PROBLEMS IN THE CONTROL OF FLEXIBLE SPACECRAFT

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Memorandum No. UCB/ERL M89/97

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Abstract

This memorandum investigates the application of boundary control techniques to a rotating flexible spacecraft. More precisely, we consider a rigid body whose center of mass is fixed in an inertial frame with a flexible beam clamped to the rigid body at one end and free at the other end. We investigate this configuration under various assumptions, depending on whether the motion takes place on a plane, or in the three-dimensional space, as well as the model we choose for the beam. In each case, we pose a stabilization problem, propose a feedback law and show that under the proposed control law, the stability of the configuration is obtained.

The memorandum is organized as follows:

In Chapter 2, we review some basic tools of Newtonian Dynamics and some recent developments in the nonlinear beam theories, namely the director theory of beams and the geometrically exact beam theory. Then, as an example, we derive the equations of motion for the configuration mentioned above.

In Chapter 3, we study the basic configuration under the planar motion assumption and we use Euler-Bernoulli beam model. We propose two control laws, each consisting of a torque applied to the rigid body and a force and a torque applied to the free end of the beam. We show that under the proposed control law, exponential stabilization is obtained.

In Chapter 4, we generalize the results of Chapter 3 to the motion of the same

configuration in three dimensions.

In Chapter 5, we first prove a stabilization result for the basic configuration using the geometrically exact beam model, without any linearization. Using this result, we generalize the results obtained in Chapter 2 to the case of planar motion of the basic configuration using the Timoshenko beam model.

List of Symbols †

$\mathbf{R}$	the set of real numbers
$\mathbf{R}^n$	vector space of ordered n-tuples in $\mathbf{R}$
$\mathbf{R}_+$	nonnegative real numbers $\{x \in \mathbf{R} \mid x \geq 0\}$
$\mathbf{C}$	complex numbers
$\mathbf{r}$, $\mathbf{n}$ etc.	vectors in $\mathbf{R}^3$
$\mathbf{N}$, $\mathbf{B}$	coordinate frames in $\mathbf{R}^3$
$\mathbf{L}^2$	$\{f : [0,L] \rightarrow R \mid \int_{x=0}^{x=L} f^2\, dx < \infty \}$
$\mathbf{H}^k$	$\{f \in \mathbf{L}^2 \mid f^i \in \mathbf{L}^2 , i=1,..,k \}$
$\mathbf{H}^k{}_0$	$\{f \in \mathbf{H}^k \mid f(0) = f^1(0) = 0 \}$
$X \times Y$	product of two sets X and Y
f_x , f_t , *etc.*	$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, resp., etc.

† some additional notation is defined where it is needed.

$. \mathbf{x} .$	standard cross-product in $\mathbf{R}^3$
$<.,.>$	standard inner product in R^3
A^T	the transpose of A
$\| \| . \| \|$	norm in a vector space

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Chapter 1

Introduction

Many mechanical systems, such as spacecrafts with flexible appendages or robot arms with flexible links, can be modeled as coupled rigid and elastic parts. Such models are basic to the control of systems which have flexible parts, their stabilization, high precision pointing, etc., and the control of such systems are becoming increasingly important in the design of lightweight, high performance systems. Thus, over the last decade there has been a growing interest in obtaining new methods for the design, dynamics and control of the systems which has flexible parts. An excellent review of research in this area can be found in [Bal.1]; also for a literature survey, see [Tza.1], and for the recent history of the subject and for additional references, see [Nur.1] and [Lik.1].

Consider a system which has coupled rigid and flexible parts. The motion of the flexible parts is usually described by a set of partial differential equations with appropriate boundary conditions. Since the motion of the rigid parts are described by a set of nonlinear ordinary differential equations, and since the rigid parts are coupled with the flexible parts, the overall equations of motion are generally a set of coupled nonlinear partial and ordinary equations. These equations can be obtained by using the standard methods of Mechanics, see, e.g., [Gol.1], [Par.1]. Although the equations of motion obtained by using different methods, such as using Lagrangians, Hamiltonians, free-body diagrams, etc., are equivalent, the form of these equations and the complexity they offer depend on a particular method used in obtaining them. For a comparison of different methods to obtain the equations of motion for mechanical systems, see [Kan.1] and [Pau.1].

In the analysis of flexible structures, particularly in engineering applications, the common practice is to resort to the finite element modeling, see [Bal.1]. In effect, one approximates the continuous structure of flexible parts by a finite number of interconnected rigid elements with well-defined structural dynamics. This approach reduces the original equations, which are coupled nonlinear partial and ordinary differential equations, to a set of coupled, nonlinear and finite, although often very large, number of ordinary differential equations. As a consequence , the infinite number of modes, in theory, associated with the original set of equations is reduced to a finite, although often very large, number of modes. However, having established a control law for this reduced set of equations does not always guarantee that the same control law will work on the original set of equations, e.g. one might encounter so-called "spillover" problems, see [Bal.2]. Also note that the actual number of modes of a flexible system, in theory, is infinite and the number of modes that should be retained is not known a priori.

Stability of systems with flexible parts, particularly flexible space structures, has been studied in the past. In [Mei.3], the Hamiltonian of the whole structure was used as a Lyapunov fuction to study the stability of a damped flexible spacecraft. A comparison of control techniques based on finite element modeling can be found in [Mei.4]. More recently, [Bai.1] and [Kri.1] studied the dynamics and stability of a *rotating* flexible structure from Lagrangian and Hamiltonian point of view, respectively. Both of these works do not resort to a finite element approximation, but they do not offer a control scheme.

Recently, [Bis.1] used a Lyapunov type approach which uses the total energy of a flexible spacecraft as a Lyapunov function candidate to prove the stability of the system under appropriate forces and torques applied to the flexible spacecraft. Their proposed control laws contain *distributed* forces applied to the flexible parts, (i.e., forces which are distributed over the flexible parts), which are proportional to the deflection velocities. Implementations of such control laws might not be easy and practical.

In recent years, boundary control of flexible systems, (i.e., controls applied to the *boundaries* of the flexible parts as opposed to the controls *distributed* over the flexible parts), has become an important research area. This idea was first applied to the systems described by wave equation, (e.g. strings) [Che.1], and recently extended to the Euler-Bernoulli beam equation [Che.2,3], and the Timoshenko beam equation [Kim.1]. In particular, in [Che.2,3], it has been proven that, in a cantilever beam, a single actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam deflections, and in [Del.1] the case where the actuator is "concentrated" to an area, as opposed to a single point actuator, is investigated. Recently, the boundary control techniques has been applied to the stabilization of a flexible spacecraft performing planar motion in [Des.2].

The purpose of this thesis is to investigate the application of boundary control techniques to *rotating* flexible structures. As a case model, we study the motion of a rigid body whose center of mass is fixed in an inertial frame, with a flexible beam clamped to the rigid body at one end and free at the other end. This basic configuration captures the essential properties of a flexible spacecraft, (see, e.g., [Bis.1,2], [Bai.2], [Kri.1], [Pos.1]), such as a flexible spacecraft in a geosynchronous orbit. We consider various cases, depending on whether the motion takes place in a plane, or in the three-dimensional space $\mathbf{R}^3$, as well as the model we choose for the flexible beam. In each case we pose a stabilization problem , propose a feedback control law and then show that under the proposed control law, the exponential stability of the whole structure is obtained.

The thesis is organized as follows :

In Chapter 2, we first review some basic tools of Newtonian Mechanics and some recent developments on the nonlinear beam theories, such as geometrically exact beam models. Then, as an example, we derive the equations of motion for the basic configuration mentioned above.

In Chapter 3, we study the planar motion of the basic configuration using the Euler-Bernoulli beam model. We first derive the equations of motion, define the rest state of the system and propose *two* control laws to stabilize this system. Each law consists of a torque applied to the rigid body, a force and a torque applied to the free end of the beam.

Then we show that under the proposed control laws, *exponentially* stabilization is obtained.

In Chapter 4, we generalize the results obtained in chapter 2, to the case of 3 dimensional motion, (i.e. motion in $\mathbf{R}^3$).

In Chapter 5, we first prove a stabilization result for the motion of the basic configuration mentioned above using the geometrically exact beam model *without any linearization*. Then using this result, we generalize the results obtained in chapter 2 to the case of the planar motion of the basic configuration using the Timoshenko beam model.

The contribution of this thesis is the application of boundary control techniques to the stabilization of *rotating* flexible structures. The results obtained here indicate that these techniques can be used in the various problems arising from the control of flexible structures; an area which needs futher investigation.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we review some basic results of Newtonian Mechanics and Elastic Beam Theories which will be used in the subsequent chapters.

In Section 2, we outline the basic equations of Newtonian particle dynamics and rotational dynamics of rigid bodies. In Section 3, we will review some recent developments on nonlinear beam theories, namely the director theories of beams as developed by Green and Naghdi [Gre.1,2], geometrically exact beam theory as developed by Simo [Sim.1], and show that the latter can be treated as a special case of the former theory. In Section 4, we obtain the equations of motion of a beam clamped to a rigid body at one end and free at the other using the formulations introduced in the previous sections. The rigid body-elastic beam configuration introduced in this section is the basic configuration we study thoroughout this thesis.

2.2 Fundamentals of Newtonian Mechanics

In this Section, we summarize and derive some of the basic equations of Newtonian Mechanics which will be used in this thesis. There are many excellent textbooks on this subject. For detailed analysis, the reader is referred to, e.g., Goldstein [Gol.1], or Pars [Par.1].

2.2.1 Particle Dynamics

It is customary to give the basic equations of Newtonian Mechanics for a particle, these equations then can be extended to the equations of more complex physical objects, such as rigid bodies. A particle is a model for a physical object whose dimensions can be neglected to describe its motion, i.e., it is represented as a point in $\mathbf{R}^3$, which is the standard 3-dimensional Euclidean space.

Let $\mathbf{N}$ be a frame in $\mathbf{R}^3$, which is specified by an origin O, and following the usual convention, 3 orthonormal, dextral (i.e., right-handed) vectors fixed in N, say $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$; the frame N is given by the quadruple $\mathbf{N} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Let $\mathbf{r} = \mathbf{OP}$ be the position vector of a particle P in the frame $\mathbf{N} = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Let m be the mass of the particle P. Of fundamental importance in Newtonian Mechanics is the (linear) momentum $\mathbf{p}$ of the particle P in the frame N, which is defined as :

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt} \quad . \tag{2.2.1}$$

Let $\mathbf{f}$ be the net force acting upon the particle P. Then Newton's second law asserts that there exist a frame in which the net force $\mathbf{f}$ and the momentum $\mathbf{p}$ are related as follows:

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} \quad . \tag{2.2.2}$$

A frame in which (2.2.2) holds is called an *inertial frame*. It is easy to show that N and M are two inertial frames if and only if M is in uniform translation with respect to N, see, e.g., [Gol.1].

Let N be an inertial frame, P be a particle in N, **r** be the position vector of P in N, and **f** be the net force acting upon P. Then the torque **T**, or the moment of the force applied to the particle P in N with respect to the origin O of the frame N is defined as :

$$\mathbf{T} = \mathbf{r} \times \mathbf{f} \quad , \tag{2.2.3}$$

whereas angular momentum **L**, or moment of momentum, of the particle P in N with respect to the origin O of N is defined as :

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad , \tag{2.2.4}$$

where **x** denotes the standard cross- product in $\mathbf{R}^3$

Assuming that the mass of the particle P is constant, by differentiating (2.2.4) with respect to time t, we obtain :

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{T} \quad . \end{aligned} \tag{2.2.5}$$

Equations (2.2.2) and (2.2.5) are the

basic equations of Newtonian Particle Mechanics. We note that the equations of Mechanics can also be obtained by using other formulations, such as Lagrangian or Hamiltonian formulation. This approach will not be taken here. Interested reader is referred to any textbook on Mechanics, see, e.g., [Lan.1], [Mei.1], [Gol.1], [Par.1].

2.2.2 Rotational Dynamics of Rigid Bodies

Kinematics

Let N be an inertial frame specified by the quadruple $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Let B_R be a rigid body in N. To specify the equations of motion of B_R in N, we define another frame fixed in the rigid body B_R. Let **B** be such a frame specified by the quadruple $(O_1, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$; the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$) are called the body axes. We will choose O_1 to be the center of mass of B_R.

The most general motion of B_R in N is the combined effect of the translation of the center of mass of B_R in N and the rotation of **B** in N. We know that if **F** is the net force applied to B_R, and if O_1 is the center of mass of B_R, then with $\mathbf{R} = \mathbf{OO}_1$, we have

$$M\ddot{\mathbf{R}} = \mathbf{F} \quad .$$

For this reason we will neglect the translational motion of B_R in N and consider the center of mass of the rigid body fixed in N. Hence the motion of B_R in N reduces to rotation of **B** in N.

We will also assume without loss of generality that the origin of N and the center of mass of the rigid body and the origin of the body frame **B**, coincide at all times.

Let **r** be a vector in N, and let r^N and r^B be its components (i.e., triplets) in N and **B**, respectively. Then there exists an ortogonal matrix R with det R =1 (i.e., $R \in SO(3)$), such that the following holds :

$$r^N = R\ r^B \quad , \tag{2.2.6}$$

or equivalently, we have :

$$\mathbf{e}_j = \sum_{i=1}^{3} R_{ij}\ \mathbf{b}_i \qquad j = 1, 2, 3 \quad , \tag{2.2.7}$$

where R_{ij} denotes the elements of the matrix R. By (2.2.7), the j^{th} column of the matrix R consists of the components of $\mathbf{e}_j$ with respect to the basis $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$.

As a result of the assumptions above, the motion of B_R in N is completely described by the matrix function $R(t)$ (defined by (2.2.6)). It is well known that :

$$\frac{dR}{dt} + \Omega R = 0 \quad , \tag{2.2.8}$$

where $\Omega \in \mathbf{R}^{3\times3}$ is defined by

$$\Omega = -\frac{dR}{dt} R^T \quad , \tag{2.2.9}$$

where the superscript T denotes a transposition; for details, see, e.g., [Par.1, p.106].

Differentiating the identity $RR^T = \mathrm{I}$ with respect to t, we obtain :

$$\frac{dR}{dt}R^T + R\left(\frac{dR}{dt}\right)^T = 0 \quad ,$$

or equivalently $\Omega = -\,\Omega^T$, i.e. Ω is skew-symmetric; thus

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad . \tag{2.2.10}$$

Since Ω is skew symmetric, we can define an axial vector ω as follows :

$$\omega = \sum_{i=1}^{3} \omega_i \mathbf{b}_i \quad , \tag{2.2.11}$$

where ω is called the angular velocity of B_R in **N**.

Upon differentiating (2.2.6) with respect to time, we obtain :

$$\frac{dR^N}{dt} = \left(\frac{dR}{dt}\right)^T r^B + R^T \frac{dr^B}{dt} \quad . \tag{2.2.12}$$

Equation (2.2.12), with the aid of (2.2.8) defines the transformation of velocities between the frames **B** and **N** : More precisely, let $r^N = (r^N{}_1 \; r^N{}_2 \; r^N{}_3)^T$ and $r^B = (r^B{}_1 \; r^B{}_2 \; r^B{}_3)^T$. Using (2.2.7) and (2.2.8) in (2.2.12), we obtain :

$$\sum_{i=1}^{3} \frac{dr^N{}_i}{dt}\mathbf{e}_i = \sum_{i=1}^{3} \frac{dr^B{}_i}{dt}\mathbf{b}_i + \omega \times \mathbf{r} \quad . \tag{2.2.13}$$

If one defines

$$\left(\frac{d\mathbf{r}}{dt}\right)_N = \sum_{i=1}^{3} \frac{dr^N{}_i}{dt}\mathbf{e}_i \qquad , \qquad \left(\frac{d\mathbf{r}}{dt}\right)_B = \sum_{i=1}^{3} \frac{dr^B{}_i}{dt}\mathbf{b}_i \qquad ,$$

then (2.2.13) can be rewritten as follows :

$$\left(\frac{d\mathbf{r}}{dt}\right)_N = \left(\frac{d\mathbf{r}}{dt}\right)_B + \boldsymbol{\omega} \times \mathbf{r} \qquad . \tag{2.2.14}$$

Dynamics

We now define the *angular momentum* of the rigid body B_R with respect to the origin O of the **B**. Recall that according to our convention, the point O is also the origin of the inertial frame N and the center of mass of the rigid body B_R.

Let dm be an infinitesimal mass element of the rigid body, **r** be its position vector in N. Then $\mathbf{L}_0$, the angular momentum of B_R with respect to O, is defined by the integral :

$$\mathbf{L}_0 := \int_{B_R} \mathbf{r} \times \left(\frac{d\mathbf{r}}{dt}\right)_N dm$$

$$= \int_{B_R} \mathbf{r} \times (\omega \times \mathbf{r}) dm \quad , \tag{2.2.15}$$

where the second equation follows from (2.2.14) ; we consider m in (2.2.15) as a real valued measure in $\mathbf{R}^3$ which defines the mass distribution of the rigid body B_R .

From vector algebra, we have the relation

$$\mathbf{r} \times (\omega \times \mathbf{r}) = <\mathbf{r}, \mathbf{r}> \omega - <\omega, \mathbf{r}> \mathbf{r} \quad ,$$

where <.,.> denotes the standard inner product in $\mathbf{R}^3$. Hence (2.2.15) reduces to

$$\mathbf{L}_0 = \left(\int_{B_R} [<\mathbf{r}, \mathbf{r}> I - \mathbf{r}.\mathbf{r}] \, dm \right) \omega \quad , \tag{2.2.16}$$

where $(\mathbf{r}.\mathbf{r})\,\omega := \mathbf{r} < \mathbf{r}, \omega >$, the first parenthesis is the *outer product* of **r** with **r** : the "column vector **r** " times the " row vector **r** ", sometimes called the dyad r.r. In (2.2.16), I is the unit matrix in $\mathbf{R}^3$.

The inertia tensor of B_R with respect to O is defined by :

$$I_R := \int_{B_R} (<\mathbf{r}, \mathbf{r}> I - \mathbf{r}.\mathbf{r})\, dm \quad . \tag{2.2.17}$$

Hence (2.2.16) takes the form :

$$\mathbf{L}_0 = I_R\, \omega \quad . \tag{2.2.18}$$

The inertia tensor I_R defined by (2.2.17) is symmetric, therefore one can always find an orthonormal set of axes such that when referred to those axes, the matrix form of I_R becomes diagonal. Such a set of axes is called *principal axes of inertia* : then the matrix I_R reads diag (I_1, I_2, I_3).

Let dm be a fixed infinitesimal mass element of the rigid body, $\mathbf{r}$ be its position vector. Then the kinetic energy T of B_R in N is given by :

$$T = \frac{1}{2}\int_{B_R} < (\frac{d\mathbf{r}}{dt})_N , (\frac{d\mathbf{r}}{dt})_N > dm$$

$$= \frac{1}{2}\int_{B_R} < \omega \times \mathbf{r} , \omega \times \mathbf{r} > dm$$

$$= \frac{1}{2}\int_{B_R} < \omega , \mathbf{r} \times (\omega \times \mathbf{r}) > dm$$

$$= \frac{1}{2} < \omega , I_R\, \omega > \quad . \tag{2.2.19}$$

Let the body axes $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ coincide with the principal axes of inertia of the rigid body, then (2.2.19) takes the following form :

$$T = \frac{1}{2}\sum_{i=1}^{3} I_i \omega_i^2 \quad , \tag{2.2.20}$$

where I_i is the principal moment of inertia of the rigid body about the axis $\mathbf{b}_i$, i = 1, 2, 3.

Let $\mathbf{T}_0$ denote the net torque applied to the rigid body B_R with respect to O. Then using (2.2.5) we obtain the following :

$$\left(\frac{d\mathbf{L}_0}{dt}\right)_N = \mathbf{T}_0 \quad . \tag{2.2.21}$$

Using (2.2.14) and (2.2.18) in (2.2.21), the latter becomes :

$$I_R \frac{d\omega}{dt} + \omega \times I_R \omega = \mathbf{T}_0 \quad . \tag{2.2.22}$$

The above equation is called Euler's equation of motion for a rigid body. With $\omega = \sum_{i=1}^{3} \omega_i \mathbf{b}_i$, $I_R = diag\,(I_1, I_2, I_3)$, and $\mathbf{T}_0 = \sum_{i=1}^{3} T_{0i} \mathbf{b}_i$, the component form of (2.2.22) along the body axes become :

$$I_1\dot{\omega}_1 + (I_2 - I_3)\omega_2\,\omega_3 = T_{01} \quad , \tag{2.2.23}$$

$$I_2\dot{\omega}_2 + (I_3 - I_1)\omega_3\,\omega_1 = T_{02} \quad , \tag{2.2.24}$$

$$I_3\dot{\omega}_3 + (I_1 - I_2)\omega_3\,\omega_1 = T_{03} \quad . \tag{2.2.25}$$

2.3 Nonlinear Beam Models

In this section, we present 2 nonlinear beam models based on a 1-dimensional continuum model called a Cosserat (or a directed) curve, see, e.g., [Ant.1]. A Cosserat curve comprises a material curve embedded in $\mathbf{R}^3$, together with a number of deformable vector fields, called directors, attached to every point of the curve. There are different versions of this approach, depending on the numbers of directors used and the constraints which are imposed on the directors.

The idea of describing a body as not only a collections of points but also of directors attached to the points of the body first proposed by Duhem [Duh.1]. The Cosserat brothers then used this idea in the representation of the twisting and the bending of rods and shells [Cos.1]. Recent interest in this approach began with the work of Ericksen and Truesdell [Eri.1], who presented a generalized version of Cosserat's work.

A nonlinear beam model which uses a Cosserat curve with two directors is first given by Green and Laws [Gre.1], and later developed by Green, Laws and Naghdi [Gre.2]. Related developments of a Cosserat curve with three directors are given by Cohen [Coh.1], and DeSilva [DeS.1]. More recently Simo [Sim.1], extending a generalization of the classical Kirchoff-Love rod model due to Antman [Ant.2], obtained the so called geometrically exact beam model, which proved to be suitable for numerical simulation [Quo.1]. For the classical Kirchoff-Love rod model, see [Lov.1].

In this section, we first give some basic results on elasticity. Then we summarize the rod theory developed by Green, Laws and Naghdi [Gre.2]. As a special case of this theory, following Green and Laws [Gre.3], we obtain the geometrically exact beam model developed by Simo [Sim.1].

2.3.1 Preliminaries

Let the material points of a 3-dimensional body B_E embedded in $\mathbf{R}^3$ be identified by a *convected coordinate system* $\theta_i, i = 1, 2, 3$. (For the definition of a convected coordinate system, see, e.g., [Mar.1, p.41]). Let **N** be an inertial frame and let $\mathbf{r}^*$ denote the position vector of a material element of B_E in N. Let $\mathbf{g}_i$ and $\mathbf{g}^i$ denote the covariant and contravariant base vectors at points of B_E at time t, respectively; and let g_{ij} and g^{ij} be the corresponding *metric tensors* , for $i, j = 1, 2, 3$. By definition we have :

$$\left.\begin{array}{l} \mathbf{g}_i = \dfrac{\partial \mathbf{r}^*}{\partial \theta_i} \\ < \mathbf{g}_i , \mathbf{g}^j > = \delta_{ij} \\ g_{ij} = < \mathbf{g}_i , \mathbf{g}_j > \\ g^{ij} = < \mathbf{g}^i , \mathbf{g}^j > \\ g = \det(g_{ij}) \end{array}\right\} \quad . \tag{2.3.1}$$

Let $m \in B_E$ be a material point and **u** be a unit vector. Let $\Delta S \subset B_E$ be an element of surface containing m. Let the unit normal to ΔS be **u**, measured from one side of ΔS which is called the negative side, to the other side of ΔS which is called the positive side. We assume that the effect of the positive side of the body on the negative side of the body is equivalent to a force $\Delta\mathbf{F}$ and a moment $\Delta\mathbf{G}$ applied to the surface ΔS. We also assume

that the vector $\frac{\Delta F}{\Delta S}$ has a limit as ΔS goes to zero and the vector t defined as :

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} \tag{2.3.2}$$

is called *the stress vector at* $m \in B_E$ *in the direction of* **u** (see, e.g., [Gre.4]).

The stress vector t defined by (2.3.2) depends on the point m, the unit vector **u** and the time t. In the sequel we assume that t does not depend on t. Using the conservation of the energy of B_E and some symmetry arguments, it can be proven that t must be linear in **u** (see, e.g., [Mar.1, p.134]). Thus, at $m \in B_E$, there exists a tensor, called the stress tensor T such that the following holds :

$$\mathbf{t}(m\ ,\mathbf{u}) = T(m)\mathbf{u} \quad . \tag{2.3.3}$$

Let V be an arbitrary volume in the body B_E and let S be the boundary of V. Then neglecting the thermal effects, the conservation of energy equation is :

$$\frac{d}{dt}\int_V \rho^* (\psi^* + \frac{1}{2} < \frac{d\mathbf{r}^*}{dt}, \frac{d\mathbf{r}^*}{dt} >)\, dV = \int_V < \mathbf{f}^*, \frac{d\mathbf{r}^*}{dt} >\, dV + \int_S < \mathbf{t}, \frac{d\mathbf{r}^*}{dt} >\, dS \quad , \tag{2.3.4}$$

where ρ^* is the *mass density*, ψ^* is the *internal potential energy per unit mass*, $\mathbf{f}^*$ is the *body force per unit mass* (such as the gravitational force), t is the *stress vector* at $m \in B_E$ and <.,.> is the standard inner product in $\mathbf{R}^3$.

By using the base vectors defined in (2.3.1), we define the component form of T

defined in (2.3.3) as follows :

$$T\mathbf{g}^i = \sum_{j=1}^{3} \tau^{ij}\mathbf{g}_j \qquad i = 1, 2, 3 \quad . \tag{2.3.5}$$

Correspondingly, we define three components of the stress vector, namely the vectors

$$\mathbf{T}_i = \sum_{j=1}^{3} \sqrt{g}\ \tau^{ij}\mathbf{g}_j \qquad i = 1, 2, 3 \quad . \tag{2.3.6}$$

Hence, if the unit normal $\mathbf{u}$ is given as $\mathbf{u} = \sum_{i=1}^{i=3} u_i \mathbf{g}^i$, then (2.3.3) reduces to :

$$\mathbf{t} = \frac{1}{\sqrt{g}} \sum_{i=1}^{3} u_i \mathbf{T}_i \quad . \tag{2.3.7}$$

By assuming the invariance under superposed motions, one can deduce from the energy equation (2.3.4) the balance of momentum and angular momentum equations (see, e.g., [Gre.4]).

For our purpose, the balance of momentum equation is :

$$\sum_{i=1}^{3} \frac{\partial \mathbf{T}_i}{\partial \theta_i} + \rho^* \sqrt{g}\,\mathbf{f}^* = \rho^* \sqrt{g} \frac{d^2\mathbf{r}^*}{dt^2} \quad , \tag{2.3.8}$$

whereas the balance of angular momentum equation is :

$$\sum_{i=1}^{3} \mathbf{g}_i \times \mathbf{T}_i = 0 \quad . \tag{2.3.9}$$

Note that (2.3.9) is equivalent to the symmetry of the matrix form of T, which is called the symmetric contravariant stress tensor [Gre.2].

2.3.2 Director Theory of Beams

In this section, we outline a theory of beams based on a Cosserat curve with two (deformable) directors, as developed by Green, Naghdi and Laws [Gre.2].

Let the material points of a body B_E embedded in $\mathbf{R}^3$ be identified by a convected coordinate system, θ_i, $i = 1, 2, 3$. Let N be an inertial frame and $\mathbf{r}^*$ be the position vector of a material point of B_E in N. We define the covariant and contravariant base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$, $i = 1, 2, 3$, respectively, and their corresponding metric tensors g_{ij}, g^{ij}, $i, j = 1, 2, 3$, respectively, as in (2.3.1).

To specify the configuration of the beam, we introduce the following objects :

(i) a curve c defined in an open interval $I \subset \mathbf{R}$ as :

$$c := \{ \theta_2 \in I \mid \theta_1 = \theta_3 = 0 \} \quad , \tag{2.3.10}$$

(ii) a family of surfaces which are parametrized by $\theta_2 = \xi \in I$ as :

$$S_\xi := \{ \theta_1 , \theta_3 \in \mathbf{R} \mid \theta_i \in B_E , i = 1, 2, 3, \theta_2 = \xi \} \quad . \tag{2.3.11}$$

We make the following basic assumptions :

Assumption 1 : Let the surface S_ξ be defined as in (2.3.11). For each $\xi \in I$, S_ξ is a *planar* surface and will be referred to as the *cross section of the beam at* ξ. Moreover, S_ξ is spanned by 2 vectors $\mathbf{d}_\alpha$, $\alpha = 1,3$, called the directors , which satisfy the following equation :

$$\mathbf{d}_\alpha := \mathbf{g}_\alpha(0, \xi, 0, t) \qquad \alpha = 1, 3 \quad , t \in \mathrm{R} \quad , \xi \in I \square \tag{2.3.12}$$

Assumption 2 : Let the curve c and the cross-sections S_ξ be defined as in (2.3.10) and (2.3.11), respectively; in addition, we require that the curve c pass through the centroids of the cross-sections, i.e.,

$$\iint_{S_\xi} \theta_\alpha \rho^* \sqrt{g}\, d\theta_1 d\theta_3 = 0 \qquad \alpha = 1, 3 \quad , \xi \in I \quad . \tag{2.3.13}$$

Hence, the curve c will be referred to as the *curve of centroids* □

Let $\mathbf{r}(\xi, t)$ denote the position vector of c at ξ, at time t, then

$$\mathbf{r}(\xi, t) := \mathbf{r}^*(0, \xi, 0, t) \quad , \tag{2.3.14}$$

and a point in the cross-section S_ξ is represented by $\mathbf{r}^*(\theta_1, \xi, \theta_3, t)$, where

$$\mathbf{r}^*(\theta_1, \xi, \theta_3, t) = \mathbf{r}(\xi, t) + \theta_1 \mathbf{d}_1(\xi, t) + \theta_3 \mathbf{d}_3(\xi, t) \quad , \tag{2.3.15}$$

since S_ξ is planar.

Next, we define the *contact force* $\mathbf{n}(\xi, t)$ and the *contact moment* $\mathbf{m}(\xi, t)$ over the beam cross-sections S_ξ as follows :

$$\mathbf{n}(\xi, t) := \iint_{S_\xi} \mathbf{T}_2\, d\theta_1 d\theta_3 \quad , \tag{2.3.16}$$

$$\mathbf{m}(\xi, t) := \iint_{S_\xi} (\mathbf{r}^* - \mathbf{r}) \times \mathbf{T}_2\, d\theta_1 d\theta_3 \quad , \tag{2.3.17}$$

where $\mathbf{T}_2$ is given by (2.3.6). To give an interpretation of the contact force $\mathbf{n}(\xi, t)$ and the contact moment $\mathbf{m}(\xi, t)$, let the line of centroids c given by (2.3.10) be defined in an interval $I=[a, b]$ and let $\xi \in$ I. Then, the effect of the part of the beam which lies on the $(\xi, b]$ segment of the beam to the part which lies on the $[0,\xi]$ segment of the beam is equivalent to the contact force $\mathbf{n}(\xi, t)$ and the contact moment $\mathbf{m}(\xi, t)$ applied to the beam cross-section S_ξ.

Using (2.3.15) in (2.3.17), we obtain :

$$\mathbf{m}(\xi, t) = \iint_{S_\xi} [\theta_1 \mathbf{d}_1(\xi, t) + \theta_3 \mathbf{d}_3(\xi, t)] \times \mathbf{T}_2 \, d\theta_1 d\theta_3$$

$$= \mathbf{d}_1(\xi, t) \times \mathbf{m}_1(\xi, t) + \mathbf{d}_3(\xi, t) \times \mathbf{m}_3(\xi, t) \quad , \qquad (2.3.18)$$

where $\mathbf{m}_1$ and $\mathbf{m}_3$ are called the *contact director force* and are defined by :

$$\mathbf{m}_\alpha := \iint_{S_\xi} \theta_\alpha \mathbf{T}_2 \, d\theta_1 d\theta_3 \qquad \alpha = 1, 3 \quad . \qquad (2.3.19)$$

We derive the equations of motion for a beam by using the momentum equation (2.3.8) and the assumptions **1** and **2**.

By integrating (2.3.8) over the cross-section S_ξ we obtain the following *equation of balance of linear momentum* :

$$\frac{\partial \mathbf{n}}{\partial \xi} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{r}}{\partial t^2} \quad , \qquad (2.3.20)$$

where $\mathbf{f}$ is the resultant force per unit length at ξ, due to the surface loads and to the body forces; ρ is the mass density per unit length at ξ . They are given by :

$$\int_{\partial S_\xi} (\mathbf{T}_3\, d\theta_1 - \mathbf{T}_1\, d\theta_3) + \iint_{S_\xi} \rho^* \sqrt{g}\; \mathbf{f}^*\, d\theta_1 d\theta_3 \quad , \tag{2.3.21}$$

$$\rho = \hat{\rho}(\xi) = \int_{\partial S_\xi} \rho^* \sqrt{g}\; d\theta_1 d\theta_3 \quad , \tag{2.3.22}$$

where ∂S_ξ denotes the boundary of the cross-section S_ξ. In calculating **f**, we use the Stoke's theorem (see, e.g., [Spi.1]). We note that by the conservation of mass, ρ defined by (2.3.22) is only a function of ξ (see, e.g., [Gre.2].

By first cross-multiplying (2.3.8) with $(\mathbf{r}^* - \mathbf{r})$ on the left, then integrating over the cross-section S_ξ, using the Stoke's theorem and (2.3.9), we obtain the following equation of *balance of the angular momentum* :

$$\frac{\partial \mathbf{m}}{\partial \xi} + \frac{\partial \mathbf{r}}{\partial \xi} \times \mathbf{n} + \mathbf{l} = \sum_{\alpha=1,3} \sum_{\beta=1,3} y_{\alpha\beta} \mathbf{d}_\alpha \times \frac{\partial^2 \mathbf{d}_\beta}{\partial t^2} \quad , \tag{2.3.23}$$

where **l** is the resultant moment per unit length at ξ due to the surface loads and to the body forces; for $\alpha, \beta = 1, 3$, $y_{\alpha\beta}$ are the inertia coefficients of the beam cross-sections. They are defined as follows:

$$\mathbf{l} := \int_{\partial S_\xi} (\mathbf{r}^* - \mathbf{r}) \times (\mathbf{T}_3\, d\theta_1 - \mathbf{T}_1\, d\theta_3) + \iint_{S_\xi} (\mathbf{r}^* - \mathbf{r}) \times \rho^* \sqrt{g}\; \mathbf{f}^*\, d\theta_1 d\theta_3 \quad , \tag{2.3.24}$$

$$y_{\alpha\beta} := \iint_{S_\xi} \theta_\alpha \theta_\beta \rho^* \sqrt{g}\; d\theta_1 d\theta_3 \qquad \alpha, \beta = 1, 3 \quad , \tag{2.3.25}$$

where ∂S_ξ denotes the boundary of the cross-section S_ξ. In calculating **l**, we use the Stoke's theorem. Using (2.3.15) in (2.3.24) we obtain :

$$\mathbf{l} = \mathbf{d}_1 \times \mathbf{l}_1 + \mathbf{d}_3 \times \mathbf{l}_3 \quad , \tag{2.3.26}$$

where $\mathbf{l}_1$ and $\mathbf{l}_3$, the *assigned director forces* per unit length at ξ, are defined as :

$$\mathbf{l}_\alpha := \int_{\partial S_\xi} \theta_\alpha (\mathbf{T}_3\, d\theta_1 - \mathbf{T}_1\, d\theta_3) + \iint_{S_\xi} \theta_\alpha \rho^* \sqrt{g}\, \mathbf{f}^*\, d\theta_1 d\theta_3 \qquad \alpha = 1, 3 \quad . \tag{2.3.27}$$

By multiplying (2.3.8) with θ_α, $\alpha = 1, 3$, then integrating over the cross-section S_ξ, and using the Stoke's theorem, we obtain :

$$\frac{\partial \mathbf{m}_\alpha}{\partial \xi} + \mathbf{l}_\alpha - \mathbf{k}_\alpha = \sum_{\beta = 1,3} y_{\alpha\beta} \frac{\partial^2 \mathbf{d}_\beta}{\partial t^2} \quad , \quad \alpha = 1, 3 \quad , \tag{2.3.28}$$

where the *contact director forces* $\mathbf{m}_\alpha$ are given in (2.3.19), the *assigned director forces* $\mathbf{l}_\alpha$ are given by (2.3.27), the inertia coefficients $y_{\alpha\beta}$ are given by (2.3.25), and the *intrinsic director forces* $\mathbf{k}_\alpha$ are defined as :

$$\mathbf{k}_\alpha := \iint_{S_\xi} \mathbf{T}_\alpha\, d\theta_1 d\theta_3 \quad , \ \alpha = 1,3 \quad . \tag{2.3.29}$$

Equations (2.3.20), (2.3.23) and (2.3.28) are the basic equations of motion for a beam modeled by a Cosserat curve with two directors, as developed by Green, Laws and Naghdi [Gre.3].

We note that by using (2.3.18),(2.3.26) and (2.3.28) in (2.3.23), the latter becomes :

$$\sum_{\alpha = 1,3} \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}_\alpha + \sum_{\alpha = 1,3} \mathbf{d}_\alpha \times \mathbf{k}_\alpha + \frac{\partial \mathbf{r}}{\partial \xi} \times \mathbf{n} = 0 \quad . \tag{2.3.30}$$

Thus, (2.3.23) and (2.3.28) become equivalent to (2.3.28) and (2.3.30).

To obtain the constitutive equations, we first reduce the conservation of energy equation (2.3.4) to a line integral by integrating (2.3.4) over the cross-section S_ξ. Then, (2.3.4) can be written as :

$$\frac{d}{dt}\int_c (\psi + \frac{1}{2} < \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial t} > + \frac{1}{2} \sum_{\alpha=1,3} \sum_{\beta=1,3} < \frac{\partial \mathbf{d}_\alpha}{\partial t}, y_{\alpha\beta} \frac{\partial \mathbf{d}_\beta}{\partial t} >) d\xi$$

$$= \int_c (< \mathbf{f}, \frac{\partial \mathbf{r}}{\partial t} > + \sum_{\alpha=1,3} < \mathbf{l}_\alpha, \frac{\partial \mathbf{d}_\alpha}{\partial t} >) d\xi$$

$$+ [< \mathbf{n}, \frac{\partial \mathbf{r}}{\partial t} > + \sum_{\alpha=1,3} < \mathbf{m}_\alpha, \frac{\partial \mathbf{d}_\alpha}{\partial t} >] \Big|_{\xi = \xi_1, \xi_2} \tag{2.3.31}$$

where c is the curve of the centroids, ψ is given by :

$$\psi := \iint_{S_\xi} \psi^* \rho^* \sqrt{g}\, d\theta_1 d\theta_3 \quad , \tag{2.3.32}$$

$[\, u\,] \Big|_{a\,,b} := u(b) - u(a)$ for any function $u : \mathbf{R} \to \mathbf{R}$ and the curve of centroids is assumed to be bounded by $\xi_1 \le \xi \le \xi_2$, namely, $I = [0, L]$. Using (2.3.20) and the fact that ρ defined in (2.3.22) only depends on ξ, (2.3.31) become :

$$\frac{d\psi}{dt} = < \mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} > + \sum_{\alpha=1,3} < \mathbf{k}_\alpha, \frac{\partial \mathbf{d}_\alpha}{\partial t} > + \sum_{\alpha=1,3} < \mathbf{m}_\alpha, \frac{\partial^2 \mathbf{d}_\alpha}{\partial \xi \partial t} > \quad . \tag{2.3.33}$$

By assuming a special form on ψ, (2.3.33) gives the desired constitutive equations. In the following section, we give an example for this approach, for details see, e.g., [Gre.2], [Ant.1].

2.3.3 Geometrically Exact Beam Model

In this section, we obtain the equations of motion of the geometrically exact beam model developed by Simo and Vu-Quoc [Sim.1,2], as a special case of the director beam model presented in the previous section.

We define the curve of the centroids c and the beam cross sections S_ξ as defined by (2.3.10) and (2.3.11), respectively. For simplicity, we assume that the *unstressed configuration* of the beam, which is taken as the *reference configuration*, is such that the curve of the centroids c is a *straight line* segment in $\mathbf{R}^3$. We also regard ξ as the arc-length of c in the reference configuration. Since in this configuration c is assumed to be a straigt line, without loss of generality it may be assumed that the directors $\mathbf{d}_1$, $\mathbf{d}_3$ defined in (2.3.12) are orthonormal in the reference configuration. Let $\mathbf{D}_1$, $\mathbf{D}_3$ denote such an orthonormal pair.

