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THE IDENTIFICATION OF GENERAL PSEUDO-RECIPROCAL VECTOR FIELDS. †

Robert Lum AND Leon O. Chua. ††

Abstract

A reciprocal vector field is a vector field for which the linearisation of the vector field is reciprocal. A vector field is called pseudo-reciprocal if it can be written as the composition of a matrix with a reciprocal vector field. The main types of pseudo-reciprocal vector fields studied in this paper are those where the matrix is invertible. Such vector fields are especially important in the field of electrical engineering due to their representation of non-linear circuits.

In this paper, the identification of such vector fields is completed for the cases when the matrix is either invertible, invertible symmetric, symmetric positive definite, diagonal positive definite or diagonal invertible. In the process of such identification, a decomposition of the original vector field as the composition of a matrix and a reciprocal vector field will ensue.

From a nonlinear circuit-theorectical point of view, this paper provides a definitive answer to the following outstanding question:

Given a state equation $\dot{x} = f(x)$, does there exist a nonlinear circuit made up of only 2-terminal, and/or reciprocal n-terminal resistors, capacitors and inductors, which realises this equation?

While pseudo-reciprocal vector fields could exhibit complicated dynamics, including chaos, a pseudo-gradient vector field behaves just like gradient vector fields and is in fact the basic building block of electronic neural networks.

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†† The authors are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA.

§0. Introduction.

Pseudo-reciprocal vector fields are of particular importance in <u>nonlinear circuit theory</u> because the state equation of any nonlinear circuit made up of only 2-terminal, and/or reciprocal n-terminal resistors, inductors, and capacitors always gives rise to a pseudo-reciprocal vector field. In particular, if only 2-terminal capacitors or only 2-terminal inductors are present in the circuit, in addition to reciprocal resistors, then the Jacobian matrix of the associated vector field always assumes the form of a product of two symmetric matrices. This special case corresponds to a pseudo-gradient vector field[1].

The first section of this paper will present the definition of a reciprocal matrix, reciprocal vector field and pseudo-reciprocal vector field. It will be shown that gradient vector fields are a subset of the reciprocal vector fields. From an experimental viewpoint, the more desirable vector fields to implement are those that are already in the form of a reciprocal vector field or have a coordinate change such that under the new coordinate system, the vector field assumes the form of a reciprocal vector field. Section 2 will determine if such a coordinate change exists. If such a coordinate change is not available, then it would be desirable that there exists a matrix $\mathbf{M}(\mathbf{x})$ such that the vector field may be written as $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where the vector field $g(\mathbf{x})$ is a reciprocal vector field. Sections 3 through to 7 discuss algorithms to determine when there exist matrices $\mathbf{M}(\mathbf{x})$ with the matrix either invertible, symmetric invertible, symmetric positive definite, diagonal positive definite or diagonal invertible. Section 8 is concerned with the analogous identification problem when the decomposition desires that $g(\mathbf{x})$ be a gradient vector field. Section 9 is concerned with the identification of pseudo-reciprocal <u>piecewise-linear</u> vector fields. Because of their structure, solutions for constant matrices $\mathbf{M}$ can be more explicitly stated. Similarly, Section 10 is concerned with the identification of pseudo-gradient <u>piecewise-linear</u> vector fields. This introduction will conclude with some examples of the applications of the algorithms. Insight into whether a vector field is pseudo-reciprocal will determine if a circuit realisation of the vector field using only reciprocal circuit elements is possible.

EXAMPLE 0.1. Let the vector field be given by

$$f\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix} = \begin{cases} \begin{bmatrix} -(G+G_b)/C_1 & G/C_1 & 0 \\ G/C_2 & -G/C_2 & 1/C_2 \\ 0 & -1/L & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} (G_a-G_b)/C_1 \\ 0 \\ 0 \end{bmatrix}, & V_1 < -1 \\ \begin{bmatrix} -(G+G_a)/C_1 & G/C_1 & 0 \\ G/C_2 & -G/C_2 & 1/C_2 \\ 0 & -1/L & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix}, & -1 \le V_1 \le 1 \\ \begin{bmatrix} -(G+G_b)/C_1 & G/C_1 & 0 \\ G/C_2 & -G/C_2 & 1/C_2 \\ 0 & -1/L & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} (G_b-G_a)/C_1 \\ 0 \\ 0 \end{bmatrix}, & 1 < V_1. \end{cases}$$

By section 3, a decomposition for constant matrix $\mathbf{M}$ and vector field $g(\mathbf{x})$ is given by

$$\mathbf{M} = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 1/C_2 & 0 \\ 0 & 0 & 1/L \end{bmatrix}$$

and

$$g\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}=\begin{cases}\begin{bmatrix}-(G+G_b) & G & 0\\ G & -G & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}G_a-G_b\\0\\0\end{bmatrix}, & V_1<-1\\ \begin{bmatrix}-(G+G_a) & G & 0\\ G & -G & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}, & -1\le V_1\le 1\\ \begin{bmatrix}-(G+G_b) & G & 0\\ G & -G & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}G_b-G_a\\0\\0\end{bmatrix}, & 1<V_1.\end{cases}$$

An electronic circuit to implement $f(\mathbf{x})$ is given in figure 1. In fact, this is the vector field of the "double scroll" as described in [4].

EXAMPLE 0.2. Let the vector field be given by

$$f\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}=\begin{cases}\begin{bmatrix}-G_b/C_1 & 0 & 1/C_1\\ 0 & -G/C_2 & 1/C_2\\ -1/L & -1/L & -R/L\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}(G_a-G_b)/C_1\\0\\0\end{bmatrix}, & V_1<-1\\ \begin{bmatrix}-G_a/C_1 & 0 & 1/C_1\\ 0 & -G/C_2 & 1/C_2\\ -1/L & -1/L & -R/L\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}, & -1\le V_1\le 1\\ \begin{bmatrix}-G_b/C_1 & 0 & 1/C_1\\ 0 & -G/C_2 & 1/C_2\\ -1/L & -1/L & -R/L\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}(G_b-G_a)/C_1\\0\\0\end{bmatrix}, & 1<V_1.\end{cases}$$

By section 5, a decomposition for constant matrix $\mathbf{M}$ and vector field $g(\mathbf{x})$ is given by

$$\mathbf{M}=\begin{bmatrix}1/C_1 & 0 & 0\\ 0 & 1/C_2 & 0\\ 0 & 0 & 1/L\end{bmatrix}$$

and

$$g\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}=\begin{cases}\begin{bmatrix}-G_b & 0 & 1\\ 0 & -G & 1\\ -1 & -1 & -R\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}G_a-G_b\\0\\0\end{bmatrix}, & V_1<-1\\ \begin{bmatrix}-G_a & 0 & 1\\ 0 & -G & 1\\ -1 & -1 & -R\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}, & -1\le V_1\le 1\\ \begin{bmatrix}-G_b & 0 & 1\\ 0 & -G & 1\\ -1 & -1 & -R\end{bmatrix}\begin{bmatrix}V_1\\V_2\\I_3\end{bmatrix}+\begin{bmatrix}G_b-G_a\\0\\0\end{bmatrix}, & 1<V_1.\end{cases}$$

An electronic circuit to implement $f(\mathbf{x})$ is given in figure 2 using only passive capacitor and inductors. The circuit equations which give rise to $f(\mathbf{x})$ are given in [4].

EXAMPLE 0.3. Let the vector field be given by $(\alpha\neq\beta)$,

$$f\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{cases}\begin{bmatrix}-\frac{1}{R_1C_1}-\frac{1}{R_2C_1} & \frac{1}{R_2C_1} & \alpha\\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2}-\frac{1}{R_3C_2} & \frac{1}{R_3C_2}\\ 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}\gamma(\alpha-\beta)\\0\\0\end{bmatrix}, & x_3<-\gamma\\ \begin{bmatrix}-\frac{1}{R_1C_1}-\frac{1}{R_2C_1} & \frac{1}{R_2C_1} & \beta\\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2}-\frac{1}{R_3C_2} & \frac{1}{R_3C_2}\\ 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, & |x_3|\le\gamma\\ \begin{bmatrix}-\frac{1}{R_1C_1}-\frac{1}{R_2C_1} & \frac{1}{R_2C_1} & \alpha\\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2}-\frac{1}{R_3C_2} & \frac{1}{R_3C_2}\\ 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}\gamma(\beta-\alpha)\\0\\0\end{bmatrix}, & \gamma<x_3.\end{cases}$$

By section 4, a decomposition for a constant symmetric invertible matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} -1-\frac{C_2}{C_1}-\frac{C_3}{C_1}-\frac{R_2C_2}{R_1C_1}-\frac{R_2C_3}{R_1C_1}-\frac{R_3C_3}{R_1C_1}-\frac{2R_3C_2C_3}{C_1^2R_1}-\frac{R_3C_2C_3}{C_1^2R_2}-\frac{R_2R_3C_2C_3}{R_1^2C_1^2} & 1+\frac{R_3C_3}{R_1C_1}+\frac{R_3C_3}{R_2C_1} & 1 \\ 1+\frac{R_3C_3}{R_1C_1}+\frac{R_3C_3}{R_2C_1} & -\frac{R_3C_3}{R_2C_2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$g\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3} \\ -\frac{1}{R_3C_3} & \frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}+\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} \\ \frac{1}{R_3C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} & \alpha-\frac{1}{R_1C_1}+\frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}-\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ \gamma(\alpha-\beta) \end{bmatrix}, & x_3 < -\gamma \\ \begin{bmatrix} 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3} \\ -\frac{1}{R_3C_3} & \frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}+\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} \\ \frac{1}{R_3C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} & \beta-\frac{1}{R_1C_1}+\frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}-\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, & |x_3| \le \gamma \\ \begin{bmatrix} 0 & \frac{1}{R_3C_3} & -\frac{1}{R_3C_3} \\ -\frac{1}{R_3C_3} & \frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}+\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} \\ \frac{1}{R_3C_3} & -\frac{R_2C_2}{R_3^2C_3^2}-\frac{R_2}{R_3^2C_3}-\frac{C_2}{R_3C_1C_3}-\frac{R_2C_2}{R_1R_3C_1C_3} & \alpha-\frac{1}{R_1C_1}+\frac{R_2C_2}{R_3^2C_3^2}+\frac{R_2}{R_3^2C_3}-\frac{1}{R_3C_3}+\frac{C_2}{R_3C_1C_3}+\frac{R_2C_2}{R_1R_3C_1C_3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ \gamma(\beta-\alpha) \end{bmatrix}, & \gamma < x_3. \end{cases}$$

Notice that the matrix $\mathbf{M}$ is not positive definite since the determinant of the minor

$$\begin{bmatrix} -\frac{R_3C_3}{R_2C_2} & 0 \\ 0 & 0 \end{bmatrix}$$

is zero. Applying the algorithm of section 6 for constant matrix $\mathbf{M}$ and vector field $g(\mathbf{x})$ one has that

$$H^1_{12}(\mathbf{X}) = x_{11}/(R_2C_1) + x_{22}/(R_2C_2)$$

$$H^1_{13}(\mathbf{X}) = \begin{cases} x_{11}\alpha, & x_3 < -\gamma \\ x_{11}\beta, & |x_3| \le \gamma \\ x_{11}\alpha, & \gamma < x_3. \end{cases}$$

$$H^1_{23}(\mathbf{X}) = x_{22}/(R_3C_2) - x_{33}/(R_3C_3).$$

If $H^1_{12}(\mathbf{X}) = H^1_{13}(\mathbf{X}) = H^1_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is diagonal and $g(\mathbf{x})$ is reciprocal of order $(1, 2)$ does not exist. Similarly,

$$H^2_{12}(\mathbf{X}) = x_{11}/(R_2C_1) - x_{22}/(R_2C_2)$$

$$H^2_{13}(\mathbf{X}) = \begin{cases} x_{11}\alpha, & x_3 < -\gamma \\ x_{11}\beta, & |x_3| \le \gamma \\ x_{11}\alpha, & \gamma < x_3. \end{cases}$$

$$H^2_{23}(\mathbf{X}) = x_{22}/(R_3C_2) + x_{33}/(R_3C_3).$$

If $H^2_{12}(\mathbf{X}) = H^2_{13}(\mathbf{X}) = H^2_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is diagonal and $g(\mathbf{x})$ is reciprocal of order $(2,1)$ does not exist. Similarly,

$$H^3_{12}(\mathbf{X}) = x_{11}/(R_2C_1) - x_{22}/(R_2C_2)$$

$$H^3_{13}(\mathbf{X}) = \begin{cases} x_{11}\alpha, & x_3 < -\gamma \\ x_{11}\beta, & |x_3| \leq \gamma \\ x_{11}\alpha, & \gamma < x_3. \end{cases}$$

$$H^3_{23}(\mathbf{X}) = x_{22}/(R_3C_2) - x_{33}/(R_3C_3).$$

If $H^3_{12}(\mathbf{X}) = H^3_{13}(\mathbf{X}) = H^3_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is diagonal and $g(\mathbf{x})$ is reciprocal of order $(3,0)$ does not exist. Our algorithm shows that it is impossible to make $\mathbf{M}$ diagonal. Hence, a circuit realisation using only linear 2-terminal capacitors and/or 2-terminal inductors, and <u>reciprocal</u> resistors does not exist. Indeed, the circuit realisation in figure 3 requires a <u>controlled</u> source, which is a <u>non-reciprocal</u> resistor. The paper [2] examines the circuit in greater detail.

