GLOBAL PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS PART I: SIMPLEST CASE IN $\Re^2$

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Abstract

Among nonlinear vector fields, the simplest of which can be studied are those which are continuous and piecewise linear. Associated with these types of vector fields are partitions of the state-space into a finite number of regions. In each region the vector field is linear. On the boundary between regions it is required that the vector field be continuous from both regions in which it is linear. This presentation is devoted to the analysis in two dimensions of the simplest possible types of continuous piecewise linear vector fields, namely linear vector fields possessing only one boundary condition. As a practical concern, the analysis will attempt to ask and answer questions raised about the existence of steady-state solutions. Since the local theory of fixed points in a linear vector field is sufficient to determine stability of fixed points in a piecewise linear vector field, most of the steady state behaviour to be studied will be towards limit cycles. The results will present sufficient conditions for the existence, or nonexistence as the case may be, for limit cycles. Particular attention will be paid to the domain of attraction whenever possible.

With these results qualitative statements may be made for piecewise linear models of many physical systems.

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§0. Introduction.

The determination of limit cycles is of great practical and theorectical importance. The work on Hilbert's 16th problem (a survey paper being that of Lloyd[3]) has shown that even for two dimensions and polynomial vector fields as simple as degree two, the maximum number of limit cycles is not known. This situation is symptomatic of the present intractability of the determination of limit cycles in the entire state space $\Re^n$. However, it may be possible in certain cases to give results on the global determination of all limit cycles. One such area has arisen from the solution of problems in electrical engineering.

With the advent of computer aided design and the subsequent increase of computer simulations of physical circuits, device modeling has emerged as an increasingly important area of research. In the modeling of electrical and electronic circuits an exemplary case of such work is the paper Chua[1] "Canonical piecewise linear modeling." In that paper a large number of electronic device models were shown to have concise representations as piecewise linear functions. The interconnection of one or more of such piecewise-linear circuit models with capacitors and inductors in feedback naturally creates a piecewise-linear dynamical system.

Conversely, nonlinear vector fields which are piecewise linear may be emulated by equivalent physical circuits. Such emulation requires the use of piecewise linear resistors, capacitors and inductors.

Once a piecewise linear representation of a circuit has been created, the computer becomes a powerful tool with which to study the original circuit. Computer work with such models has suggested the possibility of proving qualitative results about certain classes of piecewise linear vector fields arising from such modeling.

This research effort has been devoted to the examination of such qualitative properties of the simplest types of piecewise linear vector fields. The research being primarily devoted to finding attractors in the system and estimating the size of the basin of attraction. Section 1 will introduce the basic definitions and concepts to be used, then sections 2 through 9 will present the analysis of continuous piecewise linear vector fields.

To conclude this introductory section, some examples of the variety of behaviour possible in comparatively simple types of piecewise linear vector fields in $\Re^2$ will be presented. A table (Table 1) of the possible phase portraits suggests that many distinct types of behaviour can exist in piecewise linear vector fields. Then a summary of the results will end the section. The following four examples give a preview of some of the results predicted from the theorems proved in this paper, EXAMPLE 1. (Figure 1.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 0 & 2 \\ -\frac{5}{3} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{2}{3} \end{bmatrix}, & 1 < x. \end{cases}$$

This vector field does not have any limit cycles despite the fact that there are infinitely many concentric nonisolated cycles (see theorem 8.9). The only equilibrium point, $(0, 0)$, is a <u>center</u>.

EXAMPLE 2. (Figure 2.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} -5 & 1 \\ -15 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -7 \\ -14 \end{bmatrix}, & 1 < x. \end{cases}$$

This vector field does not have any cycles (see theorem 3.19). The only equilibrium point, $(0, 0)$, is an <u>unstable focus</u>.

EXAMPLE 3. (Figure 3.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} -7 & 1 \\ -19 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -9 \\ -18 \end{bmatrix}, & 1 < x. \end{cases}$$

The vector field has a unique attracting cycle (see theorem 3.7). The only equilibrium point, $(0, 0)$, in this case is an <u>unstable focus</u>. Observe that the region to the left of the line $x \equiv 1$ is the same as in the previous example, equivalence of vector fields on one linear region does not guarantee similar dynamics.

EXAMPLE 4. (Figure 4.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 4 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 5 & 0 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ -4 \end{bmatrix}, & 1 < x. \end{cases}$$

This vector field does not have any cycles. The invariant manifold parallel to the y-axis prevents cycles from forming (see theorem 9.1). The only equilibrium point, $(0, 0)$, is a <u>saddle point</u>.

Summary of main results.

Conjecture 0.1. *A continuous piecewise-linear vector field with one boundary condition has at most one limit cycle. The limit cycle, if it exists, is either attracting or repelling.*

Computer experimentation has lend weight to the above conjecture. Under this conjecture the following theorems are a summary of the results obtained in sections 2 through 9.

Theorem 0.2. *Let $0 < b, 0 < a + d, (a + d)^2/4 < ad - bc, 0 < ad - bc + dk - bl$. Let (x, y) be the induced virtual fixed point of the vector field ξ with defining constants a, b, c, d, k, l. Define*

$$X_1(x, y) = y - \frac{1}{b}(-\sqrt{1-x}(a + d) - d(1 - x) - a).$$

If $X_1(x, y) < 0$ then ξ has a globally attracting limit cycle. If $0 < X_1(x, y)$ then ξ does not have any limit cycles.

PROOF. See theorems 3.7, 3.13, 3.19 and 3.2. ∎

Theorem 0.3. *Let $0 < b, 0 < a + d, (a + d)^2/4 < ad - bc, ad - bc + dk - bl < 0$. Let (x, y) be the induced fixed point of the vector field ξ with defining constants a, b, c, d, k, l. Define*

$$X_2(x, y) = y - \chi_2(\chi_1^{-1}(x))$$

where $\chi(y) = (\chi_1(y), \chi_2(y))$ is given in lemma 3.25. If $X_2(x, y) < 0$ then ξ does not have any limit cycles. If $0 < X_2(x, y)$ then ξ has a locally attracting limit cycle.

PROOF. See theorems 3.24, 3.30 and 3.29. ∎

Theorem 0.4. *Let $0 < b, 0 < a + d, 0 \leq ad - bc \leq (a + d)^2/4$. The vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See propositions 5.1, 5.2 and 6.1. ∎

Theorem 0.5. *Let $0 < b, 0 < a + d, ad - bc < 0, ad - bc + dk - bl \leq 0$. The vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 7.1 and 7.2. ∎

Theorem 0.6. *Let $0 < b, 0 < a + d, ad - bc < 0, 0 < ad - bc + dk - bl$. If the vector field ξ with defining constants a, b, c, d, k, l does not have any homoclinic orbits then either (i) ξ has no limit cycles or, (ii) ξ has a repelling limit cycle.*

PROOF. See proposition 7.3, theorems 7.5,7.8 and 7.7. ∎

Theorem 0.7. *Let $0 < b, 0 = a + d$. The vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 8.1, theorems 8.9, 8.10 and proposition 8.11. ∎

Theorem 0.8. *Let $0 = b$. The vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 9.1. ∎

§1. Definitions.

In this section the basic definitions of the nonlinear vector fields to be studied are presented. As all the work to be presented lies in the plane, it will be taken that all vectors lie in $\Re^2$.

