CONTROL AND STABILIZATION OF FLEXIBLE SPACE STRUCTURES

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CONTROL AND STABILIZATION OF FLEXIBLE SPACE STRUCTURES

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Abstract

Research in the control and stabilization of large, flexible space structures has progressed rapidly in the last decade. It is the purpose of this thesis to investigate several control system issues arising from the use of distributed parameter formulations in the modelling of these flexible space structures.

The organization of this thesis is as follows. After a brief introduction in chapter one, chapter two focuses on flexible beam dynamics and control. Standard distributed parameter beam models are introduced. This is followed by the main result of the chapter, which gives sufficient conditions for a single collocated sensor/actuator pair to uniformly exponentially stabilize an undamped, Euler-Bernoulli beam.

In chapter three, a simple spacecraft consisting of a flexible beam attached to a rigid body is introduced. The kinematics and dynamics for this structure are derived. This is followed by several control laws which uniformly exponentially stabilize the pro-

posed spacecraft. The obtained results extend well-known results for modally truncated spacecraft models to the infinite dimensional models used here.

In chapter four, the attitude control problem is considered for the spacecraft configurations of chapter three. Attitude control laws are derived using Lyapunov techniques under various assumptions on the nonlinear, infinite dimensional spacecraft models. The control laws bear striking resemblance to well-known results obtained from finite dimensional, linearized spacecraft models.

In chapter five, the attitude control problem is again considered. Attitude control laws are again obtained for several spacecraft models derived in chapter three, but the control methodology differs from chapter four in that the methods of exact linearization are employed. The approach is novel because the methods of exact linearization are normally reserved for finite dimensional nonlinear systems, whereas the systems under consideration here are infinite dimensional, nonlinear systems. Implementation issues for the control laws are also discussed.

List of Symbols

c	distance from center of mass of the rigid body to the point of attachment of the flexible beam
$\bar{A}$	Closure of the operator A
$\mathbb{C}$	Complex numbers
$\times$	vector cross product
$\underline{a}^x$	Matrix representation of the vector cross product
D(A)	Domain of the operator A
$\partial(\bullet)$	Operator defined as $\left(\frac{\partial^4(\cdot)}{\partial z^4}, \frac{\partial^4(\cdot)}{\partial z^4}, -\frac{\partial^2(\cdot)}{\partial z^2}\right)^T$
$\partial'(\bullet)$	Operator defined as $\left(\frac{\partial^3(\cdot)}{\partial z^3}, \frac{\partial^3(\cdot)}{\partial z^3}, -\frac{\partial(\cdot)}{\partial z}\right)^T$
E	Young's Modulus
$\varepsilon(x, t)$	strain of the material at point x at time t
F_i	Force thruster at the tip of the beam with direction of thrust being $\underrightarrow{b_i}$
$\underline{F}_{bB}$	Force the beam applies to the body (written in body coordinates)
$\underline{F}_T$	Force due to the force thrusters on the rigid body (written in inertial coordinates)
H^k	$\{f \in L^2[c, c+L] \mid f, f' \ldots f^k \in L^2[c, c+L]\}$
η	Damping coefficient
I	Moment of inertia for the beam cross section
I_A	Inertia tensor associated with the momentum wheels
I_0	Inertia tensor associated with the rigid body
k	Diagonal matrix containing the damping coefficients for the flexible

	beam
L	length of flexible beam
$[\cdot, \cdot]$	$L^2[c, c+L]$ inner product
L^2	Abbreviation for $L^2[c, c+L]$, i.e. $\{f$ is measurable on $[c, c+L]$ and $\int_c^{c+L} \lvert f \rvert^2 \, dx < \infty\}$
$\hat{y}$	Laplace transform of y
m_B	Mass of the rigid body
$M(x, t)$	Internal resisting moment of the beam at point x at time t
$\underline{M}_{bB}$	Moment the beam applies to the body (written in body coordinates)
μ	Diagonal matrix containing the flexural rigidities for the flexible beam
$\lVert f \rVert$	$L^2[c, c+L]$ norm of f
$\lVert \underline{x} \rVert_{\mathbb{R}}$	Euclidean norm of the triple $\underline{x}$
O_B	Origin of the body frame
O_E	Origin of the inertial frame
$\underset{\rightarrow}{a} \perp \underset{\rightarrow}{b}$	The vector $\underset{\rightarrow}{a}$ is orthogonal (or perpendicular) to the vector $\underset{\rightarrow}{b}$
$u'(x, t)$	Partial differentiation of $u(x, t)$ with respect to the spatial variable x
$\mathbb{R}$	Real numbers
$p(x, t)$	external distributed force acting on the beam at point x at time t
ρ	mass per unit length of the beam
$\sigma(A)$	Spectrum of the operator A (finite or infinite dimensional)
$\sigma(x, t)$	stress of the material at point x at time t
$\underline{v}^T$	Transpose of the triple v
$\underline{\tau}$	torque due to the torque jets

$\underline{u}$	$\underline{u} = (u_1, u_2, z + u_3)$
$u(x, t)$	Transverse displacement of the beam at point x at time t
$V(x, t)$	Shear force at point x of the beam at time t
$\underline{\omega}$	angular velocity of the body frame with respect to the inertial frame
$\underline{\Omega}_W$	triple containing the momentum wheel angular velocity
$\underline{x}_t$	velocity of the vector $\underline{x}$ with respect to the body frame
$\dot{\underline{x}}$	velocity of the vector x with respect to the inertial frame
$\underline{\xi}$	Gibbs parameters specifying attitude
$\underline{y}$	Location of the center of mass of the body frame with respect to the inertial frame
Y	Direction cosine matrix

To the Reader

The numbering system contained in this thesis should be easy to follow. Theorems, Lemmas, Remarks, and Comments are all numbered by three digits: the first number indicates the chapter, the second number the section, and the third gives the number of the Theorem, Remark, etc. in that section. For example, Theorem 4.3.1 refers to the first Theorem, Remark, Lemma, or Comment to appear in Chapter 4, section 3. Similarly, equations are numbered by three digits. (2.1.2) refers to the second equation of Chapter 2, section 1. Finally, Figures are numbered by two digits: the first number indicates the chapter, and the second the number of the figure in that chapter. For example, Figure 2.5 is the fifth figure in Chapter 2.

Table of Contents

Chapter 1

Introduction

Research on the control and stabilization of large, flexible space structures has progressed rapidly in the last decade. This has been motivated by the many planned space missions over the next generation, e.g., space station, space telescope, SDI, etc.

Candidate designs for large space structures generally exhibit a high degree of flexibility. This is because the structural loads to be supported in space are generally not large, while there is a very high cost at lift-off for using materials heavier than necessary. Using state-of-the-art materials technology, the structures which meet these requirements are generally quite flexible. Unfortunately, elastic vibrations due to the flexible nature of the structure are highly undesirable for a number of missions: they affect telescope pointing accuracy, antenna beamwidths, delicate onboard instrumentation, etc. In addition, the lowest natural frequency of vibration often falls within the bandwidth of the attitude control system, thus reducing stability margins. Further, even environmental disturbances (gravity gradient, solar heating, etc.) are considered sufficient to excite structural bending for various missions.

Based on these considerations, designers are led to the concept of actively controlled large space structures. This is achieved by using a variety of sensors and actuators located about the structure, and operating through on-line computer controllers to tailor the performance and behavior of the system. One of the chief difficulties in designing these systems is that they are, in theory, distributed parameter systems, which means that the system model is infinite dimensional in nature. Combined with the coupling of these flexibilities with nonlinear dynamics, this means that the resulting system model is both nonlinear and infinite dimensional.

Not suprisingly, the design of nonlinear, infinite dimensional control systems is a difficult task. Because of the difficulties, most authors attempt to simplify the dynamical equations so that simpler control methods may be applied. Perhaps the most common design method in this vein for flexible structures is the reduced-order model approach. In this approach, the infinite dimensional model of the flexibility is approximated by a finite dimensional model. This is most commonly accomplished in practice by using a finite element technique. Once this reduced order model is obtained, a large body of finite dimensional control theory is available to the designer. One disadvantage of such a design technique is the difficulty in choosing the appropriate order for the reduced-order model. In fact, the unmodelled dynamics corresponding to the truncated modes (termed "spillover" by Balas in [Balas 1]) can actually cause instabilities in the closed loop system designed from the reduced order model. A great deal of research attention has been devoted to this problem. The interested reader can find many of the important references in this area in [Joh. 1], [Balas 1].

An alternate approach to this design procedure is to actually consider a distributed parameter model of the structure, and to design the control law based on this model. This of course has the advantage that the spillover problem is eliminated, but it has the disadvantage that the mathematical complexities are far greater. Nevertheless, in this thesis, a distributed parameter model of a particular flexible space structure will be employed, and the control laws will be derived from this framework.

The thesis is organized as follows:

In Chapter 2, we will consider the control of a flexible, cantilevered, beam modelled by the Euler-Bernoulli partial differential equations. The most common form of control employed for this beam is distributed control, with distributed sensing. This means that forces and moments are applied at every point of the beam (or intervals of the beam), with the readouts being the deflections or deflection velocities at every point of the beam (or intervals of the beam). In practice, however, distributed control and sensing is difficult to do. Motivated by these engineering consideration, many analysts have been studying the point sensing/actuation control of distributed parameter systems. In Chapter 2, we will consider some of the control theoretic implications of point sensing/actuation versus distributed control, and then consider the point/sensing actuation problem of an undamped beam. Sufficient conditions are developed to uniformly exponentially stabilize an undamped beam using a single sensor/actuator pair located in the *interior* of the beam. This extends a result in [Che. 2] which employs a single sensor/actuator pair at the tip of the beam to uniformly exponentially stabilize the undamped Euler-Bernoulli beam.

In Chapter 3, we first derive the equations of motion for a flexible satellite consisting of a rigid hub with an attached flexible appendage. The appendage will be modelled as an Euler-Bernoulli beam. Next, we will consider the stabilization, or detumbling, problem for this spacecraft. The stabilization problem consists of using the control actuators to stop the spacecraft from spinning, and to have the beam vibrations damp out. Two problems will be considered. First, it will be assumed that the flexible beam contains sufficient internal damping that no beam control is needed. For this particular problem, the control actuators mounted on the rigid portion of the spacecraft are sufficient to perform the stabilization maneuver. The second problem to be considered is when the flexible beam is regarded as having insufficient internal damping, assumed zero. In this case, actuators on the beam (which will be of the boundary variety) as well as actuators on the rigid hub will be needed to perform the maneuver. The validity of these control strategies is demonstrated by Lyapunov techniques. The obtained results show that many well-known stabilization schemes for finite dimensional (modally truncated) spacecraft can also be successfully applied to spacecraft modelled in an infinite dimensional way.

In Chapter 4, we consider the attitude control problem for the spacecraft model derived in Chapter 3. The attitude control problem consists of using the control actuators on the spacecraft to not only de-spin the spacecraft, but also to move the spacecraft into a specific orientation relative to another frame of reference. As in Chapter 3, we will first consider the problem when the beam damping is assumed significant, followed by the problem when the beam damping is assumed zero. The methods again employed are Lyapunov based, using a generalization of LaSalle's Invariance principle. The obtained results again show that many well-known attitude control schemes for finite dimensional (modally truncated) spacecraft can also be successfully applied to spacecraft modelled in an infinite dimensional way.

In Chapter 5, the attitude control problem for a flexible spacecraft is again considered. This chapter differs from Chapter 4 in that the techniques employed to derive the control laws are based on the methods of exact linearization. This means we seek to find nonlinear feedback and a nonlinear change of coordinates to transform the nonlinear system to an equivalent linear system. The approach employed here is novel in that the methods of exact linearization are normally applied to finite dimensional nonlinear systems, while the satellite dynamical equations here are infinite dimensional. Using the methods of exact linearization, attitude control laws for several satellite configurations are derived. Finally, implementation issues for these control laws are discussed.

Chapter 2

Flexible Beam Dynamics and Control

2.1 Introduction

In this chapter we will first derive the partial differential equations governing the motion of a uniform, flexible, beam. Two derivations are required in this thesis. In section 2.2, we derive the equations for the axial motions of a flexible beam which yields the well-known wave equation. In section 2.3 we derive the equations for transverse motions of a uniform beam. This gives the famous Euler-Bernoulli partial differential equations. The effect of beam damping on these models is discussed in both sections.

In section 2.4, the transverse control of a cantilevered, damped beam modelled by the Euler-Bernoulli partial differential equations is considered. It will be shown that if the control sensors and actuators are modelled as bounded linear operators (which is usually the case if the sensors and actuators are distributed elements), then the resulting transfer functions lie in the algebra $\hat{B}(0)$. This means that standard control factorization theory may be applied to obtain controllers for this *distributed* system.

In section 2.5, we consider the control of cantilevered, damped beam modelled by the Euler-Bernoulli partial differential equations, except we now remove the restriction that the beam is damped and that the control sensor/actuator models are bounded linear operators. In particular, we assume beam damping is zero, and that the control elements are *point* sensing/actuation. Theorem 2.5.2 gives sufficient conditions for a single sensor/actuator pair to uniformly exponentially stabilize the beam. This answers conclusively a conjecture of Chen in [Che. 2]. Finally, Lemma 2.5.10 investigates the structure of the modes of the undriven beam, in order to glean some general conclusions regarding sensor and actuator placement.

2.2 Models for Axial Motion of a Uniform Beam

Standard references for the material in this section and the following one are [Pop. 1], [Lan. 1]. Particular page locations of pertinent material for these references can be found in the text below.

Consider the *axial* motion of a uniform cantilevered (one end clamped, one end free) beam depicted in Figure 2.1. Assume that all motion takes place along the x axis.

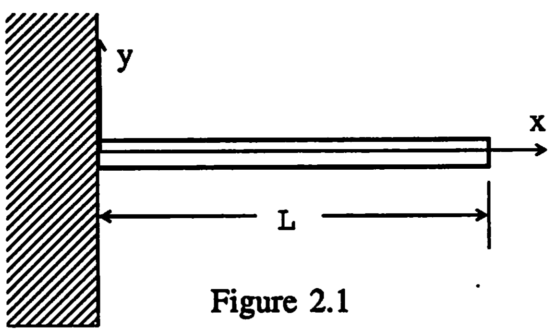

Figure 2.1

Let L denote the length of the beam, let x denote the position of the cross-section of the unstressed beam at point $x \in [0, L]$. Let $\varepsilon(x, t)$ denote the strain at cross-section x at time t, $\sigma(x, t)$ denote the stress at cross-section x at time t, and let $u(x, t)$ denote the *axial* displacement of the cross section x at time t . Finally, let A denote the cross sectional area of the beam (assumed constant for simplicity), let ρ denote the mass per unit length of the beam (again assumed constant), and let $p(x, t)$ denote the external body force per unit length applied along the x-axis.

Consider a differential element of the beam shown in Figure 2.2 Let $P(x, t)$ denote the axial force at cross section x at time t, which is numerically equal to the algebraic sum of all x-axis directed external forces acting on the isolated segment, but opposite in direction. Then balancing the forces on the differential element yields

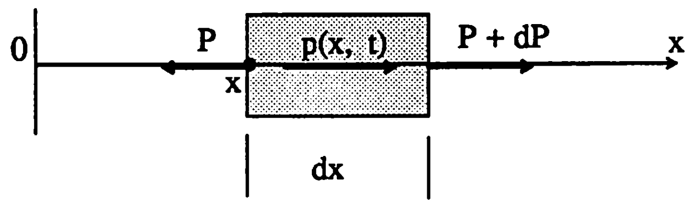

Figure 2.2

$$\Sigma F_x = 0 \rightarrow +, \qquad P + dP + p(x, t)dx - P - \rho \frac{\partial^2 u(x, t)}{\partial t^2}dx = 0. \qquad (2.2.1)$$

Dividing through by dx and taking limits as $dx \rightarrow 0$ yields

$$\frac{\partial P}{\partial x} + p(x, t) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \qquad (2.2.2)$$

Note that P(x, t) and $\sigma(x, t)$ are related by $P(x, t)A = \sigma(x, t)$. Since A is assumed constant, (2.2.2) yields

$$A \frac{\partial \sigma(x, t)}{\partial x} + p(x, t) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \qquad (2.2.3)$$

We now consider material properties. Assume that the beam material is isotropic, homogeneous, and further assume that the strains are small. (For a very brief introduction to properties of materials, see Appendix A. The reader is otherwise referred to [Pop. 1, Chapters 2-4], or [Lan. 1, Chapter 1].) If we further ignore damping and Poisson's effect, the stress $\sigma(x, t)$ and the strain $\varepsilon(x, t)$ are simply related by the Hooke's Law relation $\sigma(x, t) = E\varepsilon(x, t)$, where E is the Young's modulus of the material. Recall that by definition $\varepsilon(x, t) := \frac{\partial u(x, t)}{\partial x}$. Inserting these expressions in (2.2.3) yields

$$AE \frac{\partial^2 u(x, t)}{\partial x^2} + p(x, t) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \tag{2.2.4}$$

which the reader will recognize as the familiar wave equation, with p(x, t) acting as a source.

Finally, it is necessary to specify the boundary conditions for this beam configuration. Since the beam is clamped at x = 0, no axial displacement occurs there; hence, u(0, t) = 0, for all t > 0. If we assume that there is an applied force p(t) at the free end of the beam (x = L), this means $p(t) = P(L, t) = E\varepsilon(L, t)/A = E\frac{\partial u(L, t)}{\partial x}/A$. Thus the differential equation and boundary conditions for axial motion of a uniform beam with constant cross-section and no damping is given by

$$AE \frac{\partial^2 u(x, t)}{\partial t^2} + p(x, t) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \tag{2.2.5}$$

$$u(0, t) = 0, \quad \frac{E}{A} \frac{\partial u(L, t)}{\partial x} = p(t) \tag{2.2.6}$$

Suppose now that the beam possess Kelvin-Voight type damping. This means that each differential element can be considered to be connected to its neighbors by a parallel combination of a linear elastic spring, and a linear viscous dashpot ([Pop. 1, p. 116]). These assumptions yield a stress-strain relationship of the form

$$\sigma(x, t) = E\varepsilon(x, t) + \eta \frac{\partial \varepsilon(x, t)}{\partial t} \tag{2.2.7}$$

where η is the damping coefficient, assumed constant. Inserting this expression into the differential equation (2.2.3) and assuming $p(x, t) = 0$, then yields

$$AE \frac{\partial^2 u(x, t)}{\partial x^2} + A\eta \frac{\partial^3 u(x, t)}{\partial x^2 dt} = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \tag{2.2.8}$$

The boundary conditions for this partial differential equation are easily seen to be

$$u(0, t) = 0, \quad \frac{E}{A} \frac{\partial u(L, t)}{\partial x} + \frac{\eta}{A} \frac{\partial^2 u(L, t)}{\partial x \partial t} = p(t) \tag{2.2.9}$$

2.3 Models for Transverse Motion of a Uniform Beam

Now consider *transverse* motions of the uniform beam depicted in Figure 2.1. Assume all motion takes place in the x-y plane. Consider now an infinitesimal section of the beam at point x as shown in Figure 2.3.

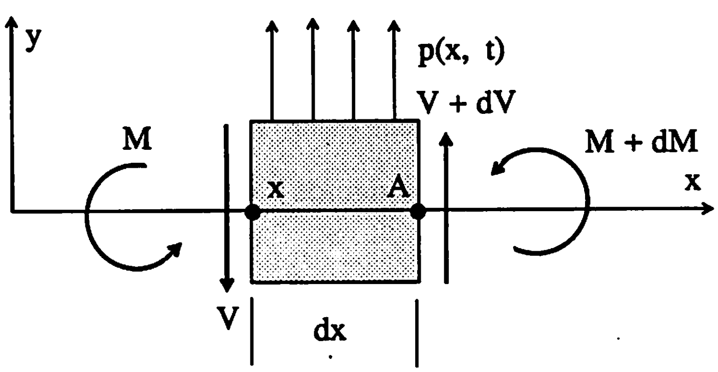

Figure 2.3

Let V(x, t) be the shear force at beam cross section x in the direction y. The shear force is numerically equal to the algebraic sum of all vertical external forces acting on the isolated segment, but is opposite in direction. Let M(x, t) denote the internal resisting moment acting on the beam cross section at x. The internal resisting moment is numerically equal to the algebraic sum of all vertical external moments acting on the isolated segment, but is opposite in direction. Let p(x, t) the distributed force acting on the beam in the y axis direction. Let ρ be the mass per unit length of the beam (assumed constant for simplicity) and let u(x, t) denote the displacement of the beam section from its neutral axis at point x. (Recall the neutral surface is the portion of the material free from stress and thus strain. The neutral axis is the intersection of the neutral surface with a right section of the beam. It will be shown in (2.3.8) that the neutral axis passes through the centroid of the cross-section area during pure bending.)

From the condition of equilibrium of vertical forces one obtains

$$\Sigma F_x = 0 \uparrow+, \qquad V + dV + p(x, t)dx - V - \rho \frac{\partial^2 u(x, t)}{\partial t^2}dx = 0. \qquad (2.3.1)$$

Dividing the expression by dx and taking limits as $dx \rightarrow 0$ yields

$$\frac{\partial V(x, t)}{\partial x} + p(x, t) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \qquad (2.3.2)$$

For equilibrium, the sum of the moments about point A also must be zero. So, upon noting that from point A the arm of the distributed force is dx/2, one has

$$\Sigma M_A = 0 \circlearrowleft +, \qquad M + dM + Vdx - M + (p(x, t)dx - \rho \frac{\partial^2 u(x, t)}{\partial t^2} dx)dx/2 = 0$$

Divide the expression by dx, and take limits as $dx \rightarrow 0$ to obtain

$$\frac{\partial M(x, t)}{\partial x} = -V(x, t) \qquad (2.3.3)$$

Note that the contribution due to the distributed forces is zero. Inserting (2.3.3) into (2.3.2) yields

$$-\frac{\partial^2 M(x,\ t)}{\partial x^2} + p(x, t) = \rho\ \frac{\partial^2 u(x,\ t)}{\partial t^2} \tag{2.3.4}$$

It thus remains to establish a relationship between the material properties and M(x, t). To do this, we now make the following kinematic assumption:

Assumption 2.3.1 - Plane sections through the beam taken normal to the x-axis remain plane during bending.

This assumption neglects shear deformation of the beam, but fortunately the deflections due to shear are small for a beam whose length is 2-3 times longer than the span.

So now consider the deformation of the beam in more detail. Suppose point A designates a point on the beam neutral axis, and the beam is deformed so that A $\rightarrow$ A$'$. (See Figure 2.4.)

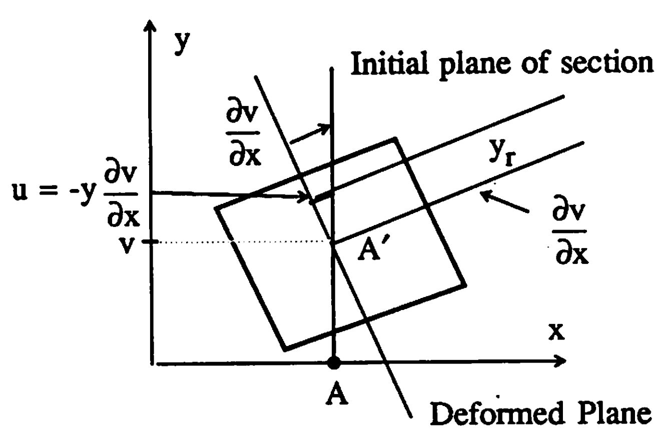

Figure 2.4

Let v(x, t) denote the deformation of point A to point A$'$ in the y direction. Let y_r denote the distance from A$'$ to the point of the beam under consideration along the direction of the deformed plane. The kinematic assumption shows that the plane through A in Figure 2.4 is transformed to a plane through A′. Note that the angle of rotation of the plane is simply the slope $\frac{\partial v(x, t)}{\partial x}$. Furthermore, since $\frac{\partial v(x, t)}{\partial x}$ is small, $\cos(\frac{\partial v(x, t)}{\partial x}) \cong 1$. Therefore, $u(x, t) = -y \frac{\partial v(x, t)}{\partial x}$. By definition of strain,

$$\varepsilon(x, t) := \frac{\partial u(x, t)}{\partial x} \tag{2.3.5}$$

Therefore,

$$\varepsilon(x, t) = -y \frac{\partial^2 v(x, t)}{\partial x^2} \tag{2.3.6}$$

Finally, we must relate the stress $\sigma(x, t)$ to the moment $M(x, t)$. Since this relationship depends on the modelled material properties, we will need to examine two cases. First we will consider the beam without damping, and afterward we will consider the beam modelled as having damping of the Kelvin-Voight type.

No Damping Present

First consider the case when damping is neglected. The kinematic assumption 2.3.1 implies that the strains in the beam vary linearly as their respective distances from the neutral axis. Thus

$$\sigma(x, t) = B(x)y \tag{2.3.7}$$

where y is the distance from the neutral surface, and B is a constant to be determined. If we assume that the beam is in simple bending due to an applied moment, then there are no forces acting in the x-direction. This implies that the net force in the x-direction on each cross-section must be zero. Hence,

$$\int_A \sigma(x, t)dA = 0 \;\Rightarrow\; \int_A y dA = 0 \;\Rightarrow\; \bar{y}A = 0 \qquad (2.3.8)$$

where $\bar{y}$ is the distance from the neutral axis to the centroid of the cross-sectional area A of the beam. This equation shows that $\bar{y}$ is actually zero (since A is not zero), which means that the neutral axis passes through the centroid of the cross-sectional area. Next, balancing moments on the cross-section yields

$$M(x, t) + \int_A \sigma(x, t)y dA = 0 \;\Rightarrow\; M = -B\int_A y^2 dA =: -BI \qquad (2.3.9)$$

where I is the moment of inertia of the cross-section about the z axis. Combining (2.3.9) with (2.3.7) then yields

$$\sigma(x, t) = -My/I \qquad (2.3.10)$$

Thus, the differential equation for the transverse motion of a uniform beam is given by

$$-Ey\,\frac{\partial^2 v(x, t)}{\partial x^2} = E\varepsilon(x, t) = \sigma(x, t) = -My/I, \quad \text{or}$$

$$EI\,\frac{\partial^2 v(x, t)}{\partial x^2} = M(x, t) \qquad (2.3.11)$$

This is the standard relationship between moment and curvature. Inserting (2.3.11) into (2.3.4) then yields

$$EI \frac{\partial^4 v(x, t)}{\partial x^4} + p(x, t) = \rho \frac{\partial^2 v(x, t)}{\partial t^2} \tag{2.3.12}$$

which is the famous Euler-Bernoulli model for transverse displacements of a uniform beam. As in the case of axial displacements, it is necessary to specify boundary conditions. The clamped conditions at x = 0 mean zero displacement and zero slope at x = 0; therefore, v(0, t) = 0, and $\frac{\partial v(0, t)}{\partial x} = 0$ for all t > 0. At the free-end of the beam (x = L), suppose there is an applied force F(t) and an applied moment M(t). Then, observing (2.3.11) we see that $EI\frac{\partial^2 v(L, t)}{\partial x^2} = M(t)$. (2.3.3) then shows $-EI \frac{\partial^3 v(L, t)}{\partial x^3} = F(t)$. Therefore, the standard Euler-Bernoulli beam model for transverse displacements is given by

$$EI \frac{\partial^4 v(x, t)}{\partial x^4} + p(x, t) = \rho \frac{\partial^2 v(x, t)}{\partial t^2} \tag{2.3.13}$$

$$EI\frac{\partial^2 v(L, t)}{\partial x^2} = M(t) \qquad -EI \frac{\partial^3 v(L, t)}{\partial x^3} = F(t) \tag{2.3.14}$$

Damping Present

Assume now that beam damping is present, and assume that it can be satisfactorily modelled as being of the Voight-Kelvin type as before. Then

$$\sigma(x, t) = E\varepsilon(x, t)y + \eta \frac{\partial \varepsilon(x, t)}{\partial t}y \tag{2.3.15}$$

First, note that equations (2.3.3), (2.3.4), and (2.3.6) are unaffected if damping is added. Therefore, $\varepsilon(x, t) = -y \frac{\partial^2 v(x, t)}{\partial x^2}$. To determine σ(x, t), note that the only change in (2.3.7) when damping is added is that $\sigma(x, t) = B(x, t)y$. Therefore, we can perform the same calculations as before (the balance of forces and moments on the cross-section) to obtain

$$\int_A \sigma(x, t)dA = 0 \Rightarrow \int_A ydA = 0 \Rightarrow \bar{y}A = 0 \Rightarrow \bar{y} = 0 \tag{2.3.16}$$

$$M(x, t) + \int_A \sigma(x, t)ydA = 0 \Rightarrow M(x, t) = -B(x, t)\int_A y^2dA =: -B(x, t)I \tag{2.3.17}$$

which are of exactly the same form as before. Using the relation $\varepsilon(x, t) = -y \dfrac{\partial^2 v(x, t)}{\partial x^2}$ we then obtain

$$M(x, t) = EI \frac{\partial^2 v(x, t)}{\partial x^2} + \frac{I\eta\partial^3 v(x, t)}{\partial x^2 \partial t} \tag{2.3.18}$$

Inserting this expression in (2.3.4) then yields

$$EI \frac{\partial^4 v(x, t)}{\partial x^4} + \eta I \frac{\partial^5 v(x, t)}{\partial x^4 \partial t} + p(x, t) = \rho \frac{\partial^2 v(x, t)}{\partial t^2} \tag{2.3.19}$$

with the easily derivable boundary conditions

$$\begin{aligned} EI\frac{\partial^2 v(L, t)}{\partial x^2} + \eta I \frac{\partial^3 v(L, t)}{\partial x^2 \partial t} &= M(t) \\ -EI \frac{\partial^3 v(L, t)}{\partial x^3} - \eta I \frac{\partial^3 v(L, t)}{\partial x^2 \partial t} &= F(t) \end{aligned} \tag{2.3.20}$$

2.4 Control of a Flexible Beam - Damping Present

Consider the cantilevered beam of Figure 2.5.

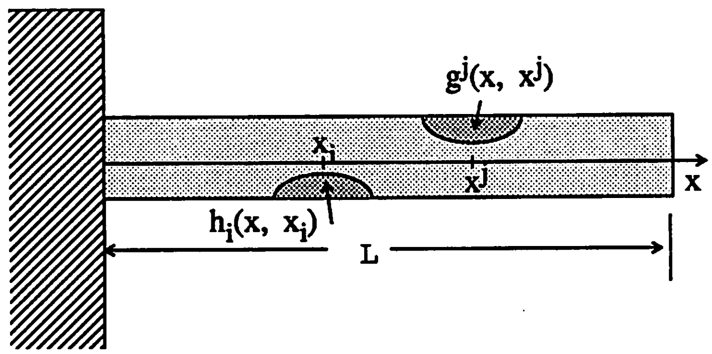

Figure 2.5

Consider the control of transverse beam motions. Let $f_j(t)$ be the control input force for the jth actuator whose influence function $g^j(x, x^j)$ is determined by the location x^j and the physical characteristics of the actuator. For example, $g^j(x, x^j)$ could be $\delta(x - x^j)$, where $\delta(x)$ is the dirac delta. This situation is called point actuation. More generally, $g^j(x, x^j)$ could be an L^2 function that approximates $\delta(x - x^j)$ in some sense. The corresponding differential equation is

$$\rho \frac{\partial^2 u(x, t)}{\partial t^2} + EI \frac{\partial^4 u(x, t)}{\partial x^4} + \eta I \frac{\partial^5 u(x, t)}{\partial x^4 \partial t} = \sum_{j=1}^{n_i} f^j(t) g^j(x, x^j) \qquad (2.4.1)$$

Assume the n_0 output sensors can be modelled by

$$y_i(t) = \int_0^L h_i(x, x_i) u(x, t) dx + \int_0^L m_i(x, x_i) \frac{\partial}{\partial t} u(x, t) dx \qquad i=1, \ldots n_0. \quad (2.4.2)$$

where x_i is the location of the ith sensor. This sensor essentially takes a weighted average of $u(x, t)$ and $\frac{\partial u(x, t)}{\partial t}$ over an interval containing x_i. Note that choosing $h_i(x, x_i) = \delta(x - x_i)$ means that we have point sensing, i.e., $y_i(t) = u(x_i, t)$.

(At this point, the reader should be somewhat familiar with the theory of linear state-space systems in infinite dimensional space (see, for example, [Bal. 1]). In particular, the reader is expected to have a brief understanding as to how the notion of a strongly continuous semigroup generalizes the notion of the state-transition matrix from finite dimensional spaces to infinite dimensional space. The reader unfamiliar with these concepts is referred to any number of excellent sources, including [Bal. 1], [Mac. 1], or [Paz. 1].

These input and output models can be used to write (2.4.1) and (2.4.2) in a state space form evolving on the Hilbert space $X = H_0^2 x L^2$, (see Appendix B for the definition of the Sobolev space $H_0^2[0, L]$) The inner product on X is given by the "energy" inner product

$$[f, g]_X = [(f_1, f_2)^T, (g_1, g_2)^T]_X = [f_1'', g_1''] + [f_2, g_2]$$

where $[\bullet, \bullet]$ denotes the ordinary inner product in $L^2[c, c+L]$. The state space system on the space X is given by

$$\begin{aligned} \dot{z} &= Az + Bf \\ y &= Cz \end{aligned} \tag{2.4.3}$$

where $A: D(A) \subset X \to X$, $B: \mathbb{R}^{ni} \to X$, $C: X \to \mathbb{R}^{no}$, are the linear operators defined by

$$A := \begin{bmatrix} 0 & 1 \\ \dfrac{-EI\partial^4(\cdot)}{\partial x^4} & \dfrac{-\eta I\partial^4(\cdot)}{\partial x^4} \end{bmatrix} \tag{2.4.4}$$

$$A : (\underline{u}, \underline{\dot{u}}) \to (\underline{\dot{u}}, EI\frac{\partial^4 u(x, t)}{\partial x^4} + \eta I \frac{\partial^5 u(x, t)}{\partial x^4 \partial t})^T$$

$$D(A) := \{(x_1\ x_2)^T | \ x_1 \in H_0^4, x_2 \in H_0^4,$$

$$x_1''(c+L) = x_2''(c+L) = 0, x_1'''(c+L)=x_2'''(c+L)=0, \} \qquad (2.4.5)$$

$$B:= \begin{bmatrix} 0 \\ g^1(x, x^1) \ldots g^{ni}(x, x^{ni}) \end{bmatrix}, B : (f_1, f_2, \ldots f_L) \to \sum_{i=1}^{L} f^i(t) g^j(x, x^j) \quad (2.4.6)$$

$$C_i := (\int_0^L h_i(x, x_i)(\cdot)dx, \int_0^L m_i(x, x_i)(\cdot)dx)^T, i=1, \ldots n_o \qquad (2.4.7)$$

$$C_i: (z_1,z_2) \to \int_0^L h_i(x, x_i)z_1(x, t)dx, + \int_0^L m_i(x, x_i)z_2(x, t)dx , i=1, \ldots n_o$$

Note that although A is defined on D(A), it is not defined on all of X. Furthermore, using the standard operator norm, A is an *unbounded* operator on X. B and C, however, are linear, bounded operators on X.

We next show that by defining the space, inner product and domain of A as above, the state space system is well-posed. (Recall that a linear partial differential equation need not admit a solution. See [John 1, Chapter 8] for an example.) To make precise the notion of well-posedness, we opt for the following definition which will be used throughout this thesis.

Definition 2.4.1 - Consider the following differential equation evolving on a Banach space X:

$$\dot{z} = Az \qquad z(0) = z_0 \qquad (2.4.8)$$

where A: $D(A) \subset X \to X$, is a (possibly) nonlinear unbounded operator and D(A) is the domain of the operator A. The system (2.4.8) is said to be well-posed if

(i) D(A) is dense in X;

(ii) For each $z_0 \in D(A)$, there exists a unique, continuously differentiable function z on $[0, \infty)$ (where the derivative is two-sided for $t > 0$, and on the right for $t = 0$) satisfying (2.4.8).

Comment 2.4.2 - Condition (i) of Definition 2.4.1 is required to insure that "nearly all" initial conditions are acceptable, while condition (ii) insures that unique solutions exist to the differential equation (2.4.8) for these initial conditions.

If A is a linear operator with non-empty resolvent in definition 2.4.1, then the system is well-posed if and only if A generates a strongly continuous semigroup [Paz. 1, Chapter 4, Thm. 1.3].

To show that the state-space system (2.4.3) is well-posed, we need one other standard result in the semi-group literature.

Theorem 2.4.3 - Suppose that A generates a strongly continuous semigroup on X. Suppose $B: X \to X$ is a bounded linear operator. Then $A + B$ generates a strongly continuous semigroup on X.

Proof of Theorem 2.4.3 - See [Paz. 1, Theorem 1.1, p. 76]. ∎

Comment 2.4.4 - This theorem merely states that if the system $\dot{z} = Az$ is well-posed, then the system $\dot{z} = (A + B)z$ is also well-posed if B is a *bounded* linear operator.

Using Theorem 5.A.2 of Chapter 5, the operator A of (2.4.4)generates a strongly continuous semigroup on X. Since we assume that $h_i(x, x_i)$ i=1, 2... n_o, $m_j(x, x_j)$ j=1, ... n_o, and $g^k(x, x^k)$ k=1, ... n_o are $L^2[0, L]$ functions, it is easy to show that B and C are *bounded* linear operators on X. Since B and C are bounded linear operators, and A generates a strongly continuous semigroup, Theorem 2.4.2 insures that the closed loop system

$$\dot{z} = (A + BFC)z \tag{2.4.9}$$

with $F \in \mathbb{R}^{nixno}$ is well-posed.

The assumptions on B and C guarantee other properties as well. We first need the following result from Appendix 5.A. (Theorem 5.A.2)

Theorem 2.4.4 - For the operator A defined in (2.4.4), the differential equation

$$\dot{z} = Az \qquad z_0 \in D(A) \tag{2.4.10}$$

has an exponentially stable solution $z(t) = T(t)z_0$. Moreover, the eigenvalues of the system are given in Figure 2.6.

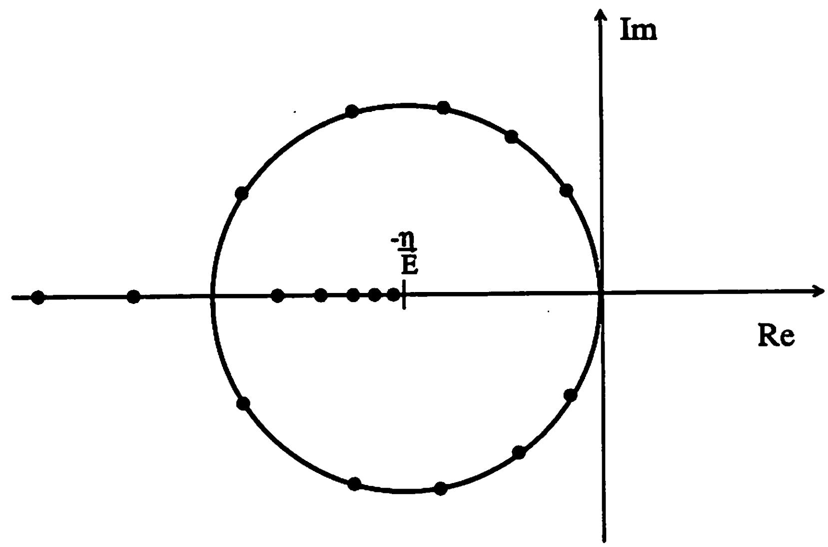

Figure 2.6 - Spectrum of the operator A given by (2.4.4)

Comment 2.4.5 - This result merely says that a beam with Voight-Kelvin damping is exponentially stable, and that the poles of the system (2.4.10) lie in the open left half plane as shown in Figure 2.6.

Since T(t) is exponentially stable, there is a M > 0 and a $\delta > 0$ such that $\|T(t)\| \leq Me^{-\delta t}$ for all t > 0. Since B and C are bounded linear operators, this means that

$$\|CT(t)B\| \leq \|C\| \, \|B\| \, Me^{-\delta t} \tag{2.4.11}$$

so that the state space system (2.4.3) is input-output stable. (If the boundedness conditions on C and B are removed, then examples exist which show that systems which are state-space stable are *not* necessarily I/O stable - a very unappealing result.) Note that the norm condition (2.4.11) implies that the Laplace transform of CT(t)B is analytic on the half-plane Re(s) > $-\delta$. This observation leads us to the following important result.

Theorem 2.4.6 - For the state-space system (2.4.3) with beam damping present (i.e., $\eta > 0$), the zero state input-output map $u \to y$ given by

$$y(t) = \int_0^t CT(t-\tau)Bu(\tau)d\tau \tag{2.4.12}$$

has its transfer function in the algebra $\hat{B}(0)$ [Vid. 1, p. 246]. A function $\hat{f} \in \hat{B}(\sigma_0)$ if for some $\sigma < \sigma_0$

(i) $\hat{f}(s)$ has a finite number of poles in the half plane (2.4.13)

$\mathbb{C}_\sigma = \{s \in \mathbb{C} \mid Re(s) \geq \sigma\}$;

(ii) The inverse transform includes - in addition to the exponentials due to the poles in $\mathbb{C}_\sigma$ -

$$f(t) = \begin{cases} 0 & t < 0 \\ \sum_{i=0}^{\infty} f_i\delta(t-t_i) + f_a(t) & t \geq 0 \end{cases} \tag{2.4.14}$$

where t_i are non-negative constants, $f_a(t)$ is a $L_1[0, \infty)$ function, and further,

$$\sum_{i=0}^{\infty} |f_i|\, e^{-\sigma t_i} < \infty \quad \text{and}$$
$$\int_0^t e^{-\sigma t}|f_a(t)|dt < \infty. \tag{2.4.15}$$

In fact, from Figure 2.6, we see that the state-space system has its transfer function in $\hat{B}(-\eta/E + \varepsilon)$, $\varepsilon > 0$, because there exists a $\sigma < -\eta/E + \varepsilon$ such there is a finite number of eigenvalues in $\mathbb{C}_\sigma$. An interesting point to note is that the transfer function is not in $\hat{B}(\sigma_0)$ for $\sigma \le -\eta/E$ because $-\eta/E$ is an accumulation point of eigenvalues.

The importance of the algebra $\hat{B}(0)$ is that every function $\hat{f} \in \hat{B}(0)$ has a coprime factorization [Vid. 2, p. 361], so one can then apply the very rich literature in algebraic control theory (see [Vid. 2], [Cal. 1]). It would take us too far afield to discuss the numerous aspects of algebraic control theory. Suffice it to say that the existence of coprime factorizations allows the determination of all stabilizing compensators, the determination of all compensators for tracking, disturbance rejection, the determination of robust stabilizing compensators, etc. We thus see that the fact that the transfer functions lie in $\hat{B}(0)$ is a very powerful condition indeed.

Before proceeding to the next section, it should also be noted that the fact that the transfer function lies in $\hat{B}(\sigma_0)$ insures that a *finite* dimensional stabilizing compensator exists ([Nett 1], [Vid. 1, p. 367]). This is perhaps not surprising since there are only a finite number of unstable poles that need to be stabilized, but the result is very important for engineering purposes. An example of a finite dimensional stabilizing compensator for the beam equations (2.4.1) is given in [Bon. 1]. The main difficulty is not in obtaining a finite dimensional compensator, but obtaining a compensator of reasonably low order. This is an area of current research interest.

2.5 Control of a Flexible Beam - No Damping Present

2.5.1 - Control Theoretic Implications of an Undamped Modelled Beam

Consider again the control of a cantilevered beam as shown in Figure 2.6. Suppose it is known that the beam damping is small, but the actual value is unknown. Since the beam damping is small, but unknown, it might be reasonable to model the beam as having no damping, i.e. $\eta = 0$ in (2.4.1).

Unfortunately, by modelling the beam in such a way we see that the transfer functions obtained from the state-space model (2.4.3) are *not* in the algebra $\hat{B}(0)$. This is because the undamped beam system is conservative, which means that the resulting transfer functions will have an infinite number of poles on the $j\omega$-axis, which violates condition (2.4.13) in the definition of $\hat{B}(0)$. It is currently unknown whether coprime factorizations exist for such systems. Thus, in order to control a beam modelled in such a way means we will have to find a stabilizing compensator based on different methods.

The first question is whether an exponentially stabilizing feedback compensator even exists for the system (2.4.1)-(2.4.2) when $\eta = 0$. (In this thesis, only exponential stabilizability will be considered. Other authors consider the strong stabilizability problem: find a control law so that the desired quantity goes to zero, but not necessarily exponentially.) If it is assumed that $h_i(x, x_i)$ $i=1, 2 \dots n_i$, $m_j(x, x_j)$ $j=1, \dots n_o$, and $g^k(x, x^k)$ $k=1, \dots n_o$ are $L^2[0, L]$ functions, then as before B and C defined by (2.4.6) and (2.4.7) are bounded linear operators on X. In addition note that C_i: $H^2 x L^2 \to L^2$ is a compact map since it is an integral map. Since B is a bounded linear map, and C is a compact linear map, this implies that BFC, for $F \in \mathbb{R}^{nixno}$, is a compact linear map [Die. 1, p. 317]. Because of these properties, there is *no* exponentially stabilizing static feedback compensator as the following theorem shows:

Theorem 2.5.1 [Gib. 2, Thm. 1, p. 312] Consider the differential equation

$$\ddot{x} + A_0 x(t) = B_1 \dot{x}(t) + B_2 x(t) \qquad x(0) = x_0 \in D(A_0) \tag{2.5.1}$$

where x(t) is in a real separable Hilbert space X, A_0 is a self-adjoint linear operator (see Definition B.3) from $D(A_0)$ (which is dense in X) to X, and B_1 and B_2 are compact linear operators mapping X to X. Then the solution to the differential equation (2.5.1), denoted $T(t)x_0$, is not uniformly exponentially stable, i.e., there is *no* M > 0 and δ > 0 such that

$$\|T(t)\| \leq Me^{-\delta t} \quad \text{for } t \geq 0. \tag{2.5.2}$$

(These conclusions were first pointed out in [Del. 1].)

One major reason for this negative result is that B and C are bounded operators, which restricts the feedback. To have any hope of obtaining an exponentially stabilizing compensator for an undamped beam, it is necessary to remove the restriction that $h_i(x, x_i)$ i=1, 2... n_i, $m_j(x, x_j)$ j=1, ... n_o, and $g^k(x, x^k)$ k=1, ... n_o are $L^2[0, L]$ functions.

Perhaps the simplest choice of unbounded linear operators for this problem is choosing a single sensor/actuator pair by $h(x, L) = \delta(x - L)$ and $m(x, L) = \delta(x - L)$, i.e., a point force at the tip of the beam of the beam, combined with point velocity sensing at the tip of the beam. (When the sensor and actuator are located at the same point, the sensor/actuator pair is commonly referred to in the literature as collocated. Thus, in this case the sensor/actuator pair is collocated at L.) It is easy to verify for this choice of $h_i(x, x_i)$ and $m_j(x, x_j)$ that C of (2.4.7) is an unbounded linear operator, while B of (2.4.6) is an unbounded linear operator by inspection (it is a delta function).

In fact, Chen et al [Che. 2] showed that by using this type of boundary control i.e., $x_j = L$, the resulting system is uniformly exponentially stable, in the sense that the beam deflections go to zero uniformly exponentially. The method of proof was by construction of an appropriate Lyapunov functional.

In the Chen paper, an open question was reported. Suppose a single collocated velocity sensor/force actuator pair is located interior to the beam, rather than at the tip. Is the system (2.4.1) also exponentially stabilized? In this section, Theorem 2.5.2, this question is partially answered. It will be seen in Theorem 2.5.2 that the system is uniformly exponentially stable if the position of the collocated pair, denoted x_1, is not located at a node of the original undriven beam, and if x_1/L is a rational number. The method of proof will be by careful eigenvalue and eigenfunction analysis. Using a theorem of Huang, [Hua. 2], these conditions for exponential stability will be derived. Afterwards, Lemma 2.5.10 will investigate the structure of these undesirable sensor/actuator locations, in order to glean some general conclusions regarding sensor and actuator placement.

The reader will undoubtedly feel that much simpler methods can be applied to prove Theorem 2.5.2. Therefore, in Section 2.5.3 we will briefly comment on the inadequacies of classical system theoretic approaches (passivity, etc.) in conjunction with this particular problem.

