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John Edmond Hauser

Memorandum No. UCB/ERL M89/99

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COVER PAGE

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Shankar Sastry
Chairman

ABSTRACT

In this dissertation, we embark on a project to make recent theoretical advances in geometric nonlinear control into a *practicable control design methodology*.

The method of input-output linearization by state feedback provides a natural framework to design controllers for systems, such as aircraft, where output tracking rather than stabilization is the control objective. Central notions include relative degree and zero dynamics. Roughly speaking, the relative degree of a system is the dimension of the part of the system that can be input-output linearized and the zero dynamics are the remaining (unobservable) dynamics. Systems with exponentially stable zero dynamics are analogous to minimum phase linear systems and can be controlled to track a rich class of output trajectories with internal stability.

While investigating the use of these methods in the control of the V/STOL Harrier aircraft, we noticed that the small forces produced when generating body moments caused the aircraft to have an unstable zero dynamics, i.e., to be nonminimum phase. However, if this coupling were zero, then the aircraft could be input-output linearized with no zero dynamics. In other words, a small change in a parameter resulted in a significant change in the system structure!

With this observation as the driving force, this dissertation studies the effects of system perturbations on the structure of the system and develops methods for tracking controller design based on approximate systems.

After reviewing the basics of geometric nonlinear control, we show that small regular perturbations in the system can result in singular perturbations in the zero dynamics. We give asymptotic formulas for the resulting fast dynamics.

Next, we develop techniques for tracking control design for systems that do not have a well defined relative degree. Using an approximate system with a well defined relative degree, we design tracking controllers that guarantee approximate tracking for the true system. This approach is shown to be superior to the usual Jacobian linearization method on a simple ball and beam system.

Returning to the aircraft control problem, we use a highly simplified planar VTOL aircraft model to illustrate the (slight) nonminimum phase characteristic of these systems and develop a controller to guarantee approximate tracking. We also develop a formal theory for this class of systems.

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In May 1985, my graduate studies were interrupted by a peculiar incident—while working in a research lab, I opened a small plastic box that contained an explosive bomb. I am sincerely grateful to Professor Mike Lieberman for his clear thinking and quick application of first aid that quite literally saved my life. I am also indebted to Dr. Joseph Togba for saving my right arm and hand through his excellent surgical skills. I would like to thank my father John G. Hauser and my brother Christopher J. Hauser for being at my side during this time. The members of the First Baptist Church of Richmond demonstrated their love and faith in supporting my family and me during this trying period. I would especially like to thank Larry and Judy Allgaier and Steven and Mary Lou Hopkins for their support and friendship.

I would like to acknowledge the generous financial support of my work here at Berkeley. My research has been supported by many sources including the United States Air Force, NASA Ames Research Center (grant number NAG2–243), the Army Research Office (grant number DAAL03–88–K–0106), the MICRO foundation, the Schlumberger Foundation, and the Berkeley Engineering Fund. I would especially like to thank Professor Karl Pister, Dean of the College of Engineering, and Professor Eugene Wong, Chairman of the Department of Electrical Engineering and Computer Science, for their help and support in this area during the years following the bombing incident.

I would like to express my deepest love and gratitude to my wife, Babbette. Her unfailing support and understanding through these years have been a source of great strength. I would like to dedicate this dissertation to our children, Jennifer, John, and Jacqueline. May we always find the world as delightful as they do.

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Introduction

There has been an explosive growth in the last ten years in the number of applications of nonlinear control techniques; examples include flight control systems in helicopters [MC80], robot manipulators [Fre82], process control [SCS87], and even drug delivery systems [CB84]. Nonetheless, it is fair to say that nonlinear control system design is not at the state of the art that it should be in industry—especially considering the fact that almost every physical system is *fundamentally nonlinear*.

A linear systems approach is often taken is to design a controller based on *linear* approximation(s) of the system about desirable operating point(s). Under reasonable conditions, the linear controller(s) can be used to stabilize and regulate the system about the operating point(s). Indeed, a large number of researchers have worked to develop *robust* methods (see, e.g., [Fra87]) to somehow enlarge the region where such a linear controller can effectively control the nonlinear system. This approach is useful for systems where *stabilization* or *regulation* is the goal. Since the controller strives to keep the system close to the nominal operating point, the approach will be effective provided the system is not *too* nonlinear and the excursions from the operating point are not *too* large. In particular, this approach has been shown to be quite effective when the neglected nonlinearities are sector bounded.

For many systems, such as aircraft, stabilization and regulation fall far short of the true control objective. Indeed, for such systems, agility and maneuverability over a large region is desirable. We may specify such a goal as the ability of the system to *track* a rich set of output trajectories.

One common approach to trajectory tracking is to linearize the system about the reference trajectory to obtain a *linear time-varying* system that is valid in a very small neighborhood of the desired trajectory. Then a time-varying compensator is designed to control the system. Unfortunately, few methods are available to design such compensators.

Another popular approach is gain scheduling. A family of linear controllers is designed—each one good at stabilizing the system around a different operating point. The system controller is then built by scheduling (perhaps interpolating between) the individual controllers based on a set of parameters (a part of the system state is often the set of parameters). Then, provided the system state and the parameters do not change too rapidly and the family of controllers is a continuous (or smooth) function of the parameters, it is hoped that the closed loop system will maintain the desirable properties of stability and be able to approximately track reasonable trajectories. Major shortcomings include the requirements that each point on the desired trajectory be close to at least one of the selected operating (i.e., equilibrium) points and that the transitions between operating regions not be too fast. Also, there are few analytical results proving the effectiveness of these methods.

Many of the difficulties experienced are simply due to the fact that control engineers are trying to fit a *round* peg (nonlinear systems) into a *square* hole (linear system design methodologies). Fortunately, a large number of these difficulties can be dealt with directly using a nonlinear system design methodology. Indeed, much of the intuition behind gain scheduling is quite intrinsic to control laws developed by nonlinear methods. This is clearly seen in succesful applications of nonlinear control in flight dynamics [MC75,MC80] and robotics [Fre75], both developed before a general theory had evolved.

From a base of more abstract system theoretic issues [Por70,SJ72,SR72,HK77, Bro78], the field of nonlinear control has seen a decade of intense activity and evolution culminating in a well-developed and understood theory (see, e.g., [Sus83,Isi85,Cla86,Isi87] for expository surveys). One of the key developments useful for the purpose of output tracking is the theory of input-output linearization of nonlinear systems by state feedback. By linearizing the input-output response of the nonlinear system, we can bring to bear the powerful methods of linear systems theory to achieve robust and stable tracking. This theory has recently achieved maturation.

The goal in this dissertation is to commence a program for the development of a practical nonlinear control design methodology. The outline of the dissertation is as follows.

In Chapter 1 we review the tools and techniques of geometric control theory that we will find useful in the sequel. In particular, we see that the nonlinear counterpart of the zeros of a system, the *zero dynamics* of a nonlinear system [BI84,IM89,BI88], provide a notion of structure for the nonlinear system. Specifically, the zero dynamics help to characterize the trajectories that the system can track.

In Chapter 2, we present theoretical results that show that the zero structure of systems (*both* linear and nonlinear) is not robust with respect to regular perturbations in the model. Roughly speaking, regular perturbations in the state space model may give rise to singular perturbations in the zero dynamics. We give asymptotic formulas for the zeros of the (linearization of) the additional *fast* zero dynamics.

In Chapter 3, we present a method for the approximate input-output linearization of systems without well defined relative degree. We show that, although the system cannot be exactly input-output linearized to achieve exact tracking of trajectories, we can often still achieve bounded error output tracking by designing the compensator based on a minimum phase nonlinear system that approximates the true system.

In Chapter 4, we present a method for the approximate input-output linearization of slightly nonminimum phase systems. Applying the methods of input-output linearization directly to a nonminimum phase system can yield exact tracking, but at the expense of unstable internal motion and, perhaps, unrealistic input requirements. As in the no relative degree case, we show that designing the compensator based on an approximate, but minimum phase system, we can achieve bounded error output tracking. These ideas are illustrated using a highly simplified planar vertical takeoff and landing (VTOL) aircraft.

These results of Chapters 3 and 4 show that, although the *structure* of the system is not robust to perturbations (cf. Chapter 2), control laws designed using systems with good structure (minimum phase, etc.) are robust to perturbations.

Much interesting work remains to be done. Particular areas indicated by this dissertation include:

- Trajectory design. For a class of invertible nonlinear systems, the method of input-output linearization is useful to guarantee that the trajectory error has an exponentially stable linear dynamics. However, in order for this approach to be effective, the desired trajectories *must* respect constraints imposed by the true system dynamics.

- Actuator limits. One of the most difficult problems in feedback control designers must face is the fact that real life actuators and systems have limits. This problem can sometimes be handled by judicious trajectory design. However, this problem can still pose major difficulties in the presence of measurement errors and external disturbances.

- System steering. Sometimes a system can be in a state where it does not possess full controllability. In this case, a natural (sub)task would be to steer the system (in allowed) directions to get into a region of the state space where normal techniques can be used.

- Tools for nonlinear systems analysis. The calculations required for even relatively simple nonlinear systems are often quite involved. Also, the estimates that we derive to prove stability and tracking performance are extremely conservative and do not reveal the true nonlinear nature of the problem.

Chapter 1

Introduction to Geometric Control Theory

In this chapter, we try to present some of the basic notions from geometric control theory. We are by no means trying to provide a complete survey of the many interesting results in this field. For a more complete review, consult one of the many excellent surveys or texts, for example, [Sus83,Isi85,Cla86,Isi87,Isi89b].

The chapter begins with a brief review of some ideas from differential geometry. We then cover much of the theory for the input-output linearization of single-input single-output nonlinear systems using state feedback. This is followed by some generalizations and algorithms for multi-input multi-output nonlinear systems. We conclude with some biographical notes.

1.1 Preliminaries

We briefly review some ideas from differential geometry. For a detailed development consult a standard text such as [Boo86,Mil76,AMR83].

Recall that a *diffeomorphism* is a smooth, bijective (one-one, onto) map with a smooth inverse. Here smooth means that the map has continuous derivatives of all orders. Sometimes we will use smooth to mean sufficiently differentiable for the task at hand.

Roughly speaking (for a precise definition see [Boo86]), a set $M \subset \mathbf{R}^n$ is a *smooth manifold* of dimension k if it is locally diffeomorphic to $\mathbf{R}^k$. Simple examples of smooth manifolds include a 2-dimensional sphere (embedded in $\mathbf{R}^3$)

$$S^2 = \left\{x \in \mathbf{R}^3 \;:\; |x|^2 = 1\right\}, \tag{1.1}$$

an $(n-1)$-dimensional hyperplane (embedded in $\mathbf{R}^n$)

$$H^{n-1} = \{x \in \mathbf{R}^n \;:\; \langle c, x\rangle = b\}, \tag{1.2}$$

and the group of proper rotations

$$SO(3) = \left\{A \in \mathbf{R}^{3\times 3} \;:\; A^T A = 1,\ \det A = 1\right\}. \tag{1.3}$$

Let U and V be open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, with $m \geq n$. Given a smooth mapping $f : U \to V$, we say that $x \in U$ is a *regular point of* f if the rank of the Jacobian Df at x is equal to n. A point $y \in f(U) \subset V$ is called a *regular value of* f if the inverse image of y under f, $f^{-1}(y) \subset U$, contains only regular points.

Note that each of the example manifolds above were given as the inverse image of a regular value of a smooth mapping. In general, since manifolds are locally diffeomorphic to $\mathbf{R}^n$, we have:

Fact 1.1 *If $f : M \to N$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

Consider the set of all smooth curves through a point x on a manifold M. The set of *tangent vectors* to these curves at x is called the *tangent space* at x, denoted T_xM. For k-dimensional manifold M embedded in $\mathbf{R}^n$, the tangent space T_xM at $x \in \mathbf{R}^n$ can be thought of as the k-dimensional hyperplane that best approximates M in the neighborhood of x. Note that this hyperplane can be specified as the inverse image of $n - k$ functions of the form $\langle c_i, x\rangle = a_i$. Also, note that the tangent space to $\mathbf{R}^n$ at $x \in \mathbf{R}^n$ is just a (different) copy of $\mathbf{R}^n$.

A *vector field* on M assigns to each $x \in M$ an element of the tangent space at x, i.e., $f(x) \in T_xM$.

The *Lie bracket* of two vector fields, f and g, denoted $[f, g]$ or $ad_f g$, is a vector field given (in coordinates) by

$$ad_f g = [f, g] = Dg \cdot f - Df \cdot g\ . \tag{1.4}$$

Repeated Lie brackets are denoted using the notation

$$\begin{aligned} ad_f^0 g &:= g\,, \\ ad_f^k g &:= [f, ad_f^{k-1} g], \qquad k > 0\,. \end{aligned} \tag{1.5}$$

The *Lie derivative* of a scalar function h along a vector field f, denoted $L_f h$, is given by

$$L_f h(x) = dh(x) \cdot f(x) \tag{1.6}$$

and is a directional derivative of h in the direction of f.

A smooth k-dimensional *distribution* Δ on a manifold M assigns (smoothly) to each point $x \in M$ a subspace of the tangent space at x, $\Delta(x) \subset T_x M$; that is, it has a *local basis* of linearly independent vector fields g_i, $i = 1, \ldots, k$, such that

$$\Delta = \operatorname{span}\{g_1\ g_2\ \cdots\ g_k\} \tag{1.7}$$

where the span is taken over the ring of C^∞ functions. A distribution Δ is called *involutive* if for all $f, g \in \Delta$, we have $[f, g] \in \Delta$, that is, Δ is closed under Lie brackets.

If Δ is a distribution on M and N is a submanifold of M such that for each $x \in N$ we have $T_x N \subset \Delta(x)$, then N is an *integral manifold* of Δ. The distribution Δ is called *completely integrable* if for each $x \in M$, there exists an integral manifold N of Δ such that $T_x N = \Delta(x)$.

The Frobenius theorem relates the concepts of involutivity and integrability:

Theorem 1.1 (Frobenius) *A distribution Δ on a manifold M is completely integrable if and only if it is involutive.*

Since a single vector field is always involutive (trivially), the Frobenius theorem guarantees the existence of solutions (locally in time) to the ordinary differential equation

$$\dot{x} = f(x) \qquad x(0) = x_0 \tag{1.8}$$

for smooth f. The solution of (1.8) for each initial condition will be a 1-dimensional integral manifold of the (trivial) distribution $\Delta = \operatorname{span}\{f\}$.

The Frobenius theorem can also be used to show that, given an independent involutive collection of vector fields, g_i, $i = 1, \ldots, k$, defined on $U \subset \mathbf{R}^n$ (thought of as a manifold), there exists a set of independent functions, η_j, $j = 1, \ldots, n-k$, such that

$$L_{g_i} \eta_j(x) = 0 \qquad x \in U \tag{1.9}$$

for all combinations of i and j. In other words, we can solve the set of partial differential equations (1.9) for the functions η_i on U.

1.2 Input-Output Linearization for a Class of SISO Nonlinear Systems

A large class of nonlinear systems can be made to have linear input-output behavior through a choice of *nonlinear state feedback* control law.

1.2.1 Introductory Concepts

Consider, at first, the single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1.10}$$

where $x \in \mathbf{R}^n$ (with $\mathbf{R}^n$ a smooth manifold), f and g are smooth vector fields, and h is a smooth nonlinear function. In this case, smooth will mean C^r with r sufficiently large. Differentiating y with respect to time, we get

$$\begin{aligned} \dot{y} &= dh \cdot f(x) + dh \cdot g(x)u \\ &= L_f h(x) + L_g h(x)u \end{aligned} \tag{1.11}$$

where $L_f h(x) : \mathbf{R}^n \to \mathbf{R}$ and $L_g h(x) : \mathbf{R}^n \to \mathbf{R}$ are the Lie derivatives of h with respect to f and g respectively. If $L_g h(x)$ is bounded away from zero for all x, the state feedback law (of the form $u = \alpha(x) + \beta(x)v$) given by

$$u = \frac{1}{L_g h}(-L_f h + v) \tag{1.12}$$

results in a linear system from v to y given by

$$\dot{y} = v \; . \tag{1.13}$$

The control law (1.12) also has the effect of rendering $(n-1)$ of the states of the system (1.10) unobservable through state feedback.

In the instance that $L_g h(x) = 0$ for all x, we differentiate (1.11) again to get

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u \; . \tag{1.14}$$

In (1.14), $L_f^2h(x)$ stands for $L_f(L_fh)(x)$ and $L_gL_fh(x) = L_g(L_fh)(x)$. Now, if $L_gL_fh(x)$ is bounded away from zero for all x, then the control law given by

$$u = \frac{1}{L_gL_fh(x)}(-L_f^2h(x) + v) \tag{1.15}$$

yields the linearized input-output system

$$\ddot{y} = v \; . \tag{1.16}$$

More generally, if γ is the *smallest* integer for which $L_gL_f^ih(x) = 0$ for all x and $i = 0, \ldots, \gamma - 2$ and $L_gL_f^{\gamma-1}h(x)$ is bounded away from zero, then the control law given by

$$u = \frac{1}{L_gL_f^{\gamma-1}h(x)}(-L_f^{\gamma}h + v) \tag{1.17}$$

yields

$$y^{(\gamma)} = v \; . \tag{1.18}$$

To make the preceding discussions more precise (and allow for the functions that are only locally zero or nonzero, i.e., on an open set rather than all of $\mathbf{R}^n$), we make the following definition for the *relative degree* of a nonlinear system:

Definition 1.1 *The SISO nonlinear system (1.10) is said to have* relative degree γ *at* x_0 *(an equilibrium point) if there exists a neighborhood* U *of* x_0 *such that, for* $x \in U$,

$$\begin{aligned} L_gL_f^ih(x) &= 0 \quad \forall\, 0 \le i < \gamma - 1 \; , \\ L_gL_f^{\gamma-1}h(x) &\neq 0 \; . \end{aligned} \tag{1.19}$$

Remarks

- This definition is compatible with the usual definition of relative degree for linear systems (as being the excess of poles over zeros).
- The relative degree of a nonlinear system (at x_0) is precisely the number of times we must differentiate the output to have the input appear explicitly.
- The last requirement in the definition of relative degree could be replaced by

  $$L_gL_f^{\gamma-1}h(x_0) \neq 0 \tag{1.20}$$

  since, by smoothness, this would imply the existence of a nonzero neighborhood.

- The relative degree of some nonlinear systems may not be defined at some points, e.g., when some $L_g L_f^i h$ is zero at x_0 but nonzero for points arbitrarily close to x_0.

□

1.2.2 Local 'Normal' Form of a SISO nonlinear system

If a SISO nonlinear system has a relative degree $\gamma \leq n$ at a point x_0, then, by a nonlinear change of coordinates, we can transform it locally into a 'normal' form. We find the transformation as follows. Define

$$\begin{aligned} \phi_1(x) &= h(x), \\ \phi_2(x) &= L_f h(x), \\ &\vdots \\ \phi_\gamma(x) &= L_f^{\gamma-1} h(x). \end{aligned} \tag{1.21}$$

To show that these functions can be used as a partial change of coordinates we need the following lemma which is interesting in its own right.

Lemma 1.2 *Suppose the system (1.10) has relative degree γ at x_0 (in a neighborhood U). Then*

$$L_{ad_f^j g} L_f^k h(x) = \begin{cases} 0 & 0 \leq j+k < \gamma - 1 \\ (-1)^j L_g L_f^{\gamma-1} h(x) & j+k = \gamma - 1 \end{cases} \tag{1.22}$$

for all $x \in U$, for all $j \leq \gamma - 1$.

Proof: By induction on j. For $j = 0$, (1.22) is equivalent to the statement that the system has relative degree γ. Suppose that (1.22) is true for $j = l$; we will show it is true for $j = l + 1$. Since (by straightforward calculation)

$$L_{ad_f g} \lambda = L_f L_g \lambda - L_g L_f \lambda \tag{1.23}$$

for all smooth functions $\lambda(x)$, we have

$$L_{ad_f^{l+1} g} L_f^k h(x) = L_f L_{ad_f^l g} L_f^k h(x) - L_{ad_f^l g} L_f^{k+1} h(x) \ . \tag{1.24}$$

We evaluate this expression for k such that $l + 1 + k \leq \gamma - 1$. The first term is zero on U since $L_{ad_f^l g} L_f^k h(x)$ is zero for $l + k < \gamma - 1$ (by assumption) and the Lie derivative of a vanishing function is zero. Using the assumption, we find that the second term is given by

$$-L_{ad_f^l g} L_f^{k+1} h(x) = \begin{cases} 0 & 0 \leq l+1+k < \gamma - 1 \\ (-1)(-1)^l L_g L_f^{\gamma-1} h(x) & l+1+k = \gamma - 1 \end{cases} \tag{1.25}$$

which shows that (1.22) holds for $j = l + 1$ and hence the lemma is true. □

Remarks

- For a linear system (c, A, b), the lemma reduces to the (somewhat trivial) statement

$$cA^k \cdot A^j b = \begin{cases} 0 & j+k < \gamma - 1 \\ cA^{\gamma-1} b & j+k = \gamma - 1 \end{cases} \tag{1.26}$$

  In other words, multiply cA^k on the right by $A^j b$ with $j + k = \gamma - 1$ to get the first nonzero Markov parameter. Note also that (c, A, b) need not be minimal since no observability assumptions are made in the definition of relative degree.

- Similarly, in the nonlinear case, the lemma tells us to multiply the differential $dL_f^k h$ on the right by $ad_f^j g$ with $j + k = \gamma - 1$ to obtain the first nonzero function. If $j + k < \gamma - 1$, the resulting function will be identically zero on U.

□

We use this result to show:

Proposition 1.3 *The functions ϕ_i, $i = 1, \ldots, \gamma$, defined in (1.21), are independent on U, that is, the differentials $d\phi_i$ are linearly independent (over the ring of smooth functions) on U.*

Proof: Suppose that the differentials are linearly dependent on U. Then there exists smooth functions $c_i(x)$, $i = 1, \ldots, \gamma$, not all identically zero such that

$$\begin{aligned} 0 &= c_1(x) d\phi_1(x) + c_2(x) d\phi_2(x) + \cdots + c_\gamma(x) d\phi_\gamma(x) \\ &= c_1 dh + c_2 dL_f h + \cdots + c_\gamma dL_f^{\gamma-1} h \end{aligned} \tag{1.27}$$

for all $x \in U$. We show, to the contrary, that each c_i must be identically zero on U. Multiply (1.27) on the right by g $(= ad_f^0 g)$ to obtain

$$\begin{aligned} 0 &= c_1 L_g h + c_2 L_g L_f h + \cdots + c_\gamma L_g L_f^{\gamma-1} h \\ &= c_\gamma L_g L_f^{\gamma-1} h \, . \end{aligned} \tag{1.28}$$

Now, since the system has relative degree γ, $L_gL_f^{\gamma-1}h$ does not vanish on U which implies that $c_\gamma(x)$ must be identically zero on U. Next, multiply (1.27) (with $c_\gamma \equiv 0$) on the right by ad_fg to get (using Lemma 1.2)

$$\begin{aligned} 0 &= c_1L_{ad_fg}h + c_2L_{ad_fg}L_fh + \cdots + c_{\gamma-1}L_{ad_fg}L_f^{\gamma-2}h \\ &= -c_{\gamma-1}L_gL_f^{\gamma-1}h \end{aligned} \tag{1.29}$$

which shows that $c_{\gamma-1}(x)$ is also identically zero on U. Continuing this process with ad_f^2g, $ad_f^3g, \cdots, ad_f^{\gamma-1}g$, we see that each c_i is identically zero on U so that the functions ϕ_i are independent on U as claimed. □

Remarks

- This proposition is quite remarkable since it says that if we find a (possibly *large*) neighborhood U on which the system has relative degree γ, then the functions ϕ_i will be independent on the whole set and therefore can be used as a partial coordinate transformation on that region.

- A *local* version of this proposition can be proved as follows. Let

$$(c, A, b) = (dh(x_0), Df(x_0), g(x_0))$$

be the coefficient matrices for the Jacobian linearized (tangent) system at the *equilibrium point* x_0. The condition that the differentials $d\phi_i$ be linearly independent in a neighborhood of x_0 is equivalent to their independence *at* x_0 (by smoothness). Since $d\phi_i(x_0) = cA^{i-1}$, we check that the matrix

$$\begin{bmatrix} d\phi_1(x_0) \\ d\phi_2(x_0) \\ \vdots \\ d\phi_\gamma(x_0) \end{bmatrix} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{\gamma-1} \end{bmatrix} \tag{1.30}$$

has full rank γ by multiplication on the right by

$$\begin{bmatrix} b \ Ab \ \cdots \ A^{\gamma-1}b \end{bmatrix} . \tag{1.31}$$

□

To complete the coordinate transformation, we need to find $(n-\gamma)$ functions η_j such that the collection ϕ_i, η_j, $i = 1, \ldots, \gamma$, $j = 1, \ldots, n-\gamma$, is independent on U. Since the single vector field g is (trivially) involutive, the Frobenius theorem guarantees the existence of $(n-1)$ independent functions λ_i, $i = 1, \ldots, n-1$, such that

$$L_g \lambda_i(x) = 0 \qquad x \in U,\ i = 1, \ldots, n-1. \tag{1.32}$$

Since the functions ϕ_i, $i = 1, \ldots, \gamma-1$, are independent and satisfy (1.32), we will use them as $(\gamma-1)$ of the functions λ_i. Now, complete the set (λ_i) with $(n-\gamma)$ more functions, η_j, $j = 1, \ldots, n-\gamma$, that satisfy (1.32) and are such that the collection ϕ_i, $i = 1, \ldots, \gamma-1$, η_j, $j = 1, \ldots, n-\gamma$ are independent on U. Then, since $L_g \phi_\gamma(x) \neq 0$ for $x \in U$, the matrix

$$d\Phi(x) = \begin{bmatrix} d\phi_1(x) \\ \vdots \\ d\phi_\gamma(x) \\ d\eta_1(x) \\ \vdots \\ d\eta_{n-\gamma}(x) \end{bmatrix} \tag{1.33}$$

has rank n for all $x \in U$. This shows that the transformation

$$\Phi : x \mapsto (\phi_1(x), \cdots, \phi_\gamma(x),\ \eta_1(x), \cdots, \eta_{n-\gamma}(x))^T \tag{1.34}$$

is a diffeomorphism of U onto $\Phi(U)$. Define coordinates (ξ, η) for the transformed state to be

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{pmatrix} := \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_\gamma(x) \\ \eta_1(x) \\ \vdots \\ \eta_{n-\gamma}(x) \end{pmatrix} = \Phi(x)\,. \tag{1.35}$$

Then, in the (ξ, η) coordinates, the nonlinear system (1.10) is given by

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_\gamma &= b(\xi, \eta) + a(\xi, \eta)u \\
\dot{\eta} &= q(\xi, \eta) \\
y &= \xi_1
\end{aligned}
\tag{1.36}
$$

where $b(\xi, \eta)$ and $a(\xi, \eta)$ are $L_f^\gamma h(x)$ and $L_g L_f^{\gamma-1} h(x)$ in (ξ, η) coordinates and $q_i(\xi, \eta)$ is $L_f \eta_i(x)$ in (ξ, η) coordinates. Thus for example,

$$
b(\xi, \eta) = L_f^\gamma h(\Phi^{-1}(\xi, \eta)) \,. \tag{1.37}
$$

Note the lack of input terms in the differential equations for η—this is due to the fact that $L_g \eta_i(x) = 0$, $i = 1, \ldots, n - \gamma$, for $x \in U$. The system description (1.36) is a local *'normal' form* of the system (1.10) [BI88,Isi89b].

1.2.3 Full State Linearization by State Feedback

Consider now the case when the system (1.10) has a relative degree of *exactly* n. In this case, we can locally transform the system into a controllable linear system. Indeed, the normal form of the system is given by

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_n &= b(\xi) + a(\xi)u \\
y &= \xi_1
\end{aligned}
\tag{1.38}
$$

so that the feedback law

$$
u = \frac{1}{a(\xi)}[-b(\xi) + v] \tag{1.39}
$$

yields a linear system with a transfer function of $1/s^n$ from v to y. The transformation from the original system (in x and u) to the linear system (in ξ and v) consists of

1. a nonlinear change of *state* coordinates

$$(\xi_1, \xi_2, \ldots, \xi_n)^T = \Phi(x) = (h(x),\, L_f h(x), \ldots,\, L_f^{n-1} h(x))^T \tag{1.40}$$

and

2. a (state-dependent) change of *input* coordinates and *state feedback*

$$v(x,u) = L_f^n h(x) + L_g L_f^{n-1} h(x) u \tag{1.41}$$

both defined in a neighborhood of x_0.

We can now place the poles of the closed loop system at the zeros of a desired polynomial $d(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0$ by choosing the state feedback

$$v = -\alpha_0 \xi_1 - \cdots - \alpha_{n-1} \xi_n \tag{1.42}$$

or, in the original coordinates

$$u = \frac{1}{L_g L_f^{n-1} h(x)} [-L_f^n h(x) - \alpha_{n-1} L_f^{n-1} h(x) - \cdots - \alpha_0 h(x)]\,. \tag{1.43}$$

Thus we see that the nonlinear system (1.10) is *equivalent* to a controllable linear system by choice of a particular set of *state* and *input* coordinates (with state feedback), namely, (1.40) and (1.41).