Definition 1.1. L is a linear † vector field $\Leftrightarrow$ there exists constants a, b, c, d, e, f such that

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} e \\ f \end{bmatrix}.$$

Definition 1.2. ξ is a continuous piecewise linear vector field $\Leftrightarrow$ there exists constants a, b, c, d, k, l with either $k \neq 0$ or $l \neq 0$, and

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x. \end{cases}$$

Unless otherwise stated, the term vector field will mean a continuous piecewise linear vector field.

A vector field is linear in each of the regions $\{(x, y) : x \leq 1\}, \{(x, y) : 1 < x\}$. As either k or l is nonzero, the vector field is nonlinear.

Definition 1.3. For the vector field ξ the function $\phi(t, (x_0, y_0))$ will denote the solution to

$$\phi(t, (x_0, y_0))' = \xi(\phi(t, (x_0, y_0)))$$
$$\phi(0, (x_0, y_0)) = (x_0, y_0).$$

Definition 1.4. The point (x_0, y_0) is called a periodic point if there is a $0 < t_0 < \infty$ for which $\phi(t_0, (x_0, y_0)) = (x_0, y_0)$. The set $\{\phi(t, (x_0, y_0)) : 0 < t \leq t_0\}$ is called a cycle.

Definition 1.5. Let (x_0, y_0) be a point on a cycle. Consider a local transversal Σ through (x_0, y_0) and Poincare map $\mathbf{P} : \Sigma \rightarrow \Sigma$. If the point (x_0, y_0) is attracting (respectively repelling) for the map $\mathbf{P}$ then the cycle is said to be attracting (respectively repelling). If it is attracting from one

† A more precise name is affine.

side in positive time and repelling from the other side in positive time then the point is said to be semi-stable.

Definition 1.6. A limit cycle is a cycle that is either attracting, repelling or semi-stable. Hence, a cycle is a limit cycle if and only if it is isolated.

Definition 1.7. Define an ordering on the set of cycles by $C_1 \prec C_2$ if the cycle C_1 lies in the interior of the cycle C_2. Let $\{\cdots \prec C_{-1} \prec C_0 \prec C_1 \prec \cdots\}$ be a maximal chain of cycles bounded below and above by the cycles $C_{-\infty}, C_\infty$. The pair $(C_{-\infty}, C_\infty)$ is an annulus with boundary cycles $C_{-\infty}, C_\infty$. The annulus will be identified with the closed region between its boundary cycles.

Definition 1.8. Let N be a set and ϕ be the solution to a vector field, then $\phi(t, N)$ is the set given by $\phi(t, N) = \{\phi(t, (x, y)) : (x, y) \in N\}$.

Definition 1.9. An attracting annulus A has a neighbourhood $N(A \subset N)$, such that for non-negative times $0 \le t_0 < t_1, A \subset \phi(t_1, N) \subset \phi(t_0, N), A = \cap_{t=0}^{\infty} \phi(t, N)$. A repelling annulus is an annulus which is attracting in reverse time. A limit annulus is an annulus that is either attracting or repelling. If the annulus is attracting from one side in positive time and repelling from the other side in positive time then it is said to be semi-stable.

Limit cycles occur often in natural phenomena. By their nature, the presence of a limit cycle points towards the presence of steady state oscillatory behaviour in the underlying system. Annuli are more general than cycles. Attracting annuli can be considered as invariant sets under the vector field that are attracting for nearby points. Under conjecture 0.1 a stronger statement can be made about annuli in vector fields.

Lemma 1.10. *Under conjecture 0.1, a vector field may have at most one attracting annulus. The annulus is an attracting limit cycle.*

PROOF. An attracting annulus is formed from a pair $(C_{-\infty}, C_\infty)$ of limit cycles. By the conjecture it follows that $C_{-\infty} = C_\infty$. The annulus is an attracting limit cycle. If there were more than one attracting annulus then there would be more than one attracting limit cycle, contradicting the conjecture. Thus, there is at most one attracting annulus, the annulus being an attracting cycle. ∎

Corollary 1.11. *Under conjecture 0.1, the vector field ξ may have at most one repelling annulus. The annulus is a repelling limit cycle.*

PROOF. Consider the vector field under reverse time. There exists at most one attracting annulus which also happens to be an attracting limit cycle. Under forward time the vector field has at most one repelling annulus which also happens to be a repelling limit cycle. ∎

Lemma 1.12. *Under conjecture 0.1, the vector field ξ does not admit a semi-stable annulus.*

PROOF. A semi-stable annulus is formed from a pair $(C_{-\infty}, C_{\infty})$ of limit cycles. By the conjecture it follows that $C_{-\infty} = C_{\infty}$. The annulus is a semi-stable limit cycle. This contradicts the conjecture that the only limit cycles are either attracting or repelling cycles, thus semi-stable annuli do not exist. ∎

§2. Simplifying assumptions for vector fields.

As the vector field ξ requires six defining constants, it would be desirable to constrain as many of the constants as possible to reduce the number of cases to consider.

Proposition 2.1. *Let ξ_1, ξ_2 be vector fields with defining constants a, b, c, d, k, l and $-a, -b, -c, -d, -k, -l$ respectively. Then the respective solutions $\phi_1(t,(x_0,y_0))$ and $\phi_2(t,(x_0,y_0))$ are related by $\phi_1(t,(x_0,y_0)) = \phi_2(-t,(x_0,y_0))$.*

PROOF. Writing $\phi_1(t,(x_0,y_0)) = (x_1(t), y_1(t))$, and $\phi_2(t,(x_0,y_0)) = (x_2(t), y_2(t))$ then

(*i*) :For $x \leq 1$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}.$$

By uniqueness of solutions, $\phi_1(t,(x_0,y_0)) = \phi_2(-t,(x_0,y_0))$.

(*ii*) :For $1 < x$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} -a-k & -b \\ -c-l & -d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} -k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}' = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions, $\phi_1(t,(x_0,y_0)) = \phi_2(-t,(x_0,y_0))$. ∎

Proposition 2.2. *Let ξ_1, ξ_2 be vector fields with defining constants a, b, c, d, k, l and a, $-b$, $-c$, d, k, $-l$ respectively. The respective solutions $\phi_1(t,(x_0,y_0)) = (x_1(t), y_1(t))$, $\phi_2(t,(x_0,y_0)) = (x_2(t), y_2(t))$ are related by $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$.*

PROOF. (*i*) :For $x \leq 1$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}.$$

By uniqueness of solutions $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$.

(ii) :For $1 < x$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$. ∎

Using proposition 2.1, one simplifying assumption that can be taken is that $0 \le a + d$. By proposition 2.2, it allows the extra lattitude of assuming that $0 \le b$. Proposition 2.2 does not affect the value of $a + d$. With these simplifying results, the constants in the vector fields can be taken to satisfy the conditions that $0 \le a + d$ and $0 \le b$.

Our analysis will proceed by fixing the four constants a, b, c, d and allowing the constants k, l to change. This reduces the number of degrees of freedom from six to two.