2.5.2 Uniform Exponential Stabilization of an Undamped Beam

Consider the cantilevered beam of Figure 2.7, with an applied point force at $x = x_1$.

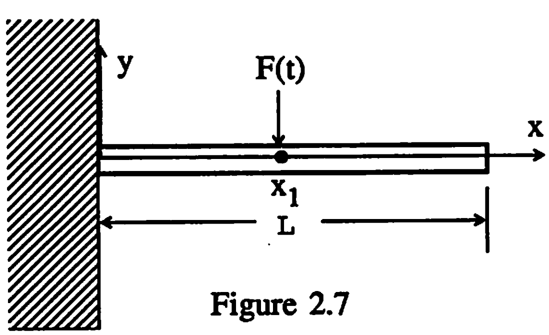

Figure 2.7

Assume for simplicity that ρ, E, and I are all unity. Also assume that no damping is present, i.e. $\eta = 0$ in (2.4.4). Let $h(x, x_1) = \delta(x - x_1)$ and $m(x, x_1) = K\delta(x - x_1)$

where $x_1 \in [0, L]$; hence we have a single collocated velocity sensor/force actuator pair located at the interior point x_1, with a control gain of K. The goal is to derive conditions on x_1 to insure that the beam deflections go to zero uniformly exponentially.

Since a point force is applied at $x = x_1$, there will be a "kink" in the shear force at that point, which implies that the solution to this problem must be broken into two parts. One differential equation will consider $x \in [0, x_1]$, and the other $x \in [x_1, L]$. Proceeding along these lines, let $y_1(x, t)$ denote the transverse deflection of the beam at time t for $x \in [0, x_1)$, and let $y_2(x, t)$ denote the transverse deflection of the beam at time t for $x \in (x_1, L]$. Using the formulation for the beam given in section 2.3, the differential equations for this configuration is given by

$$\ddot{y}_1(x, t) + y_1''''(x, t) = 0 \quad x \in (0, x_1) \tag{2.5.3}$$

$$\ddot{y}_2(x, t) + y_2''''(x, t) = 0 \quad x \in (x_1, L) \tag{2.5.4}$$

where $\dot{}$ denotes the partial derivative with respect to time, and $'$ denotes the partial derivative with respect to the spatial variable x. The initial conditions for this configuration are

$$y_1(x, 0) = \phi_1(x) \quad \dot{y}_1(x, 0) = \Psi_1(x) \quad x \in [0, x_1]. \tag{2.5.5}$$

$$y_2(x, 0) = \phi_2(x) \quad \dot{y}_2(x, 0) = \Psi_2(x) \quad x \in [x_1, L]. \tag{2.5.6}$$

Assume for simplicity that the initial conditions are C^∞ functions of x. The resulting boundary conditions are

$$y_1(x_1, t) = y_2(x_1, t) \qquad y_1'(x_1, t) = y_2'(x_1, t) \tag{2.5.7}$$

$$y_1''(x_1, t) = y_2''(x_1, t) \qquad y_1'''(x_1, t) = y_2'''(x_1, t) + K\dot{y}_1(x_1, t) \tag{2.5.8}$$

$$y_2''(L, t) = 0 \qquad y_2'''(L, t) = 0 \tag{2.5.9}$$

$$y_1(0, t) = 0 \qquad\qquad y_1'(0, t) = 0. \tag{2.5.10}$$

for all $t \geq 0$. The boundary conditions (2.5.7) indicate that at the location of the sensor/actuator pair the displacement and slope are unchanged. The first boundary condition of (2.5.8) indicates that the moments are unchanged, while the second shows that the shear force has a jump discontinuity occurring since the actuator is a point force actuator. Finally, the boundary conditions (2.5.9) and (2.5.10) are simply the boundary conditions for the cantilevered beam configuration. (See (2.3.14).)

The unbounded operator associated with system (2.5.3)-(2.5.4) is given by

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-\partial^4(\cdot)}{\partial z^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-\partial^4(\cdot)}{\partial z^4} & 0 \end{bmatrix} \tag{2.5.11}$$

Let the underlying state space for this operator be defined as

$$X := \{(y_1, y_2, y_3, y_4)^T \in H_0^2 \times L^2 \times H^2 \times L^2 \mid y_1(x_1) = y_3(x_1),\ y_1'(x_1) = y_3'(x_1)\} \tag{2.5.12}$$

(Thus, (2.5.3)-(2.5.4) can be written in compact state space form as $\dot{y} = Ay$, where $y = (y_1, \dot{y}_1, y_2, \dot{y}_2)^T$.) Let the inner product on X be $[f, g]_E$ defined as

$$[f, g]_E := [f_1'', g_1'']_1 + [f_2, g_2]_1 + [f_3'', g_3'']_2 + [f_4, g_4]_2 \tag{2.5.13}$$

where $[\cdot, \cdot]_1$ is the complex $L^2[0, x_1]$ inner product, and $[\cdot, \cdot]_2$ is the complex $L^2[x_1,$

L] inner product. Finally, let the domain of the unbounded linear operator A in (2.5.11) be defined as

$D(A) := \{(y_1, y_2, y_3, y_4)^T \in H_0^4 \times H_0^2 \times H^4 \times H^2 \mid y_1(x_1) = y_3(x_1), y_1'(x_1) = y_3'(x_1),$ $y_1''(x_1) = y_3''(x_1), y_1'''(x_1) = y_3'''(x_1) + Ky_2(x_1), y_3''(L) = y_3'''(L) = 0, y_2(x_1) =$ $y_4(x_1), y_2'(x_1) = y_4'(x_1)\}$.(2.5.14)

This leads us to the main theorem of the Chapter.

Theorem 2.5.2 - Consider the cantilevered beam equations (2.5.3) and (2.5.4) together with initial conditions (2.5.5)-(2.5.6) and the boundary conditions (2.5.7)-(2.5.10). Let K of (2.5.8) satisfy K > 0. Then the system is uniformly exponentially stable if x_1 is not located at a node of the original, undriven beam system (2.3.13), (2.3.14) (where $E=I=\rho=1$ and $p(x, t) = M(t) = F(t) = 0$), and x_1/L is a rational number.

Comment 2.5.3 - If we choose $x_1 = L$, we obtain beam boundary control as in [Che. 2]. Since it is easy to show that L is not a node of the undriven system, Theorem 2.5.5 shows that for such a placement of the sensor actuator pair, uniform exponential stability results. This agrees with [Che. 2].

The proof of Theorem 2.5.2 is very similar to a proof of a result given in [Che. 3]. The interested reader can find similar calculations to those below if difficulties arise in following the rather complex derivations given here.

To determine the exponential stability of the system, we need the following theorem due to Huang [Hua. 2, Thm. 3, p. 51]:

Theorem 2.5.4 - Let T(t) be a strongly continuous (linear) semigroup in a Hilbert space satisfying

$$\|T(t)\| \leq C \tag{2.5.15}$$

for all $t \geq 0$ and for some $C > 0$. Then T(t) is exponentially stable if and only if

$$\{i\omega \mid \omega \in \mathbb{R}\} \subset \rho(A), \quad \text{the resolvent set of A; and,} \tag{2.5.16}$$

$$B := \sup\{\|(i\omega I - A)^{-1}\|\} < \infty \tag{2.5.17}$$

are satisfied.

Thus, to prove Theorem 2.5.2 we only need to show that the conditions of Theorem 2.5.4 are satisfied. This is done by straightforward, although extremely tedious, calculation below. To perform this task, we will need several preliminary lemmas.

The first condition which must be shown is (2.5.15), which states that the semigroup generated by the operator A is a contraction semigroup (see Appendix B, Definition B.1). Because the proof of this result is unrelated to the remainder of the proof, the details are left to Appendix 2.A. The method of proof is by use of the Lumer-Phillips Theorem [Paz. 1, Chapter 1, Thm. 4.3].

Thus, to apply Theorem 2.5.4 to Theorem 2.5.2, it remains to show that the resolvent conditions (2.5.16) and (2.5.17) in Theorem 2.5.4 are satisfied. (The term resolvent refers to the operator λI - A. This is conventional terminology in the semigroup literature.) So now consider the resolvent equation for the operator A defined in (2.5.11):

$$\left[\lambda I - \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-\partial^4(\cdot)}{\partial z^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-\partial^4(\cdot)}{\partial z^4} & 0 \end{bmatrix}\right]\begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \Psi_1 \\ \phi_2 \\ \Psi_2 \end{bmatrix} \tag{2.5.18}$$

together with the boundary conditions

$$y_1(x_1) = y_2(x_1) \qquad y_1'(x_1) = y_2'(x_1) \tag{2.5.19}$$

$$y_1'''(x_1) = y_2'''(x_1) + Kz_1(x_1) \qquad y_1''(x_1) = y_1''(x_1) \tag{2.5.20}$$

$$y_2''(L) = 0, \quad y_2'''(L) = 0, \quad y_1(0) = 0, \quad y_1'(0) = 0. \tag{2.5.21}$$

If we eliminate z_1 and z_2 from the above equations we can thus rewrite (2.5.18) as

$$-\tau^4 y_1(x) + y_1''''(x) = i\tau^2\phi_1(x) + \Psi_1(x) \qquad 0 < x < x_1 \tag{2.5.22}$$

$$-\tau^4 y_2(x) + y_2''''(x) = i\tau^2\phi_2(x) + \Psi_2(x) \qquad x_1 < x < L \tag{2.5.23}$$

together with the boundary conditions

$$y_1(x_1) = y_2(x_1) \qquad y_1'(x_1) = y_2'(x_1) \tag{2.5.24}$$

$$y_1'''(x_1) = y_2'''(x_1) + Ki\tau^2 y_1(x_1) + \phi_1(x_1) \qquad y_1''(x_1) = y_1''(x_1) \tag{2.5.25}$$

$$y_2''(L) = 0, \quad y_2'''(L) = 0, \quad y_1(0) = 0, \quad y_1'(0) = 0. \tag{2.5.26}$$

The first thing we will show is that the operator A given by (2.5.18) has no $j\omega$-axis eigenvalues unless x_1 is located at a node of the undriven beam system. This is the content of the following theorem.

Theorem 2.5.5 - The operator A defined by (2.5.11) has a purely imaginary eigenvalue if and only if x_1 is located at a node for some mode for the undriven system (2.3.13)-(2.3.14) (where $E=I=\rho=1$ and $p(x, t) = M(t) = F(t) = 0$).

Proof of Theorem 2.5.5 - $\Leftarrow$ Suppose x_1 is located at a node of the undriven beam system. By definition, this means that there is some (real) modal frequency τ' satisfying $\cos\tau' L \cosh\tau' L = -1$ such that

$$(\sin\tau' L - \sinh\tau' L)(\sin\tau' x_1 - \sinh\tau' x_1) + (\cos\tau' L + \cosh\tau' L)(\cos\tau' x_1 - \cosh\tau' x_1) = 0. \tag{2.5.27}$$

Consider the mode $Y(x)$ associated with τ'

$$Y(x) := (\sin\tau' L - \sinh\tau' L)(\sin\tau' x - \sinh\tau' x) + (\cos\tau' L + \cosh\tau' L)(\cos\tau' x - \cosh\tau' x) \tag{2.5.28}$$

Note that $Y(x)$ satisfies the boundary conditions

$$Y''(L) = 0 \qquad Y'''(L) = 0 \tag{2.5.29}$$

$$Y(0) = 0 \qquad Y'(0) = 0. \tag{2.5.30}$$

Since $Y(x)$ is also a C^∞ function of x, it satisfies

$$Y(x_1-) = Y(x_1+) \qquad Y'(x_1-) = Y'(x_1+) \tag{2.5.31}$$

$$Y''(x_1-) = Y''(x_1+) \qquad Y'''(x_1-) = Y'''(x_1+) \tag{2.5.32}$$

In addition, since x_1 is a node for the modal frequency τ', this implies that $Y(x_1) = 0$, which also means that $i\tau'^2Y(x_1) = 0$. Therefore,

$$Y'''(x_1-) = Y'''(x_1+) + i\tau'^2Y(x_1) \tag{2.5.33}$$

Using $Y(x)$, define the functions $y_{10}(x)$, $z_{10}(x)$, $y_{20}(x)$, $z_{20}(x)$ by

$$\begin{aligned} y_{10}(x) &:= Y(x) \qquad & z_{10}(x) &:= i\tau'^2Y(x) \quad \text{for } x \in [0, x_1] \\ y_{20}(x) &:= Y(x) \qquad & z_{20}(x) &:= i\tau'^2Y(x) \quad \text{for } x \in [x_1, L] \end{aligned}$$

By computation, $A(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T = (i\tau'^2y_{10}(x), -\tau'^4y_{10}(x), i\tau'^2y_{20}(x), -\tau'^4y_{20}(x))^T = i\tau'^2(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T$. Since $(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T$ satisfies all the boundary conditions, it is thus an eigenfunction of the operator A corresponding to the eigenvalue $i\tau'^2$. Thus $i\tau'^2$ is an imaginary eigenvalue of the operator A.

$\underline{\Rightarrow}$ - Suppose A has an eigenvalue $\lambda_0 = i\tau_0^2$ for some $\tau_0 \in \mathbb{R}$, $\tau_0 \neq 0$ with corresponding eigenfunction $(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T \in D(A)/\{(0, 0, 0, 0)\}$. Then, by definition, there is a $\tau_0 \in \mathbb{R}$ such that $A(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T = i\tau_0^2(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T$. Further, by definition, we know that $(y_{10}(x), z_{10}(x), y_{20}(x), z_{20}(x))^T$ satisfies the boundary conditions contained in D(A), (2.5.14). Now define $y_1(x, t)$ and $y_2(x, t)$ by

$$y_1(x, t) := \exp(i\tau_0^2t)y_{10}(x) \tag{2.5.34}$$

$$y_2(x,t) := \exp(i\tau_0{}^2 t) y_{20}(x) \tag{2.5.35}$$

Differentiating (2.5.34) and (2.5.35) twice with respect to time and using the relationships from the eigenfunction equation we see that

$$\ddot{y}_1(x,t) + y_1''''(x,t) = 0 \quad x \in (0, x_1) \tag{2.5.36}$$

$$\ddot{y}_2(x,t) + y_2''''(x,t) = 0 \quad x \in (x_1, L) \tag{2.5.37}$$

where $\dot{}$ denotes the partial derivative with respect to time, and $'$ denotes the partial derivative with respect to the spatial variable x. Also note that the energy

$$E(t) := \frac{1}{2}\int_0^{x_1} (|\dot{y}_1(x,t)|^2 + |y_1''(x,t)|^2)\, dx + \frac{1}{2}\int_{x_1}^{L} (|\dot{y}_2(x,t)|^2 + |y_2''(x,t)|^2)\, dx \tag{2.5.38}$$

is *constant.* The term energy is used because the terms involving time derivatives represent the total kinetic energy of the system, while the two terms having $'$ in them represent the potential energy of the beam configuration. Next, differentiate E(t) with respect to time to yield

$$\dot{E}(t) = \int_0^{x_1} (\dot{y}_1\, \ddot{y}_1 + y_1''\, \dot{y}_1'')\, dx + \int_{x_1}^{L} (\dot{y}_2\, \ddot{y}_2 + y_2''\, \dot{y}_2'')\, dx \tag{2.5.39}$$

Use the differential equation (2.5.36) and (2.5.37), and integrate by parts to obtain

$$\dot{E}(t) = \int_0^{x_1} (-\dot{y}_1 y_1'''' + y_1'' \dot{y}_1'')\,dx + \int_{x_1}^{L} (-\dot{y}_2 y_2'''' + y_2'' \dot{y}_2'')\,dx \quad (2.5.40)$$

$$= -\dot{y}_1 y_1''' \Big|_0^{x_1} - \dot{y}_2 y_2''' \Big|_{x_1}^{L} + \int_0^{x_1} (\dot{y}_1' y_1''' + y_1'' \dot{y}_1'')\,dx + \int_{x_1}^{L} (\dot{y}_2' y_2''' + y_2'' \dot{y}_2'')\,dx \quad (2.5.41)$$

Use the boundary conditions in (2.5.14) and integrate by parts again to obtain

$$= -\dot{y}_1(x_1)\, y_1'''(x_1) - \dot{y}_1' y_1'' \Big|_0^{x_1} + \int_0^{x_1} (-\dot{y}_1'' y_1'' + y_1'' \dot{y}_1'')\,dx \; +$$

$$+ \dot{y}_2(x_1)\, y_2'''(x_1) - \dot{y}_2' y_2'' \Big|_0^{x_1} + \int_{x_1}^{L} (-\dot{y}_2'' y_2'' + y_2'' \dot{y}_2'')\,dx$$

$$= -\dot{y}_1(x_1)\, y_1'''(x_1) + \dot{y}_1'(x_1)\, y_1''(x_1) + \dot{y}_2(x_1)\, y_2'''(x_1) - \dot{y}_2'(x_1)\, y_2''(x_1)$$

or,

$$\dot{E}(t) = -\dot{y}_1(x_1)(y_1'''(x_1) - y_2'''(x_1)) \quad (2.5.42)$$

Use the boundary condition in (2.5.14) at x_1 to obtain

$$= -(K\dot{y}_1(x_1, t))^2 = 0. \quad (2.5.43)$$

where the last "= 0" term comes from the fact that energy is constant. Thus, $\dot{y}_1(x_1, t) = 0$. Since $\dot{y}_1(x_1, t) = i\tau_0^2\exp(i\tau_0^2 t) y_{10}(x_1)$, this implies that $y_{10}(x_1) = 0$. The system of equations (2.5.36) - (2.5.37) and (2.5.14) thus becomes

$$-\tau_0^4 y_{10}(x) + y_{10}''''(x) = 0 \quad x \in (0, x_1) \quad (2.5.44)$$

$$-\tau_0^4 y_{20}(x) + y_{20}''''(x) = 0 \quad x \in (x_1, L) \quad (2.5.45)$$

$$y_{10}(x_1) = y_{20}(x_1) \qquad y_{10}'(x_1) = y_{20}'(x_1) \tag{2.5.46}$$

$$y_{10}''(x_1) = y_{20}''(x_1) \qquad y_{10}'''(x_1) = y_{20}'''(x_1) \tag{2.5.47}$$

$$y_{20}''(L) = 0, \quad y_{20}'''(L) = 0, \quad y_{10}(0) = 0, \quad y_{10}'(0) = 0. \tag{2.5.48}$$

$$y_{10}(x_1) = 0 \tag{2.5.49}$$

Think of this problem as being the conditions (2.5.44) - (2.5.48), with the additional constraint $y_{10}(x_1) = 0$. The equations (2.5.44) - (2.5.48) are simply the eigenfunction equations for an undriven Euler - Bernoulli beam of length L split into two parts. The solutions to these equations are therefore (see [Mei. 2, p. 162])

$$\begin{aligned} Y_1(x) := {} & (\sin\tau_0 L - \sinh\tau_0 L)(\sin\tau_0 x - \sinh\tau_0 x) + \\ & (\cos\tau_0 L + \cosh\tau_0 L)(\cos\tau_0 x - \cosh\tau_0 x) \quad \text{for } x \in [0, x_1] \text{ and} \end{aligned}$$

$$\begin{aligned} Y_2(x) := {} & (\sin\tau_0 L - \sinh\tau_0 L)(\sin\tau_0 x - \sinh\tau_0 x) + \\ & (\cos\tau_0 L + \cosh\tau_0 L)(\cos\tau_0 x - \cosh\tau_0 x) \quad \text{for } x \in [x_1, L] \end{aligned}$$

where τ_0 satisfies the equation $\cos\tau_0 L \cosh\tau_0 L = -1$. The additional constraint $y_{10}(x_1) = 0$ implies that $Y_1(x_1) = 0$. This implies by definition that x_1 is a node corresponding to the modal frequency τ_0. This proves the Theorem. ■

Recall that we are trying to show that conditions (2.5.16) and (2.5.17) in Theorem 2.5.4 are satisfied. Theorem 2.5.5 shows that $\{i\omega \mid \omega \in \mathbb{R}\}$ is not in the point spectrum of A provided that x_1 is not located at a node for any mode of the undriven beam system. However, $\{i\omega \mid \omega \in \mathbb{R}\}$ may lie in the continuous spectrum of A. In other words, although $(i\omega I - A)^{-1}$ exists for all $\omega \in \mathbb{R}$ if x_1 is not located at a node for any mode of the undriven beam system, $(i\omega I - A)^{-1}$ may not be a bounded linear operator.

The next step in the proof is to show that the eigenvalues of A are isolated in the complex plane, with implies that there are no points of accumulation in the finite part of the complex plane. This step is necessary for the asymptotic analysis that will follow.

Lemma 2.5.6 - A^{-1} exists and is a compact operator on X. Furthermore, $\sigma(A)$ consists entirely of isolated eigenvalues.

Proof of Lemma 2.5.6 - To show that A^{-1} exists we must show that equation (2.5.18) has a solution for each $(\phi_1, \Psi_1, \phi_2, \Psi_2)$ in X when $\lambda = 0$. Therefore, set $\lambda = 0$ in equation (2.5.18). Note that the differential equation formulation (2.5.22)-(2.5.26) is an *ordinary* differential equation in x. These equations can be explicitly solved (using a state space method, for example) to yield

$$y_1(x) = y_1''(0)x^2/2! + y_1'''(0)x^3/3! + \int_0^x \frac{(x-\sigma)^3}{3!}\Psi_1(\sigma)d\sigma \qquad (2.5.50)$$

$$y_2(x) = y_2(x_1) + y_2'(x_1)(x - x_1) + y_2''(x_1)(x - x_1)^2/2!$$
$$+ y_2'''(x_1)(x - x_1)^3/3! + \int_{x_1}^x \frac{(x - x_1 - \sigma)^3}{3!}\Psi_2(\sigma)d\sigma \qquad (2.5.51)$$

To determine the constants $y_1''(0)$, $y_1'''(0)$, $y_2(x_1)$, $y_2'(x_1)$, $y_2''(x_1)$, and $y_2'''(x_1)$, we use the boundary conditions (2.5.24) - (2.5.26):

$$y_1(x_1) = y_2(x_1): \quad y_1''(0)x_1^2/2! + y_1'''(0)x_1^3/3! + \int_0^{x_1} \frac{(x-\sigma)^3}{3!}\Psi_1(\sigma)d\sigma = y_2(x_1)$$

$$y_1'(x_1) = y_2'(x_1): \quad y_1''(0)x_1 + y_1'''(0)x_1^2/2 + \int_0^{x_1} \frac{(x-\sigma)^2}{2!}\Psi_1(\sigma)d\sigma = y_2'(x_1)$$

$$y_1''(x_1) = y_1''(x_1): \quad y_1''(0) + y_1'''(0)x_1 + \int_0^{x_1}(x - \sigma)\,\Psi_1(\sigma)d\sigma = y_2''(x_1)$$

$$y_1'''(x_1) = y_2'''(x_1) + K\phi_1(x_1): \quad y_1'''(0) + \int_0^{x_1} \Psi_1(\sigma)d\sigma = y_2'''(x_1) + K\phi_1(x_1)$$

$$y_2''(L) = 0: \quad y_2''(x_1) + y_2'''(x_1)(L - x_1) + \int_{x_1}^{L} (L - x_1 - \sigma)\Psi_2(\sigma)d\sigma = 0$$

$$y_2'''(L) = 0: \quad y_2'''(x_1) + \int_{x_1}^{L} \Psi_2(\sigma)d\sigma = 0 \tag{2.5.52}$$

Note these equations can be easily solved by back substitution, starting with the last equation, (2.5.52). Thus, using these coefficients in (2.5.50) - (2.5.51) we have obtained $y_1(x)$ and $y_2(x)$. Furthermore, from the resolvent equation we see that $z_1(x) = \phi_1(x)$, and $z_2(x) = \phi_2(x)$. Thus, A^{-1} exists. Note that A^{-1} by definition maps X into $D(A) \subset H_0^4 \times H_0^2 \times H^4 \times H^2$. By the Sobolev embedding theorem [Paz. 1. p. 208] we thus see that A^{-1} is a compact map. Finally, since A^{-1} is compact it has countably many eigenvalues, with zero being the only accumulation point of these eigenvalues [Hut. 1, p. 188]. Thus, A has only countably many eigenvalues with no accumulation points in the finite part of the complex plane. This proves the lemma. ∎

We are now ready to proceed to the most difficult (and most tedious) part of the proof, which shows that the resolvent estimate (2.5.17) holds for $\lambda = i\omega$ sufficiently large, provided that the assumptions on x_1 are met.

Lemma 2.5.7 - Suppose x_1 is not located at a node of the original, undriven beam system (2.3.13), (2.3.14), and x_1/L is rational. Then the resolvent estimate (2.5.17) holds for $\lambda = i\omega$, $\omega \in \mathbb{R}$, provided that ω is sufficiently large.

Proof of Lemma 2.5.7 - Let $\lambda = i\omega =: i\tau^2$, $\omega \in \mathbb{R}$, $\tau \in \mathbb{R}$. Without loss of generality, assume that ω is positive and that the defined variable τ is also positive. Note that the differential equation formulation (2.5.22)-(2.5.26) is an *ordinary* differential equation in x. Using a state space formulation, equations (2.5.22)-(2.5.26) can be rewritten as

$$\begin{pmatrix} y_1' \\ y_1'' \\ y_1''' \\ y_1'''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tau^4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_1' \\ y_1'' \\ y_1''' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (i\tau^2\phi_1 + \Psi_1) \end{pmatrix} \tag{2.5.53}$$

or, $$\underline{y}_1' = A\underline{y}_1 + \underline{b} \tag{2.5.54}$$

where $\underline{y}_1(x, \tau) := (y_1(x), y_1'(x), y_1''(x), y_1'''(x))^T$, and the definition of A and $\underline{b}$ are obvious from (2.5.53). Note that

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tau^4 & 0 & 0 & 0 \\ 0 & \tau^4 & 0 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \tau^4 & 0 & 0 & 0 \\ 0 & \tau^4 & 0 & 0 \\ 0 & 0 & \tau^4 & 0 \end{pmatrix}, A^4 = \tau^4 I$$

By taking the power series e^{Ax} is obtained by

$$(e^{Ax})_{11} = 1 + \tau^4x^4/4! + \tau^8x^8/8! + \ldots = \frac{1}{2}(\cos(\tau x) + \cosh(\tau x))$$

All other terms of e^{Ax} can be obtained similarly:

$$(e^{Ax})_{11} = \frac{1}{2}(\cos(\tau x) + \cosh(\tau x)) \qquad (e^{Ax})_{12} = \frac{1}{2\tau}(\sin(\tau x) + \sinh(\tau x))$$

$$(e^{Ax})_{13} = \frac{1}{2\tau^2}(-\cos(\tau x) + \cosh(\tau x)) \qquad (e^{Ax})_{14} = \frac{1}{2\tau^3}(-\sin(\tau x) + \sinh(\tau x))$$

$$(e^{Ax})_{21} = \frac{\tau}{2}(-\sin(\tau x) + \sinh(\tau x)) \qquad (e^{Ax})_{22} = \frac{1}{2\tau}(\cos(\tau x) + \cosh(\tau x))$$

$$(e^{Ax})_{23} = \frac{1}{2\tau}(\sin(\tau x) + \sinh(\tau x)) \qquad (e^{Ax})_{24} = \frac{1}{2\tau^2}(\cosh(\tau x)-\cos(\tau x)) \qquad (2.5.55)$$

$$(e^{Ax})_{31} = \frac{\tau^2}{2}(-\cos(\tau x) + \cosh(\tau x)) \qquad (e^{Ax})_{32} = \frac{\tau}{2}(-\sin(\tau x) + \sinh(\tau x))$$

$$(e^{Ax})_{33} = \frac{1}{2}(\cos(\tau x) + \cosh(\tau x)) \qquad (e^{Ax})_{34} = \frac{1}{2\tau}(\sin(\tau x) + \sinh(\tau x))$$

$$(e^{Ax})_{41} = \frac{\tau^3}{2}(\sin(\tau x) + \sinh(\tau x)) \qquad (e^{Ax})_{42} = \frac{\tau^2}{2}(-\cos(\tau x) + \cosh(\tau x))$$

$$(e^{Ax})_{43} = \frac{\tau}{2}(-\sin(\tau x) + \sinh(\tau x)) \qquad (e^{Ax})_{44} = \frac{1}{2}(\cos(\tau x) + \cosh(\tau x))$$

The general solution for $\underline{y}_1(x, \tau)$ is

$$\underline{y}_1(x) = e^{Ax}\underline{y}_1(0) + \int_0^x e^{A(x-\sigma)}b(\sigma)d\sigma \qquad 0 < x < x_1 \qquad (2.5.56)$$

$$:= \underline{y}_{1h}(x) + \underline{y}_{1p}(x)$$

where $\underline{y}_{1h}(x)$ refers to the homogeneous solution, while $\underline{y}_{1p}(x)$ refers to the particular solution. The cantilevered boundary conditions (see (2.5.10)) imply that $\underline{y}_1(0) = (0, 0, y_1''(0), y_1'''(0))^T$.

Similarly, the solution for $\underline{y}_2(x) := (y_2(x), y_2'(x), y_2''(x), y_2'''(x)^T$ can be written

$$\underline{y}_2(x) = e^{A(x-x_1)}\underline{y}_2(x_1) + \int_{x_1}^{x} e^{A(x-\sigma)}b(\sigma)d\sigma \qquad x_1 < x < L \qquad (2.5.57)$$

$$:= \underline{y}_{2h}(x) + \underline{y}_{2p}(x)$$

where, as in the case of $\underline{y}_1(x)$ above, $\underline{y}_{2h}(x)$ refers to the homogeneous solution, while $\underline{y}_{2p}(x)$ refers to the particular solution. (Alternatively, the solution for $\underline{y}_2(x)$ could be written in terms of boundary conditions at $x = L$, which would take explicit advantage of the form of the boundary conditions there. Unfortunately, the resulting computations seem to be equally difficult.) Unfortunately, none of the terms of $\underline{y}_2(x_1)$ are known. We therefore have six unknown quantities: the four unknown quantities of $\underline{y}_2(x_1)$, together with the two unknowns $y_1''(0)$, $y_1'''(0)$. We also have the 6 boundary conditions given in (2.5.24) - (2.5.26). Writing out these terms explicitly, and combining them in matrix form yields

$$\begin{bmatrix} (e^{Ax_1})_{13} & (e^{Ax_1})_{14} & -1 & 0 & 0 & 0 \\ (e^{Ax_1})_{23} & (e^{Ax_1})_{24} & 0 & -1 & 0 & 0 \\ (e^{Ax_1})_{33} & (e^{Ax_1})_{34} & 0 & 0 & -1 & 0 \\ (e^{Ax_1})_{43} - & (e^{Ax_1})_{44} - & 0 & 0 & 0 & -1 \\ Ki\tau^2(e^{Ax_1})_{13} & Ki\tau^2(e^{Ax_1})_{13} & & & & \\ 0 & 0 & (e^{A(L-x_1)})_{31} & (e^{A(L-x_1)})_{32} & (e^{A(L-x_1)})_{33} & (e^{A(L-x_1)})_{34} \\ 0 & 0 & (e^{A(L-x_1)})_{41} & (e^{A(L-x_1)})_{42} & (e^{A(L-x_1)})_{43} & (e^{A(L-x_1)})_{44} \end{bmatrix} \begin{bmatrix} y_1''(0) \\ y_1'''(0) \\ y_2(x_1) \\ y_2'(x_1) \\ y_2''(x_1) \\ y_2'''(x_1) \end{bmatrix}$$

$$= - \begin{bmatrix} \frac{1}{2\tau^3}\int_0^{x_1} (-\sin\tau(x_1-\sigma)) + \sinh\tau(x_1-\sigma))(i\tau^2\phi_1(\sigma) + \Psi_1(\sigma))d\sigma \\ \frac{1}{2\tau^2}\int_0^{x_1} (-\cos\tau(x_1-\sigma)) + \cosh\tau(x_1-\sigma))(i\tau^2\phi_1(\sigma) + \Psi_1(\sigma))d\sigma \\ \frac{1}{2\tau}\int_0^{x_1} (\sin\tau(x_1-\sigma)) + \sinh\tau(x_1-\sigma))(i\tau^2\phi_1(\sigma) + \Psi_1(\sigma))d\sigma \\ \frac{1}{2}\int_0^{x_1} (\cos\tau(x_1-\sigma)) + \cosh\tau(x_1-\sigma))(i\tau^2\phi_1(\sigma) + \Psi_1(\sigma))d\sigma - Ki\tau^2 b_1 - \phi_1(x_1) \\ \frac{1}{2\tau}\int_{x_1}^{L} (\sin\tau(L-\sigma)) + \sinh\tau(L-\sigma))(i\tau^2\phi_2(\sigma) + \Psi_2(\sigma))d\sigma \\ \frac{1}{2}\int_{x_1}^{L} (\cos\tau(L-\sigma)) + \cosh\tau(L-\sigma))(i\tau^2\phi_{21}(\sigma) + \Psi_2(\sigma))d\sigma \end{bmatrix} \tag{2.5.58}$$

where $(e^{A(L-x_1)})_{ij}$ refers to the ijth element of the matrix $(e^{A(L-x_1)})$ defined in (2.5.55), and b_1 is the first row of the right hand side of the equation. Write this equation as $A^*y_0 = b^*$, where the definitions of A^*, y_0 and b^* are obvious from (2.5.58).

We now have the following lemma which explicitly gives the determinant of the matrix A*.

Lemma 2.5.8 - For the matrix A* defined in (2.5.58), det(A*) is given by

$$\det(A^*) = \frac{1}{2}(1 + \cos(\tau L)\cosh(\tau L)) -$$

$$\frac{Ki}{8\tau} \times \begin{bmatrix} (\cos\tau L \sinh\tau L + \cosh\tau L \sin\tau L)[(\sin\tau x_1 - \sinh\tau x_1)]^2 + \\ (-\cos\tau L \sinh\tau L + \cosh\tau L \sin\tau L)(\cos\tau x_1 - \cosh\tau x_1)^2 + \\ -2\sin\tau L \sinh\tau L[(\sin\tau x_1 - \sinh\tau x_1)(\cos\tau x_1 - \cosh\tau x_1) \\ + (1 + \cos(\tau L)\cosh(\tau L))(-\sinh\tau x_1 \cos\tau x_1 + \cosh\tau x_1 \sin\tau x_1) \end{bmatrix} \qquad (2.5.59)$$

Proof of Lemma 2.5.8 - (2.5.59) results from tedious calculation of the determinant of A*. The details are omitted. ■

Comment 2.5.9 - Note that for K = 0 (no feedback) the equation det(A*) = 0 reduces to that obtained for the undamped cantilevered beam (see [Mei. 2, p. 162]), as expected. In addition, if $x_1 = L$, the equation reduces to that of beam tip boundary control, also as expected.

We now proceed to show that the resolvent estimate (2.5.17) is attained for $\lambda = i\omega$ sufficiently large. In other words, we must find a constant B > 0, independent of λ, such that

$$\int_0^{x_1} (|z_1(x)|^2 + |y_1''(x)|^2)\, dx + \int_{x_1}^{L} (|z_2(x)|^2 + |y_2''(x)|^2)\, dx \quad \le$$

$$B\left(\int_0^{x_1} (|\Psi_1(x)|^2 + |\phi_1''(x)|^2)\, dx + \int_{x_1}^{L} (|\Psi_2(x)|^2 + |\phi_2''(x)|^2)\, dx\right)$$

for $\lambda = i\omega$ sufficiently large. The main observation is that $y_{1p}(x)$ and $y_{1h}(x)$ *do not* satisfy the bounds

$$\int_0^{x_1} |y_{1p}''(x)|^2 dx \leq C\{\int_0^{x_1} (|\Psi_1(x)|^2 + |\phi_1''(x)|^2)\, dx + \int_{x_1}^{L} (|\Psi_2(x)|^2 + |\phi_2''(x)|^2) dx\} \quad (2.5.60)$$

$$\int_0^{x_1} |y_{1h}''(x)|^2 dx \leq C\{\int_0^{x_1} (|\Psi_1(x)|^2 + |\phi_1''(x)|^2)\, dx + \int_{x_1}^{L} (|\Psi_2(x)|^2 + |\phi_2''(x)|^2)\, dx\} \quad (2.5.61)$$

for $\lambda = i\omega$ sufficiently large. However, the dominant terms cancel, leaving terms which do satisfy the bounds.

Estimation of $\|y_{1p}''(x)\|$ and $\|y_{2p}''(x)\|$

We will give the estimates of $\|y_{1p}''(x)\|$ in detail, and the estimates of $\|y_{2p}''(x)\|$ can be done in a similar manner. Let $\lambda = i\omega = i\tau^2$ for τ real, and $\tau > 0$. From (2.5.55)-(2.5.56) we see that

$$y_{1p}''(x) = \frac{1}{2\tau}\int_0^x (\sin\tau(x-\sigma)) + \sinh\tau(x-\sigma))(i\tau^2\phi_1(\sigma) + \Psi_1(\sigma))d\sigma$$

If we integrate the portion of the expression containing $i\tau^2\phi_1(\sigma)$ by parts twice, we obtain

$$y_{1p}''(x) = \frac{1}{2\tau}\int_0^x (\sin\tau(x-\sigma)) \quad + \quad \sinh\tau(x-\sigma))(\Psi_1(\sigma))d\sigma \; +$$

$$\frac{1}{2\tau}\int_0^x (-\sin\tau(x-\sigma)) \quad + \quad \sinh\tau(x-\sigma))(i\phi_1''(\sigma))d\sigma$$

$$= \frac{1}{2\tau}\int_0^x (\sinh\tau(x-\sigma))(i\phi_1''(\sigma) \quad + \quad \Psi_1(\sigma))d\sigma + O(\tau^{-1}(\|\phi_1''\|_1 + \|\Psi_1\|_1)) \qquad (2.5.62)$$

Simiarily, the expression for $y_{2p}''(x)$ becomes

$$y_{2p}''(x) = \frac{1}{4\tau}e^{\tau x}\int_{x_1}^x (e^{-\tau\sigma})(i\phi_2''(\sigma) \; + \; \Psi_2(\sigma))d\sigma \; +$$

$$\frac{i}{4\tau}e^{\tau(x-x_1)}(-\tau\phi_2(x_1) \quad + \quad \phi_2'(x_1) \quad) + O(\tau^{-1}(\|\phi_2''\|_2 + \|\Psi_2\|_2)) \qquad (2.5.63)$$

Next, we must compute $y_{1h}''(x)$ and $y_{2h}''(x)$. These expressions are given by (2.5.56)-(2.5.57), where the unknown coefficients are determined by the relation $A^*y_0 = b^*$. It is necessary to compute this inverse to then yield, by Cramer's Rule, $y_0 = (\det(A^*))^{-1}(\mathrm{adj}(A^*))b^*$. Before reaching any conclusions about y_0, we must first investigate the structure of $\det(A^*)$, in order to give conditions on x_1 to guarantee that $\det(A^*) \neq 0$, and to give conditions to guarantee that $\det(A^*)$ is bounded away from zero as $\tau \to \infty$.

Estimation of det(A*)

Consider again det(A*), explicitly calculated in Lemma 2.5.8. It can be verified, by brute force calculation, that det(A*) = 0 if and only if x_1 is located at a node for some mode of the original beam system. So now assume that x_1 is a not located at a node for any mode of the undriven beam system. Under this assumption we desire to give conditions on x_1 which insures that det(A*) is bounded away from zero as $\tau \to \infty$.

For τ very large det(A*) becomes

$$\det(A^*) = \frac{1}{2}e^{\tau L}\cos\tau L - \frac{Ki}{16\tau}e^{\tau L}[\sin^2\tau x_1\cos\tau L + \sin^2\tau x_1\sin\tau L - \frac{1}{2}\cos\tau L - \frac{1}{2}\sin\tau L -$$

$$\frac{1}{2}\cos\tau L + \frac{1}{2}\sin\tau L - \cos^2\tau x_1\cos\tau L + \cos^2\tau x_1\sin\tau L - 2\sin\tau x_1\cos\tau x_1\sin\tau L\,]$$

$$= \frac{1}{2}e^{\tau L}\cos\tau L - \frac{Ki}{8\tau}e^{\tau L}[\sin\tau L - \cos 2\tau x_1\cos\tau L - \sin 2\tau x_1\sin\tau L] \qquad (2.5.64)$$

Note from (2.5.64) that det(A*) is bounded away from zero as $\tau \to \infty$ if $\cos\tau L$ and $[\sin\tau L - \cos 2\tau x_1\cos\tau L - \sin 2\tau x_1\sin\tau L]$ are simultaneously bounded away from zero as $\tau \to \infty$. This is equivalent to having $\cos\tau L$ and $[1 - \sin 2\tau x_1]$ are simultaneously bounded away from zero as $\tau \to \infty$.

Consider next the set $S := \{(x, y) \mid x = \cos\tau L,\ y = 1 - \sin 2\tau x_1,\ \tau \in \mathbb{R}\}$. Note that although S does not contain the origin, if L/x_1 is irrational then S is *dense* in [-1, 1]x[0, -2]. (It is a space filling curve.) Thus, there is an infinite sequence $\{s_n\} \subset S$ such that $s_n \to 0$. This means that there may be an infinite sequence $\{\tau_n\}$ such that det(A*) goes to zero as $\tau \to \infty$. However, if L/x_1 is *rational*, then the set S is clearly not dense, and furthermore it is a *closed* set. Since $0 \notin S$, this means that there is *no* sequence $\{\tau_n\}$ such that det(A*) goes to zero as $\tau_n \to \infty$.

We have thus reached the following conclusions. First, det(A*) = 0 if and only if x_1 is located at a node for some mode of the original beam system. Secondly, if x_1 is not located at a node for some mode of the original beam system, and if L/x_1 is rational, then det(A*) is bounded away from zero as $\tau \to \infty$.

Estimate of $y_1''(x)$ and $y_2''(x)$

Assume now that the assumptions of the Lemma 2.5.7 are satisfied, i.e., x_1 is not located at a node of the original, undriven beam system (2.3.13), (2.3.14), and x_1/L is a rational number. From (2.5.58), $y_0 = (\det(A^*))^{-1}(\mathrm{adj}(A^*))b^*$. Because A* is a 6x6 matrix, we will not give the inverse of A* explicitly (to write the matrix down term by term requires about 4 typed pages), but instead we will give the conclusions of these tedious operations.

From $y_0 = (\det(A^*))^{-1}(\mathrm{adj}(A^*))b^*$, we see that $y_1''(0)$ and $y_1'''(0)$ depend on $\phi_1(x)$, $\Psi_1(x)$, and $\phi_2(x)$, $\Psi_2(x)$ (namely, in the integral terms involving b*). However, if the terms involving $\phi_2(x)$, $\Psi_2(x)$ are explicitly calculated, and inserted into the equation for $y_{1h}''(x)$, one finds that the higher order terms drop out, leaving terms of order $O(\tau^{-1}(\|\phi_2(x)\|_2^2 + \|\Psi_2(x)\|_2))$. (It should be pointed out that we have explicitly used the fact that det(A*) is bounded away from zero as $\tau \to \infty$. If det(A*) is not bounded away from zero as $\tau \to \infty$, i.e., if x_1/L is irrational, then we cannot conclude that the residual terms are $O(\tau^{-1}(\|\phi_2(x)\|_2^2 + \|\Psi_2(x)\|_2))$. This is where the proof breaks down for arbitrary x_1.) Furthermore, if the leading terms involving $\phi_1(x)$ and $\Psi_1(x)$ are explicitly calculated, one finds that these terms *precisely cancel* the terms given by (2.5.62). In other words, $y_{1h}''(x)$ is given by

$$y_{1h}''(x) = -\frac{1}{4\tau}e^{\tau x}\int_{x_1}^{x}(e^{-\tau\sigma})(i\phi_1''(\sigma) + \Psi_1(\sigma))d\sigma + O(\tau^{-1}(\|\phi_1''\|_1 + \|\Psi_1\|_1))$$
$$+ O(\tau^{-1}(\|\phi_2''\|_2 + \|\Psi_2\|_2))$$

Similarly, $y_{2h}''(x)$ contains terms involving $\phi_1(x)$, $\Psi_1(x)$, and $\phi_2(x)$, $\Psi_2(x)$ (namely, in the integral terms involving b*). Analogous to $y_{1h}''(x)$ above, the higher order terms involving $\phi_1(x)$, $\Psi_1(x)$ drop out, leaving an expression of order $O(\tau^{-1}(\|\phi_1''\|_1 + \|\Psi_1\|_1))$. Also, the leading terms of $y_{2h}''(x)$ again precisely cancel the leading terms of $y_{2p}''(x)$. In other words, $y_{2h}''(x)$ is given by

$$y_{2h}''(x) = -\frac{1}{4\tau}e^{\tau x}\int_{x_1}^{x}(e^{-\tau\sigma})(i\phi_2''(\sigma) + \Psi_2(\sigma))d\sigma -$$
$$\frac{i}{4\tau}e^{\tau(x-x_1)}(-\tau\phi_2(x_1) + \phi_2'(x_1)) + O(\tau^{-1}(\|\phi_2''\|_2 + \|\Psi_2\|_2))$$

The conclusion of this work is that $y_1''(x) = y_{1h}''(x) + y_{1p}''(x)$ and $y_2''(x) = y_{2h}''(x) + y_{2p}''(x)$ does satisfy the resolvent bound (2.5.17). In other words, there exists a constant B, independent of

$$\int_0^{x_1}|y_1''(x)|^2dx + \int_{x_1}^{L}|y_2''(x)|^2dx \leq$$

$$B\left(\int_0^{x_1} (|\Psi_1(x)|^2 + |\phi_1''(x)|^2)\,dx + \int_{x_1}^{L} (|\Psi_2(x)|^2 + |\phi_2''(x)|^2)dx \qquad (2.5.65)$$

Finally, in order to prove Lemma 2.5.7, we need to give estimates on the terms $z_1(x)$ and $z_2(x)$ in (2.5.18).