The basic *kinematic assumption* in addition to Assumptions **1** and **2** given in the previous section, is the following :

Assumption 3 : the directors $\mathbf{d}_1$, $\mathbf{d}_3$, which are taken as an orthonormal pair in the reference configuration, *remain orthonormal* at all times . □

Remark : The Assumption 3 precludes any deformation in the cros-sections S_ξ. In other words, in the geometrically exact beam model, beam cross-sections can only perform rigid motions (see [Sim.1]). □

Let the beam be initially in the reference configuration and let $\mathbf{D}_1$, $\mathbf{D}_3$ be the directors in the reference configuration. By Assumption 3 it follows that there exists a rotation matrix $\Lambda(\xi, t) \in SO(3)$, such that :

$$\mathbf{d}_\alpha = \Lambda(\xi, t)\,\mathbf{D}_\alpha \qquad \alpha = 1, 3 \quad , \xi \in I \ , \ t \in \mathbf{R} \qquad , \tag{2.3.34}$$

where I is the interval in which ξ varies. For convenience we define the vectors $\mathbf{d}_2$ and $\mathbf{D}_2$ as follows :

$$\mathbf{d}_2 := \mathbf{d}_3 \times \mathbf{d}_1 \qquad , \qquad \mathbf{D}_2 := \mathbf{D}_3 \times \mathbf{D}_1 \qquad . \tag{2.3.35}$$

From (2.3.34) it follows that :

$$\begin{aligned} \mathbf{d}_2 &= (\Lambda(\xi, t)\,\mathbf{D}_3) \times (\Lambda(\xi, t)\,\mathbf{D}_1) \\ &= \Lambda(\xi, t)\,\mathbf{D}_2 \qquad . \end{aligned} \tag{2.3.36}$$

Upon differentiating $\Lambda(\xi, t)$, similar to (2.2.8), we obtain :

$$\frac{\partial}{\partial t}\Lambda(\xi, t) = W(\xi, t)\,\Lambda(\xi, t) \qquad , \tag{2.3.37}$$

where W is a 3x3 matrix. Note that, since $\Lambda(\xi, t) \in SO(3)$, hence $\Lambda\Lambda^T = I$, we have

$$\begin{aligned} W + W^T &= (\frac{\partial}{\partial t}\Lambda)\Lambda^T + \Lambda(\frac{\partial}{\partial t}\Lambda^T) \\ &= \frac{\partial}{\partial t}(\Lambda\Lambda^T) = 0 \qquad . \end{aligned}$$

Hence W is skew-symmetric. Let the parametrization of W be given as :

$$W = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \qquad , \tag{2.3.38}$$

and define the corresponding axial vector $\mathbf{w}$ as follows :

$$\mathbf{w} := \sum_{i=1}^{3} w_i \mathbf{d}_i \qquad . \tag{2.3.39}$$

It follows that $W\mathbf{w} = 0$ and

$$W\,\mathbf{u} = \mathbf{w} \times \mathbf{u} \qquad \text{for } all \ \mathbf{u} \in \mathbf{R}^3 \qquad , \tag{2.3.40}$$

and $\mathbf{w}$ is the angular velocity of the planar cross-section S_ξ.

Similarly, upon differentiating $\Lambda(\xi, t)$ with respect to ξ, we obtain :

$$\frac{\partial}{\partial \xi}\Lambda(\xi, t) = \Omega(\xi, t)\,\Lambda(\xi, t) \qquad , \tag{2.3.41}$$

where $\Omega(\xi, t)$, similar to W given in (2.3.37), is a 3x3 skew-symmetric matrix. Let the parametrization of Ω be given as :

$$\Omega(\xi, t) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad , \tag{2.3.42}$$

and define the corresponding axial vector ω as :

$$\omega := \sum_{i=1}^{3} \omega_i \mathbf{d}_i \qquad . \tag{2.3.43}$$

It follows that $\Omega\omega = 0$ and similar to (2.3.40), we have :

$$\Omega\,\mathbf{u} = \omega \times \mathbf{u} \qquad \text{for } all \ \mathbf{u} \in \mathbf{R} \qquad . \tag{2.3.44}$$

Following Simo, [Sim.1], we define the pull-back κ of ω by Λ as follows :

$$\kappa = \Lambda^T \omega \quad . \tag{2.3.45}$$

A straightforward calculation shows that κ is the axial vector of the skew-symmetric matrix K defined as :

$$K := \Lambda^T \Omega \Lambda \quad , \tag{2.3.46}$$

and a comparison of (2.3.46) and (2.3.41) yields :

$$\frac{\partial \Lambda}{\partial \xi} = \Lambda K \quad . \tag{2.3.47}$$

Using (2.3.47) and (2.3.37), we obtain the following relation between the axial vectors ω and κ :

$$\frac{\partial \kappa}{\partial t} = \Lambda^T \frac{\partial \omega}{\partial \xi} \quad , \tag{2.3.48}$$

For the sake of clarity, we note that the quantities defined, such as the contact force **n**, the contact moment **m**, etc., and the equations derived in the previous section remain the same in this section. In particular, let the contact force **n** and the contact moment **m** be defined as in (2.3.16) and (2.3.17), respectively, and let **r** be the position vector of the curve of centroids with respect to an inertial frame N. Then the basic equations of motion are given by (2.3.20) and (2.3.23). For convenience, we repeat them here :

$$\frac{\partial \mathbf{n}}{\partial \xi} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{r}}{\partial t^2} \quad , \tag{2.3.20}$$

$$\frac{\partial \mathbf{m}}{\partial \xi} + \frac{\partial \mathbf{r}}{\partial \xi} \times \mathbf{n} + \mathbf{l} = \sum_{\alpha=1,3} \sum_{\beta=1,3} y_{\alpha\beta} \mathbf{d}_\alpha \times \frac{\partial^2 \mathbf{d}_\beta}{\partial t^2} \quad , \tag{2.3.23}$$

where the resultant force **f** per unit length is given by (2.3.21), the mass per unit length ρ is given by (2.3.22), the resultant moment per unit length **l** is given by (2.3.24) and the inertia coefficients $y_{\alpha\beta}$, $\alpha, \beta = 1, 3$ are given by (2.3.25).

The energy equation (2.3.33), which is used to obtain the constitutive equations, remains valid. In director theory of beams, the *intrinsic director forces* $\mathbf{k}_1, \mathbf{k}_3$ cannot be eliminated from the energy equation (2.3.33). Hence they must appear in the constitutive equations and a set of dynamical equations involving $\mathbf{k}_1, \mathbf{k}_3$, which are (2.3.28), must be added to the equations (2.3.20) and (2.3.23). However, we show next that, in geometrically exact beam model, due to the orthonormality of the directors $\mathbf{d}_1, \mathbf{d}_3$, the form of the energy equation (2.3.33) may be changed so that $\mathbf{k}_1, \mathbf{k}_3$ no longer appear in (2.3.33). Hence one need not introduce the *intrinsic director forces* $\mathbf{k}_1, \mathbf{k}_3$ in geometrically exact beam model.

Using (2.3.30) in the energy equation (2.3.33), we obtain :

$$\frac{d\psi}{dt} = < \mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} > + \sum_{\alpha=1,3} < \mathbf{k}_\alpha, \frac{\partial \mathbf{d}_\alpha}{\partial t} > + \sum_{\alpha=1,3} < \mathbf{m}_\alpha, \frac{\partial^2 \mathbf{d}_\alpha}{\partial \xi \partial t} >$$

$$= < \mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} > + \sum_{\alpha=1,3} < \mathbf{k}_\alpha, \mathbf{w} \times \mathbf{d}_\alpha > + \sum_{\alpha=1,3} < \mathbf{m}_\alpha, \frac{\partial}{\partial \xi} (\mathbf{w} \times \mathbf{d}_\alpha) >$$

$$= < \mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} > + \sum_{\alpha=1,3} < \mathbf{w}, \mathbf{k}_\alpha \times \mathbf{d}_\alpha > + \sum_{\alpha=1,3} < \mathbf{m}_\alpha, \frac{\partial \mathbf{w}}{\partial \xi} \times \mathbf{d}_\alpha + \mathbf{w} \times \frac{\partial \mathbf{d}_\alpha}{\partial \xi} >$$

$$= < \mathbf{n} , \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} > - < \mathbf{w} , \frac{\partial \mathbf{r}}{\partial \xi} \times \mathbf{n} + \sum_{\alpha = 1,3} \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}_\alpha >$$

$$+ \sum_{\alpha=1,3} < \frac{\partial \mathbf{w}}{\partial \xi} , \mathbf{d}_\alpha \times \mathbf{m}_\alpha > + \sum_{\alpha=1,3} < \mathbf{w} , \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}_\alpha >$$

$$= < \mathbf{n} , \frac{\partial^2 \mathbf{r}}{\partial \xi \partial t} - \mathbf{w} \times \frac{\partial \mathbf{r}}{\partial \xi} > + < \mathbf{m} , \frac{\partial \mathbf{w}}{\partial \xi} > \quad , \tag{2.3.49}$$

where we also used that

$$\frac{\partial \mathbf{d}_\alpha}{\partial t} = \mathbf{w} \times \mathbf{d}_\alpha \qquad \alpha = 1, 3 \quad ,$$

and

$$< \mathbf{a} , \mathbf{b} \times \mathbf{c} > = < \mathbf{c} , \mathbf{a} \times \mathbf{b} > \qquad \text{for all } \mathbf{a} , \mathbf{b} , \mathbf{c} \in \mathbf{R}^3 \quad .$$

We define the following strain measure :

$$\Gamma = \Lambda^T \frac{\partial \mathbf{r}}{\partial \xi} - \mathbf{D}_2 \quad . \tag{2.3.50}$$

Following Simo [Sim.1], we propose the following form for the internal energy function ψ :

$$\psi = \hat{\psi}(\Gamma , \kappa) \quad , \tag{2.3.51}$$

where κ is defined in (2.3.45).

Differentiating (2.3.51) with respect to t and using (2.3.50) and (2.3.48), we obtain :

$$\frac{\partial \psi}{\partial t} = < \frac{\partial \psi}{\partial \Gamma} , \frac{\partial \Gamma}{\partial t} > + < \frac{\partial \psi}{\partial \kappa} , \frac{\partial \kappa}{\partial t} >$$

$$=<\frac{\partial\psi}{\partial\Gamma},\frac{\partial\Lambda^T}{\partial t}\frac{\partial \mathbf{r}}{\partial\xi}+\Lambda^T\frac{\partial^2\mathbf{r}}{\partial\xi\partial t}>+<\frac{\partial\psi}{\partial\kappa},\Lambda^T\frac{\partial \mathbf{w}}{\partial\xi}>$$

$$=<\Lambda\frac{\partial\psi}{\partial\Gamma},\frac{\partial^2\mathbf{r}}{\partial\xi\partial t}-\mathbf{w}\times\frac{\partial \mathbf{r}}{\partial\xi}>+<\Lambda\frac{\partial\psi}{\partial\kappa},\frac{\partial \mathbf{w}}{\partial\xi}> \quad . \tag{2.3.52}$$

Comparing the right hand side of (2.3.49) and (2.3.52), since the contact force **n** and the contact moment **m** are independent vectors, we obtain the following constitutive equations :

$$\mathbf{n}=\Lambda\frac{\partial\psi}{\partial\Gamma} \quad , \quad \mathbf{m}=\Lambda\frac{\partial\psi}{\partial\kappa} \quad . \tag{2.3.53}$$

The right hand side of (2.3.23) deserves a special attention. Since $\frac{\partial \mathbf{d}_\alpha}{\partial t}=\mathbf{w}\times\mathbf{d}_\alpha$, $\alpha=1,3$, and since the inertia coefficients $y_{\alpha\beta}$, $\alpha,\beta=1,3$ given by (2.3.25) are symmetric, we obtain :

$$\sum_{\alpha=1,3}\sum_{\beta=1,3}\mathbf{d}_\alpha\times y_{\alpha\beta}\frac{\partial^2\mathbf{d}_\beta}{\partial t^2}=\frac{\partial}{\partial t}\sum_{\alpha=1,3}\sum_{\beta=1,3}\mathbf{d}_\alpha\times y_{\alpha\beta}(\mathbf{w}\times\mathbf{d}_\beta) \quad . \tag{2.3.54}$$

Similar to the rigid body angular momentum equation, we define the beam inertia tensor I_B as follows :

$$I_B\,\mathbf{a}=\sum_{\alpha=1,3}\sum_{\beta=1,3}\mathbf{d}_\alpha\times y_{\alpha\beta}(\mathbf{a}\times\mathbf{d}_\beta)$$

$$=\left(\sum_{\alpha=1,3}\sum_{\beta=1,3}y_{\alpha\beta}[\delta_{\alpha\beta}I-\mathbf{d}_\alpha\,.\,\mathbf{d}_\beta]\right)\mathbf{a} \qquad \mathbf{a}\in\mathbf{R}^3 \quad , \tag{2.3.55}$$

where $\delta_{\alpha\beta}=1$ if $\alpha=\beta$ and 0 otherwise; the *dyadic product* (or the outer product) $\mathbf{d}_\alpha\,.\,\mathbf{d}_\beta$ is defined as $(\mathbf{d}_\alpha\,.\,\mathbf{d}_\beta)(\mathbf{a})=<\mathbf{d}_\beta,\mathbf{a}>\mathbf{d}_\alpha$ for all $\mathbf{a}\in\mathbf{R}^3$.

Differentiating (2.3.55) with respect to t, we obtain :

$$\frac{d}{dt}[\,I_B\,\mathbf{w}\,]=-\sum_{\alpha=1,3}\sum_{\beta=1,3} y_{\alpha\beta}(\,\mathbf{w}\times\mathbf{d}_\alpha\,)<\mathbf{d}_\beta\,,\mathbf{w}>+I_B\,\dot{\mathbf{w}}$$

$$=\mathbf{w}\times(\,\sum_{\alpha=1,3}\sum_{\beta=1,3} y_{\alpha\beta}[\delta_{\alpha\beta}\,I-\mathbf{d}_\alpha\,.\,\mathbf{d}_\beta]\,)\,\mathbf{w}+\,I_B\,\dot{\mathbf{w}}$$

$$=I_B\,\dot{\mathbf{w}}+\mathbf{w}\times I_B\,\mathbf{w}\quad . \tag{2.3.56}$$

Using (2.3.55) and (2.3.56) in the moment equation (2.3.23), the latter becomes :

$$\frac{\partial\mathbf{m}}{\partial\xi}+\frac{\partial\mathbf{r}}{\partial\xi}\times\,+\mathbf{l}=I_B\,\dot{\mathbf{w}}+\mathbf{w}\times I_B\,\mathbf{w}\quad . \tag{2.3.57}$$

Equations (2.3.20), (2.3.53) and (2.3.57) are the equations of motion in the geometrically exact beam model developed by Simo and Vu-Quoc [Sim.1].

Summary :

Starting from basic equations of 3-dimensional elasticity, (i.e., (2.3.8) and (2.3.9)), we obtained the basic equation of motion for an elastic beam ,(i.e., (2.3.20) and (2.3.23)). Modeling the beam as a Cosserat curve with two directors requires one more set of equations (i.e., (2.3.28)). By using the energy equation (2.3.33), once the form of the internal energy ψ is specified, the constitutive equations can also be found and then together with the equations mentioned above, these equations form a complete set of equations of motion for a beam modeled as a Cosserat curve with two directors. By constraining the motion of directors, (see Assumption 3), these equations reduces to (2.3.20), (2.3.57), and

(2.3.53), which are the equations of motion for a geometrically exact beam.

2.3.4 An Example : Planar Motion

In this section we give the equations of motion for a beam modeled as a geometrically exact beam described in the previous section. We assume that the beam is clamped at one end, free at the other and the motion is constrained to take place in a plane.

We consider the following configuration :

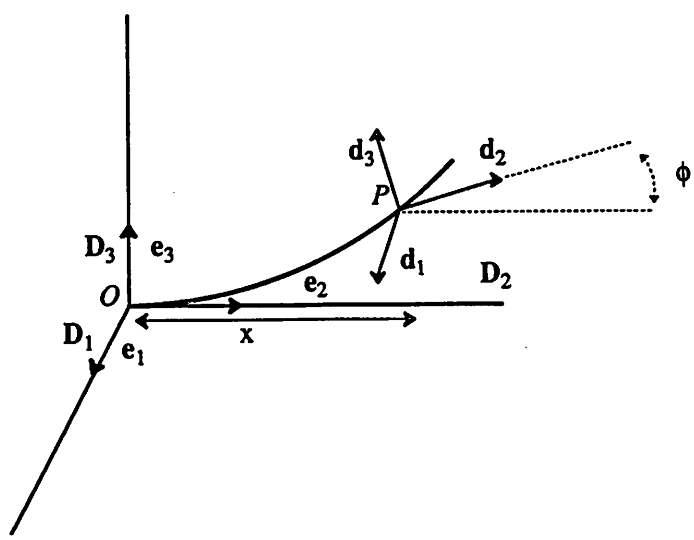

Figure 2.1 : A flexible beam

In Figure 2.1, the quadruple (O , $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) denotes the inertial frame **N**, which cincides with the frame **D** formed by the initial values of the directors, the quadruple (O , $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$). In the reference configuration, the beam is assumed to be straight along the $\mathbf{D}_2$ axis. In Figure 2.1, P is a point on the curve of the centroids and $\mathbf{r}$ is the position vector of P, the quadruple $(P, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ is the frame $\mathbf{E}$ associated with the directors at P, ϕ is the angle between the axes $\mathbf{d}_2$ and $\mathbf{D}_2$ and x is the distance between O and the initial position of P along the $\mathbf{D}_2$ axis. The motion takes place in the plane normal to the $\mathbf{D}_1$ axis.

The orthogonal transformation Λ between the frames $\mathbf{D}$ and $\mathbf{E}$ admits the following representation :

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad , \tag{2.3.58}$$

that is, we have the following :

$$\mathbf{d}_1 = \mathbf{D}_1 \quad ,$$
$$\mathbf{d}_2 = \cos\phi \, \mathbf{D}_2 + \sin\phi \, \mathbf{D}_3 \quad ,$$
$$\mathbf{d}_3 = -\sin\phi \, \mathbf{D}_2 + \cos\phi \, \mathbf{D}_3 \quad .$$

For simplicity, in the sequel we set $\xi = x$. By substituting Λ in (2.3.37), (2.3.41) and (2.3.45), we obtain :

$$\mathbf{w} = \frac{\partial\phi}{\partial t}\mathbf{d}_1 \quad , \quad \kappa = \omega = \frac{\partial\phi}{\partial x}\mathbf{d}_1 \quad . \tag{2.3.59}$$

Let u_2 and u_3 denote the deflection of the point P from the reference configuration along the axes $\mathbf{D}_2$ and $\mathbf{D}_3$, respectively. Then we have :

$$\mathbf{r} = (x + u_2)\mathbf{D}_2 + u_3 \, \mathbf{D}_3 \quad . \tag{2.3.60}$$

Let the components of the strain measure Γ defined in (2.3.50) in the frame $\mathbf{D}$ be $(0\,,\Gamma_2\,,\Gamma_3)$, i.e.,

$$\Gamma=\Gamma_2\mathbf{D}_2+\Gamma_3\mathbf{D}_3 \quad . \tag{2.3.61}$$

Then using (2.3.58), (2.3.60), (2.3.61) and (2.3.34) in (2.3.50), we obtain :

$$\Gamma_2=(1+\frac{\partial u_2}{\partial x})\cos\phi+\frac{\partial u_3}{\partial x}\sin\phi-1 \quad , \tag{2.3.62}$$

$$\Gamma_3=-(1+\frac{\partial u_2}{\partial x})\sin\phi+\frac{\partial u_3}{\partial x}\cos\phi \quad . \tag{2.3.63}$$

Using the following component form for the contact force $\mathbf{n}$ and the contact moment $\mathbf{m}$, (note that the motion takes place in the plane normal to the axis $\mathbf{e}_1=\mathbf{D}_1=\mathbf{d}_1$):

$$\mathbf{n}=n_2\mathbf{D}_2+n_3\mathbf{D}_3 \quad , \qquad \mathbf{m}=m_1\mathbf{D}_1 \quad , \tag{2.3.64}$$

and assuming that the cross-sectional inertia coefficients with different index are zero, i.e., $y_{13}=y_{31}=0$, we obtain the following component forms of the equations (2.3.20) and (2.3.23) :

$$\frac{\partial n_2}{\partial x}\cos\phi-\frac{\partial n_3}{\partial x}\sin\phi-\frac{\partial\phi}{\partial x}(n_2\sin\phi+n_3\cos\phi)+f_2=\rho\frac{\partial^2 u_2}{\partial t^2} \quad , \tag{2.3.65}$$

$$\frac{\partial n_2}{\partial x}\sin\phi+\frac{\partial n_3}{\partial x}\cos\phi+\frac{\partial\phi}{\partial x}(n_2\cos\phi-n_3\sin\phi)+f_3=\rho\frac{\partial^2 u_3}{\partial t^2} \quad , \tag{2.3.66}$$

$$\begin{aligned}&\frac{\partial m_1}{\partial x}+(1+\frac{\partial u_2}{\partial x})(n_2\sin\phi+n_3\cos\phi)\\&\qquad-\frac{\partial u_3}{\partial x}(n_2\cos\phi-n_3\sin\phi)+l_1=y_{33}\frac{\partial^2\phi}{\partial t^2} \quad ,\end{aligned} \tag{2.3.67}$$

where $\mathbf{f} = f_2\mathbf{D}_2 + f_3\mathbf{D}_3$ and $\mathbf{l} = l_1\mathbf{D}_1$ (see (2.3.20) and (2.3.23)). Finally, assuming the following quadratic form for the internal energy ψ:

$$2\,\psi = E_2\Gamma_2^{\,2} + E_3\Gamma_3^{\,2} + E_4\kappa_1^{\,2} \quad , \tag{2.3.68}$$

where E_2, E_3, E_4 are elastic coefficients, Γ_2, Γ_3 are given by (2.3.62), (2.3.63) and $\kappa_1 = \dfrac{\partial\phi}{\partial x}$; the constitutive equations (2.3.53) become :

$$n_2 = E_2\,\Gamma_2 \quad , \quad n_3 = E_3\,\Gamma_3 \quad , \quad m_1 = E_4\,\kappa_1 \quad . \tag{2.3.69}$$

As for the boundary conditions, we have the following :

$$u_2(0\,,t) = u_3(0\,,t) = 0 \quad , \quad \phi(0\,,t) = 0, \quad \text{for all } t \in \mathbf{R} \quad , \tag{2.3.70}$$

$$\mathbf{n}(L\,,t) = 0 \quad , \quad \mathbf{m}(L\,,t) = 0, \quad \text{for all } t \in \mathbf{R} \quad , \tag{2.3.71}$$

where L is the length of the undeformed beam : the equations in (2.3.70) are the clamped end conditions, whereas the equations (2.3.71) are the free end conditions.

Equations (2.3.62)-(2.3.69) are the equations of motion of a beam modeled as a geometrically exact beam performing planar motion. Together with the boundary conditions (2.3.70) and (2.3.71), they form a complete set of equations.

Special case 1 : linear inextensible beam

Let the beam be *inextensible*, i.e., $u_2 = 0$. Assuming that the deflections u_2, u_3 and ϕ are small and then neglegting the higher order terms, (2.3.62)-(2.3.69) become :

$$\Gamma_2=0 \quad , \quad \Gamma_3=\frac{\partial u_3}{\partial x}-\phi \quad , \tag{2.3.72}$$

$$\mathbf{n}=n_2\mathbf{D}_2+n_3\mathbf{D}_3 \quad , \quad \mathbf{m}=m_1\mathbf{D}_1 \quad , \tag{2.3.73}$$

$$\frac{\partial n_3}{\partial x}+f_3=\rho\frac{\partial^2 u_3}{\partial t^2} \quad , \tag{2.3.74}$$

$$\frac{\partial m_1}{\partial x}+n_3+l_1=y_{33}\frac{\partial^2 \phi}{\partial t^2} \quad , \tag{2.3.75}$$

$$n_3=E_3\Gamma_3 \quad , \quad m_1=E_4\frac{\partial \phi}{\partial x} \quad . \tag{2.3.76}$$

We note that since the beam is assumed to be inexensible, i.e., $u_2=0$, the axial component n_2 of the contact force $\mathbf{n}$ becomes indeterminable through the constitutive equations. Once the deflection u_3 and the angle ϕ are found n_2 can be found using (2.3.65).

Equations (2.3.72)-(2.3.76) are the Timoshenko beam equations [Tim.1]. By using (2.3.72), (2.3.73) and (2.3.76) in (2.3.74) and (2.3.75), the equations of motion become :

$$E_3\left(\frac{\partial^2 u_3}{\partial x^2}-\frac{\partial \phi}{\partial x}\right)=\rho\frac{\partial^2 u_3}{\partial t^2} \quad , \tag{2.3.77}$$

$$E_4\frac{\partial^2 \phi}{\partial x^2}+E_4\left(\frac{\partial u_3}{\partial x}-\phi\right)=y_{33}\frac{\partial^2 \phi}{\partial t^2} \quad , \tag{2.3.78}$$

where we assumed that the elastic coefficients E_3, E_4 do not depend on x . In the literature, $E_3=GA_3$ is called the *shear stiffness along the axis* $\mathbf{d}_3$, and $E_4=EI_1$ is called the *principal bending stiffness relative to the axis* $\mathbf{d}_1$ (see [Mei.2]).

Boundary conditions (2.3.70) and (2.3.71) now become :

$$u_3(0\,,t)=0 \quad , \quad \phi(0\,,t)=0, \quad \text{for all } t \in \mathbf{R} \quad , \tag{2.3.79}$$

$$u_{3x}(L\,,t)-\phi(L\,,t)=0 \quad , \quad \phi_{(L\,,t)}=0 \quad \text{for all } t \in \mathbf{R} \quad , \tag{2.3.80}$$

where $f_x = \dfrac{\partial f}{\partial x}$.

Special case 2 : inextensible Euler-Bernoulli beam

Assuming that the beam is inextensible and the transverse deflection u_3 is small, we can approximate the angle ϕ as $\phi = \dfrac{\partial u_3}{\partial x}$. Using this approximation in (2.3.76), we obtain :

$$m_1 = E_4 \frac{\partial^2 u_3}{\partial x^2} \quad . \tag{2.3.81}$$

Using (2.3.81) in (2.3.74) and (2.3.75), neglegting the body forces and moments (i.e. $f_2 = 0\,, l_1 = 0$), the former equation becomes :

$$E_4 \frac{\partial^4 u_3}{\partial x^4} + \rho \frac{\partial^2 u_3}{\partial t^2} = y_{33} \frac{\partial^4 u_3}{\partial x^2 \partial t^2} \quad . \tag{2.3.82}$$

This is the Euler-Bernoulli beam equation with the rotatory inertia. Neglegting the rotatory inertia, i.e., setting $y_{33} = 0$, yields the Euler-Bernoulli beam equation [Mei.2] :

$$E_4 \frac{\partial^4 u_3}{\partial x^4} + \rho \frac{\partial^2 u_3}{\partial t^2} = 0 \quad , \tag{2.3.83}$$

where we assumed that the principal bending stiffness $E_4 = EI_1$ does not depend on x.

The boundary conditions (2.3.70) and (2.3.71) now become :

$$u_3(0,t)=0 \quad , \quad u_{3x}(0,t)=0, \qquad \text{for all } t \in \mathbf{R} \quad , \tag{2.3.84}$$

$$u_{3xx}(L,t)=0 \quad , \quad u_{3xxx}(L,t)=0, \qquad \text{for all } t \in \mathbf{R} \quad . \tag{2.3.85}$$

2.4 An Elastic Beam Clamped to a Rigid Body

In this section we consider the motion of an elastic beam clamped at one end to a rigid body and free at the other end. We derive the equations of motion by considering free body diagrams.

We consider the following configuration :

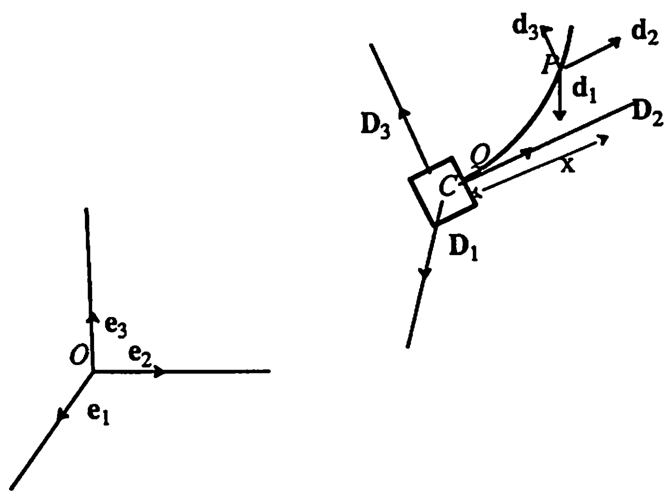

Figure 2.2 : Rigid body with a flexible beam

In Figure 2.2, the quadruple $(O\ ,\mathbf{e}_1\,,\mathbf{e}_2\,,\mathbf{e}_3)$ denotes an *inertial frame* **N**, the box and the curved line represent the rigid body B_R and the beam in the deformed configuration, respectively, C is the center of mass of the rigid body, the quadruple $(C\ ,\mathbf{D}_1\,,\mathbf{D}_2\,,\mathbf{D}_3)$ is the *body frame* **B** whose axes are also the principal axes of inertia for the rigid body, Q is the point at which the beam is clamped to the rigid body, P is a material point of the beam, the quadruple $(P\ ,\mathbf{d}_1\,,\mathbf{d}_2\,,\mathbf{d}_3)$ denotes the *frame of directors* at P. The reference configuration of the beam, which is also assumed to be the initial configuration of the beam, is a straight line along the $\mathbf{D}_2$ axis, x is the distance between the point Q and the point P in the reference configuration: hence $x = 0$ specifies the point Q.

Let $I_R = diag\ (I_1, I_2, I_3)$ be the inertia tensor for the rigid body and ω_R be the angular velocity of the body frame **B** with respect to the inertial frame **N**. Then the Euler equation (2.2.22) for the rigid body becomes :

$$I_R\ \dot{\omega}_R + \omega_R \times I_R \omega_R = \mathbf{r}\,(0\,,t) \times \mathbf{n}\,(0\,,t) + \mathbf{m}\,(0\,,t) + \mathbf{N}_c\,(t) \quad , \tag{2.4.1}$$

where $\mathbf{r}\,(x\ ,t) = \mathbf{CP}$, i.e. the position vector of the material point P in the body frame B, $\mathbf{r}\,(0\,,t) = \mathbf{CQ}$, $\mathbf{N}_c\ (t)$ is the control torque applied to the rigid body, $\mathbf{n}\,(0\,,t)$ and $\mathbf{m}\,(0\,,t)$ denote the contact force and the contact moment of the beam at Q, respectively. The first two terms on the right hand side of the equation (2.4.1) represent the torque applied by the beam to the rigid body.

The balance of linear momentum, (2.2.2), applied to the rigid body becomes :

$$M\ddot{\mathbf{R}} = \mathbf{n}\,(0\,,t) + \mathbf{f}_{gr} + \mathbf{F}\,(t) \quad , \tag{2.4.2}$$

where M is the mass of the rigid body, $\mathbf{R} = \mathbf{OC}$, (i.e., the position vector of the center of mass of the rigid body C in the inertial frame N), $\mathbf{f}_{gr}$ is the force due to the gravity acting on the rigid body B_R due to some mass located at O and $\mathbf{F}\,(t)$ is the control force applied to the rigid body.

Assuming that the beam is modeled as a geometrically exact beam, the beam equations (2.3.20) and (2.3.57) become :

$$\frac{\partial \mathbf{n}}{\partial \xi} + \mathbf{f} = \rho \left(\frac{\partial^2 (\mathbf{R}+\mathbf{r})}{\partial t^2}\right)_N$$

$$= \rho \left(\frac{\partial^2 \mathbf{R}}{\partial t^2}\right)_N + \rho\, \dot{\omega}_R \times \mathbf{r} + 2\,\rho\, \omega_R \times \left(\frac{\partial \mathbf{r}}{\partial t}\right)_B + \rho\, \omega_R \times (\omega_R \times \mathbf{r}) \quad , \tag{2.4.3}$$

$$\frac{\partial \mathbf{m}}{\partial \xi} + \frac{\partial \mathbf{r}}{\partial \xi} \times + \mathbf{l} = I_B \left(\frac{\partial (\mathbf{w}+\omega_R)}{\partial t}\right)_N + (\mathbf{w}+\omega_R) \times I_B (\mathbf{w}+\omega_R) \quad . \tag{2.4.4}$$

The constitutive equations for the beam retain their form given by (2.3.53) :

$$\mathbf{n} = \Lambda \frac{\partial \psi}{\partial \Gamma} \quad , \quad \mathbf{m} = \Lambda \frac{\partial \psi}{\partial \kappa} \quad , \tag{2.3.53}$$

where Λ is the orthogonal transformation between the body axes $\mathbf{D}_i$, $i = 1, 2, 3$ and the directors $\mathbf{d}_i$, $i = 1, 2, 3$, ψ is the internal energy function for the beam, the strain measure vectors Γ and κ are defined by (2.3.50) and (2.3.45), respectively.

The boundary conditions for the beam are :

$$\mathbf{r}(0, t) = \mathbf{CQ} \quad , \quad \Gamma(0, t) = I, \quad \text{for all } t \in \mathbf{R}^3 \quad , \tag{2.4.5}$$

$$\mathbf{n}(L, t) = 0 \quad , \quad \mathbf{m}(L, t) = 0, \quad \text{for all } t \in \mathbf{R}^3 \quad , \tag{2.4.6}$$

where L is the length of the beam in the undeformed configuration. As before, the equations (2.4.5) are the clamped end conditions and the equations (2.4.6) are the free end conditions.

Equations (2.4.1)-(2.4.6) together with (2.3.37), (2.3.41), (2.3.45), (2.3.50) and (2.3.53) form a complete set of equations that describe the motion of a flexible beam-rigid body configuration described in this section.

Chapter 3

Control of a Flexible Beam Attached to a Rigid Body : Motion in Plane

3.1 Introduction

In this chapter, we study a special case of the rigid body-clamped beam configuration introduced in the Section 2.4. We assume that the center of mass of the rigid body is fixed in an inertial frame and the whole configuration performs planar motion. The first assumption can be justified if one considers a satellite which consist of a rigid body and a flexible beam attached to it in a geosynchronous orbit; neglecting the effect of the rotation of the Earth, the center of mass of the rigid body is fixed with respect to the Earth. The second assumption simlifies the analysis presented in this chapter, but the results obtained in this chapter will be extended to the 3-dimensional motion in the Chapter 4.

In Section 2, using the Euler-Bernoulli beam model introduced in the previous chapter, (see (2.3.83)), we obtain the equations of motion and define the rest state of the system. Then we pose our control problem which is: if the system is perturbed from the rest state, find a control law which guarantees that the system is driven to the rest state.

In Section 3 we propose 2 control laws and in the remaining sections we prove that these control laws solve the control problem posed in Section 2. Moreover, we prove that the decay to the rest state is exponential.

3.2 Equations of Motion

We consider the configuration shown in the Figure 3.1 :

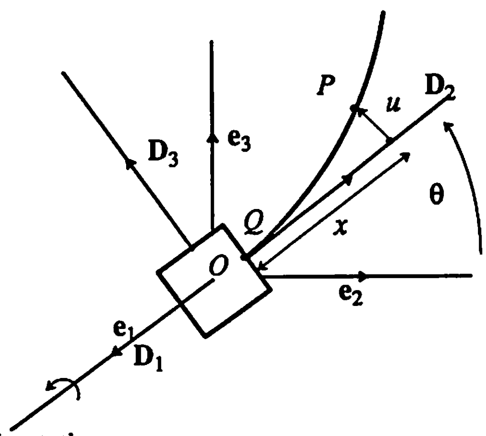

Figure 3.1 : Rigid body with flexible beam : planar case

In Figure 3.1, the quadruple $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes a dextral orthonormal *inertial* frame, which will be referred to as N, $(O, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$ denotes a dextral orthonormal frame fixed in the rigid body, which will be referred as B, where O is also the center of mass of the rigid body and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ are along the principal axes of inertia of the rigid body. The beam is clamped to the rigid body at the point Q at one end along the $\mathbf{D}_2$ axis and is free at the other end. We assume that the rigid body may rotate only about the $\mathbf{e}_1$ axis and that at all times the axes $\mathbf{e}_1$ and $\mathbf{D}_1$ coincide. Let L be the length of the beam. We assume that the mass of the rigid body is much larger than the mass of the beam, so the center of mass of the rigid body is approximately the center of mass of the whole configuration. So we assume that the point O is fixed in the inertial space throughout the motion of the whole configuration. We also assume that the beam is inextensible, (i.e., no deformation along the axis $\mathbf{D}_2$), and homogeneous with uniform cross-sections.

The beam is initially straight along the $\mathbf{D}_2$ axis. This initial configuration for the beam is also referred to as the reference configuration for the beam. Let P be a beam element whose distance from the point Q in the reference configuration is x, let u be the displacement of P along the $\mathbf{D}_3$ axis. Let $\mathbf{r}(x, t) = \mathbf{OP}$ be the position vector of P.

Neglecting gravitation, surface loads and the rotatory inertia of the beam cross-sections, the equations of motion (2.4.1), (2.4.3) and (2.4.4) now reduce to :

$$\frac{\partial \mathbf{n}}{\partial x} = \rho \left(\frac{\partial^2 \mathbf{r}}{\partial t^2}\right)_N \quad , \tag{3.2.1}$$

$$\frac{\partial \mathbf{m}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x} \times \mathbf{n} = 0 \quad , \tag{3.2.2}$$

$$I_R \, \dot{\omega}_R + \omega_R \times I_R \, \omega_R = \mathbf{r}(0, t) \times \mathbf{n}(0, t) + \mathbf{m}(0, t) + \mathbf{N}_c(t) \quad , \tag{3.2.3}$$

where $\mathbf{n}$ and $\mathbf{m}$ denote the contact force and the contact moment of the beam, respectively, ρ is the mass per unit length of the beam, which is a constant by assumption, I_R is the inertia tensor of the beam, which is diagonal, ω_R is the angular velocity of the body frame $\mathbf{B}$ with respect to the inertial frame $\mathbf{N}$, the vector $\mathbf{N}_c(t)$ is the control torque applied to the rigid body, $\mathbf{r}(x, t) = \mathbf{OP}$, i.e., the position vector of the material point P in the body frame $\mathbf{B}$, and at the clamped end we have $\mathbf{r}(0, t) = \mathbf{OQ}$. The first two terms on the right hand side of the equation (3.2.3) represent the moment effect of the beam to the rigid body.

Since the motion takes place in the plane normal to the $\mathbf{D}_1$ axis, we have the following com ponent forms of the contact force **n** , the contact moment **m** and the position vector **r** of the point P :

$$\mathbf{n} = n_2\, \mathbf{D}_2 + n_3\, \mathbf{D}_3 \quad , \quad \mathbf{m}_1 = m_1\, \mathbf{D}_1 \quad , \tag{3.2.4}$$

$$n_3 = -EI\, \frac{\partial^3 u}{\partial x^3} \quad , \quad m_1 = EI\, \frac{\partial^2 u}{\partial x^2} \quad , \tag{3.2.5}$$

$$\mathbf{r} = (b + x)\, \mathbf{D}_2 + u\, \mathbf{D}_3 \quad , \tag{3.2.6}$$

where EI is the flexural rigidity of the beam, $b = |\, OQ\, |$. Note that in (3.2.5) we assumed the Euler-Bernoulli beam model for the beam.

Since the beam is assumed to be inextensible, the axial deflection is identically zero, hence the axial component n_2 of the contact force **n** becomes indeterminable through the constitutive equations, (see [Pos.1]). The axial component of the equation (3.2.1) then determines n_2. Since this equation plays no role in the sequel, the axial component of (3.2.1) will be omitted.

Using (3.2.4)-(3.2.6) in (3.2.1)-(3.2.3), we obtain the following component forms of the equation of motion :

$$\rho\, \frac{\partial^2 u}{\partial t^2} + EI\, \frac{\partial^4 u}{\partial x^4} + \rho\, \dot{\omega}\, (b + x) - \rho\, \omega^2\, u = 0 \quad , \tag{3.2.7}$$

$$I_R\, \dot{\omega} = EI\, (-b\ u_{xxx}(0,t) + u_{xx}\, (0,t)) + N_c\, (t) \quad , \tag{3.2.8}$$

$$u\, (0\, , t) = 0 \quad , \quad u_x\, (0\, , t) = 0, \quad \text{for all } t \in \mathbf{R} \quad , \tag{3.2.9}$$

where we used $\boldsymbol{\omega} = \omega\, \mathbf{D}_1$ and $\mathbf{N}_c\, (t) = N_c\, (t)\, \mathbf{D}_1$. Equation (3.2.7) is the component of the equation (3.2.1) along the $\mathbf{D}_3$ axis, and the equations (3.2.9) are the the boundary conditions at the clamped end.