By definition, the vector field ξ always has a fixed point at the origin. The linear vector field associated with the extension of the vector field on the region $\{(x,y) : 1 < x\}$ to the whole plane, may or may not (if the matrix is singular) have a fixed point. If this linear extension has a fixed point then it is called an <u>induced fixed point</u>. A subsequent result will show an equivalence between values of k, l for which $ad - bc + dk - bl \ne 0$ and induced fixed points (x, y) for which $x \ne 1$. Moreover, the induced fixed point (x, y) is said to be <u>virtual</u> if and only if $x < 1$. More explicitly, we have:

Definition 2.3. Let ξ be a vector field with $ad - bc + dk - bl \ne 0$. The point (x, y) is called the <u>induced fixed point</u> of ξ if and only if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

Lemma 2.4. *Let $a, b, c, d, ad - bc \ne 0$ be given. Let k, l be such that $ad - bc + dk - bl \ne 0$, then for the vector field with defining constants a, b, c, d, k, l there is a unique induced fixed point $(x, y), x \ne 1$.*

Proof. As $ad - bc + dk - bl \ne 0$, there is a unique solution (x, y) given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(a+k)d - b(c+l)} \begin{bmatrix} d & -b \\ -(c+l) & a+k \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc + dk - bl} \begin{bmatrix} dk - bl \\ -ck + al \end{bmatrix}.$$

Furthermore, since $ad - bc \ne 0$, $x \ne 1$. ∎

Lemma 2.5. *Let $a, b, c, d, ad - bc \neq 0$ be given. Let $(x, y), x \neq 1$ be given. Then there exists $k, l, ad - bc + dk - bl \neq 0$ such that the vector field with defining constants a, b, c, d, k, l has (x, y) as the induced fixed point.*

PROOF. Assume (x, y) is a solution to the problem

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} kx \\ lx \end{bmatrix}.$$

From which it follows that

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{1}{1-x} \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

The vector field with defining constants a, b, c, d, k, l will have (x, y) as the induced fixed point. Furthermore, $ad - bc + dk - bl = (ad - bc)/(1 - x) \neq 0$ as claimed. ∎

Theorem 2.6. *For fixed $a, b, c, d, ad - bc \neq 0$ there exists a homeomorphism $h(k, l) = (x, y)$ from the set of parameter values k, l satisfying $ad - bc + dk - bl \neq 0$ to the set of induced fixed points (x, y) satisfying $x \neq 1$.*

PROOF. By lemma 2.4 and lemma 2.5, the function

$$h \begin{bmatrix} k \\ l \end{bmatrix} = \frac{1}{ad - bc + dk - bl} \begin{bmatrix} dk - bl \\ -ck + al \end{bmatrix}$$

with inverse given by

$$h^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1-x} \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

is the desired homeomorphism. ∎

The vector field ξ is linear to the right of the line $x \equiv 1$. The extension of this linear portion of the vector field to the whole plane is the linear vector field

$$\mathbf{L} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

By equivalence of matrices the eigenvalues of the matrix defining the linear vector field also determines the dynamics of ξ to the right of the line $x \equiv 1$. The eigenvalues of the matrix will be referred to as the eigenvalues at the induced fixed point, thus leading to the following definition and the useful corollary that follows.

Definition 2.7. The eigenvalues at the induced fixed point are the eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

Corollary 2.8. *Let ξ be a vector field with $ad - bc \neq 0, ad - bc + dk - bl \neq 0$. The product of the eigenvalues at the induced fixed point is $(ad - bc)/(1 - x)$.*

Proof. The product of the eigenvalues at the induced fixed point is given by the determinant of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}$$

which has the value $ad - bc + dk - bl$. Using the values $k = (ax + by)/(1 - x), l = (cx + dy)/(1 - x)$, the determinant becomes $(ad - bc)/(1 - x)$. ∎

The line $x \equiv 1$ as the boundary between two linear regions also has significance for the vectors that lie along the line. The following proposition outlines this significance.

Proposition 2.9. *Let ξ be a vector field with $0 < b$.. The line defined by $x \equiv 1$ is transversal to ξ at all points except $(1, y^*) = (1, -a/b)$. Moreover, the vector field points to the left for points $(1, y)$ with $y < y^*$ and to the right for points $(1, y)$ with $y^* < y$.*

Proof. For points $(1, y)$ along the line $x \equiv 1$ the value of the vector field is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} a + by \\ c + dy \end{bmatrix}.$$

If ξ is not transverse at $(1, y^*)$ then $a + by^* = 0$, i.e. at the point given by $(1, -a/b)$. For values of $y < y^*$ the x-ordinate of the vector at the point $(1, y)$ is $a + by < a + by^* = 0$ and so the vector points to the left. Similarly for values of $y^* < y$ the vector at $(1, y)$ points to the right. ∎

Because vectors along $x \equiv 1$ above the point $(1, -a/b)$ point to the right and those below point towards the left, it is then possible to define a return map for ξ along the line $x \equiv 1$.

Definition 2.10. For $v < w$, the following notation will be used,

$$L(v, w) = \{y : v < y < w\},$$

$$L(v, w] = \{y : v < y \leq w\},$$

$$L[v, w) = \{y : v \leq y < w\},$$

$$L[v, w] = \{y : v \leq y \leq w\}.$$

Definition 2.11. For the vector field ξ and solution $\phi(t, (x_0, y_0)), \pi(y)$ will denote the return map from $x \equiv 1$ to itself where

$$(1, \pi(y_0)) = \phi(t_0, (1, y_0)), t_0 = \min\{t : 0 < t, \phi(t, (1, y_0)) \cap \{(x, y) : x = 1\} \neq \emptyset\},$$

whenever it is defined. Note that the return map is continuous, in particular this implies that line segments are mapped into line segments if the end-points exists under π.

As a matter of nomenclature, in the text it will often be spoken of points in one or another of the sets L. For example, the reference of a point z in the set $L[v,w]$ will refer to the identification of z as a value and $(1,z)$ as a point in the plane.

Having defined the return map, it is then possible to give a sufficient condition for the existence of attracting annuli.

Theorem 2.12. *(Figure 5.) Let $v < w \le y^*$. If $\pi^2(L[v,w]) \subset L[v,w]$ then there exists a locally attracting annulus attracting for all points in $L[v,w]$.*

PROOF. Let $v_i = (\pi^2)^i(v), w_i = (\pi^2)^i(w)$ for $1 \le i$ be the successive images of v, w under two iterations of the return map π^2. The points v_i form a monotonically increasing sequence that is bounded above by w. Let v_0 be the limit of v_i. By continuity of π^2,

$$\begin{aligned}\pi^2(v_0) &= \pi^2(\lim_{i\to\infty} v_i)\\ &= \lim_{i\to\infty} \pi^2(v_i)\\ &= v_0.\end{aligned}$$

Thus, through the point $(1, v_0)$ lies a cycle. Similarly, through w_0, being the limit of w_i, lies a cycle. Clearly, $v_0 \le w_0$. Consider the annulus A formed by the cycles through $(1, v_0)$ and $(1, w_0)$.

Let N be the flow of the line segment $L[v,w]$ to $L[\pi^2(v), \pi^2(w)]$. Then $A \subset N$. Also, as $t \to \infty$ the sets $\phi(t, N)$ form a decreasing sequence of sets with $A \subset \phi(t_1, N) \subset \phi(t_0, N)$ for $0 \le t_0 < t_1$. By construction $A = \cap_{t=0}^{\infty}\phi(t, N)$. Thus the annulus A is attracting.

As $L[v,w] \subset N$, the annulus is attracting for all points in $L[v,w]$. ∎

§3. $0 < b$, $0 < a + d$, $(a+d)^2/4 < ad - bc$, $ad - bc + dk - bl \neq 0$.

In this sections, since $ad - bc + dk - bl \neq 0$, then by theorem 2.6 the values of k, l and the induced fixed point can be used as interchangeable concepts.

The first result will be an application of Stoke's theorem to show that limit cycles do not exist in a certain region.