Now, suppose we are given the system dynamics

$$\dot{x} = f(x) + g(x) u \tag{1.44}$$

but no output is specified. An obvious question is then: When are the dynamics of a nonlinear system of the form (1.44) equivalent to the dynamics of a controllable linear system? Necessary and sufficient conditions for this equivalence have been given by Jakubczyk and Respondek [JR80] and (independently) by Hunt, Su, and Meyer [HSM83,Su82] (and apparently also by Brockett—see note in [JR80]).

These conditions can easily be derived from our development. Indeed, we see that the nonlinear dynamics of (1.44) are equivalent to a linear dynamics if (and only if) there exists a function h for which the resulting system has relative degree exactly n. Thus the function h must be such that (by the definition of relative degree)

$$L_g L_f^i h(x) = 0 \qquad x \in U,\ i < n-1 \tag{1.45}$$

and

$$L_g L_f^{n-1} h(x) \neq 0 \quad x \in U. \tag{1.46}$$

Now, (1.45) is a set of $(n-1)$ *higher order* linear partial differential equations for the function h. Fortunately, it can be reduced to a set of *first order* linear partial differential equations using the following proposition.

Proposition 1.4 *Let f and g be smooth vector fields on an open set $U \subset \mathbf{R}^n$ and let h be a smooth function on U. The following conditions are equivalent:*

1. $L_g L_f^j h(x) = 0, \quad x \in U, 0 \leq j \leq k$.
2. $L_{ad_f^j g} h(x) = 0, \quad x \in U, 0 \leq j \leq k$.

Proof: By induction on k using the same techniques as in the proof of Lemma 1.2. □

The proposition tells us that equations (1.45) are equivalent to the set of *first order linear* partial differential equations

$$L_{ad_f^i g} h(x) = 0 \qquad x \in U, \, i < n-1 \tag{1.47}$$

which can be rewritten as

$$dh \cdot \left[g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-2} g \right] = 0 \, . \tag{1.48}$$

Proposition 1.4 also implies that equation (1.46) is equivalent to

$$L_{ad_f^{n-1} g} h(x) \neq 0 \qquad x \in U. \tag{1.49}$$

We now state a result *dual* to Proposition 1.3:

Proposition 1.5 *Suppose that (1.47) and (1.49) hold. Then the distribution*

$$span \left\{ g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-1} g \right\} \tag{1.50}$$

has dimension n on U.

Proof: Assume the vector fields are linearly dependent, that is, there exist function α_i, $i = 1, \ldots, n$, such that

$$\alpha_1 g + \alpha_2 ad_f g + \cdots + \alpha_n ad_f^{n-1} g = 0 \, . \tag{1.51}$$

Multiplying on the left by dh, we get (by Lemma 1.2)

$$\alpha_n L_{ad_f^{n-1}g} h = 0 \tag{1.52}$$

which shows that $\alpha_n(x) = 0$ for $x \in U$. Continue this procedure, multiplying by $dL_f h$, $dL_f^2 h$, etc., to see that α_{n-1}, α_{n-2}, etc., are each identically zero on U. This proves the proposition. □

We can now state the necessary and sufficient conditions for the equivalence of the system dynamics:

Theorem 1.6 [JR80,Su82,HSM83] *The dynamics of the nonlinear system*

$$\dot{x} = f(x) + g(x)u \tag{1.53}$$

are locally (on U) equivalent to the dynamics of a controllable linear system by change of state and input coordinates and state feedback if and only if

1. *the distribution (1.50) has dimension n on U, and*
2. *the distribution*

   $$span\Big\{g \; ad_f g \; \cdots \; ad_f^{n-2} g\Big\} \tag{1.54}$$

   is involutive on U.

Proof: The Frobenius theorem says that there is a function h solving (1.47) if and only if the distribution (1.54) has dimension $(n-1)$ on U and is involutive on U. This combined with the previous facts proves the theorem. □

Remarks

- The distribution (1.50) is the *controllability* distribution and is the nonlinear analog of the linear controllability matrix

  $$\Big[b \; Ab \; \cdots \; A^{n-1}b\Big] \tag{1.55}$$

  and tells us that the system is locally controllable (e.g., through its Jacobian approximation with $b = g(x_0)$ and $A = Df(x_0)$).

- The *involutivity* condition is trivially satified in the linear case but not generically satisfied in the nonlinear case. Thus, not all locally controllable nonlinear systems may be locally fully state linearized.

□

1.2.4 Zero Dynamics and Minimum Phase Nonlinear Systems

Above we saw that the relative degree of a nonlinear system was a natural extension of the relative degree of a linear system—the excess of poles over zeros. In this section, we will see that this analogy can be extended to define a nonlinear version of the system zeros, the zero dynamics.

The input-output linearizing control of (1.17) (repeated here for easy reference)

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)}(-L_f^{\gamma} h + v)$$

gives the nonlinear system (1.10) a closed loop transfer function of $1/s^{\gamma}$ from v to y (accounting for γ of the n states). The remaining $(n-\gamma)$ states of the system have been made unobservable by state feedback. To see how this happens, consider the linear case, i.e., $f(x) = Ax$, $g(x) = b$, and $h(x) = cx$. Then, the system has relative degree γ if the Markov parameters are such that

$$\begin{gathered} cb = cAb = cA^2 b = \cdots = cA^{\gamma-2} b = 0 \\ cA^{\gamma-1} b \neq 0 \end{gathered} \tag{1.56}$$

so that the control law (1.17) yields the closed loop system

$$\begin{aligned} \dot{x} &= \left[I - \frac{1}{cA^{\gamma-1}b} bcA^{\gamma-1}\right] Ax + \frac{1}{cA^{\gamma-1}b} bv \\ y &= cx \end{aligned} \tag{1.57}$$

with transfer function $1/s^{\gamma}$ from v to y. It follows that $(n-\gamma)$ of the eigenvalues of $\left[I - \frac{1}{cA^{\gamma-1}b} bcA^{\gamma-1}\right] A$ have been placed (by state feedback) at the zeros of the original system and the remaining at the origin. Thus, the input-output linearizing control law may be thought of as the nonlinear counterpart of a *zero-cancelling* control law.

There are three equivalent notions that can be used to define the zeros of an invertible linear system:

1. the system dynamics associated with the maximal controlled invariant manifold in the kernel of the output map,

2. the system dynamics under the constraint that the output be identically zero for all time, and

3. the dynamics of a minimal inverse system with the input (i.e., the original output) set to zero.

Although these notions do not always coincide for a nonlinear system. Under certain conditions (e.g., the relative degree is well defined), these notions are, in fact, equivalent [IM89].

With this fact in hand, we make the following definition:

Definition 1.2 *The* zero dynamics *of a nonlinear system (1.10) are the dynamics of the system subject to the constraint that the output be identically zero.*

To show that this definition is well-defined, we will explicitly characterize the zero dynamics of (1.10) using the local normal form (1.36). First note that

$$y(t) \equiv 0 \Longleftrightarrow \xi_1(t) = \xi_2(t) = \cdots = \xi_\gamma(t) \equiv 0 \,. \tag{1.58}$$

Also, in order to keep $\xi_\gamma \equiv 0$ we must choose the input so that

$$u(t) = -\frac{b(0,\eta(t))}{a(0,\eta(t))} \tag{1.59}$$

where $\eta(t)$ is any solution of

$$\dot{\eta} = q(0,\eta), \qquad \eta(0) \text{ arbitrary.} \tag{1.60}$$

Thus, for arbitrary $\eta(0)$ with $(0,\eta(0)) \in \Phi(U)$, the output can be held identically zero provided that the $\xi_i(0) = 0$ for $i = 1,\ldots,\gamma$. In the original coordinates, we see that the initial state $x(0)$ belongs to the manifold

$$M^* := \left\{x \in U \;:\; h(x) = L_f h(x) = \ldots = L_f^{\gamma-1}h(x) = 0\right\} \tag{1.61}$$

and the input is given as the state feedback

$$u^*(x) = -\frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1}h(x)} \,. \tag{1.62}$$

This control law is precisely (1.17) with v set to zero. Note that this feedback law renders M^* invariant, i.e, given an initial condition belonging to M^*, the entire trajectory of (1.10) will lie in M^*. Thus, the zero dynamics of (1.10) are precisely the dynamics of

$$\dot{x} = f(x) + g(x)u^*(x) \tag{1.63}$$

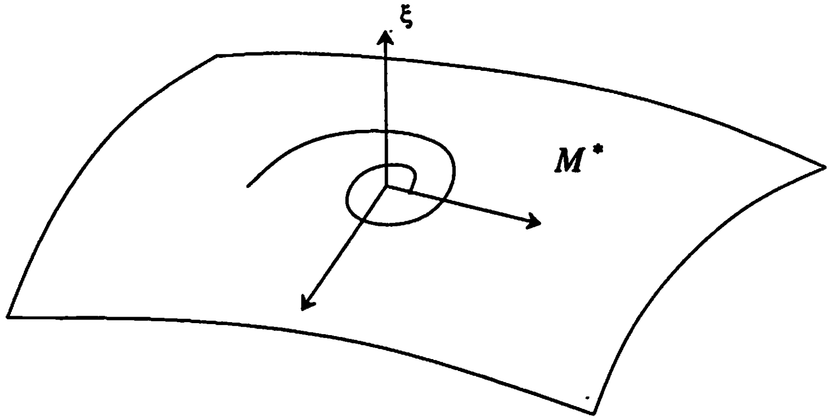

Figure 1.1: The zero dynamics manifold M^*

restricted to M^* as shown in Figure (1.1). In (ξ, η) coordinates, the zero dynamics are simply

$$\begin{aligned} \xi &= 0 \\ \dot{\eta} &= q(0, \eta) \end{aligned} \tag{1.64}$$

and are the internal dynamics of the system consistent with the constraint that $y(t) \equiv 0$.

For a linear system in normal form, the η dynamics will have the form

$$\dot{\eta} = P\xi + Q\eta \tag{1.65}$$

so that the zero dynamics are simply

$$\dot{\eta} = Q\eta \,. \tag{1.66}$$

It is easy to check that the eigenvalues of Q are indeed the zeros of the original system.

Recall that a linear system is called minimum phase if all the the system zeros lie in the open left half plane. In other words, the zero dynamics of the system are (exponentially) stable. To extend this notion to nonlinear systems we need a few more assumptions.

Recall that the normal form is defined locally around the equilibrium point x_0. Assume, without loss of generality, that the equilibrium point x_0 is mapped to $(\xi, \eta) = (0, 0)$ by the coordinate transformation Φ. Specifically, this requires that $h(x_0) = 0$ and $q(0, 0) = 0$.

Definition 1.3 *The nonlinear system (1.10) is said to be* asymptotically (exponentially) minimum phase *at x_0 if the equilibrium point $\eta = 0$ of (1.64) is locally asymptotically (exponentially) stable.*

Remarks

- It is important to note that the minimum phase property of a nonlinear system depends on the equilibrium point x_0 under consideration. Thus, a nonlinear system may be minimum phase at some points and nonminimum phase at some others.
- The stability properties of the zero dynamics are independent of the choice of η coordinates.

□

1.2.5 Stabilization and Tracking for SISO systems

With the concepts of zero dynamics and minimum phase systems in hand, we are now ready to tackle the problems of stabilization and tracking for nonlinear systems for which the relative degree is well defined. We start with stabilization:

Theorem 1.7 *Suppose the system (1.10) has relative degree γ and is locally* asymptotically *minimum phase and let $d(s) = s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \cdots + \alpha_1 s + \alpha_0$ be a Hurwitz polynomial. The state feedback law*

$$u(x) = \frac{1}{L_g L_f^{\gamma-1}h(x)}[-L_f^\gamma h(x) - \alpha_{\gamma-1}L_f^{\gamma-1}h(x) - \cdots - \alpha_1 L_f h(x) - \alpha_0 h(x)] \tag{1.67}$$

results in a (locally) asymptotically stable system.

Proof: In (ξ, η) coordinates, the closed loop system (1.10), (1.67) is given by

$$\begin{aligned}
\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_{\gamma-1} \\ \dot{\xi}_\gamma \end{bmatrix} &= \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \\ \alpha_0 & \cdots & \cdots & \alpha_{\gamma-1} \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{\gamma-1} \\ \xi_\gamma \end{bmatrix} \\
\dot{\eta} &= q(\xi, \eta)
\end{aligned} \tag{1.68}$$

or, compactly,

$$\begin{aligned}
\dot{\xi} &= A\xi \\
\dot{\eta} &= q(\xi, \eta).
\end{aligned} \tag{1.69}$$

If the zero dynamics are exponentially stable, the stability of the Jacobian linearization is sufficient to show stability. If the zero dynamics are critically stable, then center manifold theory [Car81] can be used to prove the theorem. See [BI88] for details. □

It is easy to modify the control law (1.67) in the instance that the control objective is tracking rather than stabilization. Consider the problem of tracking a given prespecified desired trajectory $y_d(t)$. Define $e \in \mathbb{R}^\gamma$ by $e_i = y^{(i-1)} - y_d^{(i-1)}$ so that e_1 is the tracking error. Note that $\xi = e + \bar{y}_d$ where $\bar{y}_d := (y_d, \dot{y}_d, \ldots, y_d^{\gamma-1})^T$. The counterpart of the stabilizing control law of (1.67) is

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} \left[-L_f^\gamma h(x) + y_d^\gamma + \alpha_{\gamma-1} e_\gamma + \cdots + \alpha_0 e_1 \right] \tag{1.70}$$

so that the closed loop system (in (e, η) coordinates with A as above) is given by

$$\begin{aligned} \dot{e} &= Ae \\ \dot{\eta} &= q(e + \bar{y}_d, \eta). \end{aligned} \tag{1.71}$$

Though it is not immediately obvious, the control law of (1.70) is a state feedback law since

$$e_i = L_f^{i-2} h(x) - y_d^{(i-1)} \tag{1.72}$$

for $i = 1, \ldots, \gamma$. The counterpart of Theorem 1.7 is the following:

Theorem 1.8 *Suppose the system (1.10) has relative degree γ and is locally* exponentially *minimum phase and, as before, let $d(s)$ be a Hurwitz polynomial. Then, if the desired trajectory $y_d(t)$ and its first $(\gamma - 1)$ derivatives are small enough, the control law (1.70) results in bounded tracking, i.e., the state x is bounded and the tracking error e_1 and its first $(\gamma - 1)$ derivatives tend to zero asymptotically.*

Proof: Clearly, from the form of the closed loop system (1.71), it is enough to show that the states remain bounded (i.e., in U). Then the stability of $\dot{e} = Ae$ will guarantee that $e \to 0$. Let b_d be a bound for y_d and its first $(\gamma - 1)$ derivatives. Note that $q(\cdot, \cdot)$ is locally Lipschitz (with constant l_q) since $\eta(\cdot)$, $f(\cdot)$, and $\Phi(\cdot)$ (and $\Phi^{-1}(\cdot)$) are smooth. Since the zero dynamics $\dot{\eta} = q(0, \eta)$ are locally exponentially stable, a converse Lyapunov theorem [Hah67] implies the existence of a Lyapunov theorem $V_2(\eta)$ such that

$$\begin{aligned} k_1|\eta|^2 \leq V_2(\eta) \leq k_2|\eta|^2 \\ \tfrac{\partial V_2}{\partial \eta} q(0, \eta) \leq -k_3|\eta|^2 \\ \left|\tfrac{\partial V_2}{\partial \eta}\right| \leq k_4|\eta| \end{aligned} \tag{1.73}$$

for some positive constants k_1, k_2, k_3, and k_4. Consider as Lyapunov function for (1.71)

$$V(e,\eta) = e^T P e + \mu V_2(\eta) \tag{1.74}$$

where $P > 0$ solves $A^T P + PA = -I$ (possible since $\dot{e} = Ae$ is stable) and $\mu > 0$ is to be determined below. Taking the time derivative of V along the trajectories of (1.71), we get

$$\begin{aligned} \dot{V} &= -|e|^2 + \mu \tfrac{\partial V_2}{\partial \eta}\left[q(0,\eta) + q(e+\bar{y}_d,\eta) - q(0,\eta)\right] \\ &\leq -|e|^2 - \mu k_3|\eta|^2 + \mu k_4 l_q |\eta|(|e| + b_d) \\ &\leq -\tfrac{3}{4}|e|^2 - \tfrac{3}{4}|\eta|^2 - (\tfrac{|e|}{2} - \mu k_4 l_q|\eta|)^2 + (\mu k_4 l_q)^2|\eta|^2 - \mu k_3(\tfrac{|\eta|}{2} - \tfrac{k_4 l_q b_d}{k_3})^2 + \mu\tfrac{(k_4 l_q)^2}{k_3} b_d^2. \end{aligned} \tag{1.75}$$

Setting $\mu = k_3/(2k_4 l_q)^2$ and dropping the squares, we get

$$\dot{V} \leq -\frac{3}{4}|e|^2 - \frac{k_3^2}{8k_4^2 l_q^2}|\eta|^2 + \frac{b_d^2}{2}. \tag{1.76}$$

Thus, $\dot{V} < 0$ whenever $|\eta|$ or $|e|$ is large which implies that $|\eta|$ and $|e|$ and, hence, $|\xi|$ and $|x|$, are bounded. The above analysis was for $x \in U$. Indeed, by choosing b_d sufficiently small and appropriate initial conditions, we can guarantee that the action remains in U. Therefore the state x remains bounded and the stability of $\dot{e} = Ae$ implies that $e \to 0$ as $t \to \infty$. □

Thus, the notions of zero dynamics and minimum phase provide useful extensions to their linear counterparts. In particular, we see that we can control minimum phase nonlinear systems effectively using a control law such as (1.70) designed using the input-output linearization methodology.

1.3 Linearization and Decoupling of MIMO Nonlinear Systems

For the multi-input multi-output case, we consider *square* systems (that is, systems with the same number of inputs as outputs) of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \\ y_1 &= h_1(x) \\ &\vdots \\ y_p &= h_m(x) \end{aligned} \tag{1.77}$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$, f, g_i, $i = 1, \ldots, m$, are smooth vector fields, and h_j, $j = 1, \ldots, m$, are smooth functions. We will sometimes abuse notation and use g and u to denote $\{g_1 \cdots g_m\}$ and $(u_1, \ldots, u_m)^T$ so that $gu = \sum_1^m g_i u_i$. There is a class of MIMO systems for which the development very closely parallels that for the SISO case of the previous section. We start with this class:

1.3.1 MIMO Systems Linearizable by Static State Feedback

As in the SISO case, we begin by differerentiating the output(s): the time derivative of the jth output y_j of the system (1.77) is

$$\dot{y}_j = L_f h_j + \sum_{i=1}^{m} (L_{g_i} h_j) u_i \ . \tag{1.78}$$

In (1.78) above, if $L_{g_i} h_j(x) = 0$ (on an open set, $U \subset \mathbf{R}^n$) for each i, then the inputs do not appear in the equation. Define γ_j to be the smallest integer such that at least one of the inputs appears in $y_j^{(\gamma_j)}$. Then

$$y_j^{(\gamma_j)} = L_f^{\gamma_j} h_j + \sum_{i=1}^{m} L_{g_i} (L_f^{\gamma_j - 1} h_j) u_i \tag{1.79}$$

with at least one of the $L_{g_i} L_f^{\gamma_j - 1} h_j(x) \neq 0$ for $x \in U$. Define the $m \times m$ matrix $A(x)$ as

$$A(x) := \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m \end{bmatrix} . \tag{1.80}$$

The matrix $A(x)$ is called the *decoupling matrix*. Using these definitions, we define the *(vector) relative degree* for MIMO systems:

Definition 1.4 *The system (1.77) is said to have* (vector) relative degree $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ *at x_0 (an equilibrium point) if there exists a neighborhood U of x_0 such that, for $x \in U$,*

(1) $L_{g_j} L_f^k h_i(x) = 0 \qquad 0 \leq k \leq \gamma_i - 1,\ 1 \leq i, j \leq m,$ *and*

(2) $A(x)$ *is nonsingular.*

With the decoupling matrix $A(x)$ defined as in (1.80), then the equations (1.79) may be written as

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1(x) \\ \vdots \\ L_f^{\gamma_m} h_m(x) \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \tag{1.81}$$

If the system (1.77) has a well defined (vector) relative degree, then $A(x) \in \mathbb{R}^{m \times m}$ is nonsingular on U and the state feedback control law

$$u = -A^{-1}(x) \begin{bmatrix} L_f^{\gamma_1} h_1(x) \\ \vdots \\ L_f^{\gamma_m} h_m(x) \end{bmatrix} + A^{-1}(x) v \tag{1.82}$$

with $v \in \mathbb{R}^m$ yields the *linear* closed loop system

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \tag{1.83}$$

Note that the system of (1.83) is, in addition, *decoupled*. Thus, decoupling is a byproduct of linearization. A useful consequence of this is that a large number of results concerning SISO nonlinear systems can be easily extended to this class of MIMO nonlinear systems. Thus, as we shall see shortly, further control objectives, such as tracking, are easily accomplished. The feedback law (1.82) is a *static state feedback* linearizing control law.

1.3.2 MIMO 'Normal' Form

If a MIMO system has (vector) relative degree $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ such that $\gamma := \gamma_1 + \cdots + \gamma_m \leq n$, we can write a normal form for the equations (1.77) as follows: Choose as coordinates

$$\begin{array}{llll} \xi_1^1 = h_1(x), & \xi_2^1 = L_f h_1(x), & \cdots & \xi_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x), \\ \xi_1^2 = h_2(x), & \xi_2^2 = L_f h_2(x), & \cdots & \xi_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x), \\ & \vdots & & \\ \xi_1^m = h_m(x), & \xi_2^m = L_f h_m(x), & \cdots & \xi_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x). \end{array} \tag{1.84}$$

Similar to the SISO case, the ξ_i^j, $j = 1, \ldots, m$, $i = 1, \ldots, \gamma_j$, qualify as a partial set of coordinates since the differentials

$$dL_f^i h_j(x) \qquad 0 \le i \le \gamma_j - 1, 1 \le j \le m \tag{1.85}$$

are linearly independent on U. Indeed, using the nonsingularity of $A(x)$, the proof of this proceeds in a manner analogous to that of Proposition 1.3. Now complete the basis choosing $n-\gamma$ more functions $\eta_1(x), \eta_2(x), \ldots, \eta_{n-\gamma}(x)$. Unlike the SISO case, it is no longer possible to guarantee that

$$L_{g_j}\eta_i(x) = 0 \qquad x \in U,\ 1 \le j \le m,\ 1 \le i \le n - \gamma \tag{1.86}$$

unless the distribution spanned by $g_1(x), \ldots, g_m(x)$ is involutive on U. Note that the transformation Φ given by $x \mapsto (\xi, \eta)$ is a local diffeomorphism of U onto $\Phi(U)$. In (ξ, η) coordinates, the system equations (1.77) are given locally by

$$\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
&\vdots \\
\dot{\xi}_{\gamma_1}^1 &= b_1(\xi,\eta) + \textstyle\sum_{j=1}^m a_j^1(\xi,\eta)u_j \\
\dot{\xi}_1^2 &= \xi_2^2 \\
&\vdots \\
\dot{\xi}_{\gamma_1}^2 &= b_2(\xi,\eta) + \textstyle\sum_{j=1}^m a_j^2(\xi,\eta)u_j \\
&\vdots \\
\dot{\xi}_1^m &= \xi_2^m \\
&\vdots \\
\dot{\xi}_{\gamma_1}^m &= b_m(\xi,\eta) + \textstyle\sum_{j=1}^m a_j^m(\xi,\eta)u_j \\
\dot{\eta} &= q(\xi,\eta) + P(\xi,\eta)u \\
y_1 &= \xi_1^1 \\
&\vdots \\
y_m &= \xi_1^m
\end{aligned} \tag{1.87}$$

where

$$b_i(\xi,\eta) = L_f^{\gamma_i} h_i(x),$$

$$a_j^i(\xi,\eta) = L_{g_j} L_f^{\gamma_i - 1} h_i(x),$$
$$q_i(\xi,\eta) = L_f \eta_i(x),$$
$$P_{ij}(\xi,\eta) = L_{g_j} \eta_i(x)$$

in the ξ, η coordinates. Note that $P \in \mathbb{R}^{n-\gamma \times m}, q \in \mathbb{R}^{n-\gamma}$ respectively. As in the SISO case the feedback law of (1.82) renders the η states unobservable.

1.3.3 Zero Dynamics and Minimum Phase MIMO Systems

In the instance that the decoupling matrix $A(x)$ is nonsingular, the zero dynamics are easily found. As in the SISO case, we find the zero dynamics by constraining the outputs to zero.

Definition 1.5 *The* zero dynamics *of a MIMO nonlinear system (1.77) are the dynamics of the system subject to the constraint that the outputs be identically zero.*

Using the normal form (1.87), we see that

$$y_i \equiv 0,\ i = 1,\ldots,m \iff \xi_j^i \equiv 0,\ j = 1,\ldots,\gamma_i,\ i = 1,\ldots,m\,. \tag{1.88}$$

In order to keep $\xi_{\gamma_i}^i \equiv 0$, $i = 1,\ldots,m$, we must choose the input by

$$u(t) = -A^{-1}(0,\eta(t))b(0,\eta(t)) \tag{1.89}$$

where $\eta(t)$ is any solution of

$$\dot{\eta} = q(0,\eta) - P(0,\eta)A^{-1}(0,\eta)b(0,\eta), \quad \eta(0) \text{ arbitrary.} \tag{1.90}$$

Thus, the output can be held identically zero provided that $\xi_j^i(0) = 0$, $j = 1,\ldots,\gamma_i$, $i = 1,\ldots,m$. In the original x coordinates, the initial conditions must be chosen to belong to the manifold

$$M^* = \left\{x \in U\ :\ h_i(x) = L_f h_i(x) = \ldots = L_f^{\gamma_i-1} h_i(x) = 0,\ 1 \le i \le m\right\}\,. \tag{1.91}$$

and the input is given by the static state feedback

$$u^*(x) = -A^{-1}(x)\begin{bmatrix} L_f^{\gamma_1} h_1(x) \\ \vdots \\ L_f^{\gamma_m} h_m(x) \end{bmatrix}\,. \tag{1.92}$$

This is the decoupling control law of (1.83) with $v(t)$ set equal to zero. Note that this feedback law renders the manifold M^* invariant. Thus, analogous to the SISO case, the zero dynamics of (1.77) are the dynamics of

$$\dot{x} = f(x) + g(x)u^*(x) \tag{1.93}$$

restricted to the *zero dynamics* manifold M^*. In the (ξ, η) coordinates, the zero dynamics are given by

$$\begin{aligned} \xi &= 0 \\ \dot{\eta} &= q(0,\eta) - P(0,\eta)A^{-1}(0,\eta)b(0,\eta) \end{aligned} \tag{1.94}$$

and are the internal dynamics consistent with the constraint that $y_i(t) \equiv 0$, $i = 1, \ldots, m$.

Recall that the MIMO normal form (1.87) is defined in a neighborhood $\Phi(U)$ of an equilibrium point $\Phi(x_0)$. Assume, without loss of generality, that $\Phi(x_0) = (0,0)$ so that $h_i(x_0) = 0$ and $\eta = 0$ is an equilibrium point of the zero dynamics (1.94). Then the notion of minumum phase parallels the SISO case.

Definition 1.6 *The MIMO nonlinear system (1.77) is said to be* asymptotically (exponentially) minimum phase *at x_0 if the equilibrium point $\eta = 0$ of (1.94) is locally asymptotically (exponentially) stable.*

1.3.4 Stabilization and Tracking for MIMO systems

The stabilization and tracking results for minimum phase SISO nonlinear systems with well-defined relative degree are easily extended to minimum phase MIMO systems with well-defined (vector) relative degree. Specific details are left to the reader. See Chapter 4 for a specific example illustrating this. The fact that the feedback law (1.82) decouples the system allows extremely simple stabilizing and tracking laws to be used.

1.3.5 Dynamic Extension of MIMO Systems

The conditions required for a MIMO nonlinear system to have a a well defined (vector) relative degree can fail in several ways. As in the SISO case, a MIMO system can fail to have a (vector) relative degree because of a control coefficient $L_{g_i}L_f^k h_j(x)$ that is neither identically zero nor bounded away from zero on U. In this case the decoupling matrix $A(x)$ is not well defined. A MIMO system, however, can also fail to have a (vector) relative degree even when $A(x)$ is well defined. Note that this cannot happen in the SISO case since a 1×1 matrix with nonzero rows is trivially nonsingular. Suppose that the decoupling matrix $A(x)$ is well defined on U but does not have full rank. If $A(x)$ has a constant rank $r < m$, then we we may be able to extend the system by adding integrators to certain input channels to obtain a system that does have a well-defined (vector) relative degree. If $A(x)$ does not have a constant rank (on U) this will not be possible. The following algorithm makes this precise.

Dynamic Extension Algorithm

Step 0 Set $\tilde{m} = m$, $\tilde{n} = n$, $\tilde{x} = x$, $\tilde{x}_0 = x_0$, $\tilde{f} = f$, $\tilde{g} = g$, $\tilde{h} = h$, and $\tilde{u} = u$.

Step 1 Calculate the decoupling matrix, $\widetilde{A}(\tilde{x})$, valid on $\widetilde{U}$, a neighborhood of $\tilde{x}_0$ (i.e., condition (1) in the definition of (vector) relative degree is satisfied on $\widetilde{U}$).

- If rank $\widetilde{A}(\tilde{x}) = m$ on $\widetilde{U}$, **stop**—the system $(\tilde{f}, \tilde{g}, \tilde{h})$ has a (vector) relative degree.
- If rank $\widetilde{A}(\tilde{x})$ is not constant in a neighborhood of $\tilde{x}_0$, **stop**—the system (1.77) cannot be extended to a system with a (vector) relative degree.
- Otherwise, set $r = \text{rank } \widetilde{A}(\tilde{x})$ and continue to Step 2.