EXAMPLE 0.4. Let the vector field be given by

$$f\begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} -m_0/C_1 & m_0/C_1 & -1/C_1 & 0 \\ m_0/C_2 & -m_0/C_2 & 0 & -1/C_2 \\ 1/L_1 & 0 & R/L_1 & 0 \\ 0 & 1/L_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} (m_1 - m_0)/C_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & V_2 - V_1 < -1 \\ \begin{bmatrix} -(2m_0 - m_1)/C_1 & (2m_0 - m_1)/C_1 & -1/C_1 & 0 \\ (2m_0 - m_1)/C_2 & -(2m_0 - m_1)/C_2 & 0 & -1/C_2 \\ 1/L_1 & 0 & R/L_1 & 0 \\ 0 & 1/L_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix}, & |V_2 - V_1| \leq 1 \\ \begin{bmatrix} -m_0/C_1 & m_0/C_1 & -1/C_1 & 0 \\ m_0/C_2 & -m_0/C_2 & 0 & -1/C_2 \\ 1/L_1 & 0 & R/L_1 & 0 \\ 0 & 1/L_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} (m_0 - m_1)/C_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & 1 < V_2 - V_1. \end{cases}$$

By section 6, a decomposition for constant matrix $\mathbf{M}$ and vector field $g(\mathbf{x})$ is given by

$$\mathbf{M} = \begin{bmatrix} 1/C_1 & 0 & 0 & 0 \\ 0 & 1/C_2 & 0 & 0 \\ 0 & 0 & 1/L_1 & 0 \\ 0 & 0 & 0 & 1/L_2 \end{bmatrix}$$

and

$$g\begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} -m_0 & m_0 & -1 & 0 \\ m_0 & -m_0 & 0 & -1 \\ 1 & 0 & R & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} m_1 - m_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & V_2 - V_1 < -1 \\ \begin{bmatrix} -(2m_0 - m_1) & (2m_0 - m_1) & -1 & 0 \\ (2m_0 - m_1) & -(2m_0 - m_1) & 0 & -1 \\ 1 & 0 & R & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix}, & |V_2 - V_1| \leq 1 \\ \begin{bmatrix} -m_0 & m_0 & -1 & 0 \\ m_0 & -m_0 & 0 & -1 \\ 1 & 0 & R & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} m_0 - m_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & 1 < V_2 - V_1. \end{cases}$$

A realisation using only uncoupled passive 2-terminal elements is given in figure 4. The derivation of $f(\mathbf{x})$ is given in [11].

EXAMPLE 0.5. Let the vector field be the Lorenz system,

$$f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ -xz+Rx-y \\ xy-Bz \end{bmatrix}.$$

Observe that

$$\mathrm{D}f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ -z+R & -1 & -x \\ y & x & -B \end{bmatrix}.$$

By the algorithm of theorem 2.5, there does not exist a coordinate change such that the vector field $f(\mathbf{x})$ is a reciprocal vector field. In applying the algorithm of section 6 for a constant matrix M, one has

$$H^1_{12}(\mathbf{X}) = x_{11}\sigma + x_{22}(R-z)$$
$$H^1_{13}(\mathbf{X}) = x_{33}y$$
$$H^1_{23}(\mathbf{X}) = -x_{22}x - x_{33}x.$$

If $H^1_{12}(\mathbf{X}) = H^1_{13}(\mathbf{X}) = H^1_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathrm{M}g(\mathbf{x})$ where M is diagonal and $g(\mathbf{x})$ is reciprocal of order $(1,2)$ does not exist. Similarly,

$$H^2_{12}(\mathbf{X}) = x_{11}\sigma - x_{22}(R-z)$$
$$H^2_{13}(\mathbf{X}) = x_{33}y$$
$$H^2_{23}(\mathbf{X}) = -x_{22}x + x_{33}x.$$

If $H^2_{12}(\mathbf{X}) = H^2_{13}(\mathbf{X}) = H^2_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathrm{M}g(\mathbf{x})$ where M is diagonal and $g(\mathbf{x})$ is reciprocal of order $(2,1)$ does not exist. Similarly,

$$H^3_{12}(\mathbf{X}) = x_{11}\sigma - x_{22}(R-z)$$
$$H^3_{13}(\mathbf{X}) = -x_{33}y$$
$$H^3_{23}(\mathbf{X}) = -x_{22}x - x_{33}x.$$

If $H^3_{12}(\mathbf{X}) = H^3_{13}(\mathbf{X}) = H^3_{23}(\mathbf{X}) = 0$ then $x_{11} = x_{22} = x_{33} = 0$. Thus, $\mathbf{X}$ is the $\mathbf{0}$ matrix and is not invertible, which implies that a decomposition with $f(\mathbf{x}) = \mathrm{M}g(\mathbf{x})$ where M is diagonal and $g(\mathbf{x})$ is reciprocal of order $(3,0)$ does not exist. Thus, the vector field may not be implemented as an electronic circuit composed of passive and uncoupled 2-terminal elements. Further information on the Lorenz system is given in [6].

§1. Definitions.

This section will present the definition of a reciprocal vector field and a pseudo-reciprocal vector field. A theorem will be presented detailing a method by which reciprocal vector fields of order (p,q) may be identified. This theorem forms the crux of the identification algorithms. The definition of a gradient vector field leads to theorem 1.6, which implies that gradient vector fields are a subset of the set of reciprocal vector fields. Finally, the set of pseudo-gradient vector fields is defined.

Definition 1.1. A matrix is reciprocal of order (p,q) if and only if it has the form

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ -\mathbf{C}^t & \mathbf{D} \end{bmatrix}$$

with $\mathbf{A}$ and $\mathbf{D}$ symmetric matrices of dimensions p and q respectively.

Remark: The term reciprocal is motivated by electrical circuit theory. Under proper reordering of the port numbers, a reciprocal n-port resistor has a hybrid matrix which satisfies the properties of Definition 1.1 [1].

Definition 1.2. A vector field $f(\mathbf{x})$ is called a reciprocal vector field of order (p,q) if and only if $\mathbf{D}f(\mathbf{x})$ is a reciprocal matrix function of order (p,q).

Definition 1.3. A vector field $f(\mathbf{x})$ is called a pseudo-reciprocal vector field of order (p,q) if and only if there exists a nonzero matrix $\mathbf{M}(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $g(\mathbf{x})$ is a reciprocal vector field of order (p,q).

Remark: An RLC circuit is reciprocal (respectively, pseudo-reciprocal) if and only if there exists some reordering of the capacitors and/or inductors such that the resulting state equations has a reciprocal (respectively, pseudo-reciprocal) vector field.

Theorem 1.4. *Let*

$$\mathbf{X}(\mathbf{x}) = \begin{bmatrix} x_{11}(\mathbf{x}) & \dots & x_{1n}(\mathbf{x}) \\ \vdots & & \vdots \\ x_{n1}(\mathbf{x}) & \dots & x_{nn}(\mathbf{x}) \end{bmatrix}$$

be an $n \times n$ matrix and

$$f\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_n(x_1,\dots,x_n) \end{bmatrix}$$

be a C^1 vector valued function. Define the functions

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = \sum_{k=1}^{n}\left(x_{ik}(\mathbf{x})\frac{\partial}{\partial x_j}f_k(\mathbf{x}) + f_k(\mathbf{x})\frac{\partial}{\partial x_j}x_{ik}(\mathbf{x}) - x_{jk}(\mathbf{x})\frac{\partial}{\partial x_i}f_k(\mathbf{x}) - f_k(\mathbf{x})\frac{\partial}{\partial x_i}x_{jk}(\mathbf{x})\right)$$

for $1 \le i < j \le p$,

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = \sum_{k=1}^{n} \left(x_{ik}(\mathbf{x}) \frac{\partial}{\partial x_j} f_k(\mathbf{x}) + f_k(\mathbf{x}) \frac{\partial}{\partial x_j} x_{ik}(\mathbf{x}) + x_{jk}(\mathbf{x}) \frac{\partial}{\partial x_i} f_k(\mathbf{x}) + f_k(\mathbf{x}) \frac{\partial}{\partial x_i} x_{jk}(\mathbf{x}) \right)$$

for $1 \le i \le p, p+1 \le j \le n$,

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = \sum_{k=1}^{n} \left(x_{ik}(\mathbf{x}) \frac{\partial}{\partial x_j} f_k(\mathbf{x}) + f_k(\mathbf{x}) \frac{\partial}{\partial x_j} x_{ik}(\mathbf{x}) - x_{jk}(\mathbf{x}) \frac{\partial}{\partial x_i} f_k(\mathbf{x}) - f_k(\mathbf{x}) \frac{\partial}{\partial x_i} x_{jk}(\mathbf{x}) \right)$$

for $p+1 \le i < j \le n$. *Then* $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ *is a reciprocal vector field of order* $(p, n-p)$ *if and only if*

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \le i < j \le n$.

Proof. If $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field then its Jacobian is a reciprocal matrix. Note that

$$\mathbf{X}(\mathbf{x})f(\mathbf{x}) = \begin{bmatrix} x_{11}(\mathbf{x})f_1(\mathbf{x}) + \ldots + x_{1n}(\mathbf{x})f_n(\mathbf{x}) \\ \vdots \\ x_{n1}(\mathbf{x})f_1(\mathbf{x}) + \ldots + x_{nn}(\mathbf{x})f_n(\mathbf{x}) \end{bmatrix}$$

has a Jacobian matrix whose ij-th entry is given by

$$\mathbf{D}\left(\mathbf{X}(\mathbf{x})f(\mathbf{x})\right)_{ij}(\mathbf{x}) = \sum_{k=1}^{n} x_{ik}(\mathbf{x}) \frac{\partial}{\partial x_j} f_k(\mathbf{x}) + f_k(\mathbf{x}) \frac{\partial}{\partial x_j} x_{ik}(\mathbf{x}).$$

Applying the conditions for a matrix to be reciprocal (Def. 1.1) to the entries of the above Jacobian matrix, we obtain the above 3 identities $G^p_{ij}(\mathbf{X})(\mathbf{x}) = 0$. ∎

Definition 1.5. A vector field $f(\mathbf{x})$ is called a gradient vector field if and only if there exists a function $F(\mathbf{x})$ such that $f(\mathbf{x}) = \nabla F(\mathbf{x})$.

Theorem 1.6. *A vector field* $f(\mathbf{x}) : \Re^n \to \Re^n$ *is a gradient vector field if and only if* $\mathbf{D}f(\mathbf{x})$ *is a symmetric matrix.*

Proof. Clearly, a gradient vector field has a Jacobian which is symmetric. Conversely, since the domain $\Re^n$ is simply connected, if the Jacobian is symmetric then it is well known that the function defined by line integrals from a fixed point gives a function $F(\mathbf{x})$ with $f(\mathbf{x}) = \nabla F(\mathbf{x})$. ∎

Definition 1.7. A vector field $f(\mathbf{x})$ is called a pseudo-gradient vector field if and only if there exists a nonzero matrix $\mathbf{M}(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $g(\mathbf{x})$ is a gradient vector field.

§2. The identification of reciprocal vector fields under coordinate changes.

Given a vector field $f(\mathbf{x})$, which is not itself a reciprocal vector field, there remains the possibility that under a permutation of coordinates, the vector field in the new coordinate system is a reciprocal vector field. This section presents an algorithm that will determine if a vector field has a permutation of coordinates to that of a reciprocal vector field. The algorithm is presented as part of a theorem, whose proof therefore validates the algorithm.

Definition 2.1. Let $\mathbf{e}_i$ denote the ith-coordinate vector.

Definition 2.2. A coordinate permutation matrix is a matrix of the form

$$\mathbf{E} = [\,\mathbf{e}_{i_1} \quad \ldots \quad \mathbf{e}_{i_n}\,]$$

where $(i_1, \ldots, i_n)$ is a permutation of the integers $(1, \ldots, n)$.

The definition of a coordinate permutation matrix corresponds to the permutation of the coordinate axes where the jth-axis is replaced by the i_jth-axis.

Definition 2.3. The matrix $\mathbf{A}$ is similar to the matrix $\mathbf{B}$ via a coordinate permutation matrix $\mathbf{E}$ if $\mathbf{E}^t\mathbf{A}\mathbf{E} = \mathbf{B}$.

Lemma 2.4. *The vector field $\xi(\mathbf{x})$ has a coordinate change via the coordinate permutation matrix* $\mathbf{E}$,

$$\mathbf{x} = \mathbf{E}\mathbf{X}$$

such that $\eta(\mathbf{X}) = \xi(\mathbf{E}\mathbf{X})$ is a reciprocal vector field if and only if $\mathbf{E}^t\mathbf{D}\xi(\mathbf{E}\mathbf{X})\mathbf{E} = \mathbf{D}\eta(\mathbf{X})$.

PROOF. Immediate. ∎

Theorem 2.5. *There exists a coordinate permutation matrix $\mathbf{E}$ such that $\mathbf{E}^t\mathbf{A}\mathbf{E}$ is reciprocal if and only if the following algorithm terminates successfully,*

Step 1: If there exists $2 \leq i \leq n$ such that $a_{1i} \neq \pm a_{i1}$ then there is no solution to the similarity problem and the algorithm terminates unsuccessfully, else let $(i_1, \ldots, i_p)$ be those indexes (in any order) for which $a_{1i} = a_{i1}$ and $(i_{p+1}, \ldots, i_n)$ be those indexes (in any order) for which $a_{1i} = -a_{i1}$.

Step 2: Let

$$\mathbf{E} = [\,\mathbf{e}_{i_1} \quad \ldots \quad \mathbf{e}_{i_p} \quad \mathbf{e}_{i_{p+1}} \quad \ldots \quad \mathbf{e}_{i_n}\,]$$

$$\mathbf{B} = \mathbf{E}^t\mathbf{A}\mathbf{E}.$$

Step 3: If $\mathbf{B}$ is not reciprocal then there is no solution to the similarity problem and the algorithm terminates unsuccessfully, else $\mathbf{A}$ is similar to $\mathbf{B}$ via the coordinate permutation matrix $\mathbf{E}$, and the algorithm terminates successfully.

PROOF. See appendix. ∎

<u>Remark:</u> The <u>necessity</u> of theorem 2.5 is quite remarkable because it asserts that the reciprocity property of a matrix **M** can be determined by examining first the entries in the <u>first</u> row and the <u>first</u> column, and then relabelling the rows and columns by grouping all entries eith $a_{1i} = a_{i1}$. If the associated electrical <u>n-port</u> is <u>reciprocal</u>, then the reordered <u>hybrid</u> matrix [1] must necessarily satisfy <u>Definition 1.1.</u> This theorem therefore reduces an otherwise cumbersome <u>combinatorial</u> problem (if n is a large number) to a very simple algorithm which can be carried out <u>by inspection</u>. Note that one does not have to carry out any matrix multiplications because the <u>reordered</u> matrix is precisely $\mathbf{E}^t\mathbf{A}\mathbf{E}$.

By the following lemma 2.6, the algorithm will determine **A** to be similar to one of $p!(n-p)!$ reciprocal matrices depending on the permutation $(i_1, \ldots, i_p), (i_{p+1}, \ldots, i_n)$ of $(1, \ldots, p), (p+1, \ldots, n)$ chosen. Furthermore, when $1 \le p < n$, the matrix **A** is similar to a further $p!(n-p)!$ reciprocal matrices corresponding to permutations $(j_1, \ldots, j_{n-p}), (j_{n-p+1}, \ldots, j_n)$ of $(p+1, \ldots, n), (1, \ldots, p)$. The total number of reciprocal matrices to which **A** is similar via coordinate permutation change matrices is $n!$ for $n = p$, and $2p!(n-p)!$ for $1 \le p < n$. In particular, when $2 \le n$, if **A** is similar to a reciprocal matrix then there are at least 2 such matrices to which **A** is similar.

