Implicitizing Rational Parametric Surfaces

Dinesh Manocha[1] *John F. Canny*[1]

Computer Science Division
University of California
Berkeley, CA 94720

Abstract: Many current geometric modeling systems use the rational parametric form to represent surfaces. Although the parametric representation is useful for tracing, rendering and surface fitting, many operations like surface intersection desire one of the surfaces to be represented implicitly. Moreover, the implicit representation can be used for testing whether a point lies on the surface boundary and to represent an object as a semi-algebraic set. In the past, resultants and Gröbner basis have been used to implicitize parametric surfaces. In particular, different formulations of resultants have been used to implicitize tensor product surfaces and triangular patches and in many cases the resulting expression consists of an extraneous factor. The separation of these extraneous factors can be a time consuming task involving multivariate factorization. Furthermore, these algorithms fail altogether if the given parametrization has base points. The parametrizations of many commonly used rational surfaces, like the quadrics and some cubics, have base points.

In this paper we present an algorithm to implicitize parametric surfaces. If a parametrization has no base points, we formulate the parametric equations in such a manner that their resultant corresponds exactly to the implicit equation without generating any extraneous factor. We also analyze the problem of implicitization in the presence of base points. In particular, we perturb the given parametrizations and use resultants or Gröbner basis of the perturbed system to compute the implicit representation. Our algorithm perturbs only one of the three equations and shows that the implicit equation is contained in the lowest degree term of the resultant of the perturbed equations (expressed in terms of the perturbing variable). The strength of the algorithm lies in the fact that it makes use of the GCD operation as opposed to multivariate factorization to extract the implicit equation out of the lowest degree term. The base points blow up to rational curves and the extraneous factors in the lowest degree term of the resultant of perturbed equations is used to compute the rational parametrizations of these curves. We also describe an efficient implementation of the algorithm.

[1]This research was supported in part by David and Lucile Packard Fellowship and in part by National Science Foundation Presidential Young Investigator Award (# IRI-8958577).

1 Introduction

Currently most geometric modeling systems use parametric and implicit forms for representing surfaces. For computational reasons, they restrict attention to rational functions for parametric representation and algebraic surfaces for implicit representation. A surface represented parametrically by rational functions is known as a *rational surface*. The parametrization of a rational surface represented in terms of homogeneous coordinates is:

$$(x, y, z, w) = (X(s,t), Y(s,t), Z(s,t), W(s,t)), \tag{1}$$

where $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are polynomials in the indeterminates s and t. An *algebraic surface* is defined as the zero set of an irreducible polynomial $F(x,y,z)$. Hereafter the parametric and implicit representations refer to rational and algebraic surfaces, respectively.

It is generally accepted that the parametric representation is best suited for generating points on the surface, thereby used for rendering and tracing, whereas the implicit form is convenient for determining whether a point lies on the surface. This property of the implicit form can be used to represent the object as a semi-algebraic set. This motivates the design of algorithms for converting from one representation to the other. Further motivation is provided by the surface intersection problem. This problem is greatly simplified if one of the surfaces is represented parametrically and the other one implicitly. This approach for surface intersection has been used in [PP88].

The set of rational surfaces is a proper subset of the set of algebraic surfaces. Thus, every rational parametric surface has a corresponding implicit representation and it is desirable to compute it. This process of converting from parametric to implicit is known as *implicitization*.

There are two known techniques for implicitization. In particular, these techniques reduce the problem of implicitizing rational surfaces to eliminating two variables from three parametric equations. The first technique involves the use of Elimination theory. In [Ho89, Chapter 5] the two variables are eliminated in succession by using the Sylvester resultant for two equations. The resulting expression does not correspond to the resultant of three parametric equations and contains an extraneous factor. The Dixon formulation for computing the resultant has been used to implicitize tensor product surfaces in [SAG84]. However it is limited to tensor product surfaces and cannot be used for implicitizing other parametrizations like the triangular patches. [BGW88] suggest the use of Macaulay's formulation for computing the resultant of three parametric equations, which is used for implicitizing. It expresses the resultant as a ratio of two determinants. Many a time, both the determinants evaluate to zero. To compute the resultant we need to perturb the equations and use limiting arguments. This corresponds to computing the characteristic polynomial of the two matrices and the resultant is expressed as the constant term of the ratio of two characteristic polynomials [Ca88]. Perturbation corresponds to introducing an additional variable and thereby increasing the symbolic complexity of the resulting expression. Later

on, we show that this technique is expensive in practice. In general, it is believed that techniques based on Elimination theory result in extraneous factors along with the implicit representation and their separation can be a difficult task [Ho88, CH89, Ho90].

The second technique utilizes Gröbner bases. It computes a canonical representation of the ideal generated by the parametric equations, by defining a suitable ordering of the variables [Bu89; Ho89, Chapter 7]. This technique is fairly expensive in practice and even for low degree parametrizations it may take a lot of time. Furthermore it requires rational coefficients in the description of parametric equations [CH89].

We analyze the problem of implicitization. In particular, we formulate the three parametric equations in such a manner that their resultant corresponds exactly to the implicit representation. These parametric equations are of the same degree and their resultant is expressed as the determinant of a matrix. Thus, we do not need to perturb the given equations or compute the characteristic polynomial of the given matrix. Moreover, the coefficients of the given parametrizations are not required to be rational numbers. We consider two types of parametrizations: total degree bounded[1] and tensor product. In each case the implicit representation corresponds to the determinant of a matrix.

All the techniques mentioned above fail when a given parametrization has base points in the parametric domain. A base point in the domain, say $s = s_0, t = t_0$, corresponds to a common solution of the following four equations

$$X(s,t) = 0, \quad Y(s,t) = 0, \quad Z(s,t) = 0, \quad W(s,t) = 0.$$

The base points also include the common solutions at infinity[2]. Substituting the base point into the parametrization, (1), results in

$$(x, y, z, w) = (0, 0, 0, 0),$$

which does not correspond to any point in the image space. It turns out that many rational parametrizations have base points. All quadric surfaces, e.g. spheres, cylinders, cones are rational surfaces. However, their parametrizations have base points and the previously known techniques cannot be used to implicitize them, unless they are represented as Steiner surfaces [SA85]. Moreover, all tensor product surfaces have base points at infinity. There is a special formulation of resultants [Di08], which has been used to implicitize tensor product surfaces [SAG84]. However this technique fails when there are base points in the affine domain or excess base points at infinity. In general, any faithful parametrization of a rational surface, whose algebraic degree is not a perfect square, has base points.

The base points blow up to rational curves on the surface (known as seam curves). We present an algorithm to implicitize rational parametrizations with base points and also compute the rational parametrizations of seam curves. In particular, we symbolically perturb the given parametric equations and show that the implicit equation is contained

[1] These parametrizations include the triangular patches.

[2] The Gröbner bases work fine in this case.

in the lowest degree term of the resultant of the perturbed system (expressed in terms of the perturbing variable). However the lowest degree term contains an extraneous factor along with the implicit equation and separating it involves multivariate factorization. To overcome this problem we consider a particular perturbation, obtained by perturbing one of the three equations and hereby denoted as the *efficient perturbation*, and show that the extraneous factor is independent of one of the variables. This allows us to compute the extraneous factor by two substitutions for that variable followed by a GCD (greatest common divisor) calculation. Moreover, it is shown that in the case of efficient perturbation the extraneous factor corresponds to the projection of the seam curves and can be used for computing the rational parametrizations of the seam curves.

The rest of the paper is organized as follows. In Section 2 we specify our notation and present some background material from algebraic geometry. In Section 3 we analyze the problem of implicitization and show how we can use resultants or Gröbner bases on a parametrization without any base points to compute the implicit representation. In Section 4 we highlight many properties of rational surfaces with base points and show why the previous techniques of implicitization using resultants or Gröbner bases fail on such parametrizations. We also consider tensor product parametrizations as a special case of rational surface consisting of base points. We perturb the given parametric equations in Section 5 and show that the implicit equation is contained in the lowest degree term of the resultant of the perturbed system. In Section 6, we consider the efficient perturbation, obtained by perturbing one of the three equations, and present an efficient algorithm to extract the implicit equation, which does not involve multivariate factorization. This perturbation is also used to compute the rational parametrizations of seam curves, the images of base points on the rational surface. Section 7 presents details of the algorithm and in Section 8 we discuss its implementation.

2 Background

A rational parametrization is a vector valued function of the form

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)). \tag{2}$$

We use lower case letters like s, t, x or y to denote scalar variables and upper case letters to represent scalar functions like $W(s,t)$ or $F(x,y,z)$ and homogeneous functions like $\overline{F}(x,y,w)$. Bold face upper case letters, like $\mathbf{F}(s,t)$, are used to represent vector valued functions and lower case bold face letters like $\mathbf{p}$ and $\mathbf{q}$ represent tuples like (s,t,u).

In (2), $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are bivariate polynomials and assumed to have *power basis* representation. All tensor product Bézier, B-spline surfaces can be converted into power basis representation. The degree of parametrization, (2) is the maximum of the degrees of $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$.

Definition: A *homogeneous* polynomial is a polynomial in which all terms are of the same degree.

Given any polynomial of the form

$$G(s,t) = a_0 s^m t^n + a_1 s^{m-1} t^n + \ldots + a_k s^m + \ldots + a_l$$

we homogenize it by adding a variable u:

$$\overline{G}(s,t,u) = a_0 s^m t^n + a_1 s^{m-1} t^n u + \ldots + a_k s^m u^n + \ldots + a_l u^{m+n}.$$

Homogeneous polynomials are used for defining objects in projective space.

Definition: A polynomial $H(x,y,z)$ is *independent* of a variable, say z, if none of its terms contains z.

The following polynomial is independent of z:

$$H(x,y,z) = b_{m,n} x^m y^n + b_{m-1,n} x^{m-1} y^n + \ldots + b_{0,n} y^n + \ldots + b_{m,0} x^m + \ldots + b_{0,0}.$$

2.1 Affine and Projective Space

In geometric modeling a surface parametrization, (2), indicates a mapping of the form

$$\mathbf{F} : \mathbf{R}^2 \rightarrow \mathbf{R}^3.$$

In fact the domain is often restricted to a finite interval, of the form $[a_1, b_1] \times [a_2, b_2]$ or a triangle. Since the field of real numbers is not algebraically closed, it is often useful to extend this definition to its algebraic closure, $\mathbf{C}$, the set of complex numbers. Hence we consider the parametrization as a vector valued function

$$\mathbf{F} : \mathbf{C}^2 \rightarrow \mathbf{C}^3.$$

Till now we have viewed our surface as a geometric object in *affine* space. Affine n-dimensional space over the complexes is the familiar n-space. Using Cartesian coordinates, a point in this space has coordinates

$$(x_1, x_2, \ldots, x_n)$$

where $x_i \in \mathbf{C}$ (and is therefore finite). However, there are a lot of advantages in considering the object in *projective* space. Projective n-dimensional space consists of the affine n-dimensional space plus the points at *infinity*. Using Cartesian coordinates, a point in this space is represented as

$$(x_1, x_2, \ldots, x_n, x_{n+1})$$

where not all x_i are zero and $x_i \in \mathbf{C}$. Moreover, for any nonzero complex number, s, $(x_1, x_2, \ldots, x_n, x_{n+1})$ and $(sx_1, sx_2, \ldots, sx_n, sx_{n+1})$ denote the same point in the space. The

variable x_{n+1} is considered a *homogenizing* variable. If $x_{n+1} = 0$ then $(x_1, \ldots, x_n, x_{n+1})$ is a point at infinity. More on affine and projective spaces is given in [Ha77, Chapter 1; Ho89, Chapter 5].

Let $s = s_0, t = t_0$ be a solution of $W(s,t) = 0$. Since (2) is a homogeneous representation of the surface, $\mathbf{F}(s_0, t_0)$ is a point at infinity. Thus, $\mathbf{F}(s,t)$ should be regarded as the following function:

$$\mathbf{F}: \quad \mathbf{C}^2 \rightarrow \mathbf{P}^3$$

where $\mathbf{P}$ denotes the complex projective space (affine complex space plus the points at infinity). It is reasonable to assume that $GCD(X(s,t), Y(s,t), Z(s,t), W(s,t)) = 1$, where GCD denotes the greatest common factor of the given polynomials. The parameters s and t can also correspond to values at infinity. For example, all tensor product surfaces have base points at infinity. Thus, the rational surface (2) should be regarded as a mapping of the form

$$\mathbf{F}: \quad \mathbf{P}^2 \rightarrow \mathbf{P}^3.$$

That is, the domain of the parameters, s and t, consists of values at infinity as well. A parameter value in the domain, $\mathbf{P}^2$, is represented by the tuple (s,t,u) and $u = 0$ corresponds to the parameter values at infinity. The rational surface $\mathbf{F}(s,t)$ should be interpreted as a representation of the form

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) \tag{3}$$

where $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials in s, t and u and each polynomial has the same degree. Moreover,

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1.$$

2.2 Algebraic Plane Curves

A plane curve of degree n, can be represented by an equation $F(x,y) = 0$, where F is a polynomial of degree n. The corresponding homogeneous representation is $\overline{F}(x,y,w) = 0$. Any point on the curve in general is a *simple point.* A few points on the curve are *multiple points* [SR85, Chapter 2].

Definition: A *multiple point* of order k (or k-fold point, $k > 1$) of a degree n curve, is a point $\mathbf{p}$ of the curve such that a generic line through $\mathbf{p}$ meets the curve in only $n - k$ further points.

Let us investigate the behavior of a curve at a multiple point. We can assume that the point under consideration is the origin, i.e. $\mathbf{p} = (0,0,1)$, else we can bring it to the origin by a suitable linear transformation. The curve can be represented as

$$\overline{F}(x,y,w) = \overline{U}_0(x,y)w^n + \overline{U}_1(x,y)w^{n-1} + \ldots + \overline{U}_{n-1}(x,y)w + \overline{U}_n(x,y) = 0,$$

where $\overline{U}_i(x,y)$ is a homogeneous polynomial of degree i in x and y. A generic line through the origin can be represented in the form $x/a = y/b$. The point of this line whose coordinates are $(ka, kb, 1)$, where k is a scalar, lies on the curve if k is any of the roots of the equation

$$\overline{U}_0(a,b) + k\overline{U}_1(a,b) + k^2\overline{U}_2(a,b) + \ldots + k^i\overline{U}_i(a,b) + \ldots + k^n\overline{U}_n(a,b) = 0. \qquad (4)$$

To make the curve have a k-fold point at the origin corresponds to making the equation, (4), have k nonzero roots for every value of the ratio a/b. This can happen, if and only if $\overline{U}_0(x,y)$, $\overline{U}_1(x,y)$, $\ldots$, $\overline{U}_{k-1}(x,y)$ vanish identically. A line corresponds to a tangent at $\mathbf{p}$, if it has $k+1$ of its intersections with the curve at $\mathbf{p}$ and the $n-k-1$ intersections at other points on the curve. All lines of the form $x/a' = y/b'$, where $\overline{U}_k(a',b') = 0$ are tangent to the curve at $\mathbf{p}$. There can be at most k such lines.

A simple version of *Bezout's theorem* is used for determining the number of intersections between a curve of degree m and that of degree n. It is assumed that the curves have no component in common. That is:

Two curves of degree m and n intersect at mn points, counted properly with respect to multiplicity.

2.3 Faithful and Unfaithful parametrizations

In many cases a rational curve or surface can be identically described by a lower degree rational parametrization. Such curves or surfaces have *unfaithful* parametrizations. In particular, a surface parametrization is *faithful* if there is a one to one relationship between the points on the surface and the parameter values, except for a finite number of points and curves on the surface. Another popular terminology for faithful and unfaithful parametrizations are proper and improper parametrizations, respectively.

Consider the following affine parametrization of the unit sphere

$$(x,y,z,w) = (1-s^2-t^2, 2s, 2t, 1+s^2+t^2).$$

Since the preimage of $(x,y,z,w) = (0,0,1,1)$ consists of a unique point in the parametric domain $(s=0, t=1)$, the given parametrization is faithful. If we reparametrize by substituting $s = uv$ and $t = u^2$, we obtain

$$(x,y,z,w) = (1-u^2v^2-u^2, 2uv, 2u^2, 1+u^2v^2+u^4),$$

which is an unfaithful parametrization. There are two points in the preimage of $(0,0,1,1)$, $(u,v) = (1,0)$ and $(u,v) = (-1,0)$.

Every rational surface has a corresponding faithful parametrization [Ca1894]. However no algorithms are known at the moment for computing the faithful parametrization of an unfaithfully parametrized rational surface. Resultants and Gröbner bases have been used

to decide whether a given parametrization is faithful [BGW88; Ho89, Chapter 7]. Our implicitization algorithm also determines whether a given parametrization is faithful or not.

2.4 Algebraic Sets

In this section we present some definitions and basic results on the dimension of algebraic sets. We use these results in the rest of the paper.

Let us consider an algebraically closed field, $\mathbf{C}$ and define a polynomial ring

$$A = \mathbf{C}[x_1, x_2, \ldots, x_m]$$

of m variables over $\mathbf{C}$. All the polynomials used in this section are assumed to be defined over this ring.

Definition: The set of common zeros of a system of polynomials $F_1, \ldots, F_n$ in $x_1, \ldots, x_m$ is called an *algebraic set* and is denoted $V(F_1, \ldots, F_n) \subset \mathbf{C}^m$. An algebraic set $V(F)$ defined by a single polynomial (which is not identically zero) is called a *hypersurface*. If F is linear, then $V(F)$ is called a *hyperplane*.

The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets [Ha77, Chapter 1].

If all the F_i are homogeneous, it is more convenient to work with the projective space $\mathbf{P}^{m-1}$, formed by identifying points in $\mathbf{C}^m$ which are scalar multiples of each other. We use the same notation, $V(\overline{F}_1, \ldots, \overline{F}_n) \subset \mathbf{P}^{m-1}$ for an algebraic set defined by homogeneous polynomials $\overline{F}_i$.