The rest state of the system given by (3.2.7) and (3.2.8) is defined as follows :

$$\left.\begin{array}{ll} \omega=0 & \\ u(x)=0 & 0\le x\le L \\ u_t(x)=0 & 0\le x\le L \end{array}\right\} \quad . \tag{3.2.10}$$

Our control problem is to find an appropriate control law $N_c\ (t)$, control force and control torque at the free end of the beam such that if the system given by (3.2.7)-(3.2.9) is perturbed from the rest state, the control law will drive the system to the rest state.

3.3 Stabilizing Control Laws

In this section we propose two stabilizing control laws to solve the control problem posed in the previous section. Each law consist of appropriate control force and torque applied at the free end of the beam and a control torque applied to the rigid body.

3.3.1 Control Law Based on Cancellation

This control law applies a force $\mathbf{n}(L, t)$ and a torque $\mathbf{m}(L, t)$ at the free end of the beam and a torque $\mathbf{N}_c(t)$ to the rigid body. They are specified as follows : we choose $\alpha > 0$, $\beta > 0$ and $k > 0$; then for all $t \geq 0$, we require the following equations:

$$-EI\, u_{xxx}(L, t) + \alpha\, u_t(L, t) = 0 \quad , \tag{3.3.1}$$

$$EI\, u_{xx}(L, t) + \beta\, u_{xt}(L, t) = 0 \quad , \tag{3.3.2}$$

$$N_c(t) = EI\,(b\, u_{xxx}(0, t) - u_{xx}(0, t)) - k\,\omega \quad , \tag{3.3.3}$$

where $\mathbf{m}(L, t) = EIu_{xx}(L, t)\mathbf{D}_1$, and $\mathbf{N}_c(t) = N_c(t)\mathbf{D}_1$. Because of the boundary condition (2.4.6) at the free end, $n_2(L, t) = 0$ and since $n_3 = -EI\, u_{xxx}$, (see (2.3.75) and (2.3.81)), it follows that $\mathbf{n}(L, t) = -EI\, u_{xxx}(L, t)\,\mathbf{D}_3$

Equation (3.3.1), {(3.3.2), resp.,} represents a transversal force, {torque, resp.,} applied at the free end of the beam in the direction of, {around, resp.,} the axis $\mathbf{D}_3$, {the axis $\mathbf{D}_1$, resp.,} whose magnitude is proportional to and whose sign is opposite to the end point deflection velocity $u_t(L,t)$, {end-point deflection angular velocity $u_{xt}(L,t)$, resp.,} of the beam. To apply the control laws given by (3.3.1)-(3.3.3), the end point deflection velocity $u_t\ (L, t)$, the end point deflection angular velocity $u_x\ (L, t)$, the rigid body angular velocity $\omega(t)$ and the moment applied by the beam to the rigid body must be measured. This moment consist of the effect of the contact force $\mathbf{n}(0, t)$ and the contact moment $\mathbf{m}(0, t)$ at the clamped end. Both can be measured by using strain rosettes and strain gauges, respectively, [Ana.1].

The control law (3.3.3) is reminiscent of a " computed torque " type control law in Robotics , [Pau.1]. When substituted in (3.2.8), it cancels the effect of the beam on the rigid body. This type of control law has been applied to the attitute control of a satellite with a flexible beam clamped to it [Ana.1].

3.3.2 Natural Control Law

This control law applies to the free end of the beam the same boundary force $\mathbf{n}(L\ , t)$ and the boundary torque $\mathbf{m}(L\ , t)$ as specified by the equations (3.3.1) and (3.3.2) , respectively; but the torque applied to the rigid body is now given by :

$$\mathrm{N}_c(t) = -\,\mathbf{r}(L\ , t) \times \mathbf{n}(L\ , t) - \mathbf{m}(L\ , t) - k\ \omega(t) \quad , \tag{3.3.4}$$

or equivalently in component form :

$$N_c(t) = EI\ (b + L)\, u_{xxx}(L\ , t) - EI\ u_{xx}(L\ , t) - k\,\omega(t) \quad . \tag{3.3.5}$$

This control law is "natural" in the sense that it enables one to choose the total energy of the whole configuration as a Lyapunov function to study the stability of the system.

The control law given by (3.3.1), (3.3.2) and (3.3.5) requires that the end point deflection velocity $u_t(L\ ,t)$, the end point deflection angular velocity $u_{xt}(L\ ,t)$ and the rigid body angular velocity $\omega(t)$ be measured. The first two could be measured by optical means and the latter by the gyros.

In the following sections, we show that the two control laws proposed in section 3.4.1 and Section 3.4.2 stabilize the system given by (3.2.1)- (3.2.3), i.e., when this system is perturbed from its rest state given by (3.2.10), these control laws drive the system to the rest state.

3.4 Stability Results for the Control Law Based on Cancellation

Consider the system given by the equations (3.2.7)-(3.2.8) and the control law given by the equations (3.3.1)-(3.3.3). After substituting (3.3.3) in (3.2.8), we get :

$$I_R\,\dot{\omega} + k\,\omega = 0 \qquad \text{for all } t \in \mathbf{R}_+ \quad . \tag{3.4.1}$$

Therefore we have the following solution for $\omega(t)$:

$$\omega(t) = \omega(0)\, e^{(\frac{-k}{I_R})t} \qquad \text{for all } t \in \mathbf{R}_+ \quad . \tag{3.4.2}$$

The remaining equation (3.2.7) can be put into the following state space form :

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{EI}{\rho}\dfrac{\partial^4}{\partial x^4} & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \omega^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\dot{\omega}\,(b+x) \end{bmatrix} \quad , \tag{3.4.3}$$

where $\omega(t)$ is given by (3.4.2). We define the following function space H in which the solutions of (3.4.3) evolve :

$$H := \{\, (u \;\; u_t)^T \mid u \in \mathrm{H}_0^{\,2},\, u_t \in \mathrm{L}^2 \}. \quad , \tag{3.4.4}$$

where the function spaces L^2, H^k and $\mathrm{H}^k{}_0$ are as defined below :

$$\mathrm{L}^2 := \{ f : [0, L] \rightarrow \mathbf{R} \mid \int_{x=0}^{x=L} f^2\, dx < \infty \} \quad , \tag{3.4.5}$$

$$\mathrm{H}^k := \{ f \in \mathrm{L}^2 \mid f^i \in \mathrm{L}^2 \,,\, i=1,\ldots,k \} \quad , \tag{3.4.6}$$

$$\mathbf{H}^k{}_0 := \{ f \in \mathbf{H}^k \mid f(0) = f^1(0) = 0 \} \quad . \tag{3.4.7}$$

In H, we define the following inner product, which is called "energy" inner product

$$< z, \hat{z} >_E := \int_{x=0}^{x=L} EI \; u_{xx} \; \hat{u}_{xx} \; + \int_{x=0}^{x=L} \rho \; u_t \; \hat{u}_t \; dx, \qquad \text{for all } z, \hat{z} \in \mathrm{H}. \tag{3.4.8}$$

where $z = (u \;\; u_t)^T$, $\hat{z} = (\hat{u} \;\; \hat{u}_t)^T$.

Note that, (3.4.8) induces a norm on H, which is called "energy norm". This norm is equivalent to a standard "Sobolev" type norm which makes H an Hilbert space (for more details, see [Paz.1] and [Che.2]).

To put (3.4.3) into an abstract equation form, (see (3.4.12) below), we define the following operators $A : H \to H$, $B : \mathrm{R}_+ \mathrm{x}\, H \to H$ and function $f : \mathrm{R}_+ \to H$,

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{EI}{\rho}\dfrac{\partial^4}{\partial x^4} & 0 \end{bmatrix} \quad , \tag{3.4.9}$$

$$B(t) = \begin{bmatrix} 0 & 0 \\ \omega^2(t) & 0 \end{bmatrix} \quad , \tag{3.4.10}$$

$$f(t) = \begin{bmatrix} 0 \\ -\dot{\omega}(t)\,(b + x) \end{bmatrix} \quad . \tag{3.4.11}$$

where the dependence on x is supressed as is usually done in abstract formulation.

3.4.1 Remark : The operator A is an unbounded linear operator on H, i.e., it is not continuous as a map on H. The operator $B(.)$ is bounded on R_+. Since ω(t) and $\dot{\omega}$(t) are exponentially decaying functions of t, (see (3.4.2)), so is $||B(t)||$, where the norm used here is the norm induced by the energy inner product given by (3.4.8). □

Using the above definitions, (3.4.3) can be put into the following abstract form :

$$\frac{dz}{dt} = A\,z + B(t)\,z + f(t) \qquad z(0) = z_0 \in H \quad , \tag{3.4.12}$$

where $z = (u \;\; u_t)^T$. The domain $D(A)$ of the operator A is defined as follows :

$$D(A) = \{(u \;\; u_t)^T : u \in \mathrm{H}_0^4, u_t \in \mathrm{H}_0^2, -EI\; u_{xxx}(L) + \alpha\, u_t(L) = 0, \\ EI\; u_{xx}(L) + \beta\, u_{xt}(L) = 0\,\} \quad , \tag{3.4.13}$$

where $\alpha > 0$ and $\beta > 0$.

It is well known that the operator $A : D(A) \subset H \to H$ defined in (3.4.9), with its domain $D(A)$ and H defined as in (3.4.13) and (3.4.4), respectively, generates an *exponentially decaying semigroup* $T(t)$, (see [Che.1, Thm.3.1]). That is, the solutions of

$$\dot{z} = A\,z \quad , \quad z(0) = z_0 \in D(A) \quad , \tag{3.4.14}$$

which are equivalent to the following equation and boundary conditions : for all $t \in \mathbf{R}_+$

$$\rho\, u_{tt} + EI\; u_{xxxx} = 0\,, \qquad x \in (0\,,L) \quad , \tag{3.4.14.1}$$

$$u\,(0\,,t) = 0 \quad , \quad u_x\,(0\,,t) = 0 \quad , \tag{3.4.14.2}$$

$$-EI\; u_{xxx}(L) + \alpha\, u_t(L) = 0 \quad , \tag{3.4.14.3}$$

$$EI\; u_{xx}(L) + \beta\, u_{xt}(L) = 0 \quad , \tag{3.4.14.4}$$

are given by $z(t) = T(t)z_0$ and there exist positive constants $M > 0$ and $\delta > 0$ such that :

$$||T(t)|| \le M e^{-\delta t} \quad , \tag{3.4.15}$$

where the norm is the norm induced by the energy inner product defined by (3.4.8).

We give some definitions and results from the semigroup theory which will be frequently used in the sequel. For details, the reader is referred to, e.g., [Paz.1], [Gol.2].

3.4.2 Definition : (*Semigroup, strongly continuous semigroup*)

Let X be a Banach space. A one parameter family $T(t): X \to X$, $0 \le t < \infty$, of bounded linear operators is a *semigroup of bounded linear operators on* X if :

(i) : $T(0) = I$, (I is the identity operator on X),

(ii) : $T(t+s) = T(t)\,T(s)$ for every $t, s \ge 0$.

A semigroup $T(t)$, $0 \le t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup of bounded linear operators if :

$$\lim_{t \to 0} T(t)x = x \qquad \text{for every} \quad x \in X \quad .$$

A strongly continuous semigroup of bounded linear operators on X will be called a C_0 semigroup. Let $T(t)$ be a C_0 semigroup. If $||T(t)|| \le 1$, then $T(t)$ is called a C_0 *semigroup of contractions*. □

3.4.3 Theorem (Hille-Yosida). A linear operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$ if and only if :

(i) A is closed and $D(A)$, the domain of A, is dense in X,

(ii) the resolvent set $\rho(A)$ of A contains R_+ and for every $\lambda > 0$,

$$||(\lambda I - A)^{-1}|| \le \frac{1}{\lambda} \quad .$$

Proof : See [Paz.1, p.8] . □

3.4.4 Theorem (Lumer-Phillips). Let A be a linear operator with a dense domain $D(A)$ in a *Hilbert* space X . If A is dissipative, i.e., $<x\ , Ax> \le 0$ for all $x \in D(A)$, and there is a $\lambda_0 > 0$ such that the range of the operator $(\lambda_0 I - A)$ is X, then A is the infinitesimal generator of a C_0 semigroups of contractions on X.

Proof : See [Paz.1, p.14]. □

3.4.5 Theorem (Pazy). Let A be the infinitesimal generator of a C_0 semigroup $T(t)$. If for some p, $1 \le p < \infty$

$$\int_0^\infty ||T(t)x||^p \, dt < \infty \qquad \text{for all } x \in X \quad ,$$

then there are constants $M \ge 1$ and $\delta > 0$ such that

$$||T(t)|| \le M e^{-\delta t} \quad .$$

Proof : See [Paz.1, p.116]. □

Consider the equation (3.4.14). Next we prove that the operator A defined by (3.4.9) generates an exponentially decaying C_0 semigroup, this result was first given in [Che.1].

3.4.6 Lemma. Consider the spaces $D(A)$ defined by (3.4.13) and H defined by (3.4.4) with the inner product given by (3.4.8). Then, $D(A)$ is dense in H.

Proof : Suppose not.Then $\overline{D(A)}$ is a proper subspace of H, and since H is a Hilbert space and $\overline{D(A)}$ is closed, there exists a $z \neq 0$, $z \in H$ such that $<z, x> = 0$ for all $x \in \overline{D(A)}$. Let $z = (z_1 \ z_2)^T$ Then for all $x = (x_1 \ x_2)^T \in D(A)$, the following holds :

$$\int_0^L EI \, z_{1xx} \, x_{1xx} \, dx + \int_0^L \rho \, z_2 x_2 \, dx = 0 \quad . \tag{3.4.16}$$

Choosing $x_2 = 0$ and noting that the class of C^∞ functions are dense in $D(A)$, it follows that $z_{1xx} = 0$ almost everywhere in $[0, L]$. By the boundary condition $z_1(0) = 0$ it follows that $z_1 = 0$ almost everywhere on $[0, L]$.

Similarly putting $x_1 = 0$ in (3.4.16) and repeating the same argument, we conclude that $z_2 = 0$ almost everywhere in $[0, L]$. Hence $z = (z_1 \ z_2)^T = 0$, which is a contradiction.

□

3.4.7 Theorem : (Existence, Uniqueness). Consider the abstract differential equation (3.4.14), where the operator A, its domain $D(A)$, and the space H are given by (3.4.9), (3.4.13), and (3.4.4), respectively. Then the operator A generates a C_0 semigroup $T(t)$ on H, (i.e., existence and uniqueness of the solutions of (3.4.14)).

Proof : We use the Lumer-Phillips theorem (e.g. theorem 3.4.4) to prove theorem 3.4.7. Note that by Lemma 3.4.6, A is a densely defined operator on H. So, to prove the theorem, we need to show that :

(a) A is dissipative, i.e.,

$$<z, Az> \leq 0 \qquad \text{for all } z \in D(A) \quad , \tag{3.4.17}$$

(b) for some $\lambda > 0$, the range of the operator $(\lambda I - A)^{-1}$ is H. That is, there exist a $\lambda > 0$ such that for any $y \in H$, the following equation has a solution $z \in D(A)$:

$$(\lambda I - A)x = y \quad . \tag{3.4.18}$$

To prove the assertion (a), we first note that since $\dot{z} = Az$, showing that (3.4.18) holds is equivalent to showing the following :

$$\frac{d}{dt} <z, z> = \frac{d}{dt}(||z||^2) \leq 0 \quad ,$$

where the norm is the norm induced by the energy inner product (3.4.8).

Consider the system defined by (3.4.14); call $E_1(t)$ its energy, then :

$$E_1(t) = \frac{1}{2} <z, z>$$

$$= \frac{1}{2} \int_0^L EI \, u_{xx}^2 \, dx + \int_0^L \rho \, u_t^2 \, dx \quad , \tag{3.4.19}$$

where $z = (u \;\; u_t)^T$. Upon differentiating (3.4.19) with respect to time t and noting that the coefficients EI and ρ are constants, we obtain :

$$\frac{dE_1(t)}{dt} = \int_0^L EI \, u_{xx} \, u_{xxt} \, dx + \int_0^L \rho \, u_t \, u_{tt} \, dx$$

$$= \int_0^L EI \, u_{xx} \, u_{xxt} \, dx - \int_0^L EI \, u_t \, u_{xxxx} \, dx$$

$$= \int_0^L EI \, u_{xx} \, u_{xxt} \, dx - EI \, u_t(L, t) \, u_{xxx}(L, t) + \int_0^L EI \, u_{xt} \, u_{xxx} \, dx$$

$$= \int_0^L EI\, u_{xx}\, u_{xxt}\, dx - EI\, u_t(L,t)\, u_{xxx}(L,t) + EI\, u_{xt}(L,t)\, u_{xx}(L,t) - \int_0^L EI\, u_{xx}\, u_{xxt}\, dx$$

$$= -\alpha u_t^{\,2}(L,t) - \beta u_{xt}^{\,2}(L,t) \quad , \tag{3.4.20}$$

where, in the second equation we used (3.4.14.1), in the third and fourth equations we used integration by parts and the boundary conditions (3.4.14.2). Then, by using the boundary controls (3.4.14.3), (3.4.14.4) we obtain (3.4.20), which proves that the operator A defined by (3.4.9) is dissipative in H.

To prove assertion (b), i.e., that the range of the operator $(\lambda I - A)$ is H for some $\lambda > 0$, we show that for any $y \in H$ there exists a $z \in D(A)$ such that the equation (3.4.18) holds, i.e. we have the following :

$$(\lambda I - A) z = y \quad .$$

Let $y = (f \quad g)^T \in H$ be given. We put $z = (u \quad w)^T$. Using (3.4.9) and without loss of generality putting $\frac{EI}{\rho} = 1$, (3.4.18) becomes equivalent to the following equations :

$$\lambda u - w = f \qquad f(0) = f_x(0) = 0 \quad , \tag{3.4.21}$$

$$u_{xxxx} + \lambda w = g \quad , \tag{3.4.22}$$

$$u(0) = u_x(0) = 0 \quad , \qquad w(0) = w_x(0) = 0 \quad , \tag{3.4.23}$$

$$-u_{xxx}(L) + \lambda w(L) = 0 \quad , \tag{3.4.24}$$

$$u_{xx}(L) + \beta w_x(L) = 0 \quad , \tag{3.4.25}$$

where (3.4.23)-(3.4.25) are the boundary conditions.

By using (3.4.21) in (3.4.22), we get :

$$u_{xxxx} + \lambda^2 u = \lambda f + g \quad . \tag{3.4.26}$$

Putting $\lambda^2 = -\tau^4$, a general solution of (3.4.26) which also satisfies the first two boundary conditions in (3.4.23), (i.e., $u(0) = u_x(0) = 0$), is given as :

$$u(x,\tau) = C_1 (\cosh\tau x - \cos\tau x) + C_2 (\sinh\tau x - \sin\tau x)$$

$$+ \frac{1}{2\tau^3} \int_0^x [\, \sinh\tau(x-\sigma) - \sin\tau(x-\sigma) \,][\, \lambda f(\sigma) + g(\sigma) \,]\, d\sigma \quad , \tag{3.4.27}$$

where C_1 and C_2 are the constants of integration which will be determined by the remaining boundary conditions (3.4.24) and (3.4.25), sinh (.), cosh (.) : $\mathbf{C} \to \mathbf{C}$ are the hyperbolic sine and the hyperbolic cosine functions, respectively.

Using (3.4.27) and (3.4.21) in the boundary conditions (3.4.24) and (3.4.25), we obtain the following matrix equation :

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad , \tag{3.4.28}$$

where

$$a_{11} = \tau^3(\sinh\tau L - \sin\tau L) + i\,\alpha\tau^2(\cosh\tau L - \cos\tau L) \quad , \tag{3.4.29}$$

$$a_{12} = \tau^3(\cosh\tau L + \cos\tau L) + i\,\alpha\tau^2(\sinh\tau L - \sin\tau L) \quad , \tag{3.4.30}$$

$$a_{21} = \tau^2(\cosh\tau L + \cos\tau L) - i\beta\tau^3(\sinh\tau L + \sin\tau L) \quad , \tag{3.4.31}$$

$$a_{22} = \tau^2(\sinh\tau L + \sin\tau L) - i\beta\tau^3(\cosh\tau L - \cos\tau L) \quad , \tag{3.4.32}$$

$$f_1 = \frac{-1}{2}\int_0^L [\,\cosh\tau(L-\sigma) + \cos\tau(L-\sigma)\,][\,-i\,\tau^2 f(\sigma) + g(\sigma)\,]\,d\sigma$$

$$\frac{-i\alpha}{2\tau}\int_0^L [\,\sinh\tau(L-\sigma) - \sin\tau(L-\sigma)\,][\,-i\,\tau^2 f(\sigma) + g(\sigma)\,]\,d\sigma \;-\alpha f(L) \quad , \tag{3.4.33}$$

$$f_2 = \frac{-1}{2\tau}\int_0^L [\,\sinh\tau(L-\sigma) + \sin\tau(L-\sigma)\,][\,-i\,\tau^2 f(\sigma) + g(\sigma)\,]\,d\sigma$$

$$\frac{+i\beta}{2}\int_0^L [\,\cosh\tau(L-\sigma) - \cos\tau(L-\sigma)\,][\,-i\,\tau^2 f(\sigma) + g(\sigma)\,]\,d\sigma \;+\beta f_x(L) \quad , \tag{3.4.34}$$

where i is the imaginary unit $\sqrt{-1}$.

Claim : For all $\lambda > 0$, $\alpha > 0$, and $\beta > 0$, the matrix $M = [a_{ij}]$, $i, j = 1, 2$, where a_{ij}, $i, j = 1, 2$ are given by (3.4.29)-(3.4.32), is nonsingular.

Proof : Suppose not. Then, for some choice of $\lambda > 0$, $\alpha > 0$, and $\beta > 0$, the matrix M defined above becomes singular. Since $\lambda > 0$ and $\lambda^2 = -\tau^4$, we have $\tau \neq 0$, hence by choosing $f = g = 0$, (3.4.33) and (3.4.34) implies that $f_1 = f_2 = 0$. Then , since the matrix M defined above is singular, it follows that (3.4.28) will have a nontrivial solution $(C_1\ C_2)^T \neq 0$. Putting this solution in (3.4.27) and using (3.4.21) one obtains a nontrivial solution $z = (u\ w)^T \neq 0$ of the equation $(\lambda I - A)z = 0$. This implies that $\lambda <z, z> = <Az, z> = 0$, but since $\lambda > 0$, $z \neq 0$ and A is dissipative, this is a contradiction. □

Therefore for all $\lambda > 0$, $\alpha > 0$, and $\beta > 0$, the equation (3.4.28) has a *unique* solution $(C_1 \ C_2)^T$. Putting this solution in (3.4.27) and (3.4.21), we obtain the solution $z = (u \ \ w)^T$ of the equations (3.4.21)-(3.4.25). This proves that for all $\lambda > 0$, the range of the operator $(\lambda I - A)$ is H. Hence by the Lumer-Phillips theorem, (see Theorem 3.4.4), the operator A defined in (3.4.9) defines a C_0 semigroup of contractions $T(t)$ on H; that is the equation (3.4.11) has a solution $z(t)$, this solution is unique and is given by :

$$z(t) = T(t) z_0 \qquad , z_0 \in D(A) \quad ,$$

(see [Paz.1, p.102, Thm.3.1]). □

3.4.8 Theorem (Exponential Decay). Consider the abstract differential equation (3.4.14), or equivalently the equations (3.4.14.1)-(3.4.14.4), where the operator A, its domain $D(A)$, and the space H are given by (3.4.9), (3.4.13), and (3.4.4), respectively. Let $T(t)$ be the C_0 semigroup generated by the operator A. Then there exist positive constants $M > 0$ and $\delta > 0$ such that the following holds :

$$|| T(t) || \leq M \, e^{-\delta t} \qquad \text{for all } t \in \mathbf{R}_+ \quad . \tag{3.4.35}$$

Proof : To prove (3.4.35), we first define the following function $V_1(t)$:

$$V_1(t) = 2(1-\varepsilon)\, t \, E_1(t) + 2 \int_0^L \rho \, x \, u_t \, u_x \, dx \quad , \tag{3.4.36}$$

where $\varepsilon \in (0,1)$ is an arbitrary constant and the energy $E_1(t)$ is defined by (3.4.19).

We prove the theorem in two steps. First we show that there exist a constant $C > 0$ such that the following estimate holds for all $t \in \mathbf{R}_+$:

$$(2\,(1-\varepsilon)\,t - C)E_1(t) \le V_1(t) \le (2\,(1-\varepsilon)\,t + C)E_1(t) \quad , \tag{3.4.37}$$

then differentiating $V_1(t)$ with respect to time, we show that there exist a $T > 0$ such that :

$$\frac{dV_1(t)}{dt} \le 0 \qquad \text{for all } t \ge T \quad . \tag{3.4.38}$$

Combining (3.4.37) and (3.4.38), it follows that

$$E_1(t) \le \frac{V_1(T)}{(2\,(1-\varepsilon)\,t - C)} \qquad \text{for all } t > T_1 \quad , \tag{3.4.39}$$

where $T_1 = \max\,\{\,T\,,\,\frac{C}{2(1-\varepsilon)}\,\}$.

Since by (3.4.20), $E_1(t)$ is bounded, from (3.4.37) it follows that $V_1(T) < \infty$. Using (3.4.39), from Pazy's theorem (Theorem 3.4.5), (3.4.35) follows.

To prove (3.4.35), we first need the following simple estimates . Since $u(0) = u_x(0) = 0$, we have :

$$u(s) = \int_0^s u_x \, dx \quad , \quad u_x(s) = \int_0^s u_{xx} \, dx \qquad s \in [0,L] \quad . \tag{3.4.40}$$

Using (3.4.40) and the Jensen's inequality, (see e.g., [Roy.1, p. 110]), we obtain the following estimates :

$$u(s)^2 \le L \int_0^L u_x{}^2 \, dx \le L^2 \int_0^L u_{xx}{}^2 \, dx \qquad , s \in [0,L] \quad . \tag{3.4.41}$$

In the sequel, we frequently use the following simple inequality :

$$a\,b \le (\delta^2 a^2 + \frac{1}{\delta^2} b^2) \qquad \text{for all } a\,,b\,,\delta \in \mathbf{R}_+\,,\delta \neq 0 \quad . \tag{3.4.42}$$

For simplicity, we define the quantity A_1 as follows :

$$A_1 := 2\int_0^L \rho\, x\, u_t\, u_x\, dx \quad . \tag{3.4.43}$$

Using (3.4.42) in (3.4.43), we obtain the following estimate :

$$|A_1| \le 2\,\rho\, L\,(\int_0^L u_t^2\, dx + \int_0^L u_{xx}^2\, dx)$$

$$\le C\, E_1(t) \quad , \tag{3.4.44}$$

where $C = \dfrac{2\,\rho\, L}{\min(\frac{\rho}{2},\frac{EI}{2})}$. Using (3.4.44) in (3.4.36), we obtain (3.4.37).

To prove (3.4.38), we first differentiate A_1 defined by (3.4.43) with respect to time :

$$\frac{dA_1}{dt} = 2\,\rho\int_0^L x\, u_{tt}\, u_x\, dx + 2\,\rho\int_0^L x\, u_t\, u_{xt}\, dx$$

$$= -2\,EI\int_0^L x\, u_{xxxx}\, u_x\, dx + 2\,\rho\int_0^L x\, u_t\, u_{xt}\, dx \quad , \tag{3.4.45}$$

where in the second equation we used (3.4.14.1). Using integration by parts, we obtain :

$$2\,EI\int_0^L x\, u_{xxxx}\, u_x\, dx = 2\,EI\, L u_x(L,t)\, u_{xxx}(L,t) - 2\,EI\, u_x(L,t)\, u_{xx}(L,t)$$

$$- EI\, L u_{xx}^2(L,t) + 3\,EI\int_0^L u_{xx}^2\, dx \quad , \tag{3.4.46}$$

$$2\,\rho\int_0^L x\, u_t\, u_{xt}\, dx = \rho\, L\, u_t^2(L,t) - \int_0^L \rho\, u_t^2\, dx \quad . \tag{3.4.47}$$

Using (3.4.45)-(3.4.47) in (3.4.36) and collecting likewise terms , we get :

$$\frac{dV_1}{dt} = 2\,(1-\varepsilon)\,t\,\frac{dE_1}{dt} + 2\,(1-\varepsilon)\,E_1 + \frac{dA_1}{dt}$$

$$= -2\,(1-\varepsilon)\,\alpha\, t\, u_t^2(L,t) - 2\,(1-\varepsilon)\,\beta\, t\, u_{xt}^2(L,t) - \varepsilon\int_0^L \rho\, u_t^2\, dx$$

$$-(2+\varepsilon)\int_0^L EI\, u_{xx}^2\, dx - 2\,EILu_x(L,t)\,u_{xxx}(L,t) + 2\,EIu_x(L,t)\,u_{xx}(L,t)$$

$$+ EI\, Lu_{xx}^2(L,t) + \rho\, L\, u_t^2(L,t) \quad , \tag{3.4.48}$$

where we have used (3.4.20).

Using the boundary controls (3.4.14.3), (3.4.14.4) and the inequality (3.4.42), we obtain the following estimates on some terms which appear in (3.4.48) :

$$-2\,EI\, Lu_x(L,t)\,u_{xxx}(L,t) = -2\,\alpha\, L\, u_x(L,t)\,u_t(L,t)$$

$$\leq 2\,\alpha\, L\,\delta_1^2\, u_x^2(L,t) + \frac{2\,\alpha\, L}{\delta_1^2}\, u_t^2(L,t) \quad , \tag{3.4.49}$$

$$2\,EI\, u_x(L,t)\,u_{xx}(L,t) = -2\,\beta u_x(L,t)\,u_{xt}(L,t)$$

$$\leq 2\,\beta\,\delta_2^2\, u_x^2(L,t) + \frac{2\,\beta}{\delta_2^2}\, u_{xt}^2(L,t) \quad . \tag{3.4.50}$$

Using (3.4.49), (3.4.50) in (3.4.48) and collecting likewise terms, the latter equation becomes :

$$\frac{dV_1}{dt} = -[2(1-\varepsilon)\alpha t - \frac{2\alpha L}{\delta_1^2} - \rho L\,]u_t^2(L,t) - \varepsilon\int_0^L \rho\, u_t^2\, dx$$

$$-[2(1-\varepsilon)\beta t - \frac{2\beta}{\delta_2^2} - \frac{L\beta^2}{EI}]u_{xt}^2(L,t)$$

$$-[(2+\varepsilon)\int_0^L EI\, u_{xx}^2\, dx - (2\alpha L\delta_1^2 + 2\beta\,\delta_2^2)u_x^2(L,t)] \quad , \qquad (3.4.51)$$

where δ_1 , δ_2 are arbitrary nonzero numbers. By choosing δ_1 and δ_2 sufficiently small and using (3.4.40) and (3.4.41), the last line in (3.4.51) can be made negative. Hence we conclude that there exists a $T > 0$ such that :

$$\frac{dV_1(t)}{dt} \le 0 \qquad \text{for all } t > T \quad ,$$

hence (3.4.35) follows from (3.4.39) and Pazy's theorem (Theorem 3.4.5). □

Now we are ready to prove that the solutions of the original equation (3.4.12) also decay exponentially. First due to the exponential decay of the angular velocity $\omega(t)$, (see (3.4.2)), there exist positive constants $c_1 > 0$, $c_2 > 0$, $\delta_1 > 0$,$\delta_2 > 0$, such that for all $t \ge 0$

$$\|B(t)\| \le c_1 e^{-\delta_1 t} \qquad (3.4.52)$$

$$\|f(t)\| \le c_2 e^{-\delta_2 t} \qquad (3.4.53)$$

where $B(t)$ and $f(t)$ are given by (3.4.10) and (3.4.11), respectively.

3.4.9 Theorem : (Exponential decay) Consider (3.4.12), where the operators A, $B(.)$ and the function $f(.)$ are defined in (3.4.9), (3.4.10), and (3.4.11) respectively. Then :

(i) for all $z_0 \in D(A)$, (3.4.12) has a unique solution $z(.)$, which is given by :

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\,B(s)\,z(s)\,ds + \int_0^t T(t-s)\,f(s)\,ds, \quad \text{for all } t \geq 0 \quad , \tag{3.4.54}$$

where $T(t)$ is the semigroup generated by the operator A,

(ii) for all $z_0 \in D(A)$, the solution $z(.)$, given by (3.4.54), decays exponentially to 0.

proof :

(i) Since $B(.)$ is globally Lipschitz on H and $||\,B(t)\,||$ is exponentially decaying to 0, (see (3.4.38)), and since the operator A generates a C_0semigroup by theorem 3.4.7, it follows that $A + B(.)$ generates a unique, globally defined (i.e., defined for all $t \in \mathbf{R}_+$), semigroup on H, (see e.g., [Paz.1, pp.185-188], [Mar.1, pp. 389-389]).

Since $f \in \mathrm{L}^1[\mathbf{R}, H]$ and is a C^∞ function of t, (see (3.4.11) and (3.4.2)), by standard theorems on nonhomogeneous partial differential equations, (see, e.g., [Paz.1, pp.105-110]), it follows that for all $z_0 \in D(A)$, (3.4.12) has unique solution defined for all $t \in \mathbf{R}_+$.

That the solution may be given as (3.4.54) can be verified by substitution of (3.4.54) in (3.4.12) and using $\frac{dT}{dt} = AT$.

(ii) By taking norms in (3.4.54) and using (3.4.17), (3.4.52), and (3.4.53) , we obtain :

$$||z(t)|| \le M\, e^{-\delta t}\, ||z_0|| + \int_0^t M\, c_1 e^{-\delta_1 s}\, e^{-\delta(t-s)} ||z(s)||\, ds$$

$$+ \int_0^t M\, c_2 e^{-\delta_2 s}\, e^{-\delta(t-s)}\, ds \quad . \tag{3.4.55}$$

Evaluating the last integral and multiplying each side of (3.4.55) by $e^{\delta t}$, we get :

$$||z(t)e^{\delta t}|| \le M\, ||z_0|| + \frac{M\, c_2}{\delta - \delta_2}\left(e^{(\delta-\delta_2)t} - 1\right)$$

$$+ \int_0^t M\, c_1\, e^{-\delta_1 s}\, ||z(s)e^{\delta s}||\, ds \quad . \tag{3.4.56}$$

Applying a general form of the Bellman-Gronwall lemma, (see, e.g., [Des.1, p.39]), we obtain the following :

$$||z(t)e^{\delta t}|| \le M\, ||z_0|| + \frac{M\, c_2}{\delta - \delta_2}\left(e^{(\delta-\delta_2)t} - 1\right)$$

$$+ \int_0^t M\, c_1\, e^{\frac{M\, c_1}{\delta_1}} \left[M\, ||z_0|| + \frac{M\, c_2}{\delta - \delta_2}\left(e^{(\delta-\delta_2)s} - 1\right)\right] e^{-\delta_1 s}\, ds$$

$$\le M\, ||z_0|| + \frac{M\, c_2}{\delta - \delta_2}\left(e^{(\delta-\delta_2)t} - 1\right) + \frac{M^2 c_1}{\delta_1}\, e^{\frac{M\, c_1}{\delta_1}} \left(||z_0|| - \frac{c_2}{\delta - \delta_2}\right)\left(1 - e^{-\delta_1 t}\right)$$

$$- \frac{M^2\, c_1\, c_2}{(\delta - \delta_2)(\delta - \delta_1 - \delta_2)}\, e^{\frac{M\, c_1}{\delta_1}} \left(1 - e^{(\delta - \delta_1 - \delta_2)t}\right) \quad . \tag{3.4.57}$$

Multiplying each side with $e^{-\delta t}$, we conclude that $z(.)$ decays exponentially to zero.

□

3.5 Stability Results for the Natural Control Law

In this section we show that the natural control law given by the equations (3.3.1), (3.3.2), and (3.3.5) stabilize the system given by the equations (3.2.1)-(3.2.3).

We define the energy $E_2(t)$ of the system given by the equations (3.2.1)-(3.2.3) as follows :

$$E_2(t) = \frac{1}{2} < \omega_R \, , I_R \, \omega_R > + \frac{1}{2} \int_0^L \rho < \mathbf{r}_t \, , \mathbf{r}_t > dx + \frac{1}{2} \int_0^L EI \; u_{xx}{}^2 \, dx \quad , \tag{3.5.1}$$

where the first term is the rotational kinetic energy of the rigid body, the second term is the kinetic energy of the beam and the third term is the potential energy of the beam, see, e.g., [Mei.2].

3.5.1 Proposition : Consider the system of equations given by (3.2.1)-(3.2.3) and the control law given by (3.3.1), (3.3.2), and (3.3.5). Then, the energy $E_2(t)$ defined in (3.5.1) is a *nonincreasing* function of time.

Proof : Upon differentiating (3.5.1) and using (3.2.1), we obtain : (here we note that the vector $I_R \, \omega_R$ is the angular momentum of the rigid body with respect to O in the body frame **B**, hence $(\frac{d}{dt}(I_R \, \omega_R))_N = I_R \, \dot{\omega}_R + \omega_R \times I_R \, \omega_R)$

$$\frac{dE_2}{dt} = < \omega_R \, , I_R \, \dot{\omega}_R + \omega_R \times I_R \, \omega_R > + \int_0^L \rho < \mathbf{r}_t \, , \mathbf{r}_{tt} > dx + \int_0^L EI \; u_{xx} u_{xxt} \, dx$$

$$= < \omega_R \, , I_R \, \dot{\omega}_R + \omega_R \times I_R \, \omega_R > + \int_0^L < (\mathbf{r}_t)_B + \omega \times \mathbf{r} \, , \mathbf{n}_x > dx + \int_0^L EI \; u_{xx} u_{xxt} \, dx$$

$$= <\omega_R\ , I_R\dot{\omega}_R + \omega_R \times I_R\omega_R> + <\omega\ , \int_0^L \mathbf{r} \times \mathbf{n}_x\, dx>$$

$$- \int_0^L EI\ u_t\ u_{xxxx}\, dx + \int_0^L EI\ u_{xx}u_{xxt}\, dx$$

$$= <\omega_R\ , I_R\dot{\omega}_R + \omega_R \times I_R\omega_R\ + \mathbf{r} \times \mathbf{n}\ \Big|_{x=0,L}\ - \int_0^L \mathbf{r}_x \times \mathbf{n}\, dx>$$

$$- \int_0^L EI\ u_t\ u_{xxxx}\, dx + \int_0^L EI\ u_{xx}u_{xxt}\, dx$$

$$= <\omega_R\ , I_R\dot{\omega}_R + \omega_R \times I_R\omega_R\ + [\mathbf{r} \times \mathbf{n} + \mathbf{m}]\ \Big|_{x=0,L}\ >$$

$$- \int_0^L EI\ u_t\ u_{xxxx}\, dx + \int_0^L EI\ u_{xx}u_{xxt}\, dx \quad , \tag{3.5.2}$$

where $[\mathbf{u}(x)]\ \Big|_{x=a,b} = \mathbf{u}(b) - \mathbf{u}(a)$ for any vector valued function $\mathbf{u} : [0,L] \rightarrow \mathbf{R}^3$. In the first equation we used the vector differentiation rule (2.2.14); in the second equation we used (2.2.14) and the balance of linear momentum equation (3.2.1); in the third equation we used (3.2.5) and (3.2.6); in the fourth equation we used integration by parts. Then, by using the balance of angular momentum equation (3.2.3) and integration by parts, (3.5.2) follows.

Using integration by parts and the boundary controls (3.2.1), (3.2.2), we obtain the following :

$$EI \int_0^L u_t\ u_{xxxx}\, dx = EI\ u_t(L,t)\, u_{xxx}(L,t) - \int_0^L EI\ u_{xt}\ u_{xxx}\, dx$$

$$= EI\ u_t(L,t)\, u_{xxx}(L,t) - EI\ u_{xt}(L,t)\, u_{xx}(L,t) + \int_0^L EI\ u_{xx}\ u_{xxt}\, dx$$

$$= \alpha u_t^{\,2}(L,t) + \beta u_{xt}^{\,2}(L,t) + \int_0^L EI\ u_{xx}\ u_{xxt}\, dx \quad , \tag{3.5.3}$$

$$= \alpha u_t^{\,2}(L,t) + \beta u_{xt}^{\,2}(L,t) + \int_0^L EI\; u_{xx}\, u_{xxt}\, dx \quad , \tag{3.5.3}$$

where in the first and second equations we used integration by parts and the boundary conditions (3.2.9). Then, using the boundary controls (3.3.1) and (3.3.2), we obtain (3.5.3).

By using (3.5.3) in (3.5.2) and by using the remaining torque control (3.3.4) and the rigid body angular momentum equation (3.2.3), we obtain the following :

$$\frac{dE_2}{dt} = -k\ \omega^2(t) - \alpha u_t^{\,2}(L,t) - \beta u_{xt}^{\,2}(L,t) \quad . \tag{3.5.4}$$

Since $\dot{E}_2(t) \le 0$, it follows that $E_2(t)$ is a nonicreasing function of time. □

Note that proposition 3.5.1 does not imply that the energy $E_2(t)$ actually decays to 0. In the next theorem we prove that the decay of $E_2(t)$ is as $O(\frac{1}{t})$ for sufficiently large t.