Estimates of $z_1(x)$ and $z_2(x)$

From (2.5.18) we see that $z_1(x)$ and $z_2(x)$ are given by

$$z_1(x) = i\tau^2 y_1(x) + \phi_1(x) \qquad (2.5.66)$$

$$z_1(x) = i\tau^2 y_1(x) + \phi_1(x) \qquad (2.5.67)$$

To get an estimate on $z_1(x)$ and $z_2(x)$ it is necessary to first get an estimate on $y_1(x)$ and $y_2(x)$. This is done by taking the inner product of both sides of (2.5.22) with $y_1(x)$ to obtain

$$[-\tau^4 y_1(x), y_1(x)]_1 + [y_1''''(x), y_1(x)]_1 = [i\tau^2\phi_1(x) + \Psi_1(x), y_1(x)]_1 \qquad (2.5.68)$$

Integrate the left hand side of this equation by parts twice, and use the boundary conditions (2.5.24) - (2.5.26), to obtain

$$y_1'''(x_1)y_1(x_1) - y_1''(x_1)y_1'(x_1) + [y_1''(x), y_1''(x)]_1 - \tau^4[y_1(x), y_1(x)]_1$$
$$= [i\tau^2\phi_1(x) + \Psi_1(x), y_1(x)]_1 \qquad (2.5.69)$$

Perform the same operations on the $y_2(x)$ equation, (2.5.23), to obtain

$$-y_2'''(x_1)y_2(x_1) + y_2''(x_1)y_2'(x_1) + [y_2''(x), y_2''(x)]_2 - \tau^4[y_2(x), y_2(x)]_2$$
$$= [i\tau^2\phi_2(x) + \Psi_2(x), y_2(x)]_2 \qquad (2.5.70)$$

If we add these two equations, (2.5.69) and (2.5.70), we obtain

$$[y_1''(x), y_1''(x)]_1 - \tau^4[y_1(x), y_1(x)]_1 + [y_2''(x), y_2''(x)]_2 - \tau^4[y_2(x), y_2(x)]_2$$
$$-y_2'''(x_1)y_2(x_1) + y_1'''(x_1)y_1(x_1) + y_2''(x_1)y_2'(x_1) - y_1''(x_1)y_1'(x_1) +$$
$$= [i\tau^2\phi_1(x) + \Psi_1(x), y_1(x)]_1 + [i\tau^2\phi_2(x) + \Psi_2(x), y_2(x)]_2 \qquad (2.5.71)$$

Using the boundary conditions (2.5.24-(2.5.26), this expression simplifies to

$$[y_1''(x), y_1''(x)]_1 - \tau^4[y_1(x), y_1(x)]_1 + [y_2''(x), y_2''(x)]_2 - \tau^4[y_2(x), y_2(x)]_2$$
$$- Ki\tau^2[y_1(x), y_1(x)]_1 + K\phi_1(x_1)y_1^*(x_1)$$
$$= [i\tau^2\phi_1(x) + \Psi_1(x), y_1(x)]_1 + [i\tau^2\phi_2(x) + \Psi_2(x), y_2(x)]_2 \qquad (2.5.72)$$

Rearrange this expression to yield

$$\tau^4[y_1(x), y_1(x)]_1 + \tau^4[y_2(x), y_2(x)]_2 = \mathrm{Re}\{-[i\tau^2\phi_2(x) + \Psi_2(x), y_2(x)]_2 +$$
$$[y_1''(x), y_1''(x)]_1 + [y_2''(x), y_2''(x)]_2 + Ki\tau^2[y_1(x), y_1(x)]_1 + K\phi_1(x_1)y_1^*(x_1)\}$$

$$= \mathrm{Re}\{-[i\tau^2\phi_2(x) + \Psi_2(x), y_2(x)]_2 + [y_1''(x), y_1''(x)]_1 +$$
$$[y_2''(x), y_2''(x)]_2 + K\phi(x_1)y_1^*(x_1)\} \qquad (2.5.73)$$

Next, use the inequality $[a, b] \le [a, a]/\delta^2 + \delta^2[b, b]$ for any $\delta \in \mathbf{R}/\{0\}$. This yields

$$\le [y_1''(x), y_1''(x)]_1 + [y_2''(x), y_2''(x)]_2 + [\phi_2(x), \phi_2(x)]_2 + \tau^4[y_2(x), y_2(x)]_2/4 +$$
$$[\phi_1(x), \phi_1(x)]_1 + \tau^4[y_1(x), y_1(x)]_1/4 + \mathrm{Re}\{K\phi_1(x_1)y_1^*(x_1)\} \qquad (2.5.74)$$

Finally, note that $\mathrm{Re}\{K\phi(x_1)y_1^*(x_1)\} \le Kx_1^2\{[\phi_1, \phi_1]_1 + [y_1, y_1]_1\}$. (The reader should consult Proposition 3.4.5 if there is difficulty in seeing this.) Inserting this, and simplifying the expression shows that there is some constant $C > 0$ such that

$$[y_1(x), y_1(x)]_1 + [y_2(x), y_2(x)]_2 \le \frac{C}{|\lambda|^2}\{[y_1''(x), y_1''(x)]_1 + [y_2''(x), y_2''(x)]_2 +$$
$$[\phi_2(x), \phi_2(x)]_2 + [\phi_1(x), \phi_1(x)]_1\}$$

Finally use this inequality in (2.5.66) and (2.5.67) to obtain

$$\|z_1(x)\|_1^2 \le |\lambda|^2 \|y_1(x)\|_1^2 + \|\phi_1(x)\|_1^2$$
$$\le C\{\|y_1(x)\|_1^2 + \|y_2(x)\|_2^2 + \|\phi_2(x)\|_2^2 + \|\phi_1(x)\|_1^2\} + \|\phi_1(x)\|_1^2 \qquad (2.5.75)$$

Similarly,

$$\|z_2(x)\|_2^2 \le C\{\|y_1(x)\|_1^2 + \|y_2(x)\|_2^2 + \|\phi_2(x)\|_2^2 + \|\phi_1(x)\|_1^2\} + \|\phi_2(x)\|_2^2 \qquad (2.5.76)$$

Proof of Lemma 2.5.7 - If we combine (2.5.65), (2.5.75), and (2.5.76) we see that for $|\lambda| = |\omega|$ sufficiently large, there is a constant B, independent of λ sufficiently large such that

$$\int_0^{x_1} |y_1''(x)|^2 dx + \int_{x_1}^{L} |y_2''(x)|^2 dx + \int_0^{x_1} |z_1(x)|^2 dx + \int_{x_1}^{L} |z_2(x)|^2 dx \le$$

$$B\Big(\int_0^{x_1} (|\Psi_1(x)|^2 + |\phi_1''(x)|^2)\, dx + \int_{x_1}^{L} (\Psi_2^2(x) + \phi_2^{2''}(x))\, dx \qquad (2.5.77)$$

This completes the proof of the lemma. ■

Proof of Theorem 2.5.2

$\Leftarrow$ To prove the result, we use the converse of the Theorem 2.5.4. Assume x_1 is not located at a node of the original, undriven beam system (2.3.13), (2.3.14) (where $E=I=\rho=1$ and $p(x, t) = M(t) = F(t) = 0$), and x_1/L is a rational number. Condition (2.5.15) has been shown in Appendix 2.A. Lemma 2.5.7 shows that there is a $0 < \omega_0 < \infty$ such that $\sup_{\omega > \omega_0}\{\|(i\omega - A)^{-1}\|\} < \infty$. Theorem 2.5.5 and Lemma 2.5.6 show that for any $\omega_0 < \infty$, $\{i\omega \mid \omega < \omega_0\} \subset \rho(A)$ and $\sup_{\omega < \omega_0}\{\|(i\omega - A)^{-1}\|\} < \infty$. Combining these two results show that $\{i\omega \mid \omega \in \mathbb{R}\} \subset \rho(A)$ and $\sup\{\|(i\omega - A)^{-1}\|\} < \infty$. This proves the theorem. ■

Although this theorem proves conclusively that exponential stability does not result for arbitrary sensor/actuator placement in the interior of the beam, there are interesting properties associated with this problem. This is the content of the following lemma.

Lemma 2.5.10 - The set of nodes of the undriven beam, given by

$$N = \{ x_1 \in [0,\ L] \mid (\sin\tau L - \sinh\tau L)(\sin\tau x_1 - \sinh\tau x_1) + (\cos\tau L + \cosh\tau L)(\cos\tau x_1 - \cosh\tau x_1) = 0, \text{ for some } \tau L \text{ satisfying } \cos\tau L\cosh\tau L = -1 \} \quad (2.5.78)$$

are dense in the beam, i.e., $\bar{N} = [0,\ L]$. In addition, $m(N) = 0$, where $m(N)$ is the Lebesgue measure of N.

Proof of Lemma 2.5.10 - Consider any closed interval [a, b] with $0 \le a < b \le L$. It will be shown that there is at least one node contained in this interval. Rewrite the node equation (2.5.78) as

$$(\sin\tau L - \sinh\tau L)\sin\tau x_1 + (\cos\tau L + \cosh\tau L)\cos\tau x_1 = (\sin\tau L - \sinh\tau L)\sinh\tau x_1 + (\cos\tau L + \cosh\tau L)\cosh\tau x_1 \quad (2.5.79)$$

Using a standard trigonometric formula on the first two terms yields

$$K\cos(\tau x_1 + \phi) = (\sin\tau L - \sinh\tau L)\sinh\tau x_1 + (\cos\tau L + \cosh\tau L)\cosh\tau x_1 \quad (2.5.80)$$

where $K^2 = (\sin\tau L - \sinh\tau L)^2 + (\cos\tau L + \cosh\tau L)^2$ and $\phi = \tan^{-1} \{(\sin\tau L - \sinh\tau L) / (\cos\tau L + \cosh\tau L) \}$. Using $\cos\tau L\cosh\tau L = -1$ and further trigonometric simplifications, K^2 can be simplified to

$$\begin{aligned} K^2 &= (\sin\tau L - \sinh\tau L)^2 + \cos^2\tau L + \cosh^2\tau L - 2 \\ &= (\sin\tau L - \sinh\tau L)^2 - \sin^2\tau L + \sinh^2\tau L \\ &= (\sin\tau L - \sinh\tau L)^2 - \sin^2\tau L + \cosh^2\tau L\sin^2\tau L \end{aligned}$$

$$= (\sin\tau L - \sinh\tau L)^2 + \sin^2\tau L \sinh^2\tau L \qquad (2.5.81)$$

Now we simplify the right hand side of (2.5.80):

$$\begin{aligned} RHS &= (\sin\tau L - \sinh\tau L)\sinh\tau x_1 - \sin^2\tau L \cosh\tau x_1/\cos\tau L \\ &= (\sin\tau L - \cosh\tau L\sin\tau L)\sinh\tau x_1 - \sinh^2\tau L \cos\tau L\cosh\tau x_1 \\ &= (1 - \cosh\tau L)\sin\tau L\sinh\tau x_1 + \sinh\tau L\sin\tau L\cosh\tau x_1 \\ &= \sin\tau L[-\cosh\tau L\sinh\tau x_1 + \sinh\tau L\cosh\tau x_1 + \sinh\tau x_1] \\ &= \sin\tau L[\sinh\tau(L-x_1) + \sinh\tau x_1] \end{aligned}$$

Finally, a few more straightforward calculations gives

$$= 2\sin\tau L\sinh(\tau L/2)\cosh\tau(x_1 - L/2) \qquad (2.5.82)$$

To verify that the equation (2.5.80) has solutions, we note that $\cosh\tau(x_1\text{-}L/2) \leq \cosh\tau(L/2)$ for $x_1 \in [0, \; L]$. Multiplying both sides by $2\sin\tau L\sinh(\tau L/2)$ yields successively

$$\begin{aligned} 2\sin\tau L\sinh(\tau L/2)\cosh\tau(x_1 - L/2) &\leq 2\sin\tau L\sinh(\tau L/2)\cosh\tau(L/2) \\ &\leq \sin\tau L\sinh(\tau L) \quad \text{for } x_1 \in [0, \; L] \qquad (2.5.83) \end{aligned}$$

by the "double angle formula" for sinhx. Using (2.5.81) and (2.5.82) in (2.5.80) yields

$$\begin{aligned} \{(\sin\tau L - \sinh\tau L)^2 + \sin^2\tau L\sinh^2\tau L\}^{0.5}\cos(\tau x_1 + \phi) \\ = 2\sin\tau L\sinh(\tau L/2)\cosh\tau(x_1 - L/2) \qquad (2.5.84) \end{aligned}$$

Inequality (2.5.83) shows that, for all $x_1 \in [0, L]$, the RHS of (2.5.84) has magnitude less than or equal to the coefficient on the $\cos(\tau x_1 + \phi)$ term. Therefore, solutions do exist. It is easy to see (see Figure 2.8) that there are at least $\lfloor \tau L/\pi \rfloor$ solutions in $[0, L]$, where $\lfloor \bullet \rfloor$ denotes the integer floor function.

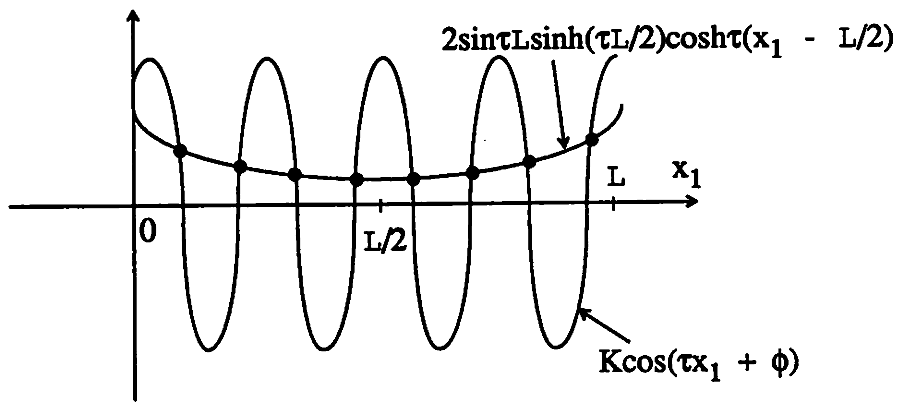

Figure 2.8 - Graph of equation (2.5.84)

In addition, for any interval $[c, d] \subset [0, L]$there is at least one solution in this interval as long as $d - c \geq 2/\lfloor \tau/\pi \rfloor$. Therefore, for the given interval [a, b], choose τ large enough so that $2/\lfloor \tau/\pi \rfloor \leq \varepsilon = a\text{-}b$. This concludes the proof of the first part of the lemma.

To show that m(N)=0, note that the number of zeros of the mode equation (contained in (2.5.78)) for each modal frequency (i.e., for each value of τ) is *finite*, and the number of modal frequencies is countable (with an obvious 1-1 correspondence with the integers). Therefore N is the union of a countable number of finite sets, hence N is countable. This implies m(N) = 0. ■

Since m(N) = 0, this means that the probability of placing the actuator on a node of any mode is zero! This means that the system is exponentially stabilized by a velocity feedback placed in the interior of the beam, with probability one.

Of course from an engineering viewpoint it is probably hopeless to attempt to uniformly stabilize the beam using velocity feedback in the interior of the beam. The fact the nodes are dense in the beam means that the uniform exponential decay constant is most likely very small. However, it is very easy to place the actuator so that a specified finite number of modes would be exponentially stabilized. Given a finite number of nodes, one would place the sensor/actuator pair at a position where the modes of interest have relatively large deflections. This is the common intuitive method used by engineers in placing actuators when the beam model is modally truncated (see [Joh.1, 508-509]).

2.5.3 Relationship to Passivity

The knowledgeable reader will probably feel that the Theorem 2.5.5 is somehow related to passivity concepts. It is the goal of this section to exhibit this relationship.

Using (2.5.42) we see that if $y_1'''(x_1) - y_2'''(x_1) = F(t)$, the applied force at point x_1, then

$$\int_{-\infty}^{t} F(t)\, \dot{y}(x_1, t)dt = \int_{-\infty}^{t} \frac{d}{dt} E(t)dt \tag{2.5.85}$$

$$= E(t) - E(-\infty) = E(t) \tag{2.5.86}$$

if we assume the energy at $t = -\infty$ is zero. Thus

$$\int_{-\infty}^{t} (\underline{\omega}^T \underline{\tau} + \underline{F}_T{}^T \dot{\underline{y}})\, dz \geq 0 \tag{2.5.87}$$

This shows that our beam plant is passive [Des. 2, p. 173]. Furthermore, our feedback law is strictly passive [Des. 2, p. 173]. (It is simply the gain K.) By the passivity theorem, [Des. 2, p. 181] we can conclude that the map

$$F(t) \rightarrow \dot{y}(x_1, t) \tag{2.5.88}$$

is L^2 stable. Since the system is linear, one can easily show that this implies the map (2.5.88) is exponentially stable.

Unfortunately, these passivity arguments do not yield the same results as Theorem 2.5.2. The main problem is that the above result only says that $\dot{y}(x_1, t)$ goes to zero exponentially, but says nothing about $\dot{y}(x, t)$ for $x \neq x_1$. One could probably use the above results, coupled with arguments involving continuity of solutions of differential equations, to obtain the results of Theorem 2.5.2, but unfortunately it doesn't seem easy.

There is one final comment that should be made regarding Theorem 2.5.2. The passivity theorem above shows that the closed loop system is input/output stable. What we really desire is that the closed loop system be state space stable. Is there any way to show that input/output stability implies that the system is state space stable? From the study of finite dimensional linear systems, we know that if a system is I/O stable, and the state space representation is minimal (i.e., controllable and observable) then the system is also state space stable. In [Jac. 1] and [Cur. 1], this concept is extended to a broad class of infinite dimensional systems. Unfortunately, the class does not include the example of Theorem 2.5.2. The primary reason for this is that the operators B and C of (2.4.6) and (2.4.7) are unbounded operators. Unfortunately, there is no obvious extension of the above mentioned papers to include such operators.

2.6 Conclusions and Future Research

In terms of future research, much still remains to be done. When beam damping is included in the Euler-Bernoulli model, it was pointed out in section 2.4 that finite dimensional compensators exist. The current method for obtaining such compensators is to approximate the infinite dimensional compensators, which usually leads to a controller of high order. From a practical viewpoint, it is desirable that compensators be of low-dimension, so that the current method is unsatisfactory. A methodology for obtaining controllers of reasonably low order would be of great practical importance, not only for flexible structure control but also for other distributed parameter systems. An example of an optimization based approach for such design considerations can be found in [Har. 1].

In section 2.5, the stabilization problem for the Euler-Bernoulli beam with no damping was considered. A uniform exponential stabilization scheme was proposed for a collocated sensor/actuator pair located at an interior point of the beam. The obtained conditions were sufficient. The natural extension of the result given here would be to find conditions under which the system is uniformly exponentially stabilized when x_1/L is irrational. Another important unresolved problem is to obtain the exponential decay rate as a function of beam parameters, especially the position of the sensor/actuator pair. This has been done for a beam with an applied moment at the tip in [Reb. 1], but the methods do not easily extend to general sensor/actuator pair locations. Another very important research topic would be to extend Theorem 2.5.2 to a beam modelled by the Timoshenko beam model (see [Russ. 1]). The Timoshenko beam model is apparently becoming something of an industry favorite, combining relatively simple formulation with reasonably accurate prediction of beam behavior. In [Kim 1], the beam tip boundary control was used to uniformly exponentially stabilize such a model, just as Chen et al [Che. 2] used this scheme to stabilize the Euler-Bernoulli model. Thus, such an extension would be a natural complement to the result obtained here.

Finally, there is still considerable work to be done in the area of multiple, perhaps non-collocated sensors and actuators for flexible structure control. A first step would be to evaluate the affects of multiple sensors and actuators on simple beam models, and then attempt to apply such methods for more complicated structures. Perhaps the greatest drawback to a distributed parameter formulation of such flexible structures is the resulting mathematical complexity (see the proof of Theorem 2.5.2!), which is probably the main reason finite dimensional approximations are used in practice. It is still an open question as to whether large scale distributed parameter system formulations applied to future large space structures will be analytically tractable at both the conceptual level and the computational level or not.

Appendix 2.A - Proof of Condition (2.5.15)

Consider again the differential operator A defined by (2.5.11). Let the space X, the energy inner product $[\cdot, \cdot]_E$, and the domain of A, D(A) be defined as in (2.5.12)-(2.5.168).

A simple calculation shows that the norm induced by the inner product (2.5.13) is equivalent to the Sobolev type norm induced by $H^2 \times L^2 \times H^2 \times L^2$. Therefore, the space X, along with the inner product (2.5.137) is a well-defined Hilbert space. Also, note that D(A) is dense in X.

To show existence and uniqueness of solutions to the differential equation (2.5.3)-

(2.5.4), it suffices to show that A generates a contraction semigroup. To prove that A generates a contraction semigroup, we will use the Lumer-Phillips Theorem [Paz. 1, Chapter 1, Theorem 4.3]. To apply this Theorem, we must show that (i) A is dissipative (see Def. B.1), and (ii) for some $\lambda > 0$ the range of $\lambda I - A$ is all of X. To show (i) first note

$$[Af, f]_E = [f_2'', f_1'']_1 + [-f_1'''', f_2]_1 + [f_4'', f_3'']_2 + [-f_3'''', f_4]_2$$

Integrate by parts, and the insert the boundary conditions to obtain

$$[Af, f]_E = [f_2'', f_1''] + [f_1''', f_2'] - f_1'''(x_1)f_2(x_1)$$
$$[f_4'', f_3''] + [f_3''', f_4'] - f_3'''(x_1)f_4(x_1)$$

Integrating by parts again, and inserting the boundary conditions yields

$$[Af, f]_E = -Kf_2^2(x_1)$$

Therefore, A is dissipative.

To show (ii), and thus complete the proof, we need only show that for some $\lambda > 0$ the range of $\lambda I - A$ is all of X. This is done in two steps: (a) For $\lambda = 1$, the range of $I - A$ is dense in X, and (b) the range of $I - A$ is closed.

Proof of (a) - Take $\lambda = 1$, and suppose $\exists\, y \in X$ such that $[(I - A)z, y]_E = 0$ for all $z \in D(A)$. If $z = (z_1\ z_2\ z_3\ z_4)^T$ and $y = (y_1\ y_2\ y_3\ y_4)^T$, then $[(I - A)z, y]_E = 0$ implies

$$[z_1'' - z_2'', y_1'']_1 + [z_1'''' + z_2, y_2]_1 + [z_3'' - z_4'', y_3'']_2 + [z_3'''' + z_4, y_4]_2 = 0 \qquad (2.A.1)$$

Set $z_2 = z_3 = z_4 = 0$. Integrate (2.A.1) by parts to obtain $[z_1'''', y_1 + y_2] = 0$. Now let z_1 be an arbitrary C^∞ function satisfying the boundary conditions $z_1(0)=0$, $z_1'(0)=0$, $z_1''(x_1)=0$, and $z_1'''(x_1)=0$. Then clearly $(z_1, 0, 0, 0)^T \in D(A)$, and the class of such elements is dense in $H_0^4 x\{0\} x\{0\} x\{0\}$. Hence, the equation (2.A.1) implies $y_1 + y_2 = 0$. The only way that this is possible is if $y_1(x) = -y_2(x)$, for all $x \in [0, x_1]$. Next, choose $z_1 = z_3 = z_4 = 0$. The equations then become

$$[-z_2'', y_1'']_1 + [z_2, y_2]_1 = 0 \tag{2.A.2}$$

Next, choose z_2 to be an approximation of y_2, where $z_2 = y_2$ except for arbitrarily small neighborhoods of the initial conditions, which may unfortunately differ. Then (2.A.2) yields $[y_2'', y_1'']_1 = [y_2, y_2]_1$, or $[y_2'', -y_2'']_1 = [y_2, y_2]_1$. The only way that this can occur is if $y_1(x) = -y_2(x) = 0$, for all $x \in [0, x_1]$. By a similar calculation, it can be shown that $y_3(x) = -y_4(x) = 0$, for all $x \in [x_1, 1]$. Hence $y=0$, and thus the range of $I - A$ is dense in X.

Proof of (b) - Let $y_n = (I - A)x_n$ converge to $y \in X$. We must show that $\exists\, x \in D(A)$ such that $y = (I - A)x$. Since A is dissipative, we have

$$\| y_n\|^2 = \|(I - A)x_n\|^2 = \| x_n\|^2 - 2(x_n, Ax_n) + \| Ax_n\|^2 \tag{2.A.3}$$

$$\geq \| x_n\|^2 + \| Ax_n\|^2 \tag{2.A.4}$$

$$\geq \| x_n\|^2 \tag{2.A.5}$$

Since y_n converges, this implies x_n converges to some value $x \in X$. (Consider a Cauchy sequence $y_n - y_m$). Hence, by (2.A.4), Ax_n converges. If $x_n = (x_{n1}\ \ x_{n2}\ \ x_{n3}\ \ x_{n4})^T$ and $x = (x_1\ \ x_2\ \ x_3\ \ x_4)^T$, then

$$Ax_n = (x_{n2} \quad -x_{n1}'''' \; x_{n3} \quad -x_{n4}'''')^T \tag{2.A.6}$$

which shows that $x_1 \in H^4$, $x_2 \in H^2$, $x_3 \in H^4$, $x_4 \in H^2$. This implies that $x \in D(A)$, from which it follows that $y = (I - A)x$. ■

CHAPTER 3

SPACECRAFT DYNAMICS AND STABILIZATION

3.1 Kinematics and Dynamics of a Flexible Spacecraft

In this section the kinematics and dynamics of a flexible spacecraft model will be developed. The structure consists of a flexible, cantilevered beam attached to a rigid body. This particular configuration is becoming popular in the literature (see [Bai. 1], [Mon. 1], [Bis. 1], [Kwa. 1]) because it reflects in many ways the hybrid rigid/flexible structures envisioned in future spacecraft. Indeed, structures such as deformable mirrors, solar panels, or radar/laser arrays attached to rigid spacecraft yield equations similar to those of the proposed model. In this thesis an infinite dimensional model of the flexible beam will be used, instead of the finite dimensional approximation that most authors prefer.

The derivation of the kinematics and dynamics of the structure will be done from an entirely classical approach: the derivations use Newton's third law of motion, and the conservation of angular momentum. Surprisingly, the method does not seem to be the norm of the literature. For example, in [Bai. 1], the equations of motion are derived from a Lagrangian viewpoint. This approach of course has the advantage that the equations of motion can be derived in the same way regardless of the generalized coordinates used. However, the resulting Frechet (Banach space) derivatives must be tediously calculated.

In a classical approach, the main disadvantage is that particular generalized coordinates are chosen, but in most applications the choice is clear. In addition, the approach is ideal for systems with external forces acting on them, which of course is the case for control system design. Finally, the computations of the equations of motion are simpler, because no Frechet derivatives are required.

With this classical approach in mind, this section is ordered as follows. Section 3.1.1 discusses change of coordinates between coordinate frames, and section 3.1.2 derives the resulting kinematics. In section 3.1.3, Gibbs parameters for attitude determination are presented, along with the resulting kinematics. In section 3.1.4, the spacecraft model is precisely defined, and the resulting dynamical equations are derived. In section 3.1.5, the beam model is introduced and the dynamics of the rotating beam are derived. Finally, in section 3.1.6, the equations are combined to give the spacecraft models used in this thesis.

3.1.1 Change of Basis

Vectors, Frames, and Triples

Consider a three dimensional Euclidean space E, which includes the notion of points, lines, distance, angle, etc. A **vector** is a magnitude and a direction. For example, given two points in the space E, O and P, the vector $\underrightarrow{OP}$ is the line segment going from O to P (hence an arrow at P). The concept of a vector is quite different than the position of a point in space. Having chosen an "origin" O, the position of the point P is uniquely specified by the vector $\underrightarrow{OP}$. Note, however, that P may be specified by different origins and different vectors.

We next introduce the concept of a frame. A **frame** is a set $\{O_B, \underrightarrow{b}_1, \underrightarrow{b}_2, \underrightarrow{b}_3\}$ where O_B is a point in E and $\underrightarrow{b}_1, \underrightarrow{b}_2, \underrightarrow{b}_3$ are a family of orthonormal, dextral (righthanded) vectors in E. The point O_B is the origin of a coordinate system determined by the axes $\underset{\rightarrow}{b}_1$, $\underset{\rightarrow}{b}_2$, $\underset{\rightarrow}{b}_3$, where the $\underset{\rightarrow}{b}_i$'s are attached to O_B.

Clearly, a vector may be expressed in terms of the basis vectors in any frame of interest. For example, a unit vector $\underset{\rightarrow}{w}$ may be resolved in terms of the vectors $\underset{\rightarrow}{b}_1$, $\underset{\rightarrow}{b}_2$, $\underset{\rightarrow}{b}_3$ of the frame $\{O_B, \underset{\rightarrow}{b}_1, \underset{\rightarrow}{b}_2, \underset{\rightarrow}{b}_3\}$ by

$$\underset{\rightarrow}{w} = w_1 \underset{\rightarrow}{b}_1 + w_2 \underset{\rightarrow}{b}_2 + w_3 \underset{\rightarrow}{b}_3 \qquad (3.1.1)$$

where w_i is the cosine of the angle between $\underset{\rightarrow}{w}$ and $\underset{\rightarrow}{b}_i$. Using this resolution, we can define the **triple** $\underline{w} = (w_1, w_2, w_3)^T$. The symbol $\underset{\rightarrow}{w}$ refers to a vector in E, which is independent of any frame, while $\underline{w}$ is a triple which presupposes a vector and a frame.

Multiple Frames

Suppose now we fix a frame $\{O_E, \underset{\rightarrow}{e}_1, \underset{\rightarrow}{e}_2, \underset{\rightarrow}{e}_3\}$, which we shall refer to as the inertial frame, and choose another frame $\{O_B, \underset{\rightarrow}{b}_1, \underset{\rightarrow}{b}_2, \underset{\rightarrow}{b}_3\}$, which will be referred to as the body frame. The basic problem is to specify the position and orientation of the body frame with respect to the inertial frame. The position of the origin of the body frame, O_B, with respect to the inertial frame is specified by a vector $\underset{\rightarrow}{y}$ defined by

$$O_B = O_E + \underset{\rightarrow}{y}, \quad \text{that is,} \quad \underset{\rightarrow}{y} = \underset{\rightarrow}{O_E O_B} \qquad (3.1.2)$$

One way to specify the orientation of the body frame is to resolve each $\underset{\rightarrow}{b}_i$, i = 1, 2, 3, along the inertial vectors $\underset{\rightarrow}{e}_1$, $\underset{\rightarrow}{e}_2$, $\underset{\rightarrow}{e}_3$, exactly as in (3.1.1):

$$\underrightarrow{b}_1 = y_{11}\,\underrightarrow{e}_1 + y_{12}\,\underrightarrow{e}_2 + y_{13}\,\underrightarrow{e}_3 \tag{3.1.3}$$

$$\underrightarrow{b}_2 = y_{21}\,\underrightarrow{e}_1 + y_{22}\,\underrightarrow{e}_2 + y_{23}\,\underrightarrow{e}_3 \tag{3.1.4}$$

$$\underrightarrow{b}_3 = y_{31}\,\underrightarrow{e}_1 + y_{32}\,\underrightarrow{e}_2 + y_{33}\,\underrightarrow{e}_3 \tag{3.1.5}$$

Let $Y = \{y_{ij}\}$, so that $[\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3] = [\,\underrightarrow{e}_1\ \ \underrightarrow{e}_2\ \ \underrightarrow{e}_3]Y$. The matrix Y is called the direction cosine matrix because the ij element of Y, $(Y)_{ij,}$ is the cosine of the angle between $\underline{b}_i$ and $\underline{e}_j$. Note that orthonormality of $\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3$ and $\underrightarrow{e}_1$, $\underrightarrow{e}_2$, $\underrightarrow{e}_3$ imply that Y is an orthogonal matrix, i.e., $Y^TY = I$. Since both frames are dextral, this means that det(Y)=1.

Note also that Y is a coordinate transformation that maps triples determined by the body vectors $\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3$ to triples determined by the inertial vectors $\underrightarrow{e}_1$, $\underrightarrow{e}_2$, $\underrightarrow{e}_3$. That is, if $\underrightarrow{R} = R_1\,\underrightarrow{b}_1 + R_2\,\underrightarrow{b}_2 + R_3\,\underrightarrow{b}_3$, then the components of $Y\underline{R}_B$ are the components of the vector $\underrightarrow{R}$ resolved along the vectors $\underrightarrow{e}_1$, $\underrightarrow{e}_2$, $\underrightarrow{e}_3$, i.e. $Y\underline{R}_B = \underline{R}_E$.

To summarize, consider a point P which is specified by the vector $\underrightarrow{r} = \underrightarrow{O_EP}$ with respect to the inertial frame. P can also be specified by a vector $\underrightarrow{R} = \underrightarrow{O_BP}$ with respect to the body frame. Thus, P can also be specified by a vector $\underrightarrow{y} + \underrightarrow{R}$ with respect to the inertial frame i.e.,

$$\underrightarrow{r} = \underrightarrow{y} + \underrightarrow{R} \tag{3.1.6}$$

If $\underline{R}$ refers to the triple of $\underrightarrow{R}$ with respect to the *body* frame, $\underline{y}$ the triple of $\underrightarrow{y}$ with respect to the *inertial* frame, and $\underline{r}$ the triple of $\underrightarrow{r}$ with respect to the *inertial* frame, then

$$\underline{r} = \underline{y} + Y\underline{R} \tag{3.1.7}$$

For further details on this material, the reader is referred to the excellent exposition of [Wer. 1, Chapter 12]. [Hug. 1], [Mort. 1] or [Kane. 1] are also good sources.

3.1.2 Kinematics

Angular Velocity

We now consider the time derivative of Y defined by (?). Since all elements of Y are scalar functions of time, the time derivative of Y, denoted $\dot{Y}$, is simply

$$\dot{Y} := \begin{pmatrix} \dot{y}_{11} & \dot{y}_{12} & \dot{y}_{13} \\ \dot{y}_{21} & \dot{y}_{22} & \dot{y}_{23} \\ \dot{y}_{31} & \dot{y}_{32} & \dot{y}_{33} \end{pmatrix} \tag{3.1.8}$$

where the overdot can be applied without fear of ambiguity. In a slight abuse of notation, we now define the matrix $\underline{\omega}^x \in \mathbb{R}^{3x3}$ by $\underline{\omega}^x := Y^T\dot{Y}$. (This notation is used in [Hug. 1].) First, note that $\underline{\omega}^x$ is skew-symmetric. This follows since $\underline{\omega}^x + (\underline{\omega}^x)^T = Y^T\dot{Y} + (Y^T\dot{Y})^T = Y^T\dot{Y} + \dot{Y}^TY = \frac{d}{dt}(Y^TY) = \dot{I} = 0$. Hence, $\underline{\omega}^x$ has the representation

$$\underline{\omega}^x := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} . \tag{3.1.9}$$

for some $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$. We then define the triple $\underline{\omega} := (\omega_1, \omega_2, \omega_3)^T$. Although $\underline{\omega}$ is not the triple of a particular vector (it is simply the components of an algebraic relation between Y and $\dot{Y}$), following tradition we nevertheless define a vector $\underset{\rightarrow}{\omega}$ by $\underset{\rightarrow}{\omega} := \omega_1 \underset{\rightarrow}{b}_1 + \omega_2 \underset{\rightarrow}{b}_2 + \omega_3 \underset{\rightarrow}{b}_3$, which we shall call the **angular velocity vector** of the body frame with respect to the inertial frame.

Kinematics of Vectors and Triples

Consider again the point P of the previous section specified by the vector $\underset{\rightarrow}{r} = \underset{\longrightarrow}{O_E P}$ with respect to the inertial frame. If the point P is moving in space, then an observer in the inertial frame and an observer in the body frame see different motions for P due to the relative motions of the frames. We denote vector time-derivatives with respect to the inertial frame by an overdot, $(\dot{\ })$, and a subscript t, $(\ )_t$ for time derivatives with respect to the body frame. Note that $\dot{\underset{\rightarrow}{e}}_i = 0$, and $\underset{\rightarrow}{b}_{it} = 0$, for i=1, 2, 3.

Let $\underset{\rightarrow}{r} = \underset{\longrightarrow}{O_E P}$ be specified by the triple $\underline{w} = (w_1, w_2, w_3)^T$. Differentiating $\underset{\rightarrow}{r}$ with respect to the inertial frame yields

$$\dot{\underset{\rightarrow}{r}} = [\ \dot{\underset{\rightarrow}{e}}_1\ \ \dot{\underset{\rightarrow}{e}}_2\ \ \dot{\underset{\rightarrow}{e}}_3]\underline{r} + [\ \underset{\rightarrow}{e}_1\ \ \underset{\rightarrow}{e}_2\ \ \underset{\rightarrow}{e}_3]\,\dot{\underline{r}} = [\ \underset{\rightarrow}{e}_1\ \ \underset{\rightarrow}{e}_2\ \ \underset{\rightarrow}{e}_3]\,\dot{\underline{r}} \qquad (3.1.10)$$

since the rate of change of the inertial frame with respect to itself is zero, i.e., $\dot{\underset{\rightarrow}{e}}_i = 0$. Note also that the time derivative of a triple, unlike the time derivative of a vector, *can have only one meaning*, so the simple overdot can be applied without ambiguity to the triple $\underline{r}$. Similarly, let $\underset{\rightarrow}{R} = [\ \underset{\rightarrow}{b}_1\ \ \underset{\rightarrow}{b}_2\ \ \underset{\rightarrow}{b}_3]\underline{R}$ then

$$\underset{\rightarrow}{R}_t = [\ \underset{\rightarrow}{b}_1\ \ \underset{\rightarrow}{b}_2\ \ \underset{\rightarrow}{b}_3]\,\dot{\underline{R}} \qquad (3.1.11)$$

Therefore, differentiating (3.1.6), $\underset{\rightarrow}{r} = \underset{\rightarrow}{Y} + \underset{\rightarrow}{R}$, yields

$$
\begin{aligned}
\dot{\underrightarrow{r}} &= \dot{\underrightarrow{y}} + [\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3]\underline{R} + [\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3]\,\dot{\underline{R}} \\
&= \dot{\underrightarrow{y}} + [\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3]\underline{R} + \underrightarrow{R}_t
\end{aligned}
\tag{3.1.12}
$$

To determine $[\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3]$, differentiate (3.1.3)-(3.1.5), $[\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3] = [\,\underrightarrow{e}_1\ \ \underrightarrow{e}_2\ \ \underrightarrow{e}_3]Y$, with respect to the inertial frame:

$$
[\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3] = [\,\dot{\underrightarrow{e}}_1\ \ \dot{\underrightarrow{e}}_2\ \ \dot{\underrightarrow{e}}_3]Y + [\,\underrightarrow{e}_1\ \ \underrightarrow{e}_2\ \ \underrightarrow{e}_3]\,\dot{Y}
\tag{3.1.13}
$$

Inserting $\dot{\underrightarrow{e}}_i = 0$, $\dot{Y} = Y\underline{\omega}^x$, and $[\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3] = [\,\underrightarrow{e}_1\ \ \underrightarrow{e}_2\ \ \underrightarrow{e}_3]Y$ into the expression (3.1.13) yields in succession

$$
\begin{aligned}
[\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3] &= [\,\underrightarrow{e}_1\ \ \underrightarrow{e}_2\ \ \underrightarrow{e}_3]Y\underline{\omega}^x \\
&= [\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3]\underline{\omega}^x \qquad \text{or,}
\end{aligned}
$$

$$
[\,\dot{\underrightarrow{b}}_1\ \ \dot{\underrightarrow{b}}_2\ \ \dot{\underrightarrow{b}}_3] = \underrightarrow{\omega} \times [\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3]
\tag{3.1.14}
$$

where $\times$ denotes the usual vector cross product. Thus, using (3.1.14)) in (3.1.12)) yields

$$
\begin{aligned}
&\dot{\underrightarrow{r}} = \dot{\underrightarrow{y}} + \underrightarrow{\omega} \times [\,\underrightarrow{b}_1\ \ \underrightarrow{b}_2\ \ \underrightarrow{b}_3]\underline{R} + \underrightarrow{R}_t, \quad \text{or} \\
&\dot{\underrightarrow{r}} = \dot{\underrightarrow{y}} + \underrightarrow{\omega} \times \underrightarrow{R} + \underrightarrow{R}_t
\end{aligned}
\tag{3.1.15}
$$

Equation (3.1.15) is a coordinate free expression that gives the relates the velocity of a point as seen by an observer in the inertial frame ($\dot{\underrightarrow{r}}$), to the velocity of the point as seen by the observer in the body frame ($\underrightarrow{R}_t$). In inertial coordinates, this calculation is similar. From (3.1.7), $\underline{r} = \underline{y} + Y\underline{R}$, we obtain by time differentiation

$$\dot{\underline{r}} = \dot{\underline{y}} + \dot{Y}\underline{R} + Y\dot{\underline{R}}$$

$$\dot{\underline{r}} = \dot{\underline{y}} + Y\underline{\omega}^{x}\underline{R} + Y\dot{\underline{R}} \tag{3.1.16}$$

where we have again used $\dot{Y} = Y\underline{\omega}^{x}$. As stated previously, there is no ambiguity with differentiation of triples with respect to time since they are not vectorial functions.

Suppose now that the point P is accelerated with respect to either frame. Then, by computation in coordinates

$$\begin{aligned}\ddot{\underline{r}} &= \ddot{\underline{y}} + \dot{Y}\underline{\omega}^{x}\underline{R} + Y\dot{\underline{\omega}}^{x}\underline{R} + Y\underline{\omega}^{x}\dot{\underline{R}} + \dot{Y}\dot{\underline{R}} + Y\ddot{\underline{R}} \\ &= \ddot{\underline{y}} + Y\underline{\omega}^{x}(\underline{\omega}^{x}\underline{R}) + Y\dot{\underline{\omega}}^{x}\underline{R} + 2Y\underline{\omega}^{x}\dot{\underline{R}} + Y\ddot{\underline{R}} \\ &= \ddot{\underline{y}} + Y(\underline{\omega}^{x}(\underline{\omega}^{x}\underline{R}) + \dot{\underline{\omega}}^{x}\underline{R} + 2\underline{\omega}^{x}\dot{\underline{R}} + \ddot{\underline{R}})\end{aligned} \tag{3.1.17}$$

As was the case with $\underset{\rightarrow}{\omega}$, define $\dot{\underset{\rightarrow}{\omega}}$ to be $\dot{\underset{\rightarrow}{\omega}} := \dot{\omega}_1 \underset{\rightarrow}{b}_1 + \dot{\omega}_2 \underset{\rightarrow}{b}_2 + \dot{\omega}_3 \underset{\rightarrow}{b}_3$, which we shall call the *angular acceleration vector* of the body frame with respect to the inertial frame. Using this definition, one can derive the vector analog of (3.1.17):

$$\ddot{\underset{\rightarrow}{r}} = \ddot{\underset{\rightarrow}{y}} + \underset{\rightarrow}{R}_{tt} + \dot{\underset{\rightarrow}{\omega}} \times \underset{\rightarrow}{R} + 2\underset{\rightarrow}{\omega} \times \underset{\rightarrow}{R}_t + \underset{\rightarrow}{\omega} \times (\underset{\rightarrow}{\omega} \times \underset{\rightarrow}{R}) \tag{3.1.18}$$

The physical interpretation of the terms in equation (3.1.18) is as follows. The first term is the acceleration of the origin of the body frame with respect to the inertial frame, the second term is the acceleration of P with respect to the body frame, the third term gives the acceleration due to the angular acceleration $\dot{\underset{\rightarrow}{\omega}}$, the fourth term is the Coriolis acceleration, and the last term is the centripetal acceleration.

3.1.3 Gibbs Parameters

Since Y defined by (3.1.3)-(3.1.5) is an orthogonal matrix, it has redundant parameters. Indeed, $Y^TY = I$ implies that y_{11}, y_{21}, and y_{31} are related by $y_{11}^2 + y_{21}^2 + y_{31}^2 = 1$, etc. Thus, to completely specify the orientation we do not need all 9 parameters. It can easily be shown that if Y is a real orthogonal matrix, with det(Y) = 1, then Y has 1 as an eigenvalue. This means there is a real triple $\underline{b}$ such that $Y\underline{b} = \underline{b}$. This gives Euler's Theorem [Kane 1, p. 14]: The most general displacement of a rigid body with one point fixed is equivalent to a rotation of the body about some axis. Thus, to completely specify the orientation, only three parameters are needed: two parameters specifying an axis of rotation, and a third parameter specifying the angle of rotation about the axis. One way of doing this is to define a triple (called the "Gibbs Parameters", or the "Rodrigues parameters") [Kane 1, p. 16] $\underline{\xi} \in \mathbb{R}^3$ by

$$\underline{\xi}=\tan(\phi/2)\underline{e} \qquad \phi \in (-\pi, \pi) \tag{3.1.19}$$

where ϕ is the angle of rotation (in radians) of the body frame about the axis of rotation $\underline{e} \in \mathbb{R}^3$. As in the case of $\underline{\omega}$, $\underline{\xi}$ is not the triple of a particular vector, but is simply a description of the rotation that must be applied to $\{O_B, \underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ to orient $\{O_B, \underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ with $\{O_E, \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$.

Since $\underline{\xi}$ and Y both specify the orientation of the body frame with respect to the inertial frame, they must be related. In fact, Y can be parametrized in terms of $\underline{\xi}$ by [Kane 1, p. 17]

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x \,] - I \tag{3.1.20}$$

where I is the 3x3 identity matrix, T denotes transpose and $\underset{\sim}{\xi}^{x}$ is the matrix representation of the cross-product with $\underset{\sim}{\xi}$ as in (3.1.9).

Gibbs Parameters Kinematics

From $\dot{Y} = Y\underline{\omega}^{x}$ and the definition of $\underset{\sim}{\xi}$ given above, along with the equation relating Y and $\underset{\sim}{\xi}$, (3.1.20), one can obtain the following differential equation for $\underset{\sim}{\xi}$: [Kane 1, p. 62]

$$\dot{\underset{\sim}{\xi}} = \frac{1}{2}[I + \underset{\sim}{\xi}\underset{\sim}{\xi}^{T} + \underset{\sim}{\xi}^{x}]\underline{\omega}. \tag{3.1.21}$$

If $t \rightarrow \underline{\omega}(t)$ is known, then this differential equation can be solved, (starting from some initial attitude $\underset{\sim}{\xi}(t_0)$), yielding $\underset{\sim}{\xi}(t)$ for all $t \geq t_0$, and hence by (3.1.20) $Y(\underset{\sim}{\xi})$ for all $t \geq t_0$.

One comment should be made at this point regarding solution of the differential equation (3.1.21). If the initial orientation of the body frame is a π rotation about some axis with respect to the inertial frame, then from (3.1.19) we see that $\underset{\sim}{\xi}(t_0) = \infty$. For such a case, one needs to redefine the inertial frame (say by rotation about an appropriate axis) so that $\underset{\sim}{\xi}(t_0) \neq \infty$. Such a procedure will also be necessary if one desires to drive the attitude to an orientation which is a π rotation about some axis with respect to the inertial frame.

There are, of course, other parametrizations of Y by attitude variables. In fact, most authors use either Euler quaternions or Euler angles for the parametrization [Dwy. 1, 3], [Mon. 1], [Vad. 1]. However, as discussed in [Dwy. 2], the Gibbs parameters are probably the best choice of kinematic variable for control synthesis in that it avoids state constraints and/or feedback singularities that are usually present when other variables are used. These problems occur because more than three parameters are used to specify the attitude, which means that the resulting control problems have constraints due to parameter redundancy. (Only three parameters are required to specify orientation.)

3.1.4 Spacecraft and Reaction Wheel Dynamics

The physical model is depicted in Figure 3.1. The structure consists of a rigid body in which a thin, flexible, cantilevered beam-like appendage of length L is attached. Assume the beam is uniform and of constant cross-section.

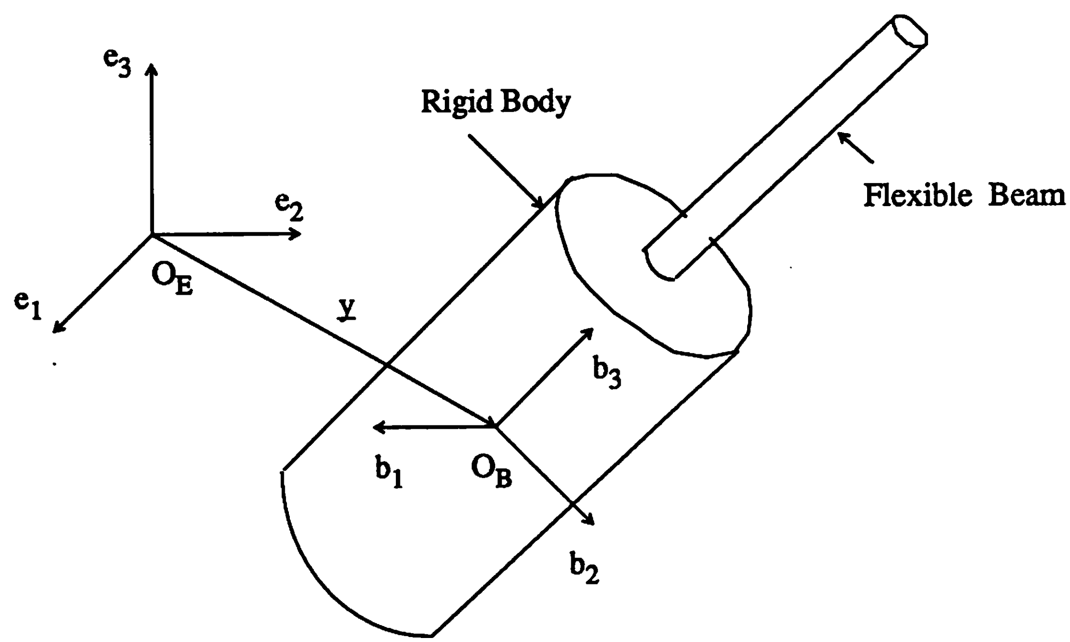

Figure 3.1 - Spacecraft Configuration

Affix the dextral body coordinate frame, denoted $\{O_B, \underset{\rightarrow}{b}_1, \underset{\rightarrow}{b}_2, \underset{\rightarrow}{b}_3\}$, to the rigid body center of mass O_B. $\{\underset{\rightarrow}{b}_1, \underset{\rightarrow}{b}_2, \underset{\rightarrow}{b}_3\}$ are orthonormal vectors which coincide with the rigid body principal axes of inertia; in addition, assume the $\underset{\rightarrow}{b}_3$ axis coincides with the centroidal axis of the undeflected beam, and that the beam is attached at $c\,\underset{\rightarrow}{b}_3$.