Definition: The *Zariski* topology on $\mathbf{P}^n$ is defined by taking the open subsets to be the complements of the algebraic sets. This is a topology, because the intersection of two open sets is an open, and the union of a family of open sets is open. Furthermore, the empty set and the whole space are both open.

An algebraic set is said to be *reducible* if it can be expressed as a finite union of proper subsets which are algebraic. Otherwise it an *irreducible* algebraic set. An irreducible algebraic set is known as a *variety*. An algebraic set can always be expressed as a finite union of irreducible algebraic subsets called *components*. Many results in algebraic geometry apply only to irreducible algebraic sets, and in much of what follows, we work with the individual components of an algebraic set.

Definition: Let Z be the intersection of n hypersurfaces in m-dimensional affine or projective space. A component W of Z is said to be *proper* if it has dimension $m - n$. A component of dimension greater than $m - n$ is said to be an *excess* component.

And in fact all components of an intersection must be either proper or excess by the following lemma [Mm76, Chapter 3]:

Lemma I: *If F_i are n non-homogeneous polynomials in m variables, (or homogeneous in $m+1$ variables), then every component of $V(F_1, \ldots, F_n)$ has dimension at least $m-n$.*

3 Implicitization

Conisder a rational surface

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where $\overline{X}(s,t,u)$, $\overline{Y}(s,t,u)$, $\overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials of degree n. $\overline{\mathbf{F}}$ is a vector valued function defined as

$$\overline{\mathbf{F}} : \mathbf{P}^2 \rightarrow \mathbf{P}^3$$

Let the image of $\overline{\mathbf{F}}$ be $\mathcal{Y}$, where $\mathcal{Y} \subset \mathbf{P}^3$. We assume that $\mathcal{Y}$ has dimension 2. For example, $\overline{\mathbf{F}}$ is not a parametrization of the form

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (s+t, s+t, s+t, u),$$

whose image is the 1-dimensional line, $x = y = z$.

A parametrization of the form $\overline{\mathbf{F}}$ corresponds to the plane representation of a rational surface in $\mathbf{P}^3$. To a generic point $\mathbf{p}$ of the plane, there corresponds a unique point of $\mathcal{Y}$. The only exceptions are the base points, which blow up to rational curves on the surface. However, there are a finite number of such base points in the parametric domain. At times, the preimage of a point in $\mathcal{Y}$ consists of a curve in the domain. Let $\mathcal{Z}$ be one such curve of the plane, where $\mathcal{Z} \subset \mathbf{P}^2$. Such curves are known as the *fundamental curves* [SR85, Chapter 6]. If $\overline{\mathbf{F}}$ is a faithful parametrization, then there are open sets $\mathcal{U} \subseteq \mathbf{P}^2$ and $\mathcal{V} \subseteq \mathcal{Y}$ such that $\mathcal{U}$ isomorphic to $\mathcal{V}$. Moreover, $\mathcal{U}$ and $\mathcal{V}$ are dense in $\mathbf{P}^2$ and $\mathcal{Y}$, respectively (A subset $\mathcal{A}$ is said to be dense in its superset $\mathcal{B}$, if it has the same dimension as that of $\mathcal{B}$). In fact

$$\overline{\mathbf{F}}(\mathcal{U}) = \mathcal{V}.$$

Thus, $\mathbf{P}^2$ and $\mathcal{Y}$ are birationally equivalent [Ha77, Chapter 1, Corollary 4.5]. For more on birational maps we recommend [Ha77, Chapter 1; Mm76, Chapter 8]. As a result there exists an inverse rational function (with respect to $\overline{\mathbf{F}}$)

$$\overline{\mathbf{F}}^{-1} : \mathcal{Y} \rightarrow \mathbf{P}^2,$$

which can be represented as

$$\overline{\mathbf{F}}^{-1}(x,y,z,w) = (s,t,u) = (\overline{S}(x,y,z,w), \overline{T}(x,y,z,w), \overline{U}(x,y,z,w)), \tag{5}$$

where $\overline{S}, \overline{T}$ and $\overline{U}$ are homogeneous polynomials in x, y, z and w. Furthermore, each of them has the same degree. Just like $\overline{\mathbf{F}}$ is not defined at the base points, $\overline{\mathbf{F}}^{-1}$ is not defined

at those values of (x, y, z, w), which correspond to the images of fundamental curves (like $\overline{\mathbf{F}}(\mathcal{Z})$)). To show the birational equivalence, we can remove such points and their images from $\mathbf{P}^2$ and $\mathcal{Y}$ and the birational map is, therefore, defined between the corresponding open sets.

Given $\overline{\mathbf{F}}$, resultants and Grőbner bases have been used to compute $\overline{\mathbf{F}}^{-1}$ in [BGW88] and [Ho89, Chapter 7], respectively. It is possible that there is more than one faithful rational parametrization of a given rational surface. All such parametrizations of a given surface are birationally equivalent.

Let $\overline{\mathbf{F}}$ be an unfaithful parametrization. Then $\overline{\mathbf{F}}$ has a corresponding faithful parametrization of the form

$$\overline{\mathbf{G}} : \mathbf{P}^2 \rightarrow \mathbf{P}^3$$

$$\overline{\mathbf{G}}(p, q, r) = (x, y, z, w) = (\overline{X}_1(p, q, r), \overline{Y}_1(p, q, r), \overline{Z}_1(p, q, r), \overline{W}_1(p, q, r)).$$

The image of $\overline{\mathbf{G}}$ is $\mathcal{Y}$. Since $\overline{\mathbf{G}}$ is a faithful parametrization, it has an inverse map

$$\overline{\mathbf{G}}^{-1} : \mathcal{Y} \rightarrow \mathbf{P}^3$$

$$\overline{\mathbf{G}}^{-1}(x, y, z, w) = (p, q, r) = (\overline{P}(x, y, z, w), \overline{Q}(x, y, z, w), \overline{R}(x, y, z, w)),$$

where $\overline{P}$, $\overline{Q}$ and $\overline{R}$ are homogeneous polynomials of same degree. Moreover,

$$(p, q, r) = \overline{\mathbf{G}}^{-1}(x, y, z, w) = \overline{\mathbf{G}}^{-1}(\overline{\mathbf{F}}(s, t, u)).$$

Let $\overline{\mathbf{G}}^{-1}\overline{\mathbf{F}}$ be $\overline{\mathbf{H}}$ and

$$(p, q, r) = \overline{\mathbf{H}}(s, t, u) = (\overline{P}_1(s, t, u), \overline{Q}_1(s, t, u), \overline{R}_1(s, t, u)),$$

where $\overline{P}_1(s, t, u) = \overline{P}(\overline{X}(s, t, u), \overline{Y}(s, t, u), \overline{Z}(s, t, u), \overline{W}(s, t, u))$. We can similarly define $\overline{Q}_1$ and $\overline{R}_1$. $\overline{\mathbf{H}}$ is a rational map of the form

$$\overline{\mathbf{H}} : \mathbf{P}^2 \rightarrow \mathbf{P}^2.$$

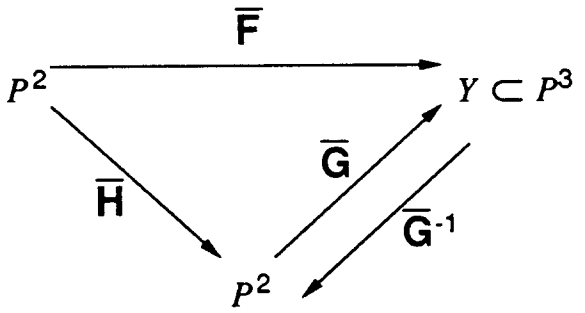

Fig. I
The relationship between the various functions

It is possible that $\overline{\mathbf{F}}$ has base points, while $\overline{\mathbf{G}}$ has no base points. For example, consider the faithful parametrization of a plane

$$\overline{\mathbf{G}}(p,q,r) = (x,y,z,w) = (p+r, 2p+q, q-3r, q+r).$$

$\overline{\mathbf{G}}$ has no base points. Substitute

$$(p,q,r) = \overline{\mathbf{H}}(s,t,u) = (st+t^2, su+tu, s^2+su),$$

and the resulting unfaithful parametrization is

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (s^2+t^2+st+su, 2t^2+2st+su+tu, -3s^2-2su+tu, s^2+tu+2su).$$

$(s,t,u) = (0,0,1)$ is a base point of $\overline{\mathbf{F}}$.

Let us consider the case when the parametrization, $\overline{\mathbf{F}}$, has *no base points* and the map $\overline{\mathbf{F}}$, is therefore, defined at all points in the domain. Since $\mathbf{P}^2$ is a closed, compact and irreducible set of dimension 2 and $\overline{\mathbf{F}}$ is a continuous rational map, the image of $\overline{\mathbf{F}}$ is a closed and irreducible set in $\mathbf{P}^3$. This can be proved formally by considering $\mathbf{P}^2$ and $\mathbf{P}^3$, the domain and range of $\overline{\mathbf{F}}$, as topological spaces with respect to Zariski topology. The image of a compact set under a continuous map is compact [Mu75, Chapter 3, Theorem 5.5]. As a result $\mathcal{Y}$ is a compact set. Furthermore, every compact subset of a Hausdorff space is closed [Mu75 , Chapter 3, Theorem 5.3]. Since $\mathbf{P}^3$ is a Hausdorff space, $\mathcal{Y}$ is therefore, a closed set. Thus, $\mathcal{Y}$ is a 2 dimensional projective variety in $\mathbf{P}^3$. The following lemma from algebraic geometry [Ha77, Chapter 1]

Lemma II: *A projective variety $Y \subset \mathbf{P}^m$ has dimension $m-1$, if and only if it is the zero set of a single irreducible and homogeneous polynomial $\overline{G}$ of positive degree.*

implies that the image of $\overline{\mathbf{F}}$ correspond to the zero set of a single irreducible and homogeneous polynomial, $\overline{G}(x,y,z,w)$. Thus, $\overline{G}(x,y,z,w)$ is the implicit representation of the given surface. It is characterized by the following property:

$$\overline{G}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 0,$$

i.e. the resulting polynomial in s, t and u, obtained after substitution, vanishes identically.

Consider the following *parametric equations*

$$\begin{aligned} \overline{F}_1(s,t,u) &= x\overline{W}(s,t,u) - w\overline{X}(s,t,u) = 0, \\ \overline{F}_2(s,t,u) &= y\overline{W}(s,t,u) - w\overline{Y}(s,t,u) = 0, \\ \overline{F}_3(s,t,u) &= z\overline{W}(s,t,u) - w\overline{Z}(s,t,u) = 0. \end{aligned} \tag{6}$$

The solution set of each equation corresponds to a 4-dimensional hypersurface in $\mathbf{P}^2 \times \mathbf{P}^3$ (spanned by (s,t,u) and (x,y,z,w), respectively). Lets consider the algebraic set, $\overline{Q} = V(\overline{F}_1, \overline{F}_2, \overline{F}_3)$, obtained by the intersection of the three hypersurfaces, obtained as the solution set of the above equations. Let $\overline{\Pi}$ be a projection function

$$\overline{\Pi} : \mathbf{P}^2 \times \mathbf{P}^3 \rightarrow \mathbf{P}^3$$

such that

$$\overline{\Pi}(s,t,u,x,y,z,w) = (x,y,z,w).$$

Lemma I implies that each component in $\overline{Q}$ has dimension at least 2. Since there are no base points, the intersection set consists of the following components:

1.

$$\overline{Q}_1 = \{(s,t,u,x,y,z,w) | x = \overline{X}(s,t,u), y = \overline{Y}(s,t,u), z = \overline{Z}(s,t,u), w = \overline{W}(s,t,u)\}.$$

$\overline{Q}_1$ is a proper component of $\overline{Q}$ and

$$\overline{\Pi}(\overline{Q}_1) = V(\overline{H}(x,y,z,w)),$$

where

$$\overline{H}(x,y,z,w) = \overline{G}(x,y,z,w)^k, \quad for \quad k \geq 1.$$

2.

$$\overline{Q}_2 = \{(s,t,u,x,y,z,w) | \overline{W}(s,t,u) = 0, w = 0\}.$$

$\overline{Q}_2$ is an excess component of $\overline{Q}$ (of dimension 3). However, $\overline{\Pi}(\overline{Q}_2)$ has dimension 2 and corresponds to the points at infinity in the (x,y,z,w) space.

We see that $\overline{\Pi}(\overline{Q})$ consists of at least two distinct components, whereas we are interested in computing $\overline{G}(x,y,z,w)$ only. We therefore, work with an affine representation of the image space and modify the parametric equations, (6), as

$$\begin{aligned} \overline{F'}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{F'}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{F'}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) = 0. \end{aligned} \tag{7}$$

This corresponds to substituting $w = 1$ in (6). Lets consider

$$Q = V(\overline{F'}_1, \overline{F'}_2, \overline{F'}_3) \subset \mathbf{P}^2 \times \mathbf{C}^3,$$

and let Π be the projection function

$$\Pi : \mathbf{P}^2 \times \mathbf{C}^3 \rightarrow \mathbf{C}^3$$

such that

$$\Pi(s,t,u,x,y,z) = (x,y,z).$$

Theorem I: *If the given parametrization has no base points then Q consists of a single component. Moreover, that component can be represented as*

$$Q_1 = \{(s,t,u,x,y,z) | x = \frac{\overline{X}(s,t,u)}{\overline{W}(s,t,u)}, y = \frac{\overline{Y}(s,t,u)}{\overline{W}(s,t,u)}, z = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}\}.$$

Proof: The fact that $\overline{Q}_1 \subset \overline{Q}$ implies that $Q_1 \subset Q$. Thus, Q_1 is a component of Q. Let us assume that Q consists of some other component, say P. Since $P \neq Q_1$, $\exists$ $\mathbf{p} = (s_1, t_1, u_1, x_1, y_1, z_1) \in P$ and $\mathbf{p} \notin Q_1$. There are two possibilities:

1. $\overline{W}(s_1, t_1, u_1) = 0$.

   We know that $\mathbf{p} \in V(\overline{F'}_1, \overline{F'}_2, \overline{F'}_3)$ and therefore

   $$\overline{F'}_1(s_1, t_1, u_1) = 0,$$

   $$\Rightarrow \overline{X}(s_1, t_1, u_1) = x_1 \overline{W}(s_1, t_1, u_1) = 0.$$

   Similarly, we can show that $\overline{Y}(s_1, t_1, u_1) = 0$ and $\overline{Z}(s_1, t_1, u_1) = 0$. This implies that (s_1, t_1, u_1) is a base point of $\overline{\mathbf{F}}$, which is contrary to our assumption.

2. $\overline{W}(s_1, t_1, u_1) \neq 0$.

   We know that $\mathbf{p} \in Q$ and therefore,

   $$\overline{F'}_1(s_1, t_1, u_1) = 0$$

   $$\Rightarrow x\overline{W}(s_1, t_1, u_1) = \overline{X}(s_1, t_1, u_1)$$

   $$\Rightarrow x_1 = \frac{\overline{X}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   Similarly we can show that

   $$y_1 = \frac{\overline{Y}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)},$$

   and

   $$z_1 = \frac{\overline{Z}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   This implies that $\mathbf{p} \in Q_1$.

Thus, all points in Q also lie in Q_1 and therefore,

$$Q = Q_1.$$

Thus, Q consist of one component.

Q.E.D.

Since Q is an irreducible algebraic set, each point in $\Pi(Q)$ lies in $\mathcal{Y}$. This follows from the representation of Q or Q_1 in Theorem I. Since Q and $\Pi(Q)$ are two dimensional algebraic sets, $\Pi(Q)$ correspond to the affine portion of the zero set of the implicit representation of $\overline{\mathbf{F}}(s, t, u)$. If the given parametrization is unfaithful, each point in $\Pi(Q)$ has more than

one preimage with respect to $\overline{\mathbf{F}}$. In this case, $\Pi(Q)$ corresponds to an algebraic set of multiplicity greater than one. Thus,

$$\Pi(Q) = V(H(x, y, z)), \tag{8}$$

where $H(x, y, z) = G(x, y, z)^k$, $k \geq 1$. $k = 1$ if and only if $\overline{F}$ is a faithful parametrization. Using Bezout's theorem it can be shown that the algebraic degree of $H(x, y, z)$ is n^2, where n is the degree of the parametrization [Sal4]. The degree of $G(x, y, z)$ is n^2/k. Moreover, k corresponds to the number of points in the (s, t, u) plane, that are the preimages of an arbitrary point in $V(G(x, y, z))$.

3.1 Resultants

Given a set of m homogeneous equations in m variables, it is always possible to combine the equations to obtain from them a single equation $R = 0$, in which these variables do not appear. We are then said to have eliminated the variables and the quantity R is the *resultant* of the system of equations. In general, the resultant of any such system of equations is a function of the coefficients, whose vanishing is the necessary and sufficient condition for the given system to have a non trivial solution. There are different formulations of computing the resultant of a system of equations. Perhaps the most general one is given by [Ma02]. If all the equations have the same degree, an improved version is given in [Ma21].

Let us consider the system of equations (7), as polynomials in s, t, and u with coefficients being functions of x, y, and z. Since $\overline{\mathbf{F}}$ has no base points, the resultant of (7), say $R(x, y, z)$, is a nonzero polynomial in x, y and z. Lets consider $V(R(x, y, z))$ and let $(x_1, y_1, z_1) \in V(R(x, y, z))$. The fact that $R(x_1, y_1, z_1) = 0$ implies that $\exists\ s_1, t_1, u_1$ such that $(x_1, y_1, z_1, s_1, t_1, u_1) \in Q$. Thus,

$$\Pi(Q) = V(R(x, y, z)),$$

and from (8), it follows that

$$R(x, y, z) = gH(x, y, z),$$

where g is a scalar.