3.5.2 Theorem : (Asymptotic Decay of Energy)

Consider the system of equations given by (3.2.1)-(3.2.6), (or equivalently consider equations (3.2.7)-(3.2.8)), the boundary conditions (3.2.9) and the control law given by equations (3.3.1), (3.3.2), and (3.3.5). Then there exists a $T > 0$ such that the energy $E_2(t)$ given by (3.5.1) decays as $O(\frac{1}{t})$ for all $t \ge T$.

Proof : The proof is analogous to the proof of Theorem 3.4.8. We first define the following function $V_2(t)$:

$$V_2(t) = 2(1-\varepsilon)\, t\, E_2(t) + 2\int_0^L \rho\, x\, (u_t + \omega(b+x))\, u_x\, dx \quad , \tag{3.5.5}$$

where $\varepsilon \in (0,1)$ is an arbitrary constant.

We prove the theorem in two steps. First we show that for some constant $C_1 > 0$, the following estimate holds :

$$(2(1-\varepsilon)t - C_1)E_2(t) \leq V_2(t) \leq (2(1-\varepsilon)t + C_1)E_2(t) \quad . \tag{3.5.6}$$

Then, differentiating $V_2(t)$ with respect to time, we show that there exists a $T > 0$ such that

$$\frac{dV_2(t)}{dt} \leq 0 \qquad \text{for all } t \geq T \quad . \tag{3.5.7}$$

Then, combining (3.5.6) and (3.5.7), it follows that :

$$E_2(t) \leq \frac{V_2(T)}{(2(1-\varepsilon)t - C_1)} \qquad \text{for all } t > T_1 \quad , \tag{3.5.8}$$

where $T_1 = \max\{T, \frac{C_1}{2(1-\varepsilon)}\}$.

Since by (3.5.4), $E_2(t)$ is bounded, from (3.5.6) it follows that $V_2(T) < \infty$. Hence, (3.5.8) proves that the energy $E_2(t)$ decays as $O(\frac{1}{t})$ for sufficiently large t.

For simplicity, we define the quantity J_1 as follows :

$$J_1 = 2 \int_0^L \rho\, x\, (u_t + \omega (b + x))\, u_x\, dx \quad . \tag{3.5.9}$$

We need the following component form for $\mathbf{r}_t$ which follows from (3.2.6) and (2.2.14) :

$$\mathbf{r}_t = -\, \omega u\, \mathbf{D}_2 + (u_t + \omega (b + x))\, \mathbf{D}_3 \quad . \tag{3.5.10}$$

We first note the following simple inequalities :

$$(a + b)^2 \leq 2\,(a^2 + b^2) \qquad a\,, b \in \mathbf{R} \quad , \tag{3.5.11}$$

$$ab \leq \delta^2 a^2 + \frac{1}{\delta^2} b^2 \qquad a\,, b\,, \delta \in \mathbf{R}\,,\ \delta \neq 0 \quad . \tag{3.5.12}$$

To obtain (3.5.6), we need the following estimate :

$$\begin{aligned} |\, J_1 | &\leq 2\, \rho\, L \int_0^L u_x^{\,2}\, dx + 2\, \rho\, L \int_0^L (u_t + \omega (b + x))^2\, dx \\ &\leq 2\, \rho\, L^2 \int_0^L u_{xx}^{\,2}\, dx + 2\, \rho\, L \int_0^L < \mathbf{r}_t\,, \mathbf{r}_t > dx \\ &\leq C_1 E_2(t) \quad , \end{aligned} \tag{3.5.13}$$

where $C_1 = 4L + \dfrac{4\rho L^2}{EI}$. The first inequality follows from (3.5.11); the second inequality follows from (3.4.41) and (3.5.10) and then (3.5.13) follows from the definition of $E_2(t)$ given in (3.5.1). Using (3.5.13) in (3.5.5), we obtain (3.5.6).

Differentiating J_1 with respect to time, we obtain :

$$\frac{dJ_1}{dt} = 2\int_0^L \rho\, x\, [u_{tt} + \dot{\omega}(b+x)]\, u_x\, dx + 2\int_0^L \rho\, x\, [u_t + \omega(b+x)]\, u_{xt}\, dx$$

$$= -2\, EI \int_0^L x\, u_x\, u_{xxxx}\, dx + 2\,\rho \int_0^L x\, \omega^2 u\, u_x\, dx + 2\,\rho \int_0^L x\, u_{xt}\, u_t\, dx$$

$$+ 2\,\rho \int_0^L x\, \omega\,(b+x)\, u_{xt}\, dx$$

$$= -2\, EI \int_0^L x\, u_x\, u_{xxxx}\, dx + \rho\, [L u_t^2(L,t) - \int_0^L u^2\, dx]\, \omega^2 + \rho\, L\, u_t^2(L,t)$$

$$- \int_0^L \rho\, u_t^2\, dx + 2\,\rho\, [\, L(b+L) u_t(L,t) - \int_0^L (b+2x) u_t\, dx\,]\omega \quad , \qquad (3.5.14)$$

where in the second equation we used (3.2.7). Then, using integration by parts (3.5.14) follows.

Using (3.5.14) in (3.5.5), we obtain :

$$\frac{dV_2}{dt} = 2\,(1-\varepsilon)\, t\, \frac{dE_2}{dt} + 2\,(1-\varepsilon) E_2 + \frac{dJ_1}{dt}$$

$$= -k\, \omega^2(t) - \alpha u_t^2(L,t) - \beta u_{xt}^2(L,t)$$

$$+ (1-\varepsilon) I_R\, \omega^2 + (1-\varepsilon)\int_0^L \rho\, \omega^2 u^2\, dx + (1-\varepsilon)\int_0^L \rho\, (u_t + \omega\,(b+x))^2\, dx$$

$$+ (1-\varepsilon)\int_0^L EI\, u_{xx}^2\, dx - 2\, EI \int_0^L x\, u_x\, u_{xxxx}\, dx$$

$$+\rho\,[Lu_t^{\,2}(L,t)-\int_0^L u^2\,dx]\,\omega^2+\rho\,L\,u_t^{\,2}(L,t)-\int_0^L \rho\,u_t^{\,2}\,dx$$

$$+2\,\rho\,[\,L(b+L)u_t(L,t)-\int_0^L (b+2x)u_t\,dx\,]\omega \quad . \tag{3.5.15}$$

To obtain a bound on some of the terms which appear in (3.5.15), we need the following estimate which follows from (3.4.40), (3.4.41), (3.5.1) and (3.5.4) :

$$u(s)^2 \le L\int_0^L u_x^{\,2}\,dx \le L^2\int_0^L u_{xx}^{\,2}\,dx \le \frac{2L^2}{EI}E_2(t) \le \frac{2L^2}{EI}E_2(0) \quad . \tag{3.5.16}$$

Using (3.5.11), (3.5.12), and (3.5.16), we obtain the following estimates for some of the terms which appear in (3.5.15) :

$$\int_0^L \rho\,(u_t+\omega\,(b+x))^2\,dx = \int_0^L \rho\,u_t^{\,2}\,dx + \int_0^L \rho\,\omega^2(b+x)^2\,dx + 2\int_0^L \rho\,\omega\,(b+x)\,u_t\,dx$$

$$\le \int_0^L \rho\,u_t^{\,2}\,dx + \rho\,\omega^2\int_0^L (b+x)^2\,dx + 2\,\delta_1^{\,2}\int_0^L \rho\,u_t^{\,2}\,dx$$

$$+\frac{2\rho\omega^2}{\delta_1^{\,2}}\int_0^L (b+x)^2\,dx \quad , \tag{3.5.17}$$

where $\delta_1 \ne 0$ is an arbitrary real number;

$$-2\,EI\int_0^L x\,u_{xxxx}\,u_x\,dx = -2\,EI\,\,Lu_x(L,t)\,u_{xxx}(L,t)+2\,EI\,\,u_x(L,t)\,u_{xx}(L,t)$$

$$+EI\,\,Lu_{xx}^{\,2}(L,t)-3\,EI\int_0^L u_{xx}^{\,2}\,dx$$

$$=2\,\alpha\,L\,\,u_x(L,t)\,u_t(L,t)-2\,\beta\,u_x(L,t)\,u_{xt}(L,t)$$

$$+\frac{L\beta^2}{EI}\,u_{xt}^{\,2}(L,t)-3\,EI\int_0^L u_{xx}^{\,2}\,dx$$

$$\leq 2\,L\,\alpha\,\delta_2^2 u_x^{\,2}(L,t)+\frac{2\,L\,\alpha}{\delta_2^2}u_t^{\,2}(L,t)+2\,\beta\,\delta_3^2 u_x^{\,2}(L,t)$$

$$+\frac{2\,\beta}{\delta_3^2}u_{xt}^{\,2}(L,t)+\frac{L\beta^2}{EI}\,u_{xt}^{\,2}(L,t)-3\,EI\int_0^L u_{xx}^{\,2}\,dx \quad , \qquad (3.5.18)$$

where the first equation follows from integration by parts and the boundary conditions (3.4.14.2), the second equation follows from the boundary control laws (3.3.1) and (3.3.2), and then (3.5.18) follows from (3.5.12), $\delta_2 \neq 0$, $\delta_3 \neq 0$ are arbitrary real numbers;

$$\rho\,[Lu_t^{\,2}(L,t)-\int_0^L u^2\,dx\,]\,\omega^2 \leq \frac{4\,\rho\,L^3 E_2(0)}{EI}\,\omega^2 \quad , \qquad (3.5.19)$$

where we used (3.5.16), (3.5.1) and (3.5.16);

$$2\,\rho\,L\,(b+L)u_t(L,t)\omega \leq 2\,\rho\,L\,(b+L)u_t^{\,2}(L,t)+2\,\rho\,L\,(b+L)\omega^2 \quad , \qquad (3.5.20)$$

$$-2\,\rho\,\omega\int_0^L (b+2x)u_t\,dx\ \leq 2\,\delta_4^2\int_0^L \rho\,u_t^{\,2}\,dx+\frac{2\rho\int_0^L (b+2x)^2\,dx}{\delta_4^2}\omega^2 \quad , \qquad (3.5.21)$$

where $\delta_4 \neq 0$ is an arbitrary real number.

Using (3.5.17)-(3.5.21) in (3.5.15), we obtain the following :

$$\frac{dV_2}{dt} \leq -[\,2(1-\varepsilon)kt-D_1\,]\omega^2+-[\,2(1-\varepsilon)\alpha t-D_2\,]u_t^{\,2}(L,t)$$

$$-[\,2(1-\varepsilon)\beta t-D_3\,]u_{xt}^{\,2}(L,t)-(\varepsilon-D_4)\int_0^L \rho\,u_t^{\,2}\,dx$$

$$-[\,(\varepsilon+2)\int_0^L EI\; u_{xx}^2\,dx \;-(2\,L\;\alpha\,\delta_2^2+2\,\beta\,\delta_3^2\,)u_x^2(L,t)\,] \quad , \tag{3.5.22}$$

where

$$D_1=(1-\varepsilon)I_R+\frac{2\,\rho\,(1-\varepsilon)L^3E_2(0)}{EI}+\frac{4\,\rho\,L^3E_2(0)}{EI}$$

$$+\rho\int_0^L(b+x)^2\,dx+\frac{2\rho\int_0^L(b+2x)^2\,dx}{\delta_1^2}+\frac{2\rho\int_0^L(b+2x)^2\,dx}{\delta_4^2} \quad , \tag{3.5.23}$$

$$D_2=\frac{2L\alpha}{\delta_2^2}+\rho\,L+2\rho L(b+L) \quad , \tag{3.5.24}$$

$$D_3=\frac{2\beta}{\delta_3^2}+\frac{L\beta^2}{EI} \quad , \tag{3.5.25}$$

$$D_4=2\delta_1^2+2\delta_4^2 \quad , \tag{3.5.26}$$

and δ_i , $i=1,\ldots,4$ are arbitrary nonzero real numbers.

Let $\varepsilon\in(0,1)$ be fixed. Using (3.5.16), we can obtain the following inequality :

$$\frac{1}{L}\,u_x^2(L,t)\le\int_0^L u_{xx}^2\;dx \quad . \tag{3.5.27}$$

Hence, by choosing δ_2 and δ_3 sufficiently small so that the following inequality holds :

$$\frac{(2\,L\;\alpha\,\delta_2^2+2\,\beta\,\delta_3^2\,)}{(\varepsilon+2)EI}<\frac{1}{L} \quad ,$$

the last line in (3.5.22) can be made negative. Also by choosing δ_1 and δ_4 sufficiently small, we can have $D_4<\varepsilon$. Then from (3.5.22) the inequality (3.5.7) follows, i.e.,

$$\frac{dV_2(t)}{dt} \le 0 \qquad \text{for all } t \ge T \quad ,$$

where $T = \max\{\frac{D_1}{2(1-\varepsilon)k}, \frac{D_2}{2(1-\varepsilon)\alpha}, \frac{D_1}{2(1-\varepsilon)\beta}\}$.

Combining (3.5.7) and (3.5.6), we obtain (3.5.8), that is :

$$E_2(t) \le \frac{V_2(T)}{(2\,(1-\varepsilon)\,t - C_1)} \qquad \text{for all } t > \max\,(T\,, \frac{C_1}{2(1-\varepsilon)}) \quad ,$$

which proves that for sufficiently large t, $E_2(t)$ decays as $O(\frac{1}{t})$. □

Remark : If one chooses $\beta = 0$, i.e., no torque control at the free end of the beam, (see (3.3.2)), then the conclusion of Theorem 3.5.2 still holds, that is for sufficiently large t, the energy $E_2(t)$ decays as $O(\frac{1}{t})$. This can be concluded by observing that $\beta = 0$ implies $D_3 = 0$, (see (3.5.25)). Therefore in (3.5.22) the term multiplying $u_x^{\,2}(L\,,t)$ becomes identically zero. But since the remaining terms are all negative, it still follows that $\frac{dV_2}{dt} \le 0$ for sufficiently large t, hence (3.5.8) follows. On the other hand, if one puts $k = 0$ and/or $\alpha = 0$, this conclusion does not follow from (3.5.22), due to the strictly positive terms D_1 and/or D_2. In other words, Theorem 3.5.2 holds for $k > 0$, $\alpha > 0$, and $\beta \ge 0$.

□

3.5.3 Existence, Uniqueness and Exponential Decay of Solutions

In the previous section, we proved that the solutions of equations (3.2.7) and (3.2.8) together with the control law (3.3.1), (3.3.2) and (3.3.5) decay as $O(\frac{1}{t})$ for sufficiently large t. In this section, we prove that the solution of the equations mentioned above exists, is unique, and decays exponentially to zero.

Without loss of generality, in the sequel we put

$$\rho = 1 \quad , \quad EI = 1 \quad , \quad I_R = 1 \quad . \tag{3.5.28}$$

Using (3.3.5) in (3.2.8), we obtain :

$$\dot{\omega} = \int_0^L (b + x)\, u_{xxxx}\, dx \; - k\, \omega \quad . \tag{3.5.29}$$

We can put (3.5.29), and (3.2.7) into the following abtract form :

$$\frac{d\hat{z}(t)}{dt} = \hat{A}\hat{z} + g(\hat{z}) \quad , \quad \hat{z}(0) \in D(\hat{A}) \quad , \tag{3.5.30}$$

where $\hat{z} = (u \;\; u_t \;\; \omega)^T \in \hat{H} := H \times \mathbf{R}$, where H is defined by (3.4.4) and $D(\hat{A}) := D(A) \times \mathbf{R}$, where $D(A)$ is defined in (3.4.13). $\hat{A} : D(\hat{A}) \subset \hat{H} \rightarrow \hat{H}$ is a linear unbounded operator whose matrix form is specified as follows :

$$\hat{A} = \{a_{ij} \mid i, j = 1, 2, 3\} \quad , \tag{3.5.31}$$

where all a_{ij} are zero except :

$$a_{12} = 1 \quad ,$$

$$a_{21} = -\frac{\partial^4}{\partial x^4} - (b + x) \int_0^L (b + x) \frac{\partial^4}{\partial x^4}\, dx \quad ,$$

$$a_{23} = k(b + x) \quad ,$$

$$a_{31} = \int_0^L (b + x) \frac{\partial^4}{\partial x^4} \, dx$$

$$a_{33} = -k \quad .$$

The nonlinear operator $g : \hat{H} \rightarrow \hat{H}$ is defined as follows :

$$g(\hat{z}) = (g_1 \;\; g_2 \;\; g_3)^T \quad , \tag{3.5.32}$$

where

$$g_1 = g_3 = 0 \quad ,$$

$$g_2 = \omega^2 u \quad .$$

In $\hat{H}$ we define the following inner product :

$$< z \, , \hat{z} >_L = \frac{1}{2} \int_0^L [u_t + \omega(b + x)] \, [\hat{u}_t + \hat{\omega}(b + x)] \, dx + \frac{1}{2} \int_0^L u_{xx} \, \hat{u}_{xx} \, dx + \frac{1}{2} \, \omega\hat{\omega} \quad , \tag{3.5.33}$$

where $z = (u \;\; u_t \;\; \omega)^T$ and $\hat{z} = (\hat{u} \;\; \hat{u}_t \;\; \hat{\omega})^T$. The corresponding "energy" norm $E_L(t)$ induced by (3.5.33) is :

$$E_L(t) = \frac{1}{2} \int_0^L [u_t + \omega(b + x)]^2 \, dx + \frac{1}{2} \int_0^L u_{xx}^{\;2} \, dx + \frac{1}{2} \, \omega^2 \quad . \tag{3.5.34}$$

In view of the energy $E_1(t)$ defined by (3.4.19), which is an approppropriate energy for the plane vibration of an Euler-Bernoulli beam without rotation, a natural extention which includes the effect of rotation is the following :

$$\hat{E}_L(t) = \frac{1}{2}\int_0^L u_t^2\,dx + \frac{1}{2}\int_0^L u_{xx}^2\,dx + \frac{1}{2}\,\omega^2 \quad . \tag{3.5.35}$$

We note that (3.5.35) is equivalent to a standard Sobolev norm which makes $\hat{H}$ a Hilbert space, see [Che.2]. Next we prove that the norms defined by (3.5.34) and (3.5.35) are equivalent.

3.5.4 Lemma : (Equivalence of Norms). Let the space H be defined as in (3.4.4) and the space $\hat{H}$ be defined as $\hat{H} = H \times \mathbf{R}$. Then the norms defined by (3.5.34) and (3.5.35) are equivalent .

Proof : By using (3.5.11) it follows that :

$$E_L(t) \le M_1\,\hat{E}_L(t) \quad , \tag{3.5.36}$$

where $M_1 = \max\{2\,,\,1+2\int_0^L (b+x)^2\,dx\}$.

To prove the inequality in the other direction, first using (3.5.12) we obtain :

$$-2\,\omega\int_0^L (b+x)u_t\,dx \;\le 2\,\delta^2\int_0^L u_t^2\,dx + \frac{2\int_0^L (b+x)^2\,dx}{\delta^2}\,\omega^2 \quad , \tag{3.5.37}$$

where $\delta \neq 0$ is an arbitrary real number. Using (3.5.37) in (3.5.34) we obtain the following :

$$2\,E_L(t) = \int_0^L u_t^{\,2}\,dx + \int_0^L (b+x)^2\,dx\;\omega^2 + 2\int_0^L \omega\,(b+x)\,u_t\,dx + \int_0^L u_{xx}^{\,2}\,dx + \omega^2$$

$$\geq (1-\delta^2)\int_0^L u_t^{\,2}\,dx + \int_0^L u_{xx}^{\,2}\,dx + (1 + C - \frac{C}{\delta^2})\,\omega^2 \quad , \tag{3.5.38}$$

where $C = 2\int_0^L (b+x)^2\,dx$.

Choosing $\delta \in (\frac{C}{C+1}\,,1)$, we have $1-\delta^2 > 0$ and $(1 + C - \frac{C}{\delta^2}) > 0$. Therefore comparing (3.5.38) and (3.5.35), we obtain :

$$E_L(t) \geq \min\{(1-\delta^2)\,,(1 + C - \frac{C}{\delta^2})\,\}\,\hat{E}_L(t) \quad . \tag{3.5.39}$$

By using (3.5.39) and (3.5.36), we conclude that the norms defined by (3.5.34) and (3.5.35) are equivalent. □

Next, we consider the linear part of the equation (3.5.30) :

$$\frac{d\hat{z}_L}{dt} = \hat{A}\hat{z}_L \quad , \quad \hat{z}_L(0) \in D(\hat{A}) \quad , \tag{3.5.40}$$

where $\hat{A}$ is defined by (3.5.31) and $\hat{z}_L = (u \;\; u_t \;\; \omega)^T$. Note that (3.5.40) is equivalent to the following system of equations : for all $t \geq 0$,

$$u_{tt} + u_{xxxx} + \dot{\omega}(b+x) = 0 \quad , \quad x \in (0,L) \quad , \tag{3.5.40.1}$$

$$\dot{\omega} = \int_0^L (b+x)u_{xxxx}\,dx \;\; k\,\omega \quad , \tag{3.5.40.2}$$

$$u(0,t) = u_x(0,t) = 0 \quad , \tag{3.5.40.3}$$

$$-u_{xxx}(L,t) + \alpha\, u_t(L,t) = 0 \quad , \tag{3.5.40.4}$$

$$u_{xx}(L,t) + \beta u_{xt}(L,t) = 0 \quad . \tag{3.5.40.5}$$

3.5.5 Theorem : Consider the equation (3.5.26) where $\hat{z}_L = (u \;\; u_t \;\; \omega)^T$, $\hat{A}$ is given by (3.5.31), $D(\hat{A}) = D(A) \times \mathbf{R}$, and $D(A)$ is given by (3.4.13). Then we have :

(i) The operator $\hat{A}$ generates a C_0 semigroup $\hat{T}(t)$,

(ii) The semigroup $\hat{T}(t)$ decays exponentially, i.e., there exist constants $M > 0$ and $\delta > 0$ such that

$$|| \hat{T}(t) || \leq M\, e^{-\delta t} \qquad \text{for all } t \in \mathbf{R} \tag{3.5.41}$$

where the norm is given by (3.5.34).

Proof :

(i) By lemma (3.4.2), $D(A)$ is dense in H. Hence $D(\hat{A}) \subset \hat{H} = H \times \mathbf{R}$ is dense in $\hat{H}$.

We use the Lumer-Phillips theorem to prove the assertion (i), see Theorem 3.4.4. Hence, we have to show that

(a) : $\hat{A}$ is dissipative,

(b) : for some $\lambda > 0$, the range of the operator $(\lambda I - \hat{A})$ is $\hat{H}$.

To prove assertion (a), we differentiate the energy $E_L(t)$, use (3.5.40), and the control law (3.3.1), (3.3.2) , and (3.3.5). Then, we obtain :

$$\frac{dE_L}{dt} = \int_0^L [u_{tt} + \dot{\omega}(b+x)]\,[u_t + \omega(b+x)]\,dx + \int_0^L u_{xx}\,u_{xxt}\,dx + \omega\dot{\omega}$$

$$= -\int_0^L u_{xxxx}\,[u_t + \omega(b+x)]\,dx + \int_0^L u_{xx}\,u_{xxt}\,dx + \omega\dot{\omega}$$

$$= -k\,\omega^2 - \alpha\,u_t^{\,2}(L,t) - \beta\,u_{xt}^{\,2}(L,t) \;\le 0 \qquad , \tag{3.5.42}$$

which proves that $\hat{A}$ is dissipative.

To prove assertion (b), we first define the following linear operator $\hat{T}_D : D(\hat{A}) \subset \hat{H} \to \hat{H}$:

$$\hat{T}_D = \{t_{ij} : i, j = 1, 2, 3 \quad ,\} \tag{3.5.43}$$

where all t_{ij} are zero except :

$$t_{21} = -(b+x)\int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \qquad ,$$

$$t_{31} = \int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \qquad ,$$

and we define the operator $\hat{A}_1 : D(\hat{A}) \subset \hat{H} \to \hat{H}$:

$$\hat{A}_1 = \hat{A} - \hat{T}_D \qquad . \tag{3.5.44}$$

We first note the following remarks :

1) $\hat{A}_1 : D(\hat{A}) \subset \hat{H} \to \hat{H}$ is a linear unbounded operator . Its domain D($\hat{A}_1$) is equal to D($\hat{A}$). By using Theorem 3.4.3 and noting the block diagonal form of $\hat{A}_1$, it follows that $\hat{A}_1$ generates an exponentially decaying C_0 semigroup. Hence, $(\lambda I - \hat{A}_1) : \hat{H} \to \hat{H}$ is an

invertible operator for all $\lambda > 0$. In fact the range of $(\lambda I - \hat{A}_1)^{-1}$ is equal to $D(\hat{A})$ and by Hille-Yosida theorem, (see Theorem 3.4.3), we have :

$$\| (\lambda I - \hat{A}_1)^{-1} \| \leq \frac{1}{\lambda} \quad \lambda > 0 \quad .$$

2) The operator $\hat{T}_D : D(\hat{A}) \subset \hat{H} \to \hat{H}$ is a degenerate linear operator relative to the $\hat{A}_1$ (see, e.g. [Kat.1, p. 245]). By definition, the range space of $\hat{T}_D$ is finite dimensional and there exist positive constants a and b such that :

$$\| \hat{T}_D \hat{z} \| \leq a \| \hat{z} \| + b \| \hat{A}_1 \hat{z} \| \qquad \text{for all } \hat{z} \in D(\hat{A}) \quad . \tag{3.5.45}$$

That the operator $\hat{T}_D$ has a finite dimensional range follows from (3.5.43).

By using (3.5.43), (3.5.44), and (3.5.30) in (3.5.33), we obtain the inequality (3.5.45) for some $a > 0$ and $b > 0$.

From the remarks 1 and 2 above it follows that $\hat{T}_D (\lambda I - \hat{A}_1)^{-1} : \hat{H} \to \hat{H}$ is a *bounded* linear operator with finite dimensional range ; hence $\| \hat{T}_D (\lambda I - \hat{A}_1)^{-1} \| \leq M$ for some $M >$ 0 and $\hat{T}_D (\lambda I - \hat{A}_1)^{-1}$ is a *compact* operator, (see, e.g., [Kat.1, p. 245]).

Next we need the following fact :

Fact : for all $\lambda > 0$, the real number 1 is not an eigenvalue of the compact operator $\hat{T}_D (\lambda I - \hat{A}_1)^{-1}$.

Proof : Suppose not. Then there exists a $\lambda > 0$ and a $y \in \hat{H}, y \neq 0$ such that the following holds :

$$y = \hat{T}_D(\lambda I - \hat{A}_1)^{-1}y. \quad . \tag{3.5.46}$$

Define $x \in D(\hat{A})$ as

$$x = (\lambda I - \hat{A}_1)^{-1}y \quad .$$

Then (3.5.46) implies that the following equation also holds :

$$(\lambda I - \hat{A}_1 - \hat{T}_D)x = 0 \quad .$$

But since $\hat{A} = \hat{A}_1 + \hat{T}_D$ is dissipative and $\lambda > 0$, it follows that $x = 0$, which implies $y = 0$, which is a contradiction. □

From the above fact it follows that the operator $I - \hat{T}_D(\lambda I - \hat{A}_1)^{-1}$ is invertible for all $\lambda > 0$. Hence we conclude that $(\lambda I - \hat{A}) : \hat{H} \to \hat{H}$ is invertible for all $\lambda > 0$ and its inverse is given by :

$$(\lambda I - \hat{A})^{-1} = (\lambda I - \hat{A}_1)^{-1}(I - \hat{T}_D(\lambda I - \hat{A}_1)^{-1})^{-1} \quad .$$

This shows that $(\lambda I - \hat{A}) : \hat{H} \to \hat{H}$ is onto for all $\lambda > 0$. From this and the fact that $\hat{A}$ is dissipative it follows that $\hat{A}$ generates a C_0 semigroup, see Theorem 3.4.4 (Lumer-Phillips theorem).

(ii) To prove the exponential decay of the semigroup $\hat{T}(t)$ generated by $\hat{A}$, as in the proof of Theorem 3.4.8, we define the following function $V_L(t)$:

$$V_L(t) = 2(1-\varepsilon)tE_L(t) + 2\int_0^L x\ [u_t + \omega(b+x)]\ u_x\ dx \quad , \tag{3.5.49}$$

where $\varepsilon \in (0,1)$ is a constant.

Using the inequalities (3.5.16) and (3.5.12), we obtain the following estimate :

$$\left| 2\int_0^L x\,[u_t+\omega(b+x)]\,u_x\,dx \right| \le 2L\int_0^L u_x^2\,dx + 2L\int_0^L [u_t+\omega(b+x)]^2\,dx$$

$$\le 2L^2\int_0^L u_x^2\,dx + 2L\int_0^L [u_t+\omega(b+x)]^2\,dx$$

$$\le K\,E_L(t) \quad , \tag{3.5.50}$$

where $K=\max(2L^2, 2L)$. Hence we have the following estimate for $V_L(t)$:

$$[2(1-\varepsilon)t-K]\,E_L(t) \le V_L(t) \le [2(1-\varepsilon)t+K]\,E_L(t) \qquad \text{for all } t \ge 0 \quad . \tag{3.5.51}$$

Differentiating (3.5.49) and using (3.5.40), we obtain the following :

$$\frac{dV_L(t)}{dt} = 2(1-\varepsilon)t\frac{dE_L(t)}{dt} + 2(1-\varepsilon)E_L(t) + 2\int_0^L x\,[u_{tt}+\dot{\omega}(b+x)]u_x\,dx$$

$$+2\int_0^L x\,[u_t+\omega(b+x)]u_{xt}\,dx$$

$$= 2(1-\varepsilon)t\frac{dE_L(t)}{dt} + 2(1-\varepsilon)E_L(t) - 2\int_0^L x\,u_{xxxx}\,u_x\,dx$$

$$+2\int_0^L x\,u_t\,u_{xt}\,dx + 2\,\omega\int_0^L x\,(b+x)u_{xt}\,dx \quad . \tag{3.5.52}$$

For the last two terms, using integration by parts and (3.5.12) we obtain the following estimates :

$$2\int_0^L x\,u_t\,u_{xt}\,dx = Lu_t^2(L,t) - \int_0^L u_t^2\,dx \quad , \tag{3.5.53}$$

$$2\omega\int_0^L x\,(b+x)u_{xt}\,dx \le \frac{4}{\delta^2}\omega^2 + \delta^2\Big(\int_0^L x\,(b+x)u_{xt}\,dx\Big)^2$$

$$\le \frac{4}{\delta^2}\omega^2 + \delta^2\Big[L(b+L)u_t(L,t) - \int_0^L (b+2x)u_t\,dx\Big]^2$$

$$\le \frac{4}{\delta^2}\omega^2 + 2\delta^2 L^2(b+L)^2 u_t^{\,2}(L,t) + 2L\,\delta^2(b+2L)^2\int_0^L u_t^{\,2}\,dx \quad , \qquad (3.5.54)$$

where we used Jensen's inequality in the last step, (see, e.g., [Roy.1, p. 110]). Using (3.4.46), (3.5.53), (3.5.34) , (3.5.42), and (3.5.54) in (3.5.53) and using the argument we used in the proof of Theorem 3.4.8, we conclude that there exists a $T \ge 0$ such that $\frac{dV_L}{dt} \le 0$, for all $t \ge T$. Combining this with (3.5.52) and (3.5.42), we conclude that

$$E_L(t) \le \frac{V_L(T)}{2(1-\varepsilon)-K} \quad ,$$

for all $t > \max\{T\,,\,\frac{K}{2(1-\varepsilon)}\}$. Hence , $\int_0^\infty E_L^{\,2}(t)\,dx < \infty$, therefore by Pazy's theorem (Theorem 3.4.5), (3.5.41) follows, i.e. the semigroup $\hat{T}(t)$ generated by $\hat{A}$ decays exponentially. □

From Theorem 3.5.5 it follows that the linear part of the equation (3.5.30) generates an exponentially decaying semigroup $\hat{T}(t)$. Using this fact we can prove the following theorem :

3.5.6 Theorem : (Existence, Uniqueness and Exponential Decay).

Consider the equation (3.5.30), where $\hat{z} = (u\,,\,\hat{u}\,,\,\omega)^T$, the operator $\hat{A}$ is given by (3.5.31) and g is given by (3.5.32). Let $\hat{T}(t)$ be the semigroup generated by $\hat{A}$. Then :

(i) For all $\hat{z}(0) \in D(\hat{A})$, equation (3.5.30) has unique solution $\hat{z}(t)$,

(ii) In terms of the semigroup $\hat{T}(t)$ generated by $\hat{A}$, this solution may be given as :

$$\hat{z}(t) = \hat{T}(t)\hat{z}(0) + \int_0^t \hat{T}(t-s) g(\hat{z}(s))\, ds \quad , \tag{3.5.55}$$

(iii) this solution $\hat{z}(t)$ decays exponentially.

Proof :

(i) Since $\hat{A}$ generates a C_0 semigroup $\hat{T}(t)$ and $g : \hat{H} \to \hat{H}$ is a C^∞ function, by the standard theorems on partial differential equations (see, e.g., [Paz.1, pp. 183-191]), it follows that (3.5.16) has a unique solution for all $\hat{z}(0) \in D(\hat{A})$, which is defined locally in time, i.e., in a time interval $(0, T)$ for some $T > 0$. But since the solutions are bounded and asymptotically decaying as $O(\frac{1}{t})$, this local solution can be extended to a global solution, i.e., defined for all $t \geq 0$.

(ii) This assertion may be proven by back substitution of (3.5.55) in (3.5.30) and using $\frac{d\hat{T}(t)}{dt} = \hat{A}\hat{T}(t)$.

(iii) Using (3.5.32) in (3.5.34), we obtain :

$$|| g(\hat{z}) || = \int_0^L \omega^4 u^2\, dx \ \leq L^2 \omega^4 \int_0^L u_{xx}{}^2\, dx \ \leq L^2 \omega^4 || g(\hat{z}) ||^2 \quad , \tag{3.5.56}$$

where in deriving the first inequality we used (3.5.16).

By taking norms in (3.5.55), using (3.5.41) and multiplying by $e^{\delta t}$ we obtain :

$$||\hat{z}(t)e^{\delta t}|| \leq M\,||\hat{z}(0)|| + \int_0^L M\,L^2\omega^4 ||\hat{z}(s)e^{\delta t}||\,ds \quad . \tag{3.5.57}$$

Since $\hat{z}(t)$ is asyptotically decaying at least as $O(\frac{1}{t})$ by Theorem (3.5.2), it follows that for any $\gamma > 0$ which satisfies $\gamma < \frac{\delta}{M}$, there exists a $T > 0$ such that

$$ML^2\omega^4(t) \leq \gamma < \frac{\delta}{M} \qquad \text{for all } t \geq T \quad . \tag{3.5.58}$$

Applying the Bellman-Gronwall lemma to (3.5.58) we obtain :

$$||\hat{z}(t)|| \leq M\,||\hat{z}(0)||\; e^{-(\delta-\gamma M)t} \qquad \text{for all } t \geq T \quad . \tag{3.5.59}$$

□

Chapter 4

Control of a Flexible Beam Attached to a Rigid Body : Motion in Space

4.1 Introduction

In Chapter 2 we introduced a rigid body-flexible beam configuration and derived the equations of motion for this configuration, (see Section 4, Chapter 2). In chapter 3 we studied a special case of this configuration; we assumed that the rigid body center of mass is fixed in an inertial frame and that the motion of the whole configuration is restricted to be a planar motion in that frame.

In this chapter we continue to study the configuration mentioned above. We assume that the center of mass of the rigid body is fixed in an inertial frame, but the motion of the whole configuration is not restricted otherwise.

In Section 2 we give the equations of motion and define the rest state of the system. Then we state the control problem, namely if the system is perturbed from the rest state to find appropriate control laws which drives the system to the rest state. We then extend the control laws proposed in Chapter 3 to solve this problem,(see Section 3, Chapter 3). In the remaining sections we show that the proposed control laws solve the control problem posed above.

4.2 Equations of Motion

We consider the following configuration : Figure 4.1 shows the rigid body (drawn as a square) and the beam ; P is a point on the beam.

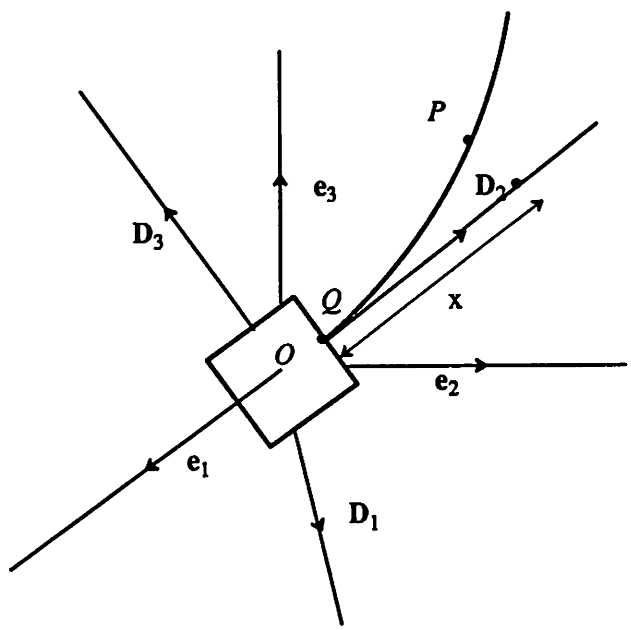

Figure 4.1 : Rigid body with flexible beam.

In Figure 4.1, the quadruple (O, e_1, e_2, e_3) denotes a dextral orthonormal *inertial* frame, which will be referred to as N, the quadruple (O, D_1, D_2, D_3) denotes a dextral orthonormal frame fixed in the rigid body, which will be referred as **B**, where O is also the center of mass of the rigid body and D_1, D_2, D_3 are along the principal axes of inertia of the rigid body. One end of the beam is clamped to the rigid body at the point Q along the D_2 axis and the other end is free. Let L be the length of the beam. We assume that the mass of the rigid body is much larger than the mass of the beam, so the center of mass of the rigid body is approximately the center of mass of the whole configuration. So we take it that the point O is fixed in the inertial space throughout the motion of the whole configuration and the rigid body may rotate arbitrarily in the inertial space.

The beam is initially straight, along the $\mathbf{D}_2$ axis. Let P be a typical beam element whose distance from Q in the undeformed configuration is x, let u_1 and u_3 be the displacement of P along the $\mathbf{D}_1$ and $\mathbf{D}_3$ axes, respectively. We assume that the beam is inextensible, that is the beam deflection u_2 along the $\mathbf{D}_2$ axis is identically zero. Let $\mathbf{r}(x,t) = \mathbf{OP}$ be the position vector of P. Let the beam be homogeneous with uniform cross-sections.

Neglecting gravitation, surface loads and rotatory inertia of the beam cross-sections, equations (2.4.1), (2.4.3) and (2.4.4) which are the equations of motion of the whole configuration, are now reduced to: for all $t \geq 0$

$$\frac{\partial \mathbf{n}}{\partial x} = \rho \frac{\partial^2 \mathbf{r}}{\partial t^2} \quad , \; 0 < x < L \quad , \tag{4.2.1}$$

$$\frac{\partial \mathbf{m}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x} \times \mathbf{n} = 0 \quad , \; 0 < x < L \quad , \tag{4.2.2}$$

$$I_R \dot{\omega} + \omega \times I_R \omega = \mathbf{r}(0,t) \times \mathbf{n}(0,t) + \mathbf{m}(0,t) + \mathbf{N}_c(t) \quad , \tag{4.2.3}$$

where $\mathbf{n}(x,t)$ and $\mathbf{m}(x,t)$ are the contact force and the contact moment , respectively, (see Section 3, Chapter 2), ρ is the mass per unit length of the beam, which is a constant by assumption, L is the length of the beam, I_R is the inertia tensor of the rigid body, which is diagonal, $\omega = \sum_{i=1}^{3} \omega_i \mathbf{D}_i$ is the angular velocity of the rigid body with respect to the inertial frame N and $N_c(t)$ is the control torque applied to the rigid body, (see, e.g., [Ant.1]).

Equation (4.2.1) and (4.2.2) state the balance of forces and the balance of moments at the beam cross sections and equation (4.2.3) is the rigid body angular momentum equation. Note that the first two terms in the right hand side of (4.2.3) represent the torque applied by the beam to the rigid body.