Control inputs mounted on the rigid body (not shown in Figure 3.1) consist of three torque jets, three force thrusters and three momentum wheels. For i=1, 2, 3, the torque jets J_i produce a pure torque τ_i about the $\underset{\to}{b}_3$ axis, the ith thruster produces a force F_{Ti}' in the direction $\underset{\to}{b}_3$, and the ith momentum wheel spins about an axis parallel to $\underset{\to}{b}_3$, thus also producing a torque τ_i' about $\underset{\to}{b}_3$.

First, some notation will be needed. Let I_0 be the rigid body inertia tensor (including torque jets and *locked* wheels) calculated with respect to the body frame $\{O_B, \underset{\to}{b}_1, \underset{\to}{b}_2, \underset{\to}{b}_3\}$. Let $I_A = \mathrm{diag}(I_{w1}, I_{w2}, I_{w3}) \in \mathbf{R}^{3x3}$ where I_{wi} is the rotary inertia of wheel i. Let $\underline{\Omega}_w = (\Omega_{w1}, \Omega_{w2}, \Omega_{w3})^T \in \mathbf{R}^3$ where Ω_{wi} denotes the angular velocity of wheel i about its axle (in body coordinates), and let m_B be the mass of the rigid body. Let $\underset{\to}{\tau} = \sum_{i=1}^{3} \tau_i \underset{\to}{b}_i$ denote the torque due to the torque jets, with $\underline{\tau} = (\tau_1, \tau_2, \tau_3)^T$ denoting the triple of $\underset{\to}{\tau}$ with respect to the body frame. Finally, $\underset{\longrightarrow}{F_{bB}} = \sum_{i=1}^{3} F_{bBi} \underset{\to}{b}_i$ is the force the beam exerts on the body at $c\underset{\to}{b}_3$, with corresponding triple $\underline{F}_{bB} = (F_{bB1}, F_{bB2}, F_{bB3})^T$ with respect to the body frame, while $\underset{\longrightarrow}{M_{bB}} = \sum_{i=1}^{3} M_{bBi} \underset{\to}{b}_i$ is the moment the beam exerts on the body at $c\underset{\to}{b}_3$, with corresponding triple $\underline{M}_{bB} = (M_{bB1}, M_{bB2}, M_{bB3})^T$ with respect to the body frame.

Rigid Body

For simplicity, all calculations will be performed in coordinates, rather than vectors. It should be stressed that nearly all the calculations can be performed using vectors. The reader is encouraged to try these calculations, using methods identical to that of section 3.1.1.

Now consider a free-body diagram drawn around the rigid body portion (excluding thrusters) of the spacecraft. Note that the triple of the angular momentum of the rigid body with respect to the body frame, calculated about O_B and, denoted by $\underline{h}$, is $I_0\underline{\omega} + I_A\underline{\Omega}_w$. Therefore, the triple of the angular momentum with respect to the inertial frame is $Y\underline{h}$. Therefore, computing the time derivative $Y\underline{h}$ yields

$$I_0\underline{\dot{\omega}} + I_A\,\underline{\dot{\Omega}}_w + \underline{\omega} x I_0 \underline{\omega} + \underline{\omega} x I_A \underline{\Omega}_w = \underline{\tau} + \underline{c}^x\,\underline{F}_{bB} + \underline{M}_{bB}. \tag{3.1.22}$$

where $\underline{c} = (0, 0, c)^T$. The right-hand side of (3.1.22) is the net torque (calculated about O_B) applied to the rigid body. It is composed of the torque due to the torque jets, and the net moment that the beam applies to the rigid body. Next, apply Newton's third law of motion to the free-body with respect to O_E, an inertial frame. Assume that the torque jets apply no net force on the rigid body. Since $\underline{y}$ gives the coordinates of O_B with respect to the inertial frame,

$$m_B\underline{\ddot{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{3.1.23}$$

where $\underline{F}_T$ is the triple of $\vec{F}_T$ of the force thrusters with respect to the *inertial* frame. (Hence, $\underline{F}_T = Y\underline{F}_T'$.)

Momentum Wheels

Now draw a free-body diagram about the momentum wheels alone. Compute the rate of change of the angular momentum associated with the momentum wheels with respect to the inertial frame, and write out the components associated with the wheel axles, to obtain, in matrix form,

$$I_A(\underline{\dot{\omega}} + \underline{\dot{\Omega}}_w) = \underline{\tau}' \tag{3.1.24}$$

where $\underline{\tau}' = (\tau_1', \tau_2', \tau_3')^T$, τ_i' is the torque exerted by the ith motor on the rotor of the ith wheel. Complete details for this calculation, and the others above, can be found in many sources, for example [Hug. 1, p. 67].

Finally, substituting (3.1.24) into (3.1.22) yields

$$(I_0 - I_A)\underline{\dot{\omega}} + \underline{\omega}^x I_0 \underline{\omega} + \underline{\omega}^x I_A \underline{\Omega}_w = \underline{\tau} - \underline{\tau}' + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}. \tag{3.1.25}$$

3.1.5 Beam Dynamics

Consider now a free-body diagram drawn around an infinitesimal section of the beam located between $z\,\underset{\rightarrow}{b}_3$ and $(z+dz)\,\underset{\rightarrow}{b}_3$. (See Figure 2.2 and Figure 2.3 in Chapter 2.) Let $\underline{u} = (u_1, u_2, u_3+z)^T$ denote the triple of $\underset{\rightarrow}{u}$ with respect to the body frame of a point p whose undeformed position is $z\,\underset{\rightarrow}{b}_3$, and let $\underline{u}_t$ denote the triple with respect to the body frame of the rate of change of $\underset{\rightarrow}{u}$ with respect to the body frame. Let $\underset{\rightarrow}{F(z)} = \sum_{i=1}^{3} F_i\,\underset{\rightarrow}{b}_i$ denote the shear force acting on the section of the beam at $z\,\underset{\rightarrow}{b}_3$, and $\underset{\rightarrow}{F(z+dz)}$ the force acting on the section of the beam at $(z+dz)\,\underset{\rightarrow}{b}_3$. Then, since the acceleration of P with respect to the inertial frame is given by (3.1.9), writing Newton's third law with respect to the inertial frame yields

$$\underline{u}_{tt} + \underline{\dot{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) - d\underline{F}(z) + Y^{-1}\underline{\ddot{y}} = 0. \tag{3.1.26}$$

Note that it is assumed the beam mass per unit length is unity, and

$$d\underline{F}(z) := \lim_{dz \to 0} \frac{\underline{F}(z + dz) - \underline{F}(z)}{dz} . \tag{3.1.27}$$

Up to this point no beam model has been employed. The term $d\underline{F}(z)$ is the only term in (3.1.26) which is model dependent. We will model the beam as an Euler-Bernoulli type beam, with Voight-Kelvin damping [Pop. 1, p.116] (often referred to as viscous damping), and for simplicity we will ignore torsion. Let μ_i ($= E_i I_i$ in Chapter 2) denote the flexural rigidity of the beam in the ith direction, and let k_i ($= \mu_i I_i$ in Chapter 2) be a positive constant reflecting the rate of energy dissipation of the beam in the ith direction, i=1, 2, 3. Assume for simplicity that the beam has its principal axes of inertia parallel to the principal axes of the rigid body, so that the expression for $\underline{F}(z)$ for an Euler-Bernoulli beam becomes (see (2.2.4) and (2.3.13))

$$\underline{F}(z) = -\mu\partial'(\underline{u}) - k\partial'(\underline{u}_t) \tag{3.1.28}$$

where $\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$, $k = \mathrm{diag}(k_1, k_2, k_3)$ and $\partial'(\cdot) := \left(\frac{\partial^3(\cdot)_1}{\partial z^3}, \frac{\partial^3(\cdot)_2}{\partial z^3}, -\frac{\partial(\cdot)_3}{\partial z}\right)^T$.
Hence, for such a beam $d\underline{F}(z)$ becomes

$$d\underline{F}(z) = -\mu\partial(\underline{u}) - k\partial(\underline{u}_t) \tag{3.1.29}$$

where $\partial(\cdot) = \left(\frac{\partial^4(\cdot)_1}{\partial z^4}, \frac{\partial^4(\cdot)_2}{\partial z^4}, -\frac{\partial^2(\cdot)_3}{\partial z^2}\right)^T$. Insert (3.1.29) into (3.1.26) to obtain

$$\underline{u}_{tt} + \underline{\dot{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\underline{\ddot{y}} = 0 \tag{3.1.30}$$

Suppose force thrusters F_i, i=1, 2, 3 are present at the end of the beam with the direction of F_i parallel to $\underline{b}_i$. The force due to thruster F_i is positive if the force exerted is in the direction $\underline{b}_i$. Then the boundary conditions for this configuration become (see Chapter 2, and [Pop. 1, pp. 385-386, 124, 128])

$$\begin{aligned}
&u_1(c) = u_2(c) = u_3(c) = 0,\ \ u_1'(c) = u_2'(c) = 0 \\
&u_1''(c + L) = u_2''(c + L) = 0 \\
&\mu_i u_i'''(c + L) + k_i u_{it}'''(c + L) = -F_i, \quad i = 1, 2. \\
&\mu_3 u_3'(c + L) + k_3 u_{3t}'(c + L) = -F_3
\end{aligned} \tag{3.1.31}$$

The first set of boundary conditions indicate that no deflections occur at the point of attachment and that there is no slope in the transverse directions at the point of attachment. The second set indicates that there is no moment at the free end of the beam, while the third and fourth set indicates that the force at the point of attachment is $F_i(t)$.

To complete the derivation of the spacecraft model, we must determine the relationship between $\underline{F}_{bB}$ in equations (3.1.23) and (3.1.25) and the beam, and between $\underline{M}_{bB}$ in equation (3.1.25) and the beam. From (3.1.28)

$$\underline{F}_{bB} = \underline{F}(c) = -\mu\partial'(\underline{u})|_c - k\partial'(\underline{u}_t)|_c. \tag{3.1.32}$$

As for the moment at the point of attachment $\underline{M}_{bB}$, note that there is no moment due to axial effects. Further, we ignore torsion. Therefore, $(\underline{M}_{bB})_3 = 0$. The transverse moments are, as in Chapter 2,

$$\underline{M}(z) = (\mu_1 u_1''(z) + k_1 u_{1t}''(z),\ \ \mu_2 u_2''(z) + k_2 u_{2t}''(z),\ \ 0)^T \tag{3.1.33}$$

Therefore,

$$\underline{M}_{bB} = \underline{M}(c) = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\ \ \mu_2 u_2''(c) + k_2 u_{2t}''(c),\ \ 0)^T \tag{3.1.34}$$

In this thesis, other formulations of $\underline{F}_{bB}$ and $\underline{M}_{bB}$ will be useful. Using the boundary conditions for a free-end beam (which occurs when $\underline{F} = \underline{0}$ at $z = L$ in (3.1.31)) we also obtain

$$\underline{F}_{bB} = \int_c^{c+L} [\mu\partial(\underline{u}) + k\partial(\underline{u}_t)\ ]dz \tag{3.1.35}$$

$$\underline{M}_{bB} = \int_c^{c+L} \left(\int_z^L \mathrm{col}[(\mu\partial(\underline{u}) + k\partial(\underline{u}_t))_{1,2},\ 0]dx \right) dz \tag{3.1.36}$$

where $\mathrm{col}(\underline{a}, \underline{b})$ is the column vector $(\underline{a}^T, \underline{b}^T)^T$, and $(\underline{a})_{1,2} = (a_1, a_2)^T$. These equations are obtained by direct integration.

There is one more formulation that will occasionally be used in this thesis. Recall that $\underline{u} = (u_1, u_2, z + u_3)^T$ where the u_i are the deflections in the direction $\underline{b}_i$. Then, using (3.1.34), (3.1.28) and the boundary conditions for the free-end beam, fixed end beam (which occurs when $\underline{F}$ is zero)

$$\underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} = \int_c^{c+L} \underline{u}^x\ [\mu\partial(\underline{u}) + k\partial(\underline{u}_t)\ ]dz \tag{3.1.37}$$

as a simple integration will show. (The right hand side is simply the summation of the infinitesimal moments about O_E.)

In this paper, all these formulations will be used extensively. It should be noted that we know of no other author that uses the formulation (3.1.20) - (3.1.21). In fact, one author [Bai. 1, p.52, Remark 4.5] claims that there is no relationship of the form (3.1.35) and (3.1.37)!

3.1.6 Spacecraft Models

Flexible Space Structure With Momentum Wheels (FSSMW)

Sections 3.1.1-3.1.5 gave the kinematics and dynamics for the flexible spacecraft illustrated in Figure 3.1. Combining (3.1.21), (3.1.25), (3.1.24), (3.1.23), (3.1.20), (3.1.32), (3.1.34), (3.1.30) and (3.1.31), we obtain the model of the flexible spacecraft used in this paper:

$$
\text{(FSSMW)}\quad
\left[
\begin{array}{ll}
\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. & (3.1.38) \\
(I_0 - I_A)\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} + \underline{\omega}^x I_A \underline{\Omega}_w = \underline{\tau} - \underline{\tau}' + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}. & (3.1.39) \\
I_A(\dot{\underline{\omega}} + \dot{\underline{\Omega}}_w) = \underline{\tau}' & (3.1.40) \\
m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T & (3.1.41) \\
Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\,I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I & (3.1.42) \\
\underline{F}_{bB} = \underline{F}(c) = -\mu\partial'(\underline{u})\big|_c - k\partial'(\underline{u}_t)\big|_c. & (3.1.43) \\
\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\ \ \mu_2 u_2''(c) + k_2 u_{2t}''(c),\ \ 0)^T & (3.1.44) \\
\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0 & (3.1.45) \\
u_1(c) = u_2(c) = u_3(c) = 0,\ \ u_1'(c) = u_2'(c) = 0 & \\
u_1''(c + L) = u_2''(c + L) = 0 & (3.1.46) \\
\mu_i u_i'''(c + L) + k_i u_{it}'''(c + L) = -F_i, \quad i = 1, 2. & \\
\mu_3 u_3'(c + L) + k_3 u_{3t}'(c + L) = -F_3 &
\end{array}
\right.
$$

As noted before, equations (3.1.43) and (3.1.44) will be occasionally modified to the relations (3.1.35) and (3.1.36). Also, the expression $\underline{c}^{x}\ \underline{F}_{bB} + \underline{M}_{bB}$ in (3.1.39) will sometimes be replaced by (3.1.37).

For simplicity, refer to the set of equations of the flexible space structure with momentum wheels as (FSSMW).

Rigid Structure With Momentum Wheels (RSMW)

In the case where the beam is absent, we also obtain the equations of motion of a rigid spacecraft with momentum wheels:

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^{T} + \underline{\xi}^{x}]\underline{\omega}. \tag{3.1.47}$$

(RSMW)

$$(I_0 - I_A)\dot{\underline{\omega}} + \underline{\omega}^{x}I_0\underline{\omega} + \underline{\omega}^{x}I_A\underline{\Omega}_w = \underline{\tau} - \underline{\tau}' \tag{3.1.48}$$

$$I_A(\dot{\underline{\omega}} + \dot{\underline{\Omega}}_w) = \underline{\tau}' \tag{3.1.49}$$

$$m_B\ddot{\underline{y}} = \underline{F}_T \tag{3.1.50}$$

Let this system of equations for the rigid structure with momentum wheels be denoted as (RSMW).

Flexible Space Structure Without Momentum Wheels (FSS)

We will often be concerned with controlling the spacecraft maneuvers with the torque jets and beam actuators only. In actual spacecraft, the torque jets are used to move the spacecraft around (due to their high achievable torque), while the momentum wheels are used to precisely cancel the effects of small environmental torques such as solar torques [Ben. 1], and make fine attitude adjustments. Thus, our standard flexible spacecraft model, denoted (FSS), is (FSSMW) with the momentum wheels locked:

(FSS)

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{3.1.51}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}. \tag{3.1.52}$$

$$m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{3.1.53}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{3.1.54}$$

$$\underline{F}_{bB} = \underline{F}(c) = -\mu\partial'(\underline{u})\big|_c - k\partial'(\underline{u}_t)\big|_c. \tag{3.1.55}$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\quad \mu_2 u_2''(c) + k_2 u_{2t}''(c),\quad 0)^T \tag{3.1.56}$$

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x\,\underline{u} + 2\underline{\omega}^x\,\underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0 \tag{3.1.57}$$

$$u_1(c) = u_2(c) = u_3(c) = 0,\quad u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0 \tag{3.1.58}$$

$$\mu_i u_i'''(c + L) + k_i u_{it}'''(c + L) = -F_i, \qquad i = 1, 2.$$

$$\mu_3 u_3'(c + L) + k_3 u_{3t}'(c + L) = -F_3$$

Rigid Structure Without Momentum Wheels (RS)

Our standard rigid spacecraft model, denoted (RS), is (RSMW) with the momentum wheels locked:

(RS)

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{3.1.59}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} \tag{3.1.60}$$

$$m_B\ddot{\underline{y}} = \underline{F}_T \tag{3.1.61}$$

Note for either rigid spacecraft model (with or without momentum wheels) the rotational and translational terms are decoupled. From a control point of view, this means we can perform maneuvers on each of the terms separately, and there is never a need to fire the force thrusters and torque jets simultaneously. For either flexible spacecraft model, these terms are coupled. This is because the beam flexes and changes the center of mass of the spacecraft, and conversely the acceleration of the center of mass of the rigid body and the rotation of the rigid body causes the beam to flex. In general, this means that both sets of actuators will be needed for each type of maneuver.

Remark 3.1.1 - It is interesting to examine what happens to (FSS) (without the momentum wheels) as the flexural rigidities μ_i, i= 1, 2, 3, go to infinity, i.e., as the structure becomes rigid. Intuitively, (3.1.52), (3.1.53), and (3.1.57) should reduce to (3.1.60) and (3.1.61). First consider (3.1.52). It can be rewritten

$$I_0\underline{\dot{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} + \int_c^{c+L} u^x \left(\underline{u}_{tt} + \underline{\dot{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\underline{\ddot{y}} \right) dz \qquad (3.1.62)$$

where (3.1.37) and (3.1.57) have been used. Since $\mu_i \to \infty$, $\underline{u}_t \to 0$ and $\underline{u}_{tt} \to 0$ (3.1.62) becomes

$$I_0\underline{\dot{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} - I_b\underline{\dot{\omega}} - \underline{\omega}^x I_b\underline{\omega}$$

where I_b is the inertia tensor of the beam calculated in the body frame about O_B. Thus (3.1.52) indeed becomes (3.1.61) in the limit. Next consider (3.1.53). Using the reformulation (3.1.35)

$$m_B\ddot{\underline{y}} = Y(\xi)\int_c^{c+L} [\mu\partial(\underline{u}) + k\partial(\underline{u}_t)]dz + \underline{F}_T$$

$$= -Y(\xi)\int_c^{c+L} (\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}})dz + \underline{F}_T$$

$$= -Y(\xi)\int_c^{c+L} (\dot{\underline{\omega}}^x\, \underline{u} + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}})dz + \underline{F}_T$$

$$= -\frac{d^2}{dt^2}\left(Y(\xi)\int_c^{c+L} \underline{u}\, dz\right) + \underline{F}_T$$

$$= -\frac{d^2}{dt^2}(Y(\xi)\underline{c}_b) + \underline{F}_T$$

where $\underline{c}_b$ is the center of mass of the beam with respect to the body frame, and where the derivatives are taken with respect to the inertial frame. Thus, (3.1.53) does reduce to (3.1.61), except now $\underline{y}$ refers to the center of mass of the structure including the beam. Finally, consider (3.1.57). Since $\mu_i \to \infty$ for i = 1, 2, 3 there is no material deformation so that $\partial(\underline{u})$ (= $\partial(\underline{u}_t)$) = 0. However, what is $\mu\partial(\underline{u})$ as $\mu_i \to \infty$? In fact, (3.1.57) - (3.1.58) are meaningless equations in the sense that $\mu\partial(\underline{u}) + k\partial(\underline{u}_t)$ becomes any value necessary in order that the rigid body equations (3.1.60) and (3.1.61) hold. So, in general, $\mu\partial(\underline{u}) + k\partial(\underline{u}_t)$ *does not* equal zero as one might expect. Thus, (FSS) does become (RS) as the flexural rigidities become infinite, as expected.

With these ideas in mind, we now develop the last spacecraft model that will be needed.

Flexible Space Structure Ignoring Axial Effects (FSS/A)

Suppose now we ignore axial effects. Clearly the formulation (3.1.55)-(3.1.56) of (FSS) is unacceptable since, as shown in Remark 3.1.1, the term $\mu_3 \frac{\partial^2(\cdot)_3}{\partial z^2}$ becomes meaningless as the structure becomes rigid. Therefore, as in Remark 3.1.1, we will use the formulation

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c),\ -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c),$$
$$-\Big[\int_c^{c+L} (\underline{\dot{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\underline{\ddot{y}}\,)_3 dz\Big]\Big)^T \tag{3.1.64}$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\ \mu_2 u_2''(c) + k_2 u_{2t}''(c),\ 0)^T \tag{3.1.65}$$

Using this formulation of $\underline{F}_{bB}$ and $\underline{M}_{bB}$, we obtain the model for the flexible spacecraft without momentum wheels, and ignoring axial effects:

$$
\textbf{(FSS/A)}\quad
\left[
\begin{array}{ll}
\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega} & (3.1.66) \\[6pt]
I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} + \underline{c}^x\,\underline{F}_{bB} + \underline{M}_{bB} & (3.1.67) \\[6pt]
m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T & (3.1.68) \\[6pt]
Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\,I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I & (3.1.69) \\[6pt]
\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c), \quad -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c), & \\
\qquad\qquad \displaystyle\int_c^{c+L} -(\dot{\underline{\omega}}^x\,\underline{u} + 2\underline{\omega}^x\,\underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_3 dz\Big)^T & (3.1.70) \\[6pt]
\underline{M}_{bB} = \Big(\mu_1 u_1''(c) + k_1 u_{1t}''(c), \;\; \mu_2 u_2''(c) + k_2 u_{2t}''(c), \;\; 0\Big)^T & (3.1.71) \\[6pt]
(\underline{u}_{tt} + \dot{\underline{\omega}}^x\,\underline{u} + 2\underline{\omega}^x\,\underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + \mu\partial(\underline{u}) + k\partial(\,\underline{u}_t\,) + Y^{-1}\ddot{\underline{y}})_{1,2} = 0 & (3.1.72) \\[6pt]
u_1(c) = u_2(c) = 0, \;\; u_1'(c) = u_2'(c) = 0 & \\[6pt]
u_1''(c + L) = u_2''(c + L) = 0 & (3.1.73) \\[6pt]
\mu_i u_i'''(c + L) + k_i u_{it}'''(c + L) = -F_i, \qquad i = 1, 2. &
\end{array}
\right.
$$

It should also be noted that since axial effects are ignored, $\underline{u} = (u_1, u_2, z)^T$, where z is the point of the beam under consideration.

3.2 Spacecraft Stabilization - Introduction

In this section and the following one, we consider the stabilization, or detumbling, problem for spacecraft. The problem consists of designing a control law such that the spacecraft stops spinning, and if there are flexible portions to the structure, the beam deflections and velocities go to zero.

For a rigid spacecraft, the problem is well known and several schemes exist. Indeed, a sample control law will be given in Theorem 3.3.1. For a flexible structure, the problem is fairly straightforward if the beam is modelled by its finite dimensional approximation. In this case, strictly passive feedback utilizing torque jets on the rigid body, and collocated sensors and actuators on the beam can be used to stabilize the system.

When the beam is modelled in an infinite dimensional form, the results in the literature are currently somewhat limited. In [Bis. 1], a stabilization scheme is proposed using *distributed control* on the beam, and an asymptotic stability result is obtained. From an engineering viewpoint, though, the value of such a result is somewhat dubious for two reasons. First, distributed control along a beam is not easy to do, nor is measuring the velocities along the whole length of the beam. Secondly, asymptotic stability means that if we start from an initial state sufficiently close to an equilibrium point, then all trajectories of this system tend to the equilibrium point. For a spacecraft, our primary interest in a detumbling maneuver is to decrease the angular velocity from a possibly large value to a small value, or zero. Thus, the control goal should be to obtain a control law that guarantees global asymptotic stability results (i.e., starting from any initial state, the system tends to the rest state), or better yet, an exponential stability result.

We now address these issues in the next three sections. In Section 3.3, Theorem 3.3.1, we propose a feedback control law for the torque jets to exponentially stabilize the rigid spacecraft. The method of attack uses Lyapunov techniques.

In section 3.4, we will consider the flexible spacecraft in section 3.1.5, where the beam damping will be assumed significant. Theorem 3.4.1 gives a control law which stabilizes this system using the torque jets and the force thrusters mounted on the rigid body. The method of proof of this result is as follows. First, we will define an energy functional E, and then a modified functional V. Using V, it will be shown that $E<K/t$, for t sufficiently large. This will give a global asymptotic stability result. By separating the linear and nonlinear parts of the pertinent differential equation, and using a Bellman-Gronwall argument, it will then be shown that the system is actually exponentially stable if the mass of the rigid portion of the spacecraft is much larger

than the mass of the flexible beam. In Theorem 3.4.5, a modified control law which accounts for possible nonlinearities in the sensors and actuators is proposed. Using methods identical to that of Theorem 3.4.1, a global asymptotic stability result is obtained.

Finally, in section 3.5, we will consider the flexible spacecraft in section 3.1.5, but this time the beam damping is assumed zero. For this problem, active beam control will also be needed to stabilize the system. The beam control will be of the boundary variety discussed in Chapter 2. Theorem 3.5.1 gives a linear control law, which results in global asymptotic stability. Theorem 3.5.2 gives a similar result but allows sensors and actuators to contain sector nonlinearities. The method of proof is identical to that of Theorem 3.4.1 in that it uses Lyapunov functionals.

3.3 Stabilization of Rigid Spacecraft

In this section we consider the problem of spacecraft stabilization described above. The main reason for discussing the rigid spacecraft first is to elucidate some of the ideas that will be used in the more general case of a flexible spacecraft. The mathematical complexities are far greater in the flexible case, but the ideas behind the control laws are very similar.

Theorem 3.3.1 - Consider the rigid spacecraft (RS) described in section 3.1.5. Since the rotational and translational motions are decoupled, we need only to consider the rotational term

$$I_0\underline{\dot{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} \tag{3.3.1}$$

Let the control law be

$$\underline{\tau} := -\,g_\omega(\underline{\omega}) \tag{3.3.2}$$

where $g_\omega(\underline{\omega})$ lies in the sector $[c_\omega,\ \infty)$, $c_\omega > 0$, (i.e., $c_\omega \| \underline{\omega} \|^2 \le \underline{\omega}^T g_\omega(\underline{\omega}) < \infty$). Then $\underline{\omega} \to 0$ exponentially, i.e., the body stops spinning exponentially.

Comment 3.3.2 - There is a simple interpretation of this result when $g_\omega(\underline{\omega}) = K_\omega \underline{\omega}$, where K_ω is a positive definite matrix. The $-K_\omega \underline{\omega}$ applied torque decreases the magnitude of $I_0\underline{\omega}$, and the $\underline{\omega}^x I_0 \underline{\omega}$ term doesn't affect the magnitude of $I_0\underline{\omega}$ since $I_0\underline{\omega} \perp \underline{\omega}^x I_0 \underline{\omega}$. For small values of $\underline{\omega}$, $\underline{\omega}^x I_0 \underline{\omega}$ is negligible. Thus, roughly speaking, (3.3.1) looks like $I_0 \dot{\underline{\omega}} = -K_\omega \underline{\omega}$, which is exponentially stable since both I_0 and K_ω are positive definite.

Proof of Theorem 3.3.1 - Consider the Lyapunov function candidate

$$E(\underline{\omega}) := \frac{1}{2} \underline{\omega}^T I_0\, \underline{\omega} \ge \frac{1}{2} \lambda_{min}(I_0) \| \underline{\omega} \|^2. \tag{3.3.3}$$

where $\lambda_{min}(I_0)$ is the minimum eigenvalue of I_0. $E(\underline{\omega})$ represents the total rotational kinetic energy of the system. Note that $E(\underline{\omega})$ is positive definite since I_0 is positive definite, and it is decrescent ([Vid.1 p. 143]) for the same reason. Differentiating $E(\underline{\omega})$ with respect to time, denoted $\dot{E}(\underline{\omega})$, along trajectories of the system we obtain

$$\dot{E}(\underline{\omega}) = \underline{\omega}^T I_0 \dot{\underline{\omega}} \tag{3.3.4}$$

$$= \underline{\omega}^T(-\underline{\omega}^x I_0 \underline{\omega} + \underline{\tau}) \tag{3.3.5}$$

$$= \underline{\omega}^T\, \underline{\tau} \tag{3.3.6}$$

Inserting the control law $\underline{\tau} := -\, g_\omega(\underline{\omega})$ we obtain

$$\dot{E}(\underline{\omega}) = -\underline{\omega}^T g_\omega(\underline{\omega}) \le -c_\omega \| \underline{\omega} \|^2 \le 0 \qquad (3.3.7)$$

where the last inequality follows form the sector condition on $g_\omega(\underline{\omega})$. Note that $\dot{E}(\underline{\omega})$ is negative definite. Since $E(\underline{\omega})$ is positive definite and decrescent, and $\dot{E}(\underline{\omega})$ is negative definite, from [Vid. 1, p.154] we can conclude that our nonlinear system is globally asymptotically stable, i.e., $\forall \underline{\omega}_0 \in \mathbf{R}^3$, $\underline{\omega} \to 0$ as $t \to \infty$. This, however, does *not* say that $\underline{\omega} \to 0$ exponentially. However, combining (3.3.3) with (3.3.7) we obtain

$$\frac{\dot{E}(\underline{\omega})}{E(\underline{\omega})} \le \frac{-c_\omega}{2\lambda_{min}(I_0)} \qquad (3.3.8)$$

Integrating form 0 to t we thus obtain

$$E(\underline{\omega}) \le E(0)\exp(-c_\omega t/2\lambda_{min}(I_0)) \qquad (3.3.9)$$

Since I_0 is positive definite and $c_\omega > 0$, this shows that $E(\underline{\omega}) \to 0$ exponentially. Therefore, $\frac{1}{2}\lambda_{min}(I_0)\| \underline{\omega} \|^2 \le E(\underline{\omega})$ implies that $\underline{\omega} \to 0$ exponentially, which proves the theorem. ■

3.4 Stabilization of Flexible Spacecraft - Beam Damping Present

The previous section gave a control law which stabilized the rigid spacecraft. In this section, the stabilization scheme is expanded to include the flexible spacecraft (FSS) described in section 3.1.5. The chief difficulty in showing these results is that the beam is modelled in an infinite dimensional form, so that most of the standard stability results in nonlinear system theory (for example, LaSalle's Theorem) do not strictly apply.

In this section, we will first consider the spacecraft model with beam damping present, and then with the damping absent. The idea in proving these results is similar to Theorem 3.3.1 for the rigid spacecraft. First, we will define an energy functional E which is positive definite, and then a modified functional V. Using V, it will be shown that $E < K/t$ for some constant K and for t sufficiently large. For the case where the mass of the rigid body is much larger than the mass of the beam (a reasonable engineering assumption), then it will be shown that the closed loop system is actually exponentially stable by using a Bellman-Gronwall type argument.

Theorem 3.4.1 - Consider the flexible spacecraft model described in section 3.1.5 and denoted (FSS/A) where axial effects have been ignored. Also assume no active beam control, i.e., $F_i(t) = 0$, $i = 1, 2, 3$ in (3.1.60). The equations then become

$$\dot{\underline{\xi}} = [I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \quad (3.4.1)$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} \quad (3.4.2)$$

$$m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \quad (3.4.3)$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\,I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}x\,] - I \quad (3.4.4)$$

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c), \quad -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c), \quad \int_c^{c+L} -(\dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_3 dz\Big)^T \quad (3.4.5)$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c), \quad \mu_2 u_2''(c) + k_2 u_{2t}''(cz), \quad 0)^T \quad (3.4.6)$$

$$(\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + \mu\partial(\underline{u}) + k\partial(\,\underline{u}_t\,) + Y^{-1}\ddot{\underline{y}})_{1,2} = 0 \quad (3.4.7)$$

$$u_1(c) = u_2(c) = 0, \;\; u_1'(c) = u_2'(c) = 0 \quad (3.4.8)$$

$$u_1''(c + L) = u_2''(c + L) = 0 \quad u_1'''(c + L) = u_2'''(c + L) = 0$$

Assume that damping is explicitly present, i.e. $k_i > 0$, i=1, 2. Let the control law be

$$\underline{\tau} := -\,K_\omega \underline{\omega} \quad (3.4.9)$$

$$\underline{F}_T := -K_{\dot{y}}\, \dot{\underline{y}} \quad (3.4.10)$$

where K_ω and $K_{\dot{y}}$ are positive definite matrices. Then for any initial conditions sufficiently smooth

(i) $\underline{\omega}$ and $\dot{\underline{y}}$ go to zero as $t \to \infty$, and $\underline{u}(x, t)$, $\underline{u}_t(x, t)$ go to zero as $t \to \infty$ in appropriate norms.

(ii) If the mass of the rigid body is much larger than the mass of the beam, then $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero exponentially as $t \to \infty$.

Before proving Theorem 3.4.1, we first prove the following simple proposition which will be used extensively in the calculations below. The proposition merely states that deflections and deflection velocities can be bounded by the higher order derivative (energy type) terms.

Proposition 3.4.2 - Let $\underline{u} = (u_1, u_2, u_3+z)^T$ denote the position (in body coordinates) of a point P whose undeformed position is $z\,\underline{b}_3$, as above. Then $\underline{u}$ satisfies

$$\text{(i)}\quad (u_i(z,t))^2 \le L\int_c^{c+L} (u_i'(z,t))^2 dz \qquad i = 1, 2, 3.$$

$$\text{(ii)}\quad (u_{it}(z,t))^2 \le L\int_c^{c+L} (u_{it}'(z,t))^2 dz \qquad i = 1, 2, 3.$$

$$\text{(iii)}\quad (u_i'(z,t))^2 \le L\int_c^{c+L} (u_i''(z,t))^2 dz \qquad i = 1, 2.$$

$$\text{(iv)}\quad (u_{it}'(z,t))^2 \le L\int_c^{c+L} (u_{it}''(z,t))^2 dz \qquad i = 1, 2.$$

$$\text{(v)}\quad (u_i(z,t))^2 \le L^2\int_c^{c+L} (u_i''(z,t))^2 dz \qquad i = 1, 2.$$

Proof of Proposition 3.4.2 - (i) Using the fundamental theorem of calculus and the boundary conditions $u_i(c, t) = 0$, for $i = 1, 2, 3$ and for all $t \in \mathbb{R}^+$,

$$u_i(z,t) = \int_c^x u_i'(z,t)dz$$

Therefore, by the Schwarz inequality we have

$$(u_i(z,t))^2 = \left(\int_c^x u_i'(z,t)dz\right)^2 \le L\int_c^{c+L} (u_i'(z,t))^2 dz$$

which proves statement (i). Statements (ii), (iii), and (iv) are proved similarly. Combining (i) and (iii) then yields statement (v). ■

Proof of Theorem 3.4.1 - The first step is to verify that our system is well - posed. That is, we must show that solutions exist and are unique. This is straightforward and is done in Appendix 3.B. Assuming existence and uniqueness of solutions, we define the energy functional as

$$E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_c^{c+L} \| (\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2} \|^2 dz$$

$$+ \frac{1}{2} m_B \| \dot{\underline{y}} \|^2 + \int_c^{c+L} (\mu_1 (u_1'')^2 + \mu_2 (u_2'')^2) dz \qquad (3.4.13)$$

(For simplicity, let E denote $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$.) The term energy is used because the first term is the rotational kinetic energy of the rigid body, the second term is the total kinetic energy of the beam, the third term is the translational kinetic energy of the rigid body, while the last term is the potential energy of the beam. Thus E represents the total energy of the spacecraft system. Next, we desire to calculate the time derivative of E, which will be denoted as $\dot{E}$. We could calculate the answer term by term, but a little thought makes the job infinitely easier. Recall that

$$\frac{d}{dt}(\text{Energy}) = \text{instantaneous power delivered to system} + \text{dissipated power of system}$$

This then implies

$$\dot{E} = \underline{\omega}^T \underline{\tau} + \underline{F}_T{}^T \dot{\underline{y}} - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 dz \} \tag{3.4.14}$$

The first term of (3.4.14) is the instantaneous power delivered by the torque jets, the second term is the instantaneous power delivered by the external thrusters, and the last term is the dissipated energy in the system due to beam damping. Inserting the values of $\underline{F}_T$ and $\underline{\tau}$ from (3.4.9) - (3.4.10) then yields

$$\dot{E} = -\underline{\omega}^T K_\omega \underline{\omega} - \dot{\underline{y}}^T K_{\dot{y}} \dot{\underline{y}} - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 dz \} \tag{3.4.15}$$

$$\leq 0$$

Unfortunately, $\dot{E}$ can be shown to be only negative semidefinite, rather than negative definite as was the case for the rigid spacecraft. Using a generalization of LaSalle's Theorem to infinite dimensional systems (see Chapter 4), we could get a global asymptotic stability result. But that is not our plan of attack here. Instead, we intend to find a modified Lyapunov type functional and perform calculations to get stronger convergence results.

So now consider the function

$$V_1 := \int_c^{c+L} [(\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2}]^T (\underline{u})_{1,2} dz \tag{3.4.16}$$

$$= \int_c^{c+L} [Y(\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2}]^T (Y\underline{u})_{1,2} dz \tag{3.4.17}$$

Recall $-2a^Tb \leq a^Ta + b^Tb = \| a \|^2 + \| b \|^2$. Therefore,

$$-2V_1 \le \int_c^{c+L} \| (\underline{u}_t + \underline{\omega}^x\, \underline{u} + Y^{-1}\, \dot{\underline{y}})_{1,2}\|^2 dz + \int_c^{c+L} \| (\,\underline{u}\,)_{1,2}\|^2\, dz \qquad (3.4.18)$$

$$\le \int_c^{c+L} \| \underline{u}_t + \underline{\omega}^x\, \underline{u} + Y^{-1}\, \dot{\underline{y}}\|^2\, dz + L^2 \int_c^{c+L} [(u_1'')^2 + (u_2'')^2] dz \qquad (3.4.19)$$

where we have used Proposition 3.4.2 to obtain (3.4.19). Therefore, $-2V_1 \le mE$ for some $m > 0$. Rewritten, this becomes $V_1 \ge -mE/2$. Next, calculating the time derivative of V_1 (use 3.4.17), again denoted $\dot{V}_1$, we obtain

$$\dot{V}_1 = \int_c^{c+L} \left([(\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_{1,2}\,]^T (\,\underline{u}\,)_{1,2} \right) dz$$

$$+ \int_c^{c+L} \|(\,\underline{u}_t + \underline{\omega}^x\, \underline{u} + Y^{-1}\, \dot{\underline{y}}\,)_{1,2}\|^2\, dz$$

$$= \int_c^{c+L} \left(-[(\,\mu\partial(\underline{u}) + k\partial(\underline{u}_t))_{1,2}]^T (\underline{u})_{1,2} + \| (\underline{u}_t + \underline{\omega}^x\, \underline{u} + Y^{-1}\, \dot{\underline{y}}\,)_{1,2}\|^2\, dz \right) \qquad (3.4.20)$$

where (3.4.6) has been used. Integrate the first term twice by parts and use the boundary conditions (3.4.7)-(3.4.8) to obtain

$$= -\int_c^{c+L} \left(\mu_1(u_1'')^2 + \mu_2(u_2'')^2 + k_1 u_1'' u_{1t}'' + k_2 u_2'' u_{2t}''\right) dz$$

$$+ \int_{c}^{c+L} \|(\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2}\|^2 dz \tag{3.4.21}$$

So define the modified Lyapunov functional as $V := Et + V_1$. The statement after (3.4.18) shows that

$$V \geq Et - mE/2 = (t-m/2)E. \tag{3.4.22}$$

Next, computing the time derivative of V, we obtain

$$\begin{aligned}
\dot{V} = \dot{E}\, t + E + \dot{V}_1 = \{ &-\underline{\omega}^T K_\omega \underline{\omega} - \dot{\underline{y}}^T K_{\dot{y}}\, \dot{\underline{y}} - \int_{c}^{c+L} (k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2\,)dz\ \}t \\
&+ \tfrac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \tfrac{1}{2} m_B \|\dot{\underline{y}}\|^2 + \tfrac{1}{2} \int_{c}^{c+L} (\mu_1(u_1'')^2 + \mu_2(u_2'')^2\,)dz \\
&+ \tfrac{1}{2} \int_{c}^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}}\|^2 dz \ + \int_{c}^{c+L} \|(\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2}\|^2 dz \\
&- \{ \int_{c}^{c+L} \mu_1(u_1'')^2 + \mu_2(u_2'')^2 + k_1 u_1'' u_{1t}'' + k_2 u_2'' u_{2t}''\, dz \}
\end{aligned} \tag{3.4.23}$$

$$\begin{aligned}
&= (\tfrac{1}{2}\underline{\omega}^T I_0 \underline{\omega} - \underline{\omega}^T K_\omega \underline{\omega} t) + (\tfrac{1}{2} m_B \|\dot{\underline{y}}\|^2 - \dot{\underline{y}}^T K_{\dot{y}}\, \dot{\underline{y}}\, t) \\
&+ \tfrac{1}{2} \int_{c}^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}}\|^2 dz \ + \int_{c}^{c+L} \|(\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}})_{1,2}\|^2 dz\ - \\
&- \tfrac{1}{2} \int_{c}^{c+L} (\mu_1(u_1'')^2 + \mu_2(u_2'')^2\,)dz \ - \left(\int_{c}^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2\, dz \right) t
\end{aligned} \tag{3.4.24}$$

$$- \int_c^{c+L} (k_1 u_1'' u_{1t}'' + k_2 u_2'' u_{2t}'')dz$$

Now note the following facts:

1. $\| \underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}} \|^2 \le \| \underline{u}_t \|^2 + \| \underline{\omega}^x \underline{u} \|^2 + \| Y^{-1} \dot{\underline{y}} \|^2$

$$\le \| \underline{u}_t \|^2 + \| \underline{\omega} \|^2 \| \underline{u} \|^2 + \| \dot{\underline{y}} \|^2.$$

Therefore,

$$\int_c^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}}\|^2 dz \le L \| \dot{\underline{y}} \|^2 + \int_c^{c+L} \| \underline{u}_t \|^2 dz + \| \underline{\omega} \|^2 \int_c^{c+L} \| \underline{u} \|^2 dz \quad (3.4.25)$$

2. $a^T b \le \delta^2 a^T a + b^T b/\delta^2$ for all $\delta \in \mathbf{R}/\{0\}$, and for all $a, b \in \mathbf{R}^n$. (3.4.26)

3. Using Fact 2 above, $- \int_c^{c+L} (k_1 u_1'' u_{1t}'' + k_2 u_2'' u_{2t}'')dz \le \delta^2 \int_c^{c+L} k_1 (u_1'')^2 + k_2 (u_2'')^2 dz$

$+ 1/\delta^2 \int_c^{c+L} (k_1 (u_{1t}'')^2 + k_2 (u_{2t}'')^2)dz$, for all $\delta \in \mathbf{R}/\{0\}$. (3.4.27)

Now using Facts 1, 2, 3, and Proposition 3.4.2 in (3.4.24) above, we obtain

$$\dot{V} \le (\tfrac{1}{2} \underline{\omega}^T I_0 \underline{\omega} + \tfrac{3}{2} \| \underline{\omega} \|^2 \int_c^{c+L} \| \underline{u} \|^2 dx - (\underline{\omega}^T K_{\omega} \underline{\omega}) t) + (\tfrac{1}{2} m_B \| \dot{\underline{y}} \|^2 + \tfrac{3}{2} L \| \dot{\underline{y}} \|^2 -$$

$$\dot{\underline{y}}^T K_{\dot{y}} \dot{\underline{y}} t) - \int_c^{c+L} [(\mu_1/2 - \delta^2 k_1)(u_1'')^2 + (\mu_2/2 - \delta^2 k_2)(u_2'')^2]dz +$$

$$+\left(-\frac{1}{2}t + 1/\delta^2 + \frac{3}{2}M''\right)\int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 dz \qquad (3.4.28)$$

Thus, if we choose δ small enough and t large enough, $\dot{V}$ can be made negative. In other words, there exists a $T > 0$ and a $\delta' > 0$ such that $\dot{V} \le 0$, for all $t \ge T$. Thus, from (3.4.22), this shows that

$$E(t) \le V(T) / t, \qquad \text{for all } t \ge T. \qquad (3.4.29)$$

Using the definition of energy (3.4.13) we thus see that $\omega(t) \to 0$ as $1/t$, $\dot{\underline{y}} \to 0$ as $1/t$, $\int_c^{c+L} \| (\underline{u}_t + \underline{\omega}^x\, \underline{u} + Y^{-1}\, \dot{\underline{y}})_{1,2} \|^2 dz \to 0$ as $1/t$ and $\int_c^{c+L} (\mu_1(u_1'')^2 + \mu_2(u_2'')^2)dz \to 0$ as $1/t$. This in turn implies $\omega(t) \to 0$ as $1/t$, $\dot{\underline{y}} \to 0$ as $1/t$, $\int_c^{c+L} \| (\underline{u}_t)_{1,2} \|^2 dz \to 0$ as $1/t$ and $\int_c^{c+L} (\mu_1(u_1'')^2 + \mu_2(u_2'')^2)dz \to 0$ as $1/t$. Using Proposition 3.4.2 (v), the latter terms shows that $u_1(x, t) \to 0$ as $1/t$ uniformly in x, and $u_2(x, t) \to 0$ as $1/t$ uniformly in x. Thus $\omega(t)$ and $\dot{\underline{y}}$ go to zero as $1/t$, $u_{1t}(x, t)$ and $u_{2t}(x, t)$ go to zero in L^2, and $u_1(x, t)$ and $u_2(x, t)$ go to zero as $1/t$ uniformly in x. This proves (i)

Proof of (ii) - First, separate the linear and nonlinear parts of the differential equation (FSS). It will be shown that the linear part generates an exponentially stable semigroup. From the first part of the Theorem, we will bound the nonlinear terms by terms that go to zero as $1/t$. Using a Bellman-Gronwall type argument will then complete the proof. The details are left to Appendix 3.A. ■

Remark 3.4.3 - It is instructive to consider this nonlinear system from an input/output point of view. If equation (3.4.14) is integrated from $-\infty$ to t, we obtain

$$\int_{-\infty}^{t} (\underline{\omega}^T \underline{\tau} + \underline{F}_T{}^T \dot{\underline{y}})\, dz - \{ \int_{-\infty}^{t} (\int_{c}^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2\, dz)\, dt \} = \int_{-\infty}^{t} \frac{d}{dt} E(t) dt \quad (3.4.38)$$

$$= E(t) - E(-\infty) = E(t)$$

if we assume the energy at $t = -\infty$ is zero. Thus

$$\int_{-\infty}^{t} (\underline{\omega}^T \underline{\tau} + \underline{F}_T{}^T \dot{\underline{y}})\, dz \geq 0 \quad (3.4.39)$$

This shows that our spacecraft system is <u>passive</u> [Des. 2, p. 173]. Furthermore, our feedback law (3.4.9) - (3.4.10) is strictly passive [Des. 2, p. 173]. By the passivity theorem, [Des. 2, p. 181] we can conclude that the map

$$\begin{pmatrix} \underline{\tau} \\ \underline{F}_T \end{pmatrix} \rightarrow \begin{pmatrix} \underline{\omega} \\ \dot{\underline{y}} \end{pmatrix} \quad (3.4.40)$$

is L^2 stable. Since the feedback law (3.4.6) - (3.4.7) is linear, one can show using arguments similar to Theorem 6.4.14 of [Vid. 1] that this implies the map (3.4.40) is globally asymptotically stable.

Despite these positive outcomes, passivity arguments are unfortunately unsuitable to complete the proof. Note that the passivity result above says nothing about the beam. Indeed, one needs to make other complicated arguments to reason that $\underline{u}$ and

$\underline{u}_t$ go to zero asymptotically. In addition, passivity theorems do not yield exponential stability results such as those obtained in Theorem 3.4.1. Again, other arguments unrelated to passivity would be needed to complete the result.