Given $R(x, y, z)$, $G(x, y, z)$ can be expressed as

$$G(x, y, z) = \frac{R(x, y, z)}{GCD(R(x, y, z), R_z(x, y, z))}, \tag{9}$$

where $R_z(x, y, z)$ is the partial derivative of $R(x, y, z)$ with respect to z.

The resultant of three equal degree homogeneous equations can be expressed as a determinant of a matrix. Such formulations are given in [Sal885; Di08;Mo25]. In particular the formulation in [Di08] constructs a matrix of order $2n^2 - n$, where n is the degree of the equations, and its determinant correspond to the resultant. Some other cases where the

resultant can be expressed as the determinant of a single matrix are given in [MC27]. Thus, we need not perturb the equations for computing the resultant. Later on we show that this formulation of resultants is an efficient and compact representation for the implicit representation.

Example I Let

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{st+1}{t^2}, \frac{s}{t^2}, \frac{s^2}{t^2}).$$

After homogenizing we obtain the following system of equations

$$xt^2 - st = 0,$$

$$yt^2 - su = 0,$$

$$zt^2 - s^2 = 0.$$

Considering these equations as polynomials in s, t and u, the resultant is

$$R(x,y,z) = y^4 - 2xy^2z + x^2z^2 - z^3,$$

whose degree is 4, since $n = 2$. Since $\mathbf{F}$ is a faithful parametrization

$$R(x,y,z) = H(x,y,z) = G(x,y,z) = y^4 - 2xy^2z + x^2z^2 - z^3.$$

3.2 Gröbner Bases

Consider a ring of polynomials

$$\mathcal{S} = \mathrm{C}[x_1, \ldots, x_m].$$

An *ideal*, $\mathcal{I}$, of $\mathcal{S}$ is its subring such that for any $F \in \mathcal{S}$, $H \in \mathcal{I}$

$$FH \in \mathcal{I}.$$

Any ideal in $\mathcal{S}$ can be generated by a finite number of polynomials and the generating set is represented as

$$\mathcal{I} = \{F_1, \ldots, F_k\}.$$

Any member of the ideal is of the form

$$A_1F_1 + A_2F_2 + \ldots + A_kF_k,$$

where $A_i \in \mathcal{S}$.

Ideals are sets of polynomials that describe elementary geometric objects symbolically, and are a natural representation for geometric objects. Typically, an ideal has many generating sets defining it and *Gröbner bases* of an ideal is one of them. Given an ordering of the

variables, $x_1, x_2, \ldots, x_m$, the Gröbner bases of an ideal, $\mathcal{I}$, is its canonical representation of the form

$$\mathcal{I} = \{G_1, \ldots, G_r\}.$$

The representation is a function of the variable ordering and allows conceptually simple algorithms to decide whether a given polynomial belongs to the ideal. For more details on Gröbner bases we recommend [Bu85; Ho89, Chapter 7].

Gröbner bases allow us to describe objects in affine space and perform geometric operations. We can always homogenize the polynomials defining the generating set and thereby, describe objects in the projective space.

Given an affine parametrization of a surface

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{X(s,t)}{W(s,t)}, \frac{Y(s,t)}{W(s,t)}, \frac{Z(s,t)}{W(s,t)}),$$

where $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are polynomials in s and t, not necessarily of the same degree. We need not homogenize these polynomials. Lets consider the ring

$$\mathcal{A} = \mathbf{C}[x,y,z,s,t]$$

and its ideal

$$\mathcal{I} = \{xW(s,t) - X(s,t), yW(s,t) - Y(s,t), zW(s,t) - Z(s,t)\}.$$

We define $\mathcal{I}$ as the ideal generated by the parametric equations. We know from theorem I that $V(\mathcal{I})$, the zero set of the ideal $\mathcal{I}$, corresponds to Q', where $Q' = Q_{u=1}$ and $Q' \subset \mathbf{C}^2 \times \mathbf{C}^3$. In other words, Q' can be represented as

$$Q' = \{(s,t,x,y,z) | x = \frac{X(s,t)}{W(s,t)}, y = \frac{Y(s,t)}{W(s,t)}, z = \frac{Z(s,t)}{W(s,t)}\}.$$

Consider the implicit form, $G(x,y,z)$. For every $(x_0, y_0, z_0, s_0, t_0) \in Q'$, $G(x_0, y_0, z_0) = 0$. This follows from the property of the implicit representation . Thus, according to Hilbert's Nullstellensatz [Ha77, Chapter 1, Theorem 1.3A]

Let $\mathbf{C}$ *be an algebraically closed field and* $\mathcal{I}$ *be an ideal in* $\mathcal{A} = \mathbf{C}[x,y,z,s,t]$ *and let* $G \in \mathcal{A}$ *be a polynomial which vanishes at all the points of* $V(\mathcal{I})$. *Then* $G^r \in \mathcal{I}$ *for some* $r > 0$.

Hence, if we order the variables as

$$z < y < x < t < s,$$

and compute the Gröbner bases of $\mathcal{I}$, say $\mathcal{H}$. The first polynomial of $\mathcal{H}$ will be independent of s and t and corresponds to the $(G(x,y,z))^r$ for some $r > 0$ and we can recover $G(x,y,z)$ from it.

Example II Let us consider the affine parametrization of the surface in example I

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{st+1}{t^2}, \frac{s}{t^2}, \frac{s^2}{t^2}).$$

In this case

$$\mathcal{I} = \{t^2x - st - 1, t^2y - s, t^2z - s^2\}.$$

Ordering the variables $z < y < x < t < s$, we obtain the Gröbner bases

$$\begin{aligned}\mathcal{H} \;=\; & \{x^2z^2 + y^4 - 2xy^2z - z^3, y^2 - xz + tyz, xy^2 - x^2z + z^2 + ty^3, txy^2 + yz + tz^2 - tx^2z, \\ & t^2y^2 - z, y + tz - t^2xy, x + ty + t^2z - t^2x^2, 1 + t^3y - t^2x, t^2y - s\}.\end{aligned}$$

The first polynomial in the Gröbner bases, $x^2z^2 + y^4 - 2xy^2z - z^3$, corresponds to the affine representation of the implicit equation.

4 Base Points

A base point is a common solution of

$$\overline{X}(s,t,u) = 0, \quad \overline{Y}(s,t,u) = 0, \quad \overline{Z}(s,t,u) = 0, \quad \overline{W}(s,t,u) = 0.$$

The solution set of any of the polynomials, say $\overline{X}(s,t,u) = 0$, corresponds to an algebraic plane curve in the $\mathbf{P}^2$ plane (denoted by homogeneous coordinates s, t and u). Each curve may have more than one component and the base point corresponds to the intersection of these curves. The *multiplicity* of each base point is equal to the multiplicity of the curves at that point. In other words, a base point has multiplicity k, if it is a k-fold point of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$. Let

$$\mathcal{S} = V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u))$$

be the set of base points. Since

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1,$$

$\mathcal{S}$ is therefore, a finite set. Let $\mathbf{p} = (s_0, t_0, u_0) \in \mathcal{S}$. Moreover,

$$\overline{\mathbf{F}}(\mathbf{p}) = \overline{\mathbf{F}}(s_0, t_0, u_0) = (0,0,0,0),$$

which does not correspond to any point in the image space. It has been known that base points blow up to rational curves on the surface (known as *seam curves*) [Cl1869; SR85, Chapter 6; Sn70, page 281; Se90]. Furthermore, the degree of the seam curve is bounded by the multiplicity of the corresponding base point. The blow up can be explained in the following manner:

Consider the rational parametrization

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (su + 2tu + s^2, su + 3tu + t^2, su + tu + 2st, su + 4tu).$$

It follows that $\mathbf{p} = (0,0,1)$ is a base point of the given parametrization. Lets consider the first neighborhood of $\mathbf{p}$ in the domain. That can be obtained by substituting $s = mt$, $u = 1$ and taking the limit $t \to 0$. That is,

$$\mathbf{G}(m) = \lim_{t \to 0} \overline{\mathbf{F}}(mt, t, 1)$$

$$\Rightarrow \mathbf{G}(m) = \lim_{t \to 0}(mt + 2t + m^2t^2, mt + 3t + t^2, mt + t + 2mt^2, mt + 4t)$$

$$\Rightarrow \mathbf{G}(m) = \lim\ t \to 0(m + 2 + m^2t, m + 3 + t, m + 1 + 2mt, m + 4)$$

$$\Rightarrow \mathbf{G}(m) = (m + 2, m + 3, m + 1, m + 4),$$

which is the rational parametrization of a straight line. Thus, each direction in the first neighborhood of the base point gives rise to a different limit point in the image. For an arbitrary parametrization with a base point at the origin, this has been shown in Fig. II. For more on blowing up and its applications we recommend [Ha77, Chapter 1; Wa50, Chapter 4]. If the parametric curves intersect tangentially at a base point, then the first neighborhood cannot be used for computing the seam curve. Later on, we present an algorithm to compute the rational parametrization of seam curves.

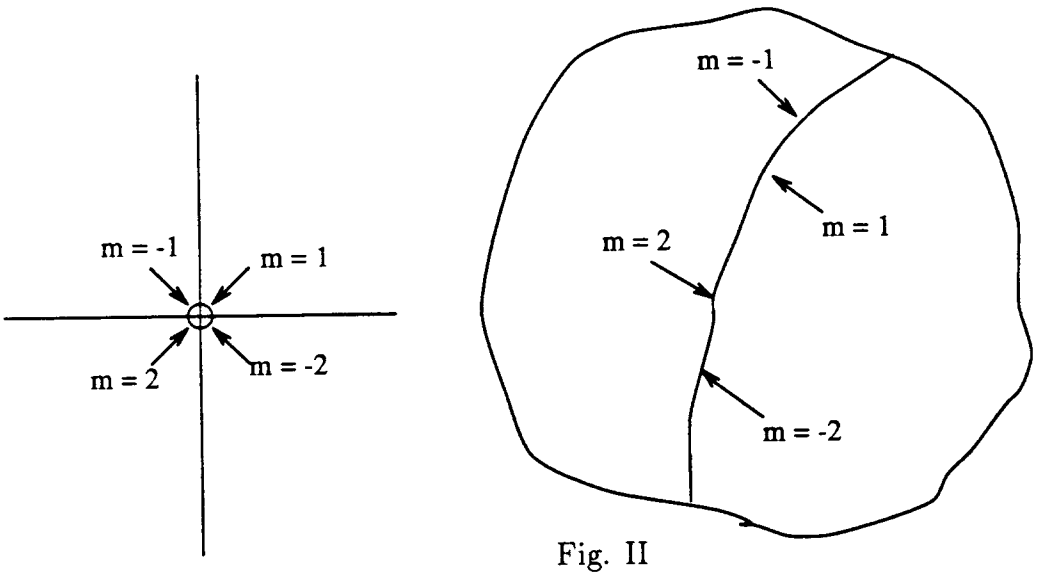

Fig. II
Blowing up of a base point at the origin.

Since $\overline{\mathbf{F}}$ is not defined at the base points, we modify its domain and define it as a mapping of the form

$$\overline{\mathbf{F}'} : \mathbf{P}^2 \setminus \mathcal{S} \to \mathbf{P}^3$$

$$\overline{\mathbf{F}'}(s,t,u) = \overline{\mathbf{F}}(s,t,u),$$

where $\mathbf{P}^2 \setminus \mathcal{S}$ represents the difference of two sets. $\mathbf{P}^2 \setminus \mathcal{S}$ is an open and irreducible set of dimension 2. Let $\mathcal{K}$ be the image of $\overline{\mathbf{F}'}$. We know that $\mathcal{K}$ is a 2-dimensional set and $\mathcal{K} \subset \mathbf{P}^3$. In general, $\mathcal{K}$ is a proper subset of an algebraic set $V(\overline{H}(x,y,z,w))$. The problem of implicitization corresponds to computing $\overline{H}(x,y,z,w)$.

The degree of the implicit representation is a function of the number of base points. If a parametrization of degree n has p simple base points, then the implicit representation has degree at most $n^2 - p$. This is assuming that the parametrization is faithful, else we divide it by a suitable k. This can be explained in the following manner [Sa14]:

The degree of the implicit equation of a parametric surface can be determined by counting the number of times it is intersected by a generic straight line. Let us assume that the parametrization is faithful. Such a line in the space can be represented as the intersection of following 2 planes:

$$a_1x + a_2y + a_3z + a_4 = 0,$$

$$b_1x + b_2y + b_3z + b_4 = 0.$$

To determine the number of intersections of this line with $\overline{\mathbf{F}}$, we substitute for x, y and z and obtain

$$\overline{F}_1(s,t,u) = a_1\overline{X}(s,t,u) + a_2\overline{Y}(s,t,u) + a_3\overline{Z}(s,t,u) + a_4\overline{W}(s,t,u) = 0,$$

$$\overline{F}_2(s,t,u) = b_1\overline{X}(s,t,u) + b_2\overline{Y}(s,t,u) + b_3\overline{Z}(s,t,u) + b_4\overline{W}(s,t,u) = 0.$$

These curves are each of degree n in s, t and u. According to Bezout's theorem the two curves intersect in n^2 points. If a parametrization has no base points, each such point of intersection in the (s,t,u) plane is a preimage of a point of intersection between the line and the surface. Thus, the surface has degree n^2. If $\overline{\mathbf{F}}$ has any base points, both $\overline{F}_1(s,t,u)$ and $\overline{F}_2(s,t,u)$ contain the base point. Thus, they intersect at that base point. The base points blow up to 1-dimensional curves on the surface $(V(\overline{H}(x,y,z,w)) \setminus \mathcal{K})$. Since there are a finite number of such curves on the surface, it is always possible to find a line which does not intersect the surface along any of these curves. In other words, the line does not intersect with any point in the set $V(\overline{H}(x,y,z,w)) \setminus \mathcal{K}$. For such a line the base point is not the preimage of any point of intersection on the surface, and the line intersects the surface at $n^2 - 1$ or less number of points. If there are p simple base points, then the degree of the implicit equation is $n^2 - p$. Base points of multiplicity k decrease the degree of the surface by at least k^2 [Sn70, page 281; Se90]. If $F_1(s,t,u)$ and $F_2(s,t,u)$ intersect tangentially along a k-fold base point, it decreases the degree by at least $k^2 + 1$.

The total number of base points (counted properly) of a parametrization of degree n is $n^2 - d$, where d is the degree of its implicit representation. Thus, a base point of multiplicity k is counted at least k^2 times.

4.0.1 Problem with Resultants

Given $\overline{\mathbf{F}}$, a parametrization with base points, we use resultants to compute the implicit equation. The corresponding parametric equations, (7), are

$$\begin{aligned}\overline{F}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0,\\ \overline{F}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0,\\ \overline{F}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) = 0.\end{aligned}$$

The resultant of these equations, by considering s, t and u as variables, is zero. This can be explained in the following manners:

1. Given $\mathbf{p} = (s_0, t_0, u_0)$, a base point in the parametrization. From the definition of a base point it follows that

   $$\overline{F}_1(s_0,t_0,u_0) = 0 \quad \overline{F}_2(s_0,t_0,u_0) = 0, \quad \overline{F}_3(s_0,t_0,u_0) = 0.$$

   Thus, the given system of equations, (7), has a non trivial solution (s_0, t_0, u_0). Moreover, this solution is independent of the coefficients, x, y and z. The resultant is therefore, identically zero.

2. The solution set of each equation is a 4-dimensional hypersurface. The resultant of these equations contains projections of each component in the intersection set to the (x, y, z) space. Due to base points, one of the component is

   $$Q' = \{(s_0, t_0, u_0, x, y, z) | (s_0, t_0, u_0) \in \mathcal{S}\}.$$

   Q' has dimension 3 and $\Pi(Q')$ is a 3-dimensional component, too. Let $R(x, y, z)$ be the resultant, a polynomial in x, y and z. Thus,

   $$\Pi(Q') \mid R(x, y, z).$$

   Since $\Pi(Q')$ has dimension 3, $R(x, y, z) = 0$. Thus, we cannot recover the implicit representation by computing the resultant of the given parametric equations.

4.0.2 Problem with Grőbner Bases

The Grőbner bases approach is based on the fact, that the implicit equation is contained in the ideal generated by the parametric equations. Thus, by a suitable ordering of the variables, a polynomial in the Grőbner bases of the ideal corresponds to the implicit equation. However, this method does not work when a given parametrization has base points in the affine domain. If there are only base points at infinity, we can use the affine formulation of the given parametric equations for defining the generating set of the ideal and obtain the implicit equation by computing its Grőbner bases.

Theorem II: *Given a parametrization with base points in the affine domain. Let $\mathcal{I}$ be the ideal generated by the parametric equations. There is no polynomial in $\mathcal{I}$, which is independent of s and t, the variables used for defining the parametric domain.*

Proof: The ideal is of the form

$$\mathcal{I} = \{xW(s,t) - wX(s,t), yW(s,t) - wY(s,t), zW(s,t) - wZ(s,t)\}.$$

Let (s_0, t_0) be a base point in the affine domain.

Let us assume that there is a polynomial $F(x,y,z)$, which is independent of s and t in the ideal $\mathcal{I}$. Than $F(x,y,z)$ can be expressed as

$$\begin{aligned} F(x,y,z) &= A_1(x,y,z,s,t)(xW(s,t) - wX(s,t)) + \\ &\quad A_2(x,y,z,s,t)(yW(s,t) - wY(s,t)) + \\ &\quad A_3(x,y,z,s,t)(zW(s,t) - wZ(s,t)). \end{aligned}$$

Lets substitute $s = s_0, t = t_0$ in the above equation. Thus,

$$\begin{aligned} F(x,y,z) &= A_1(x,y,z,s_0,t_0)(xW(s_0,t_0) - wX(s_0,t_0)) + \\ &\quad A_2(x,y,z,s_0,t_0)(yW(s_0,t_0) - wY(s_0,t_0)) + \\ &\quad A_3(x,y,z,s_0,t_0)(zW(s_0,t_0) - wZ(s_0,t_0)), \\ \Rightarrow F(x,y,z) &= 0. \end{aligned}$$

Hence, the only polynomial independent of s and t, which lies in $\mathcal{I}$ is the zero polynomial. Thus, all other polynomials have a term containing s or t.