Step 2 Calculate a smooth matrix $\beta(\tilde{x})$ of elementary column operations to *compress* the columns of $\widetilde{A}(\tilde{x})$ so that the last $(m - r)$ columns of

$$\widetilde{A}(\tilde{x})\beta(\tilde{x}) \tag{1.95}$$

are identically zero on $\widetilde{U}$. This is possible since the rank of $\widetilde{A}(\tilde{x})$ is constant on $\widetilde{U}$ (see [DM85] for a construction of such smooth elementary column operations). Furthermore, let β contain column permutations so that the first $r_1 \leq r$ columns of (1.95) consist of *all* the columns with two or more nonzero entries (thus, columns $r_1 + 1$ through r have only *one* nonzero entry.) Partition β as

$$\beta(\tilde{x}) = [\beta_1(\tilde{x}) \; \beta_2(\tilde{x})] \tag{1.96}$$

such that β_1 consists of the first r_1 columns of β.

Step 3 Extend the system by adding one integrator to each of the first r_1 (redefined) input channels. Specifically, define $z_1 \in \mathbf{R}^{r_1}$, $w_2 \in \mathbf{R}^{\tilde{m}-r_1}$ by

$$\begin{bmatrix} z_1 \\ w_2 \end{bmatrix} := \beta(\tilde{x})^{-1}\tilde{u} . \tag{1.97}$$

Then, with $w_1 \in \mathbf{R}^{r_1}$, the extended system is given by

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{\tilde{x}} \\ \dot{z}_1 \end{bmatrix}}_{\dot{\bar{x}}} &= \underbrace{\begin{bmatrix} \tilde{f} + \tilde{g}\beta_1 z_1 \\ 0 \end{bmatrix}}_{\bar{f}} + \underbrace{\begin{bmatrix} \tilde{g}\beta_2 & 0 \\ 0 & I \end{bmatrix}}_{\bar{g}} \cdot \underbrace{\begin{bmatrix} w_2 \\ w_1 \end{bmatrix}}_{\bar{u}} \\ y &= \bar{h}(\bar{x}) = \tilde{h}(\tilde{x}) \end{aligned} \tag{1.98}$$

where $\bar{x}^T = (\tilde{x}^T, z_1^T)^T$ is the extended state.

Step 4 Replace $\tilde{x}$, $\tilde{u}$, $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ by $\bar{x}$, $\bar{u}$, $\bar{f}$, $\bar{g}$, and $\bar{h}$, respectively, and return to Step 1.

□

Descusse and Moog [DM85] have shown that this algorithm will be successful (terminating in a finite number of steps) if the system is left invertible (see also [Isi86]). In fact, it can also been shown that if the algorithm is successful in extending a system, then it will terminate in at most n iterations [Isi87].

When the dynamic extension algorithm is succesful, the extended system has a well defined (vector) relative degree. The extension portion of the new system will then form the core of a dynamic compensator. Indeed, if only one iteration of the algorithm was necessary, the resulting extension would be

$$\begin{aligned} \dot{z}_1 &= w_1 , \\ u &= \beta_1(x)z_1 + \beta_2(x)w_2 . \end{aligned} \tag{1.99}$$

More generally, the dynamic compensator will have the form

$$\begin{aligned} \dot{z} &= c(x,z) + d(x,z) , \\ u &= \alpha(x,z) + \beta(x,z)w . \end{aligned} \tag{1.100}$$

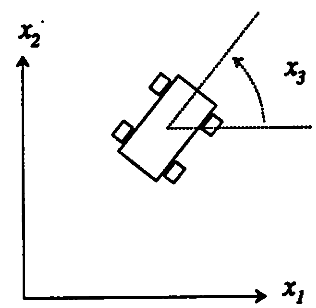

Figure 1.2: The simple planar vehicle.

1.3.6 MIMO Example

To help clarify the process of dynamic extension and to show the form of the dynamic compensator, we present a simple example. Consider the planar vehicle shown in Figure 1.2 where (x_1, x_2) is the vehicle position and x_3 is the vehicle heading. For simplicity, suppose that the inputs are the vehicle speed (u_1) and turning rate (u_2) (actual controls will normally be accelerations) and that the control objective is to steer the vehicle along a given path in x_1 and x_2. The system equations are then

$$\begin{aligned}
\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{\dot{x}} &= \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g_2} u_2 \\
y_1 &= \underbrace{x_1}_{h_1} \\
y_2 &= \underbrace{x_2}_{h_2} .
\end{aligned} \tag{1.101}$$

Differentiating each output until at least one input appears, we get (see (1.81))

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}}_{A(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{1.102}$$

Clearly, $A(x)$ has rank 1 and, fortunately, $\beta(x)$ can be chosen to be the identity. Also, $r_1 = r = 1$. Setting

$$\begin{bmatrix} z_1 \\ w_2 \end{bmatrix} = I \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.103}$$

we *extend* the system giving (as in (1.98))

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{z}_1 \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} z_1 \cos x_3 \\ z_1 \sin x_3 \\ 0 \\ 0 \end{bmatrix}}_{\bar{f}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{g}_1} w_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{g}_2} w_2 \tag{1.104}$$

with the same outputs. Differentiating the outputs until the new controls (w) appear, we find

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos x_3 & -z_1 \sin x_3 \\ \sin x_3 & z_1 \cos x_3 \end{bmatrix}}_{\bar{A}(x)} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{1.105}$$

Thus, for z_1 bounded away from zero, we can choose

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 & \sin x_3 \\ -\frac{\sin x_3}{z_1} & \frac{\cos x_3}{z_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.106}$$

to decouple and linearize the system yielding

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{1.107}$$

Then, the feedback law

$$\begin{aligned} v_1 &= \ddot{y}_{1d} + \alpha_1(\dot{y}_{1d} - \dot{y}_1) + \alpha_0(y_{1d} - y_1) \\ &= \ddot{y}_{1d} + \alpha_1(\dot{y}_{1d} - z_1 \cos x_3) + \alpha_0(y_{1d} - x_1) \\ v_2 &= \ddot{y}_{2d} + \alpha_1(\dot{y}_{2d} - \dot{y}_2) + \alpha_0(y_{2d} - y_2) \\ &= \ddot{y}_{2d} + \alpha_1(\dot{y}_{2d} - z_1 \sin x_3) + \alpha_0(y_{2d} - x_2) \end{aligned} \tag{1.108}$$

will result in stable output tracking. Note that (1.108) is a function of only the extended state (x_1, x_2, x_3, z_1) and the desired trajectory. The *dynamic compensator* is thus

$$\begin{aligned} \dot{z}_1 &= w_1 \\ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} z_1 \\ w_2 \end{bmatrix} \end{aligned} \tag{1.109}$$

where w_1 and w_2 are given by (1.106), (1.108) so that (1.109) is indeed a dynamic state feedback law.

1.4 Bibliographical Notes

Many of the tools and techniques now common to nonlinear control can be found (implicitly) in early work on differential equations by such researchers as Poincaré [Poi28] and Chow [Cho39]. With the introduction of the methods into control theory by Hermann [Her63] and others [GS68,Lob70,HH70,Sus73], the stage was set for the development of basic geometric nonlinear systems theory. Researchers investigated important notions such as decoupling [Por70], accessibility [SJ72], controllability and observability [HK77], as well as general systems theory in nontraditional settings [Bro72,Bro73,Lob73].

Also, during this time, research into the problem of system equivalence [Kre73, MC75,Bro78] led to methods for input-to-state linearization [JR80,Su82,HSM83]. Additionally, many researchers improved our understanding of the questions of nonlinear decoupling and noninteracting control [SR72,Fre75] and system inversion [Hir79a,Hir79b,Sin81].

Using the tools of differential geometry and especially the notion of controlled invariance [Hir81,IKGM81b,IKGM81a], many of the important problems in nonlinear control have been clarified and solved forming the base of a nonlinear control theory [Isi85] that is clearly the nonlinear counterpart to the *geometric* methods of linear control theory [Won74].

Indeed, new techniques and applications are being developed at a exciting rate as shown by numerous meetings devoted exclusively to nonlinear control [FH86,Isi89a].

Chapter 2

The Structure of Zero Dynamics

In this chapter we present results that show that the structure of a system—the zero dynamics—is not robust to perturbations. In particular, we show that regular perturbations of the state space descriptions of linear and nonlinear single-input single-output (SISO) systems of relative degree ≥ 2 may result in the appearance of singularly perturbed (fast) zero dynamics. In other words, perturbations in the state space descriptions may cause the migration of some zeros from ∞ to finite locations in the complex plane. Depending on the sign of the regular perturbations, some of the perturbed zeros can migrate from ∞ to the right half of the complex plane. This leads to a reconsideration of minimum phase systems of high relative degree (pole-zero excess ≥ 2) as being only *dominantly* minimum phase since small perturbations may result in right half plane zeros of large magnitude.

Our investigations in this direction were motivated in part by a study in [HSM88, HSM89] (see Chapter 4) of the linearization by nonlinear state feedback of a class of slightly non-minimum phase nonlinear systems encountered in the flight control of VTOL aircraft. Indeed, in this work, the *true* system had a small *regular* perturbation in its equations caused by the way moments were generated on the aircraft. This, in turn, manifested itself as fast time scale zero dynamics, with a saddle type equilibrium point, making the system slighty *non-minimum phase*. Though this example was a multi-input multi-output (MIMO) system, we restrict ourselves to the SISO case here and postpone the considerably more technical MIMO case.

This chapter deals with both linear and nonlinear systems—definitions of zero dynamics for nonlinear systems were introduced in [BI84] and made more precise in [IM89, Isi87]. The qualitative theory is similar for both classes of systems, though the techniques are rather different. The techniques also draw heavily from the literature on singular perturbation [KKO86,SOK84].

An outline of the chapter is as follows: In Section 2.1, we develop explicit formulas for the locations of the large magnitude zeros of linear systems under perturbation. In Section 2.2, we repeat this development for the nonlinear case. Section 2.3 collects some concluding remarks.

2.1 Linear Systems

In this section we will consider the effects of regular perturbations of b_0, c_0, A_0 on the zeros of a SISO linear system of the form

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u \\ y &= c_0 x. \end{aligned} \tag{2.1}$$

We will assume that the system (2.1) is *minimal* and has *relative degree* (excess of poles over finite zeros) γ_0, i.e.,

$$\begin{gathered} c_0 b_0 = c_0 A_0 b_0 = \cdots = c_0 A_0^{\gamma_0 - 2} b_0 = 0 \\ c_0 A_0^{\gamma_0 - 1} b_0 \neq 0. \end{gathered} \tag{2.2}$$

To exhibit its $(n - \gamma_0)$ finite zeros, it is useful to use a normal form which will also prove convenient in the nonlinear case. To this end, we define

$$\begin{bmatrix} \xi \\ \hline \eta \end{bmatrix} := \begin{bmatrix} c_0 \\ c_0 A_0 \\ \vdots \\ c_0 A_0^{\gamma_0 - 1} \\ \hline H \end{bmatrix} x =: \begin{bmatrix} \Xi \\ \hline H \end{bmatrix} x \tag{2.3}$$

with $\xi \in \mathbf{R}^{\gamma_0}$, $\eta \in \mathbf{R}^{n-\gamma_0}$ such that $(\xi^T, \eta^T)^T$ is a change of coordinates on the original state space, i.e., $\begin{bmatrix} \Xi \\ H \end{bmatrix} \in \mathbf{R}^{n \times n}$ is nonsingular. Further, from the definition of relative degree in

(2.2), we may choose H so that $Hb_0 = 0$. The linear system (2.1) can be rewritten in (ξ, η) coordinates as

$$\left[\begin{array}{c} \dot{\xi} \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & a_1^T & & \end{matrix} & \begin{matrix} 0 \\ \\ a_2^T \end{matrix} \\ \hline P & Q \end{array}\right] \left[\begin{array}{c} \xi \\ \hline \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ c_0 A_0^{\gamma_0 - 1} b_0 \\ \hline 0 \end{array}\right] u. \tag{2.4}$$

Here, $a_1 \in \mathbb{R}^{\gamma_0}$, $a_2 \in \mathbb{R}^{n-\gamma_0}$, $P \in \mathbb{R}^{n-\gamma_0 \times \gamma_0}$, $Q \in \mathbb{R}^{n-\gamma_0 \times n-\gamma_0}$, and

$$\begin{aligned} c_0 A_0^{\gamma_0 - 1} x &= a_1^T \xi + a_2^T \eta \\ H A_0 x &= P\xi + Q\eta. \end{aligned} \tag{2.5}$$

The form (2.4) is referred to as a *normal form* and it is well known that the $(n - \gamma_0)$ eigenvalues of Q are the zeros of the system (2.1). It is useful to note that the state feedback law

$$u = -\frac{1}{c_0 A_0^{\gamma_0 - 1} b_0} \left(a_1^T \xi + a_2^T \eta \right) \tag{2.6}$$

renders the η variables unobservable and furthermore, if $\xi(0) = 0$, it *zeros the output* for all t, that is,

$$y(t) = \dot{y}(t) = \cdots = y^{(\gamma_0 - 1)}(t) \equiv 0. \tag{2.7}$$

The subspace

$$\begin{aligned} \mathcal{V}_0 &= \left\{ x : c_0 x = c_0 A_0 x = \cdots = c_0 A_0^{\gamma_0 - 1} x = 0 \right\} \\ &= \{(0, \eta) : \eta \in \mathbb{R}^{n-\gamma_0}\} \end{aligned} \tag{2.8}$$

(rendered invariant by (2.6)) is referred to as the *zero dynamics subspace* of (2.1).

2.1.1 Perturbations in b

To begin with, we study the effects of perturbations in the input channel alone, i.e., systems of the form

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u + \epsilon b_1 u \\ y &= c_0 x. \end{aligned} \tag{2.9}$$

Note that (2.9) remains minimal for ϵ small. Let the relative degree of the *perturbation system* (c_0, A_0, b_1) be γ_1. The case of greatest interest* is $\gamma_1 < \gamma_0$, which, by the definition of γ_1, implies that

$$\begin{gathered} c_0 b_1 = c_0 A_0 b_1 = \cdots = c_0 A_0^{\gamma_1 - 2} b_1 = 0 \\ c_0 A_0^{\gamma_1 - 1} b_1 \neq 0 \end{gathered} \tag{2.10}$$

and that the relative degree of the *perturbed system* (2.9) will be γ_1. It is easy to obtain the form of (2.9) in the (ξ, η) coordinates defined in (2.3). It will, however, be convenient for us to decompose the ξ of (2.3) as $\xi^T = (\xi_1^T, \xi_2^T)$ with

$$\begin{aligned} \xi_1 &:= \begin{bmatrix} c_0 \\ c_0 A_0 \\ \vdots \\ c_0 A_0^{\gamma_1 - 1} \end{bmatrix} x \in \mathbf{R}^{\gamma_1}, \\ \xi_2 &:= \begin{bmatrix} c_0 A_0^{\gamma_1} \\ \vdots \\ c_0 A_0^{\gamma_0 - 1} \end{bmatrix} x \in \mathbf{R}^{\gamma_0 - \gamma_1}. \end{aligned} \tag{2.11}$$

*See the remarks after Theorem 2.1.

Note that if $\gamma_0 = \gamma_1$, ξ_2 fails to exist. Also note that we have already used the assumption that $\gamma_1 \leq \gamma_0$ in the definition (2.11). The system (2.9) is now written as

$$\left[\begin{array}{c} \dot{\xi}_1 \\ \hline \dot{\xi}_2 \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \begin{matrix} & & \\ & 0 & \\ & & \\ 1 & & \end{matrix} & 0 \\ \hline \begin{matrix} \\ 0 \\ \\ {}^1a_1^T \end{matrix} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & {}^2a_1^T & & \end{matrix} & \begin{matrix} \\ 0 \\ \\ a_2^T \end{matrix} \\ \hline {}^1P & {}^2P & Q \end{array}\right] \left[\begin{array}{c} \xi_1 \\ \hline \xi_2 \\ \hline \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \epsilon c_0 A_0^{\gamma_1 - 1} b_1 \\ \hline \epsilon c_0 A_0^{\gamma_1} b_1 \\ \vdots \\ \epsilon c_0 A_0^{\gamma_0 - 2} b_1 \\ c_0 A_0^{\gamma_0 - 1}(b_0 + \epsilon b_1) \\ \hline \epsilon H b_1 \end{array}\right] u. \tag{2.12}$$

where ${}^1a_1 \in \mathbf{R}^{\gamma_1}$, ${}^2a_1 \in \mathbf{R}^{\gamma_0 - \gamma_1}$, ${}^1P \in \mathbf{R}^{(n-\gamma_0)\times\gamma_1}$, ${}^2P \in \mathbf{R}^{(n-\gamma_0)\times(\gamma_0-\gamma_1)}$, and

$$a_1^T = ({}^1a_1^T \; {}^2a_1^T)^T, \qquad P = [{}^1P \; {}^2P]. \tag{2.13}$$

Note that, in (2.12), the perturbations appear as input terms in the equations for $\dot{\xi}_{1\gamma_1}$, $\dot{\xi}_{21}$, $\ldots$, $\dot{\xi}_{2,\gamma_0-\gamma_1}$, and $\dot{\eta}$. To find the zero dynamics of (2.12), we use the state feedback

$$u = -\frac{1}{\epsilon c_0 A_0^{\gamma_1 - 1} b_1} \xi_{21} = -\frac{1}{\epsilon c_0 A_0^{\gamma_1 - 1} b_1} c_0 A_0^{\gamma_1} x \tag{2.14}$$

to *zero the output* making the subspace

$$\begin{aligned} \mathcal{V}_1 &= \left\{ x : c_0 x = c_0 A_0 x = \cdots = c_0 A_0^{\gamma_1 - 1} x = 0 \right\} \\ &= \{ (0, \xi_2, \eta) : \xi_2 \in \mathbf{R}^{\gamma_0 - \gamma_1}, \eta \in \mathbf{R}^{n - \gamma_0} \} \end{aligned} \tag{2.15}$$

invariant and the state variables $(\xi_2, \eta) \in \mathbf{R}^{n-\gamma_1}$ unobservable. The dynamics of the (ξ_2, η) variables on $\mathcal{V}_1$ are given by

$$\left[\begin{array}{c} \dot{\xi}_2 \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{cccccc|c} -\frac{c_0 A_0^{\gamma_1} b_1}{c_0 A_0^{\gamma_1 - 1} b_1} & 1 & & & & & \\ -\frac{c_0 A_0^{\gamma_1 + 1} b_1}{c_0 A_0^{\gamma_1 - 1} b_1} & 0 & 1 & & & & \\ \vdots & \vdots & \ddots & \ddots & & & 0 \\ -\frac{c_0 A_0^{\gamma_0 - 2} b_1}{c_0 A_0^{\gamma_1 - 1} b_1} & 0 & \cdots & 0 & 1 & & \\ -\frac{c_0 A_0^{\gamma_0 - 1} b_1}{c_0 A_0^{\gamma_1 - 1} b_1} - \frac{1}{\epsilon}\frac{c_0 A_0^{\gamma_0 - 1} b_0}{c_0 A_0^{\gamma_1 - 1} b_1} + {}^2a_{11} & {}^2a_{12} & \cdots & \cdots & {}^2a_{1,\gamma_0-\gamma_1} & & a_2^T \\ \hline & & {}^2\tilde{P} & & & & Q \end{array}\right] \left[\begin{array}{c} \xi_2 \\ \hline \eta \end{array}\right] \tag{2.16}$$

where

$${}^2\tilde{P} = \left[\begin{array}{cc} -\frac{1}{c_0 A_0^{\gamma_1 - 1} b_1} H b_1 & 0 \end{array}\right] + {}^2P. \tag{2.17}$$

Thus, we see that the system (2.9) has $(n - \gamma_1) \geq (n - \gamma_0)$ zeros. To establish the structure of these zeros, we note that the ϵ-dependent term in (2.16) corresponding to $\dot{\xi}_{2,\gamma_0-\gamma_1}$ is of order $1/\epsilon$ and thus is certainly not a *regular* perturbation. This *singular* perturbation term is due to the *high-gain* form of the feedback control (2.14). The rich literature on high-gain systems (see, in particular, [Mar88,San83]) is therefore applicable to the study of perturbed zero dynamics.

Theorem 2.1 *The linear system (2.9) has $(n - \gamma_1)$ zeros which, according to their asymptotic behavior as $\epsilon \to 0$, belong to two groups:*

- *The $(\gamma_0 - \gamma_1)$ large zeros tend to ∞ asymptotically as*

$$\left(-\frac{1}{\epsilon}\frac{c_0 A_0^{\gamma_0 - 1} b_0}{c_0 A_0^{\gamma_1 - 1} b_1}\right)^{\frac{1}{\gamma_0 - \gamma_1}}. \tag{2.18}$$

- *The remaining $(n - \gamma_0)$ zeros tend to the zeros of the* unperturbed *system (2.1).*

Proof: To facilitate an asymptotic calculation of the eigenvalues of the matrix in (2.16), we transform the system into a standard singular perturbation form. To this end, we rescale ξ_2 as follows

$$\tilde{\xi}_{21} = \xi_{21},\ \tilde{\xi}_{22} = \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\xi_{22},\ \cdots,\ \tilde{\xi}_{2,\gamma_0-\gamma_1} = \epsilon^{\frac{\gamma_0-\gamma_1-1}{\gamma_0-\gamma_1}}\xi_{2,\gamma_0-\gamma_1} \tag{2.19}$$

and rewrite (2.16) as

$$\begin{aligned} \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\dot{\tilde{\xi}}_2 &= W\tilde{\xi}_2 + \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\Gamma\tilde{\xi}_2 + O\left(\epsilon^{\frac{2}{\gamma_0-\gamma_1}}\right) \\ \dot{\eta} &= {}^2\tilde{P}\tilde{\xi}_2 + Q\eta \end{aligned} \tag{2.20}$$

where

$$W := \begin{bmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ -\frac{c_0A_0^{\gamma_0-1}b_0}{c_0A_0^{\gamma_1-1}b_1} & 0 & \cdots & 0 \end{bmatrix} \text{ and } \Gamma := \begin{bmatrix} \left(-\frac{c_0A_0^{\gamma_1}b_1}{c_0A_0^{\gamma_1-1}b_1}\right) & 1 & & \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & {}^2a_{1,\gamma_0-\gamma_1} \end{bmatrix}. \tag{2.21}$$

To see that (2.20) is in the standard two-time-scale form of [KKO86], note that its right hand side is regularly perturbed by $\epsilon^{\frac{1}{\gamma_0-\gamma_1}}\Gamma$, while the matrix of the unperturbed part is block lower triangular. By inspection, the upper diagonal block is nonsingular as required for a standard form. It follows that the eigenvalues of (2.16) are asymptotically

$$\epsilon^{-\frac{1}{\gamma_0-\gamma_1}}\cdot\lambda(W)\cup\lambda(Q). \tag{2.22}$$

Clearly, the eigenvalues of Q are the $(n-\gamma_0)$ zeros of the unperturbed system. It is easy to see [Wil65, chapter 2] that the remaining $(\gamma_0-\gamma_1)$ eigenvalues are the $(\gamma_0-\gamma_1)$th roots of $(-\frac{c_0A_0^{\gamma_0-1}b_0}{c_0A_0^{\gamma_1-1}b_1})$ multiplied by $\epsilon^{-\frac{1}{\gamma_0-\gamma_1}}$, that is

$$\left(-\frac{1}{\epsilon}\frac{c_0A_0^{\gamma_0-1}b_0}{c_0A_0^{\gamma_1-1}b_1}\right)^{\frac{1}{\gamma_0-\gamma_1}}. \tag{2.23}$$

This is the asymptotic expression of the $(\gamma_0-\gamma_1)$ large zeros of the perturbed system which tend to ∞ as $\epsilon\to 0$. The remaining $(n-\gamma_0)$ tend as a set to the eigenvalues of Q (zeros of the unperturbed system). □

Remarks

- Theorem 2.1 states that if the relative degree γ_1 of the perturbation ϵb_1 is less than that of the original b_0, then $(\gamma_0 - \gamma_1)$ *of the original system's infinite zeros become finite* according to the asymptotic formula (2.23).

- We leave it to the reader to verify (by direct calculation) that, if the relative degree of the perturbation b_1 is $\gamma_1 \geq \gamma_0$, then both the perturbed and unperturbed systems have the same number of zeros and the zero locations are a smooth function of ϵ.

- Theorem 2.1 has important implications for the concepts of non-minimum phase and minimum phase. In particular, if $\gamma_0 - \gamma_1 > 2$, it follows that arbitrarily small perturbations of the form (2.9) result in non-minimum phase systems since, for $\gamma_0 - \gamma_1 > 2$, at least one of the roots of (2.18) is in the right half plane. Of course, for ϵ small enough, the non-minimum phase zeros are far off in the right half plane prompting us to think of the perturbed system as being *slightly non-minimum phase*. Nevertheless, numerous system theory results based on a *strict minimum phase assumption* should be reexamined in this light.

- Even when $\gamma_0 - \gamma_1 = 1$, the relative signs of the quantities in (2.18) *may result in right half plane zeros*. In particular, some zeros *will* be in the right half plane either when ϵ is positive or when ϵ is negative.

- Note that, if a perturbation resulting in direct feed-through ($y = c_0 x + \epsilon d_1 u$) were allowed, then γ_1 would be 0 and the asymptotes would coincide with the familiar *root locus* asymptotes for the closed loop poles that go to ∞ under increasing output feedback.

- This result is reminiscent of results in high gain feedback giving the asymptotic location of the closed loop poles as the gain $1/\epsilon$ goes to ∞. Indeed the proof techniques of [YKU77], [San83], for example, can be used to give an alternative and elegant proof of Theorem 2.1.

□

2.1.2 Perturbations in c

Consider now the effects of perturbations of (2.1) in the output channel, i.e.,

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u \\ y &= c_0 x + \epsilon c_1 x. \end{aligned} \tag{2.24}$$

As before, if γ_1 represents the relative degree of the perturbation system (c_1, A_0, b_0), it follows from considerations dual to those given above that if $\gamma_1 < \gamma_0$ the system (2.24) has $(\gamma_0 - \gamma_1)$ extra zeros given asymptotically by the $(\gamma_0 - \gamma_1)$ roots of

$$\left(-\frac{1}{\epsilon}\frac{c_0 A_0^{\gamma_0 - 1} b_0}{c_1 A_0^{\gamma_1 - 1} b_0}\right)^{\frac{1}{\gamma_0 - \gamma_1}}. \tag{2.25}$$

2.1.3 Perturbations in A

The qualitative effects of perturbations in A_0 are similar in that some of the $(n - \gamma_0)$ zeros at ∞ may become finite; the details of the proof are more subtle.

Consider

$$\begin{aligned} \dot{x} &= (A_0 + \epsilon A_1) x + b_0 u \\ y &= c_0 x. \end{aligned} \tag{2.26}$$

Further, let the *perturbed* system (2.26) have relative degree γ_1 ($\gamma_1 < \gamma_0$, as before, is the case of interest), i.e.,

$$\begin{aligned} c_0 b_0 = c_0 (A_0 + \epsilon A_1) b_0 = \cdots = c_0 (A_0 + \epsilon A_1)^{\gamma_1 - 2} b_0 = 0 \qquad & \forall \epsilon \\ c_0 (A_0 + \epsilon A_1)^{\gamma_1 - 1} b_0 \neq 0 \qquad & \text{for } \epsilon \text{ small.} \end{aligned} \tag{2.27}$$

From (2.27), it is easy to see that the relative degree γ_1 depends on A_1 in a complicated fashion. For the purpose of this paper, we will restrict our attention to the class of perturbations A_1 satisfying assumptions (2.29) and (2.30) below. Define the subpaces

$$\Delta_i := \text{ span } \left\{ b_0, A_0 b_0, \ldots, A_0^i b_0 \right\}. \tag{2.28}$$

Assume that

$$A_1 \Delta_i \subset \Delta_i \subset \text{ Ker } c_0 \qquad \text{for } i = 1, \ldots, \gamma_1 - 3 \tag{2.29}$$

and

$$A_1 \Delta_{\gamma_1 - 2} \not\subset \Delta_{\gamma_1 - 2} \text{ and } A_1 \Delta_{\gamma_1 - 2} \not\subset \text{ Ker } c_0. \tag{2.30}$$

If $\gamma_1 < 3$, assumption (2.29) is vacuous and if $\gamma_1 < 2$, then (2.30) is vacuous. We conjecture that, if the assumptions (2.29), (2.30) are violated, then the fast zero dynamics occur at a *multiplicity* of time scales. The assumptions (2.29), (2.30) guarantee that

$$\begin{aligned} c_0(A_0 + \epsilon A_1)^{\gamma_1 - 1} b_0 &= \epsilon c_0 A_1 A_0^{\gamma_1 - 2} b_0 + O(\epsilon^2) \\ &=: \epsilon \alpha_0(\epsilon). \end{aligned} \tag{2.31}$$

Note that $\alpha_0(0) = c_0 A_1 A_0^{\gamma_1 - 2} b_0$.