Remark 3.4.4 - To implement the control law (3.4.9)-(3.4.10) one must be able to determine $\underline{\omega}$ and $\dot{\underline{y}}$ by measurement. To measure $\underline{\omega}$ one simply uses rate integrating gyros mounted on the rigid body [Wer. 1, p. 199]. To determine $\dot{\underline{y}}$ one can use accelrometers attached to the rigid body.

From an engineering viewpoint, the control scheme in Theorem 3.4.3 has the drawback that the control sensors and actuators are assumed *linear*. Unfortunately, all sensors and actuators have some residual nonlinearities. The following theorem takes into account a class of such nonlinearities.

Theorem 3.4.5 - Consider the system of Theorem 3.4.3. Let the control law be

$$\underline{\tau} := - g_{\omega}(\underline{\omega}) \tag{3.4.41}$$

$$\underline{F}_T := -g_{\dot{y}}(\dot{\underline{y}}) \tag{3.4.42}$$

where $g_{\omega}(\underline{\omega})$ is a nonlinear function lying in the sector $[c_{\omega}, \infty)$, $c_{\omega} > 0$, (i.e., $c_2\| \underline{\omega} \|^2 \leq \underline{\omega}^T g(\underline{\omega}) < \infty$) and $g_{\dot{y}}(\dot{\underline{y}})$ is a nonlinear function lying in the sector $[c_{\dot{y}}, \infty)$, $c_{\dot{y}} > 0$. Assume that the system is well-posed, i.e., the closed loop system has a unique, continuously differentiable solution for all initial conditions sufficiently smooth. Then the system is globally asymptotically stable, i.e., $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero as $t \to \infty$.

Proof of Theorem 3.4.5 - The proof is almost identical to that of Theorem 3.4.3, and will thus only be sketched. Define E exactly as in (3.4.13), and compute $\dot{E}$ to obtain (3.4.14). Inserting the control law (3.4.41)-(3.4.42) then yields

$$\dot{E} = -\underline{\omega}^T g_\omega(\underline{\omega}) - \dot{\underline{y}}^T g_{\dot{y}}(\dot{\underline{y}}) - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 \, dz \}$$

$$\leq - c_\omega \| \underline{\omega} \|^2 - c_{\dot{y}} \| \dot{\underline{y}} \|^2 - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 \, dz \}$$

$$\leq 0$$

We next choose V_1 exactly as in (3.4.16). Defining $V = Et + V_1$, one can repeat the same calculations to obtain (3.4.22), i.e., $V \geq (t-m/2)E$ for all $t \geq 0$. Next, computing $\dot{V}$, note that the modified control law only affects the terms involving $\dot{E}$. Therefore, $\dot{V}$ is given by

$$\dot{V} \leq (\tfrac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \tfrac{3}{2} \| \underline{\omega} \|^2 \int_c^{c+L} \| \underline{u} \|^2 dx - c_\omega \| \underline{\omega} \|^2 t) + (\tfrac{1}{2} m_B \| \dot{\underline{y}} \|^2 + \tfrac{3}{2} L \| \dot{\underline{y}} \|^2 -$$

$$c_{\dot{y}} \| \dot{\underline{y}} \|^2 t) - \int_c^{c+L} [(\mu_1/2 - \delta^2 k_1)(u_1'')^2 + (\mu_2/2 - \delta^2 k_2)(u_2'')^2]dz \; +$$

$$+(-\tfrac{1}{2} t + 1/\delta^2 + \tfrac{3}{2} M'') \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 dz$$

Note that this expression is almost identical to that of (3.4.28), with the difference that the $c_\omega \| \underline{\omega} \|^2$ term has replaced the $\underline{\omega}^T K_\omega \underline{\omega}$ term, and the $c_{\dot{y}} \| \dot{\underline{y}} \|^2$ term has replaced the $\dot{\underline{y}}^T K_{\dot{y}} \dot{\underline{y}}$ term. Thus, if we choose δ small enough and t large enough, $\dot{V}$ can be made negative. In other words, there exists a $T > 0$ and a $\delta' > 0$ such that $\dot{V} \leq$

0, for all $t \geq T$. Thus, this shows that $E(t) \leq V(T) / t$, for all $t \geq T$, which proves the Theorem. ■

3.5 Stabilization of Flexible Spacecraft With Beam Control

The previous section gave explicit stabilization schemes for the flexible spacecraft system (FSS) when beam damping is present. However, if damping is small, the exponential decay rate guaranteed by the theorems is undoubtedly quite small. This means that the beam oscillations may occur for an undesirably long time. Since future space structures will contain flexible portions with small damping [Joh. 1], many analysts have studied the problem where beam damping is assumed zero. If the damping is zero, then Theorem 3.4.1 is probably not true, so some sort of beam control will be needed to guarantee that beam deflections go to zero. As remarked previously, in [Bis. 1], an asymptotic stability result was obtained using distributed control along the beam. From an engineering perspective, this is unfortunately unimplementable. (It should also be noted that using the methods of this section, or the passivity methods mentioned in the above remark, the [Bis. 1] can be shown to not only guarantee asymptotic stability, but exponential stability - a far stronger result.) A more reasonable engineering approach is to consider boundary control of the type discussed in Chapter 2, since only limited additional hardware is required.

This was done for the planar case in [Des. 1], and extended to the 3 - dimensional case in [Mor. 2]. Theorem 3.5.1 differs from the [Des. 1] and [Mor. 2] in that effects due to the coupling of the translational term is taken in account, whereas it is ignored in the these two papers. The price to be paid is that exponential stability is not readily apparent.

Theorem 3.5.1 - Consider the flexible spacecraft model where axial effects are ignored, described in section 3.1.5 and denoted (FSS/A). Also assume that no beam damping is present, i.e. k = 0 in (3.1.57)-(3.1.60). Then the equations become

$$\dot{\underline{\xi}} = [I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{3.5.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} \tag{3.5.1}$$

$$m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{3.5.2}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{3.5.3}$$

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c), \quad -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c), \quad \int_c^{c+L} -(\dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_3 dz\Big)^T \tag{3.5.4}$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c), \quad \mu_2 u_2''(c) + k_2 u_{2t}''(c), \quad 0)^T \tag{3.5.5}$$

$$(\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_{1,2} = 0 \tag{3.5.6}$$

$$u_1(c) = u_2(c) = 0, \quad u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0 \tag{3.5.7}$$

$$\mu_i u_i'''(c + L) = -F_i, \quad i = 1, 2.$$

Let the control law be

$$\underline{\tau} := -K_\omega \underline{\omega} \tag{3.5.8}$$

$$\underline{F}_T := -K_{\dot{y}}\, \dot{\underline{y}} \tag{3.5.9}$$

$$(\underline{F})_{1,2} := -\Lambda(\,(\,\underline{u}_t(c + L) + \omega^x\, \underline{u}(c + L) + Y^{-1}\,\dot{\underline{y}})\,)_{1,2} \tag{3.5.10}$$

where $K_\omega \in \mathbf{R}^{3x3}$ is a positive definite matrix, $K_{\dot{y}} \in \mathbf{R}^{3x3}$ is a positive definite matrix, $\alpha > 0$, $\beta > 0$, and $\Lambda \in \mathbf{R}^{2x2}$ is a positive definite matrix. Then the system is globally asymptotically stable, i.e., $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero as $t \to \infty$. In fact, $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero as 1/t as $t \to \infty$.

Proof of Theorem 3.5.1 - The proof is very similar to that of Theorem 3.4.1 and will only be sketched. Of course existence and uniqueness of solutions must be established, and this can be done similar to the proof in Appendix 3.B. Next, choose an energy functional E exactly as in (3.1.13). Compute $\dot{E}$ to obtain $\dot{E} = -\underline{\omega}^T K_\omega \underline{\omega}$ - $(\underline{u}_t(c+L) + \underline{\omega}^x\, \underline{u}(c+L) + Y^{-1}\,\dot{\underline{y}})_{1,2}{}^T\, \Lambda\, (\underline{u}_t(c+L) + \underline{\omega}^x\, \underline{u}(c+L) + Y^{-1}\,\dot{\underline{y}})_{1,2} - \dot{\underline{y}}^T K_{\dot{y}}\, \dot{\underline{y}}$, which, as before, is the instantaneous power delivered to the system. Let the modified functional be

$$V := 2(1-\varepsilon)tE + \int_c^{c+L} 2\,(z-c)\,[(\underline{u}_t + \underline{\omega}^x\,\underline{u} + Y^{-1}\,\dot{\underline{y}})_{1,2}^T]\;(\underline{u}')_{1,2}dz \qquad (3.5.11)$$

Again, we can find a m_1, m_2, and $m_3 > 0$ such that $V \geq (m_1 t - m_2)E - m_3$, for all $t \geq 0$. By tedious calculation, using methods very similar to the proof of Theorem 3.4.1, it can be shown that there exists $T > 0$ such that $\dot{V} \leq 0$, for all $t \geq T$. Thus, E(t) goes to zero as 1/t. ■

Finally, we have a theorem, analogous to Theorem 3.4.5, which allows for a class of nonlinearities in both sensors and actuators.

Theorem 3.5.2 - Consider the system described in Theorem 3.5.1, and given by equations (3.5.1)-(3.5.7). Let the control law be

$$\underline{\tau} := -g_\omega(\underline{\omega}) \qquad (3.5.12)$$

$$\underline{F}_T := -g_{\dot{y}}(\dot{\underline{y}}) \qquad (3.5.13)$$

$$(\underline{F})_{1,2} := -g_F((\, \underline{u}_t(c + L) + \omega^x\, \underline{u}(c + L) + Y^{-1}\, \dot{\underline{y}}\,))_{1,2} \tag{3.5.14}$$

where $g_\omega(\underline{\omega})$ is a nonlinear function lying in the sector $[c_\omega, \infty)$, $c_\omega > 0$, $g_{\dot{y}}(\dot{\underline{y}})$ is a nonlinear function lying in the sector $[c_{\dot{y}}, \infty)$, $c_{\dot{y}} > 0$, and $g_F((\, \underline{u}_t(c + L) + \omega^x\, \underline{u}(c + L) + Y^{-1}\, \dot{\underline{y}}\,))_{1,2}$ is a nonlinear function lying in the sector $[c_F, \infty)$, $c_F > 0$. Assume that the system is well-posed, i.e., the closed loop system has a unique, continuously differentiable solution for all initial conditions sufficiently smooth. Then the system is globally asymptotically stable, i.e., $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero as $t \to \infty$.

Proof of Theorem 3.5.2 - Identical to that of Theorem 3.5.1. The reader should also consult the proof of Theorem 3.4.5 if there are any difficulties. ■

Remark 3.5.3 - To implement this control law, $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}(c + L)$, and $\underline{u}_t(c + L)$ must be measured. Remark 3.4.4 discusses the measurement of $\underline{\omega}$ and $\dot{\underline{y}}$. To measure $\underline{u}(c + L)$, and $\underline{u}_t(c + L)$, i.e., the position and velocity of the tip of the beam, one can use optical methods.

3.6 Concluding Remarks and Future Research

Some rather encouraging trends can be gleaned from the stabilization results of sections 3.4 and 3.5, as well as the thesis of Morgul. In Morgul's thesis, he extends the stabilization result to the structure of section 3.1, where the flexible beam is modelled by the Timoshenko beam model. The control law is very similar to those obtained here. Thus it appears that the beam model employed is not crucial in the stabilizability of the rigid body/beam system. As long as the beam is exponentially stable, or can be exponentially stabilized by beam control methods, it appears that the overall structure can be stabilized by a linear control law of the form (3.5.8)-(3.5.10).

Unfortunately, this conjecture has not been proved rigorously. The way to proceed would be to consider the most complete beam model currently available, the so-called geometrically exact beam models of Naghdi et al (see [Gre. 1], [Gre. 2]), Simo (see [Sim. 1]), etc. which reduce to the other models when appropriate terms are ignored. The main difficulty is the complicated nature of the model, which would make it very difficult to establish existence and uniqueness of solutions for the resulting differential equations. In addition, one would have to dream up the appropriate modified Lyapunov functional to establish the exponential stability results. As we have seen above, the energy functional is generally *not* sufficient to establish the exponential stability of the system. If one were to establish such a result, it would encompass all the results in this chapter, and provide a broad generalization to the attitude control laws presented in the succeeding chapters.

3. A Appendix

We first need a preliminary theorem which will figure prominently in the proof below.

Theorem 3.A.1 - Consider the following differential equation evolving on a Banach space X

$$\dot{x} = Ax + f(x) \tag{3.A.1}$$

where A: $X \to X$ is a linear map, possibly unbounded, and where f: $X \to X$ is a C^1 function satisfying $\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$. Suppose it is known that (i) A generates an exponentially stable semigroup and (ii) the differential equation (3.A.1) is globally asymptotically stable, i.e., for any initial condition $x_0 \in D(A)$, the solution to the differential equation (3.A.1), denoted $x(t) = S(t)x_0$, satisfies $\|x(t)\| \to 0$. Then $\|x(t)\| \to 0$ <u>expo-</u>

nentially.

Proof of Theorem 3.A.1 - Let T(t) denote the semigroup generated by A. Using the "variation of constants" formula yields for $x_0 \in D(A)$

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\tau)f(x(\tau))d\tau$$

Take norms on both sides to obtain

$$\|x(t)\| \le \|T(t-t_0)\| \, \|x_0\| + \int_{t_0}^{t} \|T(t-\tau)\| \, \|f(x(\tau)\| \, d\tau \tag{3.A.2}$$

By assumption, $\|T(t-t_0)\| \le M\exp(-\delta(t-t_0))$ for some $M > 0$ and some $\delta > 0$. Also, with the assumption on f(x), there is a $\rho > 0$ such that $\|f(x(t))\| \le \delta\|x(t)\|/2$ for all $\|x(t)\| \le \rho$. Since the system (3.A.1) is globally asymptotically stable by assumption, there is a $t^* \ge t_0$ such that $\|x(t)\| \le \rho$ for all $t \ge t^*$. Inserting these expressions into (3.A.2) yields

$$\|x(t)\| \le Me^{-\delta(t-t^*)}\|x^*\| + \int_{t^*}^{t} \frac{\delta}{2} \, e^{-\delta(t-\tau)}\|x(\tau)\|d\tau \tag{3.A.3}$$

Define u(t) by $u(t) := \|x(t)\|e^{\delta(t-t^*)}$. Then (3.A.3) becomes

$$u(t) \le Mu(t^*) + \int_{t^*}^{t} \frac{\delta}{2} \, u(\tau)d\tau$$

Using the Bellman-Gronwall lemma [Vid. 1, p. 292] on this expression then gives

$$u(t) \le Me^{\delta(t-t^*)/2}u(t^*), \quad \text{or}$$

$$\|x(t)\| \leq Me^{-\delta(t-t^*)/2}\|x(t^*)\|$$

which proves the Theorem. ■

Proof of Theorem 3.4.1 - part (ii)

To show that the decay rate is exponential, we will first show that the linearized part of (3.4.1)-(3.4.10) is exponentially stable if the mass of the rigid body is much larger than the mass of the beam (Lemma 3.A.3). Using the results of part (i) of Theorem 3.4.1 and the Bellman - Gronwall type result Theorem 3.A.1 above will then yield the result.

For notational simplicity, axial effects will first be considered. At the end of the proof, we will show that ignoring the axial terms does not affect the proof of Theorem 3.4.1.

Consider a new differential equation $\dot{x} = Ax$ where A is the linearized portion of (3.4.1) - (3.4.10):

$$A := \begin{bmatrix} -I_0^{-1}K_\omega & 0 & \begin{array}{c} I_0^{-1}\int_c^{c+L} \underline{c}^x \, \mu\partial(\cdot)dz \; + \\ I_0^{-1}\int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(\mu\partial(\cdot))_{1,2},\ 0]dx\right)dz \end{array} & \begin{array}{c} I_0^{-1}\int_c^{c+L} \underline{c}^x \, k\partial(\cdot)dz \; + \\ I_0^{-1}\int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(k\partial(\cdot))_{1,2},\ 0]dx\right)dz \end{array} \\ 0 & -K_y/m_B & \frac{Y}{m_B}\int_c^{c+L} \mu\partial(\cdot)dz & \frac{Y}{m_B}\int_c^{c+L} k\partial(\cdot)dz \\ 0 & 0 & 0 & I \\ 0 & Y^{-1}K_y/m_B & -\mu\partial(\cdot) - \frac{1}{m_B}\int_c^{c+L}\mu\partial(\cdot)dz & -k\partial(\cdot) - \frac{1}{m_B}\int_c^{c+L} k\partial(\cdot)dz \end{bmatrix}$$

(3.A.4)

(Refer to section 2.1 for the definition of the various terms in A.) Note that Y is the direction cosine matrix evaluated at the rest state of the spacecraft. Let the space A operates on be $X := \mathbb{R}^3 x \mathbb{R}^3 x H_0{}^2 x H_0{}^2 x H_0{}^1 x L^2 x L^2 x L^2$, and let the domain of A, D(A), be defined as

$$D(A) := \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8)^T |\ x_1 \in \mathbb{R}^3,\ x_2 \in \mathbb{R}^3,\ x_3 \in H_0{}^4,\ x_4 \in H_0{}^4,\ x_5 \in H^2,\ x_6 \in H_0{}^4,\ x_7 \in H_0{}^4,\ x_8 \in H^2,\ x_5(c)=x_8(c)=0,\ x_3''(c+L) = x_4''(c+L) = x_6''(c+L) = x_7''(c+L) = 0,\ x_3'''(c+L)=x_4'''(c+L)=x_6'''(c+L)=x_7'''(c+L)=0,\ x_5'(c+L)=x_8'(c+L)=0\} \tag{3.A.5}$$

Let the inner product on X be

$$[a,b]_X := [a_1,b_1]_{\mathbb{R}} + m_B[a_2,b_2]_{\mathbb{R}} + [a_3,b_3] + [a_4,b_4] + [a_5,b_5] + [a_6,b_6] + [a_7,b_7] + [a_8,b_8] \tag{3.A.6}$$

where $[a, b]_{\mathbb{R}}$ is the ordinary inner product in $\mathbb{R}^3$, and [a, b] is the ordinary L^2 inner product.

Lemma 3.A.2 - Consider A of (3.A.4). Suppose the mass of the rigid body is much greater than the mass of the beam. Then $\bar{A}$, the closure of A, generates an analytic semigroup on X.

Proof of Lemma 3.A.2 - Define A′ as follows

$$A' := \begin{bmatrix} -I_0^{-1}K_\omega & 0 & 0 & 0 \\ 0 & -K_y/m_B & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} \tag{3.A.7}$$

Using Theorem 5.B.2, $\bar{A}'$, the smallest extension of A', generates an analytic semigroup on X. Now, let $D(\bar{A}')$ denote the domain of this closed extension. (Unfortunately, A' is not closed on $D(A')$. We must enlarge the domain to make A' closed.) Define the operator B by $B := C\,\bar{A}'$, where $C:\ X \to X$ is the operator defined by

$$C\begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \\ \underline{v}_4 \end{bmatrix} := \begin{bmatrix} I_0^{-1}\int_c^{c+L} \underline{c}^x\,\underline{v}_4 dz + I_0^{-1}\int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(\underline{v}_4)_{1,2},\ 0]dx\right)dz \\ \dfrac{Y}{m_B}\int_c^{c+L} \underline{v}_4 dz \\ 0 \\ -Y^{-1}\underline{v}_2 + \dfrac{Y}{m_B}\int_c^{c+L} \underline{v}_4 dz \end{bmatrix} \tag{3.A.8}$$

(where $v_1 \in \mathbb{R}^3$, $v_2 \in \mathbb{R}^3$, $v_3 \in H_0^2xH_0^2xH_0^1$, $v_4 \in L^2xL^2xL^2$.) Let the domain of B be $D(B) := D(\bar{A}')$. With this definition of B, we see that it is simply A - A', when A and $\bar{A}'$ are restricted to D(A) given by (3.A.5). (Recall $\bar{A}'$ restricted to $D(A')$ is simply A' from the definition of closure.) The idea behind the proof is to try to use Theorem B.10, the perturbation theorem on analytic semigroups. To use Theorem B.10, the first thing which must be done is to show that B is a closed operator. Unfortunately, B is probably not a closed operator on D(B). However, the next best thing would be that B is actually closable, so that we can define an extension of B which agrees with B on D(B).

Therefore, we will first show that B is closable. Note that for $f \in D(B) = D(\bar{A}')$

$$\| Bf \|_X^2 = \left\| \begin{matrix} I_0^{-1}[\underline{c}^x \int_c^{c+L} (\bar{A}'f)_4 dz + \int_c^{c+L} (\int_z^{L} \mathrm{col}[((\bar{A}'f)_4)_{1,2},\ 0]dx)dz] \\ \frac{Y}{m_B} \int_c^{c+L} (\bar{A}'f)_4 dz \\ 0 \\ \frac{1}{m_B} Y^{-1} K_y (\bar{A}'f)_2 + \frac{1}{m_B} \int_c^{c+L} (\bar{A}'f)_4 dz \end{matrix} \right\|_X^2 \tag{3.A.9}$$

where we have used the notation $(a)_i = a_i$. Using the definition of norm (3.A.6) this expression becomes

$$\| Bf \|_X^2 = \| I_0^{-1}[\underline{c}^x \int_c^{c+L} (\bar{A}'f)_4 dz + \int_c^{c+L} (\int_z^{L} \mathrm{col}[((\bar{A}'f)_4)_{1,2},\ 0]dx)dz] \|_{\mathbb{R}}^2 +$$

$$m_B \| \frac{1}{m_B} \int_c^{c+L} (\bar{A}'f)_4 dz \|_{\mathbb{R}}^2 + \left(\frac{1}{m_B}\right)^2 \int_c^{c+L} \| Y^{-1} K_y (\bar{A}'f)_2 + \int_c^{c+L} (\bar{A}'f)_4 dz \|_{\mathbb{R}}^2 dz$$

Using the Cauchy-Schwarz inequality and the triangle inequality repeatedly, this expression becomes

$$\leq (\sigma_{max}(I_0))^{-1}\ [c\ L\int_c^{c+L} \|(\bar{A}'f)_4\|_{\mathbb{R}}{}^2 dz + \int_c^{c+L} (\|\int_z^L col[((\bar{A}'f)_4)_{1,2},\ 0]dx\|_{\mathbb{R}}{}^2)dz] +$$

$$\frac{L}{m_B}\int_c^{c+L} \|(\bar{A}'f)_4\|^2{}_{\mathbb{R}} dz\ +\ \left(\frac{1}{m_B}\right)^2 L\,\|Y^{-1}K_y(\bar{A}'f)_2\ +\int_c^{c+L} (\bar{A}'f)_4 dz\ \|^2{}_{\mathbb{R}}$$

$$\leq \frac{L}{\sigma_{max}(I_0)}[c\int_c^{c+L} \|(\bar{A}'f)_4\|_{\mathbb{R}}{}^2 dz + L\int_c^{L} \|col[((\bar{A}'f)_4)_{1,2},\ 0]\|_{\mathbb{R}}{}^2 dx\ ] +$$

$$\frac{L}{m_B}\int_c^{c+L} \|(\bar{A}'f)_4\|^2{}_{\mathbb{R}} dz\ +\ \left(\frac{1}{m_B}\right)^2 L\left(\|K_y(\bar{A}'f)_2\|_{\mathbb{R}}{}^2\ +\int_c^{c+L} \|\ (\bar{A}'f)_4 dz\ \|^2{}_{\mathbb{R}}\right)$$

$$\leq \frac{L}{\sigma_{max}(I_0)}[c\int_c^{c+L} \|(\bar{A}'f)_4\|_{\mathbb{R}}{}^2 dz + L\int_c^{L} \|(\bar{A}'f)_4\ \|^2{}_{\mathbb{R}} dx] +$$

$$\frac{L}{m_B}\int_c^{c+L} \|(\bar{A}'f)_4\|^2{}_{\mathbb{R}} dz + \left(\frac{1}{m_B}\right)^2 L\left(\sigma_{max}{}^2(K_y)\|(\bar{A}'f)_2\|^2 + L\int_c^{c+L} \|(\bar{A}'f)_4\ \|^2{}_{\mathbb{R}} dz\ \right)$$

Finally, since $I_0 \sim m_B$ this means

$$\|\,Bf\,\|_X{}^2 \leq \frac{L}{m_B}K\|\,\bar{A}'f\,\|^2{}_X \tag{3.A.10}$$

for some appropriate K. To show that B is closable, take any sequence $x_n \to 0$, $x_n \in D(B) = D(\bar{A}')$. We must show that $Bx_n \to 0$. (See [Bal. 1, p. 221].) From (3.A.10) we see that

$$\|\,Bx_n\,\|_X{}^2 \leq \frac{L}{m_B}K\|\,\bar{A}'x_n\,\|^2{}_X$$

Since is $\overline{A}'$ closed by definition, $x_n \in D(\overline{A}') \rightarrow 0$ implies $\overline{A}'x_n \rightarrow 0$. Thus, $Bx_n \rightarrow 0$, so it is indeed closable. Let $\overline{B}$ denote this closed operator, and let $D(\overline{B})$ denote the new domain of this operator.

We now need to show that the conditions of Theorem B.10 are satisfied for $\overline{A}'$ and $\overline{B}$. This means we must show that (i) $\overline{B}$ is a closed operator (ii) $D(\overline{A}') \subset D(\overline{B})$ and (iii) there exists α sufficiently small and $\beta \geq 0$ such that for $x \in D(\overline{A}')$ $\|\overline{B}x\| \leq \alpha\| \overline{A}'x\| + \beta\|x\|$. We have just shown (i). As for (ii), note that $D(\overline{B}) \supset D(B) = D(\overline{A}')$. Thus it only remains to check the norm condition on $\overline{B}x$. From (3.A.10) we see that there is a $C > 0$ such that $\| \overline{B}x\|^2 \leq C\| \overline{A}'x \|^2$. In particular, inspection of the (3.A.10) shows that $C \sim L/m_B$. Thus, if the mass of the rigid body is sufficiently greater than the mass of the beam, the conditions on Theorem B.10 are met so we can conclude that $\overline{A} = \overline{A}' + \overline{B}$ generates an analytic semigroup on X. ■

In an abuse of notation, but for the sake of simplicity, let A denote the closed operator $\overline{A}$.

Lemma 3.A.3 - Consider the operator A of (3.A.4). Suppose the mass of the rigid body is much larger than the mass of the beam. Then the operator A generates an exponentially stable semigroup T(t) on X.

Proof of Lemma 3.A.3 - Since A generates an analytic semigroup by Lemma 3.A.2, then by Proposition B.9 of Appendix B, we know that T(t) satisfies $\|T(t)\| \leq M\exp(\omega_0(t))$, where $\omega_0 = \sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\}$. It thus suffices to verify that the spectrum of A is strictly negative and bounded away from zero. To compute the spectrum of A we first add Y^{-1}(row2) to row 4, and we know this does not change the spectrum. Performing this operation we obtain

$$\left[\begin{array}{cc:cc}
-I_0^{-1}K_\omega & 0 & \int_c^{c+L} \underline{c}^x \mu\partial(\cdot)dz + \int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(\mu\partial(\cdot))_{1,2},\ 0]dx\right)dz & \int_c^{c+L} \underline{c}^x k\partial(\cdot)dz + \int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(k\partial(\cdot))_{1,2},\ 0]dx\right)dz \\
0 & -K_y/m_B & \dfrac{Y}{m_B}\int_c^{c+L} \mu\partial(\cdot)dz & \dfrac{Y}{m_B}\int_c^{c+L} k\partial(\cdot)dz \\
\hdashline
0 & 0 & 0 & I \\
0 & 0 & -\mu\partial(\cdot) & -k\partial(\cdot)
\end{array}\right]$$

Since the matrix is block upper triangular, the spectrum is easily seen to be

$$\left\{ \sigma(-I_0^{-1}K_\omega) \cup \sigma(-K_y/m_B) \cup \sigma\begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} \right\}.$$

Since I_0, K_ω, K_y are all positive definite, the eigenvalues of the first two terms in the curly brackets are strictly negative. As for the last term, using Lemma 5.B.2, we know that the spectrum of this operator is strictly negative, and bounded away from the $j\omega$-axis. Thus, the spectrum of A is strictly negative, and bounded away from the $j\omega$-axis. From Proposition B.9, we thus conclude that A generates an exponentially stable semigroup on X. ■

To conclude the proof that $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ go to zero exponentially, we now separate (3.4.1)-(3.4.10) into its linear and nonlinear terms. The linear term is very similar to A, but not quite. In A, axial effects were considered, whereas in Theorem 3.4.1, axial effects were ignored. However, the linear part of (3.4.1)-(3.4.10), denoted A'', *does* generate an analytic, exponentially stable semigroup. This can be seen as follows. Choose the state variable to be $(\underline{\omega},\ \dot{\underline{y}},\ u_1,\ u_2,\ u_{1t},\ u_{2t})^T$. Then A'' is explicitly given by

$$\begin{bmatrix}
-I_0^{-1}K_\omega & 0 & \begin{array}{l} I_0^{-1}\int_c^{c+L} \underline{c}^x \,\mathrm{col}[(\mu\partial(\cdot) + k\partial(\cdot))_{1,2},\ 0]dz + \\ I_0^{-1}\int_c^{c+L}\left(\int_z^{L} \mathrm{col}[(\mu\partial(\cdot) + k\partial(\cdot))_{1,2},\ 0]dx\right)dz \end{array} & & & \\
0 & -\frac{K_y}{m_B} + Y\begin{pmatrix} 0 \\ 0 \\ \frac{(Y^{-1}K_y(\cdot))_3}{m_B(m_B+L)} \end{pmatrix} & \frac{Y}{m_B}\begin{pmatrix} \int_c^{c+L}\mu_1(\cdot)''''dz \\ 0 \\ 0 \end{pmatrix} & \frac{Y}{m_B}\begin{pmatrix} \int_c^{c+L}k_1(\cdot)''''dz \\ 0 \\ 0 \end{pmatrix} & \frac{Y}{m_B}\begin{pmatrix} 0 \\ \int_c^{c+L}\mu_2(\cdot)''''dz \\ 0 \end{pmatrix} & \frac{Y}{m_B}\begin{pmatrix} 0 \\ \int_c^{c+L}k_2(\cdot)''''dz \\ 0 \end{pmatrix} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & \frac{-(Y^{-1}K_y(\cdot))_{1,2}}{m_B} & -\mu_2(\cdot)'''' + \frac{1}{m_B}\int_c^{c+L}\mu_2(\cdot)''''dz & 0 & -k_2(\cdot)'''' + \frac{1}{m_B}\int_c^{c+L}k_2(\cdot)''''dz & 0 \\
0 & & 0 & -\mu_1(\cdot)'''' + \frac{1}{m_B}\int_c^{c+L}\mu_1(\cdot)''''dz & 0 & -k_1(\cdot)'''' + \frac{1}{m_B}\int_c^{c+L}k_1(\cdot)''''dz
\end{bmatrix}$$

(3.A.11)

To show that this linear operator generates an analytic semigroup can be done in exactly the same way as before, using the perturbation theorem on analytic semigroups. In fact, conceptually the proof is simpler because there are no u″ terms due to axial displacements present. (Notationally, however, the proof is much tougher!) The only thing that remains to be checked is whether the semigroup is exponentially stable. If we multiply the second row of this matrix by Y^{-1}, and then add the upper 2x2 block to the last two rows, we obtain an upper triangular matrix as before. The spectrum is therefore the union of the individual blocks. Since the matrix is block upper triangular, the spectrum is easily seen to be

$$\{ \sigma(-I_0^{-1}K_\omega) \cup \sigma(-K_y/m_B + Y_{\cdot,3}\frac{(Y^{-1}K_y)_{3,\cdot}}{m_B(m_B+L)}) \cup \sigma\begin{bmatrix} 0 & I \\ -\mu_1(\cdot)'''' & -k_1(\cdot)'''' \end{bmatrix} \cup \sigma\begin{bmatrix} 0 & I \\ -\mu_2(\cdot)'''' & -k_2(\cdot)'''' \end{bmatrix} \}. \tag{3.A.12}$$

where, in a notation specific to this proof, $Y_{\cdot,3}$ is the third column of Y, whereas $(Y^{-1}K_y)_{3,\cdot}$ denotes the third row of the matrix $Y^{-1}K_y$.

Since I_0, K_ω, K_y are all positive definite, $\sigma(-I_0^{-1}K_\omega)$ lies in the open left half-plane. From Theorem 5.B.2, the spectrum of the last two terms also lie in the open left half plane, since by assumption, $k_1 > 0$ and $k_2 > 0$. Thus, it only remains to show that all eigenvalues of the matrix

$$-K_y/m_B + Y_{\cdot,3}\frac{(Y^{-1}K_y)_{3,\cdot}}{m_B(m_B+L)} \tag{3.A.13}$$

lie in the open left half-plane.

By assumption, K_y is positive definite, and the mass of the rigid body is much larger than the mass of the beam L. Therefore, the eigenvalues of (3.A.13) are all strictly negative. ((3.A.13) essentially looks like A + εB, where A is negative definite, ε arbitrarily small, and B arbitrary.)

Hence, the spectrum of A″ given by (3.A.12) is negative and bounded away from the jω-axis. Using the fact that A″ generates an analytic semigroup, combined with Proposition B.9, shows that A″ generates an exponentially semigroup on X. Next, we want to show that the remaining terms can be bounded by $K\|(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)\|_X^2$, for some K > 0, so that Theorem 3.A.1 can be applied. The nonlinearities are

$$f(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) := \begin{bmatrix} -I_0^{-1}(\underline{\omega}^x I_0 \underline{\omega}) + I_0^{-1}\int_c^{c+L} \underline{u}^x \begin{pmatrix} 0 \\ 0 \\ (\underline{u}^x \dot{\underline{\omega}} - 2\underline{\omega}^x \underline{u}_t - \underline{\omega}^x (\underline{\omega}^x \underline{u}))_3 \end{pmatrix} dz \\ \dfrac{Y}{m_B}\begin{pmatrix} 0 \\ 0 \\ \int_c^{c+L} (\underline{u}^x \dot{\underline{\omega}} - 2\underline{\omega} x\, \underline{u}_t - \underline{\omega}^x (\underline{\omega}^x \underline{u}))_3 dz \end{pmatrix} \\ (\underline{u}^x \dot{\underline{\omega}} - 2\underline{\omega}^x \underline{u}_t - \underline{\omega}^x (\underline{\omega}^x \underline{u}))_{1,2} \end{bmatrix}$$

(3.A.14)

(3.A.14) also contains terms in $\dot{\underline{\omega}}$. However, using (3.4.1) $\dot{\underline{\omega}}$ can be replaced by terms involving $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$. The motivation here is simplicity of notation.) There is no difficulty in establishing such a bound for all terms not involving $\dot{\underline{\omega}}$, since they are clearly at least quadratic in $\underline{\omega}$, $\underline{u}$, and $\underline{u}_t$. So it only remains to show that the terms involving $\dot{\underline{\omega}}$ are also at least quadratic in the state variables. An inspection of (3.A.14) shows that it suffices to show that $\|\dot{\underline{\omega}}\|_{\mathbb{R}}$ can be bounded above by

$C\|(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)\|_X$ for some $C > 0$. So consider the equation for $\dot{\underline{\omega}}$:

$$\dot{\underline{\omega}} = -K_\omega \underline{\omega} - I_0^{-1}(\underline{\omega}^x I_0 \underline{\omega}) + I_0^{-1} \int_c^{c+L} \underline{u}^x \begin{pmatrix} \mu_1 u_1'''' + k_1 u_1'''' \\ \mu_2 u_2'''' + k_1 u_2'''' \\ (\underline{u}^x \; \dot{\underline{\omega}} \; - 2\underline{\omega}^x \; \underline{u}_t - \underline{\omega}^x \; (\underline{\omega}^x \; \underline{u}) \;)_3 \end{pmatrix} dz \quad (3.A.15)$$

$$+ \frac{I_0^{-1}}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (2\underline{\omega}^x \; \underline{u}_t + \underline{\omega}^x \; (\underline{\omega}^x \; \underline{u}) - \dfrac{2Y^{-1}K_y \dot{\underline{y}}}{m_B})_3 \end{pmatrix} dz$$

where we have used the formulation (3.1.24) for $\underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}$, and the term $(Y^{-1}\ddot{\underline{y}})_3$ has been explicitly calculated. (See the discussion in the paragraph above equation (4.3.16) if there is any difficulty in establishing this.) An inspection of (3.A.15) shows that only the terms inside the integral pose any difficulty, since $\|-I_0^{-1}(\underline{\omega} x I_0 \underline{\omega})\| \leq C'\|\underline{\omega}\|_{\mathbb{R}}^2$ for an appropriate C'. If we integrate by parts twice on the terms involving $\mu_1 u_1''''$ and $\mu_2 u_2''''$, these terms can be bounded by potential energy type terms, i.e., terms of the form $\int_c^{c+L} \mu_1 (u_1'')^2 dz$ and $\int_c^{c+L} \mu_1 (u_1'')^2 dz$. (This is primarily why we chose this formulation of $\underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}$.) Using similar means, the other terms in the integral can also be bounded by energy type terms. Hence, inserting these bounds on $\dot{\underline{\omega}}$ into (3.A.14), shows that the nonlinearity $f(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$ can be bounded as

$$\|f(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)\|_X \leq K\|(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)\|_X^2$$

for an appropriate $K > 0$.

We have thus shown the following facts: (i) The linear part of (3.4.1)-(3.4.10) generates an exponentially stable semigroup, (ii) the nonlinear part $f(\underline{\omega}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$ satisfies $\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$ (where $x = (\underline{\omega}, \dot{\underline{y}}, u_1, u_2, u_{1t}, u_{2t})^T$), and (using the first part of Theorem 3.4.1) (iii) the differential equation is globally asymptotically stable. Thus, Theorem 3.A.1 applies so we conclude that $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ go to zero *exponentially*.

■

Appendix 3.B

Proof of Existence and Uniqueness of solutions to the equations of Theorem 3.4.1

To verify that the coupled nonlinear partial differential equations given by (3.4.1)-(3.4.10) have a unique, continuously differentiable solution, we use standard semigroup theory. The idea is to first separate the differential equation into its linear and nonlinear terms. It is easily verified that the linear portion has a unique, continuously differentiable solution. Thinking of the nonlinear term as a perturbation, it remains to show that the "perturbed" system has a unique, continuously differentiable solution. This will follow from a result of Segal if, *very roughly speaking*, the perturbation is "less unbounded" then the linear portion.

First, separate the differential equation into its linear and nonlinear parts. The linear part, denoted A'', is given by (3.A.11), while the nonlinear part is given by (3.A.14).

Let the space A'' operates on be $X := \mathbb{R}^3 x \mathbb{R}^3 x H_0^2 x H_0^2 x L^2 x L^2$, and let the domain of A'', $D(A'')$, be defined as

$$D(A'') := \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T |\ x_1 \in \mathbb{R}^3, x_2 \in \mathbb{R}^3, x_3 \in H_0^4, x_4 \in H_0^4, x_5 \in H_0^4,$$
$$x_6 \in H_0^4,\ x_3''(c+L) = x_4''(c+L) = x_5''(c+L) = x_6''(c+L) = 0,$$

$x_3'''(c+L)=x_4'''(c+L)=x_5'''(c+L)=x_6'''(c+L)=0\}$

Let the inner product on X be

$$[a,b]_X := [a_1,b_1]_{\mathbb{R}} + m_B[a_2,b_2]_{\mathbb{R}} + [a_3'',b_3''] + [a_4'',b_4''] + [a_5,b_5] + [a_6,b_6]$$

where $[a, b]_{\mathbb{R}}$ is the ordinary inner product in $\mathbb{R}^3$, and $[a, b]$ is the ordinary L^2 inner product.

It is easy to verify that D(A′) is actually a Banach space when equipped with the graph norm

$$\|x\|_G^2 := \|x\|^2 + \|Ax\|^2$$

where $\|x\|$ is the L^2 norm. Let [D(A′)] denote this Banach space. Further note that the nonlinear term f is a compact operator on [D(A′)]. Indeed, f is actually C^∞ on [D(A′)]. Using [Seg. 1, Theorem 4.1], we can thus conclude local existence and uniqueness of the differential equations of 3.4.1.

From the proof of Theorem 3.4.1, we see that solutions are exponentially stable if they exist, so the local existence and uniqueness result can be extended globally, i.e. for all $t \geq 0$. This concludes the proof. ■

CHAPTER 4

ATTITUDE CONTROL VIA LYAPUNOV TECHNIQUES

4.1 - Introduction

In the remainder of this thesis, we consider the problem of satellite attitude control. In its simplest form, the problem consists of trying to move the satellite to a specified orientation with respect to the earth. This might be desirable, for instance, if the spacecraft were to be pointed at an earth station or target.

As might be suspected, this problem has been studied extensively, especially when the spacecraft is a rigid body. The standard approach to rigid body attitude control is to linearize the spacecraft equations of motion about a nominal orbit and design a linear control law for the linearized equations. (For example, see [Mork 1], [Hir.1], and [Dou. 1].) This approach is perfectly valid for small spacecraft adjustments, such as those needed to compensate for solar torque effects, gravity gradients, and other small external disturbances. For flexible spacecraft, the approach is similar: Model the flexible portions of the spacecraft by a suitable finite dimensional approximation, linearize the resulting equations of motion around a nominal orbit, and design a suitable linear control law. This design methodology assumes small perturbations, including small deflections in the elastic components of the structure. Again, these assumptions are perfectly valid and even desirable for spacecraft such as communications satellites, where large deflections of solar panels, for example, are undesirable.

Future generations of space vehicles will have entirely different requirements. For applications such as the Strategic Defense Initiative (SDI), spacecraft will be required to slew over large angles, at fast angular rates. It is easy to see that either of these two requirements completely invalidate the linear analysis of traditional spacecraft control system design, which assumed small perturbations of current orbital position. Motivated by such requirements, a myriad of nonlinear attitude control laws have been proposed for both rigid and flexible structures. (See, for example, [Dwy. 1], [Dwy. 4], [Mei. 1], [Vad. 1] and [Mon. 1].)

Almost without exception, however, the methods of these papers assume that the flexible portion of the structure (which is possibly infinite dimensional) can be suitably modelled by a finite dimensional approximation. The resulting equations are then ordinary, nonlinear differential equations, from which the attitude control law is then designed. The difficulty is choosing an appropriate finite dimensional approximation to the system. Currently, there is no systematic way of choosing such an order, and each author has his own method of verifying whether the unmodelled modes (the "spillover" effects) affect the system performance.

This thesis will dispense with such issues by considering the flexible portion of the spacecraft modelled as being of infinite dimensional form. The main disadvantage of such an approach is the corresponding mathematical difficulties, which tend to obscure the physical principles behind the control laws.

In section 4.2 attitude control laws for a rigid spacecraft are proposed. In Theorem 4.2.1, a linear attitude control law for a rigid spacecraft is proposed. The proof is by Lyapunov methods; a Lyapunov functional is constructed, and its derivative is computed to be nonpositive. From LaSalle's Invariance principle, we will conclude that the system is globally asymptotically stable. Observing that the linearization about the origin yields a linear system with strictly negative eigenvalues, combined with a theorem from Appendix B will give an exponential stability result.

In Theorem 4.2.3, another attitude control law is proposed for a rigid spacecraft.

The control law is a nonlinear control law, which would take into account possible nonlinearities in sensors and actuators. The method of proof is again by Lyapunov methods, and a global asymptotic stability result is obtained.

In section 4.3, an attitude control law for a flexible spacecraft with significant beam damping is proposed. The method is proof is identical to that of the rigid spacecraft, with the exception that we must use an infinite dimensional version of LaSalle's Theorem. The control law is linear, and an exponential stability result is obtained.

Finally, in section 4.5, we consider the attitude control of a flexible spacecraft where the beam damping of the spacecraft is assumed zero. This assumption necessitates the use of beam control, which will be boundary control of the type used in Chapter 2 and Chapter 3. The method of proof is similar to that of sections 4.2 and 4.3, where again the infinite dimensional version of LaSalle's Theorem will be used. Using the linear control law, exponential stability is once again obtained.

4.2 Lyapunov Based Attitude Control Law for a Rigid Spacecraft

To help illustrate the ideas for the following sections, we will first consider the attitude control of a rigid body. The equations of motion for the structure are

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{4.2.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} \tag{4.2.2}$$

where as in Chapter 3, section 3.1.4, $\underline{\omega}$ is the angular velocity of the spacecraft with respect to the inertial frame, I_0 is the spacecraft moment of inertia with respect to the body frame, $\underline{\tau}$ is the torque due to the torque jets, and $\underline{\xi}$ is the Gibb's vector representing the spacecraft attitude with respect to the inertial frame. Since the rigid structure rotational and translational terms are decoupled, we can assume $\dot{\underline{y}} = 0$.

In this chapter, the attitude control problem will to be to design a control law so that $\underline{\xi} \to 0$, $\underline{\omega} \to 0$, and, when appropriate, have the beam displacements and velocities go to zero. This insures that the attitude is corrected, and stays that way for ever after.

Lyapunov type control laws for rigid spacecraft have been obtained before ([Mort. 1] is the first reference known to the author), but the implementations have required nonlinear feedback. The Lyapunov control law given below will be similar in style to the above papers, but it will be implemented using <u>linear</u> feedback.

Theorem 4.2.1 - Consider the system described by (4.2.1) and (4.2.2) above. Let the control law be

$$\underline{\tau} := -k_\xi \underline{\xi} - K_\omega \underline{\omega} \tag{4.2.3}$$

where $K_\omega \in \mathbf{R}^{3x3}$ is positive definite and $k_\xi \in \mathbf{R}$, $k_\xi > 0$. Then the system is *exponentially* stable, i.e., $\underline{\omega}$ and $\underline{\xi}$ go to zero exponentially. (Physically, this means that the body stops spinning, and the attitude is corrected.)

Proof of Theorem 4.2.1 - Consider the Lyapunov functional

$$V(\underline{\omega}, \underline{\xi}) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + k_\xi \ln(1 + \underline{\xi}^T \underline{\xi}) \tag{4.2.4}$$

The first term represents the energy of the body (recall that the velocity of the center of mass of the rigid body is assumed zero), while the second term is a measure of the attitude "energy". It is easy to see that V is a positive definite function ([Vid. 1, p. 141]). Also note that

$$\dot{V}(\underline{\omega}, \xi) = \underline{\omega}^T I_0 \dot{\underline{\omega}} + (1 + \xi^T \xi)^{-1} k_\xi 2 \xi^T \dot{\xi} \tag{4.2.5}$$

$$= \underline{\omega}^T (-\underline{\omega}^x I_0 \underline{\omega} + \underline{\tau}) + (1 + \xi^T \xi)^{-1} k_\xi 2 \xi^T (\tfrac{1}{2} [I + \xi\xi^T + \xi^x] \underline{\omega}) \tag{4.2.6}$$

$$= \underline{\omega}^T \underline{\tau} + k_\xi \xi^T \underline{\omega} \tag{4.2.7}$$

$$= -\underline{\omega}^T K_\omega \underline{\omega} \leq 0 \tag{4.2.8}$$

where (4.2.7) has been obtained by noting that $\underline{\omega} \perp \underline{\omega}^x I_0 \underline{\omega}$ and $\xi \perp \xi^x \underline{\omega}$, and (4.2.8) is obtained by insertion of the control law (4.2.3) into (4.2.7).

Note that $\dot{V}(\underline{\omega}, \xi) = 0$ if and only if $\underline{\omega} = 0$, which in turn implies that $\dot{\underline{\omega}} = 0$. Using (4.2.2), $\underline{\tau} = 0$, and (4.2.3) implies that $\xi = 0$. Thus the largest invariant set [Vid. 1, p. 156] of system (4.2.1)-(4.2.2) containing $\{\xi^x\underline{\omega} \mid \underline{\omega} = 0\}$ is $\{(0, 0)\}$. Thus the conditions of LaSalle's Invariance principle [Vid. 1, p. 157] are met and we conclude that for all initial conditions $\underline{\omega}_0$ and ξ_0 the trajectories of the system (4.2.1)-(4.2.2) approach zero.