Q.E.D.

Given a parametrization with base points in the affine domain, the Gröbner bases of the ideal generated by the parametric equations does not contain the implicit equation. Thus, Gröbner bases cannot be used for implicitizing such parametrizations.

4.1 Tensor Product Surfaces

In computer graphics and geometric modeling, most surfaces are represented as tensor product surfaces. A tensor product surface is a linear combination of bivariate basis functions, where each bivariate function is formed by taking every possible pair, one from one set of univariate functions and the other from another [Pr84]. Typical univariate functions are like the Bernstein polynomials, B-spline basis functions etc. Such a surface is represented as

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)) = \sum_j \sum_i \mathbf{A}_{ij} G_i(s) H_j(t),$$

where G_i and H_j are the univariate basis functions and $\mathbf{A}_{ij}$ are 4 dimensional vectors, used to represent the scalars for each component. Any component, say $X(s,t)$, can be

represented in the power basis form as:

$$X(s,t) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j, \quad X_{mn} \neq 0,$$

which is of degree $m+n$ in s and t. However, the highest power of s in any monomial is m and that of t is n. After homogenizing we obtain

$$\overline{X}(s,t,u) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j u^{m+n-i-j}, \tag{10}$$

which is a homogeneous polynomial of degree $m+n$. Let

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)).$$

Lemma III: *Every tensor product surface of the form* $\overline{\mathbf{F}}(s,t,u)$ *has a base point of multiplicity* n *at* $(1,0,0)$ *and of multiplicity* m *at* $(0,1,0)$.

Proof Let us consider one of the components, say $\overline{X}(s,t,u)$. It follows from (10) that

$$\overline{X}(1,0,0) = 0, \quad \overline{X}(0,1,0) = 0.$$

Similarly $\overline{Y}$, $\overline{Z}$ and $\overline{W}$ vanish at these points, too. Thus, $(1,0,0)$ and $(0,1,0)$ are base points of $\overline{\mathbf{F}}$.

To analyze the multiplicity of $(1,0,0)$, we represent $\overline{X}(s,t,u)$ as

$$\overline{X}(s,t,u) = \overline{X}_0(t,u)s^{m+n} + \ldots + \overline{X}_{n-1}(t,u)s^{m+1} + \overline{X}_n(t,u)s^m + \ldots + \overline{X}_{m+n}(t,u),$$

where $\overline{X}_i(t,u)$ is a homogeneous polynomial of degree i in t and u. Since the highest degree of s in any term of $\overline{X}(s,t,u)$ is m, it follows that $\overline{X}_0(t,u), \overline{X}_1(t,u), \ldots \overline{X}_{n-1}(t,u)$ vanish identically.

Thus, $(1,0,0)$ is a base point of multiplicity n. Similarly, $(0,1,0)$ is a base point of multiplicity m.

Q.E.D.

Since a tensor product surface always has base points at infinity, the resultant of the parametric equations, considering them as total degree bounded polynomials in s, t and u is identically zero. Dixon gave a special formulation of a resultant of three quantics in two independent variables, of the form $X(s,t)$, such that the vanishing of the resultant is a necessary and sufficient condition for the system to have a *non trivial* solution [Di08]. The set of trivial solutions consists of

- $(1,0,0)$ of multiplicity n.
- $(0,1,0)$ of multiplicity m.

For total degree bounded polynomials, every base point in $\mathbf{P}^2$ is a *non trivial* base point. We refer to this formulation for tensor product surfaces as *Dixon eliminant*[3]. The term resultant would be used for total degree bounded parametrizations. The Dixon's eliminant has been used by [SAG84] to implicitize tensor product surfaces.

The degree of the homogeneous formulation of the parametric equations of $\mathbf{F}$ is $m+n$. Hence, the degree of the corresponding implicit equation can be at most $(m+n)^2$. However, base points of multiplicity n and m decrease the degree by n^2 and m^2, respectively, and the implicit representation is of degree $2mn$ or less. If the parametrization has base points in the affine domain or the curves $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{W}$ intersect tangentially at $(1,0,0)$ or $(0,1,0)$, the degree of the implicit equation is strictly less than $2mn$ and the Dixon's formulation for such equations is identically zero, too.

Let us see when do these curves, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{W}$, intersect tangentially along a base point at infinity. We will later on use these constraints for choosing a suitable perturbation. Consider the base point $(1,0,0)$. Let

$$\begin{aligned}
\overline{X}(s,t,u) &= \overline{X}_n(t,u)s^m + \ldots + \overline{X}_{m+n}(t,u), \\
\overline{Y}(s,t,u) &= \overline{Y}_n(t,u)s^m + \ldots + \overline{Y}_{m+n}(t,u), \\
\overline{Z}(s,t,u) &= \overline{Z}_n(t,u)s^m + \ldots + \overline{Z}_{m+n}(t,u), \\
\overline{W}(s,t,u) &= \overline{W}_n(t,u)s^m + \ldots + \overline{W}_{m+n}(t,u).
\end{aligned}$$

A line, $t/a = u/b$ is tangent to these curves at $(1,0,0)$ if and only if

$$\overline{X}_n(a,b) = 0, \quad \overline{Y}_n(a,b) = 0, \quad \overline{Z}_n(a,b) = 0, \quad \overline{W}_n(a,b) = 0.$$

Since $\overline{X}_n, \overline{Y}_n, \overline{Z}_n, \overline{W}_n$ are homogeneous polynomials in t and u this can happen if and only if

$$\overline{G}(t,u) = GCD(\overline{X}_n(t,u), \overline{Y}_n(t,u), \overline{Z}_n(t,u), \overline{W}_n(t,u))$$

is a homogeneous polynomial of positive degree in t and u. Moreover, (a,b) is one of the roots of $\overline{G}(t,u)$. This constraint is equivalent to saying that

$$G(t) = GCD(\sum_{j=0,n} X_{mj}t^j, \sum_{j=0,n} Y_{mj}t^j, \sum_{j=0,n} Z_{mj}t^j, \sum_{j=0,n} W_{mj}t^j)$$

is a polynomial of positive degree. $\sum_{j=0,n} X_{mj}t^j$ corresponds to the coefficient of s^m in $X(s,t)$. Similarly the curves intersect tangentially at $(0,1,0)$ if and only if

$$H(s) = GCD(\sum_{i=0,m} X_{in}s^i, \sum_{i=0,m} Y_{in}s^i, \sum_{i=0,m} Z_{in}s^i, \sum_{j=0,m} W_{in}s^i)$$

is a polynomial of positive degree. In such cases Dixon's formulation can not be directly used for implicitizing. However, Gröbner bases can still be used since the base points are at infinity. Gröbner bases fail when the tensor product surface has base points in the affine domain.

[3]Dixon gave many other formulations for computing the resultant for a set of equations, including one for total degree bounded polynomials, too. We are only referring to the one used for tensor product surfaces.

5 Perturbation

Till now we have analyzed two main techniques for implicitizing parametric surfaces: Elimination theory and Grőbner bases. Elimination theory has been used in the form of resultants for total degree bounded parametrizations and Dixon eliminant for tensor product surfaces. All these techniques fail when a parametrization has non trivial base points. For example, the resultant of the parametric equations is identically zero due to the presence of an excess component in the image space. Thus, the problem of implicitizing corresponds to: *computing the proper component in the presence of excess component.* Some similar problems have been encountered while solving system of polynomial equations and techniques for dealing with such problems have been highlighted in [Ca88, Ie89]. The technique corresponds to perturbing the given equations, such that the resulting algebraic set (in the higher dimensional space defined by adding the perturbing variable) has no excess component. The projections of the proper components of the algebraic set corresponding to the unperturbed system can be obtained from the projections of the algebraic set corresponding to the perturbed system by applying limiting arguments.

We will carry out the rest of perturbation analysis for total degree bounded parametrizations and use resultants. The results obtained are also applicable for tensor product parametrizations or using Grőbner bases instead of techniques from Elimination theory. These methods will be illustrated with some examples.

Lets consider the parametrization

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

of degree n, which has base points in the domain, represented by set $\mathcal{S}$. The resultant of the parametric equations, (7), is identically zero. Lets perturb the given system of equations and the resulting parametric equations are

$$\begin{aligned} \overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) + \lambda\overline{X}_1(s,t,u) = 0, \\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) + \lambda\overline{Y}_1(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0, \end{aligned} \tag{11}$$

where λ is the perturbing variable and $\overline{X}_1(s,t,u), \overline{Y}_1(s,t,u)$ and $\overline{Z}_1(s,t,u)$ are homogeneous polynomials of degree n such that

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{X}_1(s,t,u), \overline{Y}_1(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

In other words, the perturbed system of parametric equations, (11), has no non-trivial solutions and therefore, their resultant does not vanish. A simple procedure is to choose random polynomials, $\overline{X}_1(s,t,u), \overline{Y}_1(s,t,u)$ and $\overline{Z}_1(s,t,u)$. The resulting system of perturbed equations has a base point if and only if their resultant of $\overline{G}_i$'s is zero. This process of choosing random polynomials can be repeated until the resultant is nonzero. The probability of success is very close to 1.

Let

$$Q = V(\overline{G}_1, \overline{G}_2, \overline{G}_3) \subset \mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1,$$

and Π be the projection function

$$\Pi : \mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1 \rightarrow \mathbf{C}^3 \times \mathbf{C}^1,$$

such that

$$\Pi(s,t,u,x,y,z,\lambda) = (x,y,z,\lambda).$$

According to Lemma I every component of Q has dimension greater than or equal to 3. Let $R(x,y,z,\lambda)$ be the resultant of the perturbed system, (11), i.e.

$$R(x,y,z,\lambda) = \Pi(Q).$$

Let us express the resultant as a polynomial in λ, while the coefficients are polynomials in x, y and z:

$$R(x,y,z,\lambda) = P_i(x,y,z)\lambda^i + \ldots + P_d(x,y,z)\lambda^d. \tag{12}$$

If we specialize ($\lambda = 0$) in the parametric equations, (11), the resultant of the specialized system is zero due to the presence of base points. As a result

$$R(x,y,z,\lambda)_{\lambda=0} = 0$$

and therefore, $i > 0$ in (12).

Theorem III: *$H(x,y,z)$, the implicit representation of $\overline{\mathbf{F}}(s,t,u)$ is contained in $P_i(x,y,z)$, i.e.*

$$H(x,y,z) \mid P_i(x,y,z),$$

where $P_i(x,y,z)$ is the coefficient of the lowest degree term of $R(x,y,z,\lambda)$, expressed as a polynomial in λ.

Proof: [4] Let

$$P = V(\overline{F}_1(s,t,u), \overline{F}_2(s,t,u), \overline{F}_3(s,t,u))$$

where $\overline{F}_i(s,t,u)$ is an unperturbed parametric equation and

$$P \subset \mathbf{P}^2 \times \mathbf{C}^3.$$

Let B be component of P defined as

$$\begin{aligned} B \;=\; & \{(s,t,u,x,y,z) \mid x = \frac{\overline{F}_1(s,t,u)}{\overline{F}_4(s,t,u)}, y = \frac{\overline{F}_2(s,t,u)}{\overline{F}_4(s,t,u)}, z = \frac{\overline{F}_3(s,t,u)}{\overline{F}_4(s,t,u)},\ (s,t,u) \in \mathbf{P}^2 \setminus \mathcal{S}\} \\ & \cup \{(s,t,u,x,y,z) \mid (s,t,u) \in \mathcal{S} \text{ and } (x,y,z) \in C_{(s,t,u)}(x,y,z)\}, \end{aligned}$$

where $C_{(s,t,u)}(x,y,z)$ is the set of all points lying on the seam curve corresponding to (s,t,u). B is a proper component of P.

[4] It is very similar to the proof of Theorem 3.2 in [Ca88].

With the addition of a complex variable λ, the zero set of Q lies in $\mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1$. Since $\overline{F}_i(s,t,u)$ and $\overline{G}_i(s,t,u)$ are identical when $\lambda = 0$, it follows that

$$P \times \{0\} = Q \cap (\lambda = 0).$$

Thus, $B \times \{0\} \subset Q$. Since every component of Q has dimension greater than or equal to 3, $B \times \{0\}$ must be contained in some 3 dimensional component B' of Q. Every point of B' has a 3 dimensional neighborhood whose intersection with the hypersurface $\lambda = 0$ is a 2 dimensional set. Thus, for every point $\mathbf{q} = (s_k, t_k, u_k, x_k, y_k, z_k, 0) \in B \times \{0\}$, there is a sequence of points $\mathbf{q}_j = (s_j, t_j, u_j, x_j, y_j, z_j, \lambda_j)$ in $B' - B \times \{0\}$ which converges to $\mathbf{q}$. Moreover $R(\Pi(\mathbf{q}_j)) = 0$ for all j's. Thus, $R(x_j, y_j, z_j, \lambda_j) = 0$. Divide the polynomial throughout by $(\lambda_j)^i$ (which is non-zero) and we obtain

$$P_i(x_j, y_j, z_j) + P_{i+1}(x_j, y_j, z_j)(\lambda_j) + \ldots + P_d(x_j, y_j, z_j)(\lambda_j)^{d-i} = 0$$

for all $\mathbf{q}_j$. This is a polynomial in the coordinates of $\mathbf{q}_j$ and is, therefore, a continuous function of the coordinates. Since it is zero for $\mathbf{q}_j \rightarrow \mathbf{q}$, it must be zero at $\mathbf{q}$. But $\mathbf{q}$ is a point lying on the hypersurface $\lambda = 0$, so $P_i(x_k, y_k, z_k) = 0$. Since

$$V(H(x,y,z)) = \{(x_k, y_k, z_k) \mid \mathbf{q} = (s_k, t_k, u_k, x_k, y_k, z_k, 0) \in B \times \{0\}\},$$

$V(H(x,y,z)) \subset V(P_i(x,y,z))$. If $\overline{\mathbf{F}}$ is a faithful parametrization, $H(x,y,z)$ is an irreducible polynomial and therefore, $H(x,y,z) \mid P_i(x,y,z)$. Else let any generic point in $\mathcal{Y}$ have m preimages ($m > 1$). Thus, $H(x,y,z) = G(x,y,z)^m$. Let $(x_1, y_1, z_1) \in \mathcal{Y}$ and (s_i, t_i, u_i), $1 \leq i \leq m$ be its preimages. In other words $\mathbf{q}_i = (s_i, t_i, u_i, x_1, y_1, z_1) \in B$ for all i. As a result $\mathbf{q}_i \times 0 \in B'$ and it has a 3-dimensional neighborhood in $B' - B \times 0$ which converges to $\mathbf{q}_i \times 0$. Since $R(x,y,z,\lambda) = \Pi(Q)$, we can use the limiting argument to show that (x_1, y_1, z_1) is a point of multiplicity m in $V(P_i(x,y,z))$. Thus,

$$H(x,y,z) \mid P_i(x,y,z).$$

Q.E.D.

The same result hold when we use Dixon eliminant on tensor product parametrizations or Gröbner bases. We illustrate the techniques on the following example.

Example III Let

$$\mathbf{F}(s,t) = (x,y,z) = \left(\frac{s^2 - 1 - t^2}{s^2 + 1 + t^2}, \frac{2s}{s^2 + 1 + t^2}, \frac{2st}{s^2 + 1 + t^2}\right)$$

be the parametrization of a rational surface (a sphere in this case), which has a base point at $(s,t) = (0,i)$, where $i = \sqrt{-1}$. The ideal generated by the parametric equations is

$$\mathcal{I} = \{x(s^2 + 1 + t^2) - s^2 + 1 + t^2, y(s^2 + 1 + t^2) - 2s, z(s^2 + 1 + t^2) - 2st\}.$$

According to Theorem II, none of the polynomials in $\mathcal{I}$ is independent of s and t. Lets perturb the parametric equations and the resulting ideal is

$$\mathcal{J} = \{x(s^2 + 1 + t^2) - s^2 + 1 + t^2 - \lambda t, y(s^2 + 1 + t^2) - 2s - \lambda, z(s^2 + 1 + t^2) - 2st + \lambda s\}.$$

Compute the Gröbner bases of $\mathcal{J}$ with a variable ordering

$$z < y < x < \lambda < s < t.$$

The first polynomial in the Gröbner bases is independent of s and t. It can be expressed as a polynomial in λ as

$$\lambda(2+\lambda^2)(\lambda^2y^2(-3-x^2-y^2-4z-z^2)-2\lambda y(-x+x^3-2y^2+xy^2-z-4xz-x^2z-3y^2z-2z^2)$$
$$-2\lambda y(-3xz^2-z^3)-2(x^2+y^2+z^2-1)(1+2x+x^2+y^2+2z+2xz+z^2)),$$

whose lowest degree term is

$$-2(x^2+y^2+z^2-1)(1+2x+x^2+y^2+2z+2xz+z^2).$$

Thus, the implicit representation of the sphere, $x^2+y^2+z^2-1$, is contained in the lowest degree term.

Example IV Lets consider a tensor product parametrization

$$\mathbf{F}(s,t) = (x,y,z,w) = (st^2-t, st+s, 2s-2t, st^2),$$

which has a base point at $(s,t) = (0,0)$. The resulting parametric equations are

$$\begin{aligned} xst^2 - st^2 + t &= 0, \\ yst^2 - st - s &= 0, \\ zst^2 - 2s + 2t &= 0, \end{aligned}$$

whose Dixon eliminant is zero. Lets perturb these equations and the resulting system is

$$\begin{aligned} xst^2 - st^2 + t + \lambda(s+2) &= 0, \\ yst^2 - st - s + \lambda t^2 &= 0, \\ zst^2 - 2s + 2t + \lambda(s+4) &= 0. \end{aligned}$$

The Dixon eliminant of these equations is polynomial in x, y, z and λ and after expressing it as a polynomial in λ, the lowest degree term is

$$8(-2+2x-z)(-4x+4x^2-8y+8xy+4y^2+2z-4xz-4yz+z^2).$$

In this case, $(-2+2x-z)$ is an extraneous factor and $(-4x+4x^2-8y+8xy+4y^2+2z-4xz-4yz+z^2)$ is the implicit representation.