Lemma 3.1. *Let ξ be a vector field with $0 \le a + k + d$ then cycles may not intersect the line $x \equiv 1$.*

Proof. If such cycle exists then it must intersect the line $x \equiv 1$ transversally at some points $(1, y_1), (1, y_2)$ with $y_1 < y^* < y_2$. A cycle will then join the points $(1, y_2), (1, y_1)$ in a clockwise orientation. Let C denote the cycle. Then by Stoke's theorem,

$$\oint_C \frac{dx}{dt}dy - \frac{dy}{dt}dx = \int_{\text{int}(C)} \left[\frac{d}{dx}\left(\frac{dx}{dt}\right) + \frac{d}{dy}\left(\frac{dy}{dt}\right)\right] dx\ dy.$$

Breaking up the area integral into two parts, $A = \text{int}(C) \cap \{(x,y) : x \le 1\}$ and $B = \text{int}(C) \cap \{(x,y) : 1 < x\}$, then

$$0 = \int_A \left[\frac{d}{dx}(ax + by) + \frac{d}{dy}(cx + dy)\right] dx\ dy +$$
$$\int_B \left[\frac{d}{dx}((a+k)x + by - k) + \frac{d}{dy}((c+l)x + dy - l)\right] dx\ dy.$$

Thus,

$$0 = \int_A (a+d) dx\ dy + \int_B (a + k + d) dx\ dy.$$

As $0 < a + d, 0 \le a + k + d$ the integral on the right is nonzero. By contradiction, such cycles do not exist. ∎

Theorem 3.2. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ and*

$$\frac{1}{b}(-d(1-x) - a) \le y$$

then there are no cycles.

Proof. If a cycle existed then it must contain a fixed point. The only fixed point is the origin. As the cycle contains the origin in its interior then the cycle lies wholly in the region $\{(x,y) : x \le 1\}$ or intersects the line $x \equiv 1$. As $0 < a + d$ cycles may not lie wholly in the region $\{(x,y) : x \le 1\}$ for which ξ is strictly linear. By lemma 3.1 it is sufficient for values of k corresponding to the point (x, y) to satisfy $0 \le a + k + d$ to prove that there are no cycles. This becomes the requirement that

$$\frac{ax + by}{1 - x} = k \ge -(a + d).$$

Thus, and remembering that $x < 1$,

$$ax + by \ge ax + dx - (a + d).$$

Or,

$$y \geq \frac{-d(1-x)-a}{b}. \qquad \blacksquare$$

Some results will be now shown which determine the types of eigenvalues possible for the induced fixed point and conditions for their occurrence. This will give rise to various cases for determining some regions for which it can be said whether or not attractive cycles may exist.

Lemma 3.3. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ. Then it has complex eigenvalues* $\Leftrightarrow$

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a) < y < \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a).$$

PROOF. The characteristic equation for the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix},$$

determines the eigenvalues at the induced fixed point. This equation is given by $\lambda^2-(a+k+d)\lambda+[(a+k)d-(c+l)b]=0$. The eigenvalues are complex if and only if the discriminant is negative, or in other words when $(a+k+d)^2-4[(a+k)d-(c+l)b]<0$.Using the values $k=(ax+by)/(1-x), l=(cx+dy)/(1-x)$ the equation becomes $(a+by+d(1-x))^2<4(1-x)(ad-bc)$. Thus,

$$-2\sqrt{1-x}\sqrt{ad-bc} < a+by+d(1-x) < 2\sqrt{1-x}\sqrt{ad-bc},$$

or,

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a) < y < \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a). \qquad \blacksquare$$

If the induced fixed point does not have complex eigenvalues, then the eigenvectors corresponding to the real eigenvalues induce linear invariant manifolds. If the linear manifolds intersect the line $x \equiv 1$, more can be said about the points of intersection.

Lemma 3.4. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ. Then*

$$y \in \{\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a), \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc}-d(1-x)-a)\} \Rightarrow$$

there exists a point $(1, v)$ for which the line through the state vector at that point passes through (x, y).

PROOF. When y attains these values, the characteristic equation has a single solution for the eigenvalues at the induced fixed point. The eigenspace corresponding to the eigenvalue may or may not have dimension two. Nonetheless, in either case there is an eigenvector. The line through the induced fixed point, passing through the direction of the eigenvector is an invariant manifold for the linear vector field. If it can be shown that the invariant manifold must intersect the line $x \equiv 1$ then the point of intersection is the desired $(1, v)$.

Assume the invariant manifold does not intersect the line $x \equiv 1$, then the line has the form $x \equiv K$ for some constant K. As the line is invariant for the linear vector field, the x-ordinate of the vectors at points on the line is always 0. Thus $0 = (a+k)K + by - k$ is to hold for all values of y.As $0 < b$ the equality cannot hold independent of y, thus the invariant manifold must intersect the line $x \equiv 1$ at some point $(1, v)$. ∎

Lemma 3.5. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ. Then*

$$y \notin \{y : \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)\} \Rightarrow$$

there exist points $(1, v_1), (1, v_2)$ for which the line through the vector at those points pass through (x, y).

PROOF. By lemma 3.3, for these values of y the characteristic equation has a pair of distinct solutions. For each eigenvalue there is also a distinct eigenvector. It can be argued that for each of the eigenvectors there is a unique point of intersection with the line $x \equiv 1$.As the two invariant manifolds, one for each corresponding eigenvectors, are not collinear the points $(1, v_1), (1, v_2)$ are distinct. ∎

Proposition 3.6. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ and*

$$y \leq \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

then the points of intersection as given in lemma 3.4 and lemma 3.5 have y-ordinate less than y^* where y^* is defined in proposition 2.9.

PROOF. By corollary 2.8 the product of the eigenvalues is given by $(ad-bc)/(1-x)$. For $x < 1$ this is positive so that the eigenvalues at (x, y) have the same sign. For the given values of y it is also true that $y < (-d(1-x) - a)/b$. Thus y is outside of the region $0 \leq a + k + d$. Thus the trace satisfies $a + k + d < 0$, combined with the note on the product of the eigenvalues, the sign of the eigenvalues are both negative. Solving

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \lambda \begin{bmatrix} 1-x \\ v-y \end{bmatrix},$$

for v, it follows that $a + bv = \lambda(1-x) < 0$. Thus, it follows that $v < y^*$ for each of the point $(1, v)$ of intersection in the lemmas. ∎

With these results, the following theorem shows a sufficient condition for the existence of attracting cycles.

Theorem 3.7. *(Figure 6.) Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ with*

$$y \leq \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

then ξ has a globally attracting annulus in $\Re^2 - \{(0,0)\}$. Moreover, assuming conjecture 0.1 holds, then ξ has a globally attracting limit cycle.

PROOF. By lemma 3.4 and lemma 3.5 the induced fixed point (x, y) also induces either one or two points along the line $x \equiv 1$ for which the vector field passes in the direction of a line containing (x, y). If there are two such points $(1, v_1), (1, v_2)$ then consider the point $(1, v)$ where $v = \max\{v_1, v_2\}$. By proposition 3.6, $v < y^*$.

As $(a+d)^2 < 4(ad - bc)$ the vector field for $x \leq 1$ has non-zero rotational speed. Thus let $y^{**} = \pi(y)$. Note also that $\pi : L(-\infty, y^*] \to L[y^{**}, \infty)$ is well-defined.

As the eigenvalues of the induced fixed point are both negative the vector field for $1 < x$ will be attractive towards the induced fixed point. Thus, the points of $L(y^*, \infty)$ will eventually, under the vector field , intersect the line $L(-\infty, y^*]$. Thus, $\pi : L(y^*, \infty) \to L(-\infty, y^*]$ is well-defined.