Lemma 2.6. *Let* **A** *be similar to the reciprocal matrix* **B** *via a coordinate permutation matrix* **E**. *If* **B** *is of order* $(p, n-p)$ *and* $(i_1, \ldots, i_p), (i_{p+1}, \ldots, i_n)$ *is a permutation of* $(1, \ldots, p), (p+1, \ldots, n)$ *then* **A** *is similar to the reciprocal matrix*

$$\begin{bmatrix} \mathbf{e}_{i_1}^t \\ \vdots \\ \mathbf{e}_{i_p}^t \\ \mathbf{e}_{i_{p+1}}^t \\ \vdots \\ \mathbf{e}_{i_n}^t \end{bmatrix} \mathbf{B} \, [\mathbf{e}_{i_1} \ \cdots \ \mathbf{e}_{i_p} \ \ \mathbf{e}_{i_{p+1}} \ \cdots \ \mathbf{e}_{i_n}]$$

via a coordinate permutation matrix. Furthermore, if $1 \le p < n$ *and* $(j_1, \ldots, j_{n-p}), (j_{n-p+1}, \ldots, j_n)$ *is a permutation of* $(p+1, \ldots, n), (1, \ldots, p)$ *then* **A** *is similar to the reciprocal matrix*

$$\begin{bmatrix} \mathbf{e}_{j_1}^t \\ \vdots \\ \mathbf{e}_{j_{n-p}}^t \\ \mathbf{e}_{j_{n-p+1}}^t \\ \vdots \\ \mathbf{e}_{j_n}^t \end{bmatrix} \mathbf{B} \, [\mathbf{e}_{j_1} \ \cdots \ \mathbf{e}_{j_{n-p}} \ \ \mathbf{e}_{j_{n-p+1}} \ \cdots \ \mathbf{e}_{j_n}]$$

via a coordinate permutation matrix.

Proof. Coordinate permutation matrices are given by

$$\mathbf{F} = \mathbf{E}[\,\mathbf{e}_{i_1} \quad \cdots \quad \mathbf{e}_{i_p} \quad \mathbf{e}_{i_{p+1}} \quad \cdots \quad \mathbf{e}_{i_n}\,]$$

and

$$\mathbf{G} = \mathbf{E}[\,\mathbf{e}_{j_1} \quad \cdots \quad \mathbf{e}_{j_{n-p}} \quad \mathbf{e}_{j_{n-p+1}} \quad \cdots \quad \mathbf{e}_{j_n}\,]$$

respectively. ∎

Example 2.7. Let the matrix be given by

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ -4 & 6 & -5 \\ 2 & 5 & 3 \end{bmatrix}.$$

Here, $p = 2, i_1 = 1, i_2 = i_p = 3$, and $i_3 = i_{p+1} = 2$. Then a coordinate permutation matrix is given by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

to which $\mathbf{A}$ is similar to the reciprocal matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ -4 & -5 & 6 \end{bmatrix}.$$

§3. Pseudo-reciprocal vector fields $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x})$ is invertible.

Given a vector field $f(\mathbf{x})$, this section will determine if there exists an invertible matrix $\mathbf{M}(\mathbf{x})$ and reciprocal vector field $g(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$. If the vector field $f(\mathbf{x})$ is associated with an <u>electrical circuit</u> and

$$\begin{aligned}\mathbf{M}(\mathbf{x}) &= \begin{bmatrix}\mathbf{M}_p(\mathbf{x}_p) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n-p}(\mathbf{x}_{n-p})\end{bmatrix} \\ &= \mathbf{M}_p(\mathbf{x}_p) \oplus \mathbf{M}_{n-p}(\mathbf{x}_{n-p})\end{aligned}$$

is a block diagonal matrix where $\mathbf{M}_p(\mathbf{x}_p)$ is a $p \times p$ matrix depending only on $\mathbf{x}_p = (x_1, x_2, \ldots, x_p)$ and $\mathbf{M}_{n-p}(\mathbf{x}_{n-p})$ is an $(n-p) \times (n-p)$ matrix depending only on $\mathbf{x}_{n-p} = (x_{p+1}, x_{p+2}, \ldots, x_n)$, then $\mathbf{M}(\mathbf{x})$ can be interpreted as the Jacobian matrix associated with a p-port capacitor $\mathbf{C}_p$ and an $(n-p)$-port inductor $\mathbf{L}_{n-p}$ respectively [1]. In this case, $\mathbf{M}_p^{-1}(\mathbf{x}_p)$ is the small-signal capacitance matrix associated with the p-port capacitor $\mathbf{C}_p$, and $\mathbf{M}_{n-p}^{-1}(\mathbf{x}_{n-p})$ is the small-signal inductance matrix associated with the $(n-p)$-port inductor $\mathbf{L}_{n-p}$. In the special case where $\mathbf{M} = \mathbf{M}_p \oplus \mathbf{M}_{n-p}$ is a <u>constant</u> matrix, then $\mathbf{M}_p^{-1}$ is the capacitance matrix of a <u>linear</u> p-port capacitor $\mathbf{C}_p$ and $\mathbf{M}_{n-p}^{-1}$ is the inductance matrix of a <u>linear</u> $(n-p)$-port inductor $\mathbf{L}_{n-p}$.

Since $\mathbf{M}(\mathbf{x})$ is invertible, the problem is equivalent to finding an invertible matrix $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. Finding matrices $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field is done by application of theorem 1.4. Application of theorem 1.4 is attempted for each possible order $(p, n-p)$ for $p = 1, \ldots, n$. Once a successful determination of the matrix $\mathbf{X}(\mathbf{x})$ is achieved there still remains the problem of determining which, if any of the matrices $\mathbf{X}(\mathbf{x})$ are invertible. Determining whether or not the matrix $\mathbf{X}(\mathbf{x})$ is invertible is done by computing the determinant of the matrix. If no such invertible matrices exist then $f(\mathbf{x})$ may not be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ with $\mathbf{M}(\mathbf{x})$ invertible and $g(\mathbf{x})$ a reciprocal vector field of order $(p, n-p)$. Application of theorem 1.4 is applied to the remaining untried orders until all possible orders are exhausted. If at this point an invertible matrix $\mathbf{X}(\mathbf{x})$ does not exist then it can be concluded that a decomposition of the desired form does not exist. If however, an invertible $\mathbf{X}(\mathbf{x})$ exists for some order $(p, n-p)$, then $\mathbf{M}(\mathbf{x})$ can be set to $\mathbf{X}(\mathbf{x})^{-1}$. Then $f(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}(\mathbf{X}(\mathbf{x})f(\mathbf{x}))$ is a decomposition of the desired form.

The problem becomes one of solving (for p=1,...,n)

$$G_{ij}^p(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$ and

$$\det \mathbf{X}(\mathbf{x}) \neq 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is invertible. The following example will demonstrate the algorithm outlined above when the matrix $\mathbf{M}$ is taken to be a constant matrix.

EXAMPLE 3.1. (Figure 5.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + x_2 \\ -\frac{5}{2}x_1^2 + 2x_2^2 + 3x_2 \end{bmatrix},$$

then (with $p = 1$ and constant matrix $\mathbf{X}$),

$$\begin{aligned} G_{12}^1(\mathbf{X}) = {} & x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) + x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ & x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) + x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ = {} & x_1(x_{11} - x_{21} - 5x_{22}) + x_2(2x_{11} + x_{21} + 4x_{12}) + x_{11} + 3x_{12}. \end{aligned}$$

Thus $G_{12}^1(\mathbf{X}) = 0$ if and only if

$$\begin{aligned} x_{11} - x_{21} - 5x_{22} &= 0 \\ 2x_{11} + x_{21} + 4x_{12} &= 0 \\ x_{11} + 3x_{12} &= 0. \end{aligned}$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{1}{3}x_{11} \\ -\frac{2}{3}x_{11} & \frac{1}{3}x_{11} \end{bmatrix}.$$

If $\mathbf{X}$ is invertible then $-1/3x_{11}^2 \neq 0$, i.e. $x_{11} \neq 0$. Let $x_{11} = 3$ and define $\mathbf{M} = \mathbf{X}^{-1}$, then

$$\begin{aligned} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1^2 + 3x_1x_2 + x_2^2 \\ x_2 - \frac{3}{2}x_1^2 - 2x_1x_2 \end{bmatrix} \end{aligned}$$

is a desired decomposition for the pseudo-reciprocal vector field $f(\mathbf{x})$.

§4. Pseudo-reciprocal vector fields $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x})$ is symmetric invertible.

Given a vector field $f(\mathbf{x})$, this section will determine if there exists a symmetric invertible matrix $\mathbf{M}(\mathbf{x})$ and reciprocal vector field $g(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$.

Since $\mathbf{M}(\mathbf{x})$ is symmetric invertible, the problem is equivalent to finding a symmetric invertible matrix $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. The method is very similar to the case where the matrix $\mathbf{M}(\mathbf{x})$ is invertible. Finding matrices $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field is done by application of theorem 1.4. Application of theorem 1.4 is attempted for each possible order $(p, n-p)$ for $p = 1, \ldots, n$. Once a successful determination of the matrix $\mathbf{X}(\mathbf{x})$ is achieved there still remains the problem of determining which, if any of the matrices $\mathbf{X}(\mathbf{x})$ are symmetric and invertible. Of the possible matrices $\mathbf{X}(\mathbf{x})$, symmetry conditions on the entries are first checked for consistency. Of those matrices $\mathbf{X}(\mathbf{x})$ which are symmetric, their determinant is computed to determine which are invertible. If no such symmetric invertible matrices exist then $f(\mathbf{x})$ may not be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ with $\mathbf{M}(\mathbf{x})$ symmetric invertible and $g(\mathbf{x})$ a reciprocal vector field of order $(p, n-p)$. Application of theorem 1.4 is applied to the remaining untried orders until all possible orders are exhausted. If at this point a symmetric invertible matrix $\mathbf{X}(\mathbf{x})$ does not exist then it can be concluded that a decomposition of the desired form does not exist. If however, a symmetric invertible $\mathbf{X}(\mathbf{x})$ exists for some order $(p, n-p)$, then $\mathbf{M}(\mathbf{x})$ can be set to $\mathbf{X}(\mathbf{x})^{-1}$. Then $f(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}(\mathbf{X}(\mathbf{x})f(\mathbf{x}))$ is a decomposition of the desired form.

The problem becomes one of solving (for p=1,. . .,n)

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$,

$$x_{ij}(\mathbf{x}) = x_{ji}(\mathbf{x})$$

for $1 \leq i < j \leq n$ and

$$\det \mathbf{X}(\mathbf{x}) \neq 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is symmetric. The last condition restricts the possible matrices $\mathbf{X}(\mathbf{x})$ to those which are symmetric and invertible. The following example will demonstrate the algorithm outlined above for a constant matrix $\mathbf{M}$.

EXAMPLE 4.1. (Figure 6.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + 2x_2^2 + x_1 - 6x_2 \\ 2x_1^2 + 3x_2^2 + 2x_1 - 9x_2 \end{bmatrix},$$

then (with $p = 1$ and constant matrix $\mathbf{X}$),

$$\begin{aligned} G^1_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) + x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ &\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) + x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ &= x_1(2x_{21} + 4x_{22}) + x_2(4x_{11} + 6x_{12}) + x_{21} + 2x_{22} - 6x_{11} - 9x_{12}. \end{aligned}$$

Thus $G^1_{12}(\mathbf{X}) = 0$ if and only if

$$\begin{aligned} 2x_{21} + 4x_{22} &= 0 \\ 4x_{11} + 6x_{12} &= 0 \\ x_{21} + 2x_{22} - 6x_{11} - 9x_{12} &= 0. \end{aligned}$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{2}{3}x_{11} \\ x_{21} & -\frac{1}{2}x_{21} \end{bmatrix}.$$

The additional constraint

$$x_{12} = x_{21}$$

results in

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{2}{3}x_{11} \\ -\frac{2}{3}x_{11} & \frac{1}{3}x_{11} \end{bmatrix}.$$

If $\mathbf{X}$ is invertible then $-1/9x_{11}^2 \neq 0$, i.e. $x_{11} \neq 0$. Let $x_{11} = -3$ and define $\mathbf{M} = \mathbf{X}^{-1}$, then

$$\begin{aligned} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}^{-1}\left(\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1^2 + x_1 \\ x_2^2 - 3x_2 \end{bmatrix} \end{aligned}$$

is a desired decomposition for the pseudo-reciprocal vector field $f(\mathbf{x})$.

§5. Pseudo-reciprocal vector fields $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x})$ is symmetric positive definite. Given a vector field $f(\mathbf{x})$, this section will determine if there exists an symmetric positive definite matrix $\mathbf{M}(\mathbf{x})$ and a reciprocal vector field $g(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$. If $f(\mathbf{x})$ is associated with an electrical circuit, then a symmetric block diagonal positive definite $\mathbf{M}(\mathbf{x}) = \mathbf{M}_p(\mathbf{x}_p) \oplus \mathbf{M}_{n-p}(\mathbf{x}_{n-p})$ implies that all capacitors and inductors in the circuit are reciprocal and passive[1].

Since $\mathbf{M}(\mathbf{x})$ is symmetric positive definite, the problem is equivalent to finding a symmetric positive definite matrix $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. The method is very similar to the case where the matrix $\mathbf{M}(\mathbf{x})$ is symmetric invertible. Finding matrices $\mathbf{X}(\mathbf{x})$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field is done by application of theorem 1.4. Application of theorem 1.4 is attempted for each possible order $(p, n-p)$ for $p = 1, \ldots, n$. Once a successful determination of the matrix $\mathbf{X}(\mathbf{x})$ is achieved there still remains the problem of determining which, if any of the matrices $\mathbf{X}(\mathbf{x})$ are symmetric positive definite. Of the possible matrices $\mathbf{X}(\mathbf{x})$, symmetry conditions on the entries are first checked for consistency. Of those matrices $\mathbf{X}(\mathbf{x})$ which are symmetric, the determinants of their minors are computed to determine which matrices have minors that are strictly positive, thus ensuring that the matrix is positive definite. If no such symmetric positive matrices exist then $f(\mathbf{x})$ may not be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ with $\mathbf{M}(\mathbf{x})$ symmetric positive definite and $g(\mathbf{x})$ a reciprocal vector field of order $(p, n-p)$. Application of theorem 1.4 is applied to the remaining untried orders until all possible orders are exhausted. If at this point a symmetric positive definite matrix $\mathbf{X}(\mathbf{x})$ does not exist then it can be concluded that a decomposition of the desired form does not exist. If however, a symmetric positive definite $\mathbf{X}(\mathbf{x})$ exists for some order $(p, n-p)$, then $\mathbf{M}(\mathbf{x})$ can be set to $\mathbf{X}(\mathbf{x})^{-1}$. Then $f(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}(\mathbf{X}(\mathbf{x})f(\mathbf{x}))$ is a decomposition of the desired form.