Q.E.D.

Thus, we can perturb the given parametric equations such that the lowest degree term of the resultant of the perturbed system contains the implicit equation. However, there is always an extraneous factor present in the lowest degree term and extracting the implicit

representation involves multivariate factorization. Furthermore, we need to test each irreducible polynomial, obtained after factorization, whether it corresponds to the implicit equation. In many cases this process can be a time consuming task.

We can express $R(x, y, z, \lambda)$ as

$$R(x, y, z, \lambda) = \lambda^i(P_i(x, y, z) + \ldots + P_d(x, y, z)\lambda^{d-i}).$$

Each component of $V(R(x, y, z, \lambda)) \in \mathbf{C}^3 \times \mathbf{C}^1$ corresponds to the projection of a component of Q. Thus, $\lambda = 0$ is a component of multiplicity i of $\Pi(Q)$ and it is obtained as the projection of a component of the form

$$M = \{(s_0, t_0, u_0, x, y, z, 0) \mid (s_0, t_0, u_0) \in \mathcal{S}\}.$$

The lowest power of λ in $R(x, y, z, \lambda)$ is greater than the number of base points (counted properly with respect to multiplicity). In other words, the degree of the implicit equation is at least $n^2 - i$, where i is the lowest power of λ in the resultant of the perturbed system, expressed as a polynomial in λ.

6 Efficient Perturbation

In this section we present an efficient perturbation such that the implicit equation can be extracted from the lowest degree term of the resultant by computing the GCD of bivariate polynomials and multivariate polynomial division. It is simpler to compute the GCD of bivariate polynomials and perform trivariate polynomial division than to factorize trivariate polynomials. Moreover the extraneous factor in the lowest degree term of the perturbed system contains interesting information about the seams or blow-ups of the base points. In particular, we choose our perturbation so that we get the X-Y-projection of the blow-up curves. This is useful because the polynomial we obtain is the product of the implicit equation and a polynomial that depends on x and y only. This makes it easy to separate the implicit equation, assuming that it depends on z (which it will after a generic change of coordinates). As a result we do not need to factorize, and the GCD we compute involves only bivariate polynomials.

Before we present the efficient perturbation and carry out the analysis, we make certain assumptions on the given parametrization, $\overline{\mathbf{F}}$. They are:

- The implicit representation is not independent of z. In other words, it is not of the form $H(x, y) = 0$. Later on, we present a technique to verify this assumption and adjust the parametrization accordingly.
- $\overline{W}(s, t, u))$ does not divide $\overline{Z}(s, t, u)$. Otherwise the implicit representation is of the form $z - k = 0$, where $k = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}$ and we can compute it directly.

The base points blow up to rational curves of the form $(X(t), Y(t), Z(t), W(t))$ on the surface. Since these curves lie on the surface, they are characterized by the property that

$$H(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}, \frac{Z(t)}{W(t)}) = 0,$$

where $H(x, y, z)$ is the implicit representation of the surface. Lets consider the projection of one of these curves on the X-Y plane. The projected curve has a rational parametrization of the form $(X(t), Y(t), W(t))$ and it can be implicitly represented as the zero set of an irreducible polynomial, say $F(x, y)$. The fact that $V(H(x, y, z))$ represents a single component in space (unique up to multiplicity) and the assumption that the implicit equation is not independent of z imply that $V(F(x, y))$ is not contained in $V(H(x, y, z))$. In fact we will show that the lowest degree term of this efficient perturbation can be expressed as a product of $H(x, y, z)$ and $F(x, y)$'s (corresponding to different seam curves).

Given a parametrization with base points, let us perturb one of the three parametric equations, (7), say $\overline{F}_3(s, t, u)$ and the resulting perturbed system is

$$\begin{aligned} \overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0, \end{aligned} \tag{13}$$

where $\overline{Z}_1(s, t, u)$ is a homogeneous polynomial of degree n such that

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

We will denote this perturbed parametrization as $\overline{\mathbf{G}}$. It is still possible that for all choices of $\overline{Z}_1(s, t, u)$ the resultant of $\overline{G}_1(s, t, u), \overline{G}_2(s, t, u)$ and $\overline{G}_3(s, t, u)$ is zero. Let

$$Q = V(\overline{G}_1, \overline{G}_2, \overline{G}_3) \subset \mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1,$$

and Π be the projection function from $\mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1$ to $\mathbf{C}^3 \times \mathbf{C}^1$, as defined in the previous section.

Theorem IV: *Given a set of three equations of the form, $\overline{G}_1(s, t, u), \overline{G}_2(s, t, u)$ and $\overline{G}_3(s, t, u)$, where $\overline{Z}_1(s, t, u)$ is chosen such that*

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

The necessary and sufficient condition that the resultant of $\overline{G}_i$'s does not vanish is that

$$\overline{P}(s,t,u) = GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u))$$

is a constant.

Proof: *Necessity*

Let us assume that $\overline{P}(s,t,u)$ is a polynomial of positive degree. Let us consider the set

$$M = \{(s,t,u,x,y,z,\lambda) \mid \overline{P}(s,t,u) = 0, -\overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0\}$$

and

$$M \subset \mathbf{P}^2 \times \mathbf{C}^3 \times \mathbf{C}^1.$$

Let $\mathbf{p} = (s_1,t_1,u_1,x_1,y_1,z_1,\lambda_1) \in M$. Thus,

$$\overline{P}(s,t,u) = 0$$

and therefore

$$\begin{aligned}
\overline{G}_1(s_1,t_1,u_1) &= x_1\overline{W}(s_1,t_1,u_1) - \overline{X}(s_1,t_1,u_1) = 0 - 0 = 0, \\
\overline{G}_2(s_1,t_1,u_1) &= y_1\overline{W}(s_1,t_1,u_1) - \overline{Y}(s_1,t_1,u_1) = 0 - 0 = 0, \\
\overline{G}_3(s_1,t_1,u_1) &= z_1\overline{W}(s_1,t_1,u_1) - \overline{Z}(s_1,t_1,u_1) + \lambda_1\overline{Z}_1(s_1,t_1,u_1) \\
&= -\overline{Z}(s_1,t_1,u_1) + \lambda_1\overline{Z}_1(s_1,t_1,u_1) = 0.
\end{aligned}$$

Thus, $\mathbf{p} \in V(\overline{G}_1,\overline{G}_2,\overline{G}_3) \Rightarrow \mathbf{p} \in Q$. In other words, $M \subset Q$. M is a four dimensional set. Given any 4-tuple, $(x,y,z,\lambda) = (x_1,y_1,z_1,\lambda_1)$, one can find (s_1,t_1,u_1) such that $(s_1,t_1,u_1,x_1,y_1,z_1,\lambda_1) \in M$. Thus, M is an excess component of Q and $\Pi(M)$ is a four dimensional set, too. Therefore the resultant of $\overline{G}_1,\overline{G}_2$ and $\overline{G}_3$ is zero.

Sufficiency

Let $\overline{P}(s,t,u)$ be a constant polynomial. To prove the non-vanishing of the resultant it is sufficient to show that there is some value of x, y, z and λ such that for those values $\overline{G}_i$'s have no common solution.

First pick $x = 0$. Now choose a value of y so that $\overline{G}_2(s,t,u)$ has a finite number of intersections with $\overline{G}_1(s,t,u)_{x=0}$, i.e. $\overline{X}(s,t,u)$. Since $GCD(\overline{X}(s,t,u),\overline{Y}(s,t,u),\overline{W}(s,t,u))$ is a constant, for almost all values of y, $\overline{X}(s,t,u)$ and $\overline{G}_2(s,t,u)$ intersect in n^2 points, according to Bezout's theorem. Let these points be (s_i,t_i,u_i), $1 \le i \le n^2$.

Once there are a finite number of solutions for the $\overline{G}_1(s,t,u)$ and $\overline{G}_2(s,t,u)$, it is easy to choose z and λ such that $\overline{G}_3(s_i,t_i,u_i) \neq 0$. At any of the n^2 solution points, say (s_i,t_i,u_i), $\overline{X}(s_i,t_i,u_i) = 0$. Pick z and λ such that for each solution they satisfy the following constraint. The constraint depends on the value of $\overline{W}(s_i,t_i,u_i)$:

- Case $\overline{W}(s_i,t_i,u_i) = 0$.

  The fact that $\overline{X}(s_i,t_i,u_i) = 0$ implies that $\overline{Y}(s_i,t_i,u_i) = 0$. The polynomial $\overline{Z}_1(s,t,u)$ is chosen to be non-zero at the common roots of $\overline{X}(s,t,u)$, $\overline{Y}(s,t,u)$ and $\overline{W}(s,t,u)$ and therefore, $\overline{Z}_1(s_i,t_i,u_i) \neq 0$. In this case

$$\lambda \neq \frac{\overline{Z}(s_i,t_i,u_i)}{\overline{Z}_1(s_i,t_i,u_i)}.$$

- Case $\overline{W}(s_i, t_i, u_i) \neq 0$.

  Let λ take any value choose z such that

$$z \neq \frac{\overline{Z}(s_i, t_i, u_i) - \lambda \overline{Z}_1(s_i, t_i, u_i)}{\overline{W}(s_i, t_i, u_i)}.$$

Thus, for almost all choices of z and λ, the given equations have no common solution and therefore, the resultant does not vanish.

Q.E.D.

To circumvent this problem of vanishing resultant in certain cases we perform a change of coordinates and let the new parametrization be

$$\overline{\mathbf{F}}'(s,t,u) = (x', y', z', w') = (x, y + kz, z, w)$$

$$= (\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where k is a scalar. The corresponding parametric equations are

$$\begin{aligned}
\overline{G}_1'(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\
\overline{G}_2'(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) - k\overline{Z}(s,t,u) = 0, \\
\overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0.
\end{aligned}$$

Since $GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1$, for any generic k,

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1,$$

too. We compute the implicit representation in terms of x', y', z' and w' and substitute them to obtain an implicit equation in terms of x, y, z and w. From now onwards we assume that it is possible to choose $\overline{Z}_1(s,t,u)$ such that the resultant of $\overline{G}_1(s,t,u), \overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$, $R(x,y,z,\lambda)$, is nonzero. Moreover the resultant can be expressed as a polynomial of the form

$$R(x,y,z,\lambda) = \lambda^i S(x,y,z,\lambda), \tag{14}$$

where $S(x,y,z,0) \neq 0$.

Lemma IV: *The total number of base points (counted properly) of* $\overline{\mathbf{F}}$ *correspond to* i *in (14).*

Proof: Let $\overline{\mathbf{F}}$ has m base points (counted properly). Base points of multiplicity are counted at least k^2 times. Thus, its implicit representation has degree $n^2 - m$. $R(x,y,z,\lambda)$ is the resultant of $\overline{G}_1(s,t,u)$, $\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$. $\overline{G}_1(s,t,u)$ and $\overline{G}_2(s,t,u)$ correspond to plane curves of degree n each and according to Bezout's theorem intersect in n^2 points (counted properly). Let the points be (s_i, t_i, u_i), $1 \leq i \leq n^2$. If (s_0, t_0, u_0) is a base point of $\overline{\mathbf{F}}$, $\overline{G}_1(s_0, t_0, u_0) = \overline{G}_2(s_0, t_0, u_0) = 0$. Thus, the intersection set consist of these m base

points and $n^2 - m$ other intersections (which are functions of x and y). Let (s_j, t_j, u_j), $1 \leq j \leq m$ correspond to the base points. The resultant can be expressed as [Sa1885]

$$\begin{aligned} R(x,y,z,\lambda) &= \prod_{i=1}^{n^2} \overline{G}_3(s_i,t_i,u_i) \\ &= \prod_{i=1}^{n^2} (z\overline{W}(s_i,t_i,u_i) - \overline{Z}(s_i,t_i,u_i) + \lambda \overline{Z}_1(s_i,t_i,u_i)) \\ &= \alpha\lambda^m \prod_{i=m+1}^{n^2} \overline{G}_3(s_i,t_i,u_i), \end{aligned}$$

where $\alpha = \prod_{i=1}^{m} \overline{Z}_1(s_i,t_i,u_i) \neq 0$. Thus, the lowest degree term in λ in $R(x,y,z,\lambda)$ has degree at least m. Since the points, (s_i,t_i,u_i), $m < i \leq n^2$ do not correspond to the base points, at least $\overline{W}(s_i,t_i,u_i)$ or $\overline{Z}(s_i,t_i,u_i)$ does not vanish. Thus, the lowest degree term of the resultant has degree exactly equal to m.

Q.E.D.

For a generic choice of $\overline{Z}_1(s,t,u)$ it is possible to show that $S(x,y,z,\lambda)$ is an irreducible polynomial. This follows from the fact, that for any generic choice of $\lambda = \lambda_i$, the resulting parametrization $\overline{\mathbf{G}}$ has no base points and $R(x,y,z,\lambda_i)$ corresponds to its implicit representation. Therefore, $R(x,y,z,\lambda_i)$ is equal to some power of an irreducible polynomial and for a generic choice of $\overline{Z}_1(s,t,u)$, $R(x,y,z,\lambda_i)$ is an irreducible polynomial. Thus, $V(R(x,y,z,\lambda))$ consist of $i+1$ components and therefore, Q consists of $i+1$ components, too. i of these components are of the form

$$\{(s_0,t_0,u_0,x,y,z,0),\}$$

where (s_0,t_0,u_0) is a base point and the $(i+1)^{st}$ component can be represented as

$$Q' = \{\mathbf{q} = (s_j,t_j,u_j,x_j,y_j,z_j,\lambda_j) \mid \mathbf{q} \in Q,\ S(x_j,y_j,z_j,\lambda_j) = 0\}.$$

Let us express the resultant as a polynomial in λ, and let $P_i(x,y,z)$ be the constant term of $S(x,y,z,\lambda)$. We know from Theorem III that

$$P_i(x,y,z) = H(x,y,z)F(x,y,z),$$

where $H(x,y,z)$ corresponds to some power of the implicit equation and $F(x,y,z)$ is the extraneous factor. Our aim is to extract $H(x,y,z)$ without resorting to multivariate factorization.

Theorem V: *$F(x,y,z)$ is independent of z. In other words $F(x,y,z)$ is a bivariate polynomial in x and y. Moreover, $F(x,y,z)$ correspond exactly to the projections of the seam curves on the X-Y plane.*

Proof: Every component of Q has dimension 3. Let P and B be algebraic sets as defined in the proof of Theorem III. For every point $\mathbf{q} \in B \times \{0\}$, there is a sequence of points

$(\mathbf{q}_j) \in Q' - B \times \{0\}$ in its neighborhood, which converges to $\mathbf{q}$. Furthermore, $\mathbf{q}$ has a 3-dimensional neighborhood for defining such sequence of points. As a result we are able to show that $H(x,y,z) \mid P_i(x,y,z)$. Let (s_0,t_0,u_0) be a base point of $\overline{\mathbf{F}}(s,t,u)$ and let $\mathbf{q} = (s_0,t_0,u_0,x_0,y_0,z_0,0)$, where (x_0,y_0,z_0) is a point on the seam curve corresponding to (s_0,t_0,u_0). Let (x_j,y_j,z_j,λ_j) be a point in the neighborhood of $(x_0,y_0,z_0,0)$ such that $S(x_j,y_j,z_j,\lambda_j) = 0$. For each such (x_j,y_j,z_j,λ_j) there exists (s_j,t_j,u_j) such that $\mathbf{q}_j = (s_j,t_j,u_j,x_j,y_j,z_j,\lambda_j) \in Q' - B \times \{0\}$. As a result we are able to define a sequence of points $\mathbf{q}_j$ converging to $\mathbf{q}$. Corresponding to every point in this sequence let us consider another sequence of points $\mathbf{q}_j' = (s_j,t_j,u_j,x_j,y_j,z_j',\lambda_j')$ such that

$$\begin{aligned} z_j' &= kz_j, \\ \lambda_j' &= \frac{-kz_j\overline{W}(s_j,t_j,u_j) + \overline{Z}(s_j,t_j,u_j)}{\overline{Z}_1(s_j,t_j,u_j)}, \end{aligned}$$

where k is any arbitrary constant. The fact $\mathbf{q}_j \in Q'$ implies that $\mathbf{q}_j' \in Q'$. This follows because $\mathbf{q}_j' \in V(\overline{G}_1(s,t,u)) \cap V(\overline{G}_2(s,t,u))$. The choice of λ_j' implies that $\mathbf{q}_j' \in V(\overline{G}_3(s,t,u))$. As a result $R(x_j,y_j,kz_j,\lambda_j') = 0$.

As we take the sequence of points approaching $\mathbf{q}$, using the limiting argument it follows that $(x_0,y_0,z_0) \in V(P_i(x,y,z))$. Moreover,

$$\lim_{(s_j,t_j,u_j)\to(s_0,t_0,u_0)} \lambda_j' = \lim_{(s_j,t_j,u_j)\to(s_0,t_0,u_0)} \frac{-kz_j\overline{W}(s_j,t_j,u_j) - \overline{Z}(s_j,t_j,u_j)}{\overline{Z}_1(s_j,t_j,u_j)} = 0.$$

This is because (s_0,t_0,u_0) is a base point and therefore, $\overline{W}(s_0,t_0,u_0) = 0$, $\overline{Z}(s_0,t_0,u_0) = 0$ and $\overline{Z}_1(s_0,t_0,u_0) \neq 0$. Thus, $\mathbf{q}_j' \to (s_0,t_0,u_0,x_0,y_0,kz_0,0)$ and from the limiting arguments it follows that $P_i(x_0,y_0,kz_0) = 0$. Furthermore, (x_0,y_0,z_0) can correspond to any point on the seam curve and the choice of k is arbitrary.