To show that $\underline{\omega}$ and $\dot{\underline{\omega}}$ both go to zero exponentially, note that the linearization of the system (4.2.1), (4.2.2), and (4.2.3) about zero is

$$\dot{\xi} = \frac{1}{2} \underline{\omega}. \tag{4.2.9}$$

$$I_0 \dot{\underline{\omega}} = -k_\xi \xi - K_\omega \underline{\omega} \tag{4.2.10}$$

Since, I_0 and K_ω are positive definite, and $k_\xi > 0$, this means that the eigenvalues of this linear system are all strictly negative. Thus, the conditions of Theorem 3.A.1 of Chapter 3, Appendix A are satisfied, and we therefore conclude that the nonlinear system (4.2.1), (4.2.2) and (4.2.3) is *exponentially* stable. ■

From an engineering perspective, it would be nice if the control law (4.2.3) allowed some nonlinear terms. This is because sensors and actuators, no matter how carefully constructed, contain some residual nonlinearities. The following theorem allows a class of such nonlinearities.

Theorem 4.2.2 - Consider the system described by (4.2.1) and (4.2.2) above. Let the control law be

$$\underline{\tau} := -k(\xi^T\xi)\xi - \underline{g}(\underline{\omega}) \tag{4.2.11}$$

where $k(\xi^T\xi)$ is an arbitrary nonlinear function satisfying $\infty > k(\xi^T\xi) \geq c_1 > 0$, and $\underline{g}(\underline{\omega})$ is a nonlinear function lying in the sector $[c_2, \infty)$, i.e., for some $c_2 > 0$, $c_2\| \underline{\omega} \|^2 \leq \underline{\omega}^T\underline{g}(\underline{\omega}) < \infty$. Then the system (4.2.1)-(4.2.3) is globally asymptotically stable, i.e., $\underline{\omega}$ and ξ go to zero.

Proof of Theorem 4.2.2 - Consider the Lyapunov function candidate

$$V(\underline{\omega}, \xi) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_0^{\xi^T\xi} \frac{k(x)}{1+x}\, dx \tag{4.2.12}$$

First note that $V(\underline{\omega}, \xi)$ is a positive definite function. This follows since

$$V(\underline{\omega}, \xi) \geq \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_0^{\xi^T\xi} \frac{c_1}{1+x}\,dx = \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + c_1 \ln(1+\xi^T\xi) \qquad (4.2.13)$$

which is clearly a positive definite function. Next, by computation

$$\dot{V}(\underline{\omega}, \xi) = \underline{\omega}^T I_0 \dot{\underline{\omega}} + \frac{k(\xi^T\xi)}{(1+\xi^T\xi)} 2\xi^T \dot{\xi} \qquad (4.2.14)$$

$$= \underline{\omega}^T \underline{\tau} + \frac{k(\xi^T\xi)}{(1+\xi^T\xi)}(1+\xi^T\xi)\xi^T \underline{\omega} \qquad (4.2.15)$$

$$= \underline{\omega}^T \underline{\tau} + k(\xi^T\xi)\xi^T \underline{\omega} \qquad (4.2.16)$$

$$= -\underline{\omega}^T g(\underline{\omega}) \leq -c_2 \| \underline{\omega} \|^2 \qquad (4.2.17)$$

where (4.2.17) is obtained by insertion of (4.2.11) into (4.2.16), and the last inequality results from the sector condition on $g(\underline{\omega})$.

Note that $\dot{V}(\underline{\omega}, \xi) = 0$ if and only if $\underline{\omega} = 0$, which in turn implies that $\dot{\underline{\omega}} = 0$. Using (4.2.2) and (4.2.3) we obtain $k(\xi^T\xi)\xi = 0$ which implies that $\xi = 0$. Thus the largest invariant set of system (4.2.1)-(4.2.2) containing $\{\xi^x\underline{\omega} \mid \underline{\omega} = 0\}$ is $\{(0, 0)\}$. Thus the conditions of LaSalle's Invariance principle are met and we conclude that for all initial conditions $\underline{\omega}_0$ and ξ_0 the trajectories of the system (4.2.1)-(4.2.2) approach zero.

Remark 4.2.3 - To implement the control law (4.2.3)-(4.2.11) one must be able to determine ξ and $\underline{\omega}$. As stated in Remark 3.4.4, one can determine $\underline{\omega}$ by use of rate integrating gyros mounted on the rigid body. To determine the attitude ξ, a variety of methods can be used. Utilizing star sensors, horizon sensors, sun sensors, etc., one can estimate the direction cosine matrix Y of (3.1.3)-(3.1.5) directly. For example, if the direction of a star in body coordinates is measured as $\underline{u}$ (a unit vector), and the direction of the star in inertial coordinates is determined from a star catalog to be $\underline{U}$, then $\underline{u}$ and $\underline{U}$ are related by $\underline{U} = Y\underline{u}$. By using several measurements, a least squares estimate of the direction cosine matrix can be obtained [Wer. 1, p. 457]. Once the direction cosine matrix is obtained, the Gibb's vector $\underline{\xi}$ can be algebraically solved for by using equation (3.1.20), which relates Y and $\underline{\xi}$.

Alternatively, one could estimate $\underline{\xi}$ directly. The difficulty here is that the resulting estimation problem is nonlinear, and thus requires nonlinear filtering. An example of this type of procedure for estimating the quaternion attitude vector can be found in [Gai. 1].

4.3 Lyapunov Based Control Laws for a Flexible Spacecraft - Beam Damping Present

In this section we propose an attitude control law for a flexible spacecraft in much the same way as the rigid structure. Unfortunately, the proof of the result will be far more difficult since LaSalle's Invariance principle does not hold for infinite dimensional systems. (Its proof relies on the compactness of the unit ball in $\mathbf{R}^n$.) We will first need to introduce a generalization of LaSalle's Invariance principle, based on the theory of gradient systems [Hale 1]. It should we stressed, however, that the intuition is exactly the same as that in the rigid case.

Definition 4.3.1 - If T(t): X $\rightarrow$ X is a strongly continuous (possibly nonlinear) semigroup on a Banach space X, an <u>equilibrium point</u> of T(t) is a point x of X such that $T(t)x = x, \forall\, t \geq 0$.

Definition 4.3.2 [Hale 1, p. 20] - Let X be a Banach space, $T(t) : X \to X$ be a strongly continuous (nonlinear) semigroup. The semigroup is said to be a gradient system if

(i) Each bounded orbit is precompact (Recall a set E in a metric space is precompact if there is a finite covering of E by sets of diameter $< \varepsilon$)

(ii) There exists a Lyapunov functional for T(t); that is, there exists a continuous function $V : X \to \mathbf{R}$ with the following properties

(iia) V(x) is bounded below;

(iib) $V(x) \to \infty$ as $|x| \to \infty$;

(iic) $V(T(t)x)$ is nonincreasing in t for all $x \in X$;

(iid) If x is such that $V(T(t)x) = V(x)$ for all $t \geq 0$, then x is an equilibrium point of T(t).

Comment 4.3.3: It is easy to see that condition (i) is trivially satisfied for ordinary nonlinear differential equations since in $\mathbf{R}^n$, a set is precompact $\Leftrightarrow$ the set is bounded. The conditions contained in (ii) are exactly the same conditions required for LaSalle's Invariance principle to hold in finite dimensions.

Gradient systems yield the following generalization of LaSalle's Invariance Principle. [Hale 1, p. 20]

Theorem 4.3.4 - If T(t) is a gradient system, then the ω-limit set ω(x) ($:= \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} T(t)x, \forall x \in X$) belongs to the set of equilibrium points of T(t).

Proof of Theorem 4.3.4 - (See [Hale 1, p. 20].) Since V satisfies (iib) and (iic), it follows that the positive orbit through x is bounded. Thus, by hypothesis, it is precompact. Also, $V(T(t)x)$ has a definite limit as $t \to \infty$ since it is a nonincreasing function that is bounded below by (iia): let c denote this limit. Since the positive orbit is precompact, $\omega(x)$ is compact and invariant. The fact that V is continuous implies that $V(T(t)y) = c$ for all $y \in \omega(x)$ and for all $t \in \mathbb{R}^+$. Hypothesis (iid) implies that y is an equilibrium point. ■

Theorem 4.3.5 - Consider the flexible spacecraft model where axial effects are ignored, described in section 3.1.6 and denoted (FSS/A). Also assume no active beam control i.e., $F_i(t) = 0$, i= 1, 2, 3 in (3.1.73). The equations then become

$$\dot{\underline{\xi}} = \tfrac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{4.3.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}. \tag{4.3.2}$$

$$m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{4.3.3}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{4.3.4}$$

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c), \quad -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c), \quad \int_c^{c+L} -(\dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_3 dz\Big)^T \tag{4.3.5}$$

$$\underline{M}_{bB} = \big(\mu_1 u_1''(c) + k_1 u_{1t}''(c), \quad \mu_2 u_2''(c) + k_2 u_{2t}''(c), \quad 0\big)^T \tag{4.3.6}$$

$$(\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}})_{1,2} = 0 \tag{4.3.7}$$

$$u_1(c) = u_2(c) = 0, \quad u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0 \qquad u_1'''(c + L) = u_2'''(c + L) = 0 \tag{4.3.8}$$

Assume that damping is explicitly present, i.e. $k_i > 0$, i=1, 2. Let the control law be

$$\underline{\tau} := -k_\xi\underline{\xi} - K_\omega\underline{\omega} \tag{4.3.9}$$

$$\underline{F}_T := -K_{\dot{y}}\, \dot{\underline{y}} \tag{4.3.10}$$

where $K_\omega \in \mathbb{R}^{3x3}$ is a positive definite matrix, $k_\xi \in \mathbb{R}$ with $k_\xi > 0$, and $K_{\dot{y}}$ is a positive definite matrix. Then the system is globally asymptotically stable, i.e., $\underline{\omega} \to 0$, $\underline{\xi} \to 0$,

$\underline{u} \to 0$, $\underline{u}_t \to 0$, $\dot{\underline{y}} \to 0$ in appropriate norms. Furthermore, if the mass of the rigid body is much greater than that of the beam, then $\underline{\omega} \to 0$, $\underline{\xi} \to 0$, $\underline{u} \to 0$, $\underline{u}_t \to 0$, $\dot{\underline{y}} \to 0$ *exponentially.*

Proof of Theorem 4.3.5 - The method of proof will be to first show that our spacecraft system is a gradient system, and then use Theorem 4.3.4. This will give an global asymptotic stability result. Then applying Theorem 3.A.1 will show that the decay rate is actually exponential.

First, consider the Lyapunov function candidate

$$E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_c^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1}\dot{\underline{y}}\|^2 \, dx$$

$$+ \frac{1}{2} m_B \|\dot{\underline{y}}\|^2 + \int_c^{c+L} [\mu_1(u_1'')^2 + \mu_2(u_2'')^2] \, dx \quad + k_\xi \ln(1 + \underline{\xi}^T\underline{\xi}) \qquad (4.3.11)$$

(For simplicity, let E denote $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$, and let $\dot{E}$ denote the time derivative of $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$.) Note that $E \geq 0$ and the similarity between this Lyapunov functional and the one obtained for the rigid body in Theorem 4.2.1 and the flexible structure in section 3.4: The first three terms represent the total kinetic energy of the system, the fourth term the potential energy of the system, and the last term the measure of attitude "energy". Note that this functional satisfies requirements (iia) and (iib) in the definition 4.3.2 of the gradient system. We need to verify the rest of condition (ii), and then afterward we will show that condition (i) is achieved.

We can compute $\dot{E}$ exactly as in Theorem 4.2.1, but reasoning similar to that of Theorem 3.4.1 allows us to write the answer down by inspection. As before, recall that the rate of change of energy is the instantaneous power delivered to the system.

Since the external forces acting on the system are the torque jets, external thrusters and the forces applied to the beam, we must have

$$\dot{E} = \underline{\omega}^T\underline{\tau} + \underline{F}_T{}^T \dot{\underline{y}} - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 \, dz \} + (1 + \underline{\xi}^T\underline{\xi})^{-1} k_\xi 2\underline{\xi}^T \dot{\underline{\xi}}$$

$$= \underline{\omega}^T\underline{\tau} - \{ \int_c^{c+L} [k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2] \, dz \} + \underline{F}_T{}^T \dot{\underline{y}} + k_\xi \underline{\xi}^T \underline{\omega} \qquad (4.3.12)$$

The first term of (4.3.12) is the instantaneous power delivered by the torque jets, the second term is the instantaneous power delivered by the force thrusters (recall that the instantaneous power must be calculated with respect to the inertial frame), the third term is the dissipated power due to the beam damping, and the last term is simply the rate of change of the attitude energy which has no simple physical interpretation. Inserting the control law giving $\underline{F}$ and $\underline{\tau}$ from (4.3.9)-(4.3.10) then yields

$$\dot{E} = -\underline{\omega}^T K_\omega \underline{\omega} - \{ \int_c^{c+L} k_1(u_{1t}'')^2 + k_2(u_{2t}'')^2 \, dz \} - \dot{\underline{y}}^T K_{\dot{y}} \dot{\underline{y}} \qquad (4.3.13)$$

$$\leq 0$$

Thus, condition (iic) in the definition 4.3.2 of gradient systems is verified. To show (iid), we need to show that if $E(T(t)x) = E(x)$ for all $t \geq 0$, then x is an equilibrium point of (FSS). From (4.3.13), $E(T(t)x) = E(x)$ for all $t \geq 0$ implies that $\underline{\omega} = 0$, $\dot{\underline{y}} = 0$, and $(\underline{u}_t''(z))_{1,\,2} = 0$ in L^2. This in turn shows that $\underline{\omega} = 0$, $\dot{\underline{y}} = 0$, $\dot{\underline{\omega}} = 0$, $\ddot{\underline{y}} = 0$, and $(\underline{u}_t''(z))_{1,\,2} = 0$ in L^2. From Proposition 3.4.2, (ii) and (iv), the latter term implies that $\underline{u}_{it}(z) = 0$, $i=1, 2$. Equations (4.3.7) and (4.3.8) then reduce to

$$(\underline{u}_{tt} + \mu\partial(\underline{u}))_{1,2} = 0 \qquad (4.3.14)$$

$$u_1(c)=u_2(c)=0,\ u_1'(c)=u_2'(c)=0 \quad u_1''(c+L)=0\ \ u_2''(c+L)=0 \qquad (4.3.15)$$

$$u_1'''(c+L)=0,\ u_2'''(c+L)=0, \quad u_{it}(z)=0,\ i=1,2.$$

It is easy to show that the only solution satisfying this linear differential equation, the given boundary conditions and the conditions $\underline{u}_{it}(z) = 0$, i=1, 2 is the zero solution. Therefore, $u_1 = 0$, $u_2 = 0$, $u_{1t} = 0$ and $u_{2t} = 0$. Finally, combining these results in (4.3.12) and (4.3.9) shows that $\xi = 0$. Thus, $E(T(t)x) = E(x)$ for all $t \geq 0$, implies that $x = 0$, i.e., x is an equilibrium point of (FSS). Thus (iid) of definition 4.3.2 is satisfied.

Thus, it only remains to show that bounded orbits are precompact. To show this, we first need a lemma, followed by a lemma, again due to Hale ([Hale 1, p. 14]):

Definition 4.3.6 - A family of mappings $T(t)$, $t \geq 0$, on a Banach space X is said to be conditionally completely continuous for $t \geq t_1$ if, for each $t \geq t_1$ and each bounded set B in X for which $\{T(s)B, 0 \leq s \leq t\}$ is bounded, the set $T(t)B$ is precompact.

Lemma 4.3.7 [Hale 1 p. 14] - Let $S(t)$, $t \geq 0$, be a strongly continuous (nonlinear) semigroup on a Banach space X satisfying

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds$$

for $t \geq 0$ and all $x \in X$ where $T(t)$ is a C^0 (nonlinear) semigroup on X and B is a mapping from X to X satisfying

(i) B is compact (i.e., B is continuous and for any bounded subset E of X, $\overline{B(E)}$ is compact);

(ii) $||T(t)|| \leq c(t)$, $c: \mathbf{R}^+ \to \mathbf{R}^+$, $c(t)$ continuous with $\lim_{t\to\infty} c(t) = 0$.

Then $S(t) = T(t) + U(t)$, where $U(t)$ is conditionally completely continuous.

The reason for introducing Definition 4.3.6 and Lemma 4.3.7 is contained in the following lemma. This lemma will allow us to verify that our system satisfies condition (i) in Definition 4.3.2 for gradient systems. (Incidentally, the proof of the following result does not seem to be anywhere in the literature, although Hale says that it is "clear". For completeness the proof is performed, with no claim to its originality.)

Lemma 4.3.8 - Suppose that S(t) satisfies the conditions of Lemma 4.3.7. Then bounded orbits of S(t) are precompact.

Proof of Lemma 4.3.8 - Suppose the positive orbit through $x_0 \in X$ is bounded. To show that the bounded orbit is precompact, we must show $M = \{S(s)x_0 : 0 \leq s < \infty\}$ is precompact. By definition, M is precompact if, given $\varepsilon > 0$, there exists a <u>finite</u> subset $M_F \subset M$ such that for each $x \in M$ there is a $x_F \in M_F$ satisfying $\| x - x_F \| < \varepsilon$. By assumption M is bounded, say $\| x \| < K$, for all $x \in M$. Note that $M = M_0 \cup M_\infty$, where $M_0 = \{S(s)x_0 : 0 \leq s < t_0\}$, and $M_\infty = \{S(s)x_0 : t_0 \leq s < \infty\}$. Note also $M_\infty = S(t_0)M$. Thus, $M = M_0 \cup S(t_0)M = M_0 \cup (T(t_0) + U(t_0))M \subset M_0 \cup T(t_0)M \cup U(t_0)M$.

Since T(t) satisfies $\| T(t) \| \leq c(t)$, $c: \mathbf{R}^+ \to \mathbf{R}^+$, $c(t)$ continuous with $\lim_{t\to\infty} c(t) = 0$, then for any $x \in M$, $\| T(t)x \| \leq \| T(t) \| \| x \| \leq c(t) K$. Thus, there is a $t_0 > 0$ such that $\| T(t)x \| < \varepsilon$, for all $t \geq t_0$. Fix this value of t_0.

Next, since S(t) is strongly continuous, there exists a finite set $\{t_1, t_2, \ldots t_j\}$ such that for each $t \in [0, t_0)$ there exists $t_k \in \{t_1, t_2, \ldots t_j\}$ such that $\| S(t)x_0 - S(t_k)x_0 \| < \varepsilon$.

Finally, consider $U(t_0)M$. Since $U(t_0)$ is conditionally completely continuous by hypothesis, and since $\{U(s)x_0 : 0 \le s < t_0\}$ is bounded, this implies that $U(t_0)M$ is precompact, and hence there is a finite subset $M_U \subset U(t_0)M$ such that for each $x \in U(t_0)M$ there is a $x_U \in M_U$ satisfying $\| x_U - x \| < \varepsilon$.

So now define $M_F = M_U \cup \bigcup_{k=1}^{n} S(t_k)x_0$, finite by construction. Now, take any $y \in M$. Then, $y \in M_0$, or $y \in T(t_0)M$, or $y \in S(t_0)M$. For any of these situations, the above construction shows that there is a $y_F \in M_F$ satisfying $\| y - y_F \| < \varepsilon$. This shows that M is precompact. ■

With these results in hand, we are finally able to complete the proof of Theorem 4.3.5. We wish to write the closed loop system as the sum of an exponentially stable system plus a compact function so that Lemmas 4.3.7 and 4.3.8 apply. Before proceeding, we need the the following facts.

1. Using (4.3.3) and (4.3.5), $(\underline{F}_{bB})_3$ can be explicitly calculated to be

$$(\underline{F}_{bB})_3 = \frac{m_B}{m_B + L} \int_c^{c+L} \left(\underline{\dot{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) - \frac{Y^{-1} K_y \dot{\underline{y}}}{m_B}\right)_3 \, dz$$

(ii) Note that $(\underline{F}_{bB})_3$ does not affect (4.3.2) since $\underline{c}^x(\underline{F}_{bB})_3 = 0$. It only affects (4.3.3) in a significant way.

Using these two facts the closed loop system (4.3.1)-(4.3.10) can be written in the suggestive form

$$\dot{\underline{\xi}} = -\underline{\xi} + \boxed{\underline{\xi} + \tfrac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}} \tag{4.3.16}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = -K_\omega \underline{\omega} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} - \boxed{k_\xi \underline{\xi}} \tag{4.3.17}$$

$$m_B \ddot{\underline{y}} = -K_{\dot{y}}\, \dot{\underline{y}} + Y(\underline{\xi}) \begin{pmatrix} (\underline{F}_{bB})_1 \\ (\underline{F}_{bB})_2 \\ 0 \end{pmatrix} +$$

$$+ \frac{Y(\underline{\xi}) m_B}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) - \dfrac{Y^{-1}K_y \dot{\underline{y}}}{m_B})_3 \end{pmatrix} dz \tag{4.3.18}$$

$$+ \frac{Y(\underline{\xi}) m_B}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (\underline{u}^x [I_0^{-1}(-\underline{\omega}^x I_0 \underline{\omega} - K_\omega \underline{\omega} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB})])_3 \end{pmatrix} dz$$

$$\boxed{+ \frac{Y(\underline{\xi}) m_B}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (k_\xi I_0^{-1} \underline{\xi}^x\, \underline{u}\,)_3 \end{pmatrix} dz}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T \underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{4.3.19}$$

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c) - k_1 u_{1t}'''(c), \quad -\mu_2 u_2'''(c) - k_2 u_{2t}'''(c),$$

$$\frac{m_B}{m_B + L} \int_c^{c+L} (\dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x\, (\underline{\omega}^x\, \underline{u}) - \frac{Y^{-1}K_y \dot{\underline{y}}}{m_B})_3\, dz\Big)^T \tag{4.3.20}$$

$$\underline{M}_{bB} = \big(\mu_1 u_1''(c) + k_1 u_{1t}''(c), \;\; \mu_2 u_2''(c) + k_2 u_{2t}''(c), \;\; 0\big)^T \tag{4.3.21}$$

$$(\underline{u}_{tt} + I_0^{-1}[-\underline{\omega}^x I_0 \underline{\omega} - K_\omega \underline{\omega} + \underline{c}^x\, \underline{F}_{bB} + \underline{M}_{bB}]^x\, \underline{u} + 2\underline{\omega}^x \underline{u}_t$$

$$+ \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}}\,)_{1,2} = (\,\boxed{-k_\xi I_0^{-1} \underline{\xi}^x\, \underline{u}}\,)_{1,2} \tag{4.3.22}$$

$$u_1'(c) = u_2'(c) = 0,\; u_1'(c) = u_2'(c) = 0 \quad u_1''(c + L) = 0 \;\; u_2''(c + L) = 0 \tag{4.3.23}$$

$$u_1'''(c + L) = 0,\; u_2'''(c + L) = 0$$

Think of the three encircled terms in the equations above as the perturbation, while the remainder generates an exponentially stable semigroup by Theorem 3.4.1. To apply Lemma 4.3.8 and Lemma 4.3.9, it thus remains to show that the perturbation term is a compact map.

Lemma 4.3.9 - The perturbation term $f : \mathbf{R}^3 \times \mathbf{R}^3 \times H_0^2 \times H_0^2 \times \mathbf{R} \rightarrow \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times L^2 \times L^2$ defined by

$$f(\underline{\xi}, \underline{\omega}, u_1, u_2, z) = \begin{bmatrix} \underline{\xi} + \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega} \\ -k_\xi \underline{\xi} \\ \dfrac{Y(\underline{\xi})m_B}{m_B + L} \displaystyle\int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (k_\xi I_0^{-1} \underline{\xi}^x \, \underline{u})_3 \end{pmatrix} dz \\ -(k_\xi I_0^{-1} \underline{\xi}^x \; \underline{u})_{1,2} \end{bmatrix} \tag{4.3.24}$$

is compact

Proof of Lemma 4.3.9 - Recall that a compact function [Hut. 1, p. 207] is a continuous function that maps bounded sets to relatively compact ones. Clearly, the first two components of f are compact, since these terms are finite dimensional, continuous functions. First consider the last component of f, $-k_\xi I_0^{-1} \underline{\xi}^x \, \underline{u}$, denoted f_5, which maps $\mathbf{R}^3 \times H_0^2 \times H_0^2 \times \mathbf{R} \rightarrow L^2 \times L^2$. (Recall $\underline{u} = (u_1, u_2, z)^T$ when axial displacements are ignored.) If we write $f_5 = f' \circ g$, where $g : \mathbf{R}^3 \times H_0^2 \times H_0^2 \times \mathbf{R} \rightarrow \mathbf{R}^3 \times L^2 \times L^2 \times \mathbf{R}$ is the embedding map, and $f' : \mathbf{R}^3 \times L^2 \times L^2 \times \mathbf{R} \rightarrow L^2 \times L^2$ is defined by $f'(\underline{\xi}, \underline{\omega}, u_1, u_2, z) = (-k_\xi I_0^{-1} \underline{\xi}^x \, \underline{u})_{1,2}$. Clearly, f' is a continuous function, and by the Sobolev embedding theorem [Paz. 1, p. 208], g is a compact map. Moreover, f' is a bounded map since it is a projection of a bilinear function [Die. 1, Thm. 5.5.1]. Thus $f' \circ g$ maps bounded sets to relatively compact ones. Finally, since $f' \circ g$ is continuous, this implies that $f_5 = f' \circ g$ is compact.

Finally, consider the third component of f

$$\frac{Y(\underline{\xi})m_B}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (k_\xi I_0^{-1} \underline{\xi}^x\ \underline{u}\ )_3 \end{pmatrix} dz$$

denoted f_3, which maps $\mathbb{R}^3 \times H_0^2 \rightarrow \mathbb{R}^3$. Note that f_3 can be written as $f_3 = g_I \circ g'$ where g_I is the integral operator

$$g_I(h) := \frac{Y(\underline{\xi})m_B}{m_B + L} \int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (h)_3 \end{pmatrix} dz$$

mapping $L^2 \rightarrow \mathbb{R}^3$ and g' is the operator $(k_\xi I_0^{-1} \underline{\xi}^x\ \underline{u})_3$ mapping $\mathbb{R}^3 \times H_0^2 \times H_0^2 \times \mathbb{R} \rightarrow L^2$. By arguments exactly the same as the above paragraph it can be established that g' is compact. Clearly, g_I is a bounded linear functional. Therefore, $f_3 = g_I \circ g'$ is also compact. Thus, all components of $f(\ \underline{\xi}, \underline{\omega}, u_1, u_2, z)$ are compact, which proves the lemma. ■

With this final lemma, we are finally able to complete the proof of Theorem 4.3.5.

Completion of the proof of Theorem 4.3.5

By combining Lemma 4.3.9 and Theorem 3.4.1, we see that the conditions of Lemma 4.3.8 are satisfied. Thus, for the given control law (4.3.9) - (4.3.10), bounded orbits of (FSS) are precompact, so that (i) of Definition 4.3.2 is satisfied. Since we have previously verified condition (ii) of Definition 4.3.2, this shows that the system (FSS) is a gradient system. Using Theorem 4.3.4, all trajectories evolve toward the set of equilibrium points of T(t), which is clearly {0}. Thus (FSS) with the control law (4.3.9)-(4.3.10) is globally asymptotically stable.

To show that the decay rate is actually exponential if the mass of the rigid body is much larger than the mass of the beam, consider the linearization of the system (4.3.1)-(4.3.10) about zero:

$$\dot{\underline{\xi}} = \frac{1}{2}\underline{\omega}$$

$$I_0\dot{\underline{\omega}} = -k_\xi\underline{\xi} - K_\omega\underline{\omega} + \underline{c}^x\,\underline{F}_{bB} + \underline{M}_{bB}.$$

$$m_B\ddot{\underline{y}} = \underline{F}_{bB} - K_{\dot{y}}\,\dot{\underline{y}}$$

$$\underline{F}_{bB} = \Big(\int_c^{c+L}(\mu_1 u_1'''' + k_1 u_{1t}'''')dz,\ \int_c^{c+L}(\mu_2 u_2'''' + k_2 u_{2t}'''')dz,$$
$$- L(\ddot{\underline{y}})_3\Big)^T$$

$$\underline{M}_{bB} = \int_c^{c+L}\left(\int_z^L \mathrm{col}[(\mu\partial(\underline{u}) + k\partial(\underline{u}_t))_{1,2},\ 0]dx\right)dz$$

$$(\underline{u}_{tt} + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + \ddot{\underline{y}})_{1,2} = 0$$

$$u_1(c)=u_2(c)=0,\ u_1'(c)=u_2'(c)=0$$

$$u_1''(c + L) = 0\ \ u_2''(c + L) = 0,\ \ u_1'''(c + L) = 0,\ u_2'''(c + L) = 0$$

Note that $\underline{\xi}$ is connected to the dynamics only through $\underline{\omega}$. Therefore, the state space form of these equations is in block upper triangular form, consisting of the $\underline{\xi}$-$\underline{\omega}$ block and the block of the terms in $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$. This means that the spectrum of this linear system is the union of the spectrum of the $\underline{\xi}$-$\underline{\omega}$ block

$$\begin{bmatrix} 0 & \frac{1}{2}\underline{I} \\ -k_{\xi}\underline{I}_0^{-1} & -K_{\omega} \end{bmatrix}$$

and the spectrum of the remaining terms. The spectrum of the remainder of the dynamics are shown to be exponentially stable in the proof of Theorem 3.4.1, part(ii) (see Lemma 3.A.3) if the mass of the rigid body is much greater than the mass of the beam. Therefore, the linearization of the system (4.3.1)-(4.3.10) is exponentially stable. It is also easy to verify that the remaining nonlinear terms, denoted $f(\underline{\xi}, \underline{\omega}, \dot{\underline{y}}, u_1, u_2, u_{1t}, u_{2t})$ satisfy $\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$, where $x = (\underline{\xi}, \underline{\omega}, \dot{\underline{y}}, u_1, u_2, u_{1t}, u_{2t})^T$. (See the proof of Lemma 3.A.3 if there is difficulty in establishing this.) We have therefore obtained the following information: (i) the linear part of (4.3.1)-(4.3.10) is exponentially stable, (ii) the nonlinear portion of (4.3.1)-(4.3.10) satisfies $\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$, and (iii) (4.3.1)-(4.3.10) is globally saymptotically stable. Thus, Theorem 3.A.1 applies we conclude that $\underline{\omega} \to 0, \underline{\xi} \to 0, \underline{u} \to 0, \underline{u}_t \to 0, \dot{\underline{y}} \to 0$ *exponentially.* ■

Analogous to Theorem 4.2.2, we have the following theorem, which allows for a class of nonlinearities in the sensors and actuators.

Theorem 4.3.10 - Consider the system described in Theorem 4.3.5. Let the control law be

$$\underline{\tau} := -k(\underline{\xi}^T\underline{\xi})\underline{\xi} - g_{\omega}(\underline{\omega}) \tag{4.3.25}$$

$$\underline{F}_T := -g_{\dot{y}}(\dot{\underline{y}}) \tag{4.3.26}$$

where $k(\underline{\xi}^T\underline{\xi})$ is an arbitrary continuous nonlinear function satisfying $\infty > k(\underline{\xi}^T\underline{\xi}) \geq c_1$

> 0, $\underline{g}_{\omega}(\underline{\omega})$ is a continuous nonlinear function lying in the sector $[c_{\omega}, \infty)$, $c_{\omega} > 0$, and $\underline{g}_{\dot{y}}(\dot{\underline{y}})$ is a continuous nonlinear function lying in the sector $[c_{\dot{y}}, \infty)$, $c_{\dot{y}} > 0$. Assume that the system is well-posed, i.e., there exists a unique, continuously differentiable solution to (4.3.1)-(4.3.8) with control law (4.3.25)-(4.3.26) for all initial conditions sufficiently smooth. Then the system is globally asymptotically stable, i.e., $\underline{\omega} \to 0$, $\underline{\xi} \to 0$, $\underline{u} \to 0$, $\underline{u}_t \to 0$, $\dot{\underline{y}} \to 0$ in appropriate norms.

Proof of Theorem 4.3.10 - The method of proof will be again to verify that the system is a gradient system, and then apply Theorem 2.3.4. Consider the Lyapunov function candidate

$$E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_c^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1} \dot{\underline{y}}\|^2 dx$$

$$+ \frac{1}{2} m_B \|\dot{\underline{y}}\|^2 + \int_c^{c+L} [\mu_1(u_1'')^2 + \mu_2(u_2'')^2] \, dx \; + \int_0^{\underline{\xi}^T\underline{\xi}} \frac{k(x)}{1+x} dx \, .$$

(For simplicity, let E denote $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$, and let $\dot{E}$ denote the time derivative of $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$.) As in Theorem 4.2.2, E is a positive definite function. Therefore this functional satisfies requirements (iia) and (iib) in the definition 4.3.2 of the gradient system. By using exactly the same methods as in Theorem 4.3.5, one can verify that conditions (iic) and (iid) of definition 4.3.2 are also satisfied. Thus, the remaining task is to show that condition (i) holds, i.e., bounded orbits of the system are precompact.

If we insert the control law (4.3.25)-(4.3.26), we can separate the resulting differential equation into two parts. One part will be the globally asymptotically stable system whose equations are described in Theorem 3.4.5. This system will be perturbed, much like (4.3.16)-(4.3.23). It can be verified that the resulting perturbation is

$$f(\underline{\xi}, \underline{\omega}, u_1, u_2, z) = \begin{bmatrix} \underline{\xi} + \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega} \\ -k(\underline{\xi}^T\underline{\xi})\underline{\xi} \\ \dfrac{Y(\underline{\xi})m_B}{m_B + L} \displaystyle\int_c^{c+L} \begin{pmatrix} 0 \\ 0 \\ (k(\underline{\xi}^T\underline{\xi})I_0^{-1}\underline{\xi}^x \ \underline{u}\ )_3 \end{pmatrix} dz \\ -(k(\underline{\xi}^T\underline{\xi})I_0^{-1}\underline{\xi}^x \ \ \underline{u})_{1,2} \end{bmatrix}$$

This perturbation can be shown to be compact, exactly as in Lemma 4.3.9. (The only difference in the proof is the $k(\underline{\xi}^T\underline{\xi})$ term, which by hypothesis is continuous. Otherwise the proof is exactly the same as Lemma 4.3.9.)

Using Theorem 3.4.5, this last compactness result, and Lemma 4.3.7, we thus conclude that bounded orbits of (4.3.1)-(4.3.8) together with the control law (4.3.25)-(4.3.26) are precompact. Combining the previous results, we thus conclude that (4.3.1)-(4.3.8) together with the control law (4.3.25)-(4.3.26) is a gradient system. Using Theorem 4.3.4, all trajectories evolve toward the set of equilibrium points of T(t), which is {0}. Thus (FSS) with the control law (4.3.25)-(4.3.26) is globally asymptotically stable. ■

4.4 Lyapunov Based Attitude Control for a Flexible Spacecraft - Beam Damping Absent

Theorem 4.4.1 - Consider the flexible spacecraft model where axial effects are ignored, described in section 3.1.6 and denoted (FSS/A). Also assume that no beam damping is present, i.e. $k = 0$. Then the equations become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega} \tag{4.4.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} \tag{4.4.2}$$

$$m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{4.4.3}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{4.4.4}$$

$$\underline{F}_{bB} = \Big(-\mu_1 u_1'''(c), \quad -\mu_2 u_2'''(c), \quad \int_c^{c+L} (\dot{\underline{\omega}} \times \underline{u} + 2\underline{\omega} \times \underline{u}_t + \underline{\omega} \times (\underline{\omega} \times \underline{u}) + Y^{-1}\ddot{\underline{y}}\,)_3 dz]\Big)^T \tag{4.4.5}$$

$$\underline{M}_{bB} = \big(\mu_1 u_1''(c), \quad \mu_2 u_2''(c), \quad 0\big)^T \tag{4.4.6}$$

$$(\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x\, \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\ddot{\underline{y}})_{1,2} = 0 \tag{4.4.7}$$

$$u_1(c) = u_2(c) = 0, \quad u_1'(c) = u_2'(c) = 0 \tag{4.4.8}$$

$$u_1''(c + L) = u_2''(c + L) = 0 \qquad \mu_i u_i'''(c + L) = -F_i, \qquad i = 1, 2.$$

Let the control law be

$$\underline{\tau} := -k_\xi \underline{\xi} - K_\omega \underline{\omega} \tag{4.4.9}$$

$$(\underline{F})_{1,2} := -\Lambda(\,(\,\underline{u}_t(c + L) + \omega^x\, \underline{u}(c + L) + Y^{-1}\,\dot{\underline{y}}\,)\,)_{1,2} \tag{4.4.10}$$

$$\underline{F}_T := -K_{\dot{y}}\, \dot{\underline{y}} \tag{4.4.11}$$

where K_ω is a positive definite matrix, $k_\xi \in \mathbf{R}$ with $k_\xi > 0$, $\Lambda \in \mathbf{R}^{2x2}$ is a positive definite matrix, and $K_{\dot{y}}$ is a positive definite matrix. Then the system is globally asymptotically stable, i.e., $\underline{\omega} \to 0, \underline{\xi} \to 0, \underline{u} \to 0, \underline{u}_t \to 0, \dot{\underline{y}} \to 0$ in appropriate norms.

Proof of Theorem 4.4.1 - The method of proof will be exactly the same as before: we will show that our spacecraft system is a gradient system, and then use Theorem 4.3.4. Again, consider the Lyapunov function candidate

$$E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) := \frac{1}{2}\underline{\omega}^T I_0 \underline{\omega} + \int_c^{c+L} \|\underline{u}_t + \underline{\omega}^x \underline{u} + Y^{-1}\dot{\underline{y}}\|^2 dx$$

$$+ \frac{1}{2} m_B \|\dot{\underline{y}}\|^2 + \int_c^{c+L} [\mu_1(u_1'')^2 + \mu_1(u_2'')^2]\, dx \ + k_\xi \ln(1 + \underline{\xi}^T\underline{\xi}) \quad (4.4.11)$$

(For simplicity, let E denote $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$, and let $\dot{E}$ denote the time derivative of $E(\underline{\omega}, \underline{\xi}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$.) This functional is exactly the same one that was used in the proof of Theorem 4.3.5. Note again that this functional satisfies requirements (iia) and (iib) in the definition of the gradient system. We need to verify the rest of condition (ii), and then afterward we will show that condition (i) is achieved.

As has become habit, we can write down $\dot{E}$ by inspection. For a mechanical system, recall that the rate of change of energy is the instantaneous power delivered to the system. Since the external forces acting on the system are the torque jets, external thrusters and the forces applied to the beam, we must have

$$\begin{aligned}\dot{E} &= \underline{\omega}^T\underline{\tau} + \underline{F}^T(\underline{u}_t(c+L) + \underline{\omega}^x\,\underline{u}(c+L) + Y^{-1}\dot{\underline{y}}) + \\ &\quad \underline{F}_T{}^T\dot{\underline{y}} + (1+\underline{\xi}^T\underline{\xi})^{-1}k_\xi 2\underline{\xi}^T\dot{\underline{\xi}} \qquad (4.4.12)\\ &= \underline{\omega}^T\underline{\tau} + \underline{F}^T(\underline{u}_t(c+L) + \underline{\omega}^x\,\underline{u}(c+L) + Y^{-1}\dot{\underline{y}}) + \underline{F}_T{}^T\dot{\underline{y}} + k_\xi\underline{\xi}^T\underline{\omega}\end{aligned}$$

The first term of (4.4.12) is the instantaneous power delivered by the torque jets, the second term is the instantaneous power delivered by the force actuators on the beam (recall that the instantaneous power must be calculated with respect to the inertial frame), the third term is the instantaneous power delivered by the external thrusters, and the last term is simply the rate of change of the attitude energy which has no simple physical interpretation. Inserting the values of $\underline{F}$, $\underline{F}_T$, and $\underline{\tau}$ from (4.4.9)-(4.4.11) then yields

$$\dot{E} = -\underline{\omega}^T K_{\omega} \underline{\omega} - (\underline{u}_t(c+L) + \underline{\omega}^x \underline{u}(c+L) + Y^{-1} \dot{\underline{y}})_{1,2}{}^T \Lambda\, (\underline{u}_t(c+L) + \underline{\omega}^x \underline{u}(c+L) + Y^{-1} \dot{\underline{y}})_{1,2} - \dot{\underline{y}}^T K_{\dot{y}}\, \dot{\underline{y}} \tag{4.4.13}$$

$$\leq 0$$

Thus (iic) is verified in the definition of gradient systems. To show (iid), we need to show that if E(T(t)x) = E(x) for all t ≥ 0, then x is an equilibrium points of (FSS). From (4.4.13), E(T(t)x) = E(x) for all t ≥ 0 implies that $\underline{\omega} = 0$, $\dot{\underline{y}} = 0$, and $(\underline{u}_t(c+L) + \underline{\omega}^x \underline{u}(c+L) + Y^{-1} \dot{\underline{y}})_{1,2} = 0$. Equivalently, (4.4.13) implies that $\underline{\omega} = 0$, $\dot{\underline{y}} = 0$, and $(\underline{u}_t(c+L))_{1,2} = 0$. This in turn shows that $\underline{\omega} = 0$, $\dot{\underline{y}} = 0$, $\underline{u}_t(c+L) = 0$, $\dot{\underline{\omega}} = 0$, and $\ddot{\underline{y}} = 0$. Equations (4.4.7) and (4.4.8) then reduce to

$$(\underline{u}_{tt} + \mu\partial(\underline{u}))_{1,2} = 0 \tag{4.4.14}$$

$$u_1(c)=u_2(c)=0,\ u_1'(c)=u_2'(c)=0 \quad u_1''(c+L) = 0 \quad u_2''(c+L) = 0 \tag{4.4.15}$$

$$u_1'''(c+L) = 0,\ u_2'''(c+L) = 0,\ \underline{u}_t(c+L) = 0.$$

It is easy to show that the only solution satisfying this linear differential equation and the given boundary conditions is the zero solution. Therefore, $u_1 = u_2 = 0$ and $u_{1t} = u_{2t} = 0$. Finally, combining these results in (4.4.2) and (4.4.9) shows that $\xi = 0$. Thus, E(T(t)x) = E(x) for all t ≥ 0, implies that x = 0, i.e., x is an equilibrium point of (FSS). Thus (iid) is satisfied.

Again, we must show that bounded orbits are precompact. Let us write the closed loop system in the suggestive form

$$\dot{\underline{\xi}} = -\underline{\xi} + \boxed{\underline{\xi} + \tfrac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = -K_\omega \underline{\omega} + \underline{c}^x \underline{E}_{bB} + \underline{M}_{bB} - \boxed{k_\xi \underline{\xi}}$$

$$m_B \ddot{\underline{y}} = -K_{\dot{y}}\ \dot{\underline{y}} + Y(\underline{\xi})\begin{pmatrix} (\underline{E}_{bB})_1 \\ (\underline{E}_{bB})_2 \\ 0 \end{pmatrix} +$$

$$+ \frac{Y(\underline{\xi})m_B}{m_B + L}\int_c^{c+L}\begin{pmatrix} 0 \\ 0 \\ (2\underline{\omega}^x\ \underline{u}_t + \underline{\omega}^x\ (\underline{\omega}^x\ \underline{u}) - \dfrac{Y^{-1}K_y\ \dot{\underline{y}}}{m_B}\ )_3 \end{pmatrix}dz \qquad (4.3.18)$$

$$+ \frac{Y(\underline{\xi})m_B}{m_B + L}\int_c^{c+L}\begin{pmatrix} 0 \\ 0 \\ (\underline{u}^x\ [I_0^{-1}(\ -\underline{\omega}^x I_0 \underline{\omega} - K_\omega \underline{\omega} + \underline{c}^x\ \underline{E}_{bB} + \underline{M}_{bB})])_3 \end{pmatrix}dz$$

$$+ \boxed{\frac{Y(\underline{\xi})m_B}{m_B + L}\int_c^{c+L}\begin{pmatrix} 0 \\ 0 \\ (k_\xi I_0^{-1}\underline{\xi}^x\ \underline{u}\ )_3 \end{pmatrix}dz}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\ I + \underline{\xi}\ \underline{\xi}^T + \underline{\xi}^x\ ] - I$$

$$\underline{E}_{bB} = \Big(-\mu_1 u_1'''(c), \quad -\mu_2 u_2'''(c),$$

$$\frac{m_B}{m_B + L}\int_c^{c+L}(\dot{\underline{\omega}}^x\ \underline{u}\ + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\ \underline{u}) - \frac{Y^{-1}K_y\ \dot{\underline{y}}}{m_B}\ )_3 dz\ \Big)^T$$

$$\underline{M}_{bB} = \big(\mu_1 u_1''(c),\ \ \mu_2 u_2''(c),\ \ 0\big)^T$$

$$(\underline{u}_{tt} + (I_0^{-1}[-\underline{\omega}^x I_0 \underline{\omega} - K_\omega \underline{\omega} + \underline{c}^x\ \underline{E}_{bB} + \underline{M}_{bB}])^x\ \underline{u} + 2\underline{\omega}^x\ \underline{u}_t$$

$$+ \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\ddot{\underline{y}}\ )_{1,2} = (\ \boxed{-k_\xi I_0^{-1}\underline{\xi}^x\ \ \underline{u}}\ )_{1,2}$$

$$u_1(c) = u_2(c) = 0,\ u_1'(c) = u_2'(c) = 0 \qquad u_1''(c + L) = 0\ \ u_2''(c + L) = 0$$

$$\mu_1 u_1'''(c + L) = -F_1(t),\ \mu_2 u_2'''(c + L) = -F_2(t)$$

where $(F_1(t), F_2(t))^T = -\Lambda(\,\underline{u}_t(c + L) + \underline{\omega}^x\,\underline{u}(c + L) + Y^{-1}\,\dot{\underline{y}})_{1,2}$. The encircled portion of the equation can be thought of as the perturbation, while the remainder is globally stable by Theorem 3.5.1. Note that the perturbation is of the same form as the Theorem 4.3.5, so it is a compact map, by Lemma 4.3.9. Then, applying Lemma 4.3.6, followed by Lemma 4.3.7, we conclude that bounded orbits are precompact. Thus, conditions (i) and (ii) are verified in the Definition 4.3.2. We therefore have a gradient system, so that Theorem 4.3.4 applies. From this, we conclude that all trajectories of (4.4.1)-(4.4.11) evolve toward equilibrium points of the system, which is clearly $\{0\}$. This proves that the system is globally asymptotically stable. ■

Finally, we have the analogous result to Theorems 4.2.2, 4.3.10, which allows for a class of nonlinearities in the sensors and actuators. The proof is exactly the same as those of Theorems 4.4.1, 4.3.5 and 4.3.10, and thus will be omitted.

Theorem 4.4.2 - Consider the system described in Theorem 4.4.1. Let the control law be

$$\underline{\tau} := -\,g_\omega(\underline{\omega}) - k(\underline{\xi}^T\underline{\xi})\underline{\xi}$$

$$\underline{F}_T := -g_{\dot{y}}(\dot{\underline{y}})$$

$$(\underline{F})_{1,2} := -g_F((\,\underline{u}_t(c + L) + \underline{\omega}^x\,\underline{u}(c + L) + Y^{-1}\,\dot{\underline{y}}\,))_{1,2}$$

where $k(\underline{\xi}^T\underline{\xi})$ is an arbitrary continuous nonlinear function satisfying $\infty > k(\underline{\xi}^T\underline{\xi}) \geq c_1 > 0$, $g_\omega(\underline{\omega})$ is a nonlinear function lying in the sector $[c_\omega, \infty)$, $c_\omega > 0$, $g_{\dot{y}}(\dot{\underline{y}})$ is a nonlinear function lying in the sector $[c_{\dot{y}}, \infty)$, $c_{\dot{y}} > 0$, and $g_F((\,\underline{u}_t(c + L) + \underline{\omega}^x\,\underline{u}(c + L) + Y^{-1}\,\dot{\underline{y}}\,))_{1,2}$ is a nonlinear function lying in the sector $[c_F, \infty)$, $c_F > 0$. Assume that the system is well-posed, i.e., the closed loop system has a unique, continuously differentiable solution for all initial conditions sufficiently smooth. Then the system is globally asymptotically stable, i.e., $\underline{\omega}$, $\dot{\underline{y}}$, $\underline{u}$, and $\underline{u}_t$ all go to zero as $t \to \infty$.