The problem becomes one of solving (for p=1,...,n)

$$G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \le i < j \le n$,

$$x_{ij}(\mathbf{x}) = x_{ji}(\mathbf{x})$$

for $1 \le i < j \le n$ and

$$\det \begin{bmatrix} x_{11}(\mathbf{x}) & \ldots & x_{1i}(\mathbf{x}) \\ \vdots & & \vdots \\ x_{i1}(\mathbf{x}) & \ldots & x_{ii}(\mathbf{x}) \end{bmatrix} > 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is symmetric. The last condition restricts the possible matrices $\mathbf{X}(\mathbf{x})$ to those which are symmetric positive definite. The following example will demonstrate the method outlined above for a constant matrix $\mathbf{M}$.

EXAMPLE 5.1. (Figure 7.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sin(x_1 + x_2) \\ \sin(x_1 + x_2) + 4\sin(x_1)\cos(x_2) \end{bmatrix},$$

then (with $p = 1$ and constant matrix $\mathbf{X}$),

$$
\begin{aligned}
G^1_{12}(\mathbf{X}) = {} & x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) + x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\
& x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) + x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
= {} & \cos(x_1)\cos(x_2)(x_{11} + x_{21} + x_{12} + 5x_{22}) - \sin(x_1)\sin(x_2)(x_{11} + x_{21} + 5x_{12} + x_{22}).
\end{aligned}
$$

Thus $G^1_{12}(\mathbf{X}) = 0$ if and only if

$$x_{11} + x_{21} + x_{12} + 5x_{22} = 0$$

$$x_{11} + x_{21} + 5x_{12} + x_{22} = 0.$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{1}{6}x_{11} - \frac{1}{6}x_{21} \\ x_{21} & -\frac{1}{6}x_{11} - \frac{1}{6}x_{21} \end{bmatrix}.$$

The additional constraint

$$x_{12} = x_{21}$$

results in

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{1}{7}x_{11} \\ -\frac{1}{7}x_{11} & -\frac{1}{7}x_{11} \end{bmatrix}.$$

The conditions for a positive definite matrix,

$$x_{11} > 0$$

$$-\frac{8}{49}x_{11}^2 > 0$$

cannot both be simultaneously satisfied. With $p = 2$,

$$
\begin{aligned}
G^2_{12}(\mathbf{X}) = {} & x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\
& x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
= {} & \cos(x_1)\cos(x_2)(x_{11} - x_{21} + x_{12} - 5x_{22}) - \sin(x_1)\sin(x_2)(x_{11} - x_{21} + 5x_{12} - x_{22}).
\end{aligned}
$$

Thus $G^2_{12}(\mathbf{X}) = 0$ if and only if

$$x_{11} - x_{21} + x_{12} - 5x_{22} = 0$$

$$x_{11} - x_{21} + 5x_{12} - x_{22} = 0.$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{11} + 6x_{12} & -x_{12} \end{bmatrix}.$$

The additional constraint

$$x_{12} = x_{21}$$

results in

$$\mathbf{X} = \begin{bmatrix} x_{11} & -\frac{1}{5}x_{11} \\ -\frac{1}{5}x_{11} & \frac{1}{5}x_{11} \end{bmatrix}.$$

The conditions for a positive definite matrix,

$$x_{11} > 0$$
$$\frac{4}{25}x_{11}^2 > 0$$

can be simultaneously satisfied with $x_{11} = 5/4$. Thus

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1} \left(\begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{5} \end{bmatrix} f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \cos(x_1)\sin(x_2) \\ \sin(x_1)\cos(x_2) \end{bmatrix}$$

is a desired decomposition for the pseudo-reciprocal vector field $f(\mathbf{x})$.

§6. Pseudo-reciprocal vector fields $f(\mathbf{x}) = \mathrm{M}(\mathbf{x})g(\mathbf{x})$ where $\mathrm{M}(\mathbf{x})$ is diagonal positive definite.

This section will consider the decomposition of a vector field $f(\mathbf{x})$ as $\mathrm{M}(\mathbf{x})g(\mathbf{x})$ where the matrix $\mathrm{M}(\mathbf{x})$ is diagonal positive definite and the vector field $g(\mathbf{x})$ is a reciprocal vector field. If $f(\mathbf{x})$ is associated with an electrical circuit, then a positive definite diagonal $\mathrm{M}(\mathbf{x})$ implies that all capacitors and inductors in the circuit are passive and <u>uncoupled</u> two-terminal elements[1].

The identification problem is the same as finding a diagonal positive definite matrix $\mathrm{X}(\mathbf{x}) = \Lambda(x_{11}(\mathbf{x}), \ldots, x_{nn}(\mathbf{x}))$ such that $\mathrm{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. Once this is achieved, by setting $\mathrm{M}(\mathbf{x}) = \mathrm{X}(\mathbf{x})^{-1}$, a decomposition of the required form is $f(\mathbf{x}) = \mathrm{M}(\mathbf{x})(\mathrm{M}(\mathbf{x})^{-1}f(\mathbf{x}))$. If the vector field $f(\mathbf{x})$ does not have a decomposition of the required form then the matrix $\mathrm{X}(\mathbf{x})$ does not exist.

The following theorem gives conditions on the diagonal matrix $\mathrm{X}(\mathbf{x}) = \Lambda(x_{11}(\mathbf{x}), \ldots, x_{nn}(\mathbf{x}))$ to ensure that $\mathrm{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. Of these diagonal matrices, one searches for those that are positive definite, this entails the consideration of those matrices for which $x_{11}(\mathbf{x}), \ldots, x_{nn}(\mathbf{x}) > 0$. Thus solutions are sought to the problem (for $p = 1, \ldots, n$),

$$H_{ij}^p(\mathrm{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$ (where $H_{ij}^p(\mathrm{X}(\mathbf{x}))(\mathbf{x})$ is defined below) and

$$x_{ii}(\mathbf{x}) > 0$$

for $1 \leq i \leq n$. The first condition is to ensure that $\mathrm{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ while the second condition is to ensure that the matrix $\mathrm{X}(\mathbf{x})$ is positive definite.

Theorem 6.1. *For $1 \leq i < j \leq n$ define the functions*

$$H_{ij}^p(\mathrm{X}(\mathbf{x}))(\mathbf{x}) = x_{ii}(\mathbf{x})\frac{\partial}{\partial x_j}f_i(\mathbf{x}) + f_i(\mathbf{x})\frac{\partial}{\partial x_j}x_{ii}(\mathbf{x}) - x_{jj}(\mathbf{x})\frac{\partial}{\partial x_i}f_j(\mathbf{x}) - f_j(\mathbf{x})\frac{\partial}{\partial x_i}x_{jj}(\mathbf{x})$$

for $1 \le i < j \le p$,

$$H^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = x_{ii}(\mathbf{x})\frac{\partial}{\partial x_j}f_i(\mathbf{x}) + f_i(\mathbf{x})\frac{\partial}{\partial x_j}x_{ii}(\mathbf{x}) + x_{jj}(\mathbf{x})\frac{\partial}{\partial x_i}f_j(\mathbf{x}) + f_j(\mathbf{x})\frac{\partial}{\partial x_i}x_{jj}(\mathbf{x})$$

for $1 \le i \le p, p+1 \le j \le n$,

$$H^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = x_{ii}(\mathbf{x})\frac{\partial}{\partial x_j}f_i(\mathbf{x}) + f_i(\mathbf{x})\frac{\partial}{\partial x_j}x_{ii}(\mathbf{x}) - x_{jj}(\mathbf{x})\frac{\partial}{\partial x_i}f_j(\mathbf{x}) - f_j(\mathbf{x})\frac{\partial}{\partial x_i}x_{jj}(\mathbf{x})$$

for $p+1 \le i < j \le n$. *Then* $\mathbf{M}f(\mathbf{x})$ *is a reciprocal vector field of order* $(p, n-p)$ *if and only if*

$$H^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \le i < j \le n$.

PROOF. If $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field then its Jacobian is a reciprocal matrix. Note that since $\mathbf{X}$ is a diagonal matrix then

$$\mathbf{X}(\mathbf{x})f(\mathbf{x}) = \begin{bmatrix} x_{11}f_1(\mathbf{x}) \\ \vdots \\ x_{nn}f_n(\mathbf{x}) \end{bmatrix}$$

has Jacobian matrix with ij-th entry being

$$\mathbf{D}\left(\mathbf{X}(\mathbf{x})f(\mathbf{x})\right)_{ij}(\mathbf{x}) = x_{ii}(\mathbf{x})\frac{\partial}{\partial x_j}f_i(\mathbf{x}) + f_i(\mathbf{x})\frac{\partial}{\partial x_j}x_{ii}(\mathbf{x}).$$

The conditions for a matrix to be reciprocal are immediately applied to the entries of the Jacobian matrix. ∎

EXAMPLE 6.2. (Figure 8.) This example will be an illustration where the matrix $\mathbf{M}$ is further required to be a constant matrix. Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(x_2) + \sin(x_1) \\ -2x_1\sin(x_2) + 2x_2^2 \end{bmatrix},$$

then (with $p = 1$),

$$\begin{aligned} H^1_{12}(\mathbf{X}) = x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) + x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) + x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ = -\sin(x_2)(x_{11} + 2x_{22}). \end{aligned}$$

Thus $H^1_{12}(\mathbf{X}) = 0$ if and only if

$$x_{11} + 2x_{22} = 0.$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & 0 \\ 0 & -\frac{1}{2}x_{11} \end{bmatrix}.$$

The additional constraints

$$x_{11} > 0$$
$$-\frac{1}{2}x_{11} > 0$$

cannot both be simultaneously satisfied. With $p = 2$,

$$\begin{aligned} H_{12}^{2}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ &\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ &= -\sin(x_2)(x_{11} - 2x_{22}). \end{aligned}$$

Thus $H_{12}^{2}(\mathbf{X}) = 0$ if and only if

$$x_{11} - 2x_{22} = 0$$

Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & 0 \\ 0 & \frac{1}{2}x_{11} \end{bmatrix}.$$

The additional constraints

$$x_{11} > 0$$
$$\frac{1}{2}x_{11} > 0$$

can be simultaneously satisfied with $x_{11} = 2$. Thus

$$\begin{aligned} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(x_2) + \sin(x_1) \\ -x_1\sin(x_2) + x_2^2 \end{bmatrix} \end{aligned}$$

is a desired decomposition for the pseudo-reciprocal vector field $f(\mathbf{x})$.

§7. Pseudo-reciprocal vector fields $f(\mathbf{x}) = \mathrm{M}(\mathbf{x})g(\mathbf{x})$ where $\mathrm{M}(\mathbf{x})$ is diagonal invertible.

This section will consider the decomposition of a vector field $f(\mathbf{x})$ as $\mathrm{M}(\mathbf{x})g(\mathbf{x})$ where the matrix $\mathrm{M}(\mathbf{x})$ is diagonal invertible and the vector field $g(\mathbf{x})$ is a reciprocal vector field. If $f(\mathbf{x})$ is associated with an electrical circuit, then a diagonal $\mathrm{M}(\mathbf{x})$ implies that a circuit realisation is possible by using only 2-terminal (possibly active) capacitors and inductors in addition to resistors.

The problem is the same as finding a diagonal invertible matrix $\mathbf{X}(\mathbf{x}) = \Lambda(x_{11}(\mathbf{x}), \ldots, x_{nn}(\mathbf{x}))$ such that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field. Once this is achieved, by setting $\mathrm{M}(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}$, a decomposition of the required form is $f(\mathbf{x}) = \mathrm{M}(\mathbf{x})(\mathrm{M}(\mathbf{x})^{-1}f(\mathbf{x}))$. If the vector field $f(\mathbf{x})$ does not have a decomposition of the required form then the matrix $\mathbf{X}(\mathbf{x})$ does not exist.

Thus, solutions are sought to the problem (for $p = 1, \ldots, n$),

$$H^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \le i < j \le n$ and

$$x_{ii}(\mathbf{x}) \neq 0$$

for $1 \le i \le n$. The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ while the second condition is to ensure that the matrix $\mathbf{X}(\mathbf{x})$ is invertible.

§8. Identifying pseudo-gradient vector fields[9].

If $f(\mathbf{x})$ is associated with an electrical circuit, then a pseudo-gradient $f(\mathbf{x})$ corresponds to an RC or RL 2-element type reciprocal circuit; i.e, the circuit contains only capacitors, or only inductors, in addition to reciprocal resistors[1]. In particular, if $f(\mathbf{x})$ has a decomposition as $\mathrm{M}(\mathbf{x})g(\mathbf{x})$ where the matrix $\mathrm{M}(\mathbf{x})$ is positive definite and $g(\mathbf{x})$ is a gradient vector field then $f(\mathbf{x})$ does not admit periodic orbits. This is proved in the following lemma.