Consider now the line $L[v, y^*]$ under two iterations of the return map. Thus,

$$\pi^2(L[v, y^*]) = \pi(L[\pi(y^*), \pi(v)]) = L[\pi^2(v), \pi^2(y^*)].$$

Now $\pi^2(y^*) = \pi(y^{**}) < y^*$. Since $\phi(t, (1, y^{**}))$ is to re-enter $x \leq 1$ the x-ordinate of the vector at intersection with the line $x \equiv 1$ must be less than or equal to zero. Thus $\pi(y^{**}) \leq y^*$. If $\pi(y^{**}) = y^*$ then y^* has, as under reverse time, a pre-image under π. This is not possible as under reverse time $\phi(t, (1, y^*))$ has eigenvalues whose real parts are negative, so that the distance form the origin to any point of $\phi(t, (1, y^*))$ has length less than one and cannot intersect $x \equiv 1$. Thus $\pi^2(y^*) < y^*$.

Also $v < \pi^2(v)$. As the line through $(1, v)$ is an invariant manifold for the induced fixed point, $\pi^2(v)$ cannot cross this line, thus it follows that it must intersect $x \equiv 1$ at some point above v, i.e. $v < \pi^2(v)$.

Thus, by the two previous paragraphs $\pi^2(L[v, y^*]) \subset L[v, y^*]$. It follows by theorem 2.12 that ξ has an attracting annulus A which is attracting for all points in $L[v, y^*]$.

Consider points in $L(y^*, \infty)$, after one application of the return map $\pi(L(y^*, \infty)) \subset L[v, y^*]$. So all points in $L(y^*, \infty)$ are also attracted to A. Points in $L(-\infty, v)$ iterate to $L(\pi(v), \infty) \subset L(y^*, \infty)$, so that all these points also iterate to A.

Let (x_0, y_0) be any point in the plane $\Re^2 - \{(0,0)\}$ for which $x \neq 1$. If $x < 1$ then since the eigenvalues of ξ in the region $x \leq 1$ both have positive real parts, then under finite time $0 < t_0, \phi(t_0, (x_0, y_0)) \cap \{(x, y) : x = 1\} \neq \emptyset$. If $1 < x$ then since the eigenvalues of the induced fixed point determines the vector field, and being both negative, there is again some finite time value for which $0 < t_0$ and $\phi(t_0, (x_0, y_0)) \cap \{(x, y) : x = 1\} \neq \emptyset$. Thus, the attractive annulus is globally attracting in $\Re^2 - \{(0,0)\}$. By lemma 1.10 the attracting annulus is an attracting limit cycle. ∎

The next two lemmas present a qualitative result for induced fixed points which are also foci.

Lemma 3.8. *The general solution to the differential equation*

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

when $(a+k+d)^2 - 4(ad-bc+dk-bl) < 0$ *is given by*

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$$

$$e^{\lambda t}\begin{bmatrix} X_0\cos(\omega t) - \frac{1}{\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) \\ Y_0\cos(\omega t) + \frac{1}{\omega}[(c+l)X_0 + (\lambda-(a+k))Y_0]\sin(\omega t) \end{bmatrix} + \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix},$$

where

$$\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix},$$

and $\lambda = (a+k+d)/2, \omega = \sqrt{4(ad-bc+dk-bl)-(a+k+d)^2}/2.$

Proof. Consider the following substitution given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} + \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix}.$$

Using this substitution the above differential equation becomes

$$\begin{bmatrix} X \\ Y \end{bmatrix}' = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}, \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -cl+al \end{bmatrix}.$$

The eigenvalues of the matrix given by

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}$$

are $\lambda \pm i\omega$ where $\lambda = (a+k+d)/2, \omega = \sqrt{4(ad-bc+dl-bl)-(a+k+d)^2}/2$. Eigenvectors corresponding to the eigenvalues are given by

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} 1 \\ \frac{\lambda\pm i\omega-(a+k)}{b} \end{bmatrix} = (\lambda \pm i\omega)\begin{bmatrix} 1 \\ \frac{\lambda\pm i\omega-(a+k)}{b} \end{bmatrix}.$$

Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\lambda+i\omega-(a+k)}{b} & \frac{\lambda-i\omega-(a+k)}{b} \end{bmatrix}\begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix}.$$

On substitution the problem becomes

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix}' = \begin{bmatrix} \lambda+i\omega & 0 \\ 0 & \lambda-i\omega \end{bmatrix}\begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix}, \begin{bmatrix} \overline{X}(0) \\ \overline{Y}(0) \end{bmatrix} = \begin{bmatrix} \frac{X_0}{2} + i\frac{1}{2\omega}[(\lambda-(a+k))X_0 - bY_0] \\ \frac{X_0}{2} - i\frac{1}{2\omega}[(\lambda-(a+k))X_0 - bY_0] \end{bmatrix}.$$

The solution

$$\begin{bmatrix} \overline{X}(t) \\ \overline{Y}(t) \end{bmatrix} = e^{\lambda t}\begin{bmatrix} \frac{X_0}{2}\cos(\omega t) - \frac{1}{2\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) + \\ \frac{X_0}{2}\cos(\omega t) - \frac{1}{2\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) - \end{bmatrix}$$

$$\begin{matrix} i[\frac{X_0}{2}\sin(\omega t) + \frac{1}{2\omega}[(\lambda-(a+k))X_0 - Y_0]\cos(\omega t)] \\ i[\frac{X_0}{2}\sin(\omega t) + \frac{1}{2\omega}[(\lambda-(a+k))X_0 - Y_0]\cos(\omega t)] \end{matrix}\Bigg]$$

implies that

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} X_0\cos(\omega t) - \frac{1}{\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) \\ Y_0\cos(\omega t) + \frac{1}{\omega}[(c+l)X_0 + (\lambda-(a+k))Y_0]\sin(\omega t) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$$

$$e^{\lambda t} \begin{bmatrix} X_0\cos(\omega t) - \frac{1}{\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) \\ Y_0\cos(\omega t) + \frac{1}{\omega}[(c+l)X_0 + (\lambda-(a+k))Y_0]\sin(\omega t) \end{bmatrix} + \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix}. \quad \blacksquare$$

Lemma 3.9. *Let $(x_i, y_i), x_i < 1$ be the induced virtual fixed point of the vector field ξ for which*

$$\frac{1}{b}(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a) < y_i < \frac{1}{b}(2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a).$$

Then the solution $\phi(t,(x_0,y_0))$ through the point (x_0,y_0) satisfies

$$\phi(\pi/\omega,(x_0,y_0)) = -e^{\frac{\lambda\pi}{\omega}}((x_0,y_0)-(x_i,y_i)) + (x_i,y_i)$$

where

$$\lambda = \frac{a+by_i+d(1-x_i)}{2(1-x_i)}, \omega = \frac{\sqrt{4(ad-bc)(1-x_i)-(a+by_i+d(1-x_i))^2}}{2(1-x_i)}.$$

Proof. By lemma 3.3 when $(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a)/b < y_i < (2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i)-a)/b$ the eigenvalues of the the induced fixed point are complex. Then, by the considering the characteristic equation of the matrix,

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix},$$

$(a+k+d)^2 - 4(ad-bc+dk-bl) < 0$ where k,l are given by $k = (ax_i+by_i)/(1-x_i), l = (cx_i+dy_i)/(1-x_i)$. Using lemma 3.8, the solution through the point (x_0,y_0) is then given by the function