4.5 Conclusions and Future Research

This chapter has developed an attitude control law for a variety of spacecraft systems: a rigid spacecraft, a flexible spacecraft with significant beam damping, and a flexible spacecraft with zero beam damping. The laws were seen to be implementable by linear or nonlinear static state feedback, and global asymptotic stability was obtained for each configuration.

As for future research, most of the questions remaining have to do with practical implementation. From a theoretical viewpoint, if a stabilization result could be obtained for the so-called geometrically exact beam model (see the end of Chapter 3), then it is easy to see that a feedback law similar to those of Theorem 4.3.5 and 4.4.1 would yield similar results. The major practical problems would implementation of these feedback laws with torque thrusters which are usually only full on - full off thrusters, rather than proportional thrusters. Another practical problem would be to determine the effects of limited obtainable force and torque from the actuators.

The major disadvantage of these Lyapunov based laws is that they are essentially "infinite horizon". Exponential stability is guaranteed in the above theorems, but there is no straightforward relationship between the control parameters and the exponential time constant. This makes it difficult for the engineer to design for an a priori decay rate. In addition, one would ideally like an attitude control law that would steer from one attitude to another in a fixed time interval. Unfortunately, Lyapunov based control laws as they are known at present are not suitable for such a design goal. It is with these engineering ideas that we turn to the next chapter.

Chapter 5

ATTITUDE CONTROL USING THE METHODS OF EXACT LINEARIZATION

5.1 Introduction

It is now well known that a number of nonlinear control systems of engineering interest can be transformed by a static state feedback and a nonlinear change of coordinates into an equivalent linear system [De L. 1], [De L. 2], [Mey. 1]. In particular, in the area of attitude control, the method has proved to be quite useful. Dwyer [Dwy. 1] used this method of linearizing transformations to obtain exact nonlinear continuous time control laws for large angle rotational maneuvers for a rigid body by use of external thrusters. Similar methods are employed in [Dwy. 3] to design control laws for a rigid body controlled by both external thrusters and momentum wheels.

This method has also been successfully been used for designing a nonlinear attitude control law for a satellite with flexible appendages. In [Mon. 1], the control law was derived for a satellite with its flexible appendages modelled by their finite dimensional modal approximation. However, implementation of the control scheme required information about the beam velocities and displacements at several points of the beam. In practice, these are difficult measurements to make.

The purpose of this chapter is to outline the design and implementation of a nonlinear feedback control law for a satellite with flexible appendages without the restrictions of [Mon. 1]. The spacecraft to be considered will be a rigid body with a single flexible appendage attached to the rigid body. The appendage will be modelled as an Euler-Bernoulli type beam, rather than its finite dimensional approximation. The control law will be derived using linearizing transformations in the spirit of the above papers, but the implementation will be considerably different than [Mon. 1] in that it will not depend on the beam displacements and velocities, but rather on the forces and moments at the point of attachment. These quantities can easily be determined by the use of strain rosettes.

5.2 Exact Nonlinear Attitude Control Law for a Rigid Spacecraft

As in the previous chapter, we will first examine the rigid body to help elucidate the ideas for studying the flexible structure. The control law will be obtained by methods of exact linearization. (For a thorough explanation of this procedure, see [De L. 1], [Sas. 1], and, in particular, [Isi. 1].) More precisely, we desire to find a static state feedback and a nonlinear change of coordinates to transform the nonlinear differential equations of the rigid spacecraft (RS) into a "normal form" [Isi. 1, p. 8], i.e. a system with linear input-output dynamics, and a corresponding unobservable, possibly nonlinear subsytem. The use of exact linearization for obtaining attitude control laws for a rigid spacecraft was first obtained in [Dwy. 1] and expanded in [Dwy. 2] and [Dwy. 3]. The derivation given below is a slightly more modern method, and will serve as the basis for the design of the flexible spacecraft control law.

Theorem 5.2.1 - Consider the rigid body spacecraft model without momentum wheels described in section 3.1.6 and denoted (RS):

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{5.2.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} \tag{5.2.2}$$

Let the control law be

$$\underline{\tau} = \underline{\omega}^x I_0 \underline{\omega} + 2(1 + \xi^T \xi)^{-1} I_0 (I - \xi^x)(-\beta\xi - \gamma\dot{\xi}) - I_0(\xi^T \underline{\omega})\underline{\omega} \tag{5.2.3}$$

where $\beta > 0$ and $\gamma > 0$ then

(i) The attitude $\xi(t) \to 0$ exponentially, and $\dot{\xi}(t) \to 0$ exponentially;

(ii) The angular velocity $\underline{\omega}(t) \to 0$ exponentially, and $\dot{\underline{\omega}}(t) \to 0$ exponentially.

Proof of Theorem 5.2.1 - We follow the linearization procedure given in [Isi. 1, sec. 2.3, 3.3]. Usually, an output is present in our state space formulation, and we would then "differentiate the output until an input appears". Since no output is specified, we are free to choose it. Since we are attempting to control ξ, a logical choice is to choose $\xi = (\xi_1, \xi_2, \xi_3)^T$ to be the "dummy" output function. Differentiating ξ yields (5.2.1), which will be repeated here for convenience.

$$\dot{\xi} = \frac{1}{2}(I + \xi\xi^T + \xi^x)\underline{\omega} \tag{5.2.4}$$

Since no input appears in this expression differentiate again

$$\ddot{\xi} = \frac{d}{dt}\left(\frac{1}{2}(I + \xi\xi^T + \xi^x)\right)\underline{\omega} + \frac{1}{2}(I + \xi\xi^T + \xi^x)\frac{d\omega}{dt} \, . \tag{5.2.5}$$

The calculation of the derivative in the first term is rather tedious; after computation we plug its value in and obtain

$$\ddot{\xi} = \frac{1}{2}(\xi^T\underline{\omega})(\underline{\omega} + \xi\xi^T\underline{\omega} + \xi^x\underline{\omega}) + \frac{1}{2}(I + \xi\xi^T + \xi^x)\dot{\underline{\omega}}$$

$$= \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + \underline{\dot{\omega}}]. \tag{5.2.6}$$

Insert (5.2.2) and use the fact that I_0 is invertible to obtain

$$\ddot{\underline{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + I_0^{-1}\{-\underline{\omega}^x I_0\underline{\omega} + \underline{\tau}\}]. \tag{5.2.7}$$

Note the term outside the square brackets is nonsingular with inverse $2(1 + \underline{\xi}^T\underline{\xi})^{-1}$ $(I - \underline{\xi}^x)$: using properties of cross-product and the fact that $\underline{\xi}^x\underline{\xi} = 0$ yields

$$\frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x) \cdot 2(1 + \underline{\xi}^T\underline{\xi})^{-1}(I - \underline{\xi}^x) = (1 + \underline{\xi}^T\underline{\xi})^{-1}[I - \underline{\xi}^x + \underline{\xi}\underline{\xi}^T$$

$$- \underline{\xi}\underline{\xi}^T(\underline{\xi}^x) + \underline{\xi}^x - \underline{\xi}^x(\underline{\xi}^x)] \tag{5.2.8}$$

$$= (1 + \underline{\xi}^T\underline{\xi})^{-1}[I + \underline{\xi}\underline{\xi}^T - \underline{\xi}^x(\underline{\xi}^x)] \tag{5.2.9}$$

$$= (1 + \underline{\xi}^T\underline{\xi})^{-1}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^T\underline{\xi}I - \underline{\xi}\underline{\xi}^T] \tag{5.2.10}$$

$$= I \tag{5.2.11}$$

Then, choose the following control law (by setting the RHS of (5.2.7) equal to some new exogenous input $\underline{w}$ and solving for $\underline{\tau}$)

$$\underline{\tau} = \underline{\omega}^x I_0\underline{\omega} + I_0[\frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)]^{-1}\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega}$$

$$= \underline{\omega}^x I_0\underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})^{-1}I_0(I - \underline{\xi}^x)\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} \tag{5.2.12}$$

where $\underline{w}$ is a vector of real valued functions, and again can be thought of as a new exogenous input. Applying (5.2.12) to (5.2.7) then yields the *linear* system

$$\ddot{\underline{\xi}} = \underline{w}. \tag{5.2.13}$$

On this controllable linear system, the poles can be placed as desired. For example, let

$$\underline{w} = -\beta\xi - \gamma\dot{\xi}. \tag{5.2.14}$$

where $\beta > 0$ and $\gamma > 0$. Inserting (5.2.14) into (5.2.13) and writing (5.2.13) in state space form yields

$$\begin{bmatrix} \dot{\xi} \\ \ddot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\beta I & -\gamma I \end{bmatrix} \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} \tag{5.2.15}$$

which implies (since $\beta > 0$ and $\gamma > 0$) that $\xi(t) \to 0$ exponentially, and $\dot{\xi}(t) \to 0$ exponentially. In turn, (5.2.1) shows $\underline{\omega}(t) \to 0$ exponentially. Finally, using (5.2.3) in (5.2.2) shows that $\dot{\underline{\omega}}(t) \to 0$ exponentially. Since a combination of (5.2.14) and (5.2.12) yields (5.2.3), (i) and (ii) are proved.

Comment 5.2.2 - The reader familiar with the exact linearization literature will note from (5.2.11) and (5.2.7) that the nonlinear system (RS) with dummy output $y = \xi \in \mathbb{R}^3$ has (vector) relative degree (2, 2, 2) [Isi. 1, p. 76], so the linearizing feedback (5.2.12) is as expected. Because the original system is of order 6 (state variables ξ and $\underline{\omega}$) and the linearized system is also of order 6 (state variables ξ and $\dot{\xi}$), there are no zero dynamics. In other words, the linearization is a *global* linearization.

Comment 5.2.3 - The design using the linearized system (5.2.13) can be done in many ways other than the simple pole placement done here. One desirable form of the controller on the linearized system (5.2.13) would be to use a fixed end linear optimal control law. This would yield a controller which would steer from one attitude to another in a fixed time interval. This is a major improvement over the Lyapunov based control laws obtained in Chapter 4.

5.3 Exact Nonlinear Attitude Control Laws for a Flexible Spacecraft - Beam Damping Present

For simplicity, we will control the satellite by using the torque jets only. (See Remark 5.3.3 at the end of Theorem 5.3.1 for further comments). As noted in section 3.1.6, attitude control for a flexible spacecraft differs from that of a rigid spacecraft in that the rotational and translational terms are coupled. This means that in contrast to Theorem 5.2.1, the force thrusters on the rigid body as well as the torque jets on the rigid body will be needed to perform the maneuver. Most authors ignore the translational term entirely ([Mon. 1],), arguing that the mass of the rigid body is much larger than the mass of the beam. In Theorem 5.3.4, this assumption is rigorously justified. To this author's knowledge, this is the first direct proof that the assumption is correct to appear in the literature.

To design the control law, the method of linearizing transformations will again be used. Strictly speaking, since the flexible spacecraft model (FSS) contains partial differential equations, the methods mentioned above do not necessarily apply. However, we will proceed blindly along these lines and investigate what happens.

Theorem 5.3.1 - Consider the flexible spacecraft without momentum wheels described in section 3.1.6, and denoted (FSS). Assume explicitly that damping is present. Also assume no active control on the beam, so that $F_i(t) = 0$, i=1, 2, 3. The equations then become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{5.3.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0\underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB} \tag{5.3.2}$$

$$m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{5.3.3}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{5.3.4}$$

$$\underline{F}_{bB} = \underline{F}(c) = -\mu\partial'(\underline{u})|_c - k\partial'(\underline{u}_t)|_c$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\ \ \mu_2 u_2''(c) + k_2 u_{2t}''(c),\ \ 0)^T$$

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0 \qquad (5.3.5)$$

$$u_1(c) = u_2(c) = u_3(c) = 0,\ \ u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0 \qquad (5.3.6)$$

$$u_1'''(c + L) = 0 \quad u_2'''(c + L) = 0 \quad u_3'(c + L) = 0$$

Suppose now that we can determine $\underline{F}_{bB}(t)$ and $\underline{M}_{bB}(t)$ by on-board measurements. (See Appendix A for an example of how this might be done). Apply the control law

$$\underline{\tau} = \underline{\omega}^x I_0 \underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})^{-1} I_0 (I - \underline{\xi}^x)(-\beta\underline{\xi} - \gamma\dot{\underline{\xi}}) - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} - \underline{c}^x \underline{F}_{bB} - \underline{M}_{bB} \qquad (5.3.7)$$

$$\underline{F}_T = -Y(\underline{\xi})\underline{F}_{bB} - m_B A\dot{\underline{y}} \qquad (5.3.8)$$

where $\beta > 0$, $\gamma > 0$ and $A \in \mathbb{R}^{3x3}$ is a Hurwitz matrix. Then

(i) The attitude $\underline{\xi}(t) \to 0$ exponentially, and $\dot{\underline{\xi}}(t) \to 0$ exponentially;

(ii) The angular velocity $\underline{\omega}(t) \to 0$ exponentially, and $\dot{\underline{\omega}}(t) \to 0$ exponentially;

(iii) The velocity of the center of mass of the rigid body $\dot{\underline{y}}(t) \to 0$ exponentially, and $\ddot{\underline{y}}(t) \to 0$ exponentially;

(iv) The beam deflections $\underline{u}(x, t)$ and beam velocities $\underline{u}_t(x, t)$ both go to zero exponentially.

Proof of Theorem 5.3.1 - As in the proof of the rigid body case, we attempt to linearize the equations using an appropriate feedback and change of coordinates. It is convenient to choose the dummy output function to be $\underline{z} = \text{col}(\ \underline{\xi}\ ,\ \dot{\underline{y}})$. Following the linearization procedure we first differentiate $\underline{\xi}$. This yields (5.3.1), as before. Since no input appears, differentiate again. This yields (5.2.6) as obtained previously:

$$\ddot{\underline{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + \dot{\underline{\omega}}]. \tag{5.3.9}$$

Insert (5.3.2) and use the fact that I_0 is invertible to obtain

$$\ddot{\underline{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + I_0^{-1}\{-\underline{\omega}^x I_0\underline{\omega} + \underline{c}^x\underline{F}_{bB} + \underline{M}_{bB} + \underline{\tau}\}]. \tag{5.3.10}$$

Now set,

$$\underline{\tau} := \underline{\tilde{\tau}} - \underline{c}^x\underline{F}_{bB} - \underline{M}_{bB} \tag{5.3.11}$$

where, again, $\underline{F}_{bB}$ and $\underline{M}_{bB}$ have been measured, and $\underline{\tilde{\tau}}$ is a vector of real valued functions. ($\underline{\tilde{\tau}}$ can be thought of as the new exogenous input.) Insert (5.3.11) into (5.3.10) to obtain

$$\ddot{\underline{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + I_0^{-1}\{-\underline{\omega}^x I_0\underline{\omega} + \underline{\tilde{\tau}}\}]. \tag{5.3.12}$$

But this is exactly the form of the equation one gets for a rigid body without flexible appendages (see (5.2.7)). Since the term outside the square brackets is nonsingular (see 5.2.8), we can apply the following control law

$$\begin{aligned}\underline{\tilde{\tau}} &:= \underline{\omega}^x I_0\underline{\omega} + I_0[\tfrac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x)]^{-1}\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} \\ &= \underline{\omega}^x I_0\underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})^{-1}I_0(I - \underline{\xi}^x)\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega}\end{aligned} \tag{5.3.13}$$

where $\underline{w}$ is a vector of real valued functions, and again can be thought of as a new exogenous input. Applying (5.3.13) to (5.3.12) then yields the linear system

$$\ddot{\underline{\xi}} = \underline{w}. \tag{5.3.14}$$

We now consider the other dummy output variable, $\dot{\underline{y}}$. Differentiating $\dot{\underline{y}}$ yields from (5.3.3)

$$\ddot{\underline{y}} = [Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T]/m_B. \tag{5.3.15}$$

Therefore, choosing the control law for the force thrusters to be

$$\underline{F}_T := -Y(\underline{\xi})\underline{F}_{bB} + m_B\tilde{\underline{F}}_T \tag{5.3.16}$$

where $\tilde{\underline{F}}_T$ is the new exogenous input. Inserting this control law into (5.3.15) yields

$$\ddot{\underline{y}} = \tilde{\underline{F}}_T. \tag{5.3.17}$$

The equations (5.3.14) and (5.3.17) thus comprise decoupled, controllable linear systems which can be designed using methods of the engineers choice. For example, let

$$\underline{w} := -\beta\underline{\xi} - \gamma\dot{\underline{\xi}}. \tag{5.3.18}$$

$$\tilde{\underline{F}}_T := -A\dot{\underline{y}}. \tag{5.3.19}$$

where $\beta > 0$, $\gamma > 0$, and A is a Hurwitz matrix. Insert the control law (5.3.18) and (5.3.19) into (5.3.14) and (5.3.17). Then the nonlinear, infinite dimensional system given by (5.3.1)-(5.3.6) reduces to a set of linear, ordinary differential equations, coupled to a nonlinear differential equation:

$$\begin{bmatrix} \dot{\underline{\xi}} \\ \ddot{\underline{\xi}} \\ \ddot{\underline{y}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\beta I & -\gamma I & 0 \\ 0 & 0 & -A \end{bmatrix} \begin{bmatrix} \underline{\xi} \\ \dot{\underline{\xi}} \\ \dot{\underline{y}} \end{bmatrix}. \tag{5.3.20}$$

$$\underline{u}_{tt} + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + \underline{g}(\underline{\xi}, \dot{\underline{\xi}}, \dot{\underline{y}}, \underline{u}, \underline{u}_t) = 0 \tag{5.3.21}$$

where $\underline{g}(\underline{\xi}, \dot{\underline{\xi}}, \dot{\underline{y}}, \underline{u}, \underline{u}_t)$ is obtained from equation (5.3.5), and the various relationships between $\underline{\omega}, \dot{\underline{\omega}}$ and $\underline{\xi}, \dot{\underline{\xi}}$.

The first two components of (5.3.20) imply (since $\beta > 0$ and $\gamma > 0$) that $\underline{\xi}(t) \to 0$ exponentially, and $\dot{\underline{\xi}}(t) \to 0$ exponentially. From (5.3.1), $\underline{\omega} = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}(I - \underline{\xi}x)\dot{\underline{\xi}}$ whence it follows that $\underline{\omega}(t) \to 0$ exponentially. From (5.3.18), (5.3.14) and (5.3.9) we also have that $\dot{\underline{\omega}}(t) \to 0$ exponentially. The last component of (5.3.20) shows that $\dot{\underline{y}}(t)$ and $\ddot{\underline{y}}(t) \to 0$ exponentially. Since a combination of (5.3.19) and (5.3.16) yield (5.3.8), and (5.3.18), (5.3.13) and (5.3.11) yield (5.3.7), (i), (ii) and (iii) are proved.

To show part (iv), we must verify that the solution to the partial differential equation (5.3.21) is exponentially stable. Note we are simply verifying that the "zero dynamics" are exponentially stable. Since the proof of the result requires tedious, although straightforward, arguments from semigroup theory, the details are left to the Appendix to this chapter, Appendix 5.A. It should be intuitively clear, however, that as the rigid body stops rotating and translating, the beam vibrations damp out due to internal beam damping. This is intuition behind the proof in the Appendix . ■

Comment 5.3.2 - The interpretation of the control law is simple. First, the effect of the flexible body on the rigid body is removed by (5.3.11) and (5.3.16). We are left with decoupled translational and rotational dynamics, exactly as in the rigid spacecraft. This rigid structure is the controlled by (5.3.19), (5.3.13) and (5.3.18) which is exactly the same form of control obtained in Theorem 5.2.1 (see (5.2.12) and (5.2.14)).

Remark 5.3.3 - In the case where momentum wheels alone are used to control the structure, the control law is very similar to the one in Theorem 5.3.1. The equations of motion and kinematics for this structure are given in section 3.1.6, and denoted (FSSMW). In this case, let the control law be defined by

$$\underline{\tau} := 0 \tag{5.3.21}$$

$$\begin{aligned}\underline{\tau}' := & - \underline{\omega}^{x} I_0 \underline{\omega} - \underline{\omega}^{x} I_A \underline{\Omega}_w - 2(1 + \underline{\xi}^T \underline{\xi})^{-1} (I_0 - I_A)(I - \underline{\xi}^{x})(-\beta \underline{\xi} - \gamma \dot{\underline{\xi}}) + \\ & (I_0 - I_A)(\underline{\xi}^T \underline{\omega}) \underline{\omega} + \underline{c}^{x} \underline{F}_{bB} + \underline{M}_{bB}\end{aligned} \tag{5.3.22}$$

$$\underline{F}_T := - Y(\underline{\xi}) \underline{F}_{bB} - m_B A \dot{\underline{y}} \tag{5.3.23}$$

where $\beta > 0$, $\gamma > 0$, and $A \in \mathbb{R}^{3x3}$ is a Hurwitz matrix. For this choice of control law it is easy to verify, using exactly the same methods as in the proof of Theorem 5.3.1, that the conditions (i), (ii), (iii), and (iv) of Theorem 5.3.1 are satisfied.

One undesirable feature of the proposed control law in Theorem 5.3.1 is its complexity. In contrast to Theorem 5.2.1 for the rigid space structure, both sets of actuators are needed for the attitude maneuver. This is because, as stated previously, the rotational and translational terms are coupled. It would be very nice from a practical point of view if an attitude control law could be obtained using only one set of actuators. This is the content of the following Theorem.

Theorem 5.3.4 - Consider the flexible spacecraft without momentum wheels described in section 3.1.5, and denoted (FSS). Assume explicitly that damping is present. Also assume no active control on the beam, so that $F_i(t) = 0$, i=1, 2, 3. The equations then become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}^x]\underline{\omega}. \tag{5.3.24}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}^x I_0 \underline{\omega} = \underline{\tau} + \underline{c}^x \underline{F}_{bB} + \underline{M}_{bB}. \tag{5.3.25}$$

$$m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{5.3.26}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{5.3.27}$$

$$\underline{F}_{bB} = -\mu\partial'(\underline{u})|_c - k\partial'(\underline{u}_t)|_c$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c) + k_1 u_{1t}''(c),\ \ \mu_2 u_2''(c) + k_2 u_{2t}''(c),\ \ 0)^T$$

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x\, \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0 \tag{5.3.28}$$

$$u_1(c) = u_2(c) = u_3(c) = 0,\ \ u_1'(c) = u_2'(c) = 0$$

$$u_1''(c+L) = u_2''(c+L) = 0 \tag{5.3.29}$$

$$u_1'''(c+L) = 0 \quad u_2'''(c+L) = 0 \quad u_3'(c+L) = 0$$

Suppose again that we can determine $\underline{F}_{bB}(t)$ and $\underline{M}_{bB}(t)$ by on-board measurements. Apply the control law

$$\underline{\tau} := \underline{\omega}^x I_0 \underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})^{-1} I_0 (I - \underline{\xi}^x)(-\beta\underline{\xi} - \gamma\dot{\underline{\xi}}) - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} - \underline{c}^x \underline{F}_{bB} - \underline{M}_{bB} \tag{5.3.30}$$

where $\beta > 0$, and $\gamma > 0$. Assume that the mass of the rigid body is much larger than the mass of the beam. Then

(i) The attitude $\underline{\xi}(t) \to 0$ exponentially, and $\dot{\underline{\xi}}(t) \to 0$ exponentially;

(ii) The angular velocity $\underline{\omega} \to 0$ exponentially, and $\dot{\underline{\omega}} \to 0$ exponentially;

(iii) The beam deflections $\underline{u}$ and beam velocities $\underline{u}_t$ both go to zero exponentially.

Proof of Theorem 5.3.4 - The bulk of the proof is almost identical to that of Theorem 5.3.1. Choose the dummy output function to be ξ. Differentiating ξ twice with respect to time yields exactly the same linear system as Theorem 5.3.1. Inserting the control law yields

$$\ddot{\xi} = -\beta\xi - \gamma\dot{\xi}. \tag{5.3.32}$$

which shows (i). Observing (5.3.24) and using the fact that ξ and $\dot{\xi}$ go to zero exponentially gives $\underline{\omega} \rightarrow 0$ exponentially. Using (5.3.9) and (5.3.32) also shows that $\dot{\underline{\omega}} \rightarrow 0$ exponentially. This proves (ii).

Thus, the only difference in the proof is in showing that the beam velocities and deflections go to zero exponentially. The proof is similar to that of Theorem 5.3.1, and the details can be found in Appendix 5.A.. ■

Remark 5.3.5 - As remarked previously, this result gives a rigorous justification of an assumption used widely in the literature; namely, to design an attitude control law, one can ignore the translational term if the mass of the rigid body is much larger than the mass of the beam.

5.4 Exact Nonlinear Attitude Control Law for a Flexible Spacecraft - Beam Damping Absent

In the previous section, attitude control was obtained by decoupling the rigid body from the beam, and applying a rigid body control law. By decoupling the two components we are then left with an uncontrolled, damped beam. However, if the damping is small, or essentially negligible, then oscillations in the beam can continue for an undesirably long time. In this section we will consider the problem when the beam damping is assumed to be zero. Since there is no damping in the beam, it is easy to see that the control laws of section 5.3 will not work because the beam oscillations will not die off. Thus, if we are to employ a decoupling linearization law in the spirit of Theorem 5.3.1 or Theorem 5.3.4, beam control will be needed to stabilize the beam.

The type of beam control to be employed will be of the boundary variety discussed in Chapter 2. As stated in Chapter 2, the main reason for using boundary control is that it is far easier to implement then distributed control. The specific form of boundary control to be used are force thrusters at the tip of the beam, combined with velocity sensors also located at the tip of the beam.

With these ideas in mind, we now have the following Theorem:

Theorem 5.4.1 - Consider the flexible spacecraft without momentum wheels described in section 3.1.6, and denoted (FSS). Assume beam damping is zero, i.e. k = 0 in (3.1.57). Then the equations of (FSS) become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x]\underline{\omega}. \tag{5.4.1}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}xI_0\underline{\omega} = \underline{\tau} + \underline{c}^x\,\underline{F}_{bB} + \underline{M}_{bB}. \tag{5.4.2}$$

$$m_B\underline{\ddot{y}} = Y(\underline{\xi})\underline{F}_{bB} + \underline{F}_T \tag{5.4.3}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\,I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}^x\,] - I \tag{5.4.4}$$

$$\underline{F}_{bB} = -\mu\partial'(\underline{u})|_c$$

$$\underline{M}_{bB} = (\mu_1 u_1''(c),\ \ \mu_2 u_2''(c),\ \ 0)^T$$

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x\,\underline{u} + 2\underline{\omega}^x\underline{u}_t + \underline{\omega}^x(\underline{\omega}^x\underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\underline{\ddot{y}} = 0 \tag{5.4.5}$$

$$u_1(c) = u_2(c) = u_3(c) = 0,\ \ u_1'(c) = u_2'(c) = 0$$

$$u_1''(c+L) = u_2''(c+L) = 0 \tag{5.4.6}$$

$$\mu_i u_i'''(c+L) = -F_i, \quad i = 1, 2. \qquad \mu_3 u_3'(c+L) = -F_3$$

where $F_i(t)$, i=1, 2, 3, is the the point force actuator associated with the ith axis.

Let the control law be

$$\underline{\tau} := \underline{\omega}^x I_0 \underline{\omega} + 2(1 + \xi^T\xi)^{-1} I_0 (I - \xi^x)(-\beta\xi - \gamma\dot{\xi}) - I_0(\xi^T\underline{\omega})\underline{\omega} - \underline{c}^x \underline{F}_{bB} - \underline{M}_{bB} \tag{5.4.7}$$

$$\underline{F}_T := -Y(\xi)\underline{F}_{bB} - m_B A\dot{\underline{y}} \tag{5.4.8}$$

$$F_i(t) := -\alpha_i\, u_{it}(c+L) \quad i=1, 2, 3. \tag{5.4.9}$$

where $\beta > 0$, $\gamma > 0$, $A \in \mathbb{R}^{3x3}$ is a Hurwitz matrix, and $\alpha_i > 0$, i=1, 2, 3. Then

(i) The attitude $\xi(t) \to 0$ exponentially, and $\dot{\xi}(t) \to 0$ exponentially;

(ii) The angular velocity $\underline{\omega}(t) \to 0$ exponentially, and $\dot{\underline{\omega}}(t) \to 0$ exponentially;

(iii) The velocity of the center of mass of the rigid body $\dot{\underline{y}}(t) \to 0$ exponentially, and $\ddot{\underline{y}}(t) \to 0$ exponentially;

(iv) The beam deflections $\underline{u}(x, t)$ and beam velocities $\underline{u}_t(x, t)$ both go to zero uniformly exponentially.

Proof of Theorem 5.4.1 - See Appendix 5.A.. ■

Comment 5.4.2 - The interpretation of the control law is again simple. The rigid body torque law (5.4.7)-(5.4.8) decouples the rigid body from the beam, and then stabilizes the rigid body. The beam boundary control law (5.4.9) exponentially stabilizes the beam. Thus we are again left with two decoupled exponentially stable systems, as in Theorem 5.3.1.

Remark 5.4.3 - From the proofs of Theorem 5.3.1, 5.3.4 and 5.4.1, we see that it is only crucial for the beam configuration to be exponentially stable. In particular, the Euler-Bernoulli beam model employed was *not* crucial in the proofs of the Theorems. This means that *any* exponentially stable beam model can replace the Euler-Bernoulli model, and the conclusions of the theorems are still true. For example, a damped Timoshenko beam model would suffice, or an undamped Timoshenko beam model with beam boundary control would also work (see [Kim 1]). From an engineering viewpoint, the only difference is in the calculation of the forces and moments at the point of attachment, and the calculation of beam response during maneuvers, quantities which clearly depend on the beam model employed.

5.5 Conclusions and Future Research

This chapter has considered the attitude control problem for a flexible satellite consisting of a elastic beam clamped to a rigid hub; the former is modelled as an Euler-Bernoulli beam. Two improvements were seen over previous exact linearization based attitude control laws. First, the laws were seen to be easily implementable using strain rosettes. Second, the laws took into account an infinite dimensional beam model, rather than a finite dimensional approximation. The latter improvement means that there is no need to worry about "spillover" problems associted with finite dimensional approximations.

In terms of future research, there is still quite a bit of work to do. The main problem with these exact linearization based control laws is that they are not robust. For the stabilization problem considered here, robustness comes for free, but for general tracking problems this is not the case. This is well-known in robotics, where the "computed torque methods", also based on exact linearization, are known to exhibit tracking errors due to the inherent non-robustness of these laws (see [Sas. 1]). Thus, it would be very desirable to find ways to enhance the robustness properties of these controllers.

Other problems are still worth investigating, among them shaping of beam response, sensor and actuator placement, and implementation issues dealing with limitations on achievable torque in the control jets and on energy expenditures.

5.A Proofs of Exponential Decay

We first need the following result which will be used in all of the proofs of this appendix.

Theorem 5.A.1 - Consider the following differential equation evolving on a Banach space X:

$$\dot{x} = Ax + f(t)x + g(t) \quad x_0 \in D(A) \tag{5.A.1}$$

where A: X → X is a linear map, possible unbounded, and where f: $\mathbb{R}$ x X → X and g: $\mathbb{R}$ → X are C^1 functions. Suppose it is known that (i) The system is well-posed, i.e., there is a (strong) unique solution to (5.A.1) (ii) A generates an exponentially stable semigroup (iii) $\|f(t)\| \to 0$, and (iv) $\|g(t)\| \to 0$ exponentially. Then, for any initial condition $x_0 \in D(A)$, the solution to the differential equation (5.A.1), denoted $x(t) = S(t)x_0$, satisfies $\|x(t)\| \to 0$ exponentially.

Proof of Theorem 5.A.1 - Let T(t) denote the semigroup generated by A. Using the "variation of constants" formula yields

$$\dot{x}(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\tau)[f(\tau)x(\tau) \ + \ g(\tau)]d\tau$$

Take norms on both sides to obtain

$$\|x(t)\| \leq \|T(t-t_0)\|\|x_0\| + \int_{t_0}^{t} \|T(t-\tau)\| \ [\ \|[f(\tau)\| \ \|x(\tau)\| \ + \ \|g(\tau)\|]d\tau$$

Now use the fact that $\|T(t-t_0)\| \leq M\exp(-\delta(t-t_0))$ for some $M > 0$ and some $\delta > 0$, and $\|g(t)\| \leq M_g\exp(-\delta_g(t))$ for some $M_g > 0$ and some $\delta_g > 0$ to obtain

$$\|x(t)\| \ \leq \ M\exp(-\delta(t-t_0))\|x_0\| \ + \ \int_{t_0}^{t} M\exp(-\delta(t-\tau))[\|[f(\tau)\|\|x(\tau)\|+M_g\exp(-\delta_g(\tau))]d\tau$$

Directly integrate the terms involving the exponentials to obtain

$$\|x(t)\| \ \leq \ Me^{-\delta(t-t_0)}\|x_0\| \ + \ \frac{MM_g}{\delta - \delta_g}[e^{-\delta g(t-t_0)} - \ e^{-\delta(t-t_0)}] \ + \ \int_{t_0}^{t} Me^{-\delta(t-\tau)}\|[f(\tau)\|\|x(\tau)\|d\tau \tag{5.A.2}$$

Note next that since $\|f(t)\| \to 0$, there is a t^* such that $\|f(t)\| \leq \delta/2M$, for all $t \geq t^*$. Inserting this expression into (5.A.2) yields

$$\|x(t)\| \ \leq \ Me^{-\delta(t-t^*)}\|x_0\| \ + \frac{MM_g}{\delta - \delta_g}[e^{-\delta g(t-t^*)} - \ e^{-\delta(t-t^*)}] \ + \ \int_{t^*}^{t} \frac{\delta}{2} \ \ e^{-\delta(t-\tau)}\|x(\tau)\|d\tau$$

for all $t \geq t^*$. Now set $u(t) = \ e^{\delta(t-t^*)}\|x(t)\|$. Inserting this then yields

$$u(t) \le Mu(t_0) + \frac{MM_g}{\delta - \delta_g}[e^{(\delta-\delta_g)(t-t^*)} - 1] + \frac{\delta}{2}\int_{t^*}^{t} u(\tau)d\tau$$

Now apply the generalized Bellman-Gronwall Lemma [Des. 2, Appendix E] to obtain

$$\le Mu(t^*) + \frac{MM_g}{\delta - \delta_g}[e^{(\delta-\delta_g)(t-t^*)} - 1] + \int_{t^*}^{t}[Mu(t^*) + \frac{MM_g}{\delta - \delta_g}(e^{(\delta-\delta_g)(t-t^*)} - 1)]\ \frac{\delta}{2}\ e^{-\delta(t-\tau)/2}d\tau$$

Evaluate the integrals by direct integration to finally obtain

$$u(t) \le Mu(t^*) + \frac{MM_g}{\delta - \delta_g}[e^{(\delta-\delta_g)(t-t^*)} - 1] + Mu(t^*)(1 - e^{-\delta(t-t^*)/2}) + \frac{MM_g}{\delta - \delta_g}(e^{(\delta-\delta_g)(t-t^*)} - \quad e^{\delta(t-t^*)/2}) \tag{5.A.3}$$

To recover $\|x(t)\|$, multiply both sides of (5.A.3) by $e^{-\delta(t-t^*)}$. Note that every term is then exponentially decaying. Thus, there is some $M'' > 0$, $\delta'' > 0$ such that

$$\|x(t)\| \le M''e^{-\delta''(t-t^*)}$$

for all $t \ge t^*$, which proves the Theorem. ■

Proof of Theorem 5.3.1

To show that the beam deflections and velocities go to zero exponentially, we will first show that the linear portion of the beam dynamical equations (5.3.5)-(5.3.6) generate an analytic, exponentially stable semigroup (Theorem 5.A.2). Using the results of part (i) of the Theorem and the Bellman - Gronwall type result above (Theorem 5.A.1) will then yield the result.

Consider a new differential equation $\dot{x} = Ax$ where A is the linear portion of (5.3.5):

$$A := \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} \tag{5.A.4}$$

(Refer to section 2.1 for the definition of the various terms in A.) Let the space A operates on be $X = \mathbb{R}^3 x \mathbb{R}^3 x H_0{}^2 x H_0{}^2 x H_0{}^1 x L^2 x L^2 x L^2$, and let the domain of A, D(A), be defined as

$$D(A) = \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8)^T |\ x_1 \in \mathbb{R}^3,\ x_2 \in \mathbb{R}^3,\ x_3 \in H_0{}^4,\ x_4 \in H_0{}^4,\ x_5 \in H^2,\ x_6 \in H_0{}^4,\ x_7 \in H_0{}^4,\ x_8 \in H^2,\ x_5(c)=x_8(c)=0,\ x_3''(c+L) = x_4''(c+L) = x_6''(c+L) = x_7''(c+L) = 0,\ x_3'''(c+L)=x_4'''(c+L)=x_6'''(c+L)=x_7'''(c+L)=0,\ x_5'(c+L)=x_8'(c+L)=0\} \tag{5.A.5}$$

Let the inner product on X be

$$[a,b]_X = [a_1,b_1]_{\mathbb{R}} + m_B[a_2,b_2]_{\mathbb{R}} + [a_3,b_3] + [a_4,b_4] + [a_5,b_5] + [a_6,b_6] + [a_7,b_7] + [a_8,b_8] \tag{5.A.6}$$

where $[a, b]_{\mathbb{R}}$ is the ordinary inner product in $\mathbb{R}^3$, and [a, b] is the ordinary L^2 inner product.

Theorem 5.A.1 - Consider the differential equation

$$\ddot{y} + \rho A\dot{y} + Ay = 0$$

where A is a positive definite, self-adjoint unbounded linear operator operating on a Hilbert space X, and $\rho \in \mathbb{R}$, $\rho > 0$. Write this differential equation in state space form as

$$\dot{x} = \begin{bmatrix} 0 & I \\ -A & -\rho A \end{bmatrix} x =: Lx$$

where $x := (y, \dot{y})^T$. Let the domain of the operator L be D(A)⊕D(A). Then the closure of L, denoted $\bar{L}$, generates an analytic, exponentially stable semigroup on W = $D(A^{0.5}) \oplus X$.

Proof of Theorem 5.A.1 - See [Hua. 1, Theorem 4.1], or [Mas. 1, Theorem 1.1].

Using this result, we immediately have the following theorem.

Theorem 5.A.2 The closure of the operator A defined in (5.A.4) generates an analytic, exponentially stable semigroup T(t).

Proof of Theorem 5.A.2 - The operator consists of 3 decoupled components. Two components are due to transverse components, and one is due to axial deflections. It thus suffices to show that the operators

$$L_T := \begin{bmatrix} 0 & 1 \\ \dfrac{-\partial^4(\cdot)}{\partial z^4} & \dfrac{-k\partial^4(\cdot)}{\partial z^4} \end{bmatrix} \quad \text{and } L_A := \begin{bmatrix} 0 & 1 \\ \dfrac{-\partial^2(\cdot)}{\partial z^2} & \dfrac{-k\partial^2(\cdot)}{\partial z^2} \end{bmatrix}$$

for some $k > 0$,corresponding to transverse and axial deflections, respectively, satisfy the conditions of Theorem 5.A.1. We will consider L_T, and L_A will follow similarly.

Consider the differential operator $\frac{\partial^4(\cdot)}{\partial z^4}$. Using a simple integration by parts, it is easy to verify that for g, h $\in$ $\{ x \in H_0^4 \mid x''(c+L) = 0, x'''(c+L) = 0\}$,

$$[\frac{\partial^4 f}{\partial z^4}, g] = [g, \frac{\partial^4 f}{\partial z^4}] \text{ and} \tag{5.A.7}$$

$$[\frac{\partial^4 f}{\partial z^4}, f] = [\frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial z^2}] \tag{5.A.8}$$

where $[\cdot, \cdot]$ is the ordinary L^2 inner product. (5.A.7) shows that $\frac{\partial^4(\cdot)}{\partial z^4}$ is a self-adjoint operator on L^2 with the domain $\{x \in H_0^4 \mid x''(c+L) = 0, x'''(c+L) = 0\}$. Equation (5.A.8) combined with Proposition 3.4.2 shows that it is also a positive definite operator. Therefore, using $X = L^2$, $D(A) = \{x \in H_0^4 \mid x''(c+L) = 0, x'''(c+L) = 0\}$, and Theorem 5.A.1 shows that the closure of L_T generates an analytic, exponentially stable semigroup. By identical reasoning, the closure of L_A generates an exponentially stable semigroup. Hence we conclude that the closure of A of (5.A.4) generates an exponentially stable semigroup. ■

Proof of Theorem 5.3.1, part(iii) - Rewrite (5.3.5) in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \underline{\dot{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} - \begin{bmatrix} 0 \\ Y^{-1}\underline{\ddot{y}} \end{bmatrix} \tag{5.A.9}$$

$$\text{or, } \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + f(t) \tag{5.A.10}$$

A, B(t), f(t) are obvious from (5.A.9). From Theorem 5.A.2, the closure of A generates an exponentially stable, analytic semigroup. Since $\underline{\omega}$ and $\underline{\dot{\omega}}\rightarrow 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially, where $\| \ \|$ denotes the norm induced by the inner product (5.A.6). Finally, since $\underline{\ddot{y}}\rightarrow 0$ exponentially by design, $\|f(t)\|$ goes to zero exponentially. Thus, the conditions of Theorem 5.A.1 are met, and we conclude that $\underline{u}$ and $\underline{u}_t$ go to zero in the X (energy) norm. ■

Proof of Theorem 5.3.4, part(iii) - To show that the decay rate is exponential, we will first show that the linear part of (ii) is exponentially stable (Lemma 5.A.6). Using the results of part (i) of the Theorem and a Bellman - Gronwall type proof will then yield the result.

Consider a new differential equation $\dot{x} = Ax$ where A is the linearization of (5.3.28) at the origin:

$$A := \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) - \dfrac{1}{m_B}\displaystyle\int_c^{c+L} \mu\partial(\cdot)dz & -k\partial(\cdot) - \dfrac{1}{m_B}\displaystyle\int_c^{c+L} k\partial(\cdot)dz \end{bmatrix} \tag{5.A.11}$$

(Refer to section 2.1 for the definition of the various terms in A.) Let the space A operates on be $H_0^2 x H_0^2 x H_0^1 x L^2 x L^2 x L^2$, together with the corresponding "energy" inner product

$$[f, g]_X = [(f_1, f_2, f_3, f_4, f_5, f_6)^T, (g_1, g_2, g_3, g_4, g_5, g_6)^T] = [f_1'', g_1''] + [f_2'', g_2''] + [f_3'', g_3''] + [f_4, g_4] + [f_5, g_5] + [f_6, g_6]. \tag{5.A.12}$$

where [•, •] denotes the ordinary inner product in $L^2[c, c+L]$. The term energy is used since the first three terms of the inner product (5.A.12) represent the potential energy of the beam, while the latter 3 terms represent the kinetic energy of the beam. Let the domain of A, D(A), be defined as

$$D(A) = \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8)^T |\ x_1 \in H_0^4, x_2 \in H_0^4, x_3 \in H^2,\ x_4 \in H_0^4, x_5 \in H_0^4, x_6 \in H^2, x_3(c)=x_6(c)=0, x_1''(c+L) = x_2''(c+L) = x_4''(c+L) = x_5''(c+L) = 0, x_1'''(c+L)=x_2'''(c+L)=x_4'''(c+L)=x_5'''(c+L)=0, x_3'(c+L)=x_6'(c+L)=0\} \qquad (5.A.13)$$

One should note the strong similarity between this operator, and the operator A′ (defined by (3.A.4)) used in the proof of Theorem 3.4.1, part (ii). In fact, Lemma 3.A.2 immediately yields

Lemma 5.A.3 - Consider A of (5.A.11). Suppose the mass of the rigid body is much greater than the mass of the beam. Then $\bar{A}$, the closure of A, generates an analytic semigroup on X.

Proof of Lemma 5.A.3 - For simplicity, but in a abuse of notation, let A denote the closure of the operator A of (5.A.11). Note that A of (5.A.11) is simply A′ of (1.2) restricted to the last two components. Thus, since A′ of (1.2) generates an analytic semigroup, then so does A of (5.A.11). ■

Before proceeding, we need to compute the spectrum of the operator A of (5.A.11). This is the content of the following Proposition.

Proposition 5.A.4 - Consider the linear operator A given by equation (5.A.11). Consider the eigenvalue problem

$$Ax = \lambda x \tag{5.A.14}$$

Then A has eigenvalues

$$\lambda_i^{\pm} = -\frac{k_i v_i^4 \pm \sqrt{k_i^2 v_i^8 - 4\mu_i v_i^4}}{2} \qquad i = 1, 2 \tag{5.A.15}$$

$$\lambda_3^{\pm} = -\frac{k_3 v_3^2 \pm \sqrt{k_3^2 v_3^4 - 4\mu_3 v_3^2}}{2} \tag{5.A.16}$$

where the v_i satisfy

$$\cosh v_i L \cos v_i L + \frac{1}{v_i m_B}(\cos v_i L \sinh v_i L + \cosh v_i L \sin v_i L) = -1 \quad i = 1, 2 \tag{5.A.17}$$

$$\cos v_3 L = (\sin v_3 L) / v_i m_B. \tag{5.A.18}$$

The eigenvectors corresponding to these eigenvalues are

$$x_1^{\pm} = ([\,]_1, 0, 0, \lambda_1^{\pm}[\,]_1, 0, 0)^T$$
$$x_2^{\pm} = (0, [\,]_2, 0, 0, \lambda_2^{\pm}[\,]_2, 0)^T$$
$$x_3^{\pm} = (0, 0, [\,]_3, 0, 0, \lambda_3^{\pm}[\,]_3)^T$$

where $[\,]_i = (2c_{i2}/(v_i m_B) - c_{i1})\cos v_i(z-c) - c_{i2}(\sin v_i(z-c) + 1/(v_i m_B)) +$

$c_{i1}\cosh v_i(z-c) + c_{i2}(\sinh v_i(z-c) - 1/(v_i m_B))$, i=1, 2, and

$[\ ]_3 = (\sin v_3(z\text{-}c) - 1/(v_3 m_B)) + \cosh v_3(z\text{-}c) 1/(v_3 m_B)$. Also, the c_{ij} satisfy

$$\begin{bmatrix} \cos v_iL + \cosh v_iL & \sin v_iL + \sinh v_iL - 2\cos v_iL/(v_i m_B) \\ \sinh v_iL - \sin v_iL & \cos v_iL + \cosh v_iL + 2\sin v_iL/(v_i m_B) \end{bmatrix} \begin{bmatrix} c_{i1} \\ c_{i2} \end{bmatrix} = 0,\ i=1,2 \qquad (5.A.19)$$

Comment 5.A.5 - As $m_B \rightarrow \infty$, inspection of (5.A.17)-(5.A.18) show that the eigenvalues, modal frequencies, and eigenvectors approach that of a clamped-free beam ([Mei. 1, p. ?]). This is certainly to be expected, since as $m_B \rightarrow \infty$, the rigid body is becoming an "infinite wall".

Proof of Proposition 5.A.4: Direct computation shows that $Ax_i^{\pm} = \lambda_i^{\pm} x_i^{\pm}$. The conditions that the v_i and c_{ij} satisfy come from the beam boundary conditions

$$u_i(c) = 0,\ u'_i(c) = 0$$
$$u''_i(c+L) = 0,\ u'''_i(c+L) = 0$$

which correspond to zero deflection and velocity at the fixed end, and zero moment and force at the free end.

To obtain the c_{ij}, write an arbitrary eigenvector as a linear combination of $\cosh v_i(z\text{-}c)$, $\cos v_i(z\text{-}c)$, $\sinh v_i(z\text{-}c)$ and $\sin v_i(z\text{-}c)$, $i = 1, 2$, and similarly for the v_3 term. These combinations must satisfy the boundary conditions if they are to be eigenvectors. Since there are 4 boundary conditions and four unknown coefficients, we get a homogeneous system of 4 equations and 4 unknowns. If the coefficients are to be nonzero, the determinant of the corresponding matrix must be zero. The determinant of the system is precisely the conditions (5.A.17) and (5.A.18). Partially solving the resulting system, and inserting the partial solution results in the system (5.A.19). ■

Analogous to Theorem 5.A.2, we have the following Lemma which shows that A of (5.A.11) generates an exponentially stable semigroup.

Lemma 5.A.6 - Consider the operator A of (5.A.11). Suppose the mass of the rigid body is much larger than the mass of the beam. Then the operator A generates an exponentially stable semigroup.