The normal form for the system (2.26) is not easily obtained in the (ξ, η) coordinates of (2.3); consequently, we define

$$\left[\begin{array}{c} \xi^\epsilon \\ \\ \\ \hline \eta \end{array}\right] := \left[\begin{array}{c} c_0 \\ c_0 (A_0 + \epsilon A_1) \\ \vdots \\ c_0 (A_0 + \epsilon A_1)^{\gamma_0 - 1} \\ \hline H \end{array}\right] x. \tag{2.32}$$

The matrix in (2.32) is a perturbation of that in (2.3) and is therefore nonsingular for *small* ϵ. We partition ξ^ϵ into

$$\begin{aligned} \xi_1^\epsilon &:= \left[\begin{array}{c} c_0 \\ c_0 (A_0 + \epsilon A_1) \\ \vdots \\ c_0 (A_0 + \epsilon A_1)^{\gamma_1 - 1} \end{array}\right] x \in \mathbf{R}^{\gamma_1}, \\ \xi_2^\epsilon &:= \left[\begin{array}{c} c_0 (A_0 + \epsilon A_1)^{\gamma_1} \\ \vdots \\ c_0 (A_0 + \epsilon A_1)^{\gamma_0 - 1} \end{array}\right] x \in \mathbf{R}^{\gamma_0 - \gamma_1}. \end{aligned} \tag{2.33}$$

In these coordinates we have

$$
\left[\begin{array}{c}
\dot{\xi}_1^\epsilon \\ \hline
\dot{\xi}_2^\epsilon \\ \hline
\dot{\eta}
\end{array}\right]
=
\left[\begin{array}{c|c|c}
\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \begin{matrix} \\ 0 \\ \\ 1 \end{matrix} & 0 \\ \hline
0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} & 0 \\
{}^1a_1^T(\epsilon) & {}^2a_1^T(\epsilon) & a_2^T(\epsilon) \\ \hline
{}^1P(\epsilon) & {}^2P(\epsilon) & Q(\epsilon)
\end{array}\right]
\left[\begin{array}{c}
\xi_1^\epsilon \\ \hline
\xi_2^\epsilon \\ \hline
\eta
\end{array}\right]
+
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ \epsilon\alpha_0(\epsilon) \\ \hline
\epsilon\alpha_1(\epsilon) \\ \vdots \\ \epsilon\alpha_{\gamma_0-\gamma_1-1}(\epsilon) \\ \epsilon\alpha_{\gamma_0-\gamma_1}(\epsilon) + c_0A_0^{\gamma_0-1}b_0 \\ \hline
0
\end{array}\right] u. \tag{2.34}
$$

In (2.34) the vectors, ${}^1a_1(\epsilon)$, ${}^2a_1(\epsilon)$, $a_2(\epsilon)$, and matrices, ${}^1P(\epsilon)$, ${}^2P(\epsilon)$, $Q(\epsilon)$, are all perturbations of the corresponding entries in (2.12) and the α_i $(i \geq 1)$ are smooth functions of ϵ. Note that, with the exception of $c_0A_0^{\gamma_0-1}b_0$, all the input coefficients are multiplied by ϵ. We now leave it to the interested reader to verify that the unbounded (as functions of ϵ) zeros have the asymptotic form of the $(\gamma_0 - \gamma_1)$ roots of

$$
\left(-\frac{1}{\epsilon}\frac{c_0A_0^{\gamma_0-1}b_0}{\alpha_0(0)}\right)^{\frac{1}{\gamma_0-\gamma_1}}. \tag{2.35}
$$

Equation (2.35) is very similar to (2.18) except that $\epsilon c_0 A_0^{\gamma_1-1}b_1$ is replaced by $\epsilon\alpha_0(0)$ $(= c_0A_1A_0^{\gamma_1-2}b_0)$, the control coefficient for $\dot{\xi}_{1\gamma_1}^\epsilon$.

2.1.4 Simultaneous Perturbations in A, b, and c

We do not discuss asymptotic formulas in this case. The details are cumbersome since the relative degree of the perturbed system depends on the perturbations A_1, b_1, c_1 in a complicated fashion. In particular, in the absence of assumptions like (2.29) and (2.30) a *multiplicity* of time scales may occur. Nonetheless, if we can assert that γ_1 is the first integer at which

$$
(c_0 + \epsilon c_1)(A_0 + \epsilon A_1)^{\gamma_1-1}(b_0 + \epsilon b_1) = \epsilon\alpha_0(\epsilon) \tag{2.36}
$$

where $\alpha_0(\epsilon)$ is of $O(1)$ and not $o(1)$, then the perturbed high frequency zeros will be given by (2.35).

2.2 Nonlinear Systems

We briefly review, following the lucid presentation of [Isi87], the definition of zero dynamics for SISO nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u \\ y &= h_0(x) \end{aligned} \tag{2.37}$$

where f_0 and g_0 are smooth vector fields on $\mathbf{R}^n$ and $h : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. Let x_0 be an equilibrium point of the undriven system (i.e., $f_0(x_0) = 0$) and let $U \subset \mathbf{R}^n$ be an open neighborhood of x_0. We will assume the the system has *relative degree* γ_0 *at* x_0, i.e.,

$$\begin{gathered} L_{g_0} h_0(x) = L_{g_0} L_{f_0} h_0(x) = \cdots = L_{g_0} L_{f_0}^{\gamma_0 - 2} h_0(x) = 0 \\ L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0(x) \neq 0 \end{gathered} \tag{2.38}$$

for all $x \in U$. Note that we implicitly assume that the system *has* a relative degree! We will further assume (w.l.o.g.) that $h_0(x_0) = 0$.

To find a convenient normal form for the nonlinear system (2.37), we begin by defining

$$\begin{aligned} \xi &:= \begin{bmatrix} h_0(x) \\ L_{f_0} h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_0 - 1} h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0} \\ \eta &:= \quad \eta(x) \qquad \in \mathbf{R}^{n-\gamma_0} \end{aligned} \tag{2.39}$$

such that (ξ, η) is a diffeomorphism of x in U. From the definition of relative degree, we may choose $\eta(x)$ so that

$$L_{g_0} \eta_i(x) \equiv 0 \quad i = 1, \ldots, n - \gamma_0. \tag{2.40}$$

The *normal form* of the nonlinear system (2.37) is then written (using (ξ,η) coordinates) as

$$\left[\begin{array}{c} \dot{\xi} \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ & \ddots & \ddots & & 0 \\ & & \ddots & 1 & \\ & & & 0 & \\ \hline & & 0 & & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ \hline \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ b(\xi,\eta) \\ \hline q(\xi,\eta) \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ a(\xi,\eta) \\ \hline 0 \end{array}\right] u. \tag{2.41}$$

Here, $a(\xi,\eta) = L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0(x)$ and $b(\xi,\eta) = L_{f_0}^{\gamma_0} h_0(x)$ in the (ξ,η) coordinates and $q_i(\xi,\eta) = L_{f_0}\eta_i(x)$ in the (ξ,η) coordinates.

The *zero dynamics* of a nonlinear system are the dynamics of (2.41) consistent with the constraint that the output is held identically zero, i.e., $y(t) \equiv 0$. From the normal form (2.41), it is clear that the nonlinear state feedback

$$u = -\frac{1}{a(\xi,\eta)} b(\xi,\eta) = -\frac{1}{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0(x)} L_{f_0}^{\gamma_0} h_0(x) \tag{2.42}$$

results in $y(t) \equiv 0$. Furthermore, the control law (2.42) renders the manifold

$$\begin{aligned} \mathcal{M}_0 &= \left\{ x \in U : h_0(x) = L_{f_0} h_0(x) = \cdots = L_{f_0}^{\gamma_0 - 1} h_0(x) = 0 \right\} \\ &= \{(0,\eta) : \eta \in \eta(U)\} \end{aligned} \tag{2.43}$$

invariant and makes the η variables unobservable. Since $y(t) \equiv 0$ is locally equivalent to $\xi \equiv 0$, we find that the zero dynamics of (2.41) (hence (2.37)) evolve on the *zero dynamics manifold* $\mathcal{M}_0$ and are described by

$$\dot{\eta} = q(0,\eta) \tag{2.44}$$

in a neighborhood of $\eta = 0$. We refer to the dimension of the manifold $\mathcal{M}_0$, namely $n - \gamma_0$, as the *dimension of the zero dynamics system.* Let η_0 be the η component of x_0 (i.e., $x_0 \mapsto (0,\eta_0)$ under the change of coordinates). Then η_0 is an equilibrium point of (2.44). Further, we may associate with η_0 the (Jacobian) linearization of $q(0,\eta)$ at $\eta = \eta_0$, i.e.,

$$\frac{\partial q(0,\eta_0)}{\partial \eta} \tag{2.45}$$

The stability of (2.44) is determined by the eigenvalues of this matrix, provided that it is hyperbolic. Otherwise, a study of the full *zero dynamics system* of (2.44) is necessary.

We will now study the effects of perturbations on the normal form (2.41).

2.2.1 Perturbations in g

Consider, as in the previous section, perturbations in the input channel alone, i.e.,

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u + \epsilon g_1(x)u \\ y &= h_0(x). \end{aligned} \tag{2.46}$$

We will assume that the *perturbation system* (h_0, f_0, g_1) has a strict relative degree of γ_1, i.e.,

$$\begin{gathered} L_{g_1}h_0(x) = L_{g_1}L_{f_0}h_0(x) = \cdots = L_{g_1}L_{f_0}^{\gamma_1-2}h_0(x) \equiv 0 \quad \forall x \in U \\ L_{g_1}L_{f_0}^{\gamma_1-1}h_0(x) \neq 0. \end{gathered} \tag{2.47}$$

As before, the case of greatest interest is when $\gamma_1 < \gamma_0$. Following the previous development, we partition ξ as

$$\begin{aligned} \xi_1 &= \begin{bmatrix} h_0(x) \\ L_{f_0}h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_1-1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_1}, \\ \xi_2 &= \begin{bmatrix} L_{f_0}^{\gamma_1}h_0(x) \\ \vdots \\ L_{f_0}^{\gamma_0-1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0-\gamma_1}. \end{aligned} \tag{2.48}$$

The perturbed system (2.46) expressed in (ξ_1, ξ_2, η) coordinates looks like

$$
\left[\begin{array}{c} \dot{\xi}_1 \\ \hline \dot{\xi}_2 \\ \hline \dot{\eta} \end{array}\right]
=
\left[\begin{array}{c|c|c}
\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \begin{matrix} & \\ & 0 \\ & \\ 1 & \end{matrix} & 0 \\ \hline
0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & 0 \\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} \xi_1 \\ \hline \xi_2 \\ \hline \eta \end{array}\right]
+
\left[\begin{array}{c} 0 \\ \vdots \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ b(\xi, \eta) \\ \hline q(\xi, \eta) \end{array}\right]
+
\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \epsilon L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0 \\ \hline \epsilon L_{g_1} L_{f_0}^{\gamma_1} h_0 \\ \vdots \\ \epsilon L_{g_1} L_{f_0}^{\gamma_0 - 2} h_0 \\ a(\xi, \eta) + \epsilon L_{g_1} L_{f_0}^{\gamma_0 - 1} h_0 \\ \hline \epsilon L_{g_1} \eta_1 \\ \vdots \\ \epsilon L_{g_1} \eta_{n-\gamma_0} \end{array}\right] u. \tag{2.49}
$$

Note that, in (2.49), we have deliberately chosen not to write the $L_{g_1} L_{f_0}^{i} h_0$ terms in the (ξ, η) coordinates.

Using the nonlinear state feedback

$$
u = -\frac{1}{\epsilon L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} \xi_{21} = -\frac{1}{\epsilon L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0(x)} L_{f_0}^{\gamma_1} h_0(x) \tag{2.50}
$$

to zero the output, we make the manifold

$$
\begin{aligned}
\mathcal{M}_1 &= \left\{x \in U : h_0(x) = L_{f_0} h_0(x) = \cdots = L_{f_0}^{\gamma_1 - 1} h_0(x) = 0\right\} \\
&= \{(0, \xi_2, \eta) : \xi_2 \in \xi_2(U), \eta \in \eta(U)\}
\end{aligned}
\tag{2.51}
$$

invariant and the $(\xi_2, \eta) \in \mathbf{R}^{n-\gamma_1}$ variables unobservable. Thus, the zero dynamics of (2.46) are precisely the dynamics of (ξ_2, η) on $\mathcal{M}_1$ given by

$$
\left[\begin{array}{c} \dot{\xi}_2 \\ \hline \dot{\eta} \end{array}\right]
=
\left[\begin{array}{ccccc|c}
-\frac{L_{g_1} L_{f_0}^{\gamma_1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & 1 & & & & \\
-\frac{L_{g_1} L_{f_0}^{\gamma_1 + 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & 0 & 1 & & & \\
\vdots & \vdots & \ddots & \ddots & & 0 \\
-\frac{L_{g_1} L_{f_0}^{\gamma_0 - 2} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & 0 & \cdots & 0 & 1 & \\
-\frac{L_{g_1} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} - \frac{1}{\epsilon} \frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & 0 & \cdots & \cdots & 0 & \\
\hline
-\frac{L_{g_1} \eta_1}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & & & & & \\
\vdots & & & 0 & & 0 \\
-\frac{L_{g_1} \eta_{n-\gamma_0}}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & & & & &
\end{array}\right]
\left[\begin{array}{c} \xi_2 \\ \hline \eta \end{array}\right]
+
\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ b(\xi, \eta) \\ \hline q(\xi, \eta) \end{array}\right].
\tag{2.52}
$$

Before we state a nonlinear counterpart to Theorem 2.1, we apply the scaling specified by equation (2.19) to (2.52), namely

$$
\tilde{\xi}_{21} = \xi_{21},\ \tilde{\xi}_{22} = \epsilon^{\frac{1}{\gamma_0 - \gamma_1}} \xi_{22},\ \cdots,\ \tilde{\xi}_{2,\gamma_0 - \gamma_1} = \epsilon^{\frac{\gamma_0 - \gamma_1 - 1}{\gamma_0 - \gamma_1}} \xi_{2,\gamma_0 - \gamma_1}.
\tag{2.53}
$$

Note that the scaling of $\tilde{\xi}$ is singular at $\epsilon = 0$ as is frequently the case in such asymptotic calculations. The transformation of (2.53) renders terms in the first column of the matrix in (2.52) potentially unbounded as ϵ tends to zero. Now rewrite (2.52) as

$$\left[\begin{array}{c} \epsilon^{\frac{1}{\gamma_0-\gamma_1}}\dot{\tilde{\xi}}_2 \\ \hline \dot{\eta} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ \vdots & \ddots & \ddots & & 0 \\ 0 & & \ddots & 1 & \\ -\frac{L_{g_0}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & 0 & \cdots & 0 & \\ \hline -\frac{L_{g_1}\eta_1}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & & \\ \vdots & & 0 & & 0 \\ -\frac{L_{g_1}\eta_{n-\gamma_0}}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} & & & & \end{array}\right] \left[\begin{array}{c} \tilde{\xi}_2 \\ \hline \eta \end{array}\right] + \epsilon^{\frac{1}{\gamma_0-\gamma_1}} \left[\begin{array}{c} k(\tilde{\xi}_2,\eta,\epsilon) \\ \\ \end{array}\right]. \tag{2.54}$$

In (2.54), we will *assume* that $k(\tilde{\xi}_2,\eta,\epsilon) \in \mathbf{R}^{n-\gamma_1}$ is a smooth function of $\tilde{\xi}$, η, and ϵ. Thus, for example, we may require that for some positive $K < \infty$ that

$$\begin{aligned} \left|\frac{L_{g_1}L_{f_0}^{\gamma_1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\right| &\leq K\,|\xi_{21}|\,, \\ \left|\frac{L_{g_1}L_{f_0}^{\gamma_1+1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0}\right| &\leq K\,|\xi_{21}| + K\,|\xi_{22}|\,, \\ &\text{etc.} \end{aligned} \tag{2.55}$$

Further, we will also *assume* that

$$\beta(\xi_1,\tilde{\xi}_2,\eta) := -\lim_{\epsilon\to 0}\frac{L_{g_0}L_{f_0}^{\gamma_0-1}h_0}{L_{g_1}L_{f_0}^{\gamma_1-1}h_0} \tag{2.56}$$

is a well defined smooth function and is nonzero when $\xi_1 = 0$, $\tilde{\xi}_2 = 0$, and $\eta = \eta_0$. With these assumptions, which are unique to the nonlinear case, it may be verified that the second term in (2.54) is multiplied by $\epsilon^{\frac{1}{\gamma_0-\gamma_1}}$ in analogy with the second term on the right hand side of (2.20). Equation (2.54) shows the two time scale nature of the zero dynamics. We mention in passing that if conditions such as (2.55), (2.56) do not hold then there may be more than two time scales in the zero dynamics. These assumptions have appeared in other forms as well in the literature. For instance, the conditions (2.55) guarantee the absence of *peaking response* caused by terms in $k(\tilde{\xi}_2,\eta,\epsilon)$. The condition (2.56) guarantees that the $\tilde{\xi}_2$ variables are in fact the fast variables and the slow manifold of the zero dynamics of (2.54) is the subspace corresponding to the η variables (up to zeroth order in ϵ). The fast dynamics of the $\tilde{\xi}_2$ variables are determined by studying the scaled variables $\tilde{\xi}_2$ in the fast time, i.e., set $\tau = t/\epsilon^{1/\gamma_0-\gamma_1}$ and then set $\epsilon = 0$ to get

$$\begin{aligned} \frac{d\tilde{\xi}_2}{d\tau} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \beta(0, \tilde{\xi}_2, \eta_0) & & & 0 \end{bmatrix} \tilde{\xi}_2 \\ \eta(\tau) &\equiv \eta_0. \end{aligned} \tag{2.57}$$

In spite of its apparent linear form, it is important to note that the right hand side of equation (2.57) is a nonlinear function of $\tilde{\xi}_2$. Clearly, $\tilde{\xi}_2 = 0$ is an equilibrium point of (2.57) and the eigenvalues of (the linearization of) (2.57) about $\tilde{\xi}_2 = 0$ determine the stability properties of the *fast zero dynamics system* if they are not all on the $j\omega$ axis. These eigenvalues are precisely the $(\gamma_0 - \gamma_1)$ different roots of the lower left term of (2.57) with $\tilde{\xi}_2 = 0$, i.e.,

$$\left(- \lim_{\epsilon \to 0} \frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} \right)^{\frac{1}{\gamma_0 - \gamma_1}} \quad \text{at } (\xi_1, \tilde{\xi}_2, \eta) = (0, 0, \eta_0). \tag{2.58}$$

In the original time scale, these eigenvalues are of order $1/\epsilon^{1/\gamma_0-\gamma_1}$. From this formula, it also follows that if $\gamma_0 - \gamma_1 > 2$ that at least one of the eigenvalues is in the open right half plane resulting in unstable fast zero dynamics.

The *remarkable new feature* of the fast zero dynamics of the *nonlinear system* is that they *vary with* η_0 *on the base slow manifold* $\mathcal{M}_0$. Collecting these observations, we have the following counterpart to Theorem 2.1.

Theorem 2.2 *The zero dynamics of the perturbed nonlinear system (2.46) are of dimension* $(n - \gamma_1)$. *Suppose that* $k(\tilde{\xi}_2, \eta, \epsilon) \in \mathbb{R}^{n-\gamma_1}$ *defined in (2.54) is a smooth function of* $\tilde{\xi}$, η, *and* ϵ *and* $\beta(\xi_1, \tilde{\xi}_2, \eta)$ *defined in (2.56) is a well defined smooth function and is nonzero at* $(\xi_1, \tilde{\xi}_2, \eta) = (0, 0, \eta_0)$. *According to their asymptotic behavior as* $\epsilon \to 0$, *the zero dynamics decouple into two subsystems:*

- *The fast zero dynamics subsystem of dimension $(\gamma_0 - \gamma_1)$ is given by*

$$\epsilon^{\frac{1}{\gamma_0-\gamma_1}} \frac{d\tilde{\xi}_2}{dt} = \left[\begin{array}{cccc} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\lim\limits_{\epsilon \to 0} \frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} & & & 0 \end{array} \right] \Bigg|_{\substack{\xi_1 = 0 \\ \eta = \eta_0}} \cdot \tilde{\xi}_2 \tag{2.59}$$

- *The slow zero dynamics of dimension $(n - \gamma_0)$ is identical to the zero dynamics of the* unperturbed *system (2.37) given by (2.52).*

Proof: The preceding observations yield the first part of the theorem. The verification that as $\epsilon \to 0$ the dynamics of η in (2.54) tend to those of $\dot{\eta} = q(0, \eta)$ follows from setting $\epsilon = 0$ and $\tilde{\xi}_2 = 0$ in (2.54). (Actually, this happens in quite a subtle fashion since some terms appear as multiples of $\tilde{\xi}_{21}$ and others appear as multiples of $\epsilon^{1/\gamma_0 - \gamma_1}$!) □

Remarks

- As in the remarks after Theorem 2.1, we leave it to the reader to verify that, if the relative degree of the perturbation g_1 (i.e., γ_1) is $\geq \gamma_0$, then the zero dynamics of the perturbed and unperturbed systems have the same dimension and qualitative properties (i.e., the perturbation in the zero dynamics is *regular*).

- The remarkable additional feature found in nonlinear systems that is not present in linear systems is that the locations of the eigenvalues of the Jacobian of the fast zero dynamics subsystem in the complex plane, i.e., $1/\epsilon^{1/\gamma_0 - \gamma_1}$ times the quantities in (2.58), vary with η_0. Of course, it is easy to show that if the equilibrium point x_0 of the original system corresponds to $(0, \eta_0)$, then the eigenvalues of the linearization of the fast zero dynamics will be given by

$$\left(-\frac{1}{\epsilon} \lim_{\epsilon \to 0} \frac{L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0}{L_{g_1} L_{f_0}^{\gamma_1 - 1} h_0} \right)^{\frac{1}{\gamma_0 - \gamma_1}} \Bigg|_{(\xi_1, \tilde{\xi}_2, \eta) = (0, 0, \eta_0)} . \tag{2.60}$$

□

2.2.2 Perturbations in h

As in Section 2.1.2, the qualitative results of Section 2.2.1 hold when the nonlinear system (2.37) is perturbed in h_0. To this end, we consider the perturbed system

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u \\ y &= h_0(x) + \epsilon h_1(x) \end{aligned} \tag{2.61}$$

with relative degree $\gamma_1 < \gamma_0$. It is no longer possible to invoke duality but one may use a new set of coordinates for the normal form given by

$$\begin{aligned} \xi^\epsilon &:= \begin{bmatrix} h_0(x) + \epsilon h_1(x) \\ L_{f_0}(h_0(x) + \epsilon h_1(x)) \\ \vdots \\ L_{f_0}^{\gamma_0 - 1}(h_0(x) + \epsilon h_1(x)) \end{bmatrix} \in \mathbf{R}^{\gamma_0} \\ \eta &:= \qquad \eta(x) \qquad \in \mathbf{R}^{n-\gamma_0} \end{aligned} \tag{2.62}$$

Note that the diffeomorphism of (2.62) is a perturbation of that in (2.39). By partitioning ξ^ϵ into ξ_1^ϵ and ξ_2^ϵ and scaling ξ_2^ϵ as above, it can be shown that the $\tilde{\xi}_2^\epsilon$ variables in the time scale $\tau = t/\epsilon^{1/\gamma_0 - \gamma_1}$ satisfy

$$\frac{d\tilde{\xi}_2^\epsilon}{d\tau} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\frac{a_0(0,\tilde{\xi}_2^\epsilon,\eta)}{\alpha_0(0,\tilde{\xi}_2^\epsilon,\eta)} & & & 0 \end{bmatrix} \tilde{\xi}_2^\epsilon \tag{2.63}$$

where $\alpha_0(\xi_1^\epsilon, \tilde{\xi}_2^\epsilon, \eta)$ and $a_0(\xi_1^\epsilon, \tilde{\xi}_2^\epsilon, \eta)$ are $L_{g_0} L_{f_0}^{\gamma_1 - 1} h_1$ and $L_{g_0} L_{f_0}^{\gamma_0 - 1} h_0$ in $(\xi_1^\epsilon, \tilde{\xi}_2^\epsilon, \eta)$ coordinates. As before, we will also need to *assume* that the limit as $\epsilon \to 0$ of the quantity in the lower left hand corner of (2.63) exists and is nonzero at $\tilde{\xi}_2 = 0$, $\eta = \eta_0$. Under these conditions, the results of Theorem 2.2 will hold with (2.63) replacing (2.59).

2.2.3 Perturbations in f

The situation here is delicate and analogous to that in Section 2.1.3. Consider the *perturbed* system

$$\begin{aligned} \dot{x} &= f_0(x) + \epsilon f_1(x) + g_0(x)u \\ y &= h_0(x) \end{aligned} \tag{2.64}$$

and assume that it has relative degree $\gamma_1 < \gamma_0$. The class of perturbations $f_1(x)$ satisfy a nonlinear analog of assumptions (2.29) and (2.30). Define the distributions

$$\Delta_i := \text{ span } \left\{ g_0, ad_{f_0}g_0, \ldots, ad^i_{f_0}g_0 \right\}. \tag{2.65}$$

Assume that (the notation $\perp$ means the orthogonal distribution)

$$ad_{f_1}\Delta_i \subset \Delta_i \subset \{dh_0\}^\perp \qquad \text{for } i = 1, \ldots, \gamma_1 - 3 \tag{2.66}$$

and

$$ad_{f_1}\Delta_{\gamma_1 - 2} \not\subset \Delta_{\gamma_1 - 2} \text{ and } ad_{f_1}\Delta_{\gamma_1-2} \not\subset \{dh_0\}^\perp. \tag{2.67}$$

If $\gamma_1 < 3$, assumption (2.66) is vacuous and if $\gamma_1 < 2$, (2.67) is vacuous. As in the linear case, we conjecture that if these assumptions are violated then the fast zero dynamics may occur at a multiplicity of time scales. These assumptions guarantee that

$$\begin{aligned} L_{g_0}L^{\gamma_1-1}_{f_0+\epsilon f_1}h_0 &= \epsilon L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0 + O(\epsilon^2) \\ &=: \epsilon\alpha_0(\epsilon). \end{aligned} \tag{2.68}$$

Note that $\alpha_0(0) = L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0$.

For coordinates, one uses

$$\begin{aligned} \xi^\epsilon &:= \begin{bmatrix} h_0(x) \\ L_{f_0+\epsilon f_1}h_0(x) \\ \vdots \\ L^{\gamma_0-1}_{f_0+\epsilon f_1}h_0(x) \end{bmatrix} \in \mathbf{R}^{\gamma_0} \\ \\ \eta &:= \qquad \eta(x) \qquad \in \mathbf{R}^{n-\gamma_0} \end{aligned} \tag{2.69}$$

and the development of Sections 2.2.1 and 2.2.2 can be repeated to yield the fast dynamics of equation (2.63) with the difference that $\alpha_0(\xi^\epsilon_1, \tilde{\xi}^\epsilon_2, \eta)$ is $L_{g_0}L_{f_1}L^{\gamma_1-2}_{f_0}h_0$.

2.2.4 Perturbations in f, g, and h

The same remarks as were made in Section 2.1.4 can be made here as well. As before, if

$$L_{g_0+\epsilon g_1} L_{f_0+\epsilon f_1}^{\gamma_1-1}(h_0+\epsilon h_1) = \epsilon\alpha_0(\epsilon) \tag{2.70}$$

for some $\alpha_0(\epsilon)$, a smooth function of x, which is $O(1)$ but not $o(1)$, then there is only one time scale for the fast zeros and the development of Sections 2.2.1, 2.2.2 could be repeated.

2.3 Conclusion

In this chapter, we have shown the effects of perturbation on the zero dynamics of both linear and nonlinear SISO systems. We have shown how regular perturbations in the state space descriptions of these systems can result in the appearance of singularly perturbed or fast zero dynamics. In the linear case, we have given explicit formulas for the locations in the complex plane that the zeros at ∞ migrate to under perturbation. In the nonlinear case, we have given the formula for the fast zero dynamics subsystem under perturbation. For the most part, we have placed assumptions on the structure of the allowable perturbations so as to guarantee the appearance of fast time scale zero dynamics at one time scale alone. When these assumptions are not met, we conjecture that our qualitative results will be unaltered but that fast zero dynamics at multiple time scales will appear. Our theory bears a strong resemblance to the literature on high gain feedback and is in some sense to be thought of as a companion to that literature, since it reveals the *zero structure at* ∞ by the artifact of system perturbation.

We conclude by noting that the analysis presented in this chapter can be extended albeit in much more subtle form to the MIMO case, and involving a multiplicity of time scales.

Chapter 3

Approximate Input-Output Linearization for Nonlinear Systems without Relative Degree

3.1 Introduction

The past few years have seen the maturation of the use of differential geometric techniques in understanding input-output and full state linearization of nonlinear systems, normal forms and zero dynamics. An elegant discussion of these results is in the work of Isidori [Isi87]. The conditions for the existence of full state linearizable nonlinear systems or for that matter systems which are input-output linearizable are non-generic and it is of obvious interest to extend the results to situations where these conditions fail but do so only *slightly*. Such a program was begun by Krener in [Kre84], who gave conditions for *approximate* full state linearization of nonlinear multi-input systems. In this chapter we take this program one step forward by discussing approximate input-output linearization of single input single output systems which fail to have relative degree in the sense of Byrnes and Isidori [Isi87]. Though in the same spirit as [Kre84], it is different in detail in that the control objective is tracking: i.e., a prescribed output function is required to follow a given specific function of time. Such applications are prototypical in the flight control of aircraft where trajectory following rather than set point regulation are paramount to performance.