The fact $P_i(x_0,y_0,kz_0) = 0$ implies either $H(x_0,y_0,kz_0) = 0$ or $F(x_0,y_0,kz_0) = 0$. We have assumed that $H(x,y,z)$ is not independent of z and therefore, it is not possible that for all points (x_0,y_0,z_0) on a seam curve $(x_0,y_0,kz_0) \in V(H(x,y,z))$, for any choice of k. Therefore, $F(x_0,y_0,kz_0) = 0$ for all k. Since $V(F(x,y,z))$ is a polynomial in x, y and z, this is possible if and only if $F(x_0,y_0,z) = 0$ for all such (x_0,y_0), where x_0 and y_0 represent the x and y coordinates of a point on a seam curve. Let $\beta_j(x,y)$, $1 \leq j \leq m$ correspond to the implicit representation of the projection of seam curves (where m correspond to the number of seam curves and $m \leq i$) on the X-Y plane and therefore

$$V(\beta_j(x,y)) \subset V(F(x,y,z)), \quad for \quad 1 \leq j \leq m.$$

It is still possible that $V(P_i(x,y,z))$ may consist of some other component, besides the implicit representation and the projection of seam curves. Let that component be the zero set of $\alpha(x,y,z)$. Since $\alpha(x,y,z)$ is distinct from $H(x,y,z)$ and $\beta_j(x,y)$'s, there exist $(x_1,y_1,z_1) \in V(\alpha(x,y,z))$ such that $H(x_1,y_1,z_1) \neq 0$ and $\beta_j(x_1,y_1) \neq 0$.

Let us consider the point $\mathbf{p} = (s_0, t_0, z_0, x_1, y_1, z_1, 0)$. Since $\mathbf{p} \in Q'$, we can choose a sequence $\mathbf{p}_j = (s_j, t_j, z_j, x_j, y_j, z_j, \lambda_j)$ in the neighborhood of $\mathbf{p}$ such that $\mathbf{p}_j \in Q'$. We can similarly choose a sequence $\mathbf{p}'_j = (s_j, t_j, u_j, x_j, y_j, kz_j, \lambda'_j)$, such that $\mathbf{p}'_j \in Q'$, and from the argument used above it follows that $(x_1, y_1, kz_1) \in V(\alpha(x, y, z))$ for all k. Thus, $\alpha(x, y, z)$ is independent of z and we may represent it as $\alpha(x, y)$. Moreover $\alpha(x_1, y_1) = 0$.

A seam curve corresponding to (s_0, t_0, u_0) is the set of limit points $(\overline{x}, \overline{y}, \overline{z})$ such that $\mathbf{q} = (s_0, t_0, u_0, \overline{x}, \overline{y}, \overline{z}, 0) \in Q'$ and $\mathbf{q}$ has a 3-dimensional neighborhood in Q'. Since $S(x_1, y_1, z, 0) = 0$ and $\mathbf{p} = (s_0, t_0, u_0, x_1, y_1, z, 0) \in Q'$ for all z, there exists a sequence of points $\mathbf{p}_j \in Q'$ in the neighborhood of $\mathbf{p}$. The fact that there exists such a sequence implies that x_1, y_1 must correspond to the x and y coordinates of a point on a seam curve, which is contrary to our assumption.

Thus, $F(x, y, z)$ exactly corresponds to the projections of all the seam curves on the X-Y plane.

Q.E.D.

From now onwards we will represent the lowest degree term of the resultant of the perturbed system as

$$P_i(x, y, z) = H(x, y, z)F(x, y),$$

where $F(x, y)$ is the extraneous factor. Our aim is to extract $F(x, y)$ out of $P_i(x, y, z)$ without resorting to multivariate factorization. Let $H(x, y, z)$ be a polynomial of degree d ($d > 1$) and it can be expressed as

$$H(x, y, z) = h_d(x, y) + h_{d-1}(x, y)z + \ldots, + h_0(x, y)z^d,$$

where $h_i(x, y)$ is a polynomial of degree at most i. Since $H(x, y, z)$ corresponds to some power of an irreducible polynomial

$$GCD(h_d(x, y), h_{d-1}(x, y), \ldots, h_0(x, y)) = 1.$$

Let k be any constant and let z correspond to $d + 1$ distinct powers of k. On substituting those values in $H(x, y, z)$ we obtain

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & k & k^2 & \ldots & k^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & k^d & k^{2d} & \ldots & k^{d^2} \end{pmatrix} \begin{pmatrix} h_d(x, y) \\ h_{d-1}(x, y) \\ \vdots \\ h_0(x, y) \end{pmatrix} = \begin{pmatrix} H(x, y, 1) \\ H(x, y, k) \\ \vdots \\ H(x, y, k^d) \end{pmatrix} \tag{15}$$

The coefficient matrix is non-singular (as it is a Vandermonde matrix). Let us assume that

$$GCD(H(x, y, 1), H(x, y, k), \ldots, H(x, y, k^d)) = P(x, y)$$

is a polynomial of positive degree. Thus, for every $(x_1, y_1) \in V(P(x, y))$, the right hand side vector of the matrix equation becomes a null vector. Since the coefficient matrix is non-singular, this is possible if and only if

$$h_d(x_1, y_1) = 0, \quad h_{d-1}(x_1, y_1) = 0, \quad \ldots \; h_0(x_1, y_1) = 0,$$

which is contrary to the assumption that

$$GCD(h_d(x,y), h_{d-1}(x,y), \ldots, h_0(x,y)) = 1.$$

Thus, $P(x,y)$ is a constant and as a result

$$GCD(P_i(x,y,1), P_i(x,y,k), P_i(x,y,k^2), \ldots, P_i(x,y,k^d)) = F(x,y).$$

Hence, we can extract the extraneous factor by taking the GCD of $d+1$ bivariate polynomials.

In general, for any two distinct values of z, say z_1 and z_2, $GCD(P_i(x,y,z_1), P_i(x,y,z_2))$ would correspond to $F(x,y)$. Thus, the implicit equation can be represented as

$$H(x,y,z) = \frac{P_i(x,y,z)}{GCD(P_i(x,y,z_1), P_i(x,y,z_2))}.$$

Let the parametric equations be polynomials belonging to a ring $\mathcal{F}[s,t,u]$. **Corollary I:** *If $\mathcal{F}$ is an infinite field, there exists an implicit equation belonging to the ring $\mathcal{F}[x,y,z]$.*

Proof: Ff the parametrization has no base points, then the implicit equation corresponds to the resultant expressed as determinant of a matrix. Each entry of the matrix is of the form $ax + by + cz + d$, where $a, b, c, d \in \mathcal{F}$, and therefore the coefficients of the implicit equation belong to the same field.

If the parametrization has base points, we can always choose a perturbing polynomial $\overline{Z}_1(s,t,u) \in \mathcal{F}(s,t,u)$ and let $R(x,y,z,\lambda)$ be the resultant of the perturbed system. Each coefficient of $R(x,y,z,\lambda)$ and therefore, of $P_i(x,y,z,\lambda)$ lies in $\mathcal{F}$. The implicit equation can be expressed as ratio of two polynomials. We can always choose $k \in \mathcal{F}$ such that each coefficient of $GCD(P_i(x,y,1), P_i(x,y,k), \ldots, P_i(x,y,k^d))$ lies in $\mathcal{F}$, too. Thus, the implicit equation has the same coefficient field as the parametric equations.

Q.E.D.

6.1 Rational Parametrization of Seam Curves

In the previous section we presented the technique for computing the implicit representation from the parametrization by making use of the GCD operation. The extraneous factor correspond to the projection of seam curves on the X-Y plane. Given a parametrization, $\overline{\mathbf{F}}$, we can use efficient perturbation and perturb the equations containing the x and y variable so that we are able to compute the projections of seam curves on $Y - Z$ and $X - Z$ planes, respectively. Given these projections, we present an algorithm to compute the rational parametrizations of seam curves.

Perform a transformation on the coordinates of a parametrization and let the projections of the seam curves of the resulting parametrization be $P(x,y)$, $Q(y,z)$ and $R(x,z)$ on the

X-Y, Y-Z and X-Z planes, respectively. For a generic transformation, each of these polynomials would consist of projections of all the seam curves.

Every rational space curve is birationally equivalent to an algebraic plane curve. For a generic choice of coordinates such a birational equivalence can be established between a space curve $\mathbf{B}(t) = (x(t), y(t), z(t), w(t)$ and its projection on X-Y plane, $\mathbf{C}(t) = (x(t), y(t), w(t)$. In our case, $P(x, y)$ is the product of the implicit representations of $\mathbf{C}(t)$ corresponding to each seam curve. Thus, given $P(x, y)$ we use a factorization algorithm to decompose it into irreducible polynomials of the form

$$P(x, y) = P_1(x, y)P_2(x, y)\ldots P_m(x, y),$$

where $P_i(x, y)$ is an irreducible polynomial [Ka83]. The factorization is over the algebraic closure of the base field and in our case it corresponds to factoring over the complexes. However, the polynomials involved are bivariate.

Each plane curve, $P_i(x, y) = 0$, is a curve of *genus 0* and therefore, has a rational parametrization [Wa50]. Given any algebraic plane curve of *genus 0* techniques of computing its rational parametrization have been well known in algebraic geometry [Wa50]. The computational details are worked out in [AB88]. Thus, we are able to compute the rational parametrization, $\mathbf{C}_i(t) = (x(t), y(t), w(t))$ of the projection of each seam curve.

For the choice of coordinates it is assumed that each seam curve $\mathbf{B}_i(t) = (x_i(t), y_i(t), z_i(t), w_i(t))$ is birationally equivalent to $\mathbf{C}_i(t)$. Thus, our problem is reduced to computing the rational function

$$z = \frac{\phi(x, y)}{\psi(x, y)}$$

expressing the relation between the x, y and z coordinates of almost all the points on any seam curve.

6.1.1 Remainder Sequences

Let us treat $Q(y, z)$ and $R(x, z)$ as polynomials in z and its coefficients are in the ring $\mathcal{F}[x, y]$. Without loss of generality we assume that the degree of $R(x, z)$ is less than or equal to that of $Q(y, z)$. Let

$$S_1(z) = Q(y, z),$$

$$S_2(z) = R(x, z),$$

$$\alpha_i S_i(z) = \beta_i S_{i+1}(z) - S_{i+2}(z),$$

where $S_i(z) \in \mathcal{F}[x, y][z]$, $degree(S_{i+2}(z)) < degree(S_{i+1}(z))$ for $1 \leq i \leq d$ and α_i, $\beta_i \in \mathcal{F}[x, y][z]$ such that

$$GCD(\alpha_i, \beta_i) = 1.$$

The sequence $S_1(z), S_2(z), \ldots, S_k(z)$ is a *remainder sequence* [Lo83]. $S_k(z)$ is independent of z and corresponds to the resultant of $Q(y, z)$ and $R(y, z)$ with respect to z. Let (x_1, y_1, z_1)

be any point lying on any seam curve. Thus,

$$P(x_1, y_1) = 0; \quad Q(y_1, z_1) = 0; \quad R(x_1, z_1) = 0.$$

The fact $S_i(z_1)_{x=x_1,y=y_1} = 0$ and $S_{i+1}(z_1)_{x=x_1,y=y_1} = 0$ implies that $S_{i+2}(z_1)_{x=x_1,y=y_1} = 0$. As a result all the polynomials in the remainder sequence vanish when (x, y, z) corresponds to any point on any seam curve. Lets consider the polynomial $S_{k-1}(z)$, which is a linear function in z and can be expressed as

$$S_{k-1}(z) = \psi(x, y)z - \phi(x, y),$$

where $\phi(x, y)$ and $\psi(x, y)$ are polynomials in x and y. Since this polynomial vanishes for all points on any seam curve, the points on a seam satisfy the equation

$$z = \frac{\phi(x, y)}{\psi(x, y)}. \tag{16}$$

Thus, the rational parametrizations of the seam curves are

$$\mathbf{B}_i(t) = \left(\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)}, \frac{\phi(\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)})}{\psi(\frac{x_i(t)}{w_i(t)}, \frac{y_i(t)}{w_i(t)})}\right)$$

corresponding to each $\mathbf{C}_i(t)$.

Example V: Lets again consider the parametrization of a sphere (as used in Example III)

$$\mathbf{F}(s, t) = (x, y, z) = \left(\frac{s^2 - 1 - t^2}{s^2 + 1 + t^2}, \frac{2s}{s^2 + 1 + t^2}, \frac{2st}{s^2 + 1 + t^2}\right).$$

Since the parametrization has base points, lets perturb the given system and the corresponding parametric equations are

$$\begin{aligned}
\overline{G}_1(s, t, u) &= x(s^2 + t^2 + 1) - (s^2 - 1 - t^2) = 0, \\
\overline{G}_2(s, t, u) &= y(s^2 + t^2 + 1) - 2s = 0, \\
\overline{G}_3(s, t, u) &= y(s^2 + t^2 + 1) - 2st + \lambda(2s^2 + 3t^2 + 4) = 0,
\end{aligned}$$

The resultant, $R(x, y, z, \lambda)$ is a polynomial in the 4 variables and the lowest degree of λ is 2 (equal to the number of base points in $\overline{\mathbf{F}}$). The coefficient of λ^2 is

$$\begin{aligned}
P_2(x, y, z) &= -64 - 128x + 128x^3 + 64x^4 + 64y^2 + 128xy^2 \\
&\quad +64x^2y^2 + 64z^2 + 128xz^2 + 64x^2z^2.
\end{aligned}$$

Choose 2 generic values of z, say $z = 1$ and $z = 2$ and the extraneous factor is

$$F(x, y) = GCD(P_i(x, y, 1), P_i(x, y, 2)) = 64 + 128x + 64x^2.$$

Thus, the implicit equation is

$$H(x,y,z) = \frac{P_i(x,y,z)}{F(x,y)} = x^2 + y^2 + z^2 - 1.$$

Apply a linear transformation on the coordinates and obtain

$$\begin{aligned} x &= \overline{x} - 2\overline{y} - \overline{z} \\ y &= \overline{x} - \overline{y} - \overline{z} \\ z &= -\overline{y} - \overline{z} \end{aligned}$$

and the inverse transformation is

$$\begin{aligned} \overline{x} &= y - z \\ \overline{y} &= y - x \\ \overline{z} &= x - y - z \end{aligned} \tag{17}$$

The resulting parametrization is

$$\mathbf{F}' = (2s - 2st, 2s - s^2 + 1 + t^2, s^2 - 1 - t^2 - 2s - 2st, s^2 + t^2 + 1).$$

This parametrization has the same base points as $\overline{\mathbf{F}}$ and we perturb each of the parametric equations to obtain the following extraneous factors, which correspond to the projections of seam curves on $X' - Y'$, $Y' - Z'$ and $X' - Z'$ planes.

$$\begin{aligned} P(\overline{x},\overline{y}) &= 2 + 2\overline{x} + \overline{x}^2 - 4\overline{y} - 2\overline{x}\overline{y} + 2\overline{y}^2 \\ Q(\overline{y},\overline{z}) &= 1 - 2\overline{y} + 2\overline{y}^2 + 2\overline{y}\overline{z} + \overline{z}^2 \\ R(\overline{x},\overline{z}) &= 1 + \overline{x}^2 + 2\overline{z} + \overline{z}^2 \end{aligned}$$

$P(\overline{x},\overline{y})$ can be factorized as

$$P(\overline{x},\overline{y}) = ((y-1)(i-1) - ix)((y-1)(i+1) - ix),$$

where $i = \sqrt{-1}$. The resulting parametrizations are

$$\mathbf{C}_1(t') = (\overline{x},\overline{y},\overline{w}) = (it' - t', it' + 1, 1)$$

and

$$\mathbf{C}_2(t') = (\overline{x},\overline{y},\overline{w}) = (-it' - t', -it' + 1, 1).$$

Lets consider the polynomial remainder sequence defined as

$$S_1(\overline{z}) = Q(\overline{y},\overline{z}),$$

$$S_2(\overline{z}) = R(\overline{x},\overline{z}).$$

As a result

$$S_3(\overline{z}) = S_1(\overline{z}) - S_2(\overline{z}) = -2\overline{y} + 2\overline{y}^2 + 2\overline{yz} - \overline{x}^2 - 2\overline{z}.$$

Since $S_3(\overline{z})$ is a linear polynomial in $\overline{z}$, we are able to express the rational function from the plane curves to the space curves as

$$z = \frac{\phi(x,y)}{\psi(x,y)} = \frac{\overline{x}^2 + 2\overline{y} - 2\overline{y}^2}{2\overline{y} - 2}.$$

Thus,

$$\mathbf{B}_1(t') = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = (it' - t', it' + 1, -1 - t' - it', 1)$$

and

$$\mathbf{B}_2(t') = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = (-it' - t', -it' + 1, -1 - t' + it', 1)$$

Now we can apply the inverse transform according to (17) and obtain the parametrization of the original seam curves as

$$\mathbf{B}_1(t) = (x, y, z, w) = (-1, it, t, 1)$$

and

$$\mathbf{B}_2(t) = (x, y, z, w) = (-1, -it, t, 1)$$

These seam curves lie on the surface and we can verify that by substituting their parametrizations into the surface equation, $H(x, y, z) = 0$.

7 Algorithm

In the previous sections we have highlighted the technique used for implicitizing rational parametric surfaces. These techniques consist of computing the resultants or Gröbner bases of the parametric equations. If the parametrization has base points, the parametric equations are perturbed and we compute the resultant or Gröbner bases of the perturbed system. We showed that the lowest degree term of the resultant or first polynomial of the Gröbner bases (expressed as polynomials in the perturbing variable) contains the implicit equation. An efficient technique for recovering the implicit equation has been presented, too. Although the Gröbner bases provide us the additional flexibility of working in the affine space, we prefer resultants for their computational efficiency. In general the running time complexity of the Gröbner bases algorithm can be doubly exponential in the number of variables as compared to the singly exponential complexity of the resultants [Ca88]. As a matter of fact, multipolynomial resultant algorithms provide the most efficient methods (as far as asymptotic complexity is concerned) for solving system of polynomial equations or eliminating the variables [BGW88]. In our applications the number of variables is fixed and any argument based on the asymptotic complexity may not have much significance.