Proof of Lemma 5.A.6 - Since A generates an analytic semigroup, then by Proposition B.9 of Appendix B, we know that T(t) satisfies $\|T(t)\| \leq M\exp(\omega_0(t))$, where $\omega_0 = \sup\{Re(\lambda) \mid \lambda \in \sigma(A)\}$. It thus suffices to verify that the spectrum of A is strictly negative and bounded away from zero. From Proposition 5.A.4, examining (5.A.15)-(5.A.16) we see that this indeed the case. This proves Lemma 5.A.6. ■

Proof of Theorem 5.3.4 part(iii) - Rewrite (5.3.5) in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) - \frac{1}{m_B}\int_c^{c+L} \mu\partial(\cdot)dz & -k\partial(\cdot) - \frac{1}{m_B}\int_c^{c+L} k\partial(\cdot)dz \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \dot{\underline{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} \tag{5.A.20}$$

$$\text{or,} \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} \tag{5.A.21}$$

where A and B(t) are obvious from (5.A.20). From Lemma 5.A.6, the closure of A generates an exponentially stable, analytic semigroup. Since $\underline{\omega}$ and $\dot{\underline{\omega}}\to 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially, where $\| \ \|$ denotes the norm induced by (5.A.12). Finally, since $\ddot{\underline{y}}\to 0$ exponentially by design, $\|f(t)\|$ goes to zero exponentially. Thus, the conditions of Theorem 5.A.1 are met, and we conclude that $\underline{u}$ and $\underline{u}_t$ go to zero exponentially in the X (energy) norm. ■

Proof of Theorem 5.4.1, part (iv)

The idea of this proof is identical to the others in this appendix: separate the linear and nonlinear portions of the differential equation, verify that the linear portion generates an exponentially stable semigroup, and then apply Theorem 5.A.2.

Consider now the linear portion of (5.4.5):

$$A := \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & 0 \end{bmatrix}, \tag{5.A.22}$$

Let the space A operates on, X, be defined as

$$X := H_0^2 \times H_0^1 \times H_0^1 \times L^2 \times L^2 \times L^2 \tag{5.A.23}$$

where the H^k are defined in Appendix B, and let the corresponding "energy" inner product be

$$[f, g]_E := [(f_1, f_2, f_3, f_4, f_5, f_6)^T, (g_1, g_2, g_3, g_4, g_5, g_6)^T]_E = [f_1'', g_1''] + [f_2'', g_2''] + [f_3'', g_3''] + [f_4, g_4] + [f_5, g_5] + [f_6, g_6]. \tag{5.A.24}$$

Let the domain of A, D(A), be defined as

$D(A) = \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T \mid x_1 \in H_0^4, x_2 \in H_0^4, x_3 \in H^2,\ x_4 \in H_0^4, x_5 \in H_0^4, x_6$

$\in H^2$, $x_3(c)=x_6(c)=0$, $x_1'(c)=x_2'(c)=x_4'(c)=x_5'(c)=0$, $x_1''(c+L) = x_2''(c+L) = x_5''(c+L) = x_6''(c+L) = 0$, $x_1'''(c+L)=\alpha x_4(c+L)$, $x_2'''(c+L)=\beta x_5(c+L)$, $x_3'(c+L)=\gamma x_6(c+L)\}$ (5.A.25)

where $\alpha>0$, $\beta>0$, and $\gamma>0$.

Theorem 5.A.7 Consider the operator A of (5.A.22) together with the corresponding space (5.A.23) and inner product (5.A.24). Then A generates an exponentially stable semigroup.

Proof of Theorem 5.A.7 - This follows from Theorem 2.5.2 of Chapter 2, when x_i is chosen to be L. ■

Using this result, the proof of Theorem 5.4.1 follows easily.

Proof of Theorem 5.4.1 - The rigid body control law (5.4.7) forces $\underline{\xi}(t)\to 0$ exponentially exactly as in the proof of Theorem 5.3.1. To show that the beam velocities and deflections go to zero, use exactly the methods of the proof of Theorem 5.3.4. More explicitly, rewrite (5.4.5)-(5.4.6) in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & 0 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \dot{\underline{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} - \begin{bmatrix} 0 \\ Y^{-1}\ddot{\underline{y}} \end{bmatrix} \quad (5.A.26)$$

$$\text{or,} \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + f(t) \quad (5.A.27)$$

A, B(t), f(t) are obvious from (5.A.26). Theorem 5.A.7 shows that A generates an exponentially stable semigroup. Since $\underline{\omega}$ and $\dot{\underline{\omega}}\to 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially, where $\| \ \|$ denotes the norm induced by (5.A.24). Finally, since $\underline{\ddot{y}} \to 0$ exponentially by design, $\|f(t)\|$ goes to zero exponentially. Thus, the conditions of Theorem 5.A.1 are met, and we conclude that $\underline{u}$ and $\underline{u}_t$ go to zero exponentially in the X (energy) norm. ■

Appendix 5.B - Proof of Existence and Uniqueness of Solutions

The proof of existence and uniqueness of solutions to the closed loop systems in Theorems 5.3.1, 5.3.2, and 5.3.4 is presented below. The proofs are of a somewhat different flavor than the proofs of finite dimensional nonlinear differential equations. Technical difficulties occur because there are unbounded operators present, which are not present in most finite dimensional nonlinear differential equations. The proofs below use standard perturbation theorems in semigroup theory [Paz. 1]. The intuition behind these results is that if solutions exist to the unperturbed equations, then equations exist to the perturbed equations as long as the perturbations are sufficiently "nice", in a sense to made precise.

Existence and uniqueness of solutions to Theorem 5.3.1

Upon substitution of the control law (5.3.7) and (5.3.8), note that (5.3.1), (5.3.2), (5.3.3), (5.3.4) become ordinary differential equations. The existence and uniqueness of these differential equations is easy to verify. For, observing (5.3.20), we see that $\underline{\xi}(t)$, $\dot{\underline{\xi}}(t)$, $\underline{\omega}$, $\dot{\underline{\omega}}$ and $\dot{\underline{y}}$ satisfy linear differential equations. Thus, the only difficulty is the partial differential equation (5.3.5) and boundary conditions (5.3.6).

$$\underline{u}_{tt} + \dot{\underline{\omega}} \times \underline{u} + 2\underline{\omega} \times \underline{u}_t + \underline{\omega} \times (\underline{\omega} \times \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0 \tag{5.B.5}$$

$$u_1(c) = u_2(c) = u_3(c) = 0, \quad u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0 \tag{5.B.6}$$

$$u_1'''(c + L) = 0 \quad u_2'''(c + L) = 0 \quad u_3'(c + L) = 0$$

Rewrite (5.B.5) in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \dot{\underline{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} - \begin{bmatrix} 0 \\ Y^{-1}\ddot{\underline{y}} \end{bmatrix} \tag{5.B.7}$$

$$\text{or,} \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + f(t)$$

A, B(t), f(t) are obvious from (5.B.7). From Theorem 5.A.2, A generates an exponentially stable, analytic semigroup. Since $\underline{\omega}$ and $\dot{\underline{\omega}} \to 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially, where $\| \ \|$ denotes the norm induced by the inner product 5.A.6. Hence B(t) is a bounded linear operator. Finally, since $\ddot{\underline{y}} \to 0$ exponentially by design, $\|f(t)\|$ goes to zero exponentially.

Using [Paz. 1, Chapter 4, Corollary 2.10], A + f(t) generates a strongly continuous semigroup. It is easy to verify, using a Bellman-Gronwall type argument, that A + f(t) generates an exponentially stable semigroup. Finally, using [Paz. 1, Chapter 5, Theorem 2.3] (5.B.7) has a (strong) continuously differentiable solution, for all $\underline{u}_0 \in$ D(A). ■

Proof of Existence and Uniqueness of Solutions to Theorem 5.3.4

The proof of global existence and uniqueness is almost identical to that of Theorem 5.3.1. Upon substitution of the control law (5.3.30), note that (5.3.24), (5.3.25), (5.3.26), (5.3.27) become ordinary differential equations. The existence and uniqueness of these differential equations is easy to verify, since, as in the proof of Theorem 5.3.1, $\underline{\xi}(t)$, $\dot{\underline{\xi}}(t)$, $\underline{\omega}$, $\dot{\underline{\omega}}$ and $\dot{\underline{y}}$ satisfy linear differential equations. Thus, the only difficulty is the partial differential equation (5.3.28) and boundary conditions (5.3.29).

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\underline{u}_t) + Y^{-1}\ddot{\underline{y}} = 0$$
$$u_1(c) = u_2(c) = u_3(c) = 0, \; u_1'(c) = u_2'(c) = 0$$
$$u_1''(c+L) = u_2''(c+L) = 0$$
$$u_1'''(c+L) = 0 \quad u_2'''(c+L) = 0 \quad u_3'(c+L) = 0$$

Rewrite this in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) - \dfrac{1}{m_B}\displaystyle\int_c^{c+L} \mu\partial(\cdot)dz & -k\partial(\cdot) - \dfrac{1}{m_B}\displaystyle\int_c^{c+L} \mu\partial(\cdot)dz \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \dot{\underline{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} \tag{5.B.8}$$

$$\text{or, } \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix}$$

A, B(t) are obvious from (5.B.8). From Lemma 3.A.2, A generates an exponentially stable, analytic semigroup. Since $\underline{\omega}$ and $\dot{\underline{\omega}} \to 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially. Hence B(t) is a bounded linear operator.

Using [Paz. 1, Chapter 5, Theorem 2.3], A + B(t) generates a strongly continuous semigroup. (In fact, it generates an exponentially stable semigroup.) This shows that there is a unique, continuously differentiable solution, for all $\underline{u}_0 \in D(A)$.

Existence and uniqueness of solutions to Theorem 5.4.1

Upon substitution of the control law (5.4.7)-(5.4.9), note that (5.4.1), (5.4.2), (5.4.3), (5.4.4) become ordinary differential equations. The existence and uniqueness of these differential equations is easy to verify, since $\underline{\xi}(t)$, $\dot{\underline{\xi}}(t)$, $\underline{\omega}$, $\dot{\underline{\omega}}$ and $\dot{\underline{y}}$ satisfy linear differential equations. Thus, the only difficulty is the partial differential equation (5.4.5) and boundary conditions (5.4.6).

$$\underline{u}_{tt} + \dot{\underline{\omega}}^x \underline{u} + 2\underline{\omega}^x \underline{u}_t + \underline{\omega}^x(\underline{\omega}^x \underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\ddot{\underline{y}} = 0$$

$$u_1(c) = u_2(c) = u_3(c) = 0, \;\; u_1'(c) = u_2'(c) = 0$$

$$u_1''(c + L) = u_2''(c + L) = 0$$

$$\mu_i u_i'''(c + L) = -\alpha_i \underline{u}_{it}(c+L), \quad i = 1, 2.$$

$$\mu_3 u_3'(c + L) = -\alpha_3 u_{3t}(c+L)$$

Rewrite (5.4.5)-(5.4.6) in state space form as

$$\begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & 0 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ -\underline{\omega}^x(\underline{\omega}^x) - \dot{\underline{\omega}}^x & -2\underline{\omega}^x \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} - \begin{bmatrix} 0 \\ Y^{-1}\ddot{\underline{y}} \end{bmatrix} \qquad (5.B.9)$$

$$\text{or,} \begin{bmatrix} \underline{u}_t \\ \underline{u}_{tt} \end{bmatrix} =: A \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + B(t) \begin{bmatrix} \underline{u} \\ \underline{u}_t \end{bmatrix} + f(t)$$

A, B(t), f(t) are obvious from (5.B.9). From Theorem 2.5.5, A generates an exponentially stable semigroup. Since $\underline{\omega}$ and $\dot{\underline{\omega}} \to 0$ exponentially by design, $\|B(t)\|$ goes to zero exponentially. Hence B(t) is a bounded linear operator. Finally, since $\ddot{\underline{y}} \to 0$ exponentially by design, $\|f(t)\|$ goes to zero exponentially.

Using [Paz. 1, Chapter 4, Corollary 2.10], A + f(t) generates a strongly continuous semigroup. It is easy to verify, using a Bellman-Gronwall type argument, that A + f(t) generates an exponentially stable semigroup. Finally, using [Paz. 1, Chapter 5, Theorem 2.3] (5.B.9) has a (strong) continuously differentiable solution, for all $\underline{u}_0 \in D(A)$.

APPENDIX A - DETERMINATION OF SHEAR FORCES AND MOMENTS

Determination of beam forces and moments is highly problem specific. In this appendix, we will consider the determination of forces and moments due to a rectangular beam attached to a rigid body. See Figure A.1.

The problem with determining these quantities is they cannot be directly measured, but rather must be determined through some other quantity which can be measured. The simplest way of doing this is by use of strain gauges and rosettes. The reader unfamiliar with these devices can find a simple discussion in [Pop. 1, p. 311] or a more complete discussion in [Het. 1, chapt. 5-9].

A.1 Stress and Strain Tensors

Only a very brief discussion of material properties will be given here, mainly to fix notation. Readers interested in a more detailed exposition are referred to [Pop. 1, Chapters 3, 4] or [Lan. 1, Chapter 1].

Let the position of a particle P in the beam be $r_1\underline{i} + r_2\underline{j} + r_3\underline{k}$ (where $\underline{i}$, $\underline{j}$, $\underline{k}$ refer to the unit vectors along some x, y, z coordinate axes). Upon application of forces to the beam, deformation occurs and the point P moves to $(r_1 + u_1)\underline{i} + (r_2 + u_2)\underline{j} + (r_3 + u_3)\underline{k}$.

Let $\varepsilon_{x_i x_j}$ denote the ij-th component of the strain tensor defined as

$$\varepsilon_{x_i x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j} \frac{\partial u_j}{\partial x_k} \right) \tag{A.1.1}$$

with summation over k, where $x_1 := x$, $x_2 := y$, $x_3 := z$.

Now consider an infinitesimal cubic volume element centered about a point P of the beam, with faces of area ΔA. Let $\sigma_{x_i x_j}$ denote the ij-th member of the stress tensor defined as

$$\sigma_{x_i x_j} = \lim_{\Delta A \to 0} \frac{\Delta F_{x_i j}}{\Delta A} \quad i, j = 1,2,3 \tag{A.1.2}$$

where $\Delta F_{x_i j}$ is the x_ith component of the force acting on face j of the cube. (Faces 1 and 4 have outward normals parallel to the x and -x axes, respectively, faces 2 and 5 refer similarly to y and -y, and 3 and 6 refer to z and -z.)

By assuming homogeneous, isotropic material, and also assuming small strains, we get Hooke's Law relations between stress and strain

$$\varepsilon_{xx} = \sigma_{xx}/E - \nu\sigma_{yy}/E - \nu\sigma_{zz}/E \tag{A.1.3}$$

$$\varepsilon_{yy} = \sigma_{yy}/E - \nu\sigma_{xx}/E - \nu\sigma_{zz}/E \tag{A.1.4}$$

$$\varepsilon_{zz} = \sigma_{zz}/E - \nu\sigma_{xx}/E - \nu\sigma_{yy}/E \tag{A.1.5}$$

$$\varepsilon_{xy} = \sigma_{xy}/G \tag{A.1.6}$$

$$\varepsilon_{yz} = \sigma_{yz}/G \tag{A.1.7}$$

$$\varepsilon_{xz} = \sigma_{xz}/G \tag{A.1.8}$$

where E is the Young's modulus for the material, ν is Poisson's ratio, and G is the shear modulus.

In general, the contributions to Poisson's ratio is small and hence for simplicity it will be ignored. Then equations (A.1.3) - (A.1.5) simplify to

$$\varepsilon_{xx} = \sigma_{xx}/E \tag{A.1.9}$$

$$\varepsilon_{yy} = \sigma_{yy}/E \tag{A.1.10}$$

$$\varepsilon_{zz} = \sigma_{zz}/E \tag{A.1.11}$$

A.2 Forces and Moments Affecting Beam

Consider a rectangular beam as shown in Figure A.1.

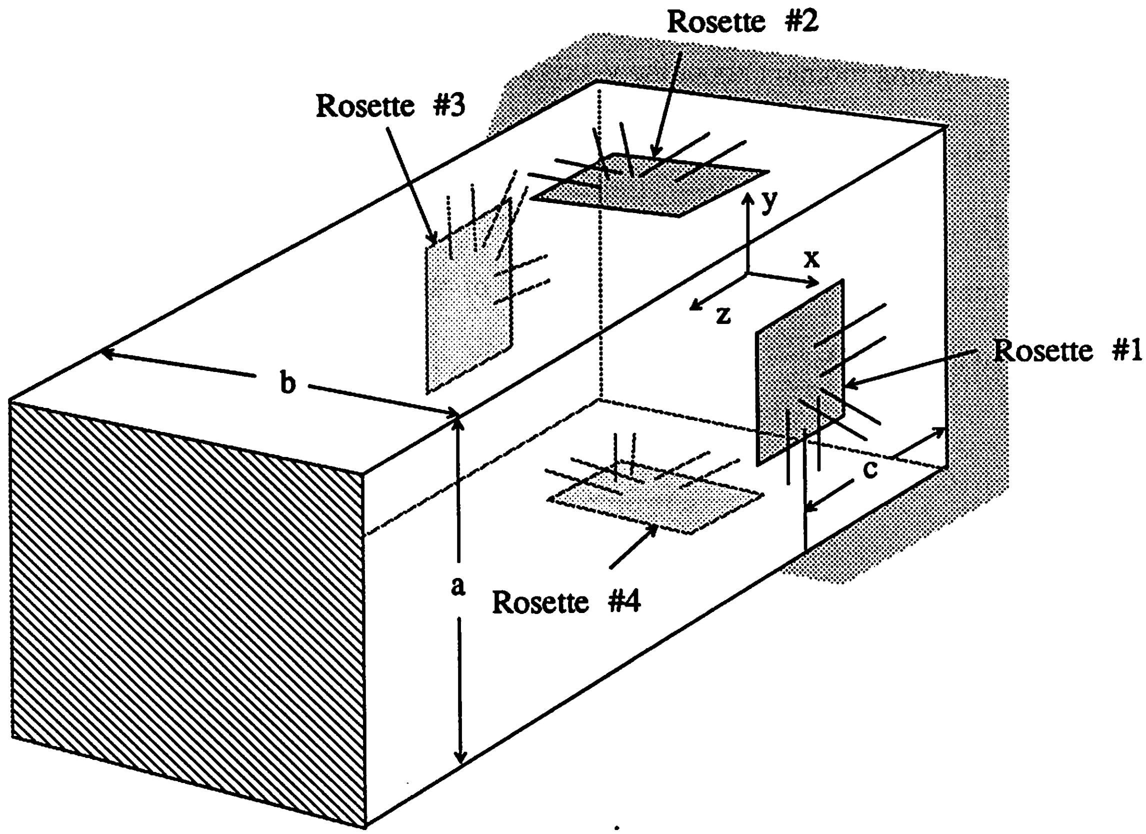

Figure A.1 - Rectangular Beam under consideration

Recall that the neutral surface (or elastic line) is the portion of the beam which does not change length during deformation. In the case shown here, it is simply the z-axis. In determining stresses due to bending moments, the fundamental assumption is that the strains vary linearly as their respective distances from the neutral surface. With such an assumption, and using equilibrium conditions for an arbitrary beam segment, it is easy to show that the bending moment about the x-axis =: M_x is [Pop. 1, p. 182]

$$M_x = I_x \sigma_{zz} / x \tag{A.2.1}$$

where $I_x = \int x^2 dA = ab^3/12$. Similarly,

$$M_y = I_y \sigma_{zz}/y \tag{A.2.2}$$

where M_y is the bending moment about the y-axis and $I_y = \int y^2 dA = ba^3/12$.

If the bar undergoes a moment M_z about the z-axis, the torsional shear distribution is somewhat difficult to compute. However, it turns out that the distribution (see Figure A.2) has a maximum occurring at the midpoint of the longest side (in this case, the side parallel to the y-axis).

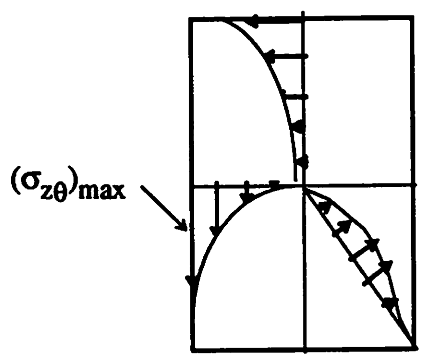

Figure A.2 - Shear Stress Distribution

The maximum shear stress turns out to be [Pop.1, p. 167]

$$(\sigma_{zx})_{max} = M_z/ba^2\alpha$$

where it is assumed that $a \geq b$, and α is a parameter depending on the b/a ratio, and M_z is the moment about the z-axis. Hence,

$$M_z = (\sigma_{zx})_{max} ba^2\alpha \tag{A.2.3}$$

By symmetry, it is also clear that $(\sigma_{zy})_{max} = M_z\alpha/b^2a$, which occurs at the midpoint of the shorter side (the side of length b). Hence we must also have

$$M_z = (\sigma_{zy})_{max} \alpha / b^2 a \tag{A.2.4}$$

To determine the shear stresses in the beam, recall that the shear distribution for a rectangular bar subjected to a shear force in the x-direction V_x is parabolic in nature and given by [Pop. 1, p. 232]

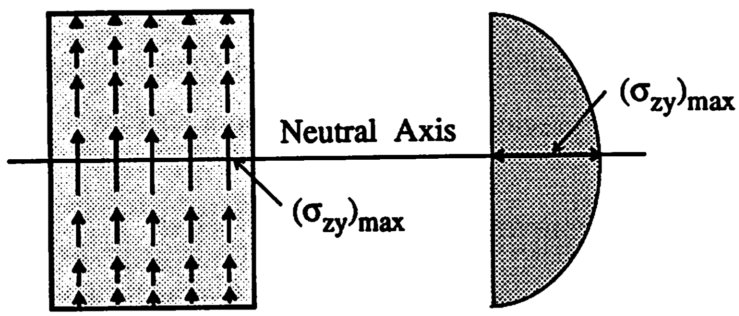

Figure A.3 - Shear Stress Distributrion due to vertical shear

$$\sigma_{zy}(x_1) = \frac{V_x}{2I_x}\left(\left(\frac{b}{2}\right)^2 - x_1^2 \right) \quad x_1 \in [0, b/2]$$

(See Figure A.3) This shows that the shear stress is zero at the boundary (x_1=b/2) and has a maximum at x_1=0 of value $(\sigma_{zy})_{max} = V_x b^2 / 8\, I_x$. Solving for V_x,

$$V_x = -\frac{8}{a^2} I_x (\sigma_{zy})_{max} = -\frac{3}{2ab} (\sigma_{zy})_{max} \tag{A.2.5}$$

Similarly,

$$V_y = -\frac{8}{b^2} I_y (\sigma_{zx})_{max} = -\frac{3}{2ab} (\sigma_{zx})_{max} \tag{A.2.6}$$

where V_y is the shear force in the y-direction.

Finally, to determine the axial stress induced by a tensile or compressive force, note that the average stress over a cross-section is simply $F_z/A = -\sigma_{zz}$ since the cross-section is constant over the length of the beam (when considering axial forces alone). Hence, F_z, the axial force in the z-direction, is

$$F_z = -A\sigma_{zz} \tag{A.2.7}$$

In the following, only small deflections will be considered, so that the principle of superposition holds. That is, the resultant strain in the system is the algebraic sum of the individual strains when applied separately. Superposition of stresses as well as strains also follows from the previous assumption of Hooke's Law.

A.3 Force and Moment Determination from Strain Rosettes

In order to determine the forces and moments affecting the beam, strain rosettes are mounted on the beam as shown in Figure A.1. With the rosettes placed as shown, the following information is obtained:

Rosette 1:
$$\left.\begin{bmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}\right|_{x=\,b/2,\ y=0,\ z=c} \tag{A.3.1}$$

Rosette 2:
$$\left.\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix}\right|_{x=\,0,\ y=a/2,\ z=c} \tag{A.3.2}$$

Rosette 3:
$$\left.\begin{bmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}\right|_{x=\,-b/2,\ y=0,\ z=c} \tag{A.3.3}$$

Rosette 4:
$$\left.\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix}\right|_{x=\,0,\ y=-a/2,\ z=c} \tag{A.3.4}$$

From these measurements, the forces and moments affecting the beam can be determined by use of equations (A.2.1)-(A.2.6), superposition, and the Hooke's Law relationships. Specifically, at rosettes 1 and 3 equations (A.2.1), (A.2.2), and (A.2.6) shows that (y=0 at both 1 and 3 so that there is no contribution due to M_x)

$$\sigma_{zz} = -F_z/A + M_y x/I_x$$

$$\Rightarrow \begin{cases} \sigma_{zz}|_1 + \sigma_{zz}|_3 = -2F_z/A \\ \sigma_{zz}|_1 - \sigma_{zz}|_3 = 2M_y b/2I_x = M_y b/I_x \end{cases}$$

$$\Rightarrow \begin{cases} F_z = -\frac{A}{2}(\sigma_{zz}|_1 + \sigma_{zz}|_3) \\ M_y = I_x/b(\sigma_{zz}|_1 - \sigma_{zz}|_3) \end{cases}$$

Similar arguments show that

$$M_x = I_y/a(\sigma_{zz}|_2 - \sigma_{zz}|_4)$$

By the Hooke's Law relationships, $\sigma_{zz} = E\varepsilon_{zz}$. Hence,

$$F_z = -\frac{A}{2E}(\varepsilon_{zz}|_1 + \varepsilon_{zz}|_3) \tag{A.3.5}$$

$$M_y = I_x/bE(\varepsilon_{zz}|_1 - \varepsilon_{zz}|_3) \tag{A.3.6}$$

$$M_x = I_y/Ea(\varepsilon_{zz}|_2 - \varepsilon_{zz}|_4) \tag{A.3.7}$$

Since the rosettes at each of these positions determine the strains in the parentheses, F_z, M_y, M_x, are determinable from the experimental data.

Finally, to determine M_z, V_x, and V_y consider Figure A.2. Since the stresses are additive at one side of the cross-section, but subtract from one another on the other side, it is easy to solve for the quantities M_z, V_x, and V_y. Proceeding along these lines, use Figure A.2 and equations (A.2.3), (A.2.4) and (A.2.5) to obtain

$$\varepsilon_{zy}|_1 G = \sigma_{zy}|_1 = +M_z\alpha/b^2a + 3V_y/2ab \qquad \text{(A.3.8)}$$

$$\varepsilon_{zy}|_3 G = \sigma_{zy}|_3 = -M_z\alpha/b^2a + 3V_y/2ab \qquad \text{(A.3.9)}$$

$$\varepsilon_{zx}|_2 G = \sigma_{zx}|_2 = +M_z/ba^2\alpha + 3V_x/2ab \qquad \text{(A.3.10)}$$

$$\varepsilon_{zx}|_4 G = \sigma_{zx}|_4 = -M_z/ba^2\alpha + 3V_x/2ab \qquad \text{(A.3.11)}$$

)

Since $\varepsilon_{zx}|_1$, $\varepsilon_{zx}|_3$, $\varepsilon_{zx}|_2$, $\varepsilon_{zx}|_4$ are known by measurement, and since b, a, and α are known, equations (A.3.8)-(A.3-11) is a system of 4 equations in 3 unknowns. From this system, a least-squares solution for M_z, V_x, and V_y can be found.

Note that if c is small, the moments and forces acting on the body by the beam are close to the corresponding values at the point of attachment. Thus,

$$\underline{M}_{bB}' \equiv \underline{M}_x i + \underline{M}_y j + \underline{M}_z k \qquad \text{(A.3.12)}$$

$$\underline{F}_{bB}' \equiv \underline{F}_x i + \underline{F}_y j + \underline{F}_z k \qquad \text{(A.3.13)}$$

If $\underline{i}$, $\underline{j}$, $\underline{k}$ are parallel to the $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ axes, respectively, then $\underline{M}_{bB}' = \underline{M}_{bB}$ and $\underline{F}_{bB}' = \underline{F}_{bB}$. Otherwise there is a (fixed) rotation matrix Q, as in the discussion of kinematics in section 3.1.1, such that $[\underline{b}_1 \;\; \underline{b}_2 \;\; \underline{b}_3]Q = [\underline{i} \;\; \underline{j} \;\; \underline{k}]$. Then the components of $\underline{M}_{bB}$ and $\underline{F}_{bB}$ are Q[components of $\underline{M}_{bB}'$] and Q[components of $\underline{F}_{bB}'$], respectively.

Remark A.3.1 - If we add Kelvin-Voight damping to our model of the form, i.e. we add damping of the form

$$\sigma_{jj} = E\varepsilon_{jj} + \eta_{jj}\frac{d\varepsilon_{jj}}{dt}$$

$$\sigma_{jk} = G\varepsilon_{jk} + \eta_{jk}\frac{d\varepsilon_{jk}}{dt} \qquad j \neq k,$$

then the formulas change very simply:

$$F_z = -\frac{A}{2E}(\varepsilon_{zz}|_1 + \varepsilon_{zz}|_3) - \frac{A}{2\eta_{zz}}(\dot{\varepsilon}_{zz}|_1 + \dot{\varepsilon}_{zz}|_3).$$

$$M_y = I_x(\varepsilon_{zz}|_1 - \varepsilon_{zz}|_3)/Eb + I_x(\dot{\varepsilon}_{zz}|_1 - \dot{\varepsilon}_{zz}|_3)/\eta_{zz}b$$

$$M_x = I_y(\varepsilon_{zz}|_2 - \varepsilon_{zz}|_4)/Ea + I_y(\dot{\varepsilon}_{zz}|_2 - \dot{\varepsilon}_{zz}|_4)/\eta_{zz}a$$

For the torsion and shear calculations, just rewrite the LHS of (A.3.8)-(A.3.11) to obtain

$$\varepsilon_{zy}|_1 G + \eta_{zy}\dot{\varepsilon}_{zy}|_1 = \sigma_{zy}|_1 = +M_z\alpha/b^2a + 3V_y/2ab$$

$$\varepsilon_{zy}|_3 G + \eta_{zy}\dot{\varepsilon}_{zy}|_3 = \sigma_{zy}|_3 = -M_z\alpha/b^2a + 3V_y/2ab$$

$$\varepsilon_{zx}|_2 G + \eta_{zx}\dot{\varepsilon}_{zx}|_2 = \sigma_{zx}|_2 = +M_z/ba^2\alpha + 3V_x/2ab$$

$$\varepsilon_{zx}|_4 G + \eta_{zx}\dot{\varepsilon}_{zx}|_4 = \sigma_{zx}|_4 = -M_z/ba^2\alpha + 3V_x/2ab$$

and again compute a least-squares solution for M_z, V_y and V_x.

All of these computations presuppose that the strain derivatives can be determined. Of course, one could get an approximation of these quantities by on-line finite differences, i.e.

$$\dot{\varepsilon}_{zz}(t) \approx \frac{\varepsilon_{zz}(t) - \varepsilon_{zz}(t - T)}{T}$$

where T is the time between strain samples.

Appendix B

Useful Facts From Semigroup Theory

Notation: Let $H^k[c, c+L]$, $k=0, 1, \ldots$ be defined as

$$H^0[c, c+L] := \{f \in L^2[c, c+L]\}$$

$$\vdots$$

$$H^k[c, c+L] := \{f \in L^2[c, c+L] \mid f, f' \ldots f^k \in L^2[c, c+L]\}$$

Also, define

$$H_0{}^1[c, c+L] := \{f \in L^2[c, c+L] \mid f, f' \in L^2[c, c+L] \text{ and } f(c) = 0\}$$

$$H_0{}^2[c, c+L] := \{f \in L^2[c, c+L] \mid f, f', f'' \in L^2[c, c+L] \text{ and } f(c) = f'(c) = 0\}$$

$$H_0{}^4[c, c+L] := \{f \in L^2[c, c+L] \mid f, f', f'', f''', f'''' \in L^2[c, c+L] \text{ and } f(c) = f'(c) = 0\}$$

For simplicity of notation H^k will denote $H^k[c, c+L]$, $H_0{}^k$ will denote $H_0{}^k[c, c+L]$, and L^2 will denote $L^2[c, c+L]$.

Before proceeding, we need to introduce some definitions and notation from the semigroup literature. The interested reader can find excellent expositions on this subject in many texts, e.g. [Paz. 1], [Kat. 1], [Bal. 1].

Definition B.1 - A linear operator A on a Hilbert space X, $A: X \supset D(A) \rightarrow X$ is said to be <u>dissipative</u> if for any $x \in D(A)$, $[Ax, x] \leq 0$.

Definition B.2 - A linear operator A on a Hilbert space X is said to be <u>closed</u> if its graph is closed.

Definition B.3 - A linear operator $A: X \supset D(A) \to X$ on a Hilbert space X with dense domain D(A) is said to be self-adjoint if $A=A^*$, where A^* denotes the adjoint operator of A.

Definition B.4 Let T(t) for all $t \in [0, \infty)$ be a bounded linear operator in a Banach space X. $\{T(t)\}$ is said to be a strongly continuous semigroup if

(i) $T(t+s) = T(t)T(s) = T(s)T(t)$, for any $t \geq 0$, any $s \geq 0$.

(ii) $T(0) = I$

(iii) $\|T(t)x - x\| \to 0$, as $t \to 0+$, for any $x \in X$.

Definition B.5 - Suppose T(t) is a strongly continuous semigroup on a Banach space X. The linear operator defined by

$$D(A) := \{x \in X \mid \lim_{t \to 0} \frac{T(t)x - x}{x} \text{ exists}\}$$

and

$$Ax := \lim_{t \to 0} \frac{T(t)x - x}{x} \text{ for } x \in D(A)$$

is called the infinitesimal generator of the semigroup T(t), and D(A) is called the domain of A.

Comment B.6 - We see from the the definition that the infinitesimal generator is the generalization of the matrix A for the matrix exponential e^{At} in finite dimensional system theory. If A is a matrix in $\mathbb{R}^{nxn}$ the domain of A is all of X since A is a continuous operator.

Definition B.7 A strongly continuous semigroup $\{T(t)\}$ satisfying $\|T(t)\| \leq 1$ for all $t \in [0, \infty)$ is called a contraction semigroup. If there exists $K>0$, $\delta>0$ such that the semigroup satisfies $\|T(t)\| \leq Ke^{-\delta t}$, then $T(t)$ is termed an exponentially stable semigroup.

Definition B.8 A semigroup $\{T(t)\}$ is said to be analytic if there exists a sector Δ of the form

$$\Delta = \{z \in C: \phi_1 < \arg(z) < \phi_2, \phi_1 < 0 < \phi_2\}$$

containing the real axis with

(i) $z \rightarrow T(z)$ is analytic in Δ.

(ii) $T(0) = I$, $\lim\limits_{\substack{z \to 0 \\ z \in \Delta}} T(z)x = x$, for any $x \in X$.

(iii) $T(z_1+z_2) = T(z_1)T(z_2) = T(z_2)T(z_1)$, for any $z_1 \in \Delta$, and any $z_2 \in \Delta$.

In the proofs in this thesis, we will often be interested in establishing that a semigroup is exponentially stable. Analogous to the finite dimensional case, one would hope that if the eigenvalues of the operator are all negative and bounded away from the $j\omega$-axis, then the semigroup is exponentially stable. Unfortunately, this is not true in general. (See [Hua. 2] for a counterexample.) Further, there are very few easy methods to determine whether a semigroup is exponentially stable. However, if A generates an analytic semigroup, then we have the following result [Tri. 1, p. 387].

Proposition B.9 - Suppose a linear operator A generates an analytic semigroup $T(t)$ on the space X. Then $T(t)$ satisfies $\|T(t)\| \leq M\exp(\omega_0(t))$, where $\omega_0 = \sup\{Re(\lambda) \,|$

$\lambda \in \sigma(A)\}$.

Proposition B.9 shows that if is often advantageous to know if one has an analytic semigroup. One way of obtaining analytic semigroups is by perturbing an analytic semigroup. One perturbation theorem for analytic semigroups is the following:

Theorem B.10 [Paz. 1, p. 80, Thm. 2.1] - Let A be the infinitesimal generator of an analytic semigroup. Let B be a closed linear operator satisfying $D(B) \supset D(A)$ and

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad \text{for } x \in D(A).$$

There exists $\delta > 0$ such that if $a \in [0, \delta]$, then A + B is the infinitesimal generator of an analytic semigroup.

REFERENCES

[Bai. 1] Baillieul, J. and Levi, M., "Rotational Elastic Dynamics," *Physica* 27D, 1987, pp. 43-62.

[Bal. 1] Balakrishnan, A.V., *Applied Functional Analysis*, 2nd Edition, Springer-Verlag New York, Inc, 1981.

[Balas 1] Balas, M. J., "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on Automatic Control*, Vol. 27, No. 3, p. 522-534, June, 1982.

[Ben. 1] Bender, D., Hughes Aircraft Company unpublished notes, July, 1987.

[Bon. 1] Bontsema, J., and Curtain, R., "A Note on Spillover and Robustness of Flexible Systems," *IEEE Transactions on Automatic Control*, Vol. 33, No. 6, p. 567-569, June, 1988.

[Cal. 1] Callier, F. M., and Desoer, C. A., *Multivariable Feedback Systems*, Springer-Verlag, 1982.

[Che. 1] Chen, G., and Russell, D. L., "A Mathematical Model for Linear Elastic Systems with Structural Damping," *Quarterly of Applied Mathematics*, January, 1982, pp. 433-454.

[Che. 2] Chen, G., Delfour, M. C., Krall, A. M., and Payre, G., "Modelling, Stabilization, and Control of Serially Connected Beams," *SIAM Journal of Con-*

trol and Optimization, Vol. 25, No. 3, May, 1987.

[Che. 3] Chen, G., Krantz, S. G., Ma, D. W., Wayne, C. E., and West, H. H., "The Euler Bernoulli Beam Equation with Boundary Energy Dissipation," in *Operator Methods for Optimal Control Problems,* edited by Lee, John Wiley & Sons, 1987.

[Cou. 1] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, John Wiley & Sons, 1953.

[Cur. 1] Curtain, R. F., "Equivalence of Input-Output Stability and Exponential Stability for Infinite Dimensional Systems," *Mathematical Systems Theory* 21, pp. 19-48, 1988.

[De L. 1] De Luca, A. and Isidori, A., "Feedback Linearization of Invertible Systems," *Colloquim in Automation and Robotics*, Duisburg, 1987.

[Del. 1] Delfour, M. C., Lagnese, J., and Polis, M. P., "Stabilization of Hyperbolic Systems Using Concentrated Sensors and Actuators," *IEEE Transactions on Automatic Control*, Vol. 31, No. 12, p. 1091-1096, December, 1986.

[Des. 1] Desoer, C. A., and Morgul, O., "Control and Stabilization of a Flexible Beam Attached to a Rigid Body: Planar Motion," *Proceedings of the 27th IEEE Conference on Decision and Control*, December, 1988.

[Des. 2] Desoer, C. A., and Vidyasagar, M., *Feedback Systems: Input/Output Properties*, Academic Press, Inc., 1975.

[Des. 3] Desoer, C. A., *Notes for a Second Course on Linear Systems*, Van Nostrand Reinhold, 1970.

[Die. 1] Dieudonne, J., *Foundations of Modern Analysis*, Academic Press, 1969.

[Dou. 1] Dougherty, H. J., Scott, E. D. and Rodden, J. J., "Analysis and Design of WHECON - An Attitude Control Concept," *AIAA 2nd Communications Satellite Systems Conference*, San Francisco, CA, 1968.

[Dwy. 1] Dwyer, T.A.W. III, "Exact Nonlinear Control of Large Angle Rotational maneuvers," *IEEE Transactions on Automatic Control*, Vol. AC-29, no. 9, pp. 769-774, 1984.

[Dwy. 2] Dwyer, T.A.W. III, "Gibbs Vector Kinematics and Inverse Dynamics for Decoupled Spacecraft Attitude maneuvers," *AIAA 24th Aerospace Sciences Meeting*, AIAA-86-0251, 1986.

[Dwy. 3] Dwyer, T.A.W. III, "Slewing maneuvers with Internal Momentum Transfer," *Journal of Guidance and Control*, March-April 1986, pp. 240-247.

[Gai. 1] Gai, E., Daly, K., Harrison, J., and Lemos, L., "Star-Sensor-Based Satellite Attitude/Attitude Rate Estimator," *Journal of Guidance and Control*, Vol.8, No. 5, pp. 560-565, Sept.-Oct., 1985.

[Gib. 1] Gibson, J. S., "An Analysis of Optimal Regulation: Convergence and Stability," *SIAM Journal on Control and Optimization*, Vol. 19, No. 5, September, 1981, pp.686-707.

[Gib. 2] Gibson, J. S., "A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Linear Feedback" *SIAM Journal on Control and Optimization*, Vol. 18, No. 3, May, 1980, pp.311-316.

[Gre. 1] Green, A. E., Naghdi, P. M., and Werner, M. L., "On the Theory of Rods I. Derivation from the three dimensional equations," *Proceedings of the Royal Society of London*, A. 337 pp. 451-483, 1974.

[Gre. 2] Green, A. E., Naghdi, P. M., and Werner, M. L., "On the Theory of Rods II. Developments by direct approach," *Proceedings of the Royal Society of London*, A. 337 pp. 485-507, 1974.

[Hale 1] Hale, J. K., "Asymptotic behavior and dynamics in infinite dimensions," in *Nonlinear Differential Equations* (Eds. Hale and Martinez-Amoros), Research Notes in Mathematics, vol. 132, Pitman Press, 1985.

[Har. 1] Harn, Y.P., and Polak, E., "On the Design of Finite Dimensional Compensators for Infinite Dimensional Feedback Systems via Semi-Infinite Optimization," *Proceedings of the 27th Conference on Decision and Control*, December, 1988.

[Het. 1] Hetenyi, M., *Handbook of Experimental Stress Analysis*, John Wiley & Sons, 1950.

[Hir. 1] Hirzinger, G., and Lange, Th., "Direct Digital Attitude Control for a Double-Gimballed Momentum Wheel," *IFAC Symposium on Automatic Control in Space*, Vol. 2, 1976, ed. E. Gottzein.

[Hua. 1] Huang, F., "On the Mathematical Model for Linear Elastic Systems with Analytic Damping," *SIAM Journal on Control and Optimization*, Vol. 26, No. 3, pp. 714-724, May 1988.

[Hua. 2] Huang, F., "Characteristic Conditions for Exponential Stability of Linear Dynamical Systems in Hilbert Spaces," *Annals of Differential Equations*, 1985.

[Hug. 1] Hughes, P.C., *Spacecraft Attitude Dynamics*, John Wiley & Sons, Inc., 1986.

[Hut. 1] Hutson, V., and Pym, J. S., *Applications of Functional Analysis and Operator Theory*, Mathematics in Science and Engineering, No. 133, Academic Press, 1980

[Isi. 1] Isidori, A., "Lectures on Nonlinear Control," unpublished notes prepared for a course at Carl Cranz Gesellschaft, July 1987.

[Jac. 1] Jacobson, C., and Nett, C., "Linear State-Space Systems in Infinite-Dimensional Space: The Role and Characterization of Joint Stabilizability/Detectability," *IEEE Transactions on Automatic Control*, Vol. 33, No. 6, June 1988.

[John 1] John, F., *Partial Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, Fourth Edition, 1982.

[Joh. 1] Johnson, T.L., "Progress in Modelling and Control of Flexible Spacecraft," *Journal of the Franklin Institute*, Vol. 315, No. 5/6, pp. 495-520, May/June 1983.

[Kane 1] Kane, T.R., Likins, P.R. and Levinson, D.A., *Spacecraft Dynamics*, MacGraw-Hill, Inc., 1983.

[Kat. 1] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag New York, Inc., 1966.

{Kim 1] Kim, J. U., and Renardy, Y., "Boundary Control of the Timoshenko Beam," *SIAM Journal of Control and Optimization*, Vol. 25, No. 6, Nov. 1987.

[Lan. 1] Landau, L. D. and Lifshitz, E.M., *Theory of Elasticity*, Pergamon Press, 1986.

[Mas. 1] Massatt, P., "Limiting Behavior for Strongly Damped Nonlinear Wave Equations," *Journal of Differential Equations*, Vol. 48, pp. 334-349, 1983.

[McB. 1] McBride, A. C., *Semigroups of Linear Operators: An Introduction*, Pitman Research Notes in Mathematics Series 156, Longman Scientific and Technical, 1987.

[Mei. 1] Meirovitch, L., and Quinn, R. D., "Maneuvering and Vibration Control of Flexible Spacecraft," *Journal of the Astronautical Sciences*, Vol. 35, No. 3, July-September 1987, pp. 301-328.

[Mei. 2] Meirovitch, L., *Analytical Methods of Vibrations*, McGraw-Hill, New York, New York, 1970.

[Mey. 1] Meyer, G., Su, R., and Hunt, R. L., "Application of Nonlinear Transformations to Automatic Flight Control," *Automatica* Vol.20, No.1, pp. 103-107, 1984.

[Mon. 1] Monaco, S. and Stornelli, S., "A Nonlinear Attitude Control Law for a Satellite with Flexible Appendages," *Proceedings of the 24th Conference on Decision and Control*," pp. 1654-1659, 1986.

[Mor. 1] Morgul, O., "Control and Stabilization of a Flexible Beam Attached to a Rigid Body," Memorandum No. UCB/ERL M89/5, Electronics Research Laboratory, University of California, Berkeley, 1989.

[Mort. 1] Mortenson, R. E., "On Systems for the Automatic Control of the Rotation of a Rigid Body," Memorandum No. UCB/ERL 63/23, Electronics Research Laboratory, University of California, Berkeley, 1963.

[Mork. 1] Mork, H. L., "Synthesis and Design of a Gimbaled Reaction Wheel Attitude Stabilization Package (GRASP)," *AIAA Guidance, Control and Flight Mechanics Conference*, Hofstra University, 1971.

[Nett 1] Nett, C. N., Jacobson, C. A., and Balas, M. J., "Fractional representation theory: Robustness with applications fo finite dimensional control of a class of linear distributed systems," *Proceedings of the IEEE Conference on Decision and Control*, pp. 268-180, 1983.

[Paz. 1] Pazy. A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences Vol. 44, Springer-Verlag, New York, Inc., 1983.

[Pop. 1] Popov, E.G., *Introduction to Mechanics of Solids*, Prentice-Hall, Inc., 1968.

[Reb. 1] Rebarber, R. L., "Spectral Determination for a Cantilever Beam," *IEEE Transactions on Automatic Control*, Vol. 34 No. 5, May 1989.

[Rus. 1] Russell, D. L., "Mathematical models for the elastic beam and their control theoretic implications," in *Semigroups, Theory, and Applications, Vol. 2*, Longmans, London, 1986.

[Sas. 1] Sastry, S.S., "Adaptive Control of Linearizable Systems," Memorandum No. UCB/ERL M87/53, Electronics Research Laboratory, University of California, Berkeley, 1987.

[Seg. 1] Segal, I., "Nonlinear Semi-Groups," *Annals of Mathematics*, Vol. 78, No. 2, 1963.

[Sim .1] Simo, J. C., and Vu-Quoc, L. "On the Dynamics of Finite Strain Rods Undergoing Large Motions - The Three Dimensional Case," Memorandum No. UCB M86/11, Electronics Research Laboratory, University of California, Berkeley, 1986.

[Tan. 1] Tanabe, H., *Equations of Evolution*, Monographs and Studies in Mathematics 6, Pitman Publishing Ltd., 1979.

[Tri. 1] Triggiani, R., "On the Stabilizability Problem in Hilbert Space, " *Journal of Mathematical Analysis and Applications*, 52, pp. 383-403, 1975.

[Vad. 1] Vadali, S.R. and Junkins, J.L., "Optimal Open-Loop and Stable Feedback Control of Rigid Body Attitude Maneuvers," *Journal of the Astronautical Sciences*, Vol. 32, No. 2, April-June 1984, pp. 105-122.

[Vid. 1] Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice-Hall, Inc., 1978.

[Vid. 2] Vidyasagar, M., *Control Systems Synthesis: A Factorization Approach*, The MIT Press, 1985.

[Wer. 1] Wertz, J. R., Editor, *Spacecraft Attitude Determination and Control*, Computer Sciences Corporation, Dordrecht, Boston, Reidel, 1978.