Lemma 8.1. *If $f(\mathbf{x}) = \mathrm{M}(\mathbf{x})g(\mathbf{x})$ where $\mathrm{M}(\mathbf{x})$ is positive definite and $g(\mathbf{x})$ is a gradient vector field then there do not exist periodic orbits for $f(\mathbf{x})$.*

PROOF. Let $G(\mathbf{x})$ be a function such that $g(\mathbf{x}) = \nabla G(\mathbf{x})$. Let $\phi(\mathbf{x}, t)$ denote the solution to

$$\phi'(\mathbf{x}, t) = f(\phi(\mathbf{x}, t))$$
$$\phi(\mathbf{x}, 0) = \mathbf{x}$$

and consider the function

$$E(\mathbf{x}, t) = -G(\phi(\mathbf{x}, t)).$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}E(\mathbf{x}, t) &= -\nabla G(\phi(\mathbf{x}, t)).\phi'(\mathbf{x}, t)\\ &= -\nabla G(\phi(\mathbf{x}, t)).f(\phi(\mathbf{x}, t))\\ &= -\nabla G(\phi(\mathbf{x}, t)).\mathrm{M}(\mathbf{x})\nabla G(\phi(\mathbf{x}, t))\\ &\le 0.\end{aligned}$$

Assume that a periodic orbit exists through the point $\mathbf{x}_0$ with period $0 < t_0$. Then

$$\begin{aligned}
0 &= G(\phi(\mathbf{x}_0, t_0)) - G(\phi(\mathbf{x}_0, 0)) \\
&= E(\mathbf{x}_0, t_0) - E(\mathbf{x}_0, 0) \\
&= \int_0^{t_0} \frac{\partial}{\partial t} E(\mathbf{x}_0, s)\, ds \\
&= \int_0^{t_0} -\nabla G(\phi(\mathbf{x}, s)).\mathbf{M}(\mathbf{x}) \nabla G(\phi(\mathbf{x}, s))\, ds \\
&\leq 0.
\end{aligned}$$

Thus, $\nabla G(\phi(\mathbf{x}_0, s)) = 0$ for $0 \leq s \leq t_0$ from which it follows that

$$\begin{aligned}
f(\mathbf{x}_0) &= \mathbf{M}(\mathbf{x}_0) g(\mathbf{x}_0) \\
&= \mathbf{M}(\mathbf{x}_0) \nabla G(\phi(\mathbf{x}_0, 0)) \\
&= 0.
\end{aligned}$$

Thus, the point $\mathbf{x}_0$ is a fixed point of $f(\mathbf{x})$ which is in contradiction to being on a periodic orbit. Thus, periodic orbits do not exist for the vector field $f(\mathbf{x})$. ∎

It is immediate that a pseudo-gradient vector field cannot admit Smale horseshoes. Such horseshoes contain infinitely many periodic orbits while lemma 8.1 rules out the possibility of periodic orbits occuring in pseudo-gradient vector fields.

Corollary 8.2. *Pseudo-gradient vector fields* $f(\mathbf{x}) = \mathbf{M}(\mathbf{x})g(\mathbf{x})$ *with positive definite* $\mathbf{M}(\mathbf{x})$ *do not admit periodic orbits.*

Proof. Immediate from lemma 8.1. ∎

Example 8.3. The condition that M be positive definite cannot be weakened to invertible. Consider the vector field

$$\begin{aligned}
f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

The vector field $f(\mathbf{x})$ admits the periodic orbits given by $(r\sin(t), r\cos(t))$ for $0 < r$.

Example 8.4. The condition that M be positive definite cannot be substituted by symmetric. Consider the vector field

$$\begin{aligned}
f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

The vector field $f(\mathbf{x})$ admits the periodic orbits given by $(r\sin(t), r\cos(t))$ for $0 < r$.

By theorem 1.6, the identification of a pseudo-gradient vector field is the same as the identification of pseudo-reciprocal vector fields where the Jacobian matrix is a reciprocal matrix of order $(n, 0)$. The identification of matrices $\mathbf{M}(\mathbf{x})$ which are invertible, symmetric invertible, symmetric positive definite, diagonal positive definite and diagonal invertible such that $\mathbf{M}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field is achieved by the following algorithms which are the restrictions of the corresponding algorithms for the pseudo-reciprocal case.

To search for invertible matrices $\mathbf{M}(\mathbf{x})$, the problem becomes one of solving for the matrix $\mathbf{X}(\mathbf{x})$ where

$$G^n_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$ and

$$\det\mathbf{X}(\mathbf{x}) \neq 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field, while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is invertible. Thus the vector field $f(\mathbf{x})$ may be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}$ is invertible and $g(\mathbf{x}) = \mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field.

To search for symmetric invertible matrices $\mathbf{M}(\mathbf{x})$, the problem becomes one of solving for the matrix $\mathbf{X}(\mathbf{x})$ where

$$G^n_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$,

$$x_{ij}(\mathbf{x}) = x_{ji}(\mathbf{x})$$

for $1 \leq i < j \leq n$ and

$$\det\mathbf{X}(\mathbf{x}) \neq 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field, while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is symmetric. The last condition restricts the possible matrices $\mathbf{X}(\mathbf{x})$ to those which are symmetric and invertible. Thus the vector field $f(\mathbf{x})$ may be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}$ is symmetric invertible and $g(\mathbf{x}) = \mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field.

To search for symmetric positive definite matrices $\mathbf{M}(\mathbf{x})$, the problem becomes one of solving for the matrix $\mathbf{X}(\mathbf{x})$ where

$$G^n_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$,

$$x_{ij}(\mathbf{x}) = x_{ji}(\mathbf{x})$$

for $1 \leq i < j \leq n$ and

$$\det \begin{bmatrix} x_{11}(\mathbf{x}) & \ldots & x_{1i}(\mathbf{x}) \\ \vdots & & \vdots \\ x_{i1}(\mathbf{x}) & \ldots & x_{ii}(\mathbf{x}) \end{bmatrix} > 0.$$

The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field, while the second condition ensures that $\mathbf{X}(\mathbf{x})$ is symmetric. The last condition restricts the possible matrices $\mathbf{X}(\mathbf{x})$ to those which are symmetric positive definite. Thus the vector field $f(\mathbf{x})$ may be written as $\mathbf{M}(\mathbf{x})g(\mathbf{x})$ where $\mathbf{M}(\mathbf{x}) = \mathbf{X}(\mathbf{x})^{-1}$ is symmetric positive definite and $g(\mathbf{x}) = \mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field.

To search for diagonal positive definite matrices $\mathbf{M}(\mathbf{x})$, the problem becomes one of solving for the diagonal matrix $\mathbf{X}(\mathbf{x})$ where

$$H^n_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$ and

$$x_{ii}(\mathbf{x}) > 0$$

for $1 \leq i \leq n$. The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field, while the second condition is to ensure that the matrix $\mathbf{X}(\mathbf{x})$ is positive definite.

To search for diagonal invertible matrices $\mathbf{M}(\mathbf{x})$, the problem becomes one of solving for the diagonal matrix $\mathbf{X}(\mathbf{x})$ where

$$H^n_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x}) = 0$$

for $1 \leq i < j \leq n$ and

$$x_{ii}(\mathbf{x}) \neq 0$$

for $1 \leq i \leq n$. The first condition is to ensure that $\mathbf{X}(\mathbf{x})f(\mathbf{x})$ is a gradient vector field, while the second condition is to ensure that the matrix $\mathbf{X}(\mathbf{x})$ is invertible.

§9. Identifying pseudo-reciprocal piecewise linear vector fields[10].

In the case of piecewise-linear vector fields, an explicit computation of the functions $G^p_{ij}(\mathbf{X}(\mathbf{x}))(\mathbf{x})$ for constant matrices $\mathbf{X}(\mathbf{x})$ is possible. This allows a more efficient algorithm to determine pseudo-reciprocity of piecewise linear vector functions. First a definition and a lemma are needed before the presentation of the algorithms. The algorithms will determine the existence of matrices $\mathbf{Y}$ such that $\mathbf{Y}\xi$ is a reciprocal vector field in the cases that $\mathbf{Y}$ is invertible, symmetric invertible, symmetric positive definite, diagonal positive definite and diagonal invertible.

Definition 9.1. Given a matrix $\mathbf{A}$, define the set

$$\chi(\mathbf{A}, p) = \{\mathbf{X} : \mathbf{XA} \text{ is reciprocal of order } (p, \dim \mathbf{A} - p)\}.$$

The matrix $\mathbf{X}$ is such that $\mathbf{XA}$ is a reciprocal matrix of order $(p,\ \dim \mathbf{A} - p)$.

Lemma [10] 2.3. *Considering a matrix* $\mathbf{X}$ *written in the form of a* $n \times n$*-tuple*

$$\begin{bmatrix} x_{1\,1} \\ \vdots \\ x_{1\,n} \\ \vdots \\ x_{n\,1} \\ \vdots \\ x_{n\,n} \end{bmatrix}$$

there exists a finite set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \Re^{n\times n}$ *such that*

$$\chi(\mathbf{A}, p) = \{t_1\mathbf{v}_1 + \ldots + t_s\mathbf{v}_s : t_1, \ldots, t_s \in \Re\}.$$

An algorithm to implement the lemma is immediate by solving a set of linear equalities that a matrix must satisfy if it is to be in the set $\chi(\mathbf{A}, p)$. Let the vector field ξ be given by

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|,$$

then an algorithm to determine the existence of invertible matrices $\mathbf{Y}$ with $\mathbf{Y}\circ\xi$ a reciprocal vector field is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}^0_1, \ldots, \mathbf{w}^0_{q_0}\}$ where the vectors $\{\mathbf{w}^0_1, \ldots, \mathbf{w}^0_{q_0}\}$ form a basis for

$$\chi\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\chi\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for $\mathrm{span}(S) \cap \mathrm{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: Form the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1, \ldots, x_{q_m})$ be the polynomial given by

$$f(x_1, \ldots, x_{q_m}) = \det \mathbf{Y}(x_1, \ldots, x_{q_m}).$$

Step 5: Determine if $f(x_1, \ldots, x_{q_m})$ is identical to the zero function. If it is then go to step 6 else choose values for $x_1, \ldots, x_{q_m}$ such that $f(x_1, \ldots, x_{q_m}) \neq 0$ and go to step 7.

Step 6: In this case, all matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are non-invertible. If $s < n$ then let $s = s + 1$ and go to step 2, otherwise there do not exist invertible matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of any order. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is invertible and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1, \ldots, x_{q_m}$ such that the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is invertible and $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ with $\mathbf{Y}^{-1}$ invertible and $\mathbf{Y} \circ \xi$ a reciprocal vector field.

An algorithm to determine the existence of invertible symmetric matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\chi\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\chi\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for $\operatorname{span}(S) \cap \operatorname{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: From the equation

$$\sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m - (\mathbf{w}_i^m)^t) = 0$$

determine a set of independent variables $x_1, \ldots, x_k$ and dependent variables $x_{k+1}, \ldots, x_{q_m}$. Form the matrix

$$\mathbf{Y}(x_1, \ldots, x_k) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1, \ldots, x_k)$ be the polynomial given by

$$f(x_1, \ldots, x_k) = \det \mathbf{Y}(x_1, \ldots, x_k).$$

Step 5: Determine if $f(x_1, \ldots, x_k)$ is identically the zero function. If it is then go to step 6 else choose values for $x_1, \ldots, x_k$ such that $f(x_1, \ldots, x_k) \neq 0$ and go to step 7.

Step 6: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are non-invertible. If $s < n$ then let $s = s+1$ and go to step 2, otherwise there do not exist invertible symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is invertible symmetric and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is invertible symmetric and $\mathbf{Y} \circ \xi$ is a reciprocal vector field of oreder $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ with $\mathbf{Y}^{-1}$ invertible symmetric and $\mathbf{Y} \circ \xi$ a reciprocal vector field.

An algorithm to determine the existence of symmetric positive definite matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\chi\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\chi\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for span$(S) \cap$ span(T).

Step 3.3: Let $S = R$.

Step 4: From the equation

$$\sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m - (\mathbf{w}_i^m)^t) = 0$$

determine a set of independent variables $x_1, \ldots, x_k$ and dependent variables $x_{k+1}, \ldots, x_{q_m}$. Form the matrix

$$\mathbf{Y}_n(x_1, \ldots, x_k) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m.$$

Define the matrices

$$\mathbf{Y}_i(x_1, \ldots, x_k) = \begin{bmatrix} \mathbf{Y}_n(x_1, \ldots, x_k)_{11} & \ldots & \mathbf{Y}_n(x_1, \ldots, x_k)_{1i} \\ \vdots & & \vdots \\ \mathbf{Y}_n(x_1, \ldots, x_k)_{i1} & \ldots & \mathbf{Y}_n(x_1, \ldots, x_k)_{ii} \end{bmatrix}$$

and let $f_i(x_1, \ldots, x_k)$ be the polynomial given by

$$f_i(x_1, \ldots, x_k) = \det \mathbf{Y}_i(x_1, \ldots, x_k)$$

for $1 \leq i \leq n$.

Step 5: Determine if there exist values $x_1, \ldots, x_k$ such that the following set of inequalities hold simultaneously,

$$\begin{aligned} f_1(x_1, \ldots, x_k) &> 0 \\ &\vdots \\ f_n(x_1, \ldots, x_k) &> 0. \end{aligned}$$

If such values do not exist then go to step 6 else go to step 7.

Step 6: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are either non-invertible or invertible and not positive definite. If $s < n$ then let $s = s+1$ and go to step 2, otherwise there do not exist symmetric positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is symmetric positive definite and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\mathbf{Y}_n(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is symmetric positive definite and $\mathbf{Y}_n \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}_n^{-1} \circ (\mathbf{Y}_n \circ \xi)$ with $\mathbf{Y}_n^{-1}$ symmetric positive definite and $\mathbf{Y}_n \circ \xi$ a reciprocal vector field.

In the case that the matrix $\mathbf{Y}$ is diagonal positive definite, a number of definitions and lemmas are necessary before allowing presentation of the algorithm. The matrix $\mathbf{Y}$ is denoted as a vector $[d_1 \ \ldots \ d_n]^t$ in the following.

Definition [10] 7.1. Define the set

$$\mathbf{E}\left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}, p\right) = \left\{ \begin{bmatrix} d_1 \\ \vdots \\ d_p \\ d_{p+1} \\ \vdots \\ d_n \end{bmatrix} : \exists \lambda \in \Re \ni \begin{bmatrix} d_1\alpha_1 \\ \vdots \\ d_p\alpha_p \\ d_{p+1}\alpha_{p+1} \\ \vdots \\ d_n\alpha_n \end{bmatrix} = \lambda \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \\ -\beta_{p+1} \\ \vdots \\ -\beta_n \end{bmatrix} \right\}.$$

Lemma [10] 7.2. *There exists vectors such that*

$$\mathbf{E}\left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}, p\right) = \left\{ \sum_{i=1}^{k} t_i \begin{bmatrix} c_{i1} \\ \vdots \\ c_{in} \end{bmatrix} : t_i \in \Re \right\}.$$

Definition [10] 7.3. Define the set

$$\mathbf{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, p\right) = \left\{ \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} : \begin{bmatrix} d_1 b_{11} & \ldots & d_1 b_{1n} \\ \vdots & & \vdots \\ d_n b_{n1} & \ldots & d_n b_{nn} \end{bmatrix} \text{ reciprocal of order } (p, n-p) \right\}.$$

Lemma [10] 7.4. *There exists vectors such that*

$$\mathbf{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, p\right) = \left\{ \sum_{i=1}^{k} t_i \begin{bmatrix} d_{i1} \\ \vdots \\ d_{in} \end{bmatrix} : t_i \in \Re \right\}.$$

Implementing algorithms for lemmas[10] 7.2, 7.4, are exercises in linear algebra. With the lemmas presented, the stage is set for the algorithm. An algorithm to determine the existence of diagonal positive definite matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathbf{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathbf{E}\left(\begin{bmatrix}\alpha_{j1}\\ \vdots \\ \alpha_{jn}\end{bmatrix}, \begin{bmatrix}\beta_{j1}\\ \vdots \\ \beta_{jn}\end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for span$(S) \cap$ span(T).