Approximate linearization of nonlinear systems has, of course, a lengthy history, starting with Jacobian linearizations and continuing with extended linearization [WR89] and pseudo-linearization [RC84]. Our approximate linearization is different in spirit in that it is specifically geared for tracking problems rather than the regulation problems that the extended or pseudo linearization techniques appear to be useful for. Also, our approximation is not an approximation by a linear system or family of linear systems but rather by a single input-output linearizable nonlinear system.

An outline of the chapter is as follows: In Section 3.2, we start with an example drawn from undergraduate control laboratories, the ball and beam experiment, and use it to study the failure of exact input-output linearization and the latitude available in our proposed technique to do approximate input-output linearization. We also compare the linearizations with the Jacobian linearized system. In Section 3.3, we present the general method motivated by Section 3.2 to define *robust relative degree* and approximate input-output linearization of SISO systems. Section 3.4 has some concluding remarks.

3.2 The Ball and Beam Example

Consider a version of the familiar ball and beam experiment found in many undergraduate control laboratories (see Figure 3.1). In this setup, the beam is symmetric and is made to rotate in a vertical plane by applying a torque at the point of rotation (the center). Rather than have the ball roll on top of the beam as usual, we restrict the ball to frictionless sliding along the beam (as a bead along a wire). Note that this allows for complete rotations and arbitrary angular accelerations of the beam without the ball losing contact with the beam. To remind us of this simplification, we shall refer to the system as the 'ball and beam' system. We shall be interested in controlling the position of the ball along the beam. However, in contrast to the usual set-point problem, we would like the ball to track an arbitrary trajectory.

In this section, we first derive the equations of motion for the 'ball and beam' system. Then, we try to apply the techniques of *input-output linearization* and *full state linearization* to develop a control law for the system and demonstrate the shortcomings of these methods as they fail on this simple nonlinear system. Finally, we demonstrate a method of control law synthesis based on *approximate input-output linearization* and compare the performance of two control laws derived using differing approximations with that derived from the standard Jacobian approximation.

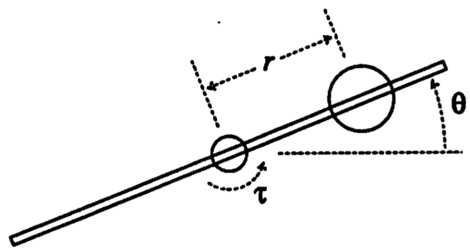

Figure 3.1: The ball and beam system.

3.2.1 Dynamics

Consider the 'ball and beam' system depicted in Figure 3.1. Let the moment of inertia of the beam be J, the mass of the ball be M, and the acceleration of gravity be G.

Choose, as generalized coordinates for this system, the angle, θ, of the beam and the position, r, of the ball. The potential energy, V, of the system is given by

$$V = MGr\sin\theta. \tag{3.1}$$

The kinetic energy of the system is given by

$$K = \underbrace{\frac{1}{2}J\dot{\theta}^2}_{\text{beam}} + \underbrace{\frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)}_{\text{ball}}. \tag{3.2}$$

Then, with the Lagrangian defined as $L = K - V$, the equations of motion are given by

$$\begin{array}{rcl} 0 & = & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} \\ \tau & = & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \end{array} \quad \text{or} \quad \begin{array}{rcl} 0 & = & \ddot{r} + G\sin\theta - r\dot{\theta}^2 \\ \tau & = & (Mr^2 + J)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MGr\cos\theta \end{array} \tag{3.3}$$

where τ is the torque applied to the beam and there is no force applied to the ball.

Once again, note that we have simplified the system by limiting the 'ball' to frictionless sliding on the 'beam'. We could easily deal with the case of the ball rolling without slipping on the beam. This would, of course, place a nontrivial restriction on the angular acceleration of the beam. Since the system would still be holonomic, the form of the equations would be the same as (3.3) with slightly modified coefficients. The true ball and beam system where the ball rolls and may slip and even lose contact with the beam is difficult and will not be considered here.

Using the invertible transformation

$$\tau = 2Mr\dot{r}\dot{\theta} + MGr\cos\theta + (Mr^2 + J)u \tag{3.4}$$

to define a new input, u, the system can be written in state space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ x_1 x_4^2 - G\sin x_3 \\ x_4 \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u \tag{3.5}$$

$$y = \underbrace{x_1}_{h(x)}$$

where $x = (x_1, x_2, x_3, x_4)^T := (r, \dot{r}, \theta, \dot{\theta})^T$ is the state and $y = h(x) := r$ is the *output* of the system (i.e., the variable that we want to control). Note that (3.4) is a *nonlinear* input transformation.

3.2.2 Exact Input-Output Linearization

We are interested in making the system output, $y(t)$, track a specified trajectory, $y_d(t)$, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.

To this end, we might try to *exactly* linearize the input-output response of the system. Following the usual procedure, we differentiate the output until the input appears:

$$\begin{aligned} y &= x_1, \\ \dot{y} &= x_2, \\ \ddot{y} &= x_1 x_4^2 - G\sin x_3, \\ y^{(3)} &= \underbrace{x_2 x_4^2 - G x_4 \cos x_3}_{b(x)} + \underbrace{2x_2 x_4}_{a(x)} u. \end{aligned} \tag{3.6}$$

At this point, if the coefficient of u, $a(x)$, were nonzero in the region of interest, we could use a control law of the form

$$u = \frac{1}{a(x)}[-b(x) + v] \tag{3.7}$$

to yield a linear input-output system described by

$$y^{(3)} = v. \tag{3.8}$$

Unfortunately, for the 'ball and beam', the control coefficient $a(x)$ is zero whenever the angular velocity $x_4 = \dot{\theta}$ or ball position $x_1 = r$ are zero. Therefore, the *relative degree* of the 'ball and beam' system *is not well defined*! This is due to the fact that

$$L_g L_f^2 h(x) = 2x_1 x_4 \tag{3.9}$$

fails to be nonzero at $x = 0$ (an equilibrium point of the undriven system) but also fails to be identically zero on a neighborhood of $x = 0$. This is a characteristic unique to *non*linear systems. Thus, when the system has nonzero angular velocity and nonzero ball position, the input acts one integrator sooner than when the angular velocity or ball position are zero.

Thus we conclude the *exact* input-output linearization does not provide a methodology for designing a trajectory tracking controller.

3.2.3 Full State Linearization

Next we try our hand at fully linearizing the state of this system, that is to say, find a set of coordinates and a feedback law such that the input-to-state behavior of the transformed system is linear. The necessary and sufficient conditions for this were given by Jakubczyk and Respondek [JR80] and, independently, by Hunt, Su, and Meyer [HSM83].

First we check the dimension of the *controllability distribution*,

$$\operatorname{span}\left\{g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-1} g\right\} \tag{3.10}$$

where $ad_f^i g$ denotes the iterated *Lie bracket* $[f,[f,\cdots[f,g]\cdots]]$. Since, the matrix

$$Q(x) = \begin{bmatrix} 0 & 0 & 2x_1x_4 & 4x_2x_4 + G\cos x_3 \\ 0 & -2x_1x_4 & -2x_2x_4 - G\cos x_3 & -4x_1x_4^3 + 3Gx_4\cos x_3 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.11}$$

has full rank at $x = 0$ ($\det Q(0) = G^2$), it follows that the 'ball and beam' system is locally controllable.

The second requirement is not generic. It is required that the distribution

$$\operatorname{span}\left\{g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-2} g\right\} \tag{3.12}$$

be involutive, that is, the Lie bracket of any two vector fields in the distribution should also be contained in the distribution.

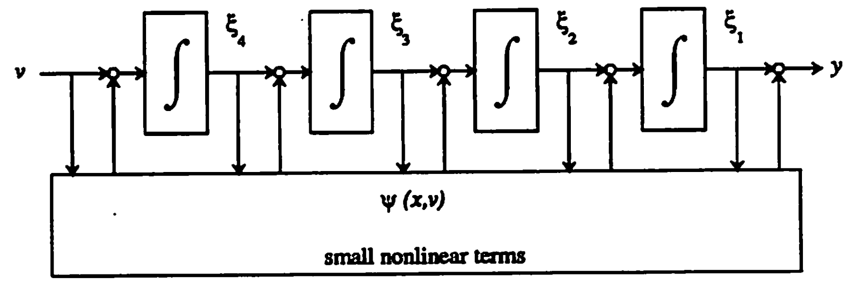

Figure 3.2: Approximate input-output linearization: a chain of intergrators perturbed by small nonlinear terms.

Checking the brackets for the 'ball and beam' system we find that

$$[g, ad_f^2 g] = (2x_1 \quad -2x_2 \quad 0 \quad 0)^T \tag{3.13}$$

does not lie within the span of the first three columns (vector fields) of (3.11).

Failing this condition, we see that it is not possible to fully linearize the 'ball and beam' system.

3.2.4 Approximate Input-Output Linearization

In this section, we see that, by appropriate choice of vector fields *close* to the system vector fields, we can design a feedback control law to achieve bounded error output tracking. The control law will, in fact, be the *exact* output tracking control law for an *approximate system* defined by these vector fields.

Ideally, we would like to find a state feedback control law, $u(x) = \alpha(x) + \beta(x)v$, that would transform the 'ball and beam' system into a linear system of the of the form $y^{(4)} = v$. Then, the system could be made to track an arbitrary (C^4) trajectory, $y_d(t)$, asymptotically by using a tracking control law of the form

$$v = y_d^{(4)}(t)+\alpha_3(y_d^{(3)}(t)-y^{(3)}(x))+\alpha_2(\ddot{y}_d(t)-\ddot{y}(x))+\alpha_1(\dot{y}_d(t)-\dot{y}(x))+\alpha_0(y_d(t)-y(x)) \tag{3.14}$$

where $s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ is a Hurwitz polynomial. Note that y, $\dot{y}$, etc., are all functions of the state x.

Unfortunately, due to the presence of the centrifugal term $r\dot{\theta}^2 = x_1 x_4^2$, the input-output response of the 'ball and beam' system cannot be exactly linearized. Here we try to find an input-output linearizable system that is *close* to the true system. We present two such approximations for the 'ball and beam' system. In each case, we will design a nonlinear change of (state) coordinates, $\xi = \Phi(x)$, and a state dependent feedback, $u(x,v) = \alpha(x) + \beta(x)v$, to make the system look like a chain of integrators (i.e., Brunovsky canonical form) perturbed by small higher order terms, $\psi(x,v)$, as depicted in Figure 3.2. We also compare the performance of these designs to a *linear* controller based on the standard Jacobian approximation to the system.

We then build an approximate tracking control law by designing u so that

$$v = y_d^{(4)}(t) + \alpha_3(y_d^{(3)}(t) - \phi_4(x)) + \alpha_2(\ddot{y}_d(t) - \phi_3(x)) + \alpha_1(\dot{y}_d(t) - \phi_2(x)) + \alpha_0(y_d(t) - \phi_1(x)) \tag{3.15}$$

making the error system into an exponentially stable linear system perturbed by small nonlinear terms.

For each approximation, we present simulation results depicting (a) the output error, $y_d(t) - \phi_1(x(t))$, (b) the neglected nonlinearity, $\psi(x,u)$, (c) the angle of the beam, $\theta(t) = x_3(t)$, and (d) the position of the ball, $r(t) = x_1(t) = y(t)$, for a desired trajectory of $y_d(t) = R\cos \pi t/30$, with $R = 5$, 10, and 15.

Approximation 1

Since the centrifugal acceleration term $x_1 x_4^2 = r\dot{\theta}^2$ causes the system to not have a well defined relative degree, we *design* our first approximate system by simply neglecting it. Let $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have (defining $\phi_i(\cdot)$ recursively)

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-G\sin x_3}_{\xi_3=\phi_3(x)} + \underbrace{x_1 x_4^2}_{\psi_2(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4\cos x_3}_{\xi_4=\phi_4(x)} \\
\dot{\xi}_4 &= \underbrace{Gx_4^2\sin x_3}_{b(x)} + \underbrace{(-G\cos x_3)}_{a(x)}u
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 + \psi_2(x) \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= b(x) + a(x)u =: v(x,u).
\end{aligned} \tag{3.16}$$

As expected, by neglecting the centrifugal term (which is higher order), we obtain an approximate system with a well defined relative degree. Note that the choice of what to neglect (i.e., $\psi_2(x)$) leads to a specification of the coordinate transformation $\Phi(x)$. In this case, the approximate system is obtained by a simple modification of the f vector field (i.e., by neglecting $\psi_2(\cdot)$).

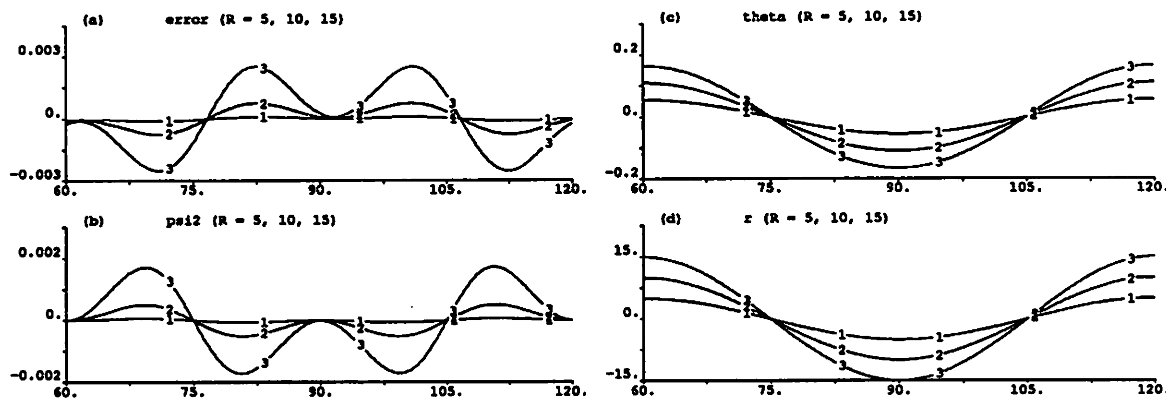

Figure 3.3: Simulation results for $y_d(t) = R\cos\pi t/30$ using the first approximation ((a) $e = y_d - \phi_1$, (b) ψ_2, (c) θ, (d) r)

The simulation results in Figure 3.3 show that the closed loop system provides good tracking. Notice that the tracking error increases in a nonlinear fashion as the amplitude of the desired trajectory increases. This is expected since the approximation error term $\psi_2(x)$ is a nonlinear function of the state. A good *a priori* estimate of the mismatch of the approximate system for a desired trajectory can be calculated using $\psi(\Phi^{-1}(y_d, \dot{y}_d, \ddot{y}_d, y_d^{(3)}))$ where $\Phi^{-1} : \xi \mapsto x$ is the inverse of the coordinate transformation. This in turn may be a useful way to define a class of trajectories that the system can track with small error.

Approximation 2

For this approximation, we will retain the centrifugal acceleration and only discard terms that we must to obtain a approximate system with a well defined relative degree. Again, let $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-G\sin x_3 + x_1 x_4^2}_{\xi_3=\phi_3(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4\cos x_3 + x_2 x_4^2}_{\xi_4=\phi_4(x)} + \underbrace{2x_2x_4u}_{\psi_3(x,u)} \\
\dot{\xi}_4 &= \underbrace{x_1x_4^4}_{b(x)} + \underbrace{(-G\cos x_3 + 2x_2x_4)}_{a(x)}u
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 + \psi_3(x,u) \\
\dot{\xi}_4 &= b(x) + a(x)u =: v(x,u).
\end{aligned}
\qquad (3.17)$$

Note that we had no choice but to discard $\psi_3(x,u) = 2x_2x_4u$ since x_2x_4 is zero at $x = 0$

but not identically zero in a neighborhood of $x = 0$. However, at this point, ψ_2 and ϕ_4 are not uniquely determined since, for example, $x_2 x_4^2$ could be included in ψ_3 rather than in ϕ_4 as we have done.

This time the approximate system is obtained by modifying the g vector field in a more subtle way. Pulling back the modified g vector field (obtained by neglecting $\psi_3(x, u)$) to the original x coordinates (using u as input) we get

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{2Gx_1x_4\cos x_3 - 4x_1x_2x_4^2}{G(G\cos^2 x_3 - 2x_2x_4\cos x_3 - x_1x_4^2\sin x_3)} \\ \frac{2x_1x_4^2\sin x_3}{G\cos^2 x_3 - 2x_2x_4\cos x_3 - x_1x_4^2\sin x_3} \end{bmatrix}}_{\Delta g(x)}. \tag{3.18}$$

The system with g modified in this manner is input-output linearizable and is an approximation to the original system since Δg is small for small angular velocity, $\dot{\theta} = x_4$.

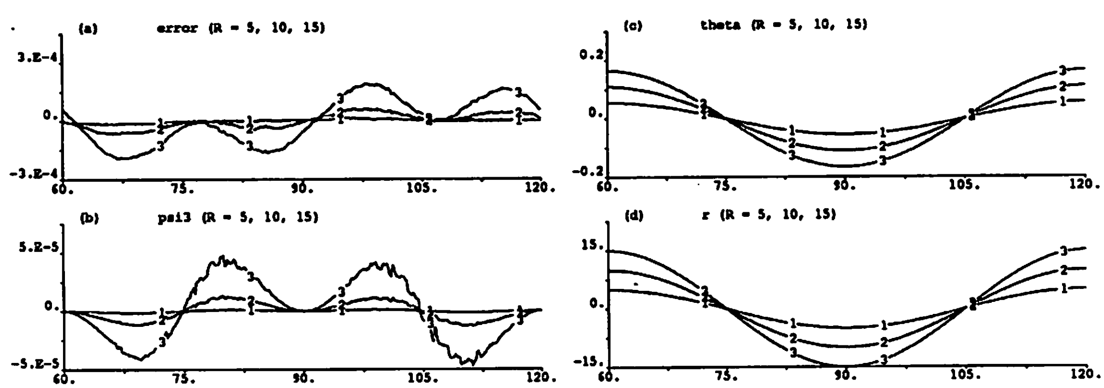

Figure 3.4: Simulation results for $y_d(t) = R\cos\pi t/30$ using the second approximation ((a) $e = y_d - \phi_1$, (b) ψ_3, (c) θ, (d) r)

The simulation results in Figure 3.4 show that the tracking error is substantially less than that obtained by the first design. Thus, in some sense, approximation 2 is closer to the true system than approximation 1.

We note that there are an endless number of approximate systems (perhaps less obvious than these two) that may result in reasonable performance. The main requirement is that the neglected terms ψ_i be higher order.

Jacobian Approximation

To provide a basis for comparison, we calculate a linear control law based on the Jacobian approximation. Previously, we used the invertible nonlinear transformation of (3.4) to simplify the form of $\dot{x}_4$. Since we are only allowed *linear* functions in the control, we must work directly with the original input τ and the true angular acceleration $\ddot{\theta} = \dot{x}_4$ given by

$$\dot{x}_4 = \frac{-MGx_1 \cos x_3 - 2Mx_1x_2x_4}{Mx_1^2 + J} + \frac{1}{Mx_1^2 + J}\tau \, . \tag{3.19}$$

We will linearize about $x = 0$, $\tau = 0$. Since the output is a *linear* function of the state, we begin with $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have (using (3.5) and (3.19))

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-Gx_3}_{\xi_3=\phi_3(x)} + \underbrace{x_1x_4^2 + G(x_3 - \sin x_3)}_{\psi_2(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4}_{\xi_4=\phi_4(x)} \\
\dot{\xi}_4 &= \underbrace{\frac{MG^2}{J}x_1}_{b(x)} + \underbrace{\frac{-G}{J}}_{a(x)}\tau + \underbrace{\frac{MG^2x_1 \cos x_3 + 2MGx_1x_2x_4}{Mx_1^2 + J} - \frac{MG^2x_1}{J} + \left(\frac{G}{J} - \frac{G}{Mx_1^2 + J}\right)\tau}_{\psi_4(x,\tau)}
\end{aligned} \tag{3.20}$$

The Jacobian approximation is, of course, obtained by replacing the f vector field by its linear approximation and the g vector field by its constant approximation.

Figure 3.5 shows the simulation results from the Jacobian approximation. Unfortunately, the control system with the linear controller is not stable for R greater than about 7. We see that the Jacobian approximation performs quite well within the somewhat limited region of validity of the approximation, but quickly loses even basic stability outside of this region.

The following table provides a direct comparison of the error $e = y_d - \phi_1$ for the three approximations:

R	Approximation 1	Approximation 2	Jacobian Approximation
5	$\pm 9.6 \cdot 10^{-5}$	$\pm 1.5 \cdot 10^{-5}$	$-4.7 \cdot 10^{-3}$ $+3.0 \cdot 10^{-3}$
10	$\pm 7.5 \cdot 10^{-4}$	$\pm 6.5 \cdot 10^{-5}$	*unstable*
15	$\pm 2.5 \cdot 10^{-3}$	$\pm 1.9 \cdot 10^{-4}$	*unstable*

Note that Approximation 2 provides better tracking for this class of inputs by about an

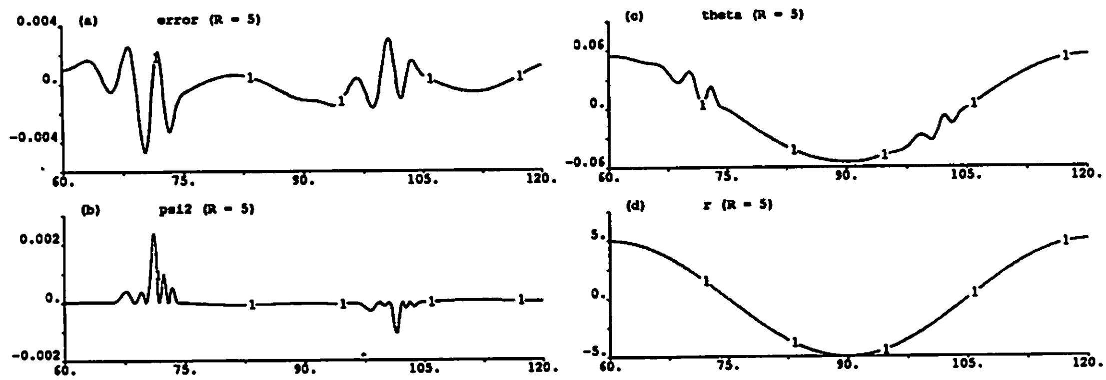

Figure 3.5: Simulation results for $y_d(t) = R\cos\pi t/30$ using the Jacobian approximation ((a) $e = y_d - \phi_1$, (b) ψ_3, (c) θ, (d) r)

order of magnitude over Approximation 1. Due to the large excursions from the origin, the Jacobian Approximation is no longer a good approximation so the system goes unstable. Of course, the other approximations will eventually go unstable as R becomes large.

In the next section, we will see that these approximations belong to a large class of approximations that provide the model to design stable closed loop control laws for approximate output tracking.

3.3 Theory for Approximate Linearization

In this section, we will consider single-input single-output systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.21}$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}$, f and g are smooth vector fields on $\mathbf{R}^n$ (i.e., $f(x) \in T_x\mathbf{R}^n = \mathbf{R}^n$, $x \in \mathbf{R}^n$), and $h : \mathbf{R}^n \to \mathbf{R}$ is a smooth function (smooth is understood to mean as differentiable as needed). We *assume* that $x = 0$ is an equilibrium point of the undriven system, i.e., $f(0) = 0$.

If the control objective is tracking, the input-output linearization proceeds as follows: differentiate the output repeatedly until the input appears for the first time on the right hand side. Thus, we obtain, for x in a neighborhood of 0,

$$\begin{aligned} \dot{y} &= L_f h(x) , \\ \ddot{y} &= L_f^2 h(x) , \\ &\vdots \\ y^{(\gamma)} &= L_f^{\gamma} h(x) + L_g L_f^{\gamma-1} h(x) u . \end{aligned} \tag{3.22}$$

Here, $L_f h(x)$ stands for the Lie derivative of $h(x)$ along f, $L_f^2 h(x)$ stands for $L_f(L_f h)(x)$ and so on. It follows that in (3.22) above, that

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{\gamma-2} h(x) \equiv 0 \quad \text{for } x \in U \tag{3.23}$$

where U is an open neighborhood of the origin. In the event that $L_g L_f^{\gamma-1} h(x) \neq 0$ for $x \in U$, the system is said to have *relative degree* γ and the control law

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} \left[-L_f^{\gamma} h(x) + v \right] \tag{3.24}$$

linearizes the system from v to y. However, it may happen that $L_g L_f^{\gamma-1} h(x) = 0$ at $x = 0$ but is not *identically zero* in a neighborhood U of $x = 0$, i.e., $L_g L_f^{\gamma-1} h(x)$ is a function which is of order $O(x)$ rather than $O(1)$. Then, the relative degree of the system is not well defined and the input-output linearizing control law of (3.24) is no longer valid.

(In the sequel we will use the O notation. Recall that a function $\delta(x)$ is said to be $O(x)^n$ if

$$\lim_{|x| \to 0} \frac{|\delta(x)|}{|x|^n} \text{ exists and is } \neq 0.$$

Also, functions which are $O(x)^0$ are referred to as $O(1)$. By abuse of notation, we will also use the notation $O(x,u)^2$ to mean functions of x, u which are sums of terms of $O(x)^2$, $O(xu)$ or $O(u)^2$. Similarly for $O(x,u)^{\rho}$.)

Failing this, we seek a set of functions of the state, $\phi_i(x)$, $i = 1, \ldots, \gamma$, that approximate the output and its derivatives in a special way. The integer γ will be determined during the approximation process.

Since our control objective is tracking, the first function, $\phi_1(x)$, should approximate the output function, that is

$$h(x) = \phi_1(x) + \psi_0(x, u) \tag{3.25}$$

where $\psi_0(x,u)$ is $O(x,u)^2$ (actually, ψ_0 does not depend on u, but for consistency below we include it). Differentiating $\phi_1(x)$ along the system trajectories we get

$$\dot{\phi}_1(x) = L_f h(x) + L_g h(x) u. \tag{3.26}$$

If $L_g h(x)$ is $O(x)$ or of higher order, we cannot effectively control the system at this level so we neglect it (and a small part of $L_f h(x)$ if we so desire) in our choice of $\phi_2(x)$:

$$L_{f+gu}\phi_1(x) = \phi_2(x) + \psi_1(x,u) \tag{3.27}$$

where $\psi_1(x,u)$ is $O(x,u)^2$. We continue this procedure with

$$L_{f+gu}\phi_i(x) = \phi_{i+1}(x) + \psi_i(x,u) \tag{3.28}$$

until at some step, say γ, the control term, $L_g\phi_\gamma(x)$, is $O(1)$, that is,

$$L_{f+gu}\phi_\gamma(x) = b(x) + a(x)u \tag{3.29}$$

where $a(x)$ is $O(1)$. Using this procedure, it looks like we have found an approximate system of relative degree γ. This motivates the following definition:

Definition 3.1 *We say that a nonlinear system (3.21) has a* robust relative degree *of* γ *about* $x = 0$ *if there exists smooth functions* $\phi_i(x)$, $i = 1, \ldots, \gamma$, *such that*

$$\begin{aligned} h(x) &= \phi_1(x) + \psi_0(x,u) \\ L_{f+gu}\phi_i(x) &= \phi_{i+1}(x) + \psi_i(x,u) \quad i = 1,\ldots,\gamma-1 \\ L_{f+gu}\phi_\gamma(x) &= b(x) + a(x)u + \psi_\gamma(x,u) \end{aligned} \tag{3.30}$$

where the functions $\psi_i(x,u)$, $i = 0, \ldots, \gamma$, *are* $O(x,u)^2$ *and* $a(x)$ *is* $O(1)$.

Remarks

- In equation (3.30) above, the dependence of ψ_i on x and u has the form

$$\begin{aligned} &\psi_0(x,u) = \psi_0^1(x)\,, \\ &\psi_i(x,u) = \psi_i^1(x) + \psi_i^2(x)u, \quad i = 1,\ldots,\gamma-1 \end{aligned} \tag{3.31}$$

  where, for $i = 0,\ldots,\gamma-1$, $\psi_i^1(x)$ is $O(x)^2$ and $\psi_i^2(x)$ is $O(x)$.

- There is considerable latitude in the definition of the $\phi_i(x)$ since each $\psi_i^1(x)$ may be chosen in a number of ways as long as it is $O(x)^2$.