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$$

$$e^{\lambda t} \begin{bmatrix} X_0\cos(\omega t) - \frac{1}{\omega}[(\lambda-(a+k))X_0 - bY_0]\sin(\omega t) \\ Y_0\cos(\omega t) + \frac{1}{\omega}[(c+l)X_0 + (\lambda-(a+k))Y_0]\sin(\omega t) \end{bmatrix} + \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix},$$

where

$$\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix},$$

and $\lambda = (a+k+d)/2, \omega = \sqrt{4(ad-bc+dk-bl)-(a+k+d)^2}/2$. Remembering by theorem 2.6 that

$$\frac{1}{ad-bc+dk-bl}\begin{bmatrix} dk-bl \\ -ck+al \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

the above becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} X_0 \cos(\omega t) - \frac{1}{\omega}[(\lambda - (a+k))X_0 - bY_0] \sin(\omega t) \\ Y_0 \cos(\omega t) + \frac{1}{\omega}[(c+l)X_0 + (\lambda - (a+k))Y_0] \sin(\omega t) \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

Writing $\phi(t, (x_0, y_0)) = (x(t), y(t))$ then computing for $t = \pi/\omega$ gives

$$\phi(\pi/\omega, (x_0, y_0)) = -e^{\frac{\lambda\pi}{\omega}}((x_0, y_0) - (x_i, y_i)) + (x_i, y_i).$$ ∎

The following results will culminate in another theorem that will augment the region of theorem 3.7 for which attracting limit cycles exist.

Lemma 3.10. *(Figure 7.) Let $\lambda \pm i\omega$ be the complex eigenvalues of the matrix given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

There exist $v_0 < y^*$ and K_0 such that for $v \leq v_0$ the return map $\pi : L(-\infty, y^*] \to L(y^*, \infty)$ satisfies

$$\pi(v) < -e^{\frac{\lambda\pi}{\omega}} v + K_0.$$

PROOF. Consider the point $(1, v)$ with $v < y^*$. Now consider the solution $\phi(t, (1, v))$ and the line that passes through this point in the direction of the vector at the point. The equation of this line through $(1, v)$ is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ v \end{bmatrix} + t \begin{bmatrix} a + bv \\ c + dv \end{bmatrix}.$$

Noticing that the solution through $(1, v)$ is tangent to the line through that point and lies above the line, then at the value $0 < t_0$ for which $\phi(t, (1, v)) = (\phi_1(t, (1, v)), \phi_2(t, (1, v)))$ first intersects $x \equiv 0$, this point of intersection T lies above the line's point of intersection with $x \equiv 0$ at S. The line above intersects $x \equiv 0$ at the point $(0, v - (c + dv)/(a + bv))$ while the solution ϕ intersects at the point $\phi(t_0, (1, v)) = (0, \phi_2(t_0, (1, v)))$. Thus,

$$v - \frac{c + dv}{a + bv} < \phi_2(t_0, (1, v)).$$

With the origin as the induced fixed point of the vector field, the conditions of lemma 3.9 are satisfied. Under the elapsing of π/ω units of time then

$$\begin{aligned} \phi(t_0 + \pi/\omega, (1, v)) &= -e^{\frac{\lambda\pi}{\omega}}(0, \phi_2(t_0, (1, v))) \\ &= (0, -e^{\frac{\lambda\pi}{\omega}} \phi_2(t_0, (1, v))) \end{aligned}$$

This point V is below the corresponding image U of the point $(0, v - (c + dv)/(a + bv))$. Thus,

$$-e^{\frac{\lambda\pi}{\omega}} \phi_2(t_0, (1, v)) < -e^{\frac{\lambda\pi}{\omega}} (v - \frac{c + dv}{a + bv}).$$

Consider the line tangent to the vector field at the point $(0, -e^{\frac{\lambda\pi}{\omega}}(v-(c+dv)/(a+bv)))$. This line has the equation given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{\frac{\lambda\pi}{\omega}}(v - \frac{c+dv}{a+bv}) \end{bmatrix} + t \begin{bmatrix} b \\ d \end{bmatrix}.$$

This line intersects the line $x \equiv 1$ at the point $(1, -e^{\frac{\lambda\pi}{\omega}}(v-(c+dv)/(a+bv)+d/b)$. This point also lies above the first intersection of the solution through $\phi(t, (0, -e^{\frac{\lambda\pi}{\omega}}\phi_2(t_0, (1, v))))$. Call this point $(1, w)$. It happens that $w = \pi(v)$. Thus, one has that

$$\begin{aligned} \pi(v) &< -e^{\frac{\lambda\pi}{\omega}}\left(v - \frac{c+dv}{a+bv}\right) + \frac{d}{b} \\ &= -e^{\frac{\lambda\pi}{\omega}}v + e^{\frac{\lambda\pi}{\omega}}\left(\frac{c+dv}{a+bv}\right) + \frac{d}{b} \\ &= -e^{\frac{\lambda\pi}{\omega}}v + e^{\frac{\lambda\pi}{\omega}}\left(\frac{d}{b} - \frac{ad-bc}{b(a+bv)}\right) + \frac{d}{b}. \end{aligned}$$

Let $v_0 = y^* - 1/b$. Then for $v \leq v_0, a+bv \leq -1$. Now $(a+d)^2/4 < ad-bc$ so that $0 \leq ad-bc$. Also $0 < b$ so that the value $-(ad-bc)/(b(a+bv))$ is bounded above by $(ad-bc)/b$. Let

$$K_0 = e^{\frac{\lambda\pi}{\omega}}\left(\frac{d}{b} + \frac{ad-bc}{b}\right) + \frac{d}{b}.$$

Then for $v \leq v_0$ one has that

$$\pi(v) < -e^{\frac{\lambda\pi}{\omega}}v + K_0.$$

∎

Lemma 3.11. *(Figure 8.) Let $(x_i, y_i), x_i < 1$ be the induced virtual fixed point of the vector field ξ such that*

$$\frac{1}{b}(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a) < y_i < \frac{1}{b}(2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a).$$

To the fixed point corresponds unique values of k, l. Let $\lambda_1 \pm i\omega_1$ be the complex eigenvalues of the matrix given by

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

There exist $y^* < v_1$ and K_1 such that for $v_1 \leq v$ the return map $\pi : L(y^*, \infty) \to L(-\infty, y^*]$ satisfies

$$-e^{\frac{\lambda_1\pi}{\omega_1}}v + K_1 < \pi(v).$$

Proof. Let $(1, v)$ be a point for which $y^* < v$. Consider a line through the point in the direction of the vector at that point. The line is tangent to the solution $\phi(t, (1, v))$ passing through $(1, v)$. The portion of the connected component of the solution bounded by the lines $x \equiv x_i, x \equiv 1$ lies below the line through $(1, v)$. This line has the equation

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ v \end{bmatrix} + t \begin{bmatrix} a+bv \\ c+dv \end{bmatrix}.$$