We use the Euler-Bernoulli beam model to give the component form of the contact force **n** and the contact moment **m** in terms of the beam deflections u_1, u_3. Assuming that the beam is inextensible, neglecting the torsion and neglecting the higher order terms, we express the contact force **n**, the contact moment **m**, and the position vector **r** in terms of u_1 and u_3 as follows: for $0 \le x \le L$, $t \ge 0$,

$$\mathbf{m} = m_1\mathbf{D}_1 + m_2\mathbf{D}_2 + m_3\mathbf{D}_3 \quad , \quad \mathbf{n} = n_1\mathbf{D}_1 + n_2\mathbf{D}_2 + n_3\mathbf{D}_3 \quad , \tag{4.2.4}$$

$$m_1 = EI_3 u_{3xx} \quad , \quad n_3 = -EI_3 u_{3xxx} \quad , \tag{4.2.5}$$

$$m_3 = -EI_1 u_{1xx} \quad , \quad n_1 = -EI_1 u_{1xxx} \quad , \tag{4.2.6}$$

$$\mathbf{r} = u_1\mathbf{D}_1 + (b + x)\mathbf{D}_2 + u_3\mathbf{D}_3 \quad , \tag{4.2.7}$$

where EI_1 and EI_3 are the flexural rigidity of the beam deflections along the axes $\mathbf{D}_1$ and $\mathbf{D}_3$, respectively, and b is the distance between the points O and Q. For more details on the constitutive equations, see [Mei.1].

Since we have neglected the axial and the torsional vibrations of the beam, the axial component n_2 of the contact force **n** and the torsion component m_2 of the contact moment **m** become indeterminable by the constitutive equations, (see [Pos.1]). Once the beam deflections u_1 and u_3 are found, the $\mathbf{D}_2$ components of the equations (4.2.1) and (4.2.2) can be used to find n_2 and m_2.

Since the beam is clamped to the rigid body at the point Q , we have (see Figure 4.1) :

$$u_i(0,t) = u_{ix}(0,t) = 0 \ , \quad t \geq 0 \ , \ i = 1,3 \quad . \tag{4.2.8}$$

The rest state of the system is by definition :

$$\left.\begin{array}{ll} \omega = 0 & \\ u_1(x) = u_3(x) = 0 & 0 \leq x \leq L \\ u_{1t}(x) = u_{3t}(x) = 0 & 0 \leq x \leq L \end{array}\right\} \quad . \tag{4.2.9}$$

We now state our

Stabilization Problem :

If the system given by the equations (4.2.1)-(4.2.8) is perturbed from the rest state defined by (4.2.9), find an appropriate control law that drives the system to the rest state.□

4.3 Proposed Control Laws :

We propose two stabilizing control laws. Each law consist of appropriate forces and torques applied to the beam at the free end and a torque applied to the rigid body. We note that these two control laws differ in the torque applied to the rigid body.

4.3.1 Control Law Based on Cancellation

This control scheme applies a force $\mathbf{n}(L,t)$ and a torque $\mathbf{m}(L,t)$ at the free end of the beam and a torque $\mathbf{N}_c\ (t)$ applied to the rigid body. They are specified as follows : we choose

$\alpha_i > 0$, $\beta_i > 0$, and a 3x3 symmetric positive definite constant matrix K, (which can be chosen diagonal); then for all $t \geq 0$, $i = 1, 3$, we require the following equations :

$$-EI_i u_{xxx}(L,t) + \alpha_i u_{it}(L,t) = 0 \quad , \tag{4.3.1}$$

$$EI_i u_{xx}(L,t) + \beta_i u_{ixt}(L,t) = 0 \quad , \tag{4.3.2}$$

$$\mathbf{N}_c\ (t) = -\mathbf{r}(0,t) \times \mathbf{n}\ (0,t) - \mathbf{m}\ (0,t) - K\,\omega(t) \quad . \tag{4.3.3}$$

Equation (4.3.1), {(4.3.2), resp.} represents a transversal force, {torque, resp.} applied at the free end of the beam in the direction of, {around, resp.} the axis $\mathbf{D}_i$ whose magnitude is proportional to and whose sign is opposite to the end point deflection velocity $u_{it}(L,t)$,{end-point deflection angular velocity $u_{ixt}(L,t)$, resp.} of the beam along the direction of $\mathbf{D}_i$ axis, for $i = 1, 3$. Also note that to apply the control laws given by (4.3.1)-(4.3.3), the end point deflection velocities $u_{it}\ (L,t)$, the end point deflection angular velocities $u_{ix}\ (L,t)$, the rigid body angular velocity vector $\omega(t)$ and the moment applied by the beam to the rigid body must be measured. This moment consist of the effect of the contact force $\mathbf{n}(0, t)$ and the contact moment $\mathbf{m}(0, t)$ at the clamped end. Both can be measured by using strain rosettes and strain gauges, respectively [Ana.1].

The control law (4.3.3) cancels the effect of the beam on the rigid body. To see this, substitute (4.3.3) into (4.2.3), then equation (4.2.3) becomes a set of nonlinear *ordinary* differential equations. Then substitute the solution $\omega(t)$ of (4.2.3) into the beam equation (4.2.1). Now the latter becomes a set of *linear* partial differential equations.

Equation (4.3.3) is reminiscent of a "computed torque" type control law in robotics, [Pau.1]. When substituted in (4.2.3), (4.3.3) cancels the effect of the beam on the rigid body. This type of control law recently has been applied to the attitude control of the flexible spacecraft [Ana.1].□

4.3.2 Natural Control Law

This control law applies the same boundary force $\mathbf{n}(L, t)$ and the moment $\mathbf{m}(L, t)$ as specified by the equations (4.3.1) and (4.3.2), respectively, but the torque applied to the rigid body is given by :

$$\mathbf{N}_c(t) = -\mathbf{r}(L, t) \times \mathbf{n}(L, t) - \mathbf{m}(L, t) - K\,\omega(t) \quad , \tag{4.3.4}$$

where K is a 3x3 positive definite constant matrix.

This control scheme is "natural" in the sense that it enables one to choose the total energy of the whole configuration as a Lyapunov function to study the stability of the system.

Unlike the control law (4.3.3), when (4.3.4) is substituted in (4.2.3), it does not cancel the effect of the beam on the rigid body. As a result of this, the equations (4.2.1)-(4.2.8), together with the control laws (4.3.1), (4.3.2), and (4.3.4) form a set of nonlinear ordinary and partial differential equations. This control law requires that the end-point deflections $u_i\ (L, t)$, the end-point deflection velocities $u_{it}\ (L, t)$, the end-point deflection angular velocities $u_{ixt}\ (L, t)$ and the rigid body angular velocity vector $\omega(t)$ be measured. The first three could be measured by optical means and the latter by gyros.

4.3.3 Assumption :

Throughout our analysis, the initial conditions $u_i\ (x, 0)$ and $u_{it}\ (x, 0)$ are assumed to be sufficiently differentiable (i.e., C^2 in t and C^4 in x) and compatible with the boundary conditions (4.2.8), (4.3.1), and(4.3.2), for $i = 1, 3$.

4.4 Stability Results for the Control Law Based on Cancellation :

After substituting (4.3.3) in (4.2.3), we obtain the following rigid body equation :

$$I_R \dot{\omega} + \omega \times I_R \omega = -K \omega \quad . \tag{4.4.1}$$

4.4.1 Proposition : Consider the equation (4.4.1). There exist a $c > 0$ and an $\alpha > 0$ such that for all initial conditions $\omega(0) \in R^3$, the solution $\omega(t)$ of (4.4.1) satisfies

$$||\,\omega(t)||^2 \le c\; e^{-\alpha t}\,||\,\omega(0)||^2 \qquad \text{for all } t \ge 0 \quad . \tag{4.4.2}$$

Proof: Consider the following "energy function" for the rigid body :

$$E_R(t) = \frac{1}{2} < \omega(t), I_R\, \omega(t) > \quad . \tag{4.4.3}$$

$E_R(t)$ is the rotational kinetic energy of the rigid body with respect to the inertial frame N. Also note that since $I_R = diag(I_1, I_2, I_3)$, we have

$$I_{\min}||\,\omega||^2 \le 2\,E_R \le I_{\max}||\,\omega||^2 \qquad \text{for all } \omega \in \mathrm{R}^3 \quad , \tag{4.4.4}$$

where $I_{\min} = \min(I_1, I_2, I_3)$ and $I_{\max} = \max(I_1, I_2, I_3)$.

Differentiating (4.4.3) and using (4.4.1) we obtain :

$$\begin{aligned} \dot{E_R}(t) &= < \omega, I_R \dot{\omega} > \\ &= -< \omega, \omega \times I_R \omega > - < \omega, K\, \omega > \\ &= -< \omega, K\, \omega > \quad . \end{aligned} \tag{4.4.5}$$

But, since K is symmetric and positive definite, there exist positive, nonzero constants λ_1 and λ_2, which may be taken as the minimum and the maximum eigenvalues of $\frac{1}{2}(K+K^T)$, respectively , such that the following holds :

$$\lambda_1 ||\,\omega\,||^2 \le <\omega, K\,\omega> \le \lambda_2 ||\,\omega\,||^2 \qquad \text{for all } \omega \in \mathbf{R}^3 \quad . \tag{4.4.6}$$

Using (4.4.4)-(4.4.6), we obtain (4.4.2) where $c = \dfrac{\max(I_1, I_2, I_3)}{\min(I_1, I_2, I_3)}$ and $\alpha = \dfrac{2\lambda_1}{\max(I_1, I_2, I_3)}$. □

Next, we obtain the component form of equation (4.2.1). After applying (2.2.14) twice, we obtain the following :

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_N = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_B + \dot{\omega} \times \mathbf{r} + 2\,\omega \times \left(\frac{d\mathbf{r}}{dt}\right)_B + \omega \times (\,\omega \times \mathbf{r}\,) \quad . \tag{4.4.7}$$

Using (4.4.7) in (4.2.1)-(4.2.8), we obtain the following equations which govern the motion of transverse beam deflections in $\mathbf{D}_1$ and $\mathbf{D}_3$ directions, including the boundary conditions : for all $t \ge 0$

$$\begin{aligned} EI_1\, u_{1xxxx} &+ \rho\, u_{1tt} + 2\,\rho\, \omega_2\, u_{3t} + \rho\,(\,\dot{\omega}_2 + \omega_1\,\omega_3\,)\, u_3 \\ &- \rho\,(\,\omega_2^2 + \omega_3^2\,)\, u_1 - \rho\,(\,\dot{\omega}_3 - \omega_1\omega_2\,)\,(\,b + x\,) = 0 \qquad 0 \le x \le L \quad , \end{aligned} \tag{4.4.8}$$

$$\begin{aligned} EI_3\, u_{3xxxx} &+ \rho\, u_{3tt} - 2\,\rho\, \omega_2\, u_{1t} - \rho\,(\,\dot{\omega}_2 - \omega_1\,\omega_3\,)\, u_1 \\ &- \rho\,(\,\omega_1^2 + \omega_2^2\,)\, u_3 + \rho\,(\,\dot{\omega}_1 + \omega_2\omega_3\,)\,(\,b + x\,) = 0 \qquad 0 \le x \le L \quad , \end{aligned} \tag{4.4.9}$$

$$u_1(L,t) = u_3(L,t) = 0 \quad , \quad u_{1x}(L,t) = u_{3x}(L,t) = 0 \quad , \tag{4.4.10}$$

$$-EI_1\, u_{1xxx}(L,t)+\alpha_1 u_{1t}(L,t)=0 \quad , \; -EI_3\, u_{3xxx}(L,t)+\alpha_3 u_{3t}(L,t)=0 \quad , \tag{4.4.11}$$

$$EI_1 u_{1xx}(L,t)+\beta_1 u_{1xt}(L,t)=0 \quad , \; EI_3 u_{3xx}(L,t)+\beta_3 u_{3xt}(L,t)=0 \quad . \tag{4.4.12}$$

Equations (4.4.8) and (4.4.9) can be rewritten in the following state space form :

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{EI_1}{\rho}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{EI_3}{\rho}\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^2+\omega_3^2 & 0 & -(\dot{\omega}_2+\omega_1\omega_3) & -2\omega_2 \\ 0 & 0 & 0 & 0 \\ \dot{\omega}_2-\omega_1\omega_3 & 2\omega_2 & \omega_1^2+\omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_{1t} \\ u_3 \\ u_{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ (\dot{\omega}_3-\omega_1\omega_2)\,(b+x) \\ 0 \\ -(\dot{\omega}_1+\omega_2\omega_3)\,(b+x) \end{bmatrix} \quad , \tag{4.4.13}$$

whose solutions evolve in the following function space H :

$$H = \{\, (u_1 \;\; u_{1t} \;\; u_3 \;\; u_{3t})^T \mid u_1 \in \mathrm{H}_0^2,\; u_3 \in \mathrm{H}_0^2,\, u_{1t} \in \mathrm{L}^2 ,\, u_{3t} \in \mathrm{L}^2 \,\} \quad , \tag{4.4.14}$$

where the function spaces L^2, H^k and $\mathrm{H}^k{}_0$ are as defined below :

$$\mathrm{L}^2 = \{f : [0,L] \to \mathbf{R} \mid \int_{x=0}^{x=L} f^2\,dx < \infty\} \quad ,$$

$$\mathrm{H}^k = \{f \in \mathrm{L}^2 \mid f^i \in \mathrm{L}^2 ,\, i=1,\dots,k\} \quad ,$$

$$\mathrm{H}^k{}_0 = \{f \in \mathrm{H}^k \mid f(0)=f^1(0)=0\} \quad .$$

In H, we define the following inner product, which is called "energy" inner product

$$< z , \hat{z} >_E := \int_0^L (\, EI_1 u_{1xx}\, \hat{u}_{1xx} + EI_3 u_{3xx}\, \hat{u}_{3xx}\,)\, dx$$

$$+\int_0^L \rho(u_{1t}\hat{u}_{1t} + u_{3t}\hat{u}_{3t})\,dx \qquad \text{for all } z, \hat{z} \in H \quad . \tag{4.4.15}$$

Note that, (4.4.15) induces a norm on H, which is called "energy norm". This norm is equivalent to a standard "Sobolev" type norm which makes H an Hilbert space.(for more details, see [Paz.1] and [Che.2]).

To put (4.4.13) into an abstract equation form, we define the following operators $A : H \to H$, $B : \mathbf{R}_+ \times H \to H$ and function $f : \mathbf{R}_+ \to H$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{EI_1}{\rho}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{EI_3}{\rho}\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix} , \tag{4.4.16}$$

$$B(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^2 + \omega_3^2 & 0 & -(\dot{\omega}_2 + \omega_1\omega_3) & -2\omega_2 \\ 0 & 0 & 0 & 0 \\ \dot{\omega}_2 - \omega_1\omega_3 & 2\omega_2 & \omega_1^2 + \omega_3^2 & 0 \end{bmatrix} , \tag{4.4.17}$$

$$f(t) = \begin{bmatrix} 0 \\ (\dot{\omega}_3 - \omega_1\omega_2)(b+x) \\ 0 \\ -(\dot{\omega}_1 + \omega_2\omega_3)(b+x) \end{bmatrix} . \tag{4.4.18}$$

4.4.2 Remark : The operator A is an unbounded linear operator on H, i.e., it is not continuous as a map on H. The operator $B(.)$ is bounded on $\mathbf{R}_+$. Since $\omega(t)$ and $\dot{\omega}(t)$ are exponentially decaying functions of t, (see proposition 4.4.1 and equation (4.4.1)), so is $\| B(t) \|$, where the norm used here is the norm induced by the energy inner product given by (4.4.15) . □

Using the above definitions, equation (4.4.13) can be put into the following abstract form :

$$\frac{dz}{dt} = A\,z + B(t)\,z + f(t) \quad , \quad z(0) = z_0 \in H \quad , \; t \geq 0 \quad , \tag{4.4.19}$$

where $z = (\,u_1 \;\; u_{1t} \;\; u_3 \;\; u_{3t}\,)^T$. The domain $D(A)$ of the operator A is defined as follows :

$$\begin{aligned} D(A) = \{\, (\,u_1 \;\; u_{1t} \;\; u_3 \;\; u_{3t})^T : \; & u_1 \in \mathrm{H}_0^4, u_{1t} \in \mathrm{H}_0^4, u_3 \in \mathrm{H}_0^2\,, u_{3t} \in \mathrm{H}_0^2\,, \\ & -EI_1\, u_{1xxx}(L) + \alpha_1\, u_{1t}(L) = 0\,, \\ & EI_1\, u_{1xx}(L) + \beta_1\, u_{1xt}(L) = 0\,, \\ & -EI_3\, u_{3xxx}(L) + \alpha_2\, u_{3t}(L) = 0\,, \\ & EI_3\, u_{3xx}(L) + \beta_2\, u_{3xt}(L) = 0 \,\} \quad . \end{aligned} \tag{4.4.20}$$

From lemma 3.4.2 it follows that $D(A)$ given by (4.4.20) is dense in H given by (4.4.14).

Next, we state the existence and uniqueness theorem of the solutions of (4.4.19).

4.4.3 Theorem : Consider equation (4.4.19) with A, B, f defined in (4.4.16)-(4.4.18), respectively; or equivalently consider equations (4.4.8)-(4.4.12). Then :

(i) The operator A generates an exponentially decaying C_0 semigroup $T(t)$ in H: that is, there exist positive constants $M > 0$ and $\delta > 0$ such that

$$||\,T(t)\,|| \leq M\, e^{-\delta t} \quad \text{for all } t \geq 0 \quad , \tag{4.4.21}$$

where for all $t \geq 0$, $T(t)$ are bounded linear maps in H;

(ii) for all $z_0 \in D(A)$, the differential equation given by (4.4.19) has unique classical solution , defined for all $t \geq 0$;

(iii) in terms of $T(t)$, that solution $z(t)$ of (4.4.19) may be written as :

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\,B(s)\,z(s)\,ds + \int_0^t T(t-s)\,f(s)\,ds, \quad \text{for all } t \geq 0 \quad . \tag{4.4.22}$$

Proof :

(i) Due to the block diagonal form of A, Assertion (i) is an easy extension of theorem 3.4.7.

(ii) Since $B(.)$ is globally lipschitz on H and $\|B(t)\|$ is exponentially decaying due to Proposition 4.4.1, (also see remark 4.4.2), it follows that $A+B(.)$ defines a unique, globally defined semigroup on H, (see, e.g., [Mar.1, pp. 388-390] , [Paz.1, pp. 185,190]). Since $f \in L^1[\mathbf{R}, H]$ and is a C^∞ function of t, (see (4.4.18)), by standard theorems on nonhomogeneous linear partial differential equations (see, e.g., [Paz.1, pp. 105-110]), it follows that (4.4.19) has unique solution in H defined for all $t \geq 0$.

(iii) That the solution may be given as (4.4.22) can be verified by substitution, using $\frac{dT}{dt} = A\ T$ and $T(0) = I$. □

Next, we prove the exponential decay of the solutions of (4.4.19).

4.4.4 Theorem : Consider equation (4.4.19), where the operators $A, B(.)$ and the function $f(.)$ are defined in (4.4.16),(4.4.17) and (4.4.18) respectively; or equivalently consider (4.4.8)-(4.4.12). Then for all $z_0 \in D(A)$, the solution $z(.)$ of (4.4.19) decays exponentially to 0.

Proof : By taking norms in (4.4.22) and using (4.4.21), we obtain : for all $t \geq 0$

$$|| z(t)|| \leq M e^{-\delta t} || z_0 || + \int_0^t M e^{-\delta(t-s)} || B(s)|| \, || z(s)|| \, ds$$

$$+ \int_0^t M e^{-\delta(t-s)} || f(s)|| \, ds \quad . \tag{4.4.23}$$

But since $\omega(t)$ and $\dot{\omega}(t)$ are decaying exponentially, it follows from (4.4.17) and (4.4.18) that there exist positive constants $c_1 > 0$, $c_2 > 0$, $\delta_1 > 0$, and $\delta_2 > 0$, such that for all $t \geq 0$

$$|| B(t)|| \leq c_1 e^{-\delta_1 t} \quad , \tag{4.4.24}$$

$$|| f(t)|| \leq c_2 e^{-\delta_2 t} \quad , \tag{4.4.25}$$

Using (4.4.24),(4.4.25) in (4.4.23), evaluating the last integral, and multiplying each side of (4.4.23) by $e^{\delta t}$, we obtain :

$$|| z(t) e^{\delta t} || \leq M || z_0 || + \frac{M c_2}{\delta - \delta_2} (e^{(\delta - \delta_2)t} - 1) + \int_0^t M c_1 e^{-\delta_1 s} || z(s) e^{\delta s} || \, ds \quad , \tag{4.4.26}$$

Now applying a general form of Bellmann- Gronwall lemma ,(see, e.g., [Des.1]), and using the following simple estimate

$$\int_{t_1}^t e^{-\delta_1 s} \, ds \leq \int_0^\infty e^{-\delta_1 s} \, ds \leq \frac{1}{\delta_1} \quad , \tag{4.4.27}$$

we obtain the following :

$$|| z(t) e^{\delta t} || \leq M || z_0 || + \frac{M c_2}{\delta - \delta_2} (e^{(\delta - \delta_2)t} - 1)$$

$$+ \int_0^t M c_1 e^{\frac{M c_1}{\delta_1}} [M || z_0 || + \frac{M c_2}{\delta - \delta_2} (e^{(\delta - \delta_2)s} - 1)] \, e^{-\delta_1 s} \, ds$$

$$\leq M\,||\,z_0\,||\,+\frac{M\;c_2}{\delta-\delta_2}\,(e^{(\delta-\delta_2)t}-1)\,+\,\frac{M^2c_1}{\delta_1}\,e^{\frac{M\,c_1}{\delta_1}}\,(||\,z_0\,||\,-\frac{c_2}{\delta-\delta_2})\,(1-e^{-\delta_1 t})$$

$$-\frac{M^2\,c_1\,c_2}{(\delta-\delta_2)(\delta-\delta_1-\delta_2)}\,e^{\frac{M\,c_1}{\delta_1}}\,(\,1-e^{(\delta-\delta_1-\delta_2)t}) \quad . \qquad (4.4.28)$$

Multiplying each side with $e^{-\delta t}$, we conclude that $z\,(.)$ decays exponentially. □

4.5 Stability Results for the Natural Control Scheme

For simplicity we will take the positive definite matrix K as $K = diag(k_1, k_2, k_3)$, (see (4.3.4)). Then equations (4.2.1)-(4.2.3) together with the boundary conditions (4.2.8) and the natural control law (4.3.1),(4.3.2), and (4.3.4) become : for all $t \geq 0$

$$EI_1 u_{1xxxx} + \rho u_{1tt} + 2\rho \omega_2 u_{3t} + \rho(\dot{\omega}_2 + \omega_1 \omega_3) u_3$$

$$-\rho(\omega_2^2 + \omega_3^2) u_1 - \rho(\dot{\omega}_3 - \omega_1\omega_2)(b + x) = 0 \qquad 0 < x < L \quad , \tag{4.5.1}$$

$$EI_3 u_{3xxxx} + \rho u_{3tt} - 2\rho \omega_2 u_{1t} - \rho(\dot{\omega}_2 - \omega_1 \omega_3) u_1$$

$$-\rho(\omega_1^2 + \omega_2^2) u_3 + \rho(\dot{\omega}_1 + \omega_2\omega_3)(b + x) = 0 \qquad 0 < x < L \quad , \tag{4.5.2}$$

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + k_1\omega_1 = EI_3\int_0^L (b + x) u_{3xxxx}\, dx \quad , \tag{4.5.3}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 + k_2\omega_2 = EI_3\int_0^L u_1 u_{3xxxx}\, dx - EI_1\int_0^L u_3 u_{1xxxx}\, dx \quad , \tag{4.5.4}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + k_3\omega_3 = -EI_1\int_0^L (b + x) u_{1xxxx}\, dx \quad , \tag{4.5.5}$$

$$u_1(L,t) = u_3(L,t) = 0 \quad , u_{1x}(L,t) = u_{3x}(L,t) = 0 \quad , \tag{4.5.6}$$

$$-EI_1 u_{1xxx}(L,t) + \alpha_1 u_{1t}(L,t) = 0 \quad , -EI_3 u_{3xxx}(L,t) + \alpha_3 u_{3t}(L,t) = 0 \quad , \tag{4.5.7}$$

$$EI_1 u_{1xx}(L,t) + \beta_1 u_{1xt}(L,t) = 0 \quad , EI_3 u_{3xx}(L,t) + \beta_3 u_{3xt}(L,t) = 0 \quad . \tag{4.5.8}$$

To prove the stability of the system given by (4.5.1)-(4.5.8), we first define the energy of the system as follows :

$$E(t)=\frac{1}{2}<\omega\,,I_R\,\omega>+\frac{1}{2}\int_0^L\rho<\mathbf{r}_t\,,\mathbf{r}_t>dx+\frac{1}{2}\int_0^L(EI_1u_{1xx}{}^2+EI_3u_{3xx}{}^2)\,dx\quad, \tag{4.5.9}$$

where <.,.> denotes the standard inner product in $\mathbf{R}^3$; the first term in (4.5.9) is the rotational kinetic energy of the rigid body, the second term is the kinetic energy of the beam, both with respect to the inertial frame **N**, and the last term is the potential energy of the beam .

4.5.1 Proposition : Consider the system given by the equations (4.5.1)-(4.5.8) . Then the energy $E(t)$ defined by (4.5.9) is a nonincreasing function of t.

Proof : By differentiating $E(t)$ with respect to t, and using (4.2.1), (2.2.14), we obtain

$$\frac{d}{dt}E(t)=<\omega\,,I_R\,\dot{\omega}+\omega\times I_R\;\omega>+\int_0^L\rho<\mathbf{r}_t\,,\mathbf{r}_{tt}>dx$$

$$+\int_0^L(EI_1u_{1xx}\,u_{1xxt}+EI_3u_{3xx}\,u_{3xxt})\,dx$$

$$=<\omega\,,I_R\,\dot{\omega}+\omega\times I_R\;\omega>+\int_0^L<\mathbf{r}_t\,,\mathbf{n}_x>dx$$

$$+\int_0^L(EI_1u_{1xx}\,u_{1xxt}+EI_3u_{3xx}u_{3xxt})\,dx$$

$$=<\omega\,,I_R\,\dot{\omega}+\omega\times I_R\;\omega>+\int_0^L<(\mathbf{r}_t)_B\,,\mathbf{n}_x>dx+\int_0^L<\omega\times\mathbf{n}_x>dx$$

$$+\int_0^L(EI_1u_{1xx}\,u_{1xxt}+EI_3u_{3xx}\,u_{3xxt})\,dx$$

$$= <\omega\,, I_R\,\dot{\omega} + \omega \times I_R\;\omega> + \int_0^L <\omega \times \mathbf{n}_x> dx - EI_1 \int_0^L u_{1t}\; u_{1xxxx}\, dx$$

$$- EI_3 \int_0^L u_{3t}\; u_{3xxxx}\, dx + EI_1 \int_0^L u_{1xx}\; u_{1xxt}\, dx + EI_3 \int_0^L u_{3xx}\; u_{3xxt}\, dx \quad , \tag{4.5.10}$$

Using integration by parts we obtain the following equation , for $i = 1, 3$:

$$EI_i \int_0^L u_{it}\; u_{ixxxx}\, dx = EI_i\, u_{ixxx}(L\,,t) u_{it}(L\,,t)$$

$$- EI_i\, u_{ixx}(L\,,t) u_{ixt}(L\,,t) + EI_i \int_0^L u_{ixx}\; u_{ixxt}\, dx \quad . \tag{4.5.11}$$

Using (4.5.11) and boundary conditions (4.5.7) and (4.5.8) in (4.5.10), we obtain

$$\dot{E}(\,t\,) = - <\omega, K\,\omega> - \alpha_1\, u_{1t}{}^2(L\,,t) - \alpha_3\, u_{3t}{}^2(L\,,t)$$

$$- \beta_1\, u_{1xt}{}^2(L\,,t) - \beta_3\, u_{3xt}{}^2(L\,,t) \leq 0 \quad . \tag{4.5.12}$$

Since the rate of change of the energy is nonpositive, it follows that the energy is a nonincreasing function of time. for all $z \in H$. □

4.5.2 Remark : If we set $\alpha_i = \beta_i = 0$, for $i = 1, 3$, and $K = 0$, (i.e no control applied to the system), we obtain $\dot{E}(t) = 0$: as expected, the total energy (given by the equation (4.5.9)) is conserved. □

4.5.3 Remark : We need an estimate, which states that if the energy given by (4.5.9) stays bounded, then so does the beam deflections $u_i(x,t)$ and their derivatives $u_{ix}(x\,,t)$,(hence so does $\mathrm{r}(x\,,t)$), for all $x \in [0, L\,]$, for $i = 1, 3$. Using the boundary condi tions and the fundamental theorem of calculus, for $i = 1, 3$ we conclude that for all $0 \leq x \leq L$, for all $t \geq 0$:

$$u_i(x,t) = \int_0^x u_{is}(s,t)\, ds \quad . \tag{4.5.13}$$

Therefore, using Jensen's inequality, (see, e.g., [Roy.1, p.110], [Mit.1]), we obtain

$$(u_i(x,t))^2 \leq L \int_0^L u_{is}{}^2(s,t)\, ds \quad . \tag{4.5.14}$$

By using the same arguments, we obtain, for all $x \in [0, L]$

$$(u_{ix}(x,t))^2 \leq L \int_0^L u_{iss}{}^2(s,t)\, ds \tag{4.5.15}$$

hence, combining (4.5.13) and (4.5.14), we obtain :

$$(u_i(x,t))^2 \leq L \int_0^L u_{is}{}^2(s,t)\, ds \leq L^2 \int_0^L u_{iss}{}^2(s,t)\, ds \quad . \square \tag{4.5.16}$$

Next, we will show that the rate of decay of the energy is at least $\frac{1}{t}$ for large t.

4.5.4 Theorem : Consider the system described by the equations (4.5.1)-(4.5.8). Then there exists a $T \geq 0$ such that the energy given by (4.5.9) is bounded above by $O(\frac{1}{t})$ for all $t \geq T$.

Proof : As in the proof of Theorem 3.4.8, we first define the following function $V(t)$:

$$V(t) = 2(1-\varepsilon)E(t) + 2\int_0^L \rho\, x\, [u_{1t} + \omega_2 u_3 - \omega_3(b+x)]\, u_{1x}\, dx$$

$$+2\int_0^L \rho\, x\, [u_{3t} + \omega_1(b+x) - \omega_2 u_1]\, u_{3x}\, dx \quad , \tag{4.5.17}$$

where $\varepsilon \in (0,1)$ is an arbitrary real number.

We prove the theorem in two steps. First we show that for some constant $C_1 > 0$, the following estimate holds :

$$[2(1-\varepsilon)t - C_1]E(t) \le V(t) \le [2(1-\varepsilon)t + C_1]E(t) \qquad t \ge 0 \quad . \tag{4.5.18}$$

Then differentiating $V(t)$, we show that there exists a $T > 0$ such that :

$$\frac{dV(t)}{dt} \le 0 \qquad t \ge T \quad . \tag{4.5.19}$$

Combining (4.5.18) and (4.5.19), we obtain :

$$E(t) \le \frac{V(T)}{2(1-\varepsilon)t - C_1} \qquad t > \max\{T, \frac{C_1}{2(1-\varepsilon)}\} \quad . \tag{4.5.20}$$

Since by Proposition 4.5.1 the energy $E(t)$ is bounded on $\mathbf{R}_+$, it follows that $V(T) < \infty$, hence (4.5.20) proves that for sufficiently large t, $E(t)$ decays as $O(\frac{1}{t})$.

For simplicity, we define the quantities J_1 and J_2 as follows :

$$J_1 = 2\int_0^L \rho\, x\, [u_{1t} + \omega_2 u_3 - \omega_3(b+x)]\, u_{1x}\, dx \quad , \tag{4.5.21}$$

$$J_2 = 2\int_0^L \rho\, x\, [u_{3t} + \omega_1(b+x) - \omega_2 u_1]\, u_{3x}\, dx \quad . \tag{4.5.22}$$

Also applying the differentiation rule (2.2.14) to (4.2.7), we obtain :

$$\mathbf{r}_t = [u_{1t} + \omega_2 u_3 - \omega_3(b+x)]\,\mathbf{D}_1 + [\omega_3 u_1 - \omega_1 u_3]\,\mathbf{D}_2$$

$$+ [u_{3t} + \omega_1(b+x) - \omega_2 u_1]\,\mathbf{D}_3 \quad . \qquad (4.5.23)$$

To obtain (4.5.18), we need the following simple inequalities :

$$(a+b)^2 \le 2\,(a^2+b^2) \qquad a\,,b \in \mathbf{R} \quad , \qquad (4.5.24)$$

$$a\,b \le \delta^2 a^2 + \frac{b^2}{\delta^2} \qquad a\,,b\,,\delta \in \mathbf{R}\,,\,\delta \ne 0 \quad . \qquad (4.5.25)$$

Using (4.5.25) in (4.5.21) and (4.5.22), we obtain the following estimates :

$$|J_1| \le 2\,\rho\,L \int_0^L [u_{1t} + \omega_2 u_3 - \omega_3(b+x)]^2\,dx + 2\,\rho\,L \int_0^L u_{1x}{}^2\,dx$$

$$\le 2\,L \int_0^L \rho < \mathbf{r}_t\,,\mathbf{r}_t > dx + 2\,\rho\,L^2 \int_0^L u_{1xx}{}^2\,dx$$

$$\le K_1 E(t) \qquad \text{for all } t \ge 0 \quad , \qquad (4.5.26)$$

where $K_1 = 4L + \dfrac{4\rho L^2}{EI_1}$. The second inequality follows from (4.5.16) and (4.5.23), and then, (4.5.26) follows from (4.5.9);

$$|J_2| \le 2\rho L \int_0^L [u_{3t} + \omega_1(b+x) - \omega_2 u_1]^2\,dx + 2\,\rho\,L \int_0^L u_{3x}{}^2\,dx$$

$$\le 2\,L \int_0^L \rho < \mathbf{r}_t\,,\mathbf{r}_t > dx + 2\,\rho\,L^2 \int_0^L u_{3xx}{}^2\,dx$$

$$\leq K_2 E(t) \qquad \text{for all } t \geq 0 \quad , \tag{4.5.27}$$

where $K_2 = 4L + \dfrac{4\rho L^2}{EI_3}$. The second inequality follows from (4.5.16) and (4.5.23), and then (4.5.27) follows from (4.5.9).

Using (4.5.26) and (4.5.27) in (4.5.17) we obtain :

$$[2(1-\varepsilon)t - K_1 - K_2]E(t) \leq V(t) \leq [2(1-\varepsilon)t + K_1 + K_2]E(t) \qquad t \geq 0 \quad ,$$

which proves (4.5.18).

To prove (4.5.19), we first differentiate J_1

$$\frac{dJ_1}{dt} = 2\int_0^L \rho\, x\, [\, u_{1tt} + \dot{\omega}_2 u_3 + \omega_2 u_{3t} - \dot{\omega}_3(b+x)\,]\, u_{1x}\, dx$$

$$+ 2\int_0^L \rho\, x\, [u_{1t} + \omega_2 u_3 - \omega_3(b+x)]\, u_{1xt}\, dx$$

$$= 2\int_0^L x\, [\, -EI_1 u_{1xxxx} - \rho\omega_2 u_{3t} - \rho\omega_1\omega_3 u_3$$

$$+ \rho(\omega_2^2 + \omega_3^2)u_1 - \rho\omega_1\omega_2(b+x)]\, u_{1x}\, dx + 2\int_0^L \rho\, x\, u_{1t}\, u_{1xt}\, dx$$

$$+ 2\omega_2 \int_0^L \rho\, x\, u_3\, u_{1xt}\, dx - 2\omega_3 \int_0^L \rho\, x\, (b+x) u_{1xt}\, dx$$

$$= -2EI_1 \int_0^L x\, u_{1x} u_{1xxxx}\, dx - 2\rho\omega_2 \int_0^L x u_{1x} u_{3t}\, dx - 2\rho\omega_1\omega_3 \int_0^L x u_3 u_{1x}\, dx$$

$$+\rho(\omega_2^2+\omega_3^2)[Lu_1^2(L,t)-\int_0^L u_1^2\,dx]-2\rho\omega_1\omega_2\int_0^L x(b+x)u_{1x}\,dx-\rho Lu_{1t}^2(L,t)$$

$$-\int_0^L u_{1t}^2\,dx+2\rho\omega_2[\,Lu_3(L,t)u_{1t}(L,t)-\int_0^L xu_{3x}u_{1t}\,dx-\int_0^L u_3u_{1t}\,dx\,]$$

$$-2\rho\omega_3[L(b+L)u_{1t}(L,t)-\int_0^L(b+2x)u_{1t}\,dx] \quad , \qquad (4.5.29)$$

where in the second equation we used (4.5.1). Then, integrating by parts and using the boundary conditions (4.5.6) we obtain (4.5.29).

Similarly, differentiating J_2, we obtain

$$\frac{dJ_2}{dt}=2\int_0^L\rho\,x\,[u_{3tt}+\dot{\omega}_1(b+x)-\dot{\omega}_2u_1-\omega_2u_{1t}]\,u_{3x}\,dx$$

$$+2\int_0^L\rho\,x\,[u_{3t}+\omega_1(b+x)-\omega_2u_1]\,u_{3xt}\,dx$$

$$=2\int_0^L x\,[-EI_3u_{3xxxx}+\rho\omega_2u_{1t}-\rho\omega_1\omega_3u_1$$

$$+\rho(\omega_1^2+\omega_2^2)u_1-\rho\omega_2\omega_3(b+x)]\,u_{3x}\,dx+2\int_0^L\rho\,x\,u_{3t}\,u_{3xt}\,dx$$

$$-2\omega_2\int_0^L\rho\,x\,u_1\,u_{3xt}\,dx+2\omega_1\int_0^L\rho\,x\,(b+x)u_{3xt}\,dx$$

$$=-2EI_3\int_0^L x\,u_{3x}u_{3xxxx}\,dx+2\rho\omega_2\int_0^L xu_{3x}u_{1t}\,dx-2\rho\omega_1\omega_3\int_0^L xu_1u_{3x}\,dx$$

$$+\rho(\omega_1^2+\omega_2^2)[Lu_3^2(L,t)-\int_0^L u_3^2\,dx]-2\rho\omega_2\omega_3\int_0^L x(b+x)u_{3x}\,dx+\rho Lu_{3t}^2(L,t)$$

$$-\int_0^L u_{3t}^2\,dx-2\rho\omega_2[\,Lu_1(L,t)u_{3t}(L,t)-\int_0^L xu_{1x}u_{3t}\,dx-\int_0^L u_1u_{3t}\,dx\,]$$

$$+2\rho\omega_1[L(b+L)u_{3t}(L,t)-\int_0^L (b+2x)u_{3t}\,dx\,] \quad , \tag{4.5.30}$$

where in the second equation we used (4.5.2). Then, integrating by parts, using the boundary conditions we obtain (4.5.30).