□

We now characterize the robust relative degree. First, define the Jacobian linearized version of the system(3.21) about $x = 0$, $u = 0$ to be

$$\begin{aligned} \dot{z} &= Az + bu \\ y &= cz \end{aligned} \tag{3.32}$$

with $A = Df(0)$, $b = g(0)$, and $c = dh(0)$. Then, we have

Theorem 3.1 *The robust relative degree of the nonlinear system (3.21) is equal to the relative degree of the Jacobian linearized system (3.32) and so is well defined.*

Proof: For $i = 1, \ldots, \gamma - 1$, we have

$$\begin{aligned} L_{f+gu}\phi_i &= \quad L_f\phi_i \quad + \quad L_g\phi_i u \\ &= \phi_{i+1} + \psi_i^1 \quad + \quad \psi_i^2 u \end{aligned} \tag{3.33}$$

so that

$$\begin{aligned} \phi_{i+1}(x) &= L_f\phi_i(x) - \psi_i^1(x)\,, \\ \psi_i^2(x) &= L_g\phi_i(x)\,. \end{aligned} \tag{3.34}$$

Also, since $\psi_i^1(x)$ is $O(x)^2$, we have, for $i = 1, \ldots, \gamma - 1$,

$$d\psi_i^1(0) = 0\,. \tag{3.35}$$

Using this and the fact that $f(0) = 0$, the differentials of the functions ϕ_i are given by

$$\begin{aligned} d\phi_1(0) &= dh(0) - d\psi_0^1(0) \\ &= c - 0, \\ d\phi_2(0) &= dL_f\phi_1(0) - d\psi_1^1(0) \\ &= d^2\phi_1(0)\cdot f(0) + d\phi_1(0)\cdot Df(0) - 0 \\ &= 0 + cA, \\ &\vdots \\ d\phi_\gamma(0) &= cA^{\gamma-1}\,. \end{aligned} \tag{3.36}$$

Calculating the control coefficients, we find

$$\begin{aligned}
\psi_1^2(0) &= d\phi_1(0)\cdot g(0) \\
&= cb, \\
\psi_2^2(0) &= cAb, \\
&\vdots \\
\psi_{\gamma-1}^2(0) &= cA^{\gamma-2}b, \\
a(0) &= cA^{\gamma-1}b.
\end{aligned} \tag{3.37}$$

Since $\psi_i^2(0) = 0$ and $a(0) \neq 0$, it follows that

$$\begin{aligned}
cb = cAb = \cdots = cA^{\gamma-2}b = 0, \\
cA^{\gamma-1}b \neq 0.
\end{aligned} \tag{3.38}$$

Thus, γ, the robust relative degree of (3.21), is equal to the relative degree of the Jacobian linearized system (3.32). From this, it is easy to see that γ is independent of the choice of the neglected functions $\psi_i(x,u)$ of order $O(x,u)^2$ and is therefore well defined. □

An immediate corollary of this theorem is

Corollary 3.2 *The approximate relative degree of a nonlinear system (3.21) is* invariant *under a state dependent change of control coordinates of the form*

$$u(x,v) = \alpha(x) + \beta(x)v \tag{3.39}$$

where α and β are smooth functions and $\alpha(0) = 0$ while $\beta(0) \neq 0$.

In order to show that this procedure produces an approximation of the true system, we need to show that the functions $\phi_i(\cdot)$ can be used as part of a (local) nonlinear change of coordinates. To this end, we prove:

Proposition 3.3 *Suppose that the nonlinear system (3.21) has approximate relative degree γ. Then the functions $\phi_i(\cdot)$, $i = 1, \ldots, \gamma$, are independent in a neighborhood of the origin.*

Proof: Since the $\phi_i(\cdot)$ are smooth, it is sufficient to check that the *constant* $\gamma \times n$ matrix

$$D\phi(0) = \begin{bmatrix} d\phi_1(0) \\ d\phi_2(0) \\ \vdots \\ d\phi_\gamma(0) \end{bmatrix} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{\gamma-1} \end{bmatrix} \tag{3.40}$$

(from (3.36)) has full rank. If we multiply $D\phi(0)$ on the right by the $n \times \gamma$ matrix

$$\left[A^{\gamma-1}b \; A^{\gamma-2}b \; \cdots \; b\right] \tag{3.41}$$

we get the nonsingular $\gamma \times \gamma$ matrix

$$\begin{bmatrix} a(0) & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & a(0) \end{bmatrix} \tag{3.42}$$

where '$*$' denotes possibly nonzero entries. This shows that $D\phi(0)$ has a rank of γ and the proposition is proved. □

With the γ independent functions, $\phi_i(\cdot)$, in hand, we can, by the Frobenius theorem, complete the nonlinear change of coordinates with a set of functions, $\eta_i(x)$, $i = 1, \ldots, n-\gamma$, such that

$$L_g \eta_i(x) = 0 \qquad x \in U. \tag{3.43}$$

Defining new coordinates, (ξ, η), by

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{bmatrix} := \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_\gamma(x) \\ \eta_1(x) \\ \vdots \\ \eta_{n-\gamma}(x) \end{bmatrix} =: \Phi(x), \tag{3.44}$$

we can rewrite the true system (3.21) as

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \psi_1(x,u) \\
&\vdots \\
\dot{\xi}_{\gamma-1} &= \xi_\gamma + \psi_{\gamma-1}(x,u) \\
\dot{\xi}_\gamma &= b(\xi,\eta) + a(\xi,\eta)u \\
\dot{\eta} &= q(\xi,\eta) \\
\\
y &= \xi_1 + \psi_0(x,u)
\end{aligned} \tag{3.45}$$

where $q(\xi,\eta)$ is $L_f\eta$ expressed in (ξ,η) coordinates.

Note that the form (3.45) is a generalization of the standard normal form of Byrnes and Isidori [Isi87,BI88] with the extra terms $\psi_i(x,u)$, $i = 0,\ldots,\gamma$ of $O(x,u)^2$. Thus the control law

$$u = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta)+v] \tag{3.46}$$

approximately linearizes the system (3.21) from the input v to the output y up to terms of $O(x,u)^2$.

If the robust relative degree of the system (3.21) is $\gamma = n$, then the system (3.21) is almost completely linearizable from input to state as well (since there will be no η state variables). This situation was investigated by Krener [Kre84] who showed that the system

$$\dot{x} = f(x) + g(x)u \tag{3.47}$$

with no output *explicitly* defined was linearizable to terms of $O(x,u)^\rho$ iff the distribution

$$\text{span}\left\{g \;\, ad_f g \;\, \cdots \;\, ad_f^{n-1} g\right\} \text{ has rank } n \tag{3.48}$$

and the distribution

$$\text{span}\left\{g \;\, ad_f g \;\, \cdots \;\, ad_f^{n-2} g\right\} \text{ is order } \rho \text{ involutive,} \tag{3.49}$$

i.e., has a basis, up to terms of $O(x)^\rho$, which is involutive up to terms of $O(x)^\rho$. Equivalently, conditions (3.48) and (3.49) guarantee (through a version of the Frobenius theorem with remainder [Kre84]) the existence of an *output* function $h(x)$ with respect to which the system (3.47) has *robust relative degree* n and further that the remainder functions $\psi_i(x,u)$

are $O(x,u)^\rho$. Our development differs somewhat from that in [Kre84] in that we are given a specific output function $y = h(x)$ and a tracking objective for this output. However, there is a happy confluence of our results and those of Krener for the ball and beam example of the previous section where it may be verified that the condition of (3.49) is satisfied for $\rho = 3$ and further more the desired output function $h(x)$ is in fact an order $\rho = 3$ integral manifold of the distribution of that equation. Consequently the ball and beam can be input-output and state space linearized up to terms of order 3.

As was remarked after Definition 3.1, there is a great deal of latitude in the choice of the functions $\psi_i^1(x)$, $i = 0, \ldots, \gamma - 1$, so long as they are $O(x)^2$. To improve the quality of the approximation, one may insist on choosing these terms to be $O(x)^\rho$ for some $\rho \geq 2$. There is less latitude in the choice of the functions $\psi_i^2(x)$. They must be neglected if they are $O(x)$ or higher and not neglected if they are $O(1)$ (this determines γ). We cannot in general guarantee that an approximation of $O(x,u)^\rho$ for $\rho > 2$ can be found. At this level of generality, it is difficult to give analytically rigorous design guidelines for the choice of the functions $\psi_i^1(x)$. However, from the ball and beam example of section 3.2, it would appear that it is advantageous to have the $\psi_i^1(x)$ be identically zero for as long (as large an i) as possible. *We conjecture that the larger the value of the first i at which either $\psi_i^1(x)$ or $\psi_i^2(x)$ are nonzero, the better the approximation.*

It is also important to note the distinction between the *nonlinear* feedback control law (3.46) which approximately linearizes the system (3.45) and the *linear* feedback control law obtained from the Jacobian linearization of the original system (3.21) given by

$$u = \frac{1}{cA^{\gamma-1}b}\left[-cA^\gamma x + v\right] \ , \tag{3.50}$$

though, as we have shown in the proof of Theorem 3.1, they agree up to first order at $x = 0$ since $cA^{\gamma-1}b = a(0)$ and $cA^\gamma = dL_f\phi_\gamma(0) = dh(0)$. It is also useful to note that the control law (3.46) is the *exact* input-output linearizing control law for the *approximate system*

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{\gamma-1} &= \xi_\gamma \\
\dot{\xi}_\gamma &= b(\xi,\eta) + a(\xi,\eta)u \\
\dot{\eta} &= q(\xi,\eta) \\
\\
y &= \xi_1
\end{aligned}
\tag{3.51}
$$

In general, we can only guarantee the existence of control laws of the form (3.46) that approximately linearize the system up to terms of $O(x,u)^2$—the Jacobian law of (3.50) is such a law. In specific applications, we see that the control law (3.46) may produce better approximations (the ball and beam of section 3.2 was linearized up to terms of $O(x,u)^3$). Furthermore, the resulting approximations may be valid on larger domains than the Jacobian linearization (also seen in the ball and beam example). We try to make this notion precise by studying the properties enjoyed by the approximately linearized system (3.21), (3.46) on a parameterized family of *operating envelopes*) defined as:

Definition 3.2 *We call $U_\epsilon \subset \mathbf{R}^n$, $\epsilon > 0$, a family of* operating envelopes *provided that*

$$U_\delta \subset U_\epsilon \text{ whenever } \delta < \epsilon \tag{3.52}$$

and

$$\sup\{\delta : B_\delta \subset U_\epsilon\} = \epsilon \tag{3.53}$$

where B_δ is a ball of radius δ centered at the origin.

Remarks

- It is not necessary that each U_ϵ be bounded (or compact) although this might be useful in some cases.
- Since the largest ball that fits in U_ϵ is B_ϵ, the set U_ϵ must get smaller in at least one direction as ϵ is decreased.

□

The functions $\psi_i(x,u)$ that are omitted in the approximation are of $O(x,u)^2$ in a neighborhood of the origin. However, if we are interested in extending the approximation to (larger) regions, say of the form of U_ϵ, we will need the following definition:

Definition 3.3 *A function* $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is said to be* uniformly higher order *on* $U_\epsilon \times B_\sigma \subset \mathbb{R}^n \times \mathbb{R}$, $\epsilon > 0$, *if, for some* $\sigma > 0$, *there exists a monotone increasing function of* ϵ, K_ϵ *such that*

$$|\psi(x,u)| \leq \epsilon K_\epsilon(|x| + |u|) \quad \textit{for } x \in U_\epsilon,\ |u| \leq \sigma. \tag{3.54}$$

Remarks

- If $\psi(x,u)$ is uniformly higher order on $U_\epsilon \times B_\sigma$ then it is $O(x,u)^2$.
- This definition is a refinement of the condition that $\psi(x,u)$ be $O(x,u)^2$ in as much as it does not allow for terms of the form $O(u)^2$.

□

Now, return to the original problem. If the approximate system (3.51) is exponentially minimum phase and the error terms ψ_i in (3.45) are uniformly higher order on $U_\epsilon \times B_\sigma$, we may use the stable tracking control law for the approximate system given by

$$u = \frac{1}{a(\xi,\eta)}\left[-b(\xi,\eta) + y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \xi_\gamma) + \cdots + \alpha_0(y_d - \xi_1)\right] \tag{3.55}$$

(with $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \cdots + \alpha_0$ a Hurwitz polynomial). We can now prove the following result:

Theorem 3.4 *Let* U_ϵ, $\epsilon > 0$, *be a family of operating envelopes and suppose that*

- *the zero dynamics of the approximate system (3.51) (i.e.,* $\dot{\eta} = q(0,\eta)$*) are* exponentially stable *and* q *is Lipschitz in* ξ *and* η *on* $\Phi(U_\epsilon)$ *for each* ϵ *and*
- *the functions* $\psi_i(x,u)$ *are* uniformly higher order *on* $U_\epsilon \times B_\sigma$.

Then, for ϵ *sufficiently small and for desired trajectories with sufficiently small values and derivatives* (y_d, $\dot{y}_d$, $\ldots$, $y_d^{(\gamma)}$)*, the states of the closed loop system (3.21), (3.55) will remain bounded and the tracking error will be* $O(\epsilon)$.

Proof: Define the trajectory error, $e \in \mathbf{R}^{\gamma}$, to be

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\gamma \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\gamma \end{bmatrix} - \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\gamma-1)} \end{bmatrix}. \tag{3.56}$$

Then, the closed loop system (3.21), (3.55) (equivalently, (3.45), (3.55)) may be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{\gamma-1} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \begin{bmatrix} \psi_1(x, u(x, \bar{y}_d)) \\ \vdots \\ \psi_{\gamma-1}(x, u(x, \bar{y}_d)) \\ \psi_\gamma(x, u(x, \bar{y}_d)) \end{bmatrix} \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \tag{3.57}$$

or, compactly,

$$\begin{aligned} \dot{e} &= Ae + \psi(x, u(x, \bar{y}_d)) \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \tag{3.58}$$

where $\bar{y}_d := (y_d, \dot{y}_d, \ldots, y_d^{(\gamma)})$. Since the zero dynamics are exponentially stable, a converse Lyapunov theorem implies the existence of a Lyapunov function (see, e.g., [Hah67]) $V_2(\eta)$ for the system

$$\dot{\eta} = q(0, \eta) \tag{3.59}$$

satisfying

$$\begin{aligned} k_1|\eta|^2 \le V_2(\eta) \le k_2|\eta|^2 \\ \tfrac{\partial V_2}{\partial \eta} q(0, \eta) \le -k_3|\eta|^2 \\ \left|\tfrac{\partial V_2}{\partial \eta}\right| \le k_4|\eta| \end{aligned} \tag{3.60}$$

for some positive constants k_1, k_2, k_3, and k_4.

We first show that e and η are bounded. To this end, consider as Lyapunov function for the error system (3.58)

$$V(e, \eta) = e^T P e + \mu V_2(\eta) \tag{3.61}$$

where $P > 0$ is chosen so that

$$A^T P + PA = -I \tag{3.62}$$

(possible since $\dot{e} = Ae$ is stable) and μ is a positive constant to be determined later.

Note that, by assumption, y_d and its first γ derivatives are bounded,

$$|\xi| \leq |e| + b_d \text{ and } |y^{(\gamma)}| \leq b_d, \tag{3.63}$$

the function, $q(\xi, \eta)$ is Lipschitz

$$|q(\xi^1, \eta^1) - q(\xi^2, \eta^2)| \leq l_q(|\xi^1 - \xi^2| + |\eta^1 - \eta^2|), \tag{3.64}$$

the function, $\psi(x, u)$, is uniformly higher order with respect to $U_\epsilon \times B_\sigma$ and $u(x, \bar{y}_d)$ locally Lipschitz in its arguments with $u(0,0) = 0$,

$$|2P\psi(x, u(x, \bar{y}_d))| \leq \epsilon K_\epsilon l_u(|x| + b_d) \quad (x, u) \in U_\epsilon \times B_\sigma, \tag{3.65}$$

and x is a local diffeomorphism of (ξ, η),

$$|x| \leq l_x(|\xi| + |\eta|). \tag{3.66}$$

Using these bounds and the properties of $V_2(\cdot)$, we have

$$\begin{aligned} \tfrac{\partial V_2}{\partial \eta} q(\xi, \eta) &= \tfrac{\partial V_2}{\partial \eta} q(0, \eta) + \tfrac{\partial V_2}{\partial \eta}(q(\xi, \eta) - q(0, \eta)) \\ &\leq -k_3|\eta|^2 + k_4 l_q |\eta|(|e| + b_d). \end{aligned} \tag{3.67}$$

Taking the derivative of $V(\cdot, \cdot)$ along the trajectories of (3.58), we find, for $(x, u) \in U_\epsilon \times B_\sigma$,

$$\begin{aligned} \dot{V} &= -|e|^2 + 2e^T P\psi(x, u(x, \bar{y}_d)) + \mu \tfrac{\partial V_2}{\partial \eta} q(\xi, \eta) \\ &\leq -|e|^2 + \epsilon |e| K_\epsilon l_x(|e| + b_d + |\eta|) + \mu(-k_3|\eta|^2 + k_4 l_q |\eta|(|e| + b_d)) \\ &\leq -(\tfrac{|e|}{2} - \epsilon K_\epsilon l_x b_d)^2 + (\epsilon K_\epsilon l_x b_d)^2 \\ &\quad -(\tfrac{|e|}{2} - (\epsilon K_\epsilon l_x + \mu k_4 l_q)|\eta|)^2 + (\epsilon K_\epsilon l_x + \mu k_4 l_q)^2 |\eta|^2 \\ &\quad -\mu k_3(\tfrac{|\eta|}{2} - \tfrac{k_4 l_q b_d}{k_3})^2 + \mu \tfrac{(k_4 l_q b_d)^2}{k_3} \\ &\quad -(\tfrac{1}{2} - \epsilon K_\epsilon l_x)|e|^2 - \tfrac{3}{4}\mu k_3 |\eta|^2 \\ &\leq -(\tfrac{1}{2} - \epsilon K_\epsilon l_x)|e|^2 - (\tfrac{3}{4}\mu k_3 - (\epsilon K_\epsilon l_x + \mu k_4 l_q)^2)|\eta|^2 \\ &\quad +(\epsilon K_\epsilon l_x b_d)^2 + \mu \tfrac{(k_4 l_q b_d)^2}{k_3}. \end{aligned} \tag{3.68}$$

Define

$$\mu_0 = \frac{k_3}{4(K_\epsilon l_x + k_4 l_q)^2}. \tag{3.69}$$

Then, for all $\mu \le \mu_0$ and all $\epsilon \le min(\mu, \frac{1}{4K_\epsilon l_x})$, we have

$$\dot{V} \le -\frac{|e|^2}{4} - \frac{\mu k_3 |\eta|^2}{2} + \frac{\mu (k_4 l_q b_d)^2}{k_3} + (\epsilon K_\epsilon l_x b_d)^2. \tag{3.70}$$

Thus, $\dot{V} < 0$ whenever $|\eta|$ or $|e|$ is large which implies that $|\eta|$ and $|e|$ and, hence, $|\xi|$ and $|x|$, are bounded. The above analysis is valid for $(x, u) \in U_\epsilon \times B_\sigma$. Indeed, by choosing b_d sufficiently small and appropriate initial conditions, we can guarantee that the state remains in U_ϵ and the input is bounded by σ. Using this fact, we may abuse notation and write the function $\psi(x, u(x, \bar{y}_d))$ as $\epsilon\psi(t)$ and note that

$$\dot{e} = Ae + \epsilon\psi(t) \tag{3.71}$$

is an exponentially stable linear system driven by an order ϵ input. Thus, we conclude that the tracking error will be $O(\epsilon)$. □

3.4 Conclusion

In this chapter, we have presented an approach for the approximate input-output linearization of nonlinear systems, particularly those for which relative degree is not well defined. We saw that there is in fact a great deal of freedom in the selection of the approximation. We have seen that, by designing a tracking controller based on the approximating system, we can achieve tracking of reasonable trajectories with small error. The approximating system is a nonlinear system, with the difference that it is input-output linearizable by state feedback. We have shown some properties of the accuracy of the approximation and in the context of the ball and beam example shown it to be far superior to the Jacobian approximation. Future research in this area will include developing methods to effectively search among the prospective approximate systems and to evaluate their accuracy.

Chapter 4

Approximate Tracking for Slightly Nonminimum Phase Systems: Application to Flight Control

The method of input-output linearization provides a natural framework for the design of tracking controllers. This technique has in fact been successfully implemented in several practical applications, such as flight control [Ass73,MC75,MC80,LS88] and the control of rigid robots by the so-called computed torque method [Fre75]. The theory is now well developed and understood [Isi85,Isi87].

One of the major obstacles to the direct application of this theory is the fact that it relies on a nonlinear version of pole-zero cancellation. Of course, the nonlinear pole-zero cancellation implicit in these techniques is only a problem when the cancellation is one involving unstable zero dynamics (introduced in [BI84] and made precise in [IM89,Isi87]). In this chapter, we focus on this problem with specific emphasis on the aircraft control problem.

While several researchers have applied the methods of nonlinear control to the aircraft problem (see [LS88] for a nice summary), most have neglected the small moment-to-force coupling without proper justification. This coupling provides dynamic effects that *cannot* be *assumed* to be *bounded*! Due to the fact that we are building a closed loop feedback system, we *must* carefully analyze the effects of this coupling to guarantee that small changes in this parameter do not result in drastic changes in behavior such as the bifurcation behavior that can result from inertial coupling [HO74] or high angles-of-attack [MKC77]. In this chapter, we provide rigorous justification for the common practice of ignoring the moment-to-force coupling in the design of the controller.

This chapter is organized as follows: Section 4.1 discusses some modeling issues for aircraft dynamics and presents a simplified planar VTOL aircraft that will be used in the main discussion. In section 4.2, we work through the details of both exact and approximate input-ouput linearization for the simplified aircraft and present illustrative simulations showing the qualitative behavior of this system. In section 4.3, we develop the rudiments of a theory for the *approximate linearization* of *slightly non-minimum phase* systems.

4.1 Aircraft Dynamics

The complete dynamics of an aircraft, taking into account flexibility of the wings and fuselage, aeroelastic effects, the (internal) dynamics of the engine and control surface actuators, and the multitude of changing variables, are quite complex and somewhat unmanageable for the purposes of control. A useful first approximation is to consider the aircraft as a rigid body upon which a set of forces and moments act.

Then, with r, R, and ω being the aircraft position, orientation (rotation matrix), and angular velocity, respectively, the equations of motion can be written as

$$m\ddot{r} = Rf_a + mg \tag{4.1}$$

$$J\dot{\omega}_a = \tau_a - \omega_a \times J\omega_a \tag{4.2}$$

$$\dot{R} = \omega \times R \tag{4.3}$$

where f_a and τ_a are the force and moment acting on the aircraft expressed in the aircraft reference frame. Here, the a subscript means that a quantity is expressed with respect to the aircraft reference frame.

Depending on the aircraft and its mode of flight, the forces and moments can be generated by aerodynamics (lift, drag, and roll-pitch-yaw moments), by momentum exchange (gross thrust vectoring and reaction controls to generate moments), or a combination of the two. The flight envelope of the aircraft is the set of flight conditions for which the pilot and/or the control system can effect the forces and moments needed to remain in the envelope and achieve the desired task.

While the function mapping the control inputs to the forces and moments is a highly nonlinear state-dependent function, it is useful to note that this function can normally be decomposed as

$$\begin{pmatrix} f_a \\ \tau_a \end{pmatrix} = \mathcal{F}(x) + \mathcal{G}(x)u(x,c) \tag{4.4}$$

where $x \in \mathbf{R}^n$ denotes the state and $c \in \mathbf{R}^m$ denotes the control input and $\mathcal{F} : \mathbf{R}^n \to \mathbf{R}^6$, $\mathcal{G} : \mathbf{R}^n \to \mathbf{R}^{6\times m}$, and $u : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$ are (continuous) functions. In particular, for each x in the function $u(x,\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ is one-to-one and hence (algebraically) invertible. The value of the function $u(\cdot,\cdot)$ can often be taken to be the components of the force and moment that the actuators were designed to produce.

As an example, consider the YAV-8B Harrier produced by McDonnell Aircraft Company [McD82,McD83] depicted in figure 4.1 (aircraft frame-**A**, runway frame-**R**). The Harrier is a single-seat transonic light attack V/STOL (vertical/short takeoff and landing) aircraft powered by a single turbo-fan engine. Four exhaust nozzles on the turbo-fan engine provide the gross thrust for the aircraft. These nozzles (two on each side of the fuselage) can be simultaneously rotated from the aft position (used for conventional wing-borne flight) forward approximately 100 degrees allowing jet-borne flight and nozzle braking. The throttle and nozzle controls thus provide two degrees of freedom of thrust vectoring within the x-z plane of the aircraft. (If the line of action of the gross thrust does not pass through the aircraft center of mass, then this thrust will also produce a net pitching moment.)

In addition to the conventional aerodynamic control surfaces (aileron, stabilator (*stabi*lizer-elev*ator*), and rudder for roll, pitch, and yaw moments, respectively), the Harrier also has a reaction control system (RCS) to provide moment generation during jet-borne and transition flight. Reaction valves in the nose, tail, and wingtips use bleed air from the high pressure compressor of the engine to produce thrust at these points and therefore moments (and forces) at the aircraft center of mass. The design of the aerodynamic and reaction controls provides complete (three degree of freedom) moment generation throughout the flight envelope of the aircraft. When moments are produced by applying a single force rather than a couple, a nonzero force (proportional to the moment) will be seen at the aircraft center of mass.

Using the throttle, nozzle, roll, pitch, and yaw controls we can produce (within physical limits) any moment and any force in the x-z plane of the aircraft. Therefore, the

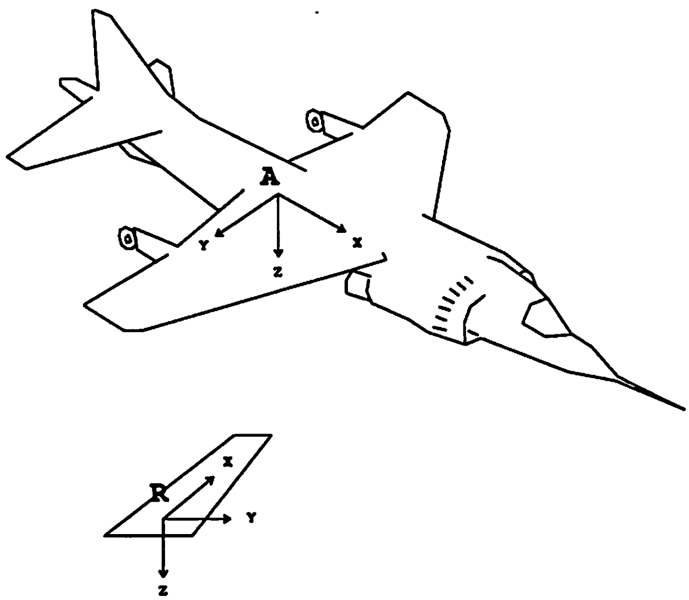

Figure 4.1: Aircraft coordinate systems (R-runway, A-aircraft)

function $u(\cdot,\cdot)$ for the Harrier can be chosen to map the control inputs to the moment and x-z force on the aircraft (with $u(x, 0) = 0$ so that the force and moment acting on the aircraft with the controls in the zero position are subsumed into $\mathcal{F}(x)$). With this choice of $u(\cdot,\cdot)$, five of the six rows of $\mathcal{G}(x)$ will be the rows of a 5×5 identity matrix. The remaining row will determine the side force f_{ay} and can easily be seen to form a (state-dependent) linear combination of the rolling and yawing moments.

Since the function $u(x,\cdot)$ can be inverted (on its range), we are free to consider u to be the input (control) rather than c. The idea of inverting the algebraic nonlinearities present in the system has been applied to real flight control problems [MC75,MC80]. With these considerations in mind, we see that the dynamics of the aircraft are of the general form

$$\dot{x} = f(x) + \sum_{1}^{m} g_i(x)u_i. \tag{4.5}$$

The small forces that are produced when moments are commanded result in some important effects. To examine these more closely, consider the geometry of the reaction control system as shown in figure 4.2. Since the roll moment reaction jets create a force that is not perpendicular to the y axis, the production of a positive rolling moment (to the pilot's right) will also produce a slight acceleration of the aircraft to the left. As we will

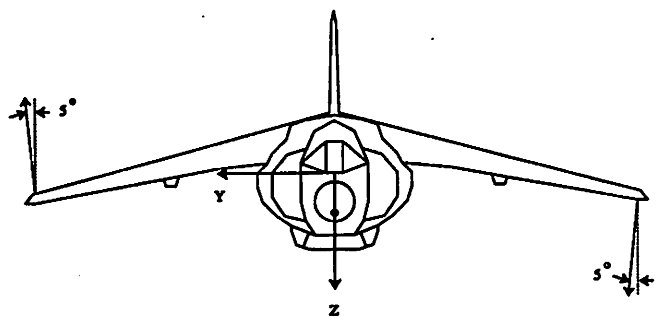

Figure 4.2: Reaction control system geometry

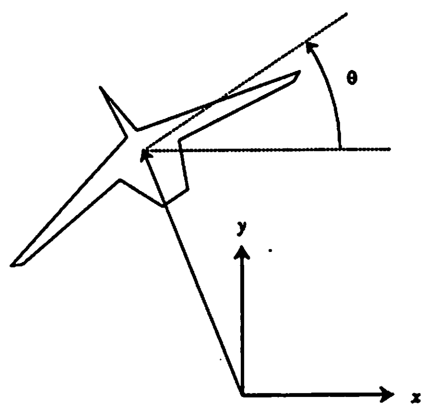

Figure 4.3: The planar vertical takeoff and landing (PVTOL) aircraft

see, this phenomenon makes the aircraft non-minimum phase.

4.1.1 A Simple Planar Aircraft

For the purpose of illustration, it is particularly useful to consider a simple toy aircraft that has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws for a real aircraft such as the Harrier. Figure 4.3 shows our prototype PVTOL (planar vertical takeoff and landing) aircraft. The aircraft state is simply the position, x, y, of the aircraft center of mass, the angle, θ, of the aircraft relative to the x-axis, and the corresponding velocities, $\dot{x}$, $\dot{y}$, $\dot{\theta}$. The control inputs, u_1, u_2, are the thrust (directed out the bottom of the aircraft) and the rolling moment.