The line intersects $x \equiv x_i$ at the point S given by $(x_i, v + (x_i - 1)(c + dv)/(a + bv))$. Let $t_0 < 0$ be the first time that $\phi(t, (1, v))$ intersects the line $x \equiv x_i$, thus $\phi(t_0, (1, v)) = (x_i, \phi_2(t, (1, v)))$. Let this point be T. Then one has that

$$\phi_2(t_0, (1, v)) < v + (x_i - 1)\frac{c + dv}{a + bv}.$$

The considerations of lemma 3.9 concerning fixed points with complex eigenvalues are satisfied. Under the elapsing of π/ω_1 units of time

$$\begin{aligned}\phi(\pi/\omega_1, (x_i, v + (x_i - 1)\frac{c + dv}{a + bv})) &= -e^{\frac{\lambda_1 \pi}{\omega_1}}((x_i, v + (x_i - 1)\frac{c + dv}{a + bv}) - (x_i, y_i)) + (x_i, y_i) \\ &= \left(x_i, -e^{\frac{\lambda_1 \pi}{\omega_1}}\left(v + (x_i - 1)\frac{c + dv}{a + bv} - y_i\right) + y_i\right).\end{aligned}$$

Also,

$$\begin{aligned}\phi(t_0 + \pi/\omega_1, (1, v)) &= -e^{\frac{\lambda_1 \pi}{\omega_1}}((x_i, \phi_2(t_0, (1, v))) - (x_i, y_i)) + (x_i, y_i) \\ &= (x_i, -e^{\frac{\lambda_1 \pi}{\omega_1}}(\phi_2(t_0, (1, v)) - y_i) + y_i).\end{aligned}$$

Using $\phi_2(t_0, (1, v)) < v + (x_i - 1)(c + dv)/(a + bv)$ then

$$-e^{\frac{\lambda_1 \pi}{\omega_1}}\left(v + (x_i - 1)\frac{c + dv}{a + bv} - y_i\right) + y_i < -e^{\frac{\lambda_1 \pi}{\omega_1}}(\phi_2(t_0, (1, v)) - y_i) + y_i.$$

Thus the solution through $\phi(t_0 + \pi/\omega_1, (1, v))$ intersects the line $x \equiv x_i$ at a point V above where $\phi(\pi/\omega_1, (x_i, v + (x_i - 1)(c + dv)/(a + bv)))$ intersects the same line at the point U. Under the linear vector field given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + k & b \\ c + l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

the line through the vector at the point $(x_i, -e^{\frac{\lambda_1 \pi}{\omega_1}}(v + (x_i - 1)(c + dv)/(a + bv) - y_i) + y_i)$ is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_i \\ -e^{\frac{\lambda_1 \pi}{\omega_1}}\left(v + (x_i - 1)\left(\frac{c+dv}{a+bv}\right) - y_i\right) + y_i \end{bmatrix} + t\begin{bmatrix} b \\ d \end{bmatrix}.$$

This line intersects the line $x \equiv 1$ at the point $(1, -e^{\frac{\lambda_1 \pi}{\omega_1}}(v + (x_i - 1)(c + dv)/(a + bv) - y_i) + y_i + (1 - x_i)d/b)$. This point lies below where the solution through $(1, v)$ intersects $x \equiv 1$ for the first positive time. Thus,

$$\begin{aligned}\pi(v) &> -e^{\frac{\lambda_1 \pi}{\omega_1}}\left(v + (x_i - 1)\left(\frac{c + dv}{a + bv}\right) - y_i\right) + y_i + (1 - x_i)\frac{d}{b} \\ &= -e^{\frac{\lambda_1 \pi}{\omega_1}}v - e^{\frac{\lambda_1 \pi}{\omega_1}}\left((x_i - 1)\left(\frac{c + dv}{a + bv}\right) - y_i\right) + y_i + (1 - x_i)\frac{d}{b} \\ &= -e^{\frac{\lambda_1 \pi}{\omega_1}}v - e^{\frac{\lambda_1 \pi}{\omega_1}}\left((x_i - 1)\left(\frac{d}{b} - \frac{ad - bc}{b(a + bv)}\right) - y_i\right) + y_i + (1 - x_i)\frac{d}{b}.\end{aligned}$$

Let $v_1 = y^* + 1/b$. Then for $v_1 \leq v$ it is true that $1 < a + bv$. Then, since $0 < b, 0 < (a+d)^2/4 < ad - bc$, the value $-(ad - bc)/(b(a + bv))$ is bounded below by $-(ad - bc)/b$. Let

$$K_1 = -e^{\frac{\lambda_1 \pi}{\omega_1}}\left((x_i - 1)\left(\frac{d}{b} - \frac{ad - bc}{b}\right) - y_i\right) + y_i + (1 - x_i)\frac{d}{b}.$$

Then for $v_1 \leq v$,

$$-e^{\frac{\lambda_1 \pi}{\omega_1}} v + K_1 < \pi(v). \qquad \blacksquare$$

Theorem 3.12. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ with*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

and $\lambda_1/\omega_1 + \lambda/\omega < 0$.Then there exists $v_2 < y^*$ such that for $v \leq v_2$ the map $\pi^2 : L(-\infty, y^*] \to L(-\infty, y^*]$ satisfies $v < \pi^2(v)$.

Proof. Let v_0, v_1, K_0, K_1 be the constants that are given by lemma 3.10 and lemma 3.11. Let $v' = \min\{v_0, \pi^{-1}(v_1)\}$. As $v_0 < y^*, y^* < v_1$ then $v' < y^*$. For $v \leq v' \leq v_0$ then $\pi(v) < -e^{\frac{\lambda \pi}{\omega}} v + K_0$. As $v_1 \leq \pi(v)$ then $\pi^2(v) > -e^{\frac{\lambda_1 \pi}{\omega_1}} \pi(v) + K_1$. Combining the two previous results gives,

$$\begin{aligned} \pi^2(v) &> e^{\frac{\lambda_1 \pi}{\omega_1}} (e^{\frac{\lambda \pi}{\omega}} v - K_0) + K_1 \\ &= e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} v - e^{\frac{\lambda_1 \pi}{\omega_1}} K_0 + K_1. \end{aligned}$$

If $\pi^2(v) > v$ then it is sufficient that

$$e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} v - e^{\frac{\lambda_1 \pi}{\omega_1}} K_0 + K_1 \geq v.$$

For $\lambda_1/\omega_1 + \lambda/\omega < 0$, then

$$K_1 - e^{\frac{\lambda_1 \pi}{\omega_1}} K_0 \geq v(1 - e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi}).$$

Thus,

$$\frac{K_1 - e^{\frac{\lambda_1 \pi}{\omega_1}} K_0}{1 - e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi}} \geq v.$$

Let $v'' = (K_1 - e^{\frac{\lambda_1 \pi}{\omega_1}} K_0)/(1 - e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi})$. Finally define $v_2 = \min\{v', v''\}$. Then for $v \leq v_2$ it happens that $v < \pi^2(v)$. $\blacksquare$

Theorem 3.13. *(Figure 9.) Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ. If*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(-\sqrt{1-x}(a+d) - d(1-x) - a)$$

then there is a globally attracting annulus in $\Re^2 - \{(0,0)\}$. Moreover, assuming conjecture 0.1 holds, then ξ has a globally attracting limit cycle.