Differentiating $V(t)$ with respect to time, using (4.5.23) to evaluate the inner product $<\mathbf{r}_t\,,\mathbf{r}_t>$ and using (4.5.29), (4.5.30) we obtain the following :

$$\frac{dV(t)}{dt}=2(1-\varepsilon)t\frac{dE(t)}{dt}+2(1-\varepsilon)E(t)+\frac{dJ_1}{dt}+\frac{dJ_2}{dt}$$

$$=-2(1-\varepsilon)t<\omega\,,K\,\omega>-2(1-\varepsilon)t\,\alpha_1u_{1t}^2(L,t)-2(1-\varepsilon)t\,\alpha_3u_{3t}^2(L,t)$$

$$-2(1-\varepsilon)t\,\beta_1u_{1xt}^2(L,t)-2(1-\varepsilon)t\,\beta_3u_{3xt}^2(L,t)$$

$$+(1-\varepsilon)<\omega\,,I_R\,\omega>+(1-\varepsilon)EI_1\int_0^L u_{1xx}^2\,dx+(1-\varepsilon)EI_3\int_0^L u_{3xx}^2\,dx$$

$$-2\,EI_1\int_0^L xu_{1x}u_{1xxxx}\,dx\;-2\,EI_3\int_0^L xu_{3x}u_{3xxxx}\,dx$$

$$-\varepsilon\int_0^L \rho u_{1t}^2\,dx-\varepsilon\int_0^L \rho u_{3t}^2\,dx+2(1-\varepsilon)\int_0^L \rho u_{1t}[\omega_2u_3-\omega_3(b+x)]\,dx$$

$$+(1-\varepsilon)\int_0^L \rho[\omega_2u_3-\omega_3(b+x)]^2\,dx+(1-\varepsilon)\int_0^L \rho[\omega_3u_1-\omega_1u_3]^2\,dx$$

$$+2(1-\varepsilon)\int_0^L \rho u_{3t}[\omega_1(b+x)] - \omega_2 u_1]\, dx + (1-\varepsilon)\int_0^L \rho[\omega_1(b+x) - \omega_2 u_1]^2\, dx$$

$$-2\rho\omega_2\int_0^L x u_{1x} u_{3t}\, dx - 2\rho\omega_1\omega_3\int_0^L x u_3 u_{1x}\, dx$$

$$+\rho(\omega_2^2+\omega_3^2)[L u_1^2(L,t) - \int_0^L u_1^2\, dx] - 2\rho\omega_1\omega_2\int_0^L x(b+x)u_{1x}\, dx$$

$$+\rho L u_{1t}^2(L,t) + 2\rho\omega_2[\, L u_3(L,t) u_{1t}(L,t) - \int_0^L x u_{3x} u_{1t}\, dx - \int_0^L u_3 u_{1t}\, dx\,]$$

$$-2\rho\omega_3[L(b+L)u_{1t}(L,t) - \int_0^L (b+2x)u_{1t}\, dx\,]$$

$$+2\rho\omega_2\int_0^L x u_{3x} u_{1t}\, dx - 2\rho\omega_1\omega_3\int_0^L x u_1 u_{3x}\, dx$$

$$+\rho(\omega_1^2+\omega_2^2)[L u_3^2(L,t) - \int_0^L u_3^2\, dx] - 2\rho\omega_2\omega_3\int_0^L x(b+x)u_{3x}\, dx$$

$$+\rho L u_{3t}^2(L,t) - 2\rho\omega_2[\, L u_1(L,t) u_{3t}(L,t) - \int_0^L x u_{1x} u_{3t}\, dx - \int_0^L u_1 u_{3t}\, dx\,]$$

$$+2\rho\omega_1[L(b+L)u_{3t}(L,t) - \int_0^L (b+2x)u_{3t}\, dx\,] \quad . \tag{4.5.31}$$

We need the following estimates for some of the terms which appear in (4.5.31):

$$k_{\min}(\omega_1^2+\omega_2^2+\omega_3^2) \le <\omega\, , K\, \omega> \le k_{\max}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \tag{4.5.32}$$

where we put $\omega = \sum_{i=1}^{3}\omega_i \mathbf{D}_i$, $K = diag\,(k_1\, , k_2\, , k_3)$ and $k_{\min} = \min(k_1\, , k_2\, , k_3)$ and

$k_{\max} = \max(k_1, k_2, k_3)$;

$$I_{\min}(\omega_1^2 + \omega_2^2 + \omega_3^2) \le <\omega , I_R\, \omega> \le I_{\max}(\omega_1^2 + \omega_2^2 + \omega_3^2) \quad , \tag{4.5.33}$$

where $I_R = diag\,(I_1, I_2, I_3)$, $I_{\min} = \min(I_1, I_2, I_3)$, and $I_{\max} = \max(I_1, I_2, I_3)$;

$$-2\,EI_i \int_0^L x\, u_{ixxxx}\, u_{ix}\, dx = -2\,EI_i\, Lu_{ix}(L,t)\, u_{ixxx}(L,t) + 2\,EI_i\, u_{ix}(L,t)\, u_{ixx}(L,t)$$

$$+ EI_i\, Lu_{ixx}^2(L,t) - 3\,EI_i \int_0^L u_{ixx}^2\, dx$$

$$= 2\,\alpha_i\, L\, u_{ix}(L,t)\, u_{it}(L,t) - 2\,\beta_i\, u_{ix}(L,t)\, u_{ixt}(L,t)$$

$$+ \frac{L\beta_i^2}{EI_i}\, u_{ixt}^2(L,t) - 3\,EI_i \int_0^L u_{ixx}^2\, dx$$

$$\le 2\,L\,\alpha_i\,\gamma_i^2 u_{ix}^2(L,t) + \frac{2\,L\,\alpha_i}{\gamma_i^2} u_{it}^2(L,t)$$

$$+ 2\,\beta_i\,\sigma_i^2 u_{ix}^2(L,t) + \frac{2\,\beta_i}{\sigma_i^2} u_{ixt}^2(L,t)$$

$$+ \frac{L\beta_i^2}{EI_i}\, u_{ixt}^2(L,t) - 3\,EI_i \int_0^L u_{ixx}^2\, dx \quad , \tag{4.5.34}$$

for $i = 1, 3$, where γ_i , σ_i , $i = 1, 3$ are arbitrary nonzero real numbers; in the first equation we used integration by parts and the boundary conditions (4.5.6), in the second equation we used the boundary controls (4.5.7) and (4.5.8). Then using (4.5.24), we obtain (4.5.34) .

Using (4.5.16), (4.5.9) and the fact that $E(t) \le E(0)$, (see proposition 4.5.1), we obtain the following estimates :

$$u_{ix}^2(s,t) \le L\int_0^L u_{ix}^2\,dx \;\le L^2\int_0^L u_{ixx}^2\,dx \;\le \frac{2L^2}{EI_i}E(0) \quad , \tag{4.5.35}$$

$$\int_0^L \rho u_{1t}[\omega_2 u_3 - \omega_3(b+x)]\,dx \le \delta_1^2\int_0^L \rho u_{1t}^2\,dx + \frac{\rho}{\delta_1^2}\int_0^L [\omega_2 u_3 - \omega_3(b+x)]^2\,dx$$

$$\le \delta_1^2\int_0^L \rho u_{1t}^2\,dx + \frac{2\rho}{\delta_1^2}\omega_2^2\int_0^L u_3^2\,dx + \frac{2\rho}{\delta_1^2}\omega_3^2\int_0^L (b+x)^2\,dx$$

$$\le \delta_1^2\int_0^L \rho u_{1t}^2\,dx + \frac{K_1}{\delta_1^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \tag{4.5.36}$$

where δ_1 is an arbitrary nonzero real number, $K_1 = \max\{\frac{4\rho L^3 E(0)}{EI_3}, 2\rho\int_0^L (b+x)^2\,dx\}$. The first inequality in (4.5.36) follows from (4.5.25), the second inequality follows from (4.5.24). Then using (4.5.35), we obtain (4.5.36).

Similarly, using (4.5.25) and (4.5.35), we obtain the following estimates :

$$\int_0^L \rho[\omega_2 u_3 - \omega_3(b+x)]^2\,dx \le 2\rho\omega_2^2\int_0^L u_3^2\,dx + 2\rho\omega_3^2\int_0^L (b+x)^2\,dx$$

$$\le K_2(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \tag{4.5.37}$$

where $K_2 = \max\{\frac{4\rho L^3 E(0)}{EI_3}, 2\rho\int_0^L (b+x)^2\,dx\}$, the first equation follows from (4.5.24), and then using (4.5.35), we obtain (4.5.37);

$$\int_0^L \rho[\omega_3 u_1 - \omega_1 u_3]^2\, dx \le K_3(\omega_1^2 + \omega_2^2 + \omega_3^2) \quad , \tag{4.5.38}$$

where $K_3 = \max\{\frac{4\rho L^3 E(0)}{EI_1}, \frac{4\rho L^3 E(0)}{EI_3}\}$;

$$\int_0^L \rho u_{3t}[\omega_1(b+x) - \omega_2 u_1]\, dx \le \delta_2^2 \int_0^L \rho u_{3t}^2\, dx + \frac{\rho}{\delta_2^2}\int_0^L [\omega_1(b+x) - \omega_2 u_1]^2\, dx$$

$$\le \delta_2^2 \int_0^L \rho u_{3t}^2\, dx + \frac{K_4}{\delta_2^2}(\omega_1^2 + \omega_2^2 + \omega_3^2) \quad , \tag{4.5.39}$$

where $K_4 = \max\{\frac{4\rho L^3 E(0)}{EI_1}, 2\rho\int_0^L (b+x)^2\, dx\}$ and δ_2 is an arbitrary nonzero real number, the first inequality follows from (4.5.24), (4.5.25) and then using (4.5.35) we obtain (4.5.39);

$$\int_0^L \rho[\omega_1(b+x) - \omega_2 u_1]^2\, dx \le K_4(\omega_1^2 + \omega_2^2 + \omega_3^2) \quad , \tag{4.5.40}$$

where K_4 is given in (3.5.39).

Similar to the estimates obtained above, using (4.5.25) and (4.5.35), we obtain the following estimates for some of the terms which appear in (4.5.31) :

$$-2\int_0^L \rho x u_{1x} u_{3t}\, dx \le 2L\,\delta_3^2 \int_0^L \rho u_{3t}^2\, dx + \frac{2\rho L}{\delta_3^2}\omega_2^2 \int_0^L u_{1x}^2\, dx$$

$$\le 2L\,\delta_3^2 \int_0^L \rho u_{3t}^2\, dx + \frac{K_5}{\delta_3^2}(\omega_1^2 + \omega_2^2 + \omega_3^2) \quad , \tag{4.5.41}$$

where $K_5 = \frac{4\rho L^2 E(0)}{EI_1}$,

$$-2\int_0^L \rho x\,\omega_1\omega_3 u_3 u_{1x}\,dx \le 2\rho L\,\omega_1^2\int_0^L u_3^2\,dx + 2\rho L\,\omega_3^2\int_0^L u_{1x}^2\,dx +$$

$$\le K_6(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \qquad (4.5.42)$$

where $K_6 = \max\{\dfrac{4\rho L^4 E(0)}{EI_3}, \dfrac{4\rho L^2 E(0)}{EI_1}\}$,

$$[\rho L u_1^2(L,t) - \rho\int_0^L u_1^2\,dx](\omega_2^2+\omega_3^2) \le \frac{4\rho L^3 E(0)}{EI_1}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \qquad (4.5.43)$$

$$-2\,\rho\int_0^L x(b+x)\omega_1\omega_2 u_{1x}\,dx \le 2\rho\omega_1^2\int_0^L x^2(b+x)^2\,dx + 2\rho\omega_2^2\int_0^L u_{1x}^2\,dx$$

$$\le K_7(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \qquad (4.5.44)$$

where $K_7 = \max\{2\rho\int_0^L x^2(b+x)^2\,dx\ ,\ \dfrac{4\rho L E(0)}{EI_1}\}$,

$$2\rho L u_3(L,t)u_{1t}(L,t)\omega_2 \le 2\rho L u_3^2(L,t)\omega_2^2 + 2\rho L u_{1t}^2(L,t)$$

$$\le \frac{4\rho L^3 E(0)}{EI_3}(\omega_1^2+\omega_2^2+\omega_3^2) + 2\rho L u_{1t}^2(L,t) \quad , \qquad (4.5.45)$$

$$-2\rho\int_0^L x\,\omega_2 u_{3x} u_{1t}\,dx \le 2L\,\delta_4^2\int_0^L \rho u_{1t}^2\,dx + \frac{2\rho L}{\delta_4^2}\omega_2^2\int_0^L u_{3x}^2\,dx$$

$$\le 2L\,\delta_4^2\int_0^L \rho u_{1t}^2\,dx + \frac{4\rho L^2 E(0)}{EI_3\delta_4^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \qquad (4.5.46)$$

$$-2\rho\int_0^L \omega_2 u_3 u_{1t}\,dx \le 2\delta_5^2\int_0^L \rho u_{1t}^2\,dx + \frac{4\rho L^3 E(0)}{EI_3\delta_5^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \qquad (4.5.47)$$

$$-2\rho L(b+L)u_{1t}(L,t)\omega_3 \le 2\rho L(b+L)u_{1t}^2(L,t) + 2\rho L(b+L)(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.48)$$

$$2\rho\int_0^L (b+2x)u_{1t}\omega_2\,dx \le 2\delta_6^2\int_0^L \rho u_{1t}^2\,dx + \frac{2\rho\int_0^L (b+2x)^2\,dx}{\delta_6^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.49)$$

$$2\rho\int_0^L x\,\omega_2 u_{1t}u_{3x}\,dx \le 2\delta_7^2\int_0^L \rho u_{1t}^2\,dx + \frac{2\rho L^2E(0)}{EI_3\delta_7^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.50)$$

$$-2\rho\int_0^L x\,\omega_1\omega_3 u_1 u_{3x}\,dx \le 2\rho L\,\omega_1^2\int_0^L u_1^2\,dx + 2\rho L\,\omega_3^2\int_0^L u_{3x}^2\,dx$$

$$\le K_8(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.51)$$

where $K_8 = \max\{\frac{4\rho L^4E(0)}{EI_1}, \frac{4\rho L^2E(0)}{EI_3}\}$,

$$[\rho L u_3^2(L,t) - \rho\int_0^L u_3^2\,dx](\omega_1^2+\omega_2^2) \le \frac{4\rho L^3E(0)}{EI_3}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.52)$$

$$2\rho L(b+L)u_{3t}(L,t)\omega_1 \le 2\rho L(b+L)u_{3t}^2(L,t) + 2\rho L(b+L)(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.53)$$

$$-2\rho\int_0^L (b+2x)u_{3t}\omega_1\,dx \le 2\delta_8^2\int_0^L \rho u_{3t}^2\,dx + \frac{2\rho\int_0^L (b+2x)^2\,dx}{\delta_8^2}(\omega_1^2+\omega_2^2+\omega_3^2) \quad , \quad (4.5.54)$$

$$-2\rho L u_1(L,t)u_{3t}(L,t)\omega_2 \le 2\rho L u_1^2(L,t)\omega_2^2 + 2\rho L u_{3t}^2(L,t)$$

$$\le \frac{4\rho L^3E(0)}{EI_1}(\omega_1^2+\omega_2^2+\omega_3^2) + 2\rho L u_{3t}^2(L,t) \quad , \quad (4.5.55)$$

$$2\rho\int_0^L x\,\omega_2 u_{1x}u_{3t}\,dx \le 2L\,\delta_9^2\int_0^L \rho u_{3t}^2\,dx + \frac{2\rho L}{\delta_9^2}\omega_2^2\int_0^L u_{1x}^2\,dx$$

$$\leq 2L\,\delta_9^{\,2}\int_0^L \rho u_{1t}^{\;2}\,dx + \frac{4\rho L^2 E(0)}{EI_1\delta_9^{\,2}}(\omega_1^{\,2}+\omega_2^{\,2}+\omega_3^{\,2}) \quad , \tag{4.5.56}$$

$$2\rho\int_0^L \omega_2 u_3 u_{1t}\,dx \leq 2\delta_{10}^{\;2}\int_0^L \rho u_{1t}^{\;2}\,dx + \frac{4\rho L^3 E(0)}{EI_1\delta_{10}^{\;2}}(\omega_1^{\,2}+\omega_2^{\,2}+\omega_3^{\,2}) \quad , \tag{4.5.57}$$

$$-2\,\rho\int_0^L x(b+x)\omega_2\omega_3 u_{3x}\,dx \leq 2\rho\omega_2^{\,2}\int_0^L x^2(b+x)^2\,dx + 2\rho\omega_3^{\,2}\int_0^L u_{3x}^{\;2}\,dx$$

$$\leq K_9(\omega_1^{\,2}+\omega_2^{\,2}+\omega_3^{\,2}) \quad , \tag{4.5.58}$$

where $K_9 = \max\{2\rho\int_0^L x^2(b+x)^2\,dx \;,\; \frac{4\rho L E(0)}{EI_3}\}$.

Using (4.5.32)-(4.5.58) in (4.5.31), and collecting likewise terms, we obtain the following estimate :

$$\frac{dV(t)}{dt} \leq -\,[2(1-\varepsilon)k_{\min}t - D_1](\omega_1^{\,2}+\omega_2^{\,2}+\omega_3^{\,2}) - [2(1-\varepsilon)\alpha_1 t - D_2]u_{1t}^{\;2}(L,t)$$

$$-\,[2(1-\varepsilon)\alpha_3 t - D_3]u_{3t}^{\;2}(L,t) - [2(1-\varepsilon)\beta_1 t - D_4]u_{1t}^{\;2}(L,t)$$

$$-\,[2(1-\varepsilon)\beta_3 t - D_5]u_{3t}^{\;2}(L,t) - (\varepsilon - D_6)\int_0^L \rho u_{1t}^{\;2}\,dx - (\varepsilon - D_7)\int_0^L \rho u_{3t}^{\;2}\,dx$$

$$-\,[\,(\varepsilon+2)EI_1\int_0^L u_{1xx}^{\;2}\,dx - (2L\,\alpha_1\gamma_1^{\,2} + 2\beta_1\sigma_1^{\,2})u_{1x}^{\;2}(L,t)]$$

$$-\,[\,(\varepsilon+2)EI_3\int_0^L u_{3xx}^{\;2}\,dx - (2L\,\alpha_3\gamma_3^{\,2} + 2\beta_3\sigma_3^{\,2})u_{3x}^{\;2}(L,t)] \quad , \tag{4.5.59}$$

where

$$D_1 = (1-\varepsilon)I_{\max} + \frac{K_1}{\delta_1^2} + K_3 + K_4 + \frac{K_5}{\delta_3^2} + K_6 + \frac{8\rho L E(0)}{EI_1}$$

$$+ K_7 + \frac{8\rho L E(0)}{EI_3} + \frac{4\rho L^2 E(0)}{EI_3 \delta_4^2} + \frac{4\rho L^3 E(0)}{EI_3 \delta_5^2} + 4\rho L(b+L)$$

$$+ \frac{2\rho \int_0^L (b+2x)^2\, dx}{\delta_6^2} + \frac{2\rho L^2 E(0)}{EI_3 \delta_7^2} + K_8 + K_9$$

$$+ \frac{2\rho \int_0^L (b+2x)^2\, dx}{\delta_8^2} + \frac{4\rho L^2 E(0)}{EI_1 \delta_9^2} + \frac{4\rho L^3 E(0)}{EI_1 \delta_{10}^2} \quad , \tag{4.5.60}$$

$$D_2 = \frac{2L\alpha_1}{\gamma_1^2} + 2\rho L + 2\rho L(b+L) \quad , \tag{4.5.61}$$

$$D_3 = \frac{2L\alpha_3}{\gamma_3^2} + 2\rho L + 2\rho L(b+L) \quad , \tag{4.5.62}$$

$$D_4 = \frac{2\beta_1}{\sigma_1^2} + \frac{L\beta_1^2}{EI_1} \quad , \tag{4.5.63}$$

$$D_5 = \frac{2\beta_3}{\sigma_3^2} + \frac{L\beta_3^2}{EI_3} \quad , \tag{4.5.64}$$

$$D_6 = \delta_1^2 + 2L\,\delta_4^2 + 2\delta_5^2 + 2\delta_6^2 + 2L\,\delta_7^2 \quad , \tag{4.5.65}$$

$$D_7 = \delta_2^2 + 2\delta_3^2 + 2\delta_8^2 + 2L\,\delta_9^2 + 2\delta_{10}^2 \quad , \tag{4.5.66}$$

and δ_i , $i = 1,\ldots, 10$ and γ_j , σ_j $j = 1, 3$ are arbitrary nonzero real numbers and

K_i , $i = 1,..., 9$ are positive numbers defined between the equations (4.5.32) and (4.5.58).

Let $\varepsilon \in (0, 1)$ be fixed. Then by choosing δ_i , $i = 1,..., 10$ sufficiently small one can have $\varepsilon > D_6$, $\varepsilon > \delta_7$, (see (4.5.65) and (4.5.66)). Also by choosing γ_j , σ_j $j = 1, 3$ sufficiently small, the last two line in (4.5.59) can be made negative, (see (4.5.15)). Then (4.5.19) follows from (4.5.59), i.e., we obtain

$$\frac{dV(t)}{dt} \leq 0 \qquad t > T \quad ,$$

where $T = \max\{\frac{D_1}{2(1-\varepsilon)k_{\min}}, \frac{D_2}{2(1-\varepsilon)\alpha_1}, \frac{D_3}{2(1-\varepsilon)\alpha_3}, \frac{D_4}{2(1-\varepsilon)\beta_1}, \frac{D_5}{2(1-\varepsilon)\beta_3}\}$.

Using (4.5.19) and (4.5.18) we obtain the following (see (4.5.20))

$$E(t) \leq \frac{V(T)}{2(1-\varepsilon)t - C_1} \quad , \qquad t > \max(T , \frac{C_1}{2(1-\varepsilon)}) \quad ,$$

which proves that for sufficiently large t, $E(t)$ decays as $O(\frac{1}{t})$. □

The existence, the uniqueness, and the exponential decay of the solutions of the equations given by (4.5.1)-(4.5.8) are presented in the following section.

4.6 Exponential Decay of the Solutions

In this section, first we give an existence and uniqueness theorem for the linear part of the equations (4.5.1)-(4.5.8), (i.e. the "natural" control scheme). Then including the nonlinear terms, we prove the exponential decay of the solutions of the same equations.

For simplicity, as in section 5, we will take the symmetric positive definite matrix K to be equal to $diag(k_1, k_2, k_3)$. For the sake of clarity, we repeat equations (4.5.1)-(4.5.8) here : for all $t \geq 0$

$$EI_1 u_{1xxxx} + \rho u_{1tt} + 2\rho \omega_2 u_{3t} + \rho(\dot{\omega}_2 + \omega_1 \omega_3) u_3$$

$$-\rho(\omega_2^2 + \omega_3^2) u_1 - \rho(\dot{\omega}_3 - \omega_1\omega_2)(b + x) = 0 \qquad 0 < x < L \quad , \tag{4.6.1}$$

$$EI_3 u_{3xxxx} + \rho u_{3tt} - 2\rho \omega_2 u_{1t} - \rho(\dot{\omega}_2 - \omega_1 \omega_3) u_1$$

$$-\rho(\omega_1^2 + \omega_2^2) u_3 + \rho(\dot{\omega}_1 + \omega_2\omega_3)(b + x) = 0 \qquad 0 < x < L \quad , \tag{4.6.2}$$

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + k_1\omega_1 = EI_3\int_0^L (b + x) u_{3xxxx}\, dx \quad , \tag{4.6.3}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 + k_2\omega_2 = EI_3\int_0^L u_1 u_{3xxxx}\, dx - EI_1\int_0^L u_3 u_{1xxxx}\, dx \quad , \tag{4.6.4}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + k_3\omega_3 = EI_1\int_0^L -(b + x) u_{1xxxx}\, dx \quad , \tag{4.6.5}$$

$$u_1(L,t) = u_3(L,t) = 0 \quad , \quad u_{1x}(L,t) = u_{3x}(L,t) = 0 \quad , \tag{4.6.6}$$

$$-EI_1\,u_{1xxx}(L,t)+\alpha_1 u_{1t}(L,t)=0 \quad , \quad -EI_3\,u_{3xxx}(L,t)+\alpha_3 u_{3t}(L,t)=0 \quad , \tag{4.6.7}$$

$$EI_1 u_{1xx}(L,t)+\beta_1 u_{1xt}(L,t)=0 \quad , \quad EI_3 u_{3xx}(L,t)+\beta_3 u_{3xt}(L,t)=0 \quad . \tag{4.6.8}$$

Let the function space H be the same as defined in (4.4.14). Define a new function space $\hat{H}$ as $\hat{H} := H \times \mathbf{R}^3$. Then, separating the linear and nonlinear parts, the equations (4.6.1)-(4.6.8) can be put into the following matrix form :

$$\frac{dz}{dt} = \hat{A}\,z + T_I(z) + g(z) \quad , \tag{4.6.9}$$

where $z = [\mathrm{u}_1 \;\; \mathrm{u}_{1t} \;\; \mathrm{u}_3 \;\; \mathrm{u}_{3t} \;\; \omega_1 \;\; \omega_2 \;\; \omega_3]^T$.

$\hat{A}:\hat{H} \rightarrow \hat{H}$ is a linear operator whose matrix form is specified by the following :

$$\hat{A} = \{m_{ij} : i = 1,\dots,7\,,\, j = 1,\dots,7\,\} \quad , \tag{4.6.10}$$

where all m_{ij} are zero except :

$$m_{12} = m_{34} = 1 \quad ,$$

$$m_{21} = -\frac{EI_1}{\rho}\frac{\partial^4}{\partial x^4} - \frac{EI_1}{I_3}(b+x)\int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \quad ,$$

$$m_{27} = -\frac{k_3}{I_3}(b+x) \quad ,$$

$$m_{43} = -\frac{EI_3}{\rho}\frac{\partial^4}{\partial x^4} - \frac{EI_3}{I_1}(b+x)\int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \quad ,$$

$$m_{45} = \frac{k_1}{I_1}(b+x) \quad ,$$

$$m_{53} = \frac{EI_3}{I_1}\int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \quad ,$$

$$m_{55} = -\frac{k_1}{I_1} \quad ,$$

$$m_{66} = -\frac{k_2}{I_2} \quad ,$$

$$m_{71} = -\frac{EI_1}{I_3}\int_0^L (b+x)\frac{\partial^4}{\partial x^4}\,dx \quad ,$$

$$m_{77} = \frac{k_3}{I_3} \quad .$$

The operator $T_I : \hat{H} \rightarrow \hat{H}$ is a nonlinear integral operator defined as :

$$T_I(z) = \begin{bmatrix} 0 \\ u_3 \int_0^L \left(-\frac{EI_1}{I_2} u_3 u_{1xxxx} + \frac{EI_3}{I_2} u_1 u_{3xxxx} \right) dx \\ 0 \\ u_1 \int_0^L \left(-\frac{EI_3}{I_2} u_1 u_{3xxxx} + \frac{EI_1}{I_2} u_3 u_{1xxxx} \right) dx \\ 0 \\ \int_0^L \left(-\frac{EI_3}{I_2} u_1 u_{3xxxx} + \frac{EI_1}{I_2} u_3 u_{1xxxx} \right) dx \\ 0 \end{bmatrix} \quad . \tag{4.6.11}$$

The operator $g : \hat{H} \rightarrow \hat{H}$ is a nonlinear operator defined as :

$$g(z) = [g_1(z), \ldots, g_7(z)]^T \quad , \tag{4.6.12}$$

where all $g_i(z)$ are defined as follows :

$$g_1(z) = g_3(z) = 0 \quad ,$$

$$g_2(z) = \frac{I_1 - I_3}{I_2} \omega_1 \omega_3 u_3 + \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 (b+x)$$
$$+ \frac{k_2}{I_2} \omega_2 u_3 - 2\, \omega_2 u_{3t} + (\omega_2^2 + \omega_3^2) u_1 - \omega_1 \omega_3 u_3 - \omega_1 \omega_2 (b+x) \quad ,$$

$$g_4(z) = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3 u_1 - \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 (b+x)$$
$$- \frac{k_2}{I_2} \omega_2 u_1 + 2\, \omega_2 u_{1t} + (\omega_1^2 + \omega_2^2) u_3 - \omega_1 \omega_3 u_1 - \omega_2 \omega_3 (b+x) \quad ,$$

$$g_5(z) = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 \quad ,$$

$$g_6(z) = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3 \quad ,$$

$$g_7(z) = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 \quad .$$

Note that $\hat{A} : \hat{H} \rightarrow \hat{H}$ is an unbounded linear operator and its domain $D(\hat{A})$ is defined as $D(\hat{A}) := D(A) \times \mathbf{R}^3$, where $D(A)$ is defined in (4.4.20). Since $D(A)$ is dense in H, it follows that $D(\hat{A})$ is dense in $\hat{H}$.

In $\hat{H}$ we define the following "energy" inner product:

$$
\begin{aligned}
< z , \hat{z} >_1 = I_1\omega_1\hat{\omega}_1 + I_2\omega_2\hat{\omega}_2 + I_3\omega_3\hat{\omega}_3 \\
+ \int_0^L \rho[u_{1t} - \omega_3(b+x)]\,[\hat{u}_{1t} - \hat{\omega}_3(b+x)]\, dx \\
+ \int_0^L \rho[u_{3t} + \omega_1(b+x)]\,[\hat{u}_{3t} + \hat{\omega}_1(b+x)]\, dx \\
+ \int_0^L (EI_1 u_{1xx}\hat{u}_{1xx} + EI_3 u_{3xx}\hat{u}_{3xx})\, dx \quad .
\end{aligned}
$$

This inner product induces a norm on $\hat{H}$, which is given below :

$$
\begin{aligned}
(\| z \|_1)^2 = 2\,\hat{E}(t) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \int_0^L (EI_1 u_{1xx}^2 + EI_3 u_{3xx}^2)\, dx \\
+ \int_0^L \rho(\,[u_{1t} - \omega_3(b+x)\,]^2 + [u_{3t} + \omega_1(b+x)]^2\,)\, dx \quad .
\end{aligned}
\tag{4.6.13}
$$

Note that the usual "Sobolev" type norm which makes $\hat{H}$ a Banach space is given by :

$$
\begin{aligned}
\| z \|^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \int_0^L (u_1^2 + u_{1x}^2 + u_{1xx}^2)\, dx \\
+ \int_0^L (u_3^2 + u_{3x}^2 + u_{3xx}^2)\, dx + \int_0^L (u_{1t}^2 + u_{3t}^2)\, dx \quad .
\end{aligned}
\tag{4.6.14}
$$

But, from Lemma 3.5.4 it follows that the norms given by (4.6.13) and (4.6.14) are equivalent.

4.6.1 Theorem : Consider the linear operator $\hat{A}:\hat{H}\rightarrow\hat{H}$ given by (4.6.10). Then :

(i) $\hat{A}$ generates a C_0 semigroup $\hat{T}(t)$;

(ii) there exist positive constants $M>0$ and $\delta>0$ such that the following holds :

$$||\hat{T}(t)|| \leq M\, e^{-\delta t} \quad , \quad t\geq 0 \quad . \tag{4.6.15}$$

Proof :

(i) We will use the Lumer-Phillips theorem to prove (i),(see thorem 3.4.4). Thus, we have to show that $\hat{A}$ is dissipative and the operator $(\lambda I-\hat{A}):\hat{H}\rightarrow\hat{H}$ is onto for some $\lambda>0$.

To prove that $\hat{A}$ is dissipative, consider the following equation :

$$\frac{dz}{dt}=\hat{A}z \quad , \quad z(0)\in D(\hat{A}) \quad . \tag{4.6.16}$$

Then, differentiating (4.6.13) and using (4.6.16) and (4.6.10), we obtain the following :

$$\begin{aligned}\frac{d\hat{E}}{dt} &= I_1\omega_1\dot{\omega}_1+I_2\omega_2\dot{\omega}_2+I_3\omega_3\dot{\omega}_3+\int_0^L\rho[u_{1t}-\omega_3(b+x)]\,[u_{1tt}-\dot{\omega}_3(b+x)]\,dx \\ &\qquad +\int_0^L\rho[u_{3t}+\omega_1(b+x)]\,[u_{3tt}+\dot{\omega}_1(b+x)]\,dx \\ &\qquad +\int_0^L(EI_1u_{1xx}u_{1xxt}+EI_3u_{3xx}u_{3xxt})\,dx \\ &= -k_1\omega_1^2-k_2\omega_2^2-k_3\omega_3^2-\alpha_1u_{1t}^{\,2}(L,t) \quad . \\ &\qquad -\alpha_3u_{3t}^{\,2}(L,t)-\beta_1u_{1xt}^{\,2}(L,t)-\beta_3u_{3xt}^{\,2}(L,t)\leq 0 \quad .\end{aligned} \tag{4.6.17}$$

This proves that $\hat{A}$ is dissipative.

To prove that the linear operator $(\lambda I - \hat{A}):\hat{H} \to \hat{H}$ is onto for some $\lambda > 0$, we decompose the operator $\hat{A}$ as follows :

$$\hat{A} = A_1 + T_D \quad , \tag{4.6.18}$$

where $A_1: \hat{H} \to \hat{H}$ is defined as :

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EI_1}{\rho}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{EI_3}{\rho}\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_1}{I_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{k_2}{I_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_3}{I_3} \end{bmatrix} \quad , \tag{4.6.19}$$

and the operator $T_D: \hat{H} \to \hat{H}$ is defined as :

$$T_D = \hat{A} - A_1 \quad . \tag{4.6.20}$$

We first note the following remarks :

1) The operator $A_1 : \hat{H} \to \hat{H}$ is a linear unbounded operator . Its domain $D(A_1)$ is equal to $D(\hat{A})$. By using Theorem 3.4.7 it can be shown that A_1 generates an C_0 contraction semigroup. Hence, $(\lambda I - A_1) : \hat{H} \to \hat{H}$ has an inverse which is a bounded linear operator on $\hat{H}$. In fact, the range of $(\lambda I - A_1)^{-1}$ is equal to $D(A_1)$, and by Hille-Yosida theorem, (see Theorem 3.4.3), we have :

$$\| (\lambda I - A_1)^{-1} \| \le \frac{1}{\lambda} \quad , \quad \lambda > 0, \ \lambda \in R \quad .$$

2) The operator $T_D : \hat{H} \to \hat{H}$ is a degenerate linear operator relative to the A_1, (see [Kat.1, p. 245]). By definition, the range space of T_D is finite dimensional and there exist positive constants a and b such that :

$$||T_D z|| \le a||z|| + b||A_1 z|| \quad , \quad \text{for all } z \in D(A_1) \quad . \tag{4.6.21}$$

That the operator T_D has a finite dimensional range follows from (4.6.20),(4.6.10) and (4.6.19).

By using (4.6.20) and (4.6.14), it can be shown that (4.6.21) holds for some positive a and b, (see Theorem 3.5.5).

From Remarks 1 and 2 above, it follows that $T_D(\lambda I - A_1)^{-1} : \hat{H} \to \hat{H}$ is a bounded linear operator with finite dimensional range ; hence $||T_D(\lambda I - A_1)^{-1}|| \le M$ for some $M > 0$ and $T_D(\lambda I - A_1)^{-1}$ is a compact operator, (see [Kat.1, p. 245]).

Next we need the following fact :

Fact : for all $\lambda > 0$, the real number 1 is not an eigenvalue of the compact operator $T_D(\lambda I - A_1)^{-1}$.

Proof : Suppose not. Then there exists a $\lambda > 0$ and a $y \in \hat{H}, y \neq 0$ such that the following holds :

$$y = T_D(\lambda I - A_1)^{-1} y \quad . \tag{4.6.22}$$

Define $x \in D(A_1)$ as

$$x = (\lambda I - A_1)^{-1} y \quad .$$

Then (4.6.22) implies that the following equation also holds :

$$(\lambda I - A_1 - T_D)x = 0 \quad .$$

But since $\hat{A} = A_1 + T_D$ is dissipative and $\lambda > 0$, it follows that $x = 0$, which implies $y = 0$, which is a contradiction. □

From the above fact it follows that the operator $I - T_D(\lambda I - A_1)^{-1}$ is invertible for all $\lambda > 0$. Hence we conclude that $(\lambda I - A_1 - T_D) : \hat{H} \to \hat{H}$ is invertible for all $\lambda > 0$ and its inverse is given by :

$$(\lambda I - A_1 - T_D)^{-1} = (\lambda I - A_1)^{-1}(I - T_D(\lambda I - A_1)^{-1})^{-1}$$

This shows that $(\lambda I - A_1 - T_D) : \hat{H} \to \hat{H}$ is onto for all $\lambda > 0$. Then, the assertion (i) follows from the Lumer-Phillips theorem, (see Theorem 3.4.4).

(ii) To prove that the semigroup $\hat{T}(t)$ generated by $\hat{A}$ is exponentially decaying, we first follow a similar argument we made in proving Theorem 4.5.4, (see theorem 3.5.5). We first define the following function $\hat{V}(t)$: for all $t \geq 0$

$$\hat{V}(t) = 2\,(1-\varepsilon)\,t\,\hat{E}(t) + 2\int_0^L \lambda x\,(u_{1t} - \omega_3(b+x))u_{1x}\,dx$$
$$+ 2\int_0^L \rho x\,(u_{3t} - \omega_1(b+x))u_{3x}\,dx \quad , \tag{4.6.23}$$

where $\varepsilon \in (0, 1)$ is arbitrary.

Following the arguments made in the proof of Theorem 4.5.4, it can be shown that there exists a $K > 0$ such that the following estimate holds :

$$(2(1-\varepsilon)t-K)\,\hat{E}(t) \le \hat{V}(t) \le (2(1-\varepsilon)t+K)\,\hat{E}(t) \quad .$$

Differentiating $\hat{V}(t)$ with respect to t, using (4.6.1)-(4.6.8) and following the line of the proof of Theorem 4.5.4, we can conclude that there exists a $T>0$ such that $\hat{V}(t)$ is bounded above for all $t \ge T$. Therefore $\hat{E}(t)$ is bounded above by $O(\frac{1}{t})$, for all $t \ge T$. Hence for some $M>0$,

$$\int_0^\infty \hat{E}^2(t)\, dt \le M \quad .$$

The assertion (ii) then follows from a theorem due to Pazy, (see Theorem 3.4.5). □

We now establish the existence and uniqueness of the solutions of (4.6). The main difficulty is the fact that the nonlinear operator $T_l(z) : \hat{H} \to \hat{H}$ defined by (4.6.11) is also unbounded, i.e., it is not defined for all $z \in \hat{H}$. But, with an appropriate norm defined on $D(\hat{A})$, (see (4.6.24) below), $T_l(z) : D(\hat{A}) \to \hat{H}$ becomes an C^∞ operator.

In the sequel, for simplicity we will assume that $EI_1 = EI_3 =: EI$, this requires that the beam cross sections have certain symmetry about $\mathbf{D}_2$ axis.

4.6.2 Theorem : Consider the system given by (4.6.9), where the operators $\hat{A}$, T_D, and g are defined in (4.6.10)-(4.6.12), respectively. Then:

(i) for all initial conditions $z(0) \in D(\hat{A})$, (4.6.9) has unique classical solution $z(.)$ defined for all $t>0$;

(ii) in terms of the semigroup $\hat{T}(t)$ generated by the linear operator $\hat{A}$, this solution can be written as :

$$z(t) = \hat{T}(t)z(0) + \int_0^t \hat{T}(t-s)T_l(z(s))\,ds + \int_0^t \hat{T}(t-s)g(z(s))\,ds \quad ;$$

(iii) the solution of (4.6.9) decays to 0 exponentially.

Proof :

(i) Following [Seg.1], we define the following norm on $D(\hat{A})$:

$$|||z||| = ||\hat{A}z|| \quad , \quad z \in D(\hat{A}) \quad , \tag{4.6.24}$$

where $||.||$ is defined in (4.6.14).

A simple calculation shows that this norm is equivalent to a standard Sobolev norm for $D(\hat{A})$. Hence $D(\hat{A})$ with this norm becomes a Banach space. Let us call this space $[D(\hat{A})]$. Then $T_l : [D(\hat{A})] \to \hat{H}$ becomes an C^∞ operator, since its components are linear combinations of products and integrals of the components of z over $[0, L]$, (see (4.6.8)).

Also note that $g : \hat{H} \to \hat{H}$, as defined by (4.6.12), is a C^∞ map, since its components are products of the components of z. Therefore it follows from a theorem due to Segal, [Seg.1, p. 351, Thm. 2], that (4.6.9) has unique classical solution for all initial conditions $z(0) \in D(\hat{A})$, defined in $[0, \delta]$ for some $\delta > 0$. (Segal considers this case as a "singular" case and shows that, as long as the perturbation term T_l in (4.6.9) is a Lipschitz operator from $[D(\hat{A})]$ into $\hat{H}$, the standard methods for finding the solution of (4.6.9) still work). But since Theorem 4.5.4 shows that the solutions are decaying to 0, this local existence theorem can be extended globally (i.e., for all $t > 0$).

(ii) This may be proven by substitution in (4.6.9);

(iii) Since by Theorem 4.5.4 the solutions of (4.6.9) are decaying to 0 in $\hat{H}$, it follows that the positive orbits $O_0^+(t) = \{z(t) \in \hat{H} \mid z(0) = z_0,\ t > 0\}$ belong to a compact set in $\hat{H}$. Therefore by a generalization of LaSalle's invariance argument to the infinite dimensional spaces , (see e.g., [Hal.1]), and by the energy decay estimate (4.5.12) it follows that asymptotically the rate of change of the energy given in (4.5.12) decays to 0. That is, $u_{it}(L,t)$, $u_{ixt}(L,t)$, $i = 1, 3$ and $\omega(t)$ decay to 0, as $t \to \infty$.

Using the norm defined in (4.6.13) and the operator $T_l : \hat{H} \to \hat{H}$ defined in (4.6.11), we obtain the following :

$$
\begin{aligned}
(\| T_l(z) \|_1)^2 = I_2 \Big[\int_0^L \Big(-\frac{EI}{I_2} u_1 u_{3xxxx} + \frac{EI}{I_2} u_3 u_{1xxxx} \Big)\, dx \Big]^2 \\
+ \rho \Big[\int_0^L \Big(-\frac{EI}{I_2} u_3 u_{1xxxx} + \frac{EI}{I_2} u_1 u_{3xxxx} \Big)\, dx \Big]^2 \int_0^L u_3^2\, dx \\
+ \rho \Big[\int_{x=0}^{x=L} \Big(-\frac{EI}{I_2} u_1 u_{3xxxx} + \frac{EI}{I_2} u_3 u_{1xxxx} \Big)\, dx \Big]^2 \int_0^L u_1^2\, dx \quad . \qquad (4.6.25)
\end{aligned}
$$

Defining the following quantity as J_l , using integration by parts and (4.6.6)-(4.6.8), we obtain the following:

$$
\begin{aligned}
J_l &:= \int_0^L (EI\ u_3 u_{1xxxx} - EI u_1 u_{3xxxx})\, dx \\
&= EI u_{1xxx}(L,t) u_3(L,t) - \int_0^L EI u_{1xxx} u_{3x}\, dx \\
&\qquad - EI u_{3xxx}(L,t) u_1(L,t) + \int_0^L EI u_{3xxx} u_{1x}\, dx \\
&= \alpha_1 u_{1t}(L,t) u_3(L,t) - \alpha_3 u_{3t}(L,t) u_1(L,t) \\
&\qquad + \beta_1 u_{1xt}(L,t) u_{3x}(L,t) - \beta_3 u_{3xt}(L,t) u_{1x}(L,t) \quad . \qquad (4.6.26)
\end{aligned}
$$

Using the inequality (4.5.19) in (4.6.26) we obtain the following estimate for J_l :

$$J_I^{\,2} = \left[\int_0^L (EI\ u_3 u_{1xxxx} - EIu_1 u_{3xxxx})\,dx\right]^2$$

$$\le \alpha_1^2 u_3^2(L,t)\,u_{1t}^2(L,t) + \alpha_3^2 u_1^2(L,t)\,u_{3t}^2(L,t)$$

$$+ \beta_1^2 u_{3x}^2(L,t)\,u_{1xt}^2(L,t) + \beta_3^2 u_{1x}^2(L,t)\,u_{3xt}^2(L,t) \quad . \qquad (4.6.27)$$

Finally, using (4.6.27), (4.5.6), and (4.5.7) in (4.6.25), we obtain the following estimate :

$$||\,T_I(z(t))\,|| \le \gamma_1(t)\,||\,z(t)\,|| \quad , \qquad (4.6.28)$$

where $\gamma_1(t) : \mathbf{R} \to \mathbf{R}$ is asymptotically decaying to 0, by the LaSalle's invariance argument, (see above).