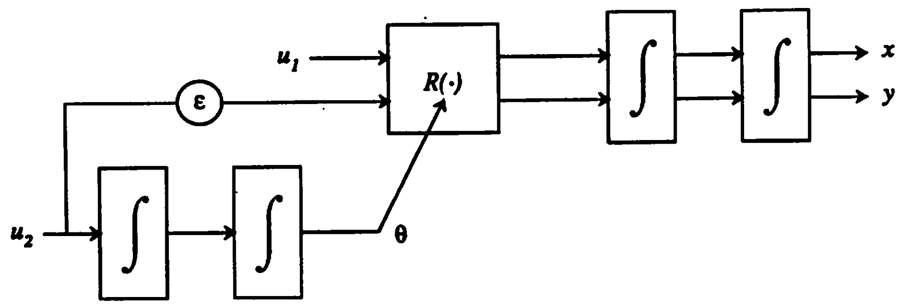

Figure 4.4: Block diagram of the PVTOL aircraft system

The equations of motion for our PVTOL aircraft are given by

$$\left.\begin{aligned} \ddot{x} &= -\sin\theta u_1 + \epsilon\cos\theta u_2 \\ \ddot{y} &= \cos\theta u_1 + \epsilon\sin\theta u_2 - 1 \\ \ddot{\theta} &= u_2 \end{aligned}\right\} \tag{4.6}$$

where '-1' is the gravitational acceleration and ϵ is the (small) coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft. Note that $\epsilon > 0$ means that applying a (positive) moment to roll left produces an acceleration to the right (positive x). Figure 4.4 provides a block diagram representation of this dynamical system.

The PVTOL aircraft system is the natural restriction of V/STOL aircraft to jet-borne operation (e.g., hover) in a vertical plane. The study of this simple planar model provides important insight that extends naturally to the more complicated six degree-of-freedom aircraft.

4.2 Linearization by State Feedback

4.2.1 Exact Input-Output Linearization of the PVTOL Aircraft System

Consider the PVTOL aircraft system given by (4.6). Since we are interested in controlling the aircraft *position*, we choose x and y as the outputs to be controlled. We seek a (possibly dynamic) state feedback law of the form

$$u = a(z) + b(z)v \tag{4.7}$$

such that, for some $\gamma = (\gamma_1, \gamma_2)^T$,

$$\begin{aligned} x^{(\gamma_1)} &= v_1 \\ y^{(\gamma_2)} &= v_2. \end{aligned} \tag{4.8}$$

Here, v is our new input and z is used to denote the entire state of the system (including compensator states, if necessary).

Proceeding in the usual way, we differentiate each output until at least one of the inputs appears. This occurs after differentiating twice and is given by (rewriting the first two equations of (4.6))

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{4.9}$$

Since the matrix operating on u (the so-called *decoupling* matrix) is nonsingular (barely—its determinant is $-\epsilon$!), we can linearize (and decouple) the system by choosing the static state feedback law

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{\cos\theta}{\epsilon} & \frac{\sin\theta}{\epsilon} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right). \tag{4.10}$$

The resulting system is

$$\left.\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{\theta} &= \tfrac{1}{\epsilon}(\sin\theta + \cos\theta v_1 + \sin\theta v_2) \end{aligned}\right\} \tag{4.11}$$

This feedback law makes our input-output map linear, but has the unfortunate side-effect of making the dynamics of θ unobservable. In order to guarantee the internal stability of the system, it is not sufficient to look at input-output stability, we must also show that all internal (unobservable) modes of the system are stable as well.

The first step in analyzing the internal stability of the system (4.11) is to look at the *zero dynamics* [BI84,IM89,Isi87] of the system. The zero dynamics of a nonlinear system are the internal dynamics of the system subject to the constraint that the outputs (and, therefore, all derivatives of the outputs) are set to zero for all time.

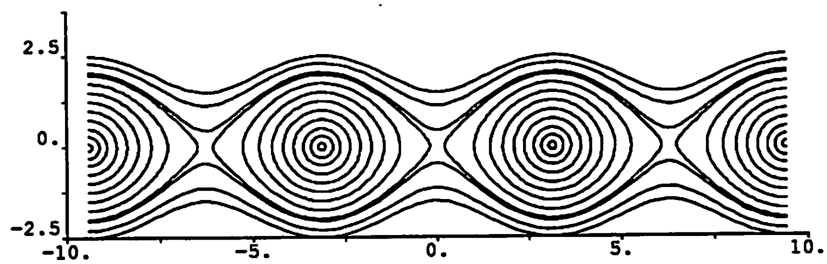

Figure 4.5: Phase portrait of an undamped pendulum ($\dot{\theta}$ vs. θ, $\epsilon = 1$)

Constraining the outputs and their derivatives to zero by setting $v_1 = v_2 = 0$ (and using appropriate initial conditions), we find the zero dynamics of (4.11) to be

$$\ddot{\theta} = \frac{1}{\epsilon} \sin \theta. \tag{4.12}$$

Equation (4.12) is simply the equation of an undamped pendulum. Figure 4.5 shows the phase portrait ($\dot{\theta}$ vs θ) of the pendulum (4.12) with $\epsilon = 1$. The phase portrait for $\epsilon < 0$ is simply a horizontal π-translate of figure 4.5. Thus, for $\epsilon > 0$, the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is unstable and the equilibrium point $(\pi, 0)$ is stable but not asymptotically stable and is surrounded by a family of periodic orbits with periods ranging from $2\pi\sqrt{\epsilon}$ to ∞. Outside of these periodic orbits is a family of unbounded trajectories. Thus, depending on the initial condition, the aircraft will either rock from side to side forever or roll continuously in one direction (except at the isolated equilibria).

Nonlinear systems, such as (4.11), with zero dynamics that are not asymptotically stable are called *non-minimum phase*. Figure 4.6 shows the response of the system (4.11) when (v_1, v_2) is chosen (by a stable feedback law) so that x will track a smooth trajectory from $x = 0$ to $x = 1$ with y remaining at zero. The bottom section of the figure shows snapshots of the PVTOL aircraft's position and orientation at 0.2 second intervals. From the phase portrait of θ (figure 4.6e), we see that the zero dynamics certainly exhibit pendulum like behavior. Initially, the aircraft rolls left (positive θ) to almost 2π. Then, it rolls right through four revolutions before settling into a periodic motion about the -3π equilibrium point. Since v_1 and v_2 are zero after $t = 5$, the aircraft continues rocking approximately $\pm\pi$ from the inverted position.

From the above analysis and simulations, it is clear that exact input-output linearization of a system such as (4.6) can produce undesirable results. The source of the

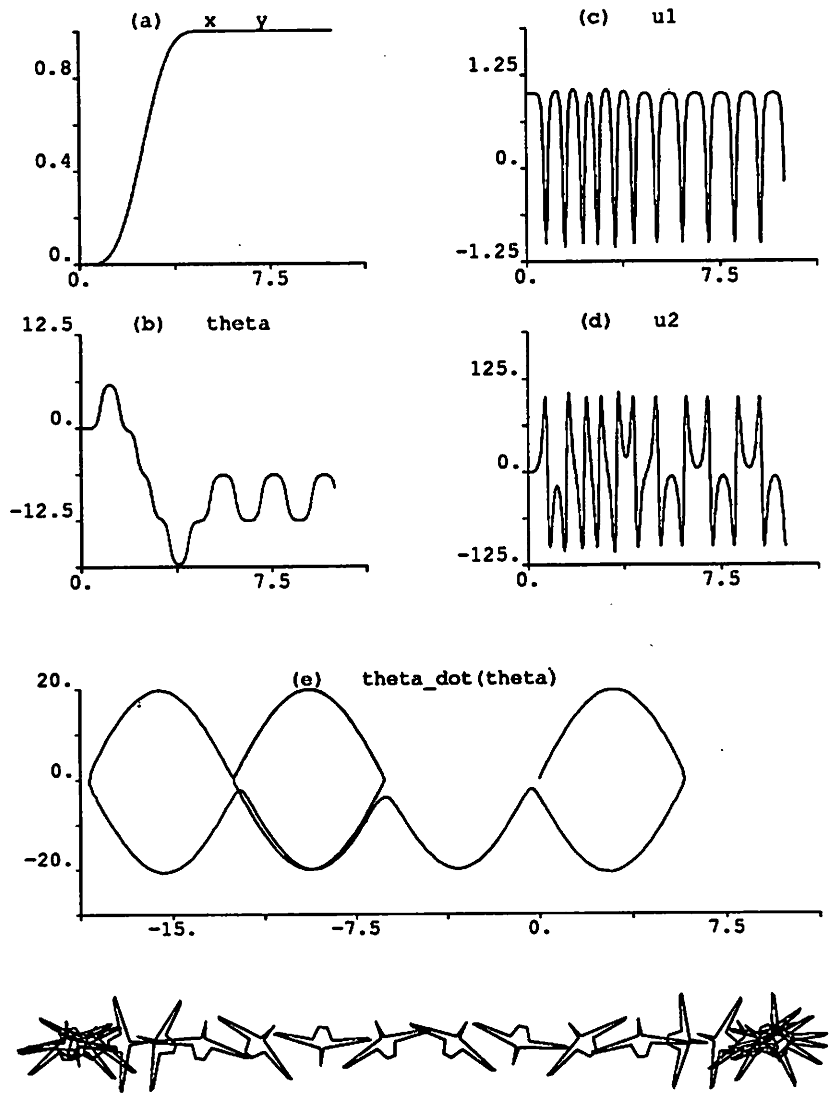

Figure 4.6: Response of non-minimum phase system to smooth *step* input

problem lies in trying to control modes of the system using inputs that are weakly (ϵ) coupled rather than controlling the system in the way it was designed to be controlled and accepting a performance penalty for the parasitic (ϵ) effects. For our simple PVTOL aircraft, we should control the linear acceleration by vectoring the thrust vector (using moments to control this vectoring) and adjusting its magnitude using the throttle.

4.2.2 Approximate Linearization of the PVTOL Aircraft System using a Simplified Model

In the last section we (exactly) linearized input-output map of the PVTOL aircraft system (4.6). However, due to the small coupling between rolling moments and lateral acceleration, the linearized system had unstable zero dynamics. Thus, while the outputs (the x and y position) can be tracked perfectly, the internal behavior (the aircraft attitude) is not regulated and exhibits unstable behavior.

In this section, we propose controlling the system as if there were no coupling between rolling moments and lateral acceleration (i.e., $\epsilon = 0$). Using this approach to control the true system (4.6), we expect to see a loss of performance due to the *unmodeled* dynamics present in the system. In particular, we see that we can guarantee stable asymptotic tracking of constant velocity trajectories and bounded tracking for trajectories with bounded higher order derivatives.

We now *model* the PVTOL aircraft as ((4.6) with $\epsilon = 0$)

$$\left.\begin{array}{rcl} \ddot{x}_m & = & -\sin\theta u_1 \\ \ddot{y}_m & = & \cos\theta u_1 - 1 \\ \ddot{\theta} & = & u_2 \end{array}\right\} \tag{4.13}$$

so that there is no coupling between rolling moments and lateral acceleration. Differentiating the model system outputs, x_m and y_m, we get (analogous to (4.9))

$$\begin{bmatrix} \ddot{x}_m \\ \ddot{y}_m \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{4.14}$$

Now, however, the matrix multiplying u is singular which implies that there is no *static* state feedback that will linearize (4.13). Since u_2 comes into the system (4.13) through $\ddot{\theta}$, we must differentiate (4.14) at least two more times. Let u_1 and $\dot{u}_1$ be states (in effect,

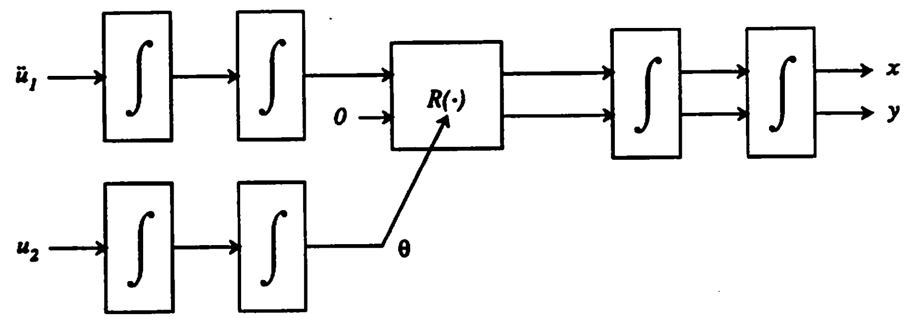

Figure 4.7: Block diagram of the augmented model PVTOL aircraft system

placing two integrators before the u_1 input) and differentiate (4.14) twice giving

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} \sin\theta\dot{\theta}^2 u_1 - 2\cos\theta\dot{\theta}\dot{u}_1 \\ -\cos\theta\dot{\theta}^2 u_1 - 2\sin\theta\dot{\theta}\dot{u}_1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & -\cos\theta u_1 \\ \cos\theta & -\sin\theta u_1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}. \tag{4.15}$$

The matrix operating on our new inputs, $(\ddot{u}_1, u_2)^T$, has determinant equal to u_1 and therefore is invertible as long as the thrust, u_1, is nonzero. This fact agrees well with our intuition since we know that no amount of rolling will affect the motion of the PVTOL aircraft if there is no thrust to effect an acceleration. Figure 4.7 shows a block diagram of the model system with u_1 and $\dot{u}_1$ considered as states. Note that each *input* must go through four integrators to get to the output. Thus, we linearize (4.13) using the *dynamic* state feedback law

$$\begin{aligned} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{u_1} & -\frac{\sin\theta}{u_1} \end{bmatrix} \left(\begin{bmatrix} -\sin\theta\dot{\theta}^2 u_1 + 2\cos\theta\dot{\theta}\dot{u}_1 \\ \cos\theta\dot{\theta}^2 u_1 + 2\sin\theta\dot{\theta}\dot{u}_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} \dot{\theta}^2 u_1 \\ -\frac{2\dot{\theta}\dot{u}_1}{u_1} \end{bmatrix} + \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{u_1} & -\frac{\sin\theta}{u_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned} \tag{4.16}$$

resulting in

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{4.17}$$

Unlike the previous case (equation (4.11)), the linearized model system does not contain any unobservable (zero) dynamics. Thus, using a stable tracking law for v, we can track an arbitrary trajectory and guarantee that the (model) system will be stable.

Of course, the natural question that comes to mind is: will a control law based on the model system (4.13) work well when applied to the true system (4.6)? In the next section, we will show (in a more general setting) that, if ϵ is *small* enough, then the system will have reasonable properties (such as stability and bounded tracking).

How small is small enough? Figure 4.8 shows the response of the true system with epsilon ranging from 0 to 0.9 (0.01 is typical during jet-borne flight, i.e., hover, for the Harrier). As in section 4.2.1, the desired trajectory is a smooth lateral motion from $x = 0$ to $x = 1$ with the altitude (y) held constant at 0. The figure also shows snapshots of the PVTOL aircraft's position orientation at 0.2 second intervals for $\epsilon = 0.0$, 0.1, and 0.3. Since the snapshots were taken at uniform intervals, the spacing between successive pictures gives a clue of the aircraft velocity and acceleration. The computer graphics movie of the trajectories provides an even better sense of the system response.

Interestingly, the x response is quite similar to the step response of a non-minimum phase linear system. Note that for ϵ less than approximately 0.6, the oscillations are reasonably damped. Although performance is certainly worse at higher values of ϵ, stability does not appear to be lost until ϵ is in the neighborhood of 0.9. A value of 0.9 for ϵ means that the aircraft will experience almost $1g$ (the acceleration of gravity) in the wrong direction when a rolling acceleration of one radian per second per second is applied. For the range of ϵ values that will normally be expected, the performance penalty due to approximation is small, almost imperceptible.

Note that, while the PVTOL aircraft system (4.6) with the approximate control (4.16) is stable for a large range of ϵ, this control allows the PVTOL aircraft to have a bounded but unacceptable altitude (y) deviation. Since the ground is hard and quite unforgiving and vertical takeoff and landing aircraft are designed to be maneuvered in close proximity to the ground, it is extremely desirable to find a control law that provides exact tracking of altitude if possible. Now, ϵ enters the system dynamics (4.6) in only one (state-dependent) direction. We therefore expect that one should be able to modify the system (by manipulating the inputs) so that the effects of the ϵ-coupling between rolling moments and aircraft lateral acceleration do not appear in the y output of the system.

Consider the decoupling matrix of the true PVTOL system (4.6) given in (4.9) as

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix}. \tag{4.18}$$

To make the y output independent of ϵ requires that the last row of this decoupling matrix be independent of ϵ. The only legal way to do this is by multiplication on the right (i.e.,

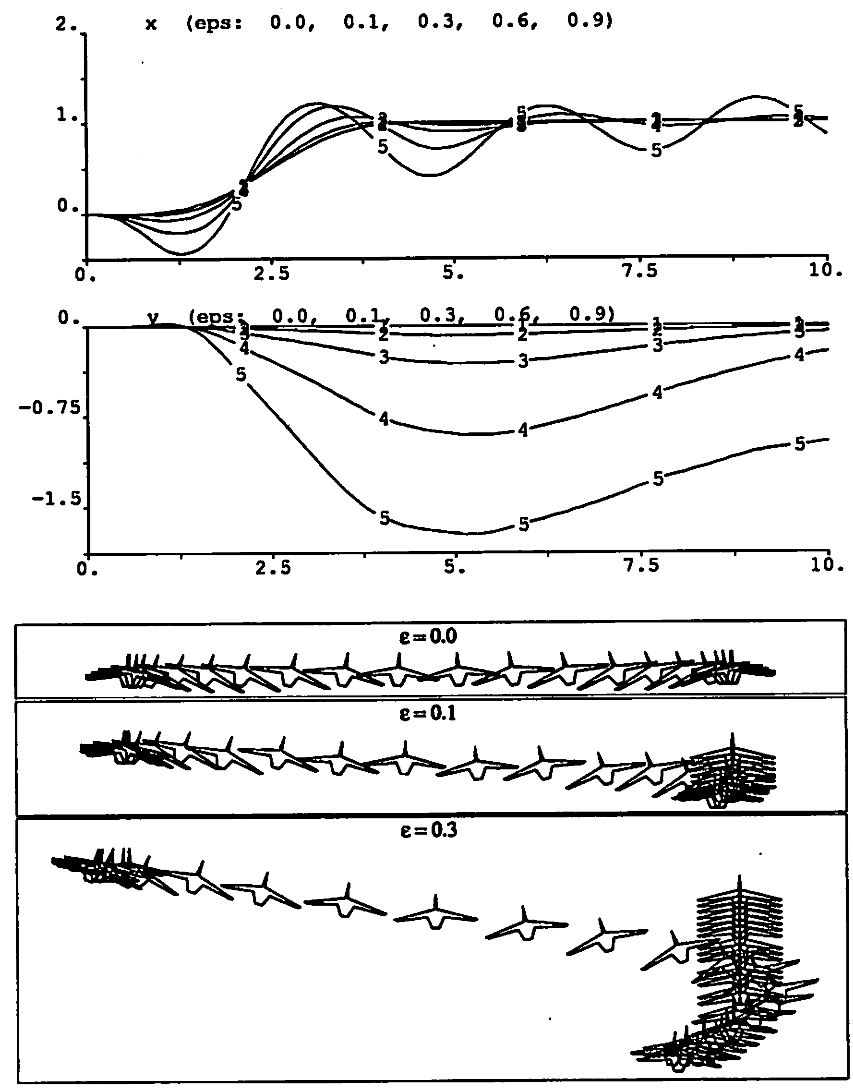

Figure 4.8: Response of the true PVTOL aircraft system under the approximate control

column operations) by a nonsingular matrix V which corresponds to multiplying the inputs by V^{-1}. In this case, we see that

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \begin{bmatrix} 1 & -\epsilon\tan\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \frac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix} \tag{4.19}$$

is the desired transformation. Defining new inputs, $\tilde{u}$, as

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon\tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.20}$$

we see that (4.9) becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & \frac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}. \tag{4.21}$$

Following the previous analysis, we set $\epsilon = 0$ and linearize the resulting approximate system using the dynamic feedback law

$$\begin{bmatrix} \ddot{\tilde{u}}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2\tilde{u}_1 \\ -\frac{2\dot{\theta}\dot{\tilde{u}}_1}{\tilde{u}_1} \end{bmatrix} + \begin{bmatrix} -\sin\theta & \cos\theta \\ -\frac{\cos\theta}{\tilde{u}_1} & -\frac{\sin\theta}{\tilde{u}_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.22}$$

Note that this control law will approximately linearize the true system. The true system inputs are then calculated as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon\tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}. \tag{4.23}$$

Figure 4.9 shows the response of the true system using the control law specified by equations (4.22) and (4.23) for the same desired trajectory. With this control law, our PVTOL aircraft maintains the altitude as desired and provides stable, bounded lateral (x) tracking for ϵ up to at least 0.6. Note, however, that the system is decidedly unstable for $\epsilon = 0.9$. Since we have forced the error into one direction (i.e., the x-channel), we expect the approximation to be more sensitive to the value of ϵ. In particular, compare the second column of the decoupling matrices of (4.9) and (4.21), i.e.,

$$\begin{bmatrix} \epsilon\cos\theta \\ \epsilon\sin\theta \end{bmatrix} \text{ and } \begin{bmatrix} \frac{\epsilon}{\cos\theta} \\ 0 \end{bmatrix}. \tag{4.24}$$

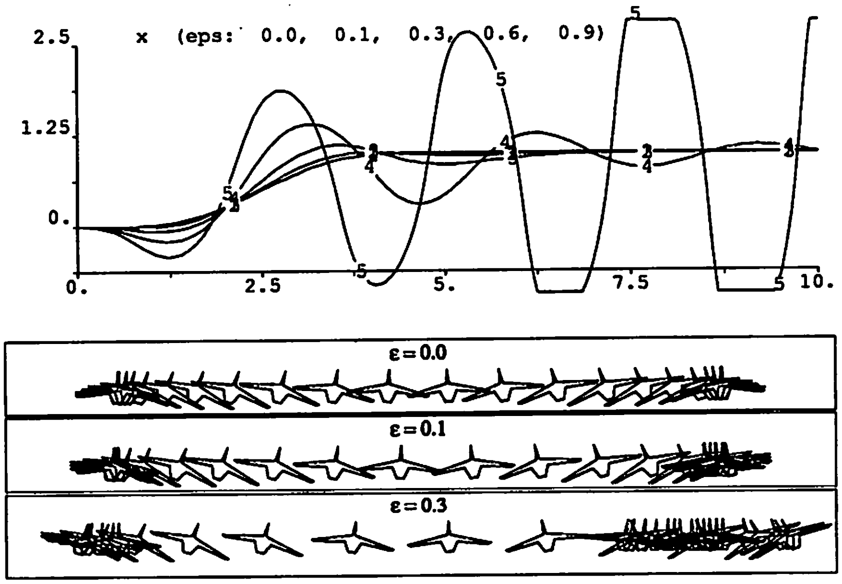

Figure 4.9: Response of the true PVTOL aircraft system under the approximate control with input transformation

Notice that the first is simply ϵ times a bounded function of θ while the second contains ϵ times an unbounded function of θ (i.e., $1/\cos\theta$). Thus, for (4.21) with $\epsilon = 0$ to be a good approximation to (4.21) with non-zero ϵ requires that θ be bounded away from $\pm\pi/2$. This is not a completely unreasonable requirement since most V/STOL aircraft do not have a large enough thrust to weight ratio to maintain level flight with a large roll angle. Since the physical limits of the aircraft usually place constraints on the achievable trajectories, a control law analogous to that defined by (4.22) and (4.23) can be used for systems with small ϵ on reasonable trajectories.

4.3 A Formal Approach to the Control of Slightly Non-minimum Phase Systems

In this section we will take a more formal approach to the control of systems that are slightly non-minimum phase.

Consider the class of nonlinear systems of the form

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}\right\} \tag{4.25}$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$, and $f : \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^{n \times m}$ are smooth vector fields and $h : \mathbf{R}^n \to \mathbf{R}^m$ is a smooth function with $h(0) = 0$.

In the sequel, we will assume that the origin is an equilibrium point of (4.25), i.e., $f(0) = 0$, and will consider x in an open neighborhood, U, of the origin, i.e., the analysis will be *local*. All statements that we make, such as the existence of certain diffeomorphisms, will be assumed merely to hold in U. Also, when we say that a function is zero, it vanishes on U, and when we say it is non-zero, we mean that it is bounded away from zero on U.

While we will not precisely define *slightly non-minimum phase* systems, the concept is easy enough to explain. The reader may wish to review the definition of the zero dynamics for non-linear systems (and the concept of minimum phase) in Isidori and Moog [IM89].

4.3.1 Single-Input Single-Output (SISO) Case

Consider first the single-input single-output (SISO) case. Suppose that $L_g h(x) = \epsilon\psi(x)$ for some scalar function $\psi(x)$ with $\epsilon > 0$ small. In other words, the *relative degree* of the system is one, but is very close to being greater than one. Here, $L_g h(x)$ is the *Lie derivative* of $h(\cdot)$ along $g(\cdot)$ and is defined to be

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x). \tag{4.26}$$

Now, define two systems in *normal* form (see Byrnes and Isidori [BI88]) using the following two sets of local diffeomorphisms of $x \in \mathbf{R}^n$

$$(\xi^T, \eta^T)^T = (\xi_1 := h(x), \eta_1(x), \ldots, \eta_{n-1}(x))^T, \tag{4.27}$$

and

$$(\tilde{\xi}^T, \tilde{\eta}^T)^T = (\tilde{\xi}_1 := h(x), \tilde{\xi}_2(x) := L_f h(x), \tilde{\eta}_1(x), \ldots, \tilde{\eta}_{n-2}(x))^T, \tag{4.28}$$

with

$$\frac{\partial \eta_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-1 \tag{4.29}$$

and

$$\frac{\partial \tilde{\eta}_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-2. \tag{4.30}$$

System 1 (true system)

$$\left.\begin{aligned} \dot{\xi}_1 &= L_f h(x) + L_g h(x) u \\ \dot{\eta} &= q(\xi, \eta) \end{aligned}\right\} \tag{4.31}$$

System 2 (approximate system)

$$\left.\begin{aligned}\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 &= L_f^2 h(x) + L_g L_f h(x) u \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})\end{aligned}\right\} \tag{4.32}$$

Note that the system (4.31) represents the system (4.25) in *normal form* and the dynamics of $q(0, \eta)$ represent the *zero dynamics* of the system (4.25). System (4.32) does not represent the system (4.25), since in the $(\tilde{\xi}, \tilde{\eta})$ coordinates of (4.28), the dynamics of (4.25) are given by

$$\left.\begin{aligned}\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + L_g h(x) u \\ \dot{\tilde{\xi}}_2 &= L_f^2 h(x) + L_g L_f h(x) u \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})\end{aligned}\right\}. \tag{4.33}$$

Informally, we call the system (4.25) *slightly* non-minimum phase *if the true system (4.31) (with nonzero* ϵ*) is non-minimum phase but the approximate system (4.32) (with* $\epsilon = 0$*) is minimum phase.* Since $L_g h(x) = \epsilon\psi(x)$, we may think of the system (4.32) as a perturbation of the system (4.31) ($\equiv$ (4.33)).

Of course, there are two difficulties with exact input-output linearization of (4.31):

- The input-output linearization requires a large control effort since the linearizing control is

$$u^*(x) = \frac{1}{L_g h(x)}(-L_f h(x) + v) = \frac{-L_f h(x) + v}{\epsilon\psi(x)}. \tag{4.34}$$

  This could present difficulties in the instance that there is saturation at the control inputs.

- If (4.31) is non-minimum phase, a tracking control law producing a linear input-output response may result in unbounded η states.

Our prescription for the *approximate* input-output linearization of the system (4.31) is to use the input-output linearizing control law for the approximate system (4.32); namely

$$u_a^* = \frac{1}{L_g L_f h(x)}(-L_f^2 h(x) + v) \tag{4.35}$$

where v is chosen depending on the control task. For instance, if y is required to track y_d, we choose v as

$$\begin{aligned} v &= \ddot{y}_d + \alpha_1(\dot{y}_d - \tilde{\xi}_2) + \alpha_0(y_d - \tilde{\xi}_1) && (4.36) \\ &= \ddot{y}_d + \alpha_1(\dot{y}_d - L_f h(x)) + \alpha_0(y_d - h(x)). && (4.37) \end{aligned}$$

Using (4.35) and (4.36) in (4.33) along with the definitions

$$\begin{aligned} e_1 &= \tilde{\xi}_1 - y_d \\ e_2 &= \tilde{\xi}_2 - \dot{y}_d \end{aligned} \qquad (4.38)$$

yields

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \qquad\qquad + \epsilon\psi(x)u_a^*(x) \\ \dot{e}_2 &= -\alpha_1 e_2 - \alpha_0 e_1 \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}). \end{aligned}\right\} \qquad (4.39)$$

As we will see below, exponential stability of the zero dynamics of the approximate system (i.e., $\dot{\tilde{\eta}} = \tilde{q}(0, \tilde{\eta})$) combined with the designed stability of the error system will guarantee overall stability of the system and yield approximate tracking.

The preceding discussion may be generalized to the case when the difference in the relative degrees between the true system and the approximate system is greater than one. For example, if

$$\begin{aligned} L_g h(x) &= \epsilon\psi_1(x) \\ L_g L_f h(x) &= \epsilon\psi_2(x) \\ &\vdots \\ L_g L_f^{\gamma-2} h(x) &= \epsilon\psi_{\gamma-1}(x) \end{aligned} \qquad (4.40)$$

but $L_g L_f^{\gamma} h(x)$ is not of order ϵ, we define

$$(\tilde{\xi}^T, \tilde{\eta}^T) = (h(x), L_f h(x), \ldots, L_f^{\gamma-1} h(x), \tilde{\eta}^T)^T \in \mathbf{R}^n \qquad (4.41)$$

and note that the true system is

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \epsilon\psi_1(x)u \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma + \epsilon\psi_{\gamma-1}(x)u \\
\dot{\tilde{\xi}}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}).
\end{aligned} \tag{4.42}$$

The approximate (minimum phase) system (with $\epsilon = 0$) is given by

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma \\
\dot{\tilde{\xi}}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned} \tag{4.43}$$

The approximate tracking control law for (4.43) is

$$u_a = \frac{1}{L_g L_f^{\gamma-1} h(x)}(-L_f^\gamma h(x) + y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - L_f^{\gamma-1} h(x)) + \cdots + \alpha_0(y_d - y)). \tag{4.44}$$

The following theorem provides a bound for the performance of this control when applied to the true system.