Step 3.3: Let $S = R$.

Step 4: Determine if there exist values $x_1, \ldots, x_{q_m}$ such that the following set of inequalities hold simultaneously,

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &> 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &> 0. \end{aligned}$$

If such values do not exist then go to step 5 else go to step 6.

Step 5: In this case, all diagonal matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are either non-invertible or invertible and not positive definite. If $s < n$ then let $s = s+1$ and go to step 2, otherwise there do not exist diagonal positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is diagonal positive definite and ζ is a reciprocal vector field.

Step 6: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\Lambda(y_1, \ldots, y_n)$$

with

$$y_j = \sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m)_j$$

is diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \Lambda(y_1, \ldots, y_n)^{-1} \circ (\Lambda(y_1, \ldots, y_n) \circ \xi)$ with $\Lambda(y_1, \ldots, y_n)^{-1}$ diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ a reciprocal vector field.

Note that if there exists a solution $y_1, \ldots, y_{q_m}$ to

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &= \epsilon_1 > 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &= \epsilon_n > 0 \end{aligned}$$

then there exists a solution $y_1', \ldots, y_{q_m}'$ to

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &\geq 1 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &\geq 1 \end{aligned}$$

by scaling the original values $y_1, \ldots, y_{q_m}$ with a sufficiently large constant. Decompose the variables $x_1, \ldots, x_{q_m}$ as $x_i = x_i^1 - x_i^2$ for $i = 1, \ldots, q_m$. It then follows that $y_1^1 = y_1', \ldots, y_{q_m}^1 = y_{q_m}', y_1^2 = 0, \ldots, y_{q_m}^2 = 0$ is an optimal solution to the linear programming problem of

$$\text{minimise} \sum_{i=1}^{n} v_i$$

subject to

$$\begin{aligned} x_1^1(\mathbf{w}_1^m)_1 - x_1^2(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_1 - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_1 - u_1 &= 1 \\ &\vdots \\ x_1^1(\mathbf{w}_1^m)_n - x_1^2(\mathbf{w}_1^m)_n + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_n - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_n - u_n &= 1 \\ x_1^1, x_1^2, \ldots, x_{q_m}^1, x_{q_m}^2 &\geq 0 \\ u_1, \ldots, u_n, v_1, \ldots, v_n &\geq 0. \end{aligned}$$

Conversely, given an optimal solution to the above linear programming problem, if $0 \leq u_i - v_i$ for $i = 1, \ldots, n$ then $x_i = x_i^1 - x_i^2$ is a solution to the original problem

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &> 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &> 0. \end{aligned}$$

An algorithm to determine the existence of diagonal invertible matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathbf{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathbf{E}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}, \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for $\text{span}(S) \cap \text{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: Determine if there exist values $x_1, \ldots, x_{q_m}$ such that the following set of inequalities hold simultaneously,

$$x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 \neq 0$$

$$\vdots$$

$$x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n \neq 0.$$

If such values do not exist then go to step 5 else go to step 6.

Step 5: In this case, all diagonal matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are non-invertible. If $s < n$ then let $s = s+1$ and go to step 2, otherwise there do not exist diagonal invertible matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is diagonal invertible and ζ is a reciprocal vector field.

Step 6: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\mathbf{A}(y_1, \ldots, y_n)$$

with

$$y_j = \sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m)_j$$

is diagonal invertible and $\mathbf{A}(y_1, \ldots, y_n) \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{A}(y_1, \ldots, y_n)^{-1} \circ (\mathbf{A}(y_1, \ldots, y_n) \circ \xi)$ with $\mathbf{A}(y_1, \ldots, y_n)^{-1}$ diagonal invertible and $\mathbf{A}(y_1, \ldots, y_n) \circ \xi$ a reciprocal vector field.

§10. Identifying pseudo-gradient piecewise linear vector fields[10].

In the case of piecewise linear vector fields, a more explicit computation of the functions $G_{ij}^n(\mathbf{X}(\mathbf{x}))$ is possible. This allows a more efficient algorithm to determine pseudo-gradiency of piecewise linear vector functions. As pseudo-gradient piecewise linear vector fields are a proper subset of pseudo-reciprocal piecewise linear vector fields, is suffices to note that algorithms for the identification of pseudo-gradient piecewise linear vector fields are immediately obtained from the algorithms for pseudo-reciprocal piecewise linear vector fields by restricting the pseudo-reciprocal piecewise linear vector fields of interest to those which are of order $(n, 0)$.

Appendix: Proof of theorem 2.5.

Clearly, if the algorithm terminates successfully then a co-ordinate permutation matrix $\mathbf{E}$ is constructed such that $\mathbf{E}^t\mathbf{A}\mathbf{E}$ is reciprocal. Conversly, assume there exists a coordinate permutation matrix $\mathbf{E}$ such that $\mathbf{B} = \mathbf{E}^t\mathbf{A}\mathbf{E}$ is reciprocal. It may be taken that

$$\mathbf{E} = [\mathbf{e}_{i_1} \quad \ldots \quad \mathbf{e}_{i_n}]$$

and

$$\mathbf{B} = \begin{bmatrix} b_{1\,1} & \ldots & b_{1\,p} & b_{1\,p+1} & \ldots & b_{1\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{1\,p} & \ldots & b_{p\,p} & b_{p\,p+1} & \ldots & b_{p\,n} \\ -b_{1\,p+1} & \ldots & -b_{p\,p+1} & b_{p+1\,p+1} & \ldots & b_{p+1\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -b_{1\,n} & \ldots & -b_{p\,n} & b_{p+1\,n} & \ldots & b_{n\,n} \end{bmatrix}.$$

Note that $\mathbf{E}^{-1}$ is also a coordinate permutation matrix. Thue $\mathbf{E}^{-1} = \mathbf{F}$ where

$$\mathbf{F} = [\mathbf{e}_{j_1} \quad \ldots \quad \mathbf{e}_{j_n}]$$

with $(j_1, \ldots, j_n)$ the inverse permutation to $(i_1 \ldots, i_n)$. The matrix

$$\mathbf{A} = (\mathbf{E}^{-1})^t\mathbf{B}\mathbf{E}^{-1} = \mathbf{F}^t\mathbf{B}\mathbf{F}$$

has first row

$$[b_{j_1j_1} \quad \ldots \quad b_{j_1j_n}]$$

and first column

$$\begin{bmatrix} b_{j_1j_1} \\ \vdots \\ b_{j_nj_1} \end{bmatrix}.$$

Since $b_{j_1i} = \pm b_{ij_1}$, the first step of the algorithm is successfully completed.

If $1 \leq j_1 \leq p$, there exist $(k_1, \ldots, k_p, k_{p+1}, \ldots, k_n)$ such that

$$\begin{aligned} b_{j_1j_{k_l}} &= a_{1k_l} \\ &= a_{k_l1} \\ &= b_{j_{k_l}j_1} \end{aligned}$$

for $1 \leq l \leq p$ and

$$\begin{aligned} b_{j_1j_{k_l}} &= a_{1k_l} \\ &= a_{k_l1} \\ &= -b_{j_{k_l}j_1} \end{aligned}$$

for $p+1 \leq i \leq n$. Thus $(j_{k_1}, \ldots, j_{k_p}) = (m_1, \ldots, m_p)$ and $(j_{k_{p+1}}, \ldots, j_{k_n}) = (m_{p+1}, \ldots, m_n)$ where $(m_1, \ldots, m_p)$ and $(m_{p+1}, \ldots, m_n)$ are permutations of the integers $(1, \ldots, p)$ and $(p+1, \ldots, n)$ respectively. Define

$$\mathbf{G} = [\mathbf{e}_{k_1} \quad \ldots \quad \mathbf{e}_{k_p} \quad \mathbf{e}_{k_{p+1}} \quad \ldots, \mathbf{e}_{k_n}].$$

This is the second step of the algorithm, then

$$\begin{aligned}\mathbf{FG} &= [\mathbf{e}_{j_1} \ \cdots \ \mathbf{e}_{j_n}][\mathbf{e}_{k_1} \ \cdots \ \mathbf{e}_{k_p} \ \ \mathbf{e}_{k_{p+1}} \ \ \ldots, \mathbf{e}_{k_n}] \\ &= [\mathbf{e}_{j_{k_1}} \ \cdots \ \mathbf{e}_{j_{k_p}} \ \ \mathbf{e}_{j_{k_{p+1}}} \ \ \mathbf{e}_{j_{k_n}}] \\ &= [\mathbf{e}_{m_1} \ \cdots \ \mathbf{e}_{m_p} \ \ \mathbf{e}_{m_{p+1}} \ \cdots \ \mathbf{e}_{m_n}],\end{aligned}$$

or,

$$\mathbf{G} = \mathbf{E}[\mathbf{e}_{m_1} \ \cdots \ \mathbf{e}_{m_p} \ \ \mathbf{e}_{m_{p+1}} \ \cdots \ \mathbf{e}_{m_n}].$$

Thus

$$\begin{aligned}\mathbf{G}^t\mathbf{A}\mathbf{G} &= \begin{bmatrix} \mathbf{e}^t_{m_1} \\ \vdots \\ \mathbf{e}^t_{m_p} \\ \mathbf{e}^t_{m_{p+1}} \\ \vdots \\ \mathbf{e}^t_{m_n} \end{bmatrix} \mathbf{E}^t\mathbf{A}\mathbf{E}[\mathbf{e}_{m_1} \ \cdots \ \mathbf{e}_{m_p} \ \ \mathbf{e}_{m_{p+1}} \ \cdots \ \mathbf{e}_{m_n}] \\ &= \begin{bmatrix} b_{m_1\ m_1} & \cdots & b_{m_1\ m_p} & b_{m_1\ m_{p+1}} & \cdots & b_{m_1\ m_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{m_1\ m_p} & \cdots & b_{m_p\ m_p} & b_{m_p\ m_{p+1}} & \cdots & b_{m_p\ m_n} \\ -b_{m_1\ m_{p+1}} & \cdots & -b_{m_p\ m_{p+1}} & b_{m_{p+1}\ m_{p+1}} & \cdots & b_{m_{p+1}\ m_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -b_{m_1\ m_n} & \cdots & -b_{m_p\ m_n} & b_{m_{p+1}\ m_n} & \cdots & b_{m_n\ m_n} \end{bmatrix}\end{aligned}$$

is a reciprocal matrix, the final step of the algorithm is successfully completed and the algorithm terminates successfully.

If $p+1 \le j_1 \le n$, there exist $(k_1, \ldots, k_{n-p}, k_{n-p+1}, \ldots, k_n)$ such that

$$\begin{aligned}b_{j_1 j_{k_l}} &= a_{1k_l} \\ &= a_{k_l 1} \\ &= b_{j_{k_l} j_1}\end{aligned}$$

for $1 \le l \le n-p$ and

$$\begin{aligned}b_{j_1 j_{k_l}} &= a_{1k_l} \\ &= a_{k_l 1} \\ &= -b_{j_{k_l} j_1}\end{aligned}$$

for $n-p+1 \le i \le n$. Thus $(j_{k_1}, \ldots, j_{k_p}) = (m_1, \ldots, m_{n-p}), (j_{k_{n-p+1}}, \ldots, j_{k_n}) = (m_{n-p+1}, \ldots, m_n)$ where $(m_1, \ldots, m_{n-p})$ and $(m_{n-p+1}, \ldots, m_n)$ are permutations of the integers $(1, \ldots, n-p)$ and $(n-p+1, \ldots, n)$ respectively. Define

$$\mathbf{G} = [\mathbf{e}_{k_1} \ \cdots \ \mathbf{e}_{k_{n-p}} \ \ \mathbf{e}_{k_{n-p+1}} \ \ \ldots, \mathbf{e}_{k_n}].$$

This is the second step of the algorithm, then

$$\begin{aligned}\mathbf{FG} &= [\mathbf{e}_{j_1} \ \cdots \ \mathbf{e}_{j_n}][\mathbf{e}_{k_1} \ \cdots \ \mathbf{e}_{k_{n-p}} \ \ \mathbf{e}_{k_{n-p+1}} \ \ \ldots, \mathbf{e}_{k_n}] \\ &= [\mathbf{e}_{j_{k_1}} \ \cdots \ \mathbf{e}_{j_{k_{n-p}}} \ \ \mathbf{e}_{j_{k_{n-p+1}}} \ \ \mathbf{e}_{j_{k_n}}] \\ &= [\mathbf{e}_{m_1} \ \cdots \ \mathbf{e}_{m_{n-p}} \ \ \mathbf{e}_{m_{n-p+1}} \ \cdots \ \mathbf{e}_{m_n}],\end{aligned}$$

or,

$$\mathbf{G} = \mathbf{E}[\,\mathbf{e}_{m_1} \;\; \cdots \;\; \mathbf{e}_{m_{n-p}} \;\; \mathbf{e}_{m_{n-p+1}} \;\; \cdots \;\; \mathbf{e}_{m_n}\,].$$

Thus

$$\mathbf{G}^t\mathbf{A}\mathbf{G} = \begin{bmatrix} \mathbf{e}^t_{m_1} \\ \vdots \\ \mathbf{e}^t_{m_{n-p}} \\ \mathbf{e}^t_{m_{n-p+1}} \\ \vdots \\ \mathbf{e}^t_{m_n} \end{bmatrix} \mathbf{E}^t\mathbf{A}\mathbf{E}[\,\mathbf{e}_{m_1} \;\; \cdots \;\; \mathbf{e}_{m_{n-p}} \;\; \mathbf{e}_{m_{n-p+1}} \;\; \cdots \;\; \mathbf{e}_{m_n}\,]$$

$$= \begin{bmatrix} b_{m_1\, m_1} & \cdots & b_{m_1\, m_{n-p}} & -b_{m_{n-p+1}\, m_1} & \cdots & -b_{m_n\, m_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{m_1\, m_{n-p}} & \cdots & b_{m_{n-p}\, m_{n-p}} & -b_{m_{n-p+1}\, m_{n-p}} & \cdots & -b_{m_n\, m_{n-p}} \\ b_{m_{n-p+1}\, m_1} & \cdots & b_{m_{n-p+1}\, m_{n-p}} & b_{m_{m-p+1}\, m_{n-p+1}} & \cdots & b_{m_{n-p+1}\, m_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{m_n\, m_1} & \cdots & b_{m_n\, m_{n-p}} & b_{m_{n-p+1}\, m_n} & \cdots & b_{m_n\, m_n} \end{bmatrix}$$

is a reciprocal matrix, the final step of the algorithm is successfully completed and the algorithm terminates successfully. ∎

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Figure captions.