Similarly, using the norm given by (4.6.13) and the operator $g : \hat{H} \to \hat{H}$ and the inequalities (4.5.18), (4.5.19), we obtain the following estimate similar to (4.6.28) :

$$||\,g(z(t))\,|| \le \gamma_2(t)\,||\,z(t)\,|| \quad , \qquad (4.6.29)$$

where $\gamma_2(t) : \mathbf{R} \to \mathbf{R}$ is asymptotically decaying to zero, by LaSalle's invariance argument.

Using the estimates (4.6.28), (4.6.29) and following the arguments made in the proof of Theorem 4.4.4, (i.e. using a generalized version of the Bellman-Gronwall lemma), we conclude that the solutions of (4.6.9) are decaying exponentially to 0. □

Chapter 5

Control of a Timoshenko Beam Attached to a Rigid Body : Planar Motion

5.1 Introduction

In this chapter we continue to study the motion of the rigid body clamped beam configuration introduced in chapter 2, section 4. We assume that, as in previous chapters, the center of mass of the rigid body is fixed in an inertial frame and the flexible beam is clamped at one end to the rigid body and free at the other end. In Chapters 2 and 3, we studied this configuration using the Euler-Bernoulli beam model to obtain the equations of motion of the flexible beam. In this chapter we use the so called geometrically exact beam model and its appropriate linearization to study the motion of the flexible beam rigid body configuration.

In Section 2, we derive the equations of motion of the whole system and state the control problem. In Section 3, we propose a "natural" control law to solve the control problem posed in Section 2. In Section 4, we first show that, without using any linearization, the proposed control law stabilizes the system introduced in Section 2, though one cannot easily extend this result to obtain asymptotic or exponential stability. Then using appropriate linearization of the geometrically exact beam model and assuming that the whole motion takes place in a plane, we prove that the system is asymptotically stable. We note that the assumptions stated above yield to a Timoshenko beam, clamped to a rigid body at one end and free at the other, (see Chapter 1, Section 3). Then, in Section 5, we prove the exponential stability of the whole motion.

5.2 Equations of Motion

In this section, we consider a rigid body whose center of mass is fixed in an inertial frame and a flexible beam, clamped to the rigid body at one end and free at the other. In previous chapters we used the Euler-Bernoulli theory to model the flexible beam. In this section, we use the geometrically exact beam model given in Section 3 of Chapter 2 and show that under appropriate control laws applied at the free end of the beam and a torque control applied to the rigid body, the energy of the whole configuration becomes a nonincreasing function of time, i.e., the proposed control laws stabilize the system.

We consider the following configuration : Figure 5.1 shows the rigid body (drawn as a square) and the beam; P is a point on the beam:

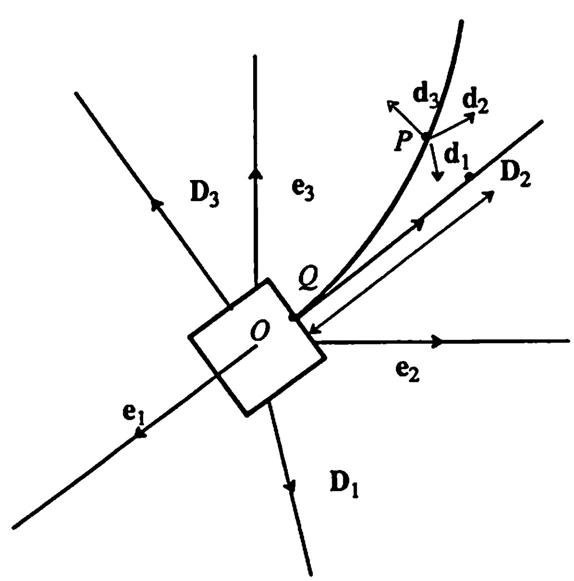

Figure 5.1 : Rigid Body with Flexible Beam

In Figure 5.1, the quadruple $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes a dextral orthonormal *inertial* frame, which will be referred to as **N**, the quadruple $(O, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$ denotes a dextral orthonormal frame fixed in the rigid body, which will be referred as **B**, where O is also the center of mass of the rigid body and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ are along the principal axes of inertia of the rigid body. The beam is clamped to the rigid body at the point Q at one end along the $\mathbf{D}_2$ axis and is free at the other end. Let L be the length of the beam. We assume that the mass of the rigid body is much larger than the mass of the beam, so that the center of mass of the rigid body is approximately the center of mass of the whole configuration. Hence, we assume that the point O is fixed in the inertial space throughout the motion of the whole configuration. We also assume that the beam is inextensible, (i.e. no deformation along the axis $\mathbf{D}_2$), and homogeneous with uniform cross-section.

The beam is initially straight along the $\mathbf{D}_2$ axis. Let P be a point on the curve of centroids whose distance from Q in the undeformed configuration is x, (i.e., when the beam is straight along the $\mathbf{D}_2$ axis), let the quadruple $(P, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ denote the frame of *directors* located at P, where $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ are the directors at x, (see Subsection 3.2 of Chapter 2). Let $\mathbf{r}(x, t) = \mathbf{OP}$ be the position vector of P. We also assume that Assumptions **1** and **2** of Subsection 3.2 of Chapter 2 and Assumption **3** of Subsection 3.3 of Chapter 2 hold.

Let, as in Chapter 2, $\Lambda(x, t) \in SO(3)$ be the orthogonal transformation between the body frame and the frame of directors; more precisely we have :

$$\mathbf{d}_i(x, t) = \Lambda(x, t)\, \mathbf{D}_i \qquad , \quad i = 1, 2, 3 \quad , \quad t \geq 0 \quad , \quad x \ \ [0, L] \quad . \tag{5.2.1}$$

Since $\Lambda(x,t) \in SO(3)$, it follows that there exist 3x3 skew symmetric matrices $\Omega(x,t)$ and $W(x,t)$ such that the following holds (see (2.3.42) and (2.3.38)) :

$$\frac{\partial \Lambda(x,t)}{\partial x} = \Omega(x,t)\Lambda(x,t) \quad , \quad \frac{\partial \Lambda(x,t)}{\partial t} = W(x,t)\Lambda(x,t) \quad . \tag{5.2.2}$$

Let $\omega(x,t)$ and $\mathbf{w}(x,t)$ be the axial vectors corresponding to the skew-symmetric matrices Ω and W, respectively: $\omega(x,t)$ determines the rate of change of the rotation matrix $\Lambda(x,t)$ as a function of x; $\mathbf{w}(x,t)$ determines the rate of change of the rotation matrix $\Lambda(x,t)$ as a function of t. The strain measures used in the geometrically exact beam model are the vectors Γ and κ defined as :

$$\Gamma = \Lambda^T \frac{\partial \mathbf{r}}{\partial x} - \mathbf{D}_2 \quad , \quad \kappa = \Lambda^T \omega \quad . \tag{5.2.3}$$

For additional information, see [Quo.1], [Sim.1].

Neglecting gravitation, surface loads and assuming that the center of mass of the rigid body is fixed in the inertial frame N, the equations of motion for the beam, (2.4.1), (2.4.3), and (2.4.4), the constitutive equation for the beam, (2.3.53), and the boundary conditions for the beam, (2.4.5), are now reduced to :

$$\frac{\partial \mathbf{n}}{\partial x} = \rho \left(\frac{\partial^2 \mathbf{r}}{\partial t^2}\right)_N \quad , \tag{5.2.4}$$

$$\frac{\partial \mathbf{m}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x} \times \mathbf{n} = I_B \left(\frac{\partial(\mathbf{w}+\omega_R)}{\partial t}\right)_B + (\mathbf{w}+\omega_R) \times I_B(\mathbf{w}+\omega_R) \quad , \tag{5.2.5}$$

$$I_R\, \dot{\omega}_R + \omega_R \times I_R \omega_R = \mathbf{r}(0,t) \times \mathbf{n}(0,t) + \mathbf{m}(0,t) + \mathbf{N}_c(t) \quad , \tag{5.2.6}$$

$$\mathbf{n} = \Lambda \frac{\partial \psi}{\partial \Gamma} \quad , \quad \mathbf{m} = \Lambda \frac{\partial \psi}{\partial \kappa} \quad , \tag{5.2.7}$$

$$\mathbf{r}(0,t) = \mathbf{OQ} \quad , \quad \Lambda(0,t) = I \quad , \tag{5.2.8}$$

where $\mathbf{n}(x,t)$ and $\mathbf{m}(x,t)$ are the contact force and the contact moment of the beam, respectively; ρ is the mass per unit length of the beam; I_B is the inertia tensor of beam cross-sections, which is constant by assumption; ω_R is the angular velocity of the rigid body in the inertial frame N; I_R is the inertia tensor of the rigid body, which is a constant diagonal matrix by assumption; $\mathbf{N}_c(t)$ is the control torque applied to the rigid body; $\psi(\Gamma, \kappa)$ is the internal energy (i.e. potential energy) per unit length of the beam, which at the moment need not be a quadratic function of its arguments.

We note that, (5.2.4) and (5.2.5) state the balance of forces and the balance of moments at the beam cross-sections, (5.2.6) is the rigid body angular momentum equation, (5.2.7) is the constitutive equation of the beam and (5.2.8) gives the boundary conditions at the clamped end.

We define the rest state of the system given by (5.2.4)-(5.2.8) as follows :

$$\omega_R = \mathbf{0} \quad , \tag{5.2.9}$$

$$\frac{\partial \mathbf{r}}{\partial x} = \mathbf{D}_2 \quad , \quad x \in [0, L] \quad , \tag{5.2.10}$$

$$\Lambda(x) = I \quad , \quad x \in [0, L] \quad . \tag{5.2.11}$$

It is easy to see that (5.2.9) holds for all $t \in \mathbf{R}_+$ if and only if the rigid body does not spin the inertial frame N.

Let the curve of centroids be represented by :

$$\mathbf{r}(x,t) = u_1\,\mathbf{D}_1 + (|\,OQ| + x + u_2)\,\mathbf{D}_2 + u_3\,\mathbf{D}_3 \quad , \tag{5.2.12}$$

and, by (5.2.8), $u_1(0,t) = u_2(0,t) = u_3(0,t) = 0$ for all $t \geq 0$. Then (5.2.10) holds for all $t \in \mathbf{R}_+$ if and only if the beam displacements u_1, u_2, u_3 are identically zero.

If (5.2.10) holds for all $t \in \mathbf{R}_+$, then the beam deflections u_1, u_2, u_3 do not depend on time, hence by the first boundary condition in (5.2.8) they are identically zero on $[0,L] \times \mathbf{R}_+$. Conversely, if u_1, u_2, u_3 are identically zero on $[0,L] \times \mathbf{R}_+$, then (5.2.10) trivially follows from (5.2.12).

Also note that, (5.2.11) holds if and only if the strain measure κ defined in (5.2.3) is identically zero on $[0,L] \times \mathbf{R}_+$. If (5.2.10) holds, then the first equation in (5.2.2) implies that the skew-symmetric matrix $\Omega(x,t)$ is identically zero on $[0,L] \times \mathbf{R}_+$, which then implies that the corresponding axial vector ω and hence the strain measure κ are all identically zero on $[0,L] \times \mathbf{R}_+$. Conversely, if κ is identically zero on $[0,L] \times \mathbf{R}_+$, then so are the axial vector ω and the corresponding skew-symmetric matrix Ω. Then (5.2.2), implies that Λ do not depend on x, hence by using the boundary condition (5.2.8), we obtain (5.2.11). Furthermore, if (5.2.10) holds, (5.2.11) implies that the other strain measure Γ defined in (5.2.3) is also identically zero on $[0,L] \times \mathbf{R}_+$.

Stabilization Problem : Our stabilization problem is stated as before : Let the system given by (5.2.4)-(5.2.8) be disturbed from the rest state given by (5.2.9)-(5.2.11); find appropriate control laws which drive the system back to the rest state.

5.3 Natural Control Law

This control law applies a *force* $\mathbf{n}(L,t)$ and a *torque* $\mathbf{m}(L,t)$ at the free end of the beam and a *torque* $\mathbf{N}_c(t)$ to the rigid body. They are specified as follows : we choose 3x3 symmetric positive definite matrices K , L , M (which all can be chosen diagonal); then for all $t \geq 0$ the "natural control law " requires :

$$\mathbf{n}(L,t) = -L\,(\mathbf{r}_t\,(L,t))_B \quad , \tag{5.3.1}$$

$$\mathbf{m}(L,t) = -M\,\mathbf{w}(L,t) \quad , \tag{5.3.2}$$

$$\mathbf{N}_c(t) = -\mathbf{r}(L,t) \times \mathbf{n}(L,t) - \mathbf{m}(L,t) - K\,\omega_R \quad , \tag{5.3.3}$$

where $\mathbf{r}$ is the position vector of P with respect to O, the subscript B in (5.3.1) denotes that the time differentiation is carried out in the body frame **B**, (see Section 2.2), w is the axial vector associated with the skew-symmetric matrix W introduced in (5.2.2) and ω_R is the angular velocity of the rigid body in the inertial frame **N**.

The force $\mathbf{n}(L,t)$ given in (5.3.1) represents a *transversal force* acting at the free end of the beam whose magnitude depends linearly on the end-point deflection velocity : Using previous notation, let n_1 , n_2 , n_3 denote the components of the contact force in the body frame **B**, (i.e., $\mathbf{n} = \sum_{i=1}^{3} n_i \mathbf{D}_i$); let u_1 , u_2 , u_3 denote the beam deflections along the axes $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$. If the symmetric positive definite matrix L is diagonal, $L = diag(\alpha_1 , \alpha_2 , \alpha_3)$, the component form of (5.3.1) now reads : for $i = 1, 2, 3$, $t \geq 0$

$$n_i(L,t) + \alpha_1\,u_{it}(L,t) = 0 \quad , \tag{5.3.4}$$

which justifies the interpretation of (5.3.1) stated above.

Roughly speaking, (5.3.2) specifies a torque $\mathbf{m}(L,t)$ applied at the free end of the beam whose magnitude depends linearly on the "deflection angular velocity " at the free end. The relation between (5.3.2) and the corresponding torque control at the free end of the beam stated in previous chapters, (see, e.g., (4.3.2)), depends on a particular parametrization of the orthogonal transformation matrix $\Lambda(x,t)$, hence the parametrization of the vector $\mathbf{w}(x,t)$. To see that in special cases (5.3.4) reduces to, say (4.3.2), which is the corresponding torque control law at the free end of the beam when we use the Euler-Bernoulli beam model, let us consider the planar motion introduced in Section 3 of Chapter 2. Then Λ has a particular parametrization given by the equation (2.3.58) :

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad ,$$

where ϕ is the angle between the director axis $\mathbf{d}_2$ and the body axis $\mathbf{D}_2$, (see Figure 2.1). Then (5.2.2) yields $\mathbf{w} = \frac{\partial\phi}{\partial t}\mathbf{D}_1$, (see (2.3.59)). To a first order approximation we have $\phi = \frac{\partial u}{\partial x}$ where to simplify notation we denote here by u the beam deflection along the $\mathbf{D}_3$ axis. If $\mathbf{m} = m_1\,\mathbf{D}_1$, then the component form of (5.3.2) reads, for all $t \geq 0$

$$m_1(L,t) + \beta_1\, u_{xt}(L,t) = 0 \qquad .$$

Neglecting gravitation, a generalization of this equation leads to (4.3.2).

5.4 Stability Results for the Natural Control Law

Consider the system given by (5.2.4)-(5.2.8) together with the control law (5.3.1)-(5.3.3). To study the stability of this system, as we did in the previous chapters, we define the energy of the system as follows : for all $t \geq 0$

$$E(t) = \frac{1}{2} <\omega_R , I_R \, \omega_R> + \frac{1}{2}\int_0^L \rho <\mathbf{r}_t , \mathbf{r}_t> dx$$

$$+ \frac{1}{2}\int_0^L <(\omega_R + \mathbf{w}) , I_B \, (\omega_R + \mathbf{w})> dx + \frac{1}{2}\int_0^L \psi(\Gamma , \kappa) \, dx \quad , \tag{5.4.1}$$

where <.,.> denotes the standard inner-product in $\mathbf{R}^3$, and $\mathbf{r}_t$ is the abbreviation for $(\mathbf{r}_t(x , t))_N$. The first term in (5.4.1) represents the rotational kinetic energy of the rigid body, the second term represents the kinetic energy of the beam in the inertial frame N, the third term represents the rotational kinetic energy of the beam cross-sections and the last term represents the potential energy of the beam.

5.4.1 Proposition : Consider the system given by (5.2.4)-(5.2.8) and (5.3.1)-(5.3.3). Then the energy $E(t)$ defined in (5.4.1) is a nonincreasing function of time.

Proof : Differentiating (5.4.1) with respect to time t, we obtain :

$$\frac{dE(t)}{dt} = <\omega_R , \frac{d}{dt}(I_R \, \omega_R)> + \int_0^L \rho <\mathbf{r}_t , \mathbf{r}_{tt}> dx + \int_0^L <(\omega_R + \mathbf{w}) , \frac{d}{dt} [\, I_B \, (\omega_R + \mathbf{w}) \,]> dx$$

$$+ \int_0^L <\frac{\partial \psi}{\partial \Gamma} , \frac{\partial \Gamma}{\partial t}> dx + \int_0^L <\frac{\partial \psi}{\partial \kappa} , \frac{\partial \kappa}{\partial t}> dx$$

$$= <\omega_R \, , (I_R \dot{\omega}_R) + \omega_R \times I_R \omega_R> + \int_0^L <(\mathbf{r}_t)_B + \omega_R \times \mathbf{r} \, , \mathbf{n}_x> dx$$

$$+ \int_0^L <(\omega_R + \mathbf{w}) \, , I_B \frac{\partial}{\partial t} (\omega_R + \mathbf{w})_B + (\omega_R + \mathbf{w}) \times I_B \, (\omega_R + \mathbf{w})> dx$$

$$+ \int_0^L <\frac{\partial \psi}{\partial \Gamma} \, , \frac{\partial \Gamma}{\partial t}> dx + \int_0^L <\frac{\partial \psi}{\partial \kappa} \, , \frac{\partial \kappa}{\partial t}> dx \quad , \tag{5.4.2}$$

where in the second equation we use (5.2.4) and the relation between the time derivation in two frames (see (2.2.14)).

Using integration by parts, we calculate various integrals in (5.4.2) as follows :

$$\int_0^L <(\mathbf{r}_t)_B \, , \mathbf{n}_x> dx = <(\mathbf{r}_t)_B \, , \mathbf{n}> \Big|_{x=0,L} - \int_0^L <(\mathbf{r}_t)_B \, , \mathbf{n}> dx \quad , \tag{5.4.3}$$

$$\int_0^L <\omega_R \times \mathbf{r} \, , \mathbf{n}_x> dx = <\omega_R \, , \int_0^L \mathbf{r} \times \mathbf{n}_x \, dx>$$

$$= <\omega_R \, , \mathbf{r} \times \mathbf{n}_x \Big|_{x=0,L} > - <\omega_R \, , \int_0^L \mathbf{r}_x \times \mathbf{n} \, dx> \quad , \tag{5.4.4}$$

$$\int_0^L <(\omega_R + \mathbf{w}) \, , I_B \frac{\partial}{\partial t} (\omega_R + \mathbf{w})_B + (\omega_R + \mathbf{w}) \times I_B \, (\omega_R + \mathbf{w})> dx$$

$$= \int_0^L <(\omega_R + \mathbf{w}) \, , \mathbf{m}_x + \mathbf{r}_x \times \mathbf{n}> dx \qquad (by\ (5.2.5))$$

$$= \int_0^L <\mathbf{w} \, , \mathbf{m}_x> dx + <\omega_R \, , \int_0^L \mathbf{m}_x \, dx >$$

$$+ \int_0^L <\mathbf{w} \, , \mathbf{r}_x \times \mathbf{n}> dx + \int_0^L <\omega_R \, , \mathbf{r}_x \times \mathbf{n}> dx \qquad (\omega_R \textit{ does not depend on } x)$$

$$= <\mathbf{w}, \mathbf{m}> \Big|_{x=0,L} - \int_0^L <\mathbf{w}_x, \mathbf{m}> dx + <\omega_R, \mathbf{m} \Big|_{x=0,L} >$$

$$+ \int_0^L <\mathbf{w} \times \mathbf{r}_x, \mathbf{n}> dx + <\omega_R, \int_0^L \mathbf{r}_x \times \mathbf{n}\, dx > \quad , \tag{5.4.5}$$

where $\mathbf{u} \Big|_{x=a,b} := \mathbf{u}(b) - \mathbf{u}(a)$ for any $\mathbf{u} : \mathbf{R} \to \mathbf{R}^3$.

Using (5.4.3)-(5.4.5) in (5.4.2), we obtain :

$$\frac{dE(t)}{dt} = <\omega_R, I_R \dot{\omega}_R + \omega_R \times I_R \omega_R + \mathbf{r} \times \mathbf{n} \Big|_{x=0,L} + \mathbf{m} \Big|_{x=0,L} >$$

$$+ <(\mathbf{r}_t)_B, \mathbf{n}> \Big|_{x=0,L} + <\mathbf{w}, \mathbf{m}> \Big|_{x=0,L}$$

$$- \int_0^L <(\mathbf{r}_{xt})_B - \mathbf{w} \times \mathbf{r}_x, \mathbf{n}> dx - \int_0^L <\mathbf{w}_x, \mathbf{m}> dx$$

$$+ \int_0^L < \frac{\partial \psi}{\partial \Gamma}, \frac{\partial \Gamma}{\partial t} > dx + \int_0^L < \frac{\partial \psi}{\partial \kappa}, \frac{\partial \kappa}{\partial t} > dx \quad . \tag{5.4.6}$$

Differentiating (5.2.3) with respect to time t and noting that the internal energy ψ of the beam is invariant under the rigid body motions (i.e., ψ measured in the body frame **B** is equal to ψ measured in the inertial frame **N**, for details, e.g. see [Mar.1, p.194], [Gre.5]), we obtain

$$(\frac{\partial \Gamma}{\partial t})_B = \frac{\partial}{\partial t}(\Lambda^T)\frac{\partial \mathbf{r}}{\partial x} + \Lambda^T (\frac{\partial^2 \mathbf{r}}{\partial t \partial x})_B$$

$$= - \Lambda^T W \frac{\partial \mathbf{r}}{\partial x} + \Lambda^T (\frac{\partial^2 \mathbf{r}}{\partial t \partial x})_B$$

$$=\Lambda^T[(\mathbf{r}_{xt})_B - \mathbf{w} \times \mathbf{r}_x] \quad , \tag{5.4.7}$$

where in the second equation we have used (5.2.2) and the skew-symmetry of W. Then, using the definition of the axial vector w associated with W, we obtain (5.4.7).

Using (2.3.48), which is stated below :

$$(\frac{\partial \kappa}{\partial t})_B = \Lambda^T \frac{\partial \mathbf{w}}{\partial x} \quad ,$$

and using (5.4.7), (5.2.6), and (5.2.7) in (5.4.6), we obtain :

$$\frac{dE(t)}{dt} = <\omega_R \,, \mathbf{r} \times \mathbf{n} \Big|_{x=L} + \mathbf{m} \Big|_{x=L} + \mathbf{N}_c(t)>$$

$$- \int_0^L <(\mathbf{r}_{xt})_B - \mathbf{w} \times \mathbf{r}_x \,, \mathbf{n}> dx - \int_0^L <\mathbf{w}_x \,, \mathbf{m}> dx$$

$$+ \int_0^L <\Lambda^T \mathbf{n} \,, \Lambda^T[(\mathbf{r}_{xt})_B - \mathbf{w} \times \mathbf{r}_x]> dx + \int_0^L <\Lambda^T \mathbf{m} \,, \Lambda^T \mathbf{w}_x> dx$$

$$+ <(\mathbf{r}_t)_B \,, \mathbf{n}> \Big|_{x=0,L} + <\mathbf{w} \,, \mathbf{m}> \Big|_{x=0,L} \quad . \tag{5.4.8}$$

Since Λ is an orthogonal matrix, the second and the third lines cancel each other. Also by the boundary conditions (5.2.8), upon differentiating with respect to time t, at the clamped end we have :

$$(\mathbf{r}_t(0,t))_B = 0 \quad , \quad \mathbf{w}(0,t) = 0 \quad \text{for all } t \geq 0 \quad . \tag{5.4.9}$$

Using (5.4.9) and the control law (5.3.1)-(5.3.3) in (5.4.8), we obtain :

$$\frac{dE(t)}{dt} = -\langle \omega_R , K\, \omega_R \rangle - \langle (\mathbf{r}_t(L,t)) , L\, (\mathbf{r}_t(L,t)) \rangle$$

$$- \langle \mathbf{w}(L,t) , M\, \mathbf{w}(L,t) \rangle \quad . \tag{5.4.10}$$

Since by our choice the matrices K, L, M are positive definite, it follows from (5.4.10) that the energy $E(t)$ defined in (5.4.1) is a nonincreasing function of time. □

5.4.2 Remark : In the derivation of (5.4.10) we have used the nonlinear equations (5.2.4)-(5.2.8) without any *linearization*. Futhermore, we have not imposed any restriction on the internal energy ψ of the beam, other than the assumption that it depends on the strain measures Γ and κ, and the assumption that it is invariant under rigid body motions, which is a standard assumption in theory of elasticity, (see, e.g. [Mar.1, p.194]). From this assumption it follows that the rate of change of the internal energy ψ as observed in the inertial frame **N** and as observed in the body frame **B** must be the same, since these two frames differ only by a rotation *which does not depend on the spatial coordinate x.*

Special Case : Let us assume that the internal energy ψ is an uncoupled quadratic function of Γ and κ, which leads to the standard linear constitutive equation, (see e.g. [Sim.1]); that is we have the following :

$$\psi(\Gamma, \kappa) = \frac{1}{2} \langle \Gamma , C_1 \Gamma \rangle + \frac{1}{2} \langle \kappa , C_2 \kappa \rangle \quad , \tag{5.4.11}$$

where C_1 and C_2 are diagonal constant matrices with positive elements. Upon differentiating (5.4.11) we obtain :

$$\frac{d\psi}{dt} = \frac{1}{2} < (\Gamma_t)_B + \omega_R \times \Gamma , C_1 \Gamma > + \frac{1}{2} < \Gamma , C_1 (\Gamma_t)_B + \omega_R \times C_1 \Gamma >$$

$$+ \frac{1}{2} < (\kappa_t)_B + \omega_R \times \kappa , C_2 \kappa > + \frac{1}{2} < \kappa , C_2 (\kappa_t)_B + \omega_R \times C_2 \kappa >$$

$$= < (\Gamma_t)_B , C_1 \Gamma > + < (\kappa_t)_B , C_2 \kappa >$$

$$+ \frac{1}{2} < \omega_R \times \Gamma , C_1 \Gamma > + \frac{1}{2} < \Gamma , \omega_R \times C_1 \Gamma >$$

$$+ \frac{1}{2} < \omega_R \times \kappa , C_2 \kappa > + \frac{1}{2} < \kappa , \omega_R \times C_2 \kappa > \quad .$$

By using the following equation

$$< \mathbf{a} , \mathbf{b} \times \mathbf{c} > = < \mathbf{a} \times \mathbf{b} , \mathbf{c} > \quad \text{for all } \mathbf{a} , \mathbf{b} , \mathbf{c} \in \mathbf{R}^3 \quad ,$$

it follows that the sum of the each of the last two lines in the above equation are zero. Since $\frac{\partial \psi}{\partial \Gamma} = C_1 \Gamma$ and $\frac{\partial \psi}{\partial \kappa} = C_2 \kappa$, it follows that

$$\frac{d\psi}{dt} = < \frac{\partial \psi}{\partial \Gamma} , (\Gamma_t)_B > + < \frac{\partial \psi}{\partial \kappa} , (\kappa_t)_B > \quad .$$

□

5.4.3 Remark : Since in deriving (5.4.10) we have not used any particular parametrization of the transformation matrix Λ and any particular form of the internal energy function ψ, (5.4.10) is a generalization of previously obtained rate of energy equations, such as (4.5.12). If we use small deformation assumption and the Euler-Bernoulli beam model, then (5.4.10) reduces to (4.5.12). To see this, let u_1 , u_2 , u_3 denote the beam deflections along the axes $\mathbf{D}_1 , \mathbf{D}_2 , \mathbf{D}_3$. Furthermore let us neglect the axial deformation (i.e., $u_2 = 0$ on $[0, L] \times \mathbf{R}_+$), and the torsion of the beam (i.e., no rotation of the beam cross-sections about $\mathbf{D}_2$ axis). Then, small deformation assumption leads to :

$$\mathbf{w} = u_{3xt}\mathbf{D}_1 - u_{1xt}\mathbf{D}_3 \quad .$$

Using the above equation, (5.4.10) leads to (5.4.12) :

$$\frac{dE(t)}{dt} = -<\omega_R \, , K \; \omega_R> - \alpha_1 u_{1t}{}^2(L, t) - \alpha_3 u_{3t}{}^2(L, t) \quad ,$$

$$- \beta_1 u_{1xt}{}^2(L, t) - \beta_3 u_{3xt}{}^2(L, t) \qquad (5.4.12)$$

where, for simplicity, we have chosen $L = diag(\alpha_1, \alpha_2, \alpha_3)$, and $M = diag(\beta_1, \beta_2, \beta_3)$. □

To prove that the solutions of (5.2.4)-(5.2.8) with the control law (5.3.1)-(5.3.3) decay to the rest state defined by (5.2.9)-(5.2.11), we need to parametrize the orthogonal transformation matrix Λ and specify the form of the internal energy function ψ.

In the sequel we will assume that the whole motion takes place in a plane whose unit normal is the inertial axis $\mathbf{e}_1$. More precisely, we consider the configuration given in Figure 5.1 with the following assumptions :

(i) The axes $\mathbf{e}_1$, $\mathbf{D}_1$, $\mathbf{d}_1$ coincide at all times and rigid body may rotate only about the axis $\mathbf{e}_1$;

(ii) The whole motion of the beam takes place in the plane spanned by the axes $\mathbf{e}_2$, $\mathbf{e}_3$;

(iii) The axial deflection (i.e., along the $\mathbf{D}_2$ axis) and the torsion (i.e., the rotation of the beam cross-sections about $\mathbf{D}_2$ axis) is negligible.

The orthogonal transformation matrix Λ between the body frame N and the frame of directors, now admits the following representation :

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad , \tag{5.4.13}$$

where ϕ is the angle between the director $\mathbf{d}_2$ and the body axis $\mathbf{D}_2$, (see Figure 2.1).

Using (5.4.13) in (5.2.2), we obtain the following :

$$\mathbf{w} = \frac{\partial \phi}{\partial t}\,\mathbf{D}_1 \quad , \quad \kappa = \omega = \frac{\partial \phi}{\partial x}\,\mathbf{D}_1 \quad . \tag{5.4.14}$$

Let $u := u_3$ denote the beam deflection along the $\mathbf{D}_3$ axis and let Γ_2 , Γ_3 denote the components of Γ given by (5.2.3). Then assuming small deflections for ϕ and u and neglecting higher order terms, (5.2.3) reduces to :

$$\Gamma_2 = 0 \quad , \quad \Gamma_3 = \frac{\partial u}{\partial x} - \phi \quad . \tag{5.4.15}$$

Following [Sim.1] and the standard linear theory, we assume the following quadratic form for the internal energy function for the beam :

$$\begin{aligned} 2\psi(\Gamma , \kappa) &= GA\,\Gamma_3{}^2 + EI\,\kappa_1{}^2 \\ &= GA\,(u_x - \phi)^2 + EI\,\phi_x{}^2 \quad , \end{aligned} \tag{5.4.16}$$

where G is the shear modulus and A is the cross-sectional area along the axis $\mathbf{D}_3$, EI is the principal bending stiffness relative to the axis $\mathbf{D}_1$, and κ_1 is the component of κ along the axis $\mathbf{D}_1$, (see (5.4.14)).

With these assumptions and neglecting the higher order terms, the relevant component forms of equations (5.2.4)-(5.2.8) now reduce to :

$$GA\,(u_{xx} - \phi_x) = \rho u_{tt} + \rho\ddot{\theta}(b + x) - \rho\dot{\theta}^2 u \quad , \tag{5.4.17}$$

$$EI\,\phi_{xx} + GA\,(u_x - \phi) = I_B\ (\phi_{tt} + \ddot{\theta}) \quad , \tag{5.4.18}$$

$$I_R\ddot{\theta} = bGA\,[u_x(0,t) - \phi\,(0,t)] + EI\,\phi_x(0,t) + N_c(t) \quad , \tag{5.4.19}$$

$$u\ (0,t) = 0 \quad , \quad \phi\,(0,t) = 0 \qquad \text{for all } t \geq 0 \quad , \tag{5.4.20}$$

where $b = |\,OQ\,|$, θ is the angle of rotation of the rigid body about the axis $\mathbf{e}_1$, hence we have,

$$\omega_R = \omega_1\,\mathbf{D}_1 = \dot{\theta}\,\mathbf{D}_1 \quad , \tag{5.4.21}$$

I_B is the principal moment of inertia of the beam cross-sections about the axis $\mathbf{d}_1$ and I_R is the principal moment of inertia of the rigid body about the axis $\mathbf{D}_1$.

The component form of the natural control law (5.3.1)-(5.3.3) now become :

$$GA\,[u_x(L,t) - \phi(L,t)] + \alpha\, u(L,t) = 0 \quad , \tag{5.4.22}$$

$$EI\,\phi_x(L,t) + \beta\,\phi_t(L,t) = 0 \quad , \tag{5.4.23}$$

$$N_c(t) = -(b+L)GA[u_x(L,t) - \phi(L,t)] - EI\phi_x(L,t) - k\dot{\theta} \quad , \tag{5.4.24}$$

where $\alpha > 0$, and $\beta > 0$ are arbitrary positive numbers.

The total energy $E(t)$ given by (5.4.1), now becomes :

$$E(t) = \frac{1}{2}I_R\dot{\theta}^2 + \frac{1}{2}\int_0^L \rho \langle \mathbf{r}_t , \mathbf{r}_t \rangle \, dx + \frac{1}{2}\int_0^L I_B(\phi_t + \dot{\theta})^2 \, dx$$

$$+ \frac{1}{2}\int_0^L GA(u_x - \phi)^2 \, dx + \frac{1}{2}\int_0^L EI\phi_x{}^2 \, dx \quad , \tag{5.4.25}$$

and the rate of change of $E(t)$ given by (5.4.10) now reduces to :

$$\frac{dE(t)}{dt} = -k\,\dot{\theta}^2 - \alpha\, u_t{}^2(L,t) - \beta\,\phi_t{}^2(L,t) \quad . \tag{5.4.26}$$

5.4.4 Theorem : Consider the system given by (5.4.17)-(5.4.20) together with the control law (5.4.22)-(5.4.24). Then there exists a $T \geq 0$ such that for $t \geq T$, the energy $E(t)$ given by (5.4.25) decays as $O(\frac{1}{t})$.

5.4.5 Remark : Equations (5.4.17)-(5.4.20) are the component forms of equations (5.2.4)-(5.2.8) under the assumptions :

(i) the motion takes place in the plane normal to the axis $\mathbf{e}_1$, the axes $\mathbf{e}_1$, $\mathbf{D}_1$, $\mathbf{d}_1$ coincide at all times and the rigid body rotates about this common axis,

(ii) the axial deformatin and the torsion are negligible,

(iii) higher order terms are negligible.

As a result of these assumptions, (5.4.17)-(5.4.20) represent the equations of motion for the planar motion of a rigid body whose center of mass is fixed in an inertial frame, with a beam modeled as a Timoshenko beam, clamped to it. □

5.4.6 Remark : If we use the conclusion of Theorem 5.4.4 in the expansion of $E(t)$, (5.4.25), and $\frac{dE(t)}{dt}$, (5.4.26), then we obtain :

$$
\begin{array}{llll}
\dot{\theta} \to 0 & & as\ t \to \infty & , \\
\phi_t(x,t) \to 0 & , \text{ for all } x \in [0, L] \ , & \text{as } t \to \infty & , \\
\phi_x(x,t) \to 0 & , \text{ for all } x \in [0, L] \ , & \text{as } t \to \infty & , \\
u_t(x,t) \to 0 & , \text{ for all } x \in [0, L] \ , & \text{as } t \to \infty & , \\
u_x(x,t) \to 0 & , \text{ for all } x \in [0, L] \ , & \text{as } t \to \infty & .
\end{array}
$$

These limiting behaviors imply that the solutions tend as $t \to \infty$ to the rest state defined by (5.2.9)-(5.2.11). □

Proof : We define the following function $V(t)$:

$$
\begin{aligned}
V(t) = 2(1-\varepsilon)tE(t) &+ 2\int_0^L \rho x\,[u_t + \dot{\theta}(b+x)]u_x\, dx \\
&+ 2\int_0^L I_B\, x\, \phi_x(\phi_t + \dot{\theta})\, dx + \delta\int_0^L I_B\, \phi(\phi_t + \dot{\theta})\, dx \\
&- \delta\int_0^L \rho u\,[u_t + \dot{\theta}(b+x)]\, dx \quad ,
\end{aligned}
\tag{5.4.27}
$$

where $\varepsilon \in (0, 1)$ and $\delta > 0$ are constants yet to be determined.

To prove the theorem, we first show that there exists a constant $C > 0$ such that the following estimate holds for all $t \geq 0$:

$$[2(1-\varepsilon)t - C]\,E(t) \leq V(t) \leq [2(1-\varepsilon)t + C]\,E(t) \quad . \tag{5.4.28}$$

Then we prove that there exists a $T_1 \geq 0$ such that

$$\frac{dV(t)}{dt} \leq 0 \qquad \text{for all } t \geq T_1 \quad . \tag{5.4.29}$$

Combining (5.4.28) and (5.4.29) we obtain :

$$E(t) \leq \frac{V(T_1)}{2(1-\varepsilon)t - C} \qquad , t > T \quad , \tag{5.4.30}$$

where $T = \max\,\{T_1\,,\,\frac{C}{2(1-\varepsilon)}\,\}$.

Since $E(t)$ is nonincreasing by (5.4.26), from (5.4.28) it follows that $V(T_1) < \infty$, and (5.4.30) proves that for sufficiently large t, $E(t)$ decays as $O(\frac{1}{t})$.