However we justify our choice with an implementation using resultants and compare its performance with Gröbner bases implementations. Our motivation for choosing resultants stemmed from the fact that most implicit representations are dense polynomials in x, y and z. Gröbner bases seem to perform well on sparse systems and for dense polynomials resultants are considered faster in practice.

Any polynomial parametrization has no base points in the affine domain. Consider the case when such a parametrization has base points at infinity. To implicitize using resultants we need to perturb the system whereas such a problem does not arise while using Gröbner bases. Although perturbation turns out to be relatively expensive in practice, it may be faster to implicitize using resultants of the perturbed system as compared to using the Gröbner bases approach.

If the given parametrization has no base points and is faithful, the resultant and therefore, the implicit representation corresponds to the determinant of a matrix. We may choose not to compute the determinant and represent the implicit equation as the determinant of a matrix. Not only is the resulting implicitization algorithm fast in practice, it also provides a compact representation of the dense polynomial corresponding to the implicit equation. Details of such a representation and its usage are given in the next section.

7.1 Checking for Base Points

Given a parametrization of degree n,

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

the base points correspond to common roots of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$. In general 4 such polynomials have no roots in common. Consider the intersection of this surface with a line represented as the intersection of the following 2 planes

$$a_1x + a_2y + a_3z + a_4w = 0$$
$$b_1x + b_2y + b_3z + b_4w = 0,$$

where a_i's and b_i's can be considered as symbolic variables. To determine the number of intersections of the line with the given surface, we substitute for x, y, z and w and obtain

$$\overline{F}_1(s,t,u) = a_1\overline{X}(s,t,u) + a_2\overline{Y}(s,t,u) + a_3\overline{Z}(s,t,u) + a_4\overline{W}(s,t,u) = 0$$
$$\overline{F}_2(s,t,u) = b_1\overline{X}(s,t,u) + b_2\overline{Y}(s,t,u) + b_3\overline{Z}(s,t,u) + b_4\overline{W}(s,t,u) = 0.$$

The given parametrization has base points, if and only if, $\overline{F}_1(s,t,u)$ and $\overline{F}_2(s,t,u)$ have common roots for all values of a_i's and b_i's. The common roots correspond to the base points. They can be determined by computing the u-resultant of $\overline{F}_1$ and $\overline{F}_2$ [Wd50]. The u-resultant is the resultant of

$$\begin{aligned} \overline{F}_1(s,t,u) &= 0 \\ \overline{F}_2(s,t,u) &= 0 \\ \alpha s + \beta t + \gamma u &= 0. \end{aligned} \tag{18}$$

The u-resultant is a homogeneous polynomial of degree n^2 in α, β and γ and it decomposes into linear factors of the form $(s_1\alpha + t_1\beta + u_1\gamma)$, where s_1, t_1 and u_1 are functions of a_i's and b_i's. The given parametrization has a base point if and only if there is a factor the form $(s'\alpha + t'\beta + u'\gamma)$, where s', t' and u' are scalar constants and independent of a_i's and b_i's. In this case (s', t', u') is a base point of the given parametrization. The number of base points (counted properly with respect to multiplicity) is given by the number of factors of the form $(s'\alpha + t'\beta + u'\gamma)$. If there are n^2 base points, the given parametrization is a degenerate parametrization and its image is not a 2-dimensional surface.

The 3 equations (18) are of degree n, n and 1. Their resultant can be expressed as the determinant of a matrix [MC27]. We prefer this formulation of resultant over Macaulay's formulation [Ma02], since it involves computing a single determinant and we do not have to perturb the given equations.

The u-resultant of $\overline{F}_1$ and $\overline{F}_2$ is a homogeneous polynomial of degree $n^2 + 2n$ in 11 variables (a_i's, b_i's, α, β and γ). In practice, this computation can be very time consuming even for low degree parametrizations. Furthermore, we need to factorize the resultant into linear factors and thereby adding to the complexity of the computation. For practical applications we present a probabilistic algorithm.

Consider 4 generic combinations of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ of the form

$$\overline{G}_i(s,t,u) = x_i\overline{X}(s,t,u) + y_i\overline{Y}(s,t,u) + z_i\overline{Z}(s,t,u) + w_i\overline{W}(s,t,u), \quad 1 \leq i \leq 4,$$

where x_i's, y_i's, z_i's and w_i's are random numbers. We use these combinations for computing the following u-resultants

$$\overline{R}_1(\alpha,\beta,\gamma) = Resultant(\overline{G}_1(s,t,u), \overline{G}_2(s,t,u), \alpha s + \beta t + \gamma u),$$

$$\overline{R}_2(\alpha,\beta,\gamma) = Resultant(\overline{G}_3(s,t,u), \overline{G}_4(s,t,u), \alpha s + \beta t + \gamma u).$$

$\overline{R}_1(\alpha,\beta,\gamma)$ and $\overline{R}_2(\alpha,\beta,\gamma)$ are homogeneous polynomials of degree n^2 in α, β and γ. Let

$$\overline{P}(\alpha,\beta,\gamma) = GCD(\overline{R}_1(\alpha,\beta,\gamma), \overline{R}_2(\alpha,\beta,\gamma)). \tag{19}$$

Let d be the degree of $\overline{P}(\alpha,\beta,\gamma)$. If $d = 0$, the given parametrization has no base points else for each base point (s', t', u'), $(s'\alpha + t'\beta + u'\gamma) \mid \overline{P}(\alpha,\beta,\gamma)$. For almost all combinations, $\overline{G}_1, \ldots, \overline{G}_4$, each linear factor of $\overline{P}(\alpha,\beta,\gamma)$ corresponds to a base point. Therefore d corresponds to the number of base points in the given parametrization and the degree of the implicit representation is $n^2 - d$. To reduce the symbolic complexity of the computation, we may specialize α, β or γ (one or two at a time.)

Our implicitization algorithm only needs to know whether the given parametrization has any base points (and not the actual number of base points) and in such cases it perturbs the parametric equations. A simple algorithm to check for the existence of base points is obtained in the following manner. Consider any three generic combinations, $\overline{G}_1(s,t,u)$,

$\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$, and let R be their resultant. For almost all such combinations, R is zero if and only if the given parametrization has base points. The resultant can be expressed as the determinant of a matrix and all entries of the matrix are numeric. The computation in this case is purely numeric and very fast in practice.

7.1.1 Tensor Product Surfaces

A tensor product surface can either have base points in the affine domain or excess base points at infinity. Given a tensor product surface

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)),$$

where any component, say $X(s,t)$, is of the form

$$X(s,t) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j, \quad X_{mn} \neq 0.$$

According to Lemma IV, it has a base point of multiplicity n at $(s,t,u) = (1,0,0)$ and of multiplicity m at $(s,t,u) = (0,1,0)$.

Let us homogenize the given parametrization and take its 4 generic combinations to compute $\overline{P}(\alpha,\beta,\gamma)$, (19). Since the parametrization has base points at infinity, $\overline{P}(\alpha,\beta,\gamma)$ can be expressed as

$$\overline{P}(\alpha,\beta,\gamma) = \alpha^{n^2}\ \beta^{m^2}\ \overline{Q}(\alpha,\beta,\gamma),$$

where $\overline{Q}(\alpha,\beta,\gamma)$ is a homogeneous polynomial. Let d be the degree of $\overline{Q}(\alpha,\beta,\gamma)$. The given parametrization has a base point in the affine domain or an excess base point at infinity, if and only if $d > 0$. Only in such cases the Dixon eliminant fails to compute the implicit representation and we need to perturb the parametric equations to compute the implicit representation. The degree of the implicit representation is $2mn - d$.

The technique used to check for the existence of base points is obtained by considering three generic combinations of X, Y, Z and W, say $G_1(s,t)$, $G_2(s,t)$ and $G_3(s,t)$. Let R be the dixon eliminant of G_1, G_2 and G_3. For almost all combinations, R is zero if and only if the given parametrization has base points in the affine domain or excess base points at infinity.

7.2 Form of Implicit Representation

In this section, we present a simple algorithm to verify whether the implicit representation, $H(x,y,z)$ is independent of z. This algorithm is required before considering the efficient perturbation and thereby perturbing the parametric equation containing the z variable (as shown in (13)).

The implicit representation is independent of z, if any line represented as the intersection of the following two planes

$$w_1 x = x_1 w$$

$$w_1 y = y_1 w,$$

where $(x_1, y_1, w_1) = (\overline{X}(s_1, t_1, u_1), \overline{Y}(s_1, t_1, u_1), \overline{W}(s_1, t_1, u_1))$ lies on the surface. In other words the given line intersects the surface at an infinite number of points. This should hold for all (s_1, t_1, u_1), where (s_1, t_1, u_1) correspond to a point in the domain. A simple probabilistic algorithm to check for this property is given below.

Let (s_1, t_1, u_1) correspond to a random point in the domain and consider the equations

$$\overline{F}_1(s,t,u) = \overline{X}(s_1,t_1,u_1)\overline{W}(s,t,u) - \overline{W}(s_1,t_1,u_1)\overline{X}(s,t,u) = 0$$

$$\overline{F}_2(s,t,u) = \overline{Y}(s_1,t_1,u_1)\overline{W}(s,t,u) - \overline{W}(s_1,t_1,u_1)\overline{Y}(s,t,u) = 0$$

and

$$\overline{G}(s,t,u) = GCD(\overline{F}_1(s,t,u), \overline{F}_2(s,t,u)).$$

If $\overline{G}(s,t,u)$ is a constant, then the implicit representation of $\overline{\mathbf{F}}$ is not independent of z. For almost all choices of (s_1, t_1, u_1), the fact that $\overline{G}(s,t,u)$ is a polynomial of positive degree implies that the implicit equation is independent of z.

7.3 Choice of Perturbing Polynomial

If a parametrization has base points, we perturb the polynomials (as shown in (13)) to compute the implicit representation. The only constraint on the perturbing polynomial $Z_1(s,t,u)$ is imposed by theorem IV, i.e.

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

To simplify the symbolic complexity of the resulting computation we choose a perturbing polynomial of the form

$$\overline{Z}_1(s,t,u) = a_1 s^n + a_2 t^n + a_3 u^n,$$

where a_1, a_2 and a_3 are random numbers. For almost all choices of a_1, a_2 and a_3 this polynomial will satisfy the constraint of the theorem IV. Else the resultant of the perturbed system is identically zero.

For tensor product surfaces, $\mathbf{F}(s,t)$, where the highest degree of s in any monomial is m and the highest degree of t in any monomial is n, we choose a perturbation polynomial of the form

$$Z_1(s,t) = a_1 s^m t^n + a_2 s^m + a_3 t^n + a_4,$$

where a_1, a_2, a_3 and a_4 are random numbers. In this case we compute the Dixon eliminant of the perturbed system and extract the implicit equation from its lowest degree term after expressing it as a polynomial in the perturbing variable.

7.4 Algorithm

Given a parametrization

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

an algorithm for implicitization is

1. Let
$$\overline{P}(s,t,u) = GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)).$$
If $\overline{P}(s,t,u)$, the common factor, is a polynomial of positive degree, cancel out the common factor from each component of the parametrization.

2. Check whether the given parametrization has base points.

3. Check whether the implicit representation, $H(x,y,z)$, is independent of z. If it is independent of z, interchange $\overline{Y}(s,t,u)$ and $\overline{Z}(s,t,u)$, such that the new parametrization is equivalent to
$$(x,y,z,w) = (\overline{X}(s,t,u), \overline{Z}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u)),$$
and its implicit equation will be a function of x and z only. Substitute y for z and the resulting representation would correspond to the implicit representation of $\overline{\mathbf{F}}$.

4. If the parametrization has no base points, compute the resultant of the parametric equations. Let the resultant be $H(x,y,z)$ and the implicit representation, $G(x,y,z)$, can be computed as
$$G(x,y,z) = \frac{H(x,y,z)}{GCD(H(X,Y,Z), H_z(x,y,z))}.$$

5. Let $k = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}$. If k is a constant, the implicit representation is given by $z = k$.

6. Let
$$\overline{P}(s,t,u) = GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u)).$$
If $\overline{P}(s,t,u)$ is a polynomial of positive degree perform a coordinate transformation (follows from Theorem IV) and let the new parametrization be
$$\overline{\mathbf{F}}'(s,t,u) = (x',y',z',w') = (x, y+kz, z, w) =$$
$$(\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$
where k is a random number. Rest of the algorithm is used to implicitize $\overline{\mathbf{F}}'$. Given $G(x',y',z')$, the implicit representation of $\overline{\mathbf{F}}'$, the implicit representation of $\overline{\mathbf{F}}$ corresponds to $G(x, y-kz, z)$.

7. Compute the resultant of the following system of equations

$$\begin{aligned}\overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0,\\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0,\\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0,\end{aligned}$$

where $\overline{Z}_1(s,t,u)$ is a perturbing polynomial. Express the resultant as a polynomial in λ and let $P_i(x,y,z)$ correspond to its lowest degree term.

8. Choose 2 generic values of z, $z = z_1$ and $z = z_2$, and let

$$H(x,y,z) = \frac{P_i(x,y,z)}{GCD(P_i(x,y,z_1), P_i(x,y,z_2))}.$$

The implicit representation is computed as

$$G(x,y,z) = \frac{H(x,y,z)}{GCD(H(X,Y,Z), H_z(x,y,z))}.$$

8 Implementation

The algorithm in the previous section can be easily implemented on top of any computer algebra system. The main operations are computing the matrix entries for resultant formulation, expanding symbolic and numeric determinants, GCD of multivariate polynomials. Most computer algebra systems support these operations. However when it comes to practice, expanding a symbolic determinant becomes a time consuming task. Consider the problem of implicitizing bicubic tensor product Bézier surfaces (with no base points in the affine domain or excess base points at infinity). The implicit representation corresponds to the determinant of a matrix of order 18, where each matrix entry is a linear polynomial in x, y and z. However the computer algebra systems on commonly available workstations (Sun-3's, Sun-4's) are not able to compute such determinants in a reasonable amount of time and space. Some experiments with the implementations of Gröbner bases and resultants in Macsyma 414.62 on a Symbolics lisp machine (with 16MB main memory and 120MB virtual memory) are described in [Ho90]. For many cases of bicubic surfaces, these systems are unable to implicitize such surfaces and fail due to insufficient virtual memory. Only a new algorithm for basis conversion is able to implicitize such surfaces, however it takes about 10^5 seconds, which would be considered impractical for most applications [Ho90].

Let us analyze some properties of the implicit representation. A polynomial equation of degree d in 3 variables can have up to M monomials, where

$$M = \binom{d+3}{d}.$$

For a polynomial of degree 18 that comes to 1330 terms. In practice the implicit representations are dense polynomials. This is not difficult to see, since a coordinate transformation of the form

$$x = \alpha_1\overline{x} + \alpha_2\overline{y} + \alpha_3\overline{z}$$
$$y = \beta_1\overline{x} + \beta_2\overline{y} + \beta_3\overline{z}$$
$$z = \gamma_1\overline{x} + \gamma_2\overline{y} + \gamma_3\overline{z}$$

would result in a very dense polynomial in $\overline{x}, \overline{y}$ and $\overline{z}$ for almost all choices of α_i, β_i and γ_i. Furthermore the coefficients of the implicit representation are much larger than those of the parametrization. If the absolute magnitude of the coefficients of parametric equations is $|N|$, the coefficients of the implicit representation can have magnitude of the order of $|N|^d$. This follows from the algorithms used for implicitization. Each entry in the matrix has coefficient size equivalent to that of the parametrization and the order of the matrix is equal to the degree of the implicit representation (for tensor product surfaces). For general parametrizations, the order of the matrix is $2n^2 - n$, whereas the implicit equation has degree atmost n^2. This property can have significant impact on the numerical stability of the algorithms, when the coefficients of the parametrization and implicit representation are floating point numbers. In particular, any approach based on implicitization can reduce a well-conditioned problem to an ill-conditioned problem [Ho89; Ho90]. As a result we restrict ourselves to exact arithmetic. In any case, algorithms based on Gröbner bases use rational arithmetic [Ho89].

There are many reasons for the failure and bad performance of implicitization algorithms implemented within the framework of computer algebra systems. Most computer algebra systems use sparse representation for multivariate polynomials and the computations become slow whenever the polynomials generated are dense. Moreover, the algorithms are symbolic in nature and large intermediate expressions may be generated. Their implementations in lisp-like environments may require a large amount of virtual memory and thereby slowing down the computations. Furthermore, these systems use exact arithmetic and represent the coefficients of intermediate expressions as *bignums*. As a result the cost of arithmetic operations is quadratic in the coefficient size. The coefficient size is proportional to the degree of the polynomial expressions being generated and tends to grow exponentially with the degree.

The bottleneck in our algorithm is the symbolic expansion of determinants. The eest of the computations are fast enough in the computer algebra systems. We therefore implemented our algorithm within the framework of a computer algebra system and used a separate implementation for determinant expansion. The main idea is to guess the form of the determinant, say a polynomial of degree d in 3 or 4 variables, corresponding to unperturbed and perturbed parametric equations, respectively, and use Vandermonde interpolation for computing the coefficients. The resulting problem is equivalent to that of interpolating a univariate polynomial which has the same number of coefficients as the multivariate polynomial. As a result, the algorithm only involves numeric computations

and no intermediate symbolic expressions are generated. Since the implicit representations are dense polynomials, we use a dense representation for the resultant. To circumvent the problem of coefficient growth and its impact on arithmetic computation, we chose to work in finite fields. The order of the finite fields is about 2^{31} and thereby making the best use of hardware implementations of 32 bit integer computations (used on most of the work-stations). The main problem is in getting a tight bound on the coefficients of the implicit equations. Since the resultant of the parametric equations (perturbed or unperturbed) corresponds to a determinant, we can use Hadamard's bound for computing a bound on the coefficients of the resultant. However, the bound so obtained is rather loose and we use a probabilistic algorithm for computing the coefficients of the implicit equation. The algorithm involves computing the coefficients modulo various primes and use chinese remainder theorem to compute the corresponding bignums. If for two successive prime sequences (the second one contains one more prime than the first sequence) same bignums are obtained, than those bignums correspond to the coefficients of the implicit equation. More details of interpolation based algorithms to compute resultants and the probabilistic algorithm for computing the coefficients of the resultant are given in [MC90]. The complexity of the algorithm is output sensitive and is given by $O(|C|M(logM)^2)$, where $|C|$ corresponds to the size of resultant coefficients, and M is the number of terms that can be present in the resultant [MC90]. If M is small, a simpler algorithm of complexity $O(|C|M^2)$ may be used.