Theorem 4.1 *Suppose that*

- *the zero dynamics of the approximate system (4.43) are locally* exponentially stable *and*
- *the functions $\psi(x)u_a(x)$ are locally Lipschitz continuous.*

Then, for ϵ sufficiently small and for desired trajectories with sufficiently small values and derivatives (y_d, $\dot{y}_d$, ..., $y_d^{(\gamma)}$), the states of the system (4.42) will be bounded and the tracking error

$$|e_1| := |\tilde{\xi}_1 - y_d| \leq k\epsilon \tag{4.45}$$

for some $k < \infty$.

Proof: Define the trajectory error, $e \in \mathbf{R}^\gamma$, to be

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\gamma \end{bmatrix} = \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_\gamma \end{bmatrix} - \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\gamma-1)} \end{bmatrix}. \tag{4.46}$$

Then, the system (4.42) with the approximate tracking control (4.44) may be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{\gamma-1} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{\gamma-1}(x) \\ 0 \end{bmatrix} u_a(x) \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{aligned} \tag{4.47}$$

or, compactly,

$$\begin{aligned} \dot{e} &= Ae + \epsilon\psi(x)u_a(x) \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}). \end{aligned} \tag{4.48}$$

Since the zero dynamics are assumed to be exponentially stable, a converse Lyapunov theorem implies the existence of a Lyapunov function (see, e.g., [Hah67]) $v_2(\tilde{\eta})$ for the system

$$\dot{\tilde{\eta}} = \tilde{q}(0, \tilde{\eta}) \tag{4.49}$$

satisfying

$$\begin{gathered} k_1|\tilde{\eta}|^2 \le v_2(\tilde{\eta}) \le k_2|\tilde{\eta}|^2 \\ \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0, \tilde{\eta}) \le -k_3|\tilde{\eta}|^2 \\ \left|\frac{\partial v_2}{\partial \tilde{\eta}}\right| \le k_4|\tilde{\eta}| \end{gathered} \tag{4.50}$$

for some positive constants k_1, k_2, k_3, and k_4.

We first show that e and $\tilde{\eta}$ are bounded. To this end, consider as Lyapunov function for the error system (4.48)

$$v(e, \tilde{\eta}) = e^T Pe + \mu v_2(\tilde{\eta}) \tag{4.51}$$

where $P > 0$ is chosen so that

$$A^T P + PA = -I \tag{4.52}$$

(possible since $\dot{e} = Ae$ is stable) and μ is a positive constant to be determined later.

Note that, by assumption, y_d and its first $(\gamma - 1)$ derivatives are bounded,

$$|\tilde{\xi}| \leq |e| + b_d, \tag{4.53}$$

the functions, $\tilde{q}(\tilde{\xi}, \tilde{\eta})$ and $\psi(x)u_a(x)$ are locally Lipschitz with $\psi(0)u_a(0) = 0$,

$$|\tilde{q}(\tilde{\xi}^1, \tilde{\eta}^1) - \tilde{q}(\tilde{\xi}^2, \tilde{\eta}^2)| \leq l_q(|\tilde{\xi}^1 - \tilde{\xi}^2| + |\tilde{\eta}^1 - \tilde{\eta}^2|), \tag{4.54}$$

$$|2P\psi(x)u_a(x)| \leq l_u|x|, \tag{4.55}$$

and x is a local diffeomorphism of $(\tilde{\xi}, \tilde{\eta})$,

$$|x| \leq l_x(|\tilde{\xi}| + |\tilde{\eta}|). \tag{4.56}$$

Using these bounds and the properties of $v_2(\cdot)$, we have

$$\begin{aligned} \tfrac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(\tilde{\xi}, \tilde{\eta}) &= \tfrac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0, \tilde{\eta}) + \tfrac{\partial v_2}{\partial \tilde{\eta}}(\tilde{q}(\tilde{\xi}, \tilde{\eta}) - \tilde{q}(0, \tilde{\eta})) \\ &\leq -k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d). \end{aligned} \tag{4.57}$$

Taking the derivative of $v(\cdot, \cdot)$ along the trajectories of (4.48), we find

$$\begin{aligned} \dot{v} &= -|e|^2 + 2\epsilon e^T P\psi(x)u_a(x) + \mu\tfrac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(\tilde{\xi}, \tilde{\eta}) \\ &\leq -|e|^2 + \epsilon|e|l_u l_x(|e| + b_d + |\tilde{\eta}|) + \mu(-k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d)) \\ &\leq -(\tfrac{|e|}{2} - \epsilon l_u l_x b_d)^2 + (\epsilon l_u l_x b_d)^2 \\ &\quad -(\tfrac{|e|}{2} - (\epsilon l_u l_x + \mu k_4 l_q)|\tilde{\eta}|)^2 + (\epsilon l_u l_x + \mu k_4 l_q)^2|\tilde{\eta}|^2 \\ &\quad -\mu k_3(\tfrac{|\tilde{\eta}|}{2} - \tfrac{k_4 l_q b_d}{k_3})^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3} \\ &\quad -(\tfrac{1}{2} - \epsilon l_u l_x)|e|^2 - \tfrac{3}{4}\mu k_3|\tilde{\eta}|^2 \\ &\leq -(\tfrac{1}{2} - \epsilon l_u l_x)|e|^2 - (\tfrac{3}{4}\mu k_3 - (\epsilon l_u l_x + \mu k_4 l_q)^2)|\tilde{\eta}|^2 \\ &\quad +(\epsilon l_u l_x b_d)^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3}. \end{aligned} \tag{4.58}$$

Define

$$\mu_0 = \frac{k_3}{4(l_u l_x + k_4 l_q)^2}. \tag{4.59}$$

Then, for all $\mu \leq \mu_0$ and all $\epsilon \leq min(\mu, \frac{1}{4 l_u l_x})$, we have

$$\dot{v} \leq -\frac{|e|^2}{4} - \frac{\mu k_3|\tilde{\eta}|^2}{2} + \frac{\mu(k_4 l_q b_d)^2}{k_3} + (\epsilon l_u l_x b_d)^2. \tag{4.60}$$

Thus, $\dot{v} < 0$ whenever $|\tilde{\eta}|$ or $|e|$ is large which implies that $|\tilde{\eta}|$ and $|e|$ and, hence, $|\tilde{\xi}|$ and $|x|$, are bounded. The above analysis is valid in a neighborhood of the origin. By choosing b_d sufficiently small and with appropriate initial conditions, we can guarantee the state will remain in a small small neighborhood. Using the boundedness of x and the continuity of $\psi(x)u_a(x)$, we see that

$$\dot{e} = Ae + \epsilon\psi(x)u_a(x) \tag{4.61}$$

is an exponentially stable linear system driven by an order ϵ input. Thus, we conclude that the tracking error, e, converges to a ball of order ϵ. □

When the control objective is stabilization and the approximate system has no zero dynamics we can do much better. In this case, one can show then that the control law that stabilizes the approximate system also stabilizes the original system.

Suppose that the approximate system has no zero dynamics, i.e.,

$$\begin{aligned} L_g h(x) &= \epsilon\psi_1(x) \\ L_g L_f h(x) &= \epsilon\psi_2(x) \\ &\vdots \\ L_g L_f^{n-2} h(x) &= \epsilon\psi_{n-1}(x) \end{aligned} \tag{4.62}$$

Define

$$\tilde{\xi} = (h(x), L_f h(x), \ldots, L_f^{n-1} h(x))^T \in \mathbf{R}^n \tag{4.63}$$

and write the approximate system

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\ &\vdots \\ \dot{\tilde{\xi}}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{aligned} \tag{4.64}$$

and the stabilizing control law

$$u_s(x) = \frac{1}{L_g L_f^{n-1} h(x)}(-L_f^n h(x) - \alpha_{n-1}\tilde{\xi}_{n-1} - \cdots - \alpha_0\tilde{\xi}_1) \tag{4.65}$$

$$= \frac{1}{L_g L_f^{n-1} h(x)}(-L_f^n h(x) - \alpha_{n-1} L_f^{n-1} h(x) - \cdots - \alpha_0 h(x)). \tag{4.66}$$

The true system in these coordinates is given by

$$
\begin{aligned}
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \epsilon\psi_1(x)u \\
&\vdots \\
\dot{\tilde{\xi}}_{n-1} &= \tilde{\xi}_n + \epsilon\psi_{n-1}(x)u \\
\dot{\tilde{\xi}}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u
\end{aligned}
\tag{4.67}
$$

Using $u_s(x)$ (from (4.65)) in (4.67) yields

$$
\begin{bmatrix} \dot{\tilde{\xi}}_1 \\ \vdots \\ \dot{\tilde{\xi}}_{n-1} \\ \dot{\tilde{\xi}}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_{n-1} \\ \tilde{\xi}_n \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{n-1}(x) \\ 0 \end{bmatrix} u_s(x). \tag{4.68}
$$

Letting $\psi(x) = (\psi_1(x), \ldots, \psi_{n-1}(x), 0)^T$, we can state the following:

Theorem 4.2 *Suppose that $\psi(x)u_s(x)$ is Lipschitz in x and that $\psi(0)u_s(0) = 0$. Then, the system (4.68) is exponentially stable for ϵ sufficiently small.*

Proof: The stabilized system (4.68) can be compactly written as

$$
\dot{\tilde{\xi}} = A\tilde{\xi} + \epsilon\psi(x)u_s(x). \tag{4.69}
$$

Choose as Lyapunov function $v = \tilde{\xi}^T P \tilde{\xi}$ with $A^T P + PA = -I$. Then, using the bounds analogous to (4.55) and (4.56), the derivative of v along trajectories of (4.69) is given by

$$
\begin{aligned}
\dot{v} &= -|\tilde{\xi}|^2 + 2\epsilon P\psi(x)u_s(x) \\
&<= -(1 - \epsilon l_u l_x)|\tilde{\xi}|^2.
\end{aligned}
\tag{4.70}
$$

Thus, for all $\epsilon < \epsilon_0 := \frac{1}{l_u l_x}$, the system (4.69) is exponentially stable. □

4.3.2 Generalization to MIMO Systems

We now consider MIMO systems of the form (4.25) which, for the sake of convenience, we rewrite as

$$\left.\begin{array}{rcl} \dot{x} & = & f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \\ y_1 & = & h_1(x) \\ & \vdots & \\ y_m & = & h_m(x) \end{array}\right\} \qquad (4.71)$$

Let γ_i be the relative degree of the ith output, i.e., we need to differentiate y_i at least γ_i times before at least one of the inputs appears in the right hand side. Then, we have

$$y_i^{(\gamma_i)} = L_f^{\gamma_i} h_i + L_{g_1} L_f^{\gamma_i - 1} h_i u_1 + \cdots + L_{g_m} L_f^{\gamma_i - 1} h_i u_m \quad i = 1, \ldots, m. \qquad (4.72)$$

The decoupling matrix is defined to be $A(x) \in \mathbf{R}^{m \times m}$ with

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m \end{bmatrix} \qquad (4.73)$$

so that

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \qquad (4.74)$$

If the decoupling matrix $A(x)$ is non-singular, the control law

$$u(x) = A(x)^{-1} \left(- \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + v \right) \qquad (4.75)$$

with $v \in \mathbf{R}^m$ linearizes (and decouples) the system (4.71) resulting in

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \qquad (4.76)$$

To take up the ideas of Section 4.3.1, we will first consider the case when $A(x)$ is non-singular but is *close to being singular*, that is, its smallest singular value is uniformly small for $x \in U$. Definitions of zero dynamics for MIMO systems are considerably more subtle than those for SISO systems and the reader may wish to review them in [IM89,Isi87] before proceeding further. Since $A(x)$ is close to being singular, i.e., it is close in norm to a matrix of rank $m-1$, we may transform $A(x)$ using elementary column operations to get

$$\bar{A}^0(x) = A(x)V^0(x) = \left[\begin{array}{cccc} \bar{a}^0_{.1}(x) & \cdots & \bar{a}^0_{.,m-1}(x) & \epsilon\bar{a}^0_{.m}(x) \end{array} \right] \tag{4.77}$$

where each $\bar{a}^0_{.i}$ is a column of $\bar{A}^0$. This corresponds to redefining the inputs to be

$$\begin{bmatrix} \bar{u}^0_1 \\ \vdots \\ \bar{u}^0_m \end{bmatrix} = (V^0(x))^{-1} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \tag{4.78}$$

Now, the *normal* form of the system (4.71) is given by defining the following local diffeomorphism of $x \in \mathbf{R}^n$,

$$\begin{aligned} (\xi^T, \eta^T) = (\quad & \xi^1_1 = h_1(x), \ldots, \xi^1_{\gamma_1} = L_f^{\gamma_1 - 1} h_1(x), \\ & \xi^2_1 = h_2(x), \ldots, \xi^2_{\gamma_2} = L_f^{\gamma_2 - 1} h_2(x), \\ & \quad \vdots \\ & \xi^m_1 = h_m(x), \ldots, \xi^m_{\gamma_m} = L_f^{\gamma_m - 1} h_m(x), \\ & \eta^T \qquad\qquad\qquad\qquad\qquad) \end{aligned} \tag{4.79}$$

and noting that

$$\left. \begin{aligned} \dot{\xi}^1_1 &= \xi^1_2 \\ &\vdots \\ \dot{\xi}^1_{\gamma_1} &= b_1(\xi, \eta) + \sum_{j=1}^{m-1} \bar{a}^0_{1j}\bar{u}^0_j + \epsilon\bar{a}^0_{1m}\bar{u}^0_m \\ \dot{\xi}^2_1 &= \xi^2_2 \\ &\vdots \\ \dot{\xi}^m_{\gamma_m} &= b_m(\xi, \eta) + \sum_{j=1}^{m-1} \bar{a}^0_{mj}\bar{u}^0_j + \epsilon\bar{a}^0_{mm}\bar{u}^0_m \\ \dot{\eta} &= q(\xi, \eta) + P(\xi, \eta)\bar{u}^0 \end{aligned} \right\} \tag{4.80}$$

where $b_i(\xi,\eta)$ is $L_f^{\gamma_i}h_i(x)$ for $i = 1,\ldots,m$ in (ξ,η) coordinates. The zero dynamics of the system are the dynamics of the η coordinates in the subspace $\xi = 0$ with the linearizing control law of (4.75) (with $v = 0$) substituted, i.e.,

$$\dot\eta = q(0,\eta) - P(0,\eta)(\bar A^0(0,\eta))^{-1}b(0,\eta). \tag{4.81}$$

We will assume that (4.71) is non-minimum phase, that is to say that the origin of (4.81) is not stable.

Now, an approximation to the system is obtained by setting $\epsilon = 0$ in (4.80). The resultant decoupling matrix is singular and the procedure for linearization by (dynamic) state feedback (the so-called *dynamic extension* process) proceeds by differentiating (4.80) and noting that

$$\dot x = f(x) + \bar g_1^0(x)\bar u_1^0 + \cdots + \bar g_m^0(x)\bar u_m^0 \tag{4.82}$$

where

$$\begin{bmatrix} \bar g_1^0(x) & \cdots & \bar g_m^0(x) \end{bmatrix} = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix} V^0(x). \tag{4.83}$$

We then get

$$\begin{aligned} \begin{bmatrix} y_1^{(\gamma_1+1)} \\ \vdots \\ y_{m-1}^{(\gamma_{m-1}+1)} \\ y_m^{(\gamma_m+1)} \end{bmatrix} &= b^1(x,\bar u_1^0,\ldots,\bar u_{m-1}^0) + A^1(x,\bar u_1^0,\ldots,\bar u_{m-1}^0)\begin{bmatrix} \dot{\bar u}_1^0 \\ \vdots \\ \dot{\bar u}_{m-1}^0 \\ \bar u_m^0 \end{bmatrix} && (4.84) \\ &= b^1(x^1) + A^1(x^1)u^1 && (4.85) \end{aligned}$$

where

$$u^1 = (\dot{\bar u}_1^0,\ldots,\dot{\bar u}_{m-1}^0,\bar u_m^0)^T \tag{4.86}$$

is the *new* input and

$$x^1 = (x^T,\bar u_1^0,\ldots,\bar u_{m-1}^0)^T \tag{4.87}$$

is the *extended* state. Note the appearance of terms of the form $\dot{\bar u}_1^0,\ldots,\dot{\bar u}_{m-1}^0$ in (4.84). The system (4.84) is linearizable (and decouplable) if $A^1(x^1)$ is nonsingular. We will assume that the singular values of A^1 are all of order 1 (i.e., A^1 is uniformly nonsingular) so that (4.84) is linearizable. The normal form for the approximate system is determined by obtaining a local diffeomorphism of the states $x, \bar{u}_1^0, \ldots, \bar{u}_{m-1}^0$ ($\in \mathbb{R}^{n+m-1}$) given by

$$
\begin{aligned}
(\tilde{\xi}^T, \tilde{\eta}^T) = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.88) \\
& (\tilde{\xi}_1^1 = h_1(x), \ldots, \tilde{\xi}_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x), \tilde{\xi}_{\gamma_1+1}^1 = L_f^{\gamma_1} h_1(x) + \sum_{j=1}^{m-1} \bar{a}_{1j}^0 \bar{u}_j^0, \\
& \tilde{\xi}_1^2 = h_2(x), \ldots, \tilde{\xi}_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x), \tilde{\xi}_{\gamma_2+1}^2 = L_f^{\gamma_2} h_2(x) + \sum_{j=1}^{m-1} \bar{a}_{2j}^0 \bar{u}_j^0, \\
& \vdots \\
& \tilde{\xi}_1^m = h_m(x), \ldots, \tilde{\xi}_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x), \tilde{\xi}_{\gamma_m+1}^m = L_f^{\gamma_m} h_m(x) + \sum_{j=1}^{m-1} \bar{a}_{mj}^0 \bar{u}_j^0, \\
& \eta^T \qquad)
\end{aligned}
$$

Note that $\tilde{\xi} \in \mathbb{R}^{\gamma_1+\cdots+\gamma_m+m}$ and $\tilde{\eta} \in \mathbb{R}^{n-\gamma_1-\cdots-\gamma_m-1}$ as compared with $\xi \in \mathbb{R}^{\gamma_1+\cdots+\gamma_m}$ and $\eta \in \mathbb{R}^{n-\gamma_1-\cdots-\gamma_m}$. With these coordinates, the *true* system (4.71) is given by

$$
\left.
\begin{aligned}
\dot{\tilde{\xi}}_1^1 &= \tilde{\xi}_2^1 \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma_1}^1 &= \tilde{\xi}_{\gamma_1-1}^1 + \epsilon \bar{a}_{1m}^0 \bar{u}_m^1 \\
\dot{\tilde{\xi}}_{\gamma_1+1}^1 &= b_1^1(\tilde{\xi}, \tilde{\eta}) + a_{1\cdot}^1(\tilde{\xi}, \tilde{\eta}) u^1 \\
\dot{\tilde{\xi}}_1^2 &= \tilde{\xi}_2^2 \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma_m}^m &= \tilde{\xi}_{\gamma_m-1}^m + \epsilon \bar{a}_{mm}^0 \bar{u}_m^1 \\
\dot{\tilde{\xi}}_{\gamma_m+1}^m &= b_m^1(\tilde{\xi}, \tilde{\eta}) + a_{m\cdot}^1(\tilde{\xi}, \tilde{\eta}) u^1 \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) + \tilde{P}(\tilde{\xi}, \tilde{\eta}) u^1
\end{aligned}
\right\} \tag{4.89}
$$

In (4.89) above, $b_i^1(\tilde{\xi}, \tilde{\eta})$ and $a_{i\cdot}^1(\tilde{\xi}, \tilde{\eta})$ are the ith element and row of b^1 and A^1, respectively, in (4.84) above (in the $\tilde{\xi}$, $\tilde{\eta}$ coordinates). The approximate system used for the design of the linearizing control is obtained from (4.89) by setting $\epsilon = 0$. The zero dynamics for the approximate system are obtained in the $\tilde{\xi} = 0$ subspace by linearizing the approximate system using

$$
u_*^1(\tilde{\xi}, \tilde{\eta}) = -(A^1(\tilde{\xi}, \tilde{\eta}))^{-1} \begin{bmatrix} b_1^1(\tilde{\xi}, \tilde{\eta}) \\ \vdots \\ b_m^1(\tilde{\xi}, \tilde{\eta}) \end{bmatrix} \tag{4.90}
$$

to get

$$\dot{\tilde{\eta}} = \tilde{q}(0,\tilde{\eta}) + \tilde{P}(0,\tilde{\eta})u_*^1(0,\tilde{\eta}). \tag{4.91}$$

Note that the dimension of $\tilde{\eta}$ is *one less* than the dimension of η in (4.81). It would appear that we are actually determining the zero dynamics of the approximation to system (4.71) with dynamic extension—that is to say with integrators appended to the first $m-1$ inputs $\bar{u}_1^0, \bar{u}_2^0, \ldots, \bar{u}_{m-1}^0$. While this is undoubtedly true, it has been shown in Byrnes and Isidori [Isi87] that the zero dynamics of systems are unchanged by dynamic extension. Thus, the zero dynamics of (4.91) are those of the approximation to system (4.71).

The system (4.71) is said to be *slightly non-minimum phase* if the equilibrium $\eta = 0$ of (4.81) is not asymptotically stable, but the equilibrium $\tilde{\eta} = 0$ of (4.91) is.

It is also easy to see that the preceding discussion may be iterated if it turns out that $A^1(\tilde{\xi},\tilde{\eta})$ has some small singular values. At each stage of the *dynamic extension* process $m-1$ *integrators* are added to the dynamics of the system and the act of approximation reduces the dimension of the zero dynamics by *one*. Also, if at any stage of this dynamic extension process, there are two, three, ... singular values of order ϵ, the dynamic extension involves $m-2$, $m-3$, ... integrators.

If the objective is tracking, the approximate tracking control law is

$$\begin{aligned} u_a^1(\tilde{\xi},\tilde{\eta}) = (A^1(\tilde{\xi},\tilde{\eta}))^{-1} \Bigg(& -\begin{bmatrix} b_1^1(\tilde{\xi},\tilde{\eta}) \\ \vdots \\ b_m^1(\tilde{\xi},\tilde{\eta}) \end{bmatrix} \\ & + \begin{bmatrix} y_{d1}^{(\gamma_1+1)} + \alpha_{\gamma_1}^1(y_{d1}^{(\gamma_1)} - \tilde{\xi}_{\gamma_1+1}^1) + \cdots + \alpha_0^1(y_{d1} - \tilde{\xi}_1^1) \\ \vdots \\ y_{dm}^{(\gamma_m+1)} + \alpha_{\gamma_1}^m(y_{dm}^{(\gamma_m)} - \tilde{\xi}_{\gamma_m+1}^m) + \cdots + \alpha_0^m(y_{dm} - \tilde{\xi}_1^m) \end{bmatrix} \Bigg) \end{aligned} \tag{4.92}$$

with the polynomials

$$s^{\gamma_i+1} + \alpha_{\gamma_i}^i s^{\gamma_i} + \cdots + \alpha_0^i, \quad i = 1,\ldots,m, \tag{4.93}$$

chosen Hurwitz.

The following theorem is the analog of Theorem 4.1 in terms of providing a bound for the system performance when the control law (4.92) is applied to the true system (4.71).

Theorem 4.3 *Suppose that*

- *the zero dynamics (4.91) of the approximate system are locally* exponentially stable *and* $\tilde{q} + \tilde{P}u_*^1$ *is locally Lipschitz in* $\tilde{\xi}$ *and* $\tilde{\eta}$ *and*
- *the functions* $\bar{a}_{im}^0 u_m^1$ *are locally Lipschitz continuous for* $i = 1, \ldots, m$.

Then, for ϵ *sufficiently small and for desired trajectories with sufficiently small values and derivatives* $(y_{id},\ \dot{y}_{id},\ \ldots,\ y_{id}^{(\gamma_i+1)})$, *the states of the system (4.89) are bounded and the tracking errors satisfy*

$$\begin{aligned}
|e_1| &= |\tilde{\xi}_1^1 - y_{d1}| &\leq k\epsilon \\
|e_2| &= |\tilde{\xi}_1^2 - y_{d2}| &\leq k\epsilon \\
&\vdots \\
|e_m| &= |\tilde{\xi}_1^m - y_{dm}| &\leq k\epsilon
\end{aligned} \tag{4.94}$$

for some $k < \infty$.

Proof: Similar to that of Theorem 4.1. $\square$

As in the SISO case, the stronger conclusions of Theorem 4.2 can be stated when the control objective is stabilization and the approximate system has no zero dynamics.

Conclusion

In this chapter, we have described the application of techniques of exact input-output linearization of nonlinear control systems to the flight control of vertical take-off and landing aircraft. We saw that the application of the theory to this example is not straightforward. In particular, the direct application of the theory yielded an undesirable controller. We remedied the situation by neglecting the coupling between the rolling moment input to the aircraft dynamics and the dynamics along the y axes.

The example of the vertical takeoff and landing aircraft is an example of a system which is slightly non-minimum phase. Thus, the exact linearization technique resulted in a system which was internally unstable. We generalized the lessons learned from this application to define, informally, *slightly non-minimum phase* systems and gave methods to *linearize* them *approximately*.

Conclusion

In this dissertation, we have seen that it is possible to achieve reasonable tracking performance for a large class of nonlinear systems, including systems that are not invertible and systems that have unstable inverses. The main requirement for this is that the true system be *close* to an exponentially minimum phase nonlinear system. Designing the exact tracking control law for the approximate system then yields bounded tracking for the true system.

This result is fortunate since, as we have seen, the structure of the zero dynamics of a system system (i.e., whether it is minimum phase or not) is not robust to system perturbations. In particular, the tracking control results show that controller design based on exponentially minimum phase systems *is* indeed robust to system perturbations.

This dissertation is just a small step in a larger project to bring nonlinear control theory into a practical control design methodology. In order to bring these interesting techniques into use we must work to understand and soften the many of the restrictions and assumptions currently required by the theory.

Much interesting work remains to be done. Particular areas indicated by this dissertation include:

- Trajectory design. For a class of invertible nonlinear control systems, the method of input-output linearization is useful to guarantee that the trajectory error has an exponentially stable linear dynamics. In order for input-output linearization to be effective, the desired trajectories must *respect constraints imposed by the true system dynamics*.

- Actuator limits. One of the most difficult problems in feedback control designers must face is the fact that real life actuators and systems have limits. This problem can sometimes be handled by judicious trajectory design. However, this problem can still pose major difficulties in the presence of measurement errors and external disturbances.

- System steering. Sometimes a system can be in a state where it does not possess full controllability. In this case, a natural (sub)task would be to steer the system (in allowed) directions to get into a region of the state space where normal techniques can be used.

- Tools for nonlinear systems analysis. The calculations required for even relatively simple nonlinear systems are often quite involved. Also, the estimates that we derive to prove stability and tracking performance are extremely conservative and do not reveal the true nonlinear nature of the problem.

We conclude with a very simple example as a warning of the pitfalls that nonlinear systems may contain. Consider a very simple system with a smooth saturating nonlinearity given by

$$\begin{aligned} \dot{x} &= u \\ y &= \tanh x \end{aligned} \tag{4.95}$$

where the control objective is to track a given desired output y_d. Clearly, since

$$\sup_{x \in \mathbf{R}} |y(x)| = 1,$$

we cannot track just any trajectory. Following the usual procedure for input-output linearization, we differentiate the output to get

$$\dot{y} = (1 - \tanh^2 x)u. \tag{4.96}$$

Since $(1 - \tanh^2 x) \neq 0$ for $x \in \mathbf{R}$, we use the control law

$$u = \frac{1}{1 - \tanh^2 x}\left[\dot{y}_d + \alpha_0(y_d - y)\right] \tag{4.97}$$

to give a closed loop system of

$$\dot{y} = \dot{y}_d + \alpha_0(y_d - y) \tag{4.98}$$

or

$$\dot{e} + \alpha_0 e = 0. \tag{4.99}$$

Given this error equation, it would appear that our control law will make the system track any desired trajectory. Indeed, as long as the system states remain bounded, the closed loop system will have the linear error dynamics given by (4.99). However, when the desired trajectory is outside the range of $\tanh x$, the system state will become unbounded in finite time! For example, let the desired trajectory be $y_d \equiv 2$. Then, the closed loop dynamics are given by (with $\alpha_0 = 4$)

$$\dot{x} = \frac{4(2 - \tanh x)}{1 - \tanh^2 x} = e^{2x} + 4 + 3e^{-2x}. \tag{4.100}$$

For any finite initial condition, the state x of (4.100) will become unbounded in finite time. The simplicity of the linear error equation (4.99) (valid for every *finite* x) has hidden the danger in the underlying differential equation (4.100) (finite escape time for this particular desired trajectory).

This very simple and somewhat contrived example illustrates an important point—we must verify, at each step, that we are not violating any important assumptions. In this regard, nonlinear systems seem to be much less forgiving than linear systems.

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