Figure 1. An electronic realisation of the vector field in example 0.1. Note that this is a circuit corresponding to a third order system.

Figure 2. An electronic realisation of the vector field in example 0.2.

Figure 3. An electronic realisation of the vector field in example 0.3.

Figure 4. An electronic realisation of the vector field in example 0.4. Note that this is a circuit corresponding to a fourth order system.

Figure 5. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + x_2 \\ -\frac{5}{2}x_1^2 + 2x_2^2 + 3x_2 \end{bmatrix}.$$

This vector field may be written as the product of an invertible matrix M and a reciprocal vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 6. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + 2x_2^2 + x_1 - 6x_2 \\ 2x_1^2 + 3x_2^2 + 2x_1 - 9x_2 \end{bmatrix}.$$

This vector field may be written as the product of an invertible symmetric matrix M and a reciprocal vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 7. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sin(x_1 + x_2) \\ \sin(x_1 + x_2) + 4\sin(x_1)\cos(x_2) \end{bmatrix}.$$

This vector field may be written as the product of a symmetric positive definite matrix M and a reciprocal vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 8. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(x_2) + \sin(x_1) \\ -2x_1\sin(x_2) + 2x_2^2 \end{bmatrix}.$$

This vector field may be written as the product of a diagonal positive definite matrix M and a reciprocal vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

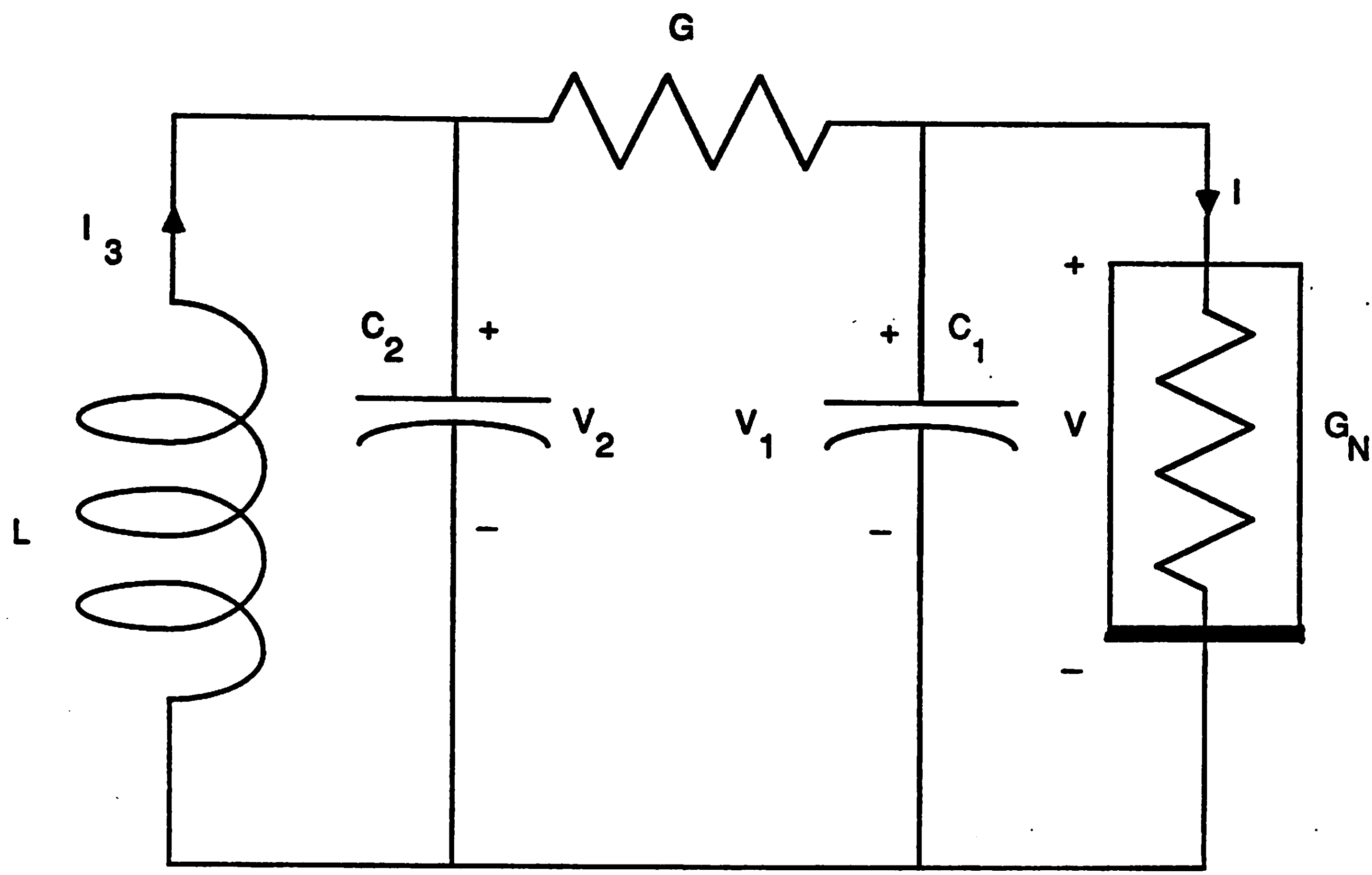


Figure 1

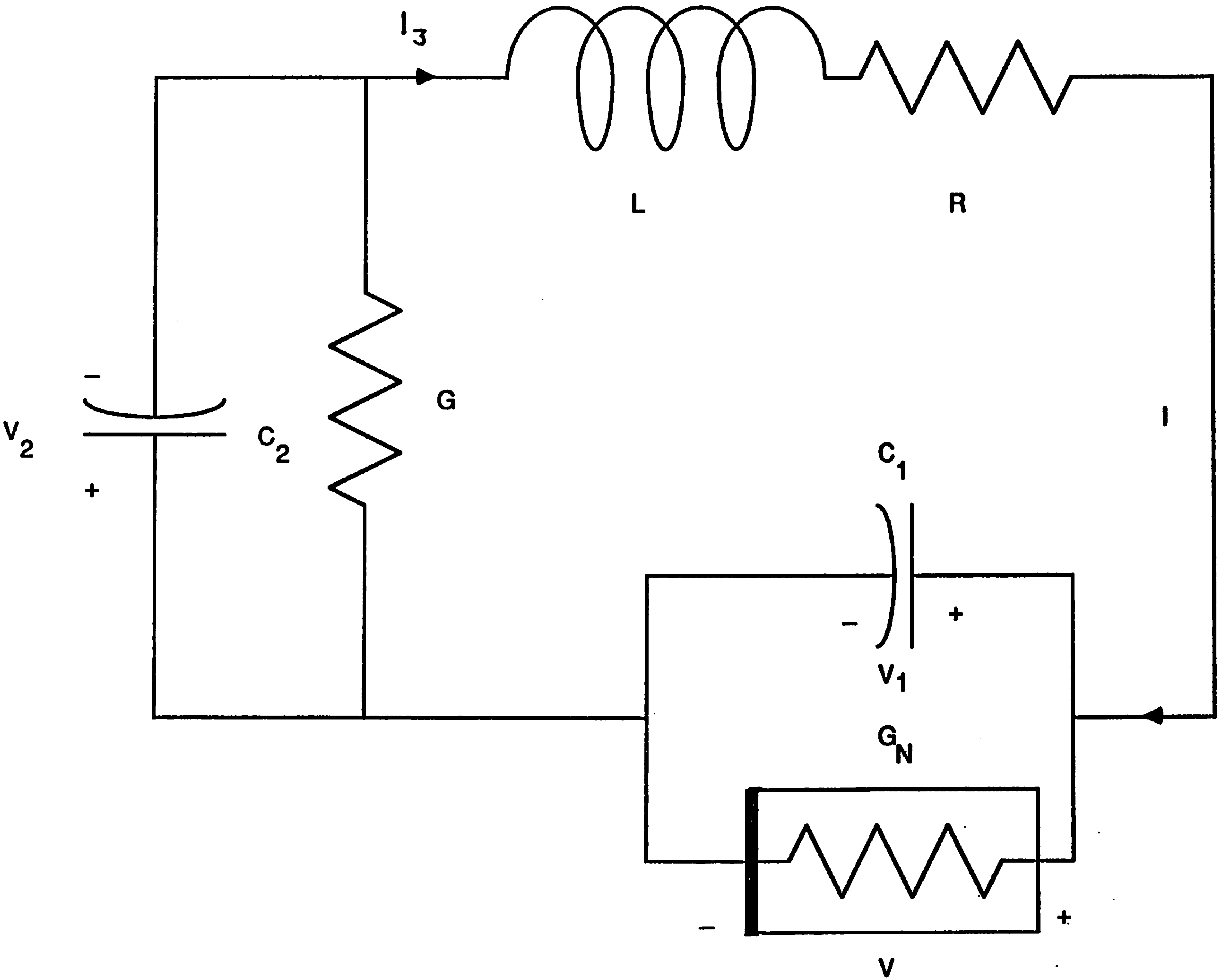


Figure 2

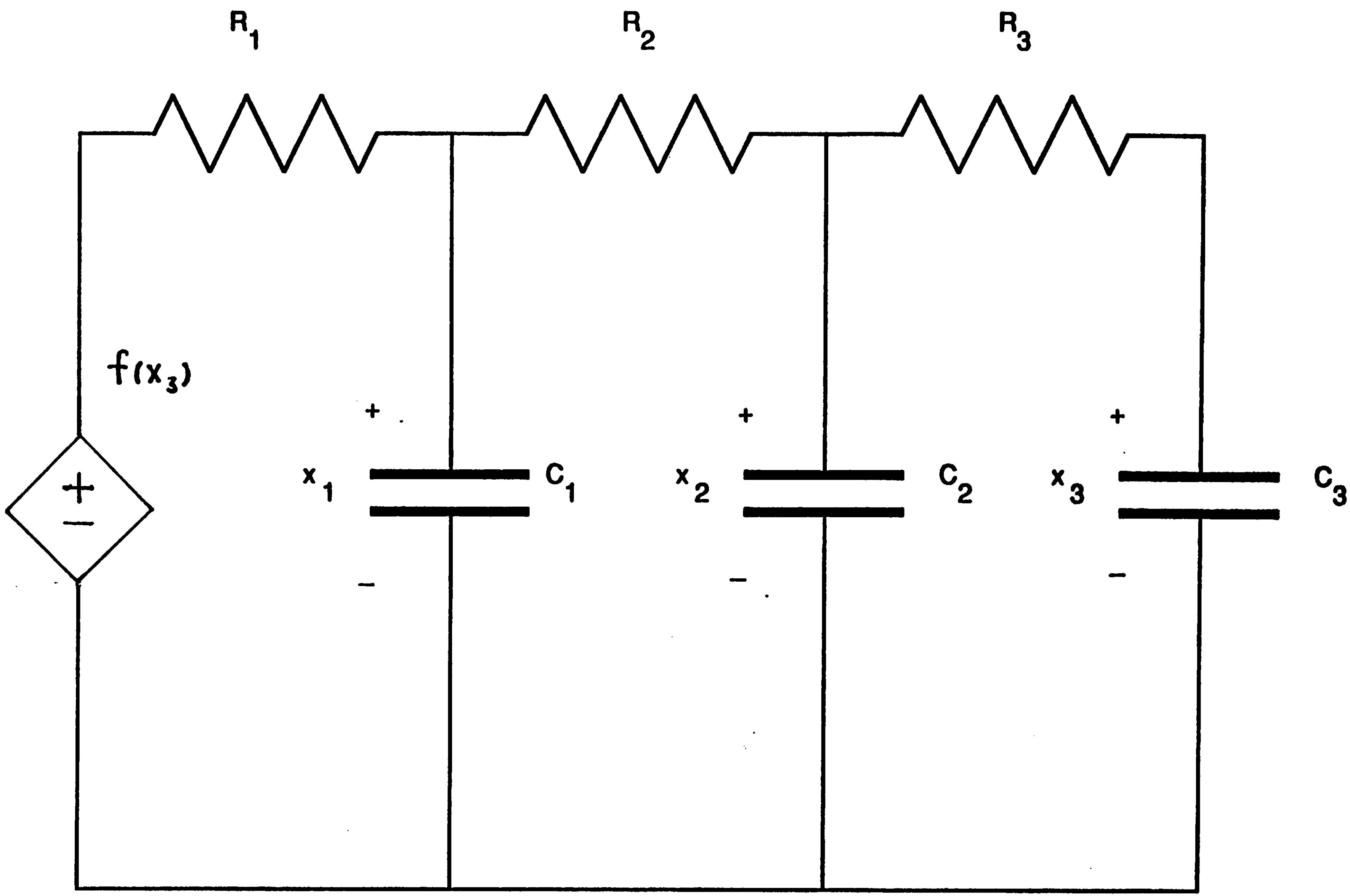
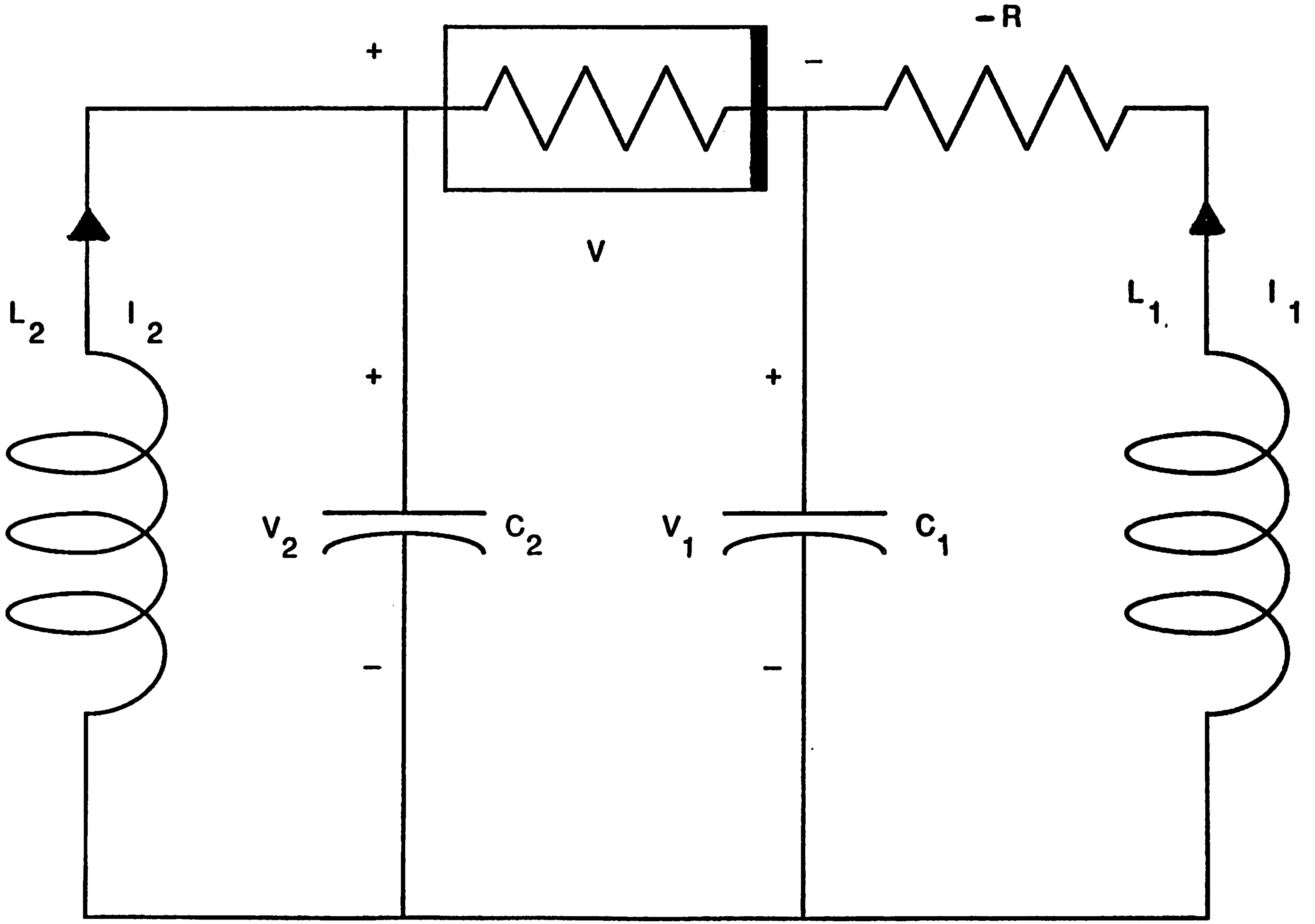