8.1 Interpolation

Lets consider the case when we do not perturb the given equations and each entry of matrix is a linear polynomial of the form $a_1x + a_2y + a_3z + a_4$. Let $D(x_1, y_1, z_1)$ be the determinant of the matrix, when x, y and z are specialized to x_1, y_1 and z_1, respectively. We choose 3 distinct primes, say p_1, p_2 and p_3 and compute M determinants of the form $D(p_1, p_2, p_3), D({p_1}^2, {p_2}^2, {p_3}^2), \ldots, D({p_1}^M, {p_2}^M, {p_3}^M)$, where M corresponds to the number of monomials in the implicit representation. Given M, we use algorithms for Vandermonde interpolation to compute the coefficients of the polynomial $H(x, y, z)$ and thereby reducing the problem to linear interpolation. If we are computing the resultant of perturbed parametric equations, then the resultant is a polynomial of the form $R(x, y, z, \lambda)$. We therefore, use 4 distinct primes and their powers for interpolation. In this case each entry of the matrix is of the form

$$a_1x + a_2y + a_3z + \lambda(a_4 + a_5x + a_6y) + a_7.$$

All the computations are performed in a finite field.

Our algorithm requires the value of M. Since it is difficult to compute the actual value, we use an upper bound corresponding to the number of monomials that the polynomials of degree d in 3 or 4 variables can have. Though there are many algorithms available for sparse interpolation, in practice they either require a tight bound on the number of monomials that the polynomials may have or actually figure out the actual monomials first and than

compute their coefficients [BT88; KL88]. In the former case, the problem is similar to that of ours and in the latter case, these algorithms seem to take more time in figuring out the actual monomials present in the polynomial and as a result may be slower as compared to the dense interpolation algorithms. Furthermore, the implicit equations and resultants of perturbed systems are generally dense polynomials.

If the resultant or Dixon eliminant of the unperturbed parametric equations does not vanish,

$$M = \begin{pmatrix} d+3 \\ d \end{pmatrix},$$

where d is the degree of the implicit representation ($d = 2mn$ for tensor product surfaces of the form $s^m t^n$ and $d = n^2$ for triangular patches of degree n). If the parametrization has base points then the determinant of the perturbed system is a polynomial of degree $2d$ (where $d = 2mn$ or $d = n^2$ depending on the case) in x, y, z and λ. In general such a polynomial can have up to

$$\begin{pmatrix} 2d+4 \\ 4 \end{pmatrix}$$

monomials. However, we can use some properties of the resultant of the perturbed parametric equations to improve this bound.

The resultant is a polynomial of degree d in the coefficients of each equation. As a result the sum of the degrees of z and λ in any monomial does not exceed d. According to Lemma IV, the resultant can be expressed as

$$R(x, y, z, \lambda) = \lambda^i S(x, y, z, \lambda),$$

where $S(x, y, z, 0) \neq 0$ and i corresponds to the total number of base points in the given parametrization. As a result the resultant can be expressed as

$$R(x, y, z, \lambda) = \lambda^i P_i(x, y, z) + \lambda^{i+1} P_{i+1}(x, y, z) + \ldots + \lambda^d P_d(x, y, z),$$

where $P_j(x, y, z)$ is a polynomial of the form

$$P_j(x, y, z) = Q_d(x, y) + zQ_{d-1}(x, y) + \ldots + z^{d-j} Q_j(x, y),$$

and $Q_k(x, y)$ is a polynomial of degree k in x and y. Thus,

$$\begin{aligned} M &= \sum_{j=i}^{d} \left(\sum_{k=j}^{d} \begin{pmatrix} k+2 \\ 2 \end{pmatrix} \right) \\ &= \sum_{j=i}^{d} \left(\begin{pmatrix} d+3 \\ 3 \end{pmatrix} - \begin{pmatrix} j+2 \\ 3 \end{pmatrix} \right). \end{aligned}$$

Some deterministic and probabilistic algorithms to compute i are presented in Section 7.1. We make use of Lemma IV to present a simple and probabilistic algorithm.

Let $(x, y, z) = (x_1, y_1, z_1)$ correspond to a random point in space and compute the determinant $R(x_1, y_1, z_1, \lambda)$, which is a polynomial of degree d in λ. We can use Vandermonde interpolation to compute it. For almost all choices of (x_1, y_1, z_1) the lowest degree of λ in $R(x_1, y_1, z_1, \lambda)$ corresponds to i.

8.2 Performance

The symbolic determinant computation has been implemented in C++ on a Sun-4 (a 10 MIPS machine) and IBM RS/6000 (a 34 MIPS machine). The rest of the algorithm has been implemented on top of Mathematica. The bottleneck of the computation is the determinant computation. In Fig. III and Fig. IV its performance for different parametrizations is given. The timings correspond to a single iteration of the finite field computation. The total number of iterations is a function of the coefficient size of the output. Since the coefficient size is proportional to the degree of the implicit representation, more iterations are needed for higher degree implicit equations. We use finite fields of order 2^{31} and therefore, the number of iterations is bounded by $k + 1$, where k is the minimum integer satisfying the relation

$$|N| < 2^{30k}$$

and $|N|$ is the magnitude of the coefficient of maximum size of the resultant.

Parametrization	Implicit Degree	M	Sun-4	IBM RS/6000
$s^2 + t^2$	4	10	1 sec.	1 sec.
$s^3 + t^3$	9	220	6 sec.	3 sec.
s^2t^2	8	165	4 sec.	2 sec.
s^3t^3	18	1330	100 sec.	23 sec.
s^3t^4	24	2925	430 sec.	118 sec.

Fig. III
The performance of implicitization algorithm for parametrizations without base points (a single iteration over a finite field).

Parametrization	Base Points	Implicit Degree	M	Sun-4	IBM RS/6000
$s^2 + t^2$	2	2	74	2 sec.	1 sec.
$s^3 + t^3$	3	6	1064	52 sec.	16 sec.
st^3	2	4	295	4 sec.	2 sec.
s^2t^2	4	4	510	13 sec.	4 sec.
s^3t^3	3	15	15300	4700 sec.	1180 sec.

Fig. IV
The performance of implicitization algorithm for parametrizations with base points (a single iteration over a finite field).

Thus, we see that the algorithm does not perform well for bicubic parametrizations with base points. In general $3-4$ iterations are needed and even on a fast machine like IBM RS/6000 that amounts to 4500 seconds. The main problem is in the perturbation technique, which increases the symbolic complexity of the resultant.

9 A Proposition

For problems like surface intersection, the implicitization approach becomes unattractive due to the algebraic degree of the intersection curve. In particular, plane curves of order 108 are obtained in the case of two bicubic Bézier surfaces. Not only is the computation of such curves a time consuming task, algorithms to trace them are numerically unstable (in general).

Lets assume that the given parametrizations has no base points. In this case, the implicit representation corresponds to the determinant of a matrix. We propose not to expand the determinant and use the matrix as the implicit representation. Let us see how can such a representation be incorporated in a geometric modeling system. Let the matrix be $\mathcal{M}(x,y,z)$. Each of its entry is a linear polynomial in x, y and z.

- Given a point (x_1,y_1,z_1), we would like to know whether the points is inside, outside or on the surface. Substitute for x, y and z and compute the determinant of $\mathcal{M}(x_1,y_1,z_1)$. From the value and sign of the numeric determinant we can infer about the point.

- Consider the problem of intersection of two Bézier surfaces. Implicitize one of the surfaces and let $\mathcal{M}(x,y,z)$ be the resulting matrix. Substitute the parametric equations of the other surface and we obtain a matrix of the form

$$\mathcal{P}(s,t) = \mathcal{M}(\frac{X(s,t)}{W(s,t)}, \frac{Y(s,t)}{W(s,t)}, \frac{Z(s,t)}{W(s,t)}).$$

  Let $G(s,t)$ correspond to the representation of the intersection curve. This curve has degree 108 in the case of intersection of two bicubic surfaces. Thus,

$$G(s,t) = Det(\mathcal{P}(s,t)).$$

  Each term of the matrix is a rational function in s and t. However their degree is equal to the degree of one of the parametrization. Given (s_1,t_1), substitute them into the matrix and we obtain a numeric matrix of the form $\mathcal{P}(s_1,t_1)$. Furthermore, the algorithms for the evaluation of the entries of $\mathcal{P}(s_1,t_1)$ are numerically stable since they involve evaluating low degree polynomials. Given $\mathcal{P}(s_1,t_1)$ we can use Guass elimination or QR methods to compute the determinants. The process of determinant computation can be made numerically stable by pivoting [Wi63]. As a result we

may be able to accurately compute $G(s_1, t_1)$. Furthermore, we can use properties of straight line programs to compute $G_s(s_1, t_1)$, where $G_s(s, t)$ represents the partial derivative with respect to s. However, it is not clear whether the computation of partial derivatives is accurate and numerically stable.

Although this may look like an interesting representation for geometric modeling systems, much work needs to be done to show its effectiveness. We should be able to accurately compute the functions of the form $G(s_1, t_1)$ and their higher order partials. Furthermore, this analysis is only limited to parametrizations with no base points. We therefore, need algorithms to represent all implicit representations as determinants of a matrix. Finally we need algorithms which can make use of such a representation for tracing plane curves or performing algebraic operations on the implicit representation.

10 Conclusion

In this paper we analyzed the problem of implicitizing rational parametric surfaces. If a parametrization has no base points, the resultant or Dixon eliminant of the parametric equation corresponds exactly to the implicit equation. The base points decrease the degree of the implicit equations and blow up to rational curves on the algebraic surface. We use the technique of efficient perturbation to compute the implicit equations of parametrizations with base points. The strength of our technique lies in the fact that we use GCD of bivariate polynomials to extract the implicit equation from the lowest degree term of the resultant of the perturbed system and show that the extraneous factor in the lowest degree term corresponds exactly to the projection of blow-up curves. As a result we are able to compute the rational parametrization of these curves. Although, similar analysis hold for Gröbner bases, we recommend resultants for efficiency reasons.

The algorithms implemented in the framework of computer algebra systems (available on commonly available workstations) are unable to implicitize parametric surfaces like bicubic patches. We present an algorithm for computing symbolic determinants and as a result achieve a significance performance improvement as compared to the previous implementations of the implicitization algorithms. If a parametrization has no base points (or a tensor product surface has no base points in the affine domain or excess base points at infinity), the algorithm is very fast on machines like IBM RS/6000. Furthermore, the algorithm can be easily parallelized and thereby achieve speedups proportional to the level of parallelism. However, perturbation increases the symbolic complexity of the resultants and the resulting algorithm to implicitize parametric surfaces with base points is slow. We therefore, need efficient algorithms to implicitize parametric surfaces with base points.

11 References

[AB88] Abhyankar, S. and Bajaj, C. (1988) "Computations with Algebraic Curves", *Lecture Notes in Computer Sciecne,* vol. **358**, pp. 279-284, Springer Verlag.

[BGW88] Bajaj, C., Garrity, T. and Warren, J. (November 1988) "On the applications of multi-equational resultants", Tech. report CSD-TR-826, Computer Science Deptt., Purdue University.

[BT88] Ben-Or, M. and Tiwari, P. (1988) "A deterministic algorithm for sparse multivariate polynomial interpolation", *20th Annual ACMSym. Theory Comp..* pp. 301-309.

[Bu85] Buchberger, B. (1987) "Gröbner bases: An algorithmic method in polynomial ideal theory", in *Multidimensional Systems Theory,* edited by N.K. Bose, pp. 184-232, D. Reidel Publishing Co..

[Bu89] Buchberger, B. (1989) "Applications of Gröbner bases in Non-linear Computational Geometry", in *Geometric Reasoning,* eds. D. Kapur and J. Mundy, pp. 415-447, MIT Press.

[Ca88] Canny, J. F. (1988) "Generalized Characteristic Polynomials", *Lecture Notes in Computer Science,* vol. **358**, pp. 293-299, Springer Verlag.

[Cal894] Castelnuovo, G. (1894) " Sulla razionalità delle involuzioni piane", *Mathematische Annalen,* vol. **44**, pp. 125-155, (in Italian).

[CH89] Chuang, J.H. and Hoffmann, C.M. (October 1989) "On Local Implicit Approximations and Its Applications", *ACM Transactions on Graphics,* vol. **8**, no. **4**, pp. 298-324.

[Cl1868] Clebsch, A. (1868) "Ueber Die Abbildung Algebraischer Flächen Insbesondere Der Vierten and Fünften Ordnung", *Mathematische Annalen,* vol. **1**, pp. 253-316, (in German).

[Di08] Dixon, A.L. (1908) "The eliminant of three quantics in two independent variables", *Proceedings of London Mathematical Society,* vol. **6**, pp. 49-69, 473-492.

[Ha77] Hartshorne, R. (1977) *Algebraic Geometry,* Springer-Verlag.

[Ho88] Hoffmann, C. (1988) "A Dimensionality Paradigm for Surface Interrogations", Tech. report CSD-TR-837, Department of Computer Science, Purdue University.

[Ho89] Hoffmann, C. (1989) *Geometric and Solid Modeling: An Introduction,* Morgan Kaufmann Publishers Inc..

[Ho90] Hoffmann, C. (1990) "Algebraic and Numeric Techniques for Offsets and Blends", in *Computation of Curves and Surfaces,* eds. W. Dahmen et. al., pp. 499-529, Kluwer Academic Publishers.

[Ie89] Ierardi D. (1989) "Quantifier Elimination in the Theory of Algebraically closed Field", Proceedings of 21^{st} *Annual ACM Symposium on Theory of Computing,* pp. 138-147 Seattle.

[Ka83] Kaltofen, E. (1983) "Factorization of Polynomials", in *Computer Algebra, Symbolic and Algebraic Computation,* Buchberger et. al. eds., pp. 94-113, second edition, Wien, New York.

[KL88] Kaltofen, E. and Yagiti, L. (1988) "Improved sparse multivariate polynomial in-

terpolation algorithms", *Lecture Notes in Computer Science,* vol. **358**, Springer-Verlag.

[Lo83] Loos, R. (1983) "Generalized Polynomial Remainder Sequences", in *Computer Algebra, Symbolic and Algebraic Computation,* Buchberger et. al. eds., pp. 115-137, second edition, Wien, New York.

[Ma02] Macaulay, F. S. (May 1902) "On some formula in elimination", *Proceedings of London Mathematical Society,* pp. 3-27.

[Ma21] Macaulay, F. S. (June 1921) "Note on the resultant of a number of polynomials of the same degree", *Proceedings of London Mathematical Society,* pp. 14-21.

[MC27] Morley, F. and Coble, A.B. (1927) "New Results in Elimination", *American Journal of Mathematics,* vol. **49**, pp. 463-488.

[MC90] Manocha, D. and Canny, J. (1990) "Multipolynomial Resultant Algorithms", in preparation.

[Mm76] Mumford, D. (1976) *Algebraic Geometry I: Complex Projective Varieties,* Springer-Verlag.

[Mo25] Morley, F. (1925) "The Eliminant of a net of Curves", *American Journal of Mathematics,* vol. **47**, pp. 91-97.

[Mu75] Munkres, J.R. (1975) *Topology A First Course,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

[PP88] Prakash, P.V. and Patrikalakis, N.M. (1988) "Surface-to-Surface Intersections for Geometric Modeling", Tech, report MITSG 88-8, MIT Sea Grant College Program, MIT.

[Pr86] Pratt, M. J. (1986) "Parametric Curves and Surfaces as used in computer aided design", in *The Mathematics of Surfaces,* ed. by J.A. Gregory, Claredon Press, Oxford..

[Sa1885] Salmon, G. (1885) *Lessons Introductory to the Modern Higher Algebra,* G.E. Stechert & Co., New York.

[Sa14] Salmon, G. (1914) *A Treatise on the Analytic Geometry of Three Dimensions,* Longmans, Green, London.

[SR85] Semple, J.G. and Roth, L. (1985) *Introduction to Algebraic Geometry,* Claredon Press, Oxford, Great Britian.

[SAG84] Sederberg, T.W., Anderson, D.C. and Goldman, R.N. (1984) "Implicit representation of parametric curves and surfaces", *Computer Vision, Graphics and Image Processing,* vol. **28**, pp. 72-84.

[SA85] Sederberg, T.W. and Anderson, D.C. (May 1985) "Steiner Surface Patches", *IEEE CG&A,* pp. 23-36.

[Se90] Sederberg, T.W. (July 1990) "Techniques for Cubic Algebraic Surfaces", *IEEE CG&A,* pp. 14-25.

[Sn70] Snyder, Virgil et. al. (1970) *Selected Topics in Algebraic Geometry,* Chelsea Publishing Company, Bronx, New York.

[Wa50] Walker, Robert J. (1950) *Algebraic Curves,* Princeton University Press, New Jersey.

[Wd50] van der Waerden B. L. (1950) *Modern Algebra,* (third edition) F. Ungar Publishing Co., New York.

[Wi63] Wilkinson, J.H. (1963) *Rounding Erros in Algebraic Processes,* Prentice-Hall,

Englewood Cliffs, NJ.