)$. Note that the $z_i(\lambda)$ are a continuous function of λ except in a neighborhood of $\lambda = -a_n/a_{n-1}$.

Note that these zeros never cross the $j\omega$-axis, i.e., there is no $\lambda \in I$ and no $i \in \{1, ...n - M\}$ such that $Re(z_i(\lambda)) = 0$. In addition, note that at $\lambda = -a_n/a_{n-1}$, the degree of $N(s, \lambda)$ drops by precisely one. Thus, precisely one member of $\{z_i(\lambda), i=1, ...n - M\}$, say $z_J(\lambda)$, goes to infinity as $\lambda \rightarrow -a_n/a_{n-1}$, and it tends to infinity along the real axis, as an asymptotic expansion shows. This means that there is a closed interval $I_J \subset I$ with $|z_J(\lambda)| < \infty$ for all $\lambda \in I_J$, satisfying $\min_{i \in \{1, ...M\}} \min_{\lambda \in I}(|z_J(\lambda) - j\omega_i|) = \min_{i \in \{1, ...M\}} \min_{\lambda \in I_J}(|z_J(\lambda) - j\omega_i|)$. This latter expression has an achievable non-zero minimum, since the locus $z_J(I_J)$ is closed and bounded, and never crosses the imaginary axis. Call this minimum distance R_J.

So now consider $z(I) := \{ z_i(\lambda) | i=1, ...n - M, i \neq J, \lambda \in I \} \subset \mathbb{C}$, the locus of all off-imaginary-axis zeros of $N(s, \lambda)$ (except for $z_J(\lambda)$). Since I is closed and bounded, this locus is a closed and bounded set. Therefore, $\min_{i \in \{1, ...M\}} |z(I) - j\omega_i|$ is a finite, non-zero constant, denoted R. Let $R^* = \min(R_J, R) > 0$. We can thus make the radius of the indentation about each $j\omega$-axis zero $j\omega_i$ equal to $R^*/2$, which proves (ii).

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