figure 3



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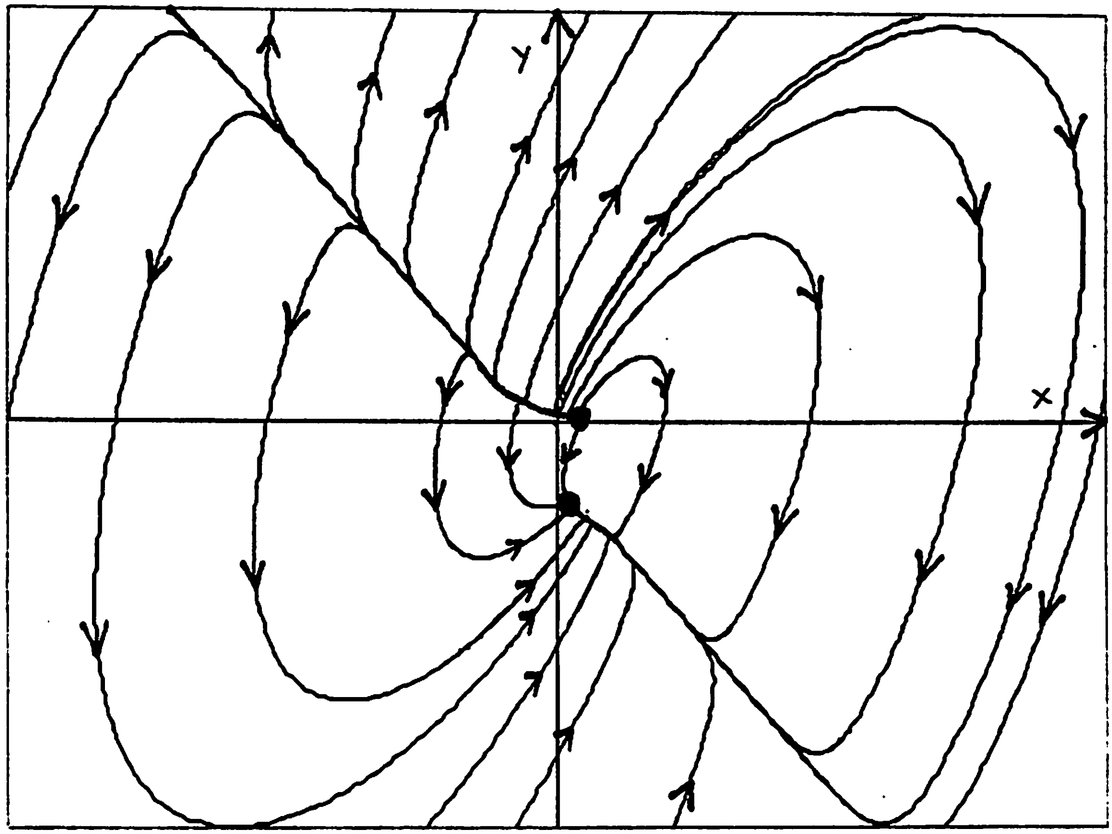

Figure 5

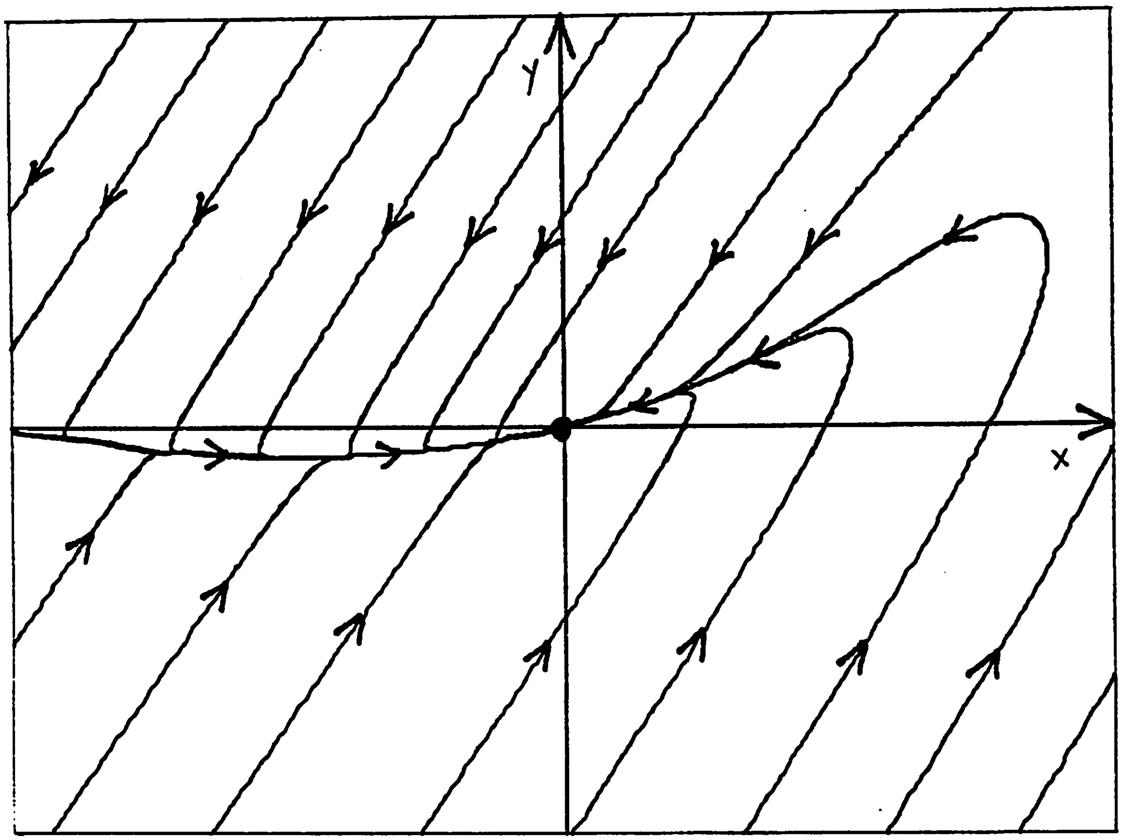

Figure 6

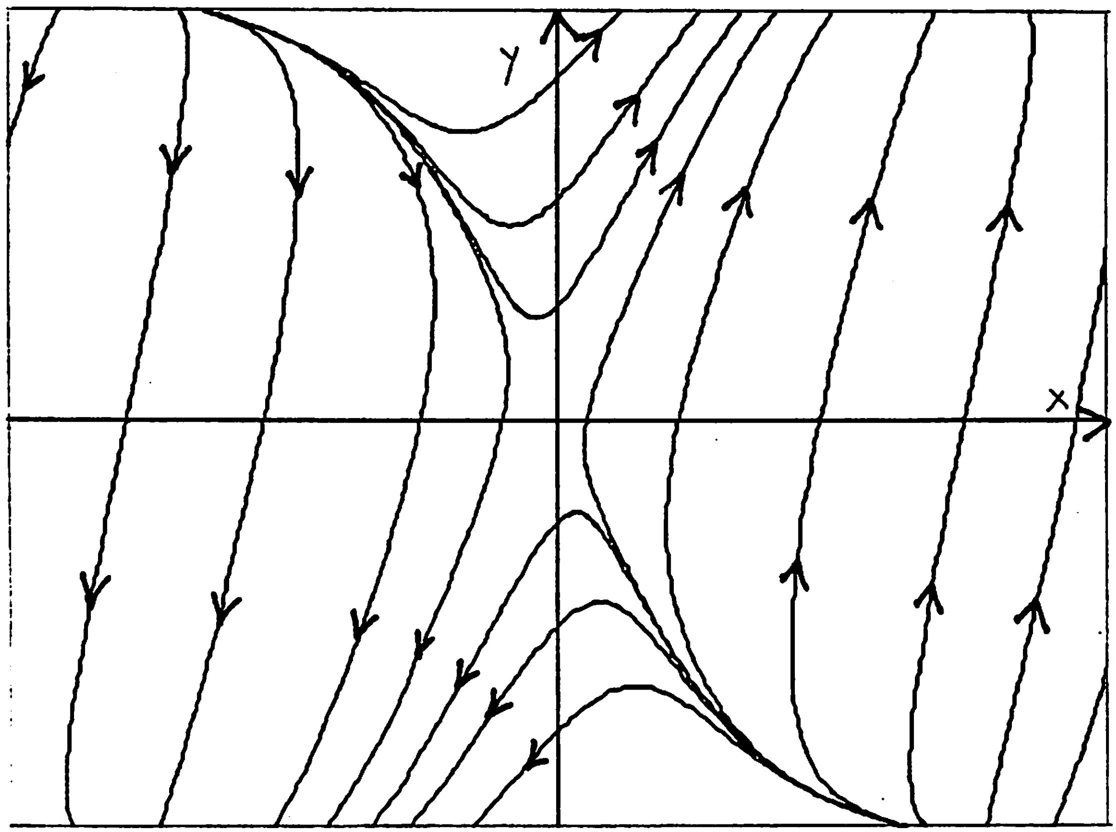

Figure 7

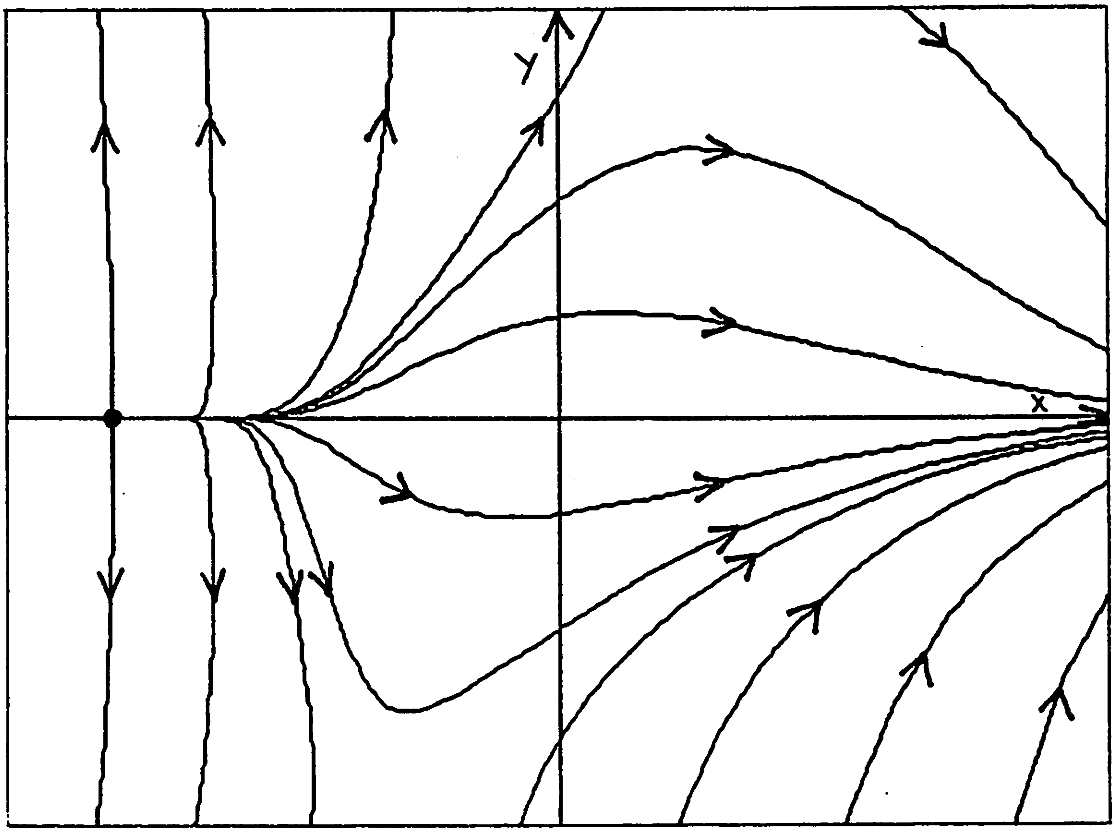

Figure 8