Due to the boundary conditions $u(0,t) = 0$, $\phi(0,t) = 0$ for all $t \geq 0$, similar to (4.5.14) we obtain the following estimates which follows from the Jensen's inequality (see, e.g., [Roy.1, p. 110]) :

$$\phi^2(x,t) \leq L\int_0^L \phi_s{}^2\,ds \qquad x \in [0, L] \quad , \tag{5.4.31}$$

$$u^2(x,t) \leq L\int_0^L u_{ss}{}^2\,ds \qquad x \in [0, L] \quad . \tag{5.4.32}$$

Using (5.4.31), we obtain the following estimate :

$$\int_0^L u_x{}^2\, dx = \int_0^L (u_x - \phi + \phi)^2\, dx \leq 2\int_0^L (u_x - \phi)^2\, dx + 2\,L^2 \int_0^L \phi_x{}^2\, dx \quad . \tag{5.4.33}$$

For simplicity, we define the quantities A_1, A_2, A_3, and A_4 which appear in (5.4.27) as follows :

$$A_1 := 2\int_0^L \rho x\,[u_t + \dot{\theta}(b + x)]u_x\, dx \quad , \tag{5.4.34}$$

$$A_2 := 2\int_0^L I_B\, x\, \phi_x (\phi_t + \dot{\theta})\, dx \quad , \tag{5.4.35}$$

$$A_3 := \delta \int_0^L I_B\, \phi(\phi_t + \dot{\theta})\, dx \quad , \tag{5.4.36}$$

$$A_4 := -\,\delta \int_0^L \rho u\,[u_t + \dot{\theta}(b + x)]\, dx \quad . \tag{5.4.37}$$

Since $\mathbf{r}(x, t) = (b + x)\,\mathbf{D}_2 + u\;\mathbf{D}_3$, using the differentiation rule (2.2.14), we obtain :

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_N = -\,\dot{\theta}u\;\mathbf{D}_2 + [u_t + \dot{\theta}(b + x)]\,\mathbf{D}_3 \quad . \tag{5.4.38}$$

Using (5.4.31)-(5.4.33) and (5.4.38), we obtain the following estimates :

$$|A_1| \leq \rho L \int_0^L u_x{}^2\, dx + \rho L \int_0^L [u_t + \dot{\theta}(b + x)]^2\, dx$$

$$\leq 2\rho L \int_0^L (u_x - \phi)^2\, dx + 2\rho L^3 \int_0^L \phi_x{}^2\, dx + L \int_0^L \rho \langle \mathbf{r}_t\,, \mathbf{r}_t \rangle\, dx$$

$$\leq K_1 E(t) \quad , \tag{5.4.39}$$

where $K_1 = \dfrac{\max\{2\rho L\ , 2\rho L^3\ , L\ \}}{\min\{\frac{I_R}{2}\ , \frac{1}{2}\ , \frac{GA}{2}\ , \frac{EI}{2}\}}$,

$$|A_2| \leq I_B L\int_0^L \phi_x{}^2\,dx + I_B L\int_0^L (\phi_t + \dot{\theta})^2\,dx$$

$$\leq K_2 E(t) \quad , \tag{5.4.40}$$

where $K_2 = \dfrac{\max\{I_B L\ , L\ \}}{\min\{\frac{I_R}{2}\ , \frac{1}{2}\ , \frac{GA}{2}\ , \frac{EI}{2}\}}$,

$$|A_3| \leq \delta I_B \int_0^L \phi^2\,dx + \delta I_B \int_0^L (\phi_t + \dot{\theta})^2\,dx$$

$$\leq \delta I_B L^2\int_0^L \phi_x{}^2\,dx + \delta I_B \int_0^L (\phi_t + \dot{\theta})^2\,dx$$

$$\leq K_3 E(t) \quad , \tag{5.4.41}$$

where $K_3 = \dfrac{\max\{\delta I_B L^2\ , \delta\ \}}{\min\{\frac{I_R}{2}\ , \frac{1}{2}\ , \frac{GA}{2}\ , \frac{EI}{2}\}}$,

$$|A_4| \leq \delta\rho\int_0^L u^2\,dx + \delta\rho\int_0^L [u_t + \dot{\theta}(b + x)]^2\,dx$$

$$\leq 2\delta\rho L^2\int_0^L (u_x - \phi)^2\,dx + 2\delta\rho L^4\int_0^L \phi_x{}^2\,dx + \delta\int_0^L \rho\langle \mathbf{r}_t\ , \mathbf{r}_t\rangle\,dx$$

$$\leq K_4 E(t) \quad , \tag{5.4.42}$$

where $K_4 = \dfrac{\max\{2\delta\rho L^2, 2\delta\rho L^4, \delta\}}{\min\{\frac{I_R}{2}, \frac{1}{2}, \frac{GA}{2}, \frac{EI}{2}\}}$.

Using (5.4.39)-(5.4.42) in (5.4.27), we obtain the following :

$$[2(1-\varepsilon)t - C]\,E(t) \le V(t) \le [2(1-\varepsilon)t + C]\,E(t) \quad , \tag{5.4.43}$$

where $C = K_1 + K_2 + K_3 + K_4$. This proves (5.4.28).

To prove (5.4.29), we first differentiate A_1 with respect to time:

$$\frac{dA_1}{dt} = 2\int_0^L \rho x\,[u_t + \dot\theta(b+x)]u_{xt}\,dx + 2\int_0^L \rho x\,[u_{tt} + \ddot\theta(b+x)]u_x\,dx$$

$$= 2\int_0^L \rho x u_t u_{xt}\,dx + 2\int_0^L \rho x u_{xt}\dot\theta(b+x)\,dx$$

$$+ 2\int_0^L GAxu_x(u_{xx} - \phi_x)\,dx + 2\dot\theta^2\int_0^L \rho x u u_x\,dx$$

$$= \rho L u_t^2(L,t) - \int_0^L \rho u_t^2\,dx + 2[\,\rho L(b+L)u_t(L,t) - \int_0^L \rho(b+2x)u_t\,dx\,]\dot\theta$$

$$+ GA\;Lu_x^2(L,t) - GA\int_0^L u_x^2\,dx - 2GA\int_0^L xu_x\phi_x\,dx$$

$$[\,\rho L u^2(L,t) - \rho\int_0^L u^2\,dx\,]\dot\theta^2 \quad , \tag{5.4.44}$$

where in the second equation we used (5.4.17). Then, using integration by parts and the fact that $\dot\theta$ does not depend on x, we obtain (5.4.44).

Since $\theta(.)$ does not depend on x, A_2 is equivalent to the following :

$$A_2 = 2\int_0^L I_B x(\phi+\theta)_x(\phi+\theta)_t \, dx \quad . \tag{5.4.45}$$

Upon differentiating (5.4.45) with respect to time, we obtain :

$$\frac{dA_2}{dt} = 2\int_0^L I_B x(\phi+\theta)_{xt} \, dx + 2\int_0^L x I_B(\phi_{tt} + \ddot{\theta})(\phi+\theta)_x \, dx$$

$$= I_B L(\phi_t + \dot{\theta})^2 \Big|_{x=L} - \int_0^L I_B(\phi_t + \dot{\theta})^2 \, dx + 2\int_0^L EIx\,\phi_x \phi_{xx} \, dx$$

$$+ 2\int_0^L GAx\,\phi_x(u_x - \phi) \, dx$$

$$= I_B L(\phi_t + \dot{\theta})^2 \Big|_{x=L} - \int_0^L I_B(\phi_t + \dot{\theta})^2 \, dx + EI\ L\phi_x{}^2 \Big|_{x=L}$$

$$- \int_0^L EI\,\phi_x{}^2 \, dx + 2GA\int_0^L x\phi_x u_x \, dx - GA\ L\phi^2 \Big|_{x=L} + GA\int_0^L \phi^2 \, dx \quad , \tag{5.4.46}$$

where in the second equation we used integration by parts and (5.4.18). Then, using integration by parts, we obtain (5.4.46).

Upon differentiating A_3, we obtain :

$$\frac{dA_3}{dt} = \delta\int_0^L I_B(\phi_t + \dot{\theta})^2 \, dx - \delta\int_0^L I_B\dot{\theta}(\phi_t + \dot{\theta}) \, dx + \delta\int_0^L I_B\phi(\phi_{tt} + \ddot{\theta}) \, dx$$

$$= \delta\int_0^L I_B(\phi_t + \dot{\theta})^2 \, dx - \delta\int_0^L I_B\dot{\theta}(\phi_t + \dot{\theta}) \, dx\, \delta\int_0^L \phi[EI\,\phi_{xx} + GA(u_x - \phi)] \, dx$$

$$= \delta\int_0^L I_B(\phi_t + \dot{\theta})^2\, dx - \delta\int_0^L I_B\,\dot{\theta}(\phi_t + \dot{\theta})\, dx + \delta EI\,\phi\phi_x \Big|_{x=L} - \delta EI\int_0^L \phi_x^{\,2}\, dx$$

$$+ \delta GA\,\phi u \Big|_{x=L} - \delta GA\int_0^L \phi_x u\, dx - \delta GA\int_0^L \phi^2\, dx \quad , \tag{5.4.47}$$

where in the first equation we added and subtracted $\dot{\theta}$, in the second equation we used (5.4.18). Then, using integration by parts and the boundary conditions (5.4.20), we obtain (5.4.47).

Similarly, upon differentiating A_4, we obtain :

$$\frac{dA_4}{dt} = -\delta\int_0^L \rho u_t[u_t + \dot{\theta}(b+x)]\, dx - \delta\int_0^L \rho u[u_{tt} + \ddot{\theta}(b+x)]\, dx$$

$$= -\delta\int_0^L \rho u_t^{\,2}\, dx - \delta\dot{\theta}\int_0^L \rho(b+x)u_t\, dx - \delta\int_0^L u[GA(u_{xx} - \phi_x)]\, dx - \delta\dot{\theta}^2\int_0^L \rho u^2\, dx$$

$$= -\delta\int_0^L \rho u_t^{\,2}\, dx - \delta\dot{\theta}\int_0^L \rho(b+x)u_t\, dx - \delta GA uu_x \Big|_{x=L}$$

$$+ \delta GA\int_0^L u_x^{\,2}\, dx + \delta GA\int_0^L u\,\phi_x\, dx - \delta\dot{\theta}^2\int_0^L \rho u^2\, dx \quad , \tag{5.4.48}$$

where in the second equation we used (5.4.17). Then, integrating by parts and using the boundary conditions (5.4.20), we obtain (5.4.48).

Differentiating $V(t)$ with respect to time and using (5.4.44)-(5.4.48), we obtain the following :

$$\frac{dV(t)}{dt} = 2(1-\varepsilon)t\frac{dE(t)}{dt} + 2(1-\varepsilon)E(t) + \sum_{i=1}^{3}\frac{dA_i}{dt}$$

$$= -2(1-\varepsilon)kt\dot{\theta}^2 - 2(1-\varepsilon)\alpha t u_t^{\,2}(L,t) - 2(1-\varepsilon)\beta t\,\phi_t^{\,2}(L,t)$$

$$+(1-\varepsilon)I_R\dot{\theta}^2 + (1-\varepsilon)\int_0^L \rho<\mathbf{r}_t\,,\mathbf{r}_t>dx + (1-\varepsilon)\int_0^L I_B(\phi_t+\dot{\theta})^2\,dx$$

$$+(1-\varepsilon)GA\int_0^L (u_x-\phi)^2\,dx + (1-\varepsilon)EI\int_0^L \phi_x^{\,2}\,dx + \rho L u_t^{\,2}(L,t)$$

$$-\int_0^L \rho u_t^{\,2}\,dx + 2[\,\rho L(b+L)u_t(L,t) - \int_0^L \rho(b+2x)u_t\,dx\,]\dot{\theta} + GA\;Lu_x^{\,2}(L,t)$$

$$-GA\int_0^L u_x^{\,2}\,dx - 2GA\int_0^L x u_x\phi_x\,dx + \rho\,[Lu^2(L,t) - \int_0^L u^2\,dx\,]\dot{\theta}^2$$

$$+I_B L(\phi_t+\theta)^2\Big|_{x=L} - \int_0^L I_B(\phi_t+\dot{\theta})^2\,dx + EI\;L\phi_x^{\,2}\Big|_{x=L}$$

$$-\int_0^L EI\,\phi_x^{\,2}\,dx + 2GA\int_0^L x\phi_x u_x\,dx - GA\;L\phi^2\Big|_{x=L} + GA\int_0^L \phi^2\,dx$$

$$+\delta\int_0^L I_B(\phi_t+\dot{\theta})^2\,dx - \delta\int_0^L I_B\dot{\theta}(\phi_t+\dot{\theta})\,dx + \delta EI\,\phi\phi_x\Big|_{x=L}$$

$$-\delta EI\int_0^L \phi_x^{\,2}\,dx + \delta GA\,\phi u\Big|_{x=L} - \delta GA\int_0^L \phi_x u\,dx - \delta GA\int_0^L \phi^2\,dx$$

$$-\delta\int_0^L \rho u_t^{\,2}\,dx - \delta\dot{\theta}\int_0^L \rho(b+x)u_t\,dx - \delta GA u u_x\Big|_{x=L}$$

$$+\delta GA\int_0^L u_x^{\,2}\,dx + \delta GA\int_0^L u\,\phi_x\,dx - \delta\dot{\theta}^2\int_0^L \rho u^2\,dx \quad , \tag{5.4.49}$$

where in the first equation we used (5.4.27) and (5.4.34)- (5.4.37). Then, using (5.4.26), (5.4.25) and (5.4.44)- (5.4.48), we obtain (5.4.49).

Using (5.4.38), the integral associated with the inner product $<\mathbf{r}_t , \mathbf{r}_t>$ can be written as :

$$\int_0^L \rho<\mathbf{r}_t , \mathbf{r}_t> dx = \dot{\theta}^2 \int_0^L \rho u^2 \, dx + \int_0^L \rho u_t^{\,2} \, dx + 2\dot{\theta} \int_0^L \rho(b+x) u_t \, dx + \dot{\theta}^2 \int_0^L (b+x)^2 \, dx \quad , \qquad (5.4.50)$$

After cancellations, using (5.4.50) and collecting likewise terms, (5.4.49) becomes :

$$\begin{aligned}
\frac{dV}{dt} = -\Big[\, & 2(1-\varepsilon)kt - (1-\varepsilon)I_R - \int_0^L \rho(u^2 + (b+x)^2) \, dx \\
& - (\rho L u^2(L,t) - \int_0^L \rho u^2 \, dx + \delta \int_0^L \rho u^2 \, dx \ \Big] \dot{\theta}^2 \\
& - [\varepsilon + \delta] \int_0^L \rho u_t^{\,2} \, dx - [\varepsilon + \delta] \int_0^L EI \, \phi_x^{\,2} \, dx - [\varepsilon - \delta] \int_0^L I_B (\phi_t + \dot{\theta})^2 \, dx \\
& + (\delta - 1) GA \int_0^L u_x^{\,2} \, dx + (1-\delta) GA \int_0^L \phi^2 \, dx + (1-\varepsilon) GA \int_0^L (u_x - \phi)^2 \, dx \\
& + [2(1-\varepsilon) - \delta] \dot{\theta} \int_0^L \rho(b+x) u_t \, dx - \delta \int_0^L I_B \, \dot{\theta} (\phi_t + \dot{\theta}) \, dx \\
& - [2(1-\varepsilon)\alpha t - \rho L] u_t^{\,2}(L,t) - 2(1-\varepsilon)\beta t \, \phi_t^{\,2}(L,t) \\
& + 2[\rho L (b+L) u_t(L,t) - \rho \int_0^L (b+2x) u_t \, dx \ ] \dot{\theta} + GAL u_x^{\,2}(L,t) \\
& + I_B L (\phi_t(L,t) + \dot{\theta}(t))^2 + EI \, \phi_x^{\,2}(L,t) - GAL \phi^2(L,t)
\end{aligned}$$

$$+\delta EI\,\phi(L,t)\phi_x(L,t)-\delta GA\,[(u_x(L,t)-\phi(L,t)]u(L,t) \quad . \tag{5.4.51}$$

Using the following simple inequalities

$$ab \le \delta^2 a^2+\frac{b^2}{\delta^2} \qquad a\,,b\,,\delta\in \mathbf{R}\,,\ \delta\neq 0 \quad , \tag{5.4.52}$$

$$(a+b)^2 \le 2(a^2+b^2) \qquad a\,,b\in\mathbf{R} \quad , \tag{5.4.53}$$

boundary controls (5.4.22), (5.4.23) and the fact that the energy $E(t)$ stays bounded, we obtain the following estimates for some of the terms which appear in (5.4.51) :

$$\int_0^L u^2\,dx \le L^2\int_0^L u_x{}^2\,dx$$

$$\le 2L^2\int_0^L (u_x-\phi)^2\,dx+2L^4\int_0^L \phi_x{}^2\,dx$$

$$\le M_1\,E(0) \quad , \tag{5.4.54}$$

where $M_1=\dfrac{\max\{2L^2\,,2L^4\}}{\min\{\frac{I_R}{2}\,,\frac{1}{2}\,,\frac{GA}{2}\,,\frac{EI}{2}\}}$,

$$u^2(L,t)\le M_2\,E(0) \quad , \tag{5.4.55}$$

where $M_2=\dfrac{\max\{2L\,,2L^3\}}{\min\{\frac{I_R}{2}\,,\frac{1}{2}\,,\frac{GA}{2}\,,\frac{EI}{2}\}}$, (see (5.4.32) and (5.4.33)),

$$\int_0^L \phi^2\,dx \le L^2\int_0^L \phi_x{}^2\,dx \quad , \ (by\ (5.4.31)) \ , \tag{5.4.56}$$

$$\int_0^L (u_x - \phi)^2\,dx \le 2\int_0^L u_x^{\,2}\,dx + 2\int_0^L \phi^2\,dx$$

$$\le 2\int_0^L u_x^{\,2}\,dx + 2\int_0^L \phi_x^{\,2}\,dx \quad , \ (by\ (5.4.33)) \ , \tag{5.4.57}$$

$$\int_0^L \rho(b+x)u_t\dot{\theta}\,dx \le \delta_1^{\,2}\int_0^L \rho u_t^{\,2}\,dx + \frac{\rho\int_0^L (b+x)^2\,dx}{\delta_1^{\,2}}\dot{\theta}^2 \quad , \ (by\ (5.4.52)) \ , \tag{5.4.58}$$

$$\int_0^L I_B\,\dot{\theta}(\phi_t + \dot{\theta})\,dx \le \delta_2^{\,2}\int_0^L I_B(\phi_t + \dot{\theta})^2\,dx + \frac{I_B L}{\delta_2^{\,2}}\dot{\theta}^2 \quad , \ (by\ (5.4.52)) \ , \tag{5.4.59}$$

$$u_t(L,t)\dot{\theta} \le \delta_3^{\,2}u_t^{\,2}(L,t) + \frac{1}{\delta_3^{\,2}}\dot{\theta}^2 \quad , \ (by\ (5.4.52)) \ , \tag{5.4.60}$$

$$\int_0^L \rho(b+2x)\,dx \le \delta_4^{\,2}\int_0^L \rho u_t^{\,2}\,dx + \frac{\rho\int_0^L (b+2x)^2\,dx}{\delta_4^{\,2}}\dot{\theta}^2 \quad , \ (by\ (5.4.52)) \ , \tag{5.4.61}$$

$$GA\ Lu_x^{\,2}(L,t) = GA\ L\phi^2(L,t) - 2\alpha L\phi(L,t)u_t(L,t) + \frac{\alpha^2 L}{GA}u_t^{\,2}(L,t)$$

$$\le GA\ L\phi^2(L,t) + 2\alpha L^2\delta_5^{\,2}\int_0^L \phi_x^{\,2}\,dx$$

$$+\left(\frac{2\alpha^2 L}{\delta_5^{\,2}} + \frac{\alpha^2 L}{GA}\right)u_t^{\,2}(L,t) \quad , \ (by\ (5.4.22)) \ , \tag{5.4.62}$$

$$I_B L(\phi_t(L,t) + \dot{\theta})^2 \le 2I_B L\phi_t^{\,2}(L,t) + 2I_B L\dot{\theta}^2 \quad , \ (by\ (5.4.23)) \ , \tag{5.4.63}$$

$$EI\,\phi_x{}^2(L,t) \le \frac{\beta^2}{EI}\phi_t{}^2(L,t) \quad , \tag{5.4.64}$$

$$\delta EI\,\phi(L,t)\phi_x(L,t) = -\delta\beta\phi(L,t)\phi_t(L,t)$$

$$\le \delta\beta\delta_6{}^2\phi^2(L,t) + \frac{\delta\beta}{\delta_6{}^2}\phi_t{}^2(L,t)$$

$$\le \delta\beta L\,\delta_6{}^2\int_0^L \phi_x{}^2\,dx + \frac{\delta\beta}{\delta_6{}^2}\phi_t{}^2(L,t) \quad , \ (by\ (5.4.52), (5.4.31)) \ , \tag{5.4.65}$$

$$-\delta GA\,[u_x(L,t) - \phi(L,t)]\,u_t(L,t) = \delta\alpha u_t(L,t)u(L,t)$$

$$\le \delta\alpha L\,\delta_7{}^2\int_0^L u_x{}^2\,dx + \frac{\delta\alpha}{\delta_7{}^2}u_t{}^2(L,t) \quad , \ (by\ (5.4.22)) \tag{5.4.66}$$

where δ_i , $i = 1,\ldots, 7$ are any nonzero real numbers.

Using the estimates (5.4.54)-(5.4.64) in (5.4.51), the latter becomes :

$$\frac{dV(t)}{dt} \le -[2(1-\varepsilon)kt - D_1]\dot{\theta}^2(t) - [\varepsilon + \delta - (2(1-\varepsilon))\delta_1{}^2 - 2\delta_4{}^2]\int_0^L \rho u_t{}^2\,dx$$

$$-[2(1-\varepsilon)\alpha t - D_2]u_t{}^2(L,t) - [2(1-\varepsilon)\beta t - D_3]\phi_t{}^2(L,t)$$

$$-[\varepsilon + \delta - (3 - \delta - 2\varepsilon)GAL^2 - 2\alpha L^2\delta_5{}^2 - \delta\beta L\,\delta_6{}^2\,]\int_0^L \phi_x{}^2\,dx$$

$$-[\varepsilon - \delta\,\delta_2{}^2]\int_0^L I_B(\phi_t + \dot{\theta})^2\,dx - [(2\varepsilon - \delta)GA - \delta\alpha\delta_7{}^2]\int_0^L u_x{}^2\,dx \quad , \tag{5.4.67}$$

where

$$D_1 = (1-\varepsilon)I_R + (2\rho + \delta)M_1 + \frac{I_B}{\delta_2^2} + \rho L M_2 + 2I_B L + \frac{2L(b+L)}{\delta_3^2}$$

$$+\rho\int_0^L (b+x)^2\,dx + \frac{(2(1-\varepsilon)-\delta)\rho\int_0^L (b+x)^2\,dx}{\delta_1^2} + \frac{2\rho\int_0^L (b+2x)^2\,dx}{\delta_4^2} \quad , \qquad (5.4.68)$$

$$D_2 = \rho L + 2\rho L(b+L)\delta_3^2 + \frac{2\alpha L}{\delta_5^2} + \frac{\alpha^2 L}{GA} + \frac{\delta\alpha}{\delta_7^2} \quad , \qquad (5.4.69)$$

$$D_3 = 2I_B L + \frac{\beta^2}{EI} + \frac{\delta\beta}{\delta_6^2} \quad . \qquad (5.4.70)$$

By choosing ε and δ sufficiently close to but smaller than 1 and by choosing δ_i , i=1,..., 7 small enough, each term multiplying the integral terms in (5.4.67) can be made negative. To see this, define $\hat{\varepsilon}$ and $\hat{\delta}$ as follows :

$$\hat{\varepsilon} := 1-\varepsilon \quad , \quad \hat{\delta} := 1-\delta \quad . \qquad (5.4.71)$$

Then sufficient conditions to make the coefficients of the integral terms in (5.4.67) negative are :

$$(1+GAL^2)\hat{\delta} + (1+2GAL^2)\hat{\varepsilon} < 2 \quad , \qquad (5.4.72)$$

$$2\hat{\varepsilon} < \hat{\delta} < 1 \quad . \qquad (5.4.73)$$

It is easy to see that one can find $\hat{\varepsilon}$ and $\hat{\delta}$ sufficiently small which satisfy (5.4.72) and (5.4.73), (e.g., choose $\hat{\varepsilon}=\frac{1}{8(1+2GAL^2)}$,and $\hat{\delta}=\frac{1}{2(1+2GAL^2)}$) . Then choosing δ_i , $i=1,\ldots,7$ small enough, the coefficients of each integral term in (5.4.67) become negative. Then, from (5.4.67) it follows that :

$$\frac{dV(t)}{dt} \le 0 \qquad , t > T_1 \quad ,$$

where $T_1 = \max\{\frac{D_1}{2(1-\varepsilon)k}, \frac{D_2}{2(1-\varepsilon)\alpha}, \frac{D_3}{2(1-\varepsilon)\beta}\}$, which proves (5.4.29). Then combining (5.4.29) and (5.4.28) we obtain

$$E(t) \le \frac{V(T_1)}{2(1-\varepsilon)t - C} \qquad , t > T \quad ,$$

where $T = \max\ \{T_1, \frac{C}{2(1-\varepsilon)}\}$, which proves that for large t, the energy $E(t)$ decays as $O(\frac{1}{t})$. □

5.5 Existence, Uniqueness and Exponential Decay of Solutions

In previous section, we proved that the solutions of the equations of motion, i.e., (5.4.17)-(5.4.24), decay at least as $O(\frac{1}{t})$ for large t. In this section we establish an existence and uniqueness theorem for the solutions of the equations mentioned above, and then prove that solutions actually decay exponentially. We use the same techniques used in the proofs of relevant theorems in previous chapters, such as Theorem 3.5.5 or Theorem 4.6.1; hence here without giving detailed calculations, we give brief sketches of proofs and refer to the relevant equations or theorems, when appropriate.

We repeat the equations of motion we studied in previous section, namely (5.4.17)-(5.4.24) : for all $t \geq 0, x \in (0, L)$

$$u_{tt} = \frac{GA}{\rho}(u_{xx} - \phi_x) + \frac{GA}{I_R}(b+x)\int_0^L (b+x)(u_{xx} - \phi_x)\, dx - \frac{GA}{I_R}(b+x)\int_0^L (u_x - \phi)\, dx$$

$$+ \frac{EI}{I_R}(b+x)\int_0^L \phi_{xx}\, dx + k(b+x)\dot{\theta} + \rho\dot{\theta}^2 u \quad , \tag{5.5.1}$$

$$\phi_{tt} = \frac{EI}{I_B}\phi_{xx} + \frac{GA}{I_B}(u_x - \phi) + \frac{GA}{I_R}\int_0^L (b+x)(u_{xx} - \phi_x)\, dx$$

$$- \frac{GA}{I_R}\int_0^L (u_x - \phi)\, dx + \frac{EI}{I_R}\int_0^L \phi_{xx}\, dx + k\dot{\theta} \quad , \tag{5.5.2}$$

$$\ddot{\theta} = -\frac{GA}{I_R}\int_0^L (b+x)(u_{xx} - \phi_x)\, dx + \frac{GA}{I_R}\int_0^L (u_x - \phi)\, dx - \frac{EI}{I_R}\int_0^L \phi_{xx}\, dx - k\dot{\theta} \quad , \tag{5.5.3}$$

$$u(0,t)=0 \quad , \quad \phi(0,t)=0 \quad , \tag{5.5.4}$$

$$GA\,[u_x(L,t)-\phi(L,t)]+\alpha\,u(L,t)=0 \quad , \tag{5.5.5}$$

$$EI\,\phi_x(L,t)+\beta\,\phi_t(L,t)=0 \quad . \tag{5.5.6}$$

We define the function space H_T in which the solutions of (5.5.1)-(5.5.6) evolve, as follows :

$$H_T := \{(u \;\; u_t \;\; \phi \;\; \phi_t \;\; \dot{\theta})^T \mid u \in \mathbf{H}^1{}_0 , \phi \in \mathbf{H}^1{}_0 , u_t \in \mathbf{L}^2 , \phi_t \in \mathbf{L}^2 , \dot{\theta} \in \mathbf{R} \} \quad , \tag{5.5.7}$$

where the function spaces $\mathbf{L}^2$, $\mathbf{H}^k$, $\mathbf{H}^k{}_0$ are introduced in (4.4.14).

Equations (5.5.1)-(5.5.3) can be put in the following form :

$$\frac{dz}{dt} = Az + g(z) \quad , \tag{5.5.8}$$

where $z = (u \;\; u_t \;\; \phi \;\; \phi_t \;\; \dot{\theta})^T \in H_T$, the operator $A : H_T \to H_T$ is a linear unbounded operator whose matrix form is specified as follows :

$$A = \{ m_{ij} : i,j = 1,\dots,6 \} \quad , \tag{5.5.9}$$

where all m_{ij} are zero except :

$$m_{12} = m_{34} = 1 \quad ,$$

$$m_{21} = \frac{GA}{I_R}\frac{\partial^2}{\partial x^2} + \frac{GA}{I_R}(b+x)\int_0^L (b+x)\frac{\partial^2}{\partial x^2}\,dx - \frac{GA}{I_R}(b+x)\int_0^L \frac{\partial}{\partial x}\,dx \quad ,$$

$$m_{23} = -\frac{GA}{\rho}\frac{\partial}{\partial x} - \frac{GA}{I_R}(b+x)\int_0^L (b+x)\frac{\partial}{\partial x}\,dx + \frac{GA}{I_R}(b+x)\int_0^L dx + \frac{EI}{I_R}(b+x)\int_0^L \frac{\partial^2}{\partial x^2}\,dx \quad ,$$

$$m_{25} = k(b+x) \quad ,$$

$$m_{41} = \frac{GA}{I_B}\frac{\partial}{\partial x} + \frac{GA}{I_R}\int_0^L (b+x)\frac{\partial^2}{\partial x^2}\,dx - \frac{GA}{I_R}\int_0^L \frac{\partial}{\partial x}\,dx \quad ,$$

$$m_{43} = \frac{EI}{I_B}\frac{\partial^2}{\partial x^2} - \frac{GA}{I_B} - \frac{GA}{I_R}\int_0^L (b+x)\frac{\partial}{\partial x}\,dx + \frac{GA}{I_R}\int_0^L dx + \frac{EI}{I_R}\int_0^L \frac{\partial^2}{\partial x^2}\,dx \quad ,$$

$$m_{45} = k \quad ,$$

$$m_{51} = -\frac{GA}{I_R}\int_0^L (b+x)\frac{\partial^2}{\partial x^2}\,dx + \frac{GA}{I_R}\int_0^L \frac{\partial}{\partial x}\,dx \quad ,$$

$$m_{53} = \frac{GA}{I_R}\int_0^L (b+x)\frac{\partial}{\partial x}\,dx - \frac{GA}{I_R}\int_0^L dx - \frac{EI}{I_R}\int_0^L \frac{\partial^2}{\partial x^2}\,dx \quad ,$$

$$m_{55} = -k \quad ,$$

the operator $g : H_T \rightarrow H_T$ is a nonlinear operator defined as :

$$g(z) = (g_1 \ \cdots \ g_5)^T \quad , \tag{5.5.10}$$

where all g_i are zero, except :

$$g_2(z) = \dot{\theta}^2 u \quad .$$

Note that for all $r > 0$, the operator $g(.)$ is Lipschitz in z in the ball $B(0,r)$.

The domain of the operator A is defined as :

$$D(A) := \{(u \ \ u_t \ \ \phi \ \ \phi_t \ \ \dot{\theta})^T \in H^T \mid u \in \mathrm{H}^2{}_0 , \phi \in \mathrm{H}^2{}_0 , u_t \in \mathrm{H}^1{}_0 , \phi_t \in \mathrm{H}^1{}_0 , \dot{\theta} \in \mathrm{R} ,$$

$$GA\,[u_x(L,t) - \phi(L,t)] + \alpha u_t(L,t) = 0,$$

$$EI\,\phi_x(L,t) + \beta\phi_t(L,t) = 0 \ \} \quad . \tag{5.5.11}$$

In H_T we define the following inner-product :

$$<z\ ,\hat{z}> = \frac{1}{2} I_R \dot{\theta}\hat{\dot{\theta}} + \frac{1}{2}\int_0^L \rho[u_t + \dot{\theta}(b+x)]\,[\hat{u}_t + \hat{\dot{\theta}}(b+x)]\,dx + \frac{1}{2}\int_0^L EI\,\phi_x \hat{\phi}_x\,dx$$

$$+\frac{1}{2}\int_0^L I_B(\phi_t + \dot{\theta})(\hat{\phi}_t + \hat{\dot{\theta}})\,dx + \frac{1}{2}\int_0^L GA\,(u_x - \phi)(\hat{u}_x - \hat{\phi})\,dx \quad , \qquad (5.5.12)$$

where $z = (u \;\; u_t \;\; \phi \;\; \phi_t \;\; \dot{\theta})^T \in H_T$ and $\hat{z} = (\hat{u} \;\; \hat{u}_t \;\; \hat{\phi} \;\; \hat{\phi}_t \;\; \hat{\dot{\theta}})^T \in H_T$.

Note the standard Sobolev norm which makes H_T a Banach space is :

$$||\,z\,||_1{}^2 = \int_0^L u^2\,dx + \int_0^L u_x{}^2\,dx + \int_0^L u_t{}^2\,dx + \int_0^L \phi^2\,dx + \int_0^L \phi_x{}^2\,dx + \dot{\theta}^2 \quad , \qquad (5.5.13)$$

but, by using inequalities (5.4.31)-(5.4.33), (5.4.52), and (5.4.53), it can be shown that the norm induced by (5.5.12) is equivalent to the norm defined by (5.5.13), (the proof of this fact is similar to the proof of Lemma 3.5.4).

5.5.1 Theorem : Consider the linear unbounded operator $A : H_T \rightarrow H_T$ given by (5.5.9). Then :

(i) A generates a C_0 semigroup $T(t)$;

(ii) there exist positive constants $M > 0$ and $\delta > 0$ such that the following holds :

$$||\,T(t)\,|| \leq M\,e^{-\delta t} \qquad t \geq 0 \quad , \qquad (5.5.14)$$

where the norm is the norm induced by the inner-product defined in (5.5.12).

Proof :

(i) We use Lumer-Phillips theorem to prove the assertion (i), (see Theorem 3.4.4). Hence, one has to prove that A is dissipative and the operator $\lambda I - A : H_T \to H_T$ is onto for some $\lambda > 0$.

As before, differentiating the norm induced by (5.5.12) we obtain : for all $z \in H_T$,

$$\frac{d}{dt} = 2 <z \ , Az>$$

$$= -k\dot{\theta}^2 - \alpha u_t{}^2(L, t) - \beta\phi_t{}^2(L, t) \ \leq 0 \quad , \tag{5.5.15}$$

which is the energy estimate (5.4.26). This proves that A is dissipative.

To prove that the linear operator $\lambda I - A : H_T \to H_T$ is onto for some $\lambda > 0$, we decompose the operator A as follows :

$$A = A_1 + T_D \quad , \tag{5.5.16}$$

where $A_1 : H_T \to H_T$ is a linear unbounded operator defined as :

$$A_1 := \{n_{ij} : i, j = 1, \ldots, 5\} \quad , \tag{5.5.17}$$

where all n_{ij} are zero except :

$$n_{12} = n_{34} = 1 \quad ,$$
$$n_{21} = \frac{GA}{I_R}\frac{\partial^2}{\partial x^2} \quad ,$$
$$n_{23} = -\frac{GA}{\rho}\frac{\partial}{\partial x} \quad ,$$
$$n_{25} = k(b + x) \quad ,$$
$$n_{41} = \frac{GA}{I_B}\frac{\partial}{\partial x} \quad ,$$

$n_{43} = \frac{EI}{I_R} \frac{\partial^2}{\partial x^2}$,

$n_{45} = k$,

$n_{55} = -k$.

The operator $A_1 : H_T \to H_T$ is a linear unbounded operator whose domain $D(A_1)$ is equal to $D(A) \subset H_T$ defined by (5.5.11). It is known that A_1 generates a C_0 semigroup in H_T, (see [Kim.1, Lemma 1.1]). Hence $\lambda I - A : H_T \to H_T$ is an invertible operator for all $\lambda > 0$.

The operator $T_D : H_T \to H_T$ is a degenerate linear operator relative to A_1, (see Theorem (3.5.5)). Hence, as proven in theorem 3.5.5, it follows that for all $\lambda > 0$, $I - T_D(\lambda I - A_1)^{-1} : H_T \to H_T$ is an invertible linear operator and we have the following :

$$(\lambda I - A)^{-1} = (\lambda I - A^1)^{-1}(I - T_D(\lambda I - A_1)^{-1})^{-1} \quad ,$$

which proves that $(\lambda I - A) : H_T \to H_T$ is onto for all $\lambda > 0$.

This, together with the fact that A is dissipative proves that A generates an C_0 semigroup in H_T.

(ii) To prove the exponential decay of the semigroup generated by the operator A defined in (5.5.9), we first define the energy $E_1(t)$ associated with the inner-product (5.5.12), that is :

$$E_1(t) = \langle z(t), z(t) \rangle \quad . \tag{5.5.18}$$

Similar to (5.4.27), we define the following function $V_1(t)$:

$$V_1(t) = 2(1-\varepsilon)tE_1(t) + 2\int_0^L \rho x[u_t + \dot{\theta}(b+x)]u_x\,dx + 2\int_0^L I_B x \phi_x(\phi_t + \dot{\theta})\,dx$$

$$+ \delta\int_0^L I_B \phi(\phi_t + \dot{\theta})\,dx - \delta\int_0^L \rho u(u_t + \dot{\theta}(b+x))\,dx \quad . \tag{5.5.19}$$

Following exactly the same proof of Theorem (5.4.4), we obtain the result that $E_1(t)$ decays as $O(\frac{1}{t})$ for large t, (see (5.4.30)). Then exponential decay follows from Pazy's theorem (see Theorem 3.4.5) cited in the proof of theorem 3.5.5. □

Next we prove the exponential decay of the solutions of (5.5.8) :

5.5.2 Theorem : Consider (5.5.8) where the linear operator $A : D(A) \subset H_T \to H_T$ is given by (5.5.9) and the nonlinear operator $g : H_T \to H_T$ is given by (5.5.10). Let $T(t)$ be the C_0 semigroup generated by the linear operator A. Then :

(i) for all $z_0 \in D(A)$, (5.5.8) has a unique solution $z(t)$;

(ii) in terms of the semigroup $T(t)$ generated by A, this solution can be written as :

$$z(t) = T(t)z_0 + \int_0^t T(t-s)g(z(s))\,ds \quad , \tag{5.5.20}$$

(iii) this solution $z(t)$ decays exponentially.

Proof :

(i) Since A generates a C_0 semigroup $T(t)$ and $g : H_T \to H_T$ is a C^∞ function, (see (5.5.10)), it follows that for all $z_0 \in D(A)$, (5.5.8) has a unique solution defined locally in time. But since for sufficiently large t, $T(t) = O(\frac{1}{t})$ by Theorem 5.4.4, it follows that the solution is in fact defined for all $t \geq 0$.

(ii) This may be proven by back substitution of (5.5.20) into (5.5.8) and by using $\frac{dT}{dt} = AT$.

(iii) From (5.5.10) and (5.5.12) it follows that

$$||g(z)||^2 = \frac{1}{2}\int_0^L \rho\dot{\theta}^4 u^2\, dx$$

$$\leq \frac{1}{2}\rho\dot{\theta}^4 M_1 ||z||^2 \tag{5.5.21}$$

where $M_1 = \dfrac{\max\{2L^2, 2L^4\}}{\min\{\frac{I_R}{2}, \frac{1}{2}, \frac{GA}{2}, \frac{EI}{2}\}}$, (see (5.4.54)). Since $\dot{\theta}^2$ is decaying at least as $O(\frac{1}{t})$ and since $||T(t)|| \leq M\, e^{-\delta t}$, applying the Bellman-Gronwall lemma to (5.5.20), (see the proof of the assertion of (iii) of theorem (3.5.6)), we conclude that the solution of (5.5.8) is decaying exponentially. □

Chapter 6

Conclusion

In this thesis we dealt with the stabilization of flexible spacecraft. We consider a rigid body-flexible beam configuration as a case model, (see Section 3.2). We assumed that the center of mass of the rigid body is fixed in an inertial frame, and the flexible beam is clamped to the rigid body at one end and free at the other end. We considered three cases : In Chapter 3, we studied the motion of the basic configuration in plane with the beam modeled as Euler-Bernoulli beam; in Chapter 4 we removed the planar motion assumption and in Chapter 5 we studied the motion of the basic configuration in plane with the beam modeled as Timoshenko beam. In each case we proposed appropriate force and torque laws, which include boundary forces and torques applied at the free end of the beam, to stabilize the configuration in question.

In Chapter 3, we proposed two control laws to stabilize the basic configuration performing planar motion. The first control law is based on cancellation, (see Section 3.4), where an appropriate control law applied to the rigid body cancels the effect of beam on the rigid body, (see (4.3.3)). This law enables one to study the rigid body and the flexible beam separately. The second control law is not not based on a cancellation and enables one to use the energy of the whole configuration as a Lyapunov function. The stabilization problem we dealt with here is the stabilization of angular velocity of the rigid body and the beam deflections. The results obtained and the techniques used here can be used to study the other problems encountered in the control of flexible structures, such as attitude control, orientation, tracking, etc. , (see [Ana.1])

In Chapter 4, we extended the results obtained in chapter 3 to the case of the motion in $\mathbf{R}^3$ of the basic configuration, and we proved some results similar to the ones obtained in chapter 3.

In Chapter 5, we consideder the basic configuration with the beam modeleded, first, as a geometrically exact beam and then as a Timoshenko beam. In the first case, generalizing the control laws proposed in previous chapters we obtained a stability result, without using any linearization, (see Proposition 5.4.1). As a future work, generalization of this stability result to a possible exponential stability result might be useful. In the remainder of chapter 5, we considered the planar motion of the basic configuration with the beam modeled as a Timoshenko beam, which comes from appropriate linearization of geometrically exact beam model. Then using previous results we obtained some results similar to the ones obtained in Chapter 3 and Chapter 4.

The results in this thesis shows that the boundary control techniques can be applied to the control of flexible structures. Applications of these techniques to the control of configurations other than the one we used here, (e.g. dual-spin spacecraft with flexible attachements); also applications to different control problems, such as tracking, pointing, etc., can be useful.

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