Proof. Consider the vector field given by

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

for k, l corresponding to the the induced fixed point (x, y). The eigenvalues $\lambda \pm i\omega, \lambda_1 \pm i\omega_1$, being the complex eigenvalues of the origin and at the induced fixed point respectively, are given by

$$\begin{aligned}
\lambda_1 &= \frac{a+k+d}{2} \\
\omega_1 &= \frac{\sqrt{4(ad-bc+dk-bl)-(a+k+d)^2}}{2} \\
\lambda &= \frac{a+d}{2} \\
\omega &= \frac{\sqrt{4(ad-bc)-(a+d)^2}}{2}.
\end{aligned}$$

With $k = (ax+by)/(1-x), l = (cx+dy)/(1-x)$ the equations for λ_1, ω_1 reduce to

$$\begin{aligned}
\lambda_1 &= \frac{a+by+d(1-x)}{2(1-x)} \\
\omega_1 &= \frac{\sqrt{4(ad-bc)(1-x)-(a+by+d(1-x))^2}}{2(1-x)}.
\end{aligned}$$

Then

$$\frac{\lambda_1}{\omega_1} = \frac{a+by+d(1-x)}{\sqrt{4(ad-bc)(1-x)-(a+by+d(1-x))^2}}, \quad \frac{\lambda}{\omega} = \frac{a+d}{\sqrt{4(ad-bc)-(a+d)^2}}.$$

As $y < (-\sqrt{1-x}(a+d) - d(1-x) - a)/b$ then $a + by + d(1-x) < -\sqrt{1-x}(a+d) < 0$. Then the implications hold,

$$\begin{aligned}
& (a+d)^2(1-x) < (a+by+d(1-x))^2 \\
\Rightarrow \quad & \frac{1}{(a+by+d(1-x))^2} < \frac{1}{(a+d)^2(1-x)} \\
\Rightarrow \quad & \frac{4(ad-bc)(1-x)}{(a+by+d(1-x))^2} < \frac{4(ad-bc)}{(a+d)^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{4(ad-bc)(1-x)-(a+by+d(1-x))^2}{(a+by+d(1-x))^2} < \frac{4(ad-bc)-(a+d)^2}{(a+d)^2} \\
\Rightarrow \quad & \frac{(a+d)^2}{4(ad-bc)-(a+d)^2} < \frac{(a+by+d(1-x))^2}{4(ad-bc)(1-x)-(a+by+d(1-x))^2} \\
\Rightarrow \quad & \frac{a+d}{\sqrt{4(ad-bc)-(a+d)^2}} < \frac{-(a+by+d(1-x))^2}{\sqrt{4(ad-bc)-(a+by+d(1-x))^2}} \\
\Rightarrow \quad & \frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega} < 0.
\end{aligned}$$

By theorem 3.12 it is possible to choose v_0 such that for $v \leq v_0, v < \pi^2(v)$. Consider the line segment $L[v_0, y^*]$. Then $\pi^2(L[v_0, y^*]) = L[\pi^2(v_0), \pi^2(y^*)] \subset L[v_0, y^*]$. Thus, by theorem 2.12 there is an attracting annulus A attracting for all points in in $L[v_0, y^*]$. Say $v < v_0$, again consider the line segment $L[v, y^*]$. Then again $\pi^2(L[v, y^*]) = L[\pi^2(v), \pi^2(y^*)] \subset L[v, y^*]$, and by theorem 2.12 there is an attracting annulus A' attracting for points in $L[v, y^*]$. The annulus A may be characterised by its boundary cycles C_1, C_2. The boundary cycles intersect the line $x \equiv 1$ at the points $(1, v_1), (1, v_2)$ where $v_2 \leq v_1 \leq y^*$. By construction in theorem 2.12 the point $(1, y^*)$ approaches the point $(1, v_1)$ and the point $(1, v)$ approaches the point $(1, v_2)$, both limits under iteration of π^2. Similarly, the annulus A' may also be characterised by its boundary cycles, the boundary cycles intersecting the line $x \equiv 1$ at the points $(1, v_1'), (1, v_2')$ where $v_2' \leq v_1' \leq y^*$. As before, the point $(1, v_1')$ is the limit for the point $(1, y^*)$, as is the point $(1, v_2')$ the limit for the point $(1, v)$.

The point $(1, y^*)$ can have only one limit under restriction to the line $x \equiv 1$. Thus $(1, v_1) = (1, v_1')$, from which it follows that $v_1 = v_1'$. If the point $(1, v_2')$ is to lie on a cycle then $v_2' \in L(v_0, y^*]$. But now the point $(1, v_0)$ has both $(1, v_2)$ and $(1, v_2')$ as limits. Thus $v_2 = v_2'$. Thus means that the annulus A' is formed by the same boundary cycles as the annulus A, the two annuli are identical. Thus, the annulus A is attracting for all points along $L(-\infty, y^*]$. As $\pi(L(y^*, \infty)) \subset L(-\infty, y^*]$, the annulus is attracting for $x \equiv 1$.

Let $(x_0, y_0) \in \Re^2 - \{(0,0)\}$. Then there is some $0 \leq t_0 < \infty$ for which $\phi(t_0, (x_0, y_0))$ intersects the line $x \equiv 1$. Thus, the point $\phi(t_0, (x_0, y_0))$ iterates to the annulus A, as so does the original point (x_0, y_0). By lemma 1.10 the attracting annulus A is an attracting cycle. Thus, there exists a globally attracting limit cycle in $\Re^2 - \{(0,0)\}$. ∎

The following four results end in a theorem claiming a region in which cycles do not exist. The first four lemmas are approximation results for the return map. With these results, the subsequent theorems prove the nonexistence of cycles.

Lemma 3.14. *Let $\phi(t, (x_0, y_0))$ be the solution through (x_0, y_0) for the differential equation*

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Let the eigenvalues of the origin be $\lambda \pm i\omega$. For $0 < t$ the point $\phi(t, (x_0, y_0))$ lies outside of the ellipse given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \cos(\omega t) - \frac{1}{\omega}[(\lambda - a)x_0 - by_0] \sin(\omega t) \\ y_0 \cos(\omega t) + \frac{1}{\omega}[cx_0 + (\lambda - a)y_0] \sin(\omega t) \end{bmatrix}.$$

Proof. The solution throught the point (x_0, y_0) is given by

$$\phi(t, (x_0, y_0)) = e^{\lambda t} \begin{bmatrix} x_0 \cos(\omega t) - \frac{1}{\omega}[(\lambda - a)x_0 - by_0] \sin(\omega t) \\ y_0 \cos(\omega t) + \frac{1}{\omega}[cx_0 + (\lambda - a)y_0] \sin(\omega t) \end{bmatrix}.$$

As $\lambda = (a + d)/2 > 0$, then for $0 < t$ the vector $\phi(t, (x_0, y_0))$ has a longer length while still being in the same direction as the vector

$$\begin{bmatrix} x_0 \cos(\omega t) - \frac{1}{\omega}[(\lambda - a)x_0 - by_0] \sin(\omega t) \\ y_0 \cos(\omega t) + \frac{1}{\omega}[cx_0 + (\lambda - a)y_0] \sin(\omega t) \end{bmatrix}.$$

Thus, for $0 < t$ the point $\phi(t,(x_0,y_0))$ lies outside of the ellipse given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0\cos(\omega t) - \frac{1}{\omega}[(\lambda-a)x_0 - by_0]\sin(\omega t) \\ y_0\cos(\omega t) + \frac{1}{\omega}[cx_0 + (\lambda-a)y_0]\sin(\omega t) \end{bmatrix}.$$

∎

Lemma 3.15. *Let $\lambda \pm i\omega$ be the complex eigenvalues of the matrix given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

There exists $v_0 < y^*$ and K_0 such that for $v \le v_0$ the return map $\pi : L(-\infty, y^*] \to L(y^*, \infty)$ satisfies

$$-e^{\frac{\lambda\pi}{\omega}}v + K_0 < \pi(v).$$

Proof. Let $v < y^*$ and consider the point $(1,v)$. By lemma 3.14 the solution $\phi(t,(1,v))$ for $0 < t$ lies outside of the ellipse given by the equations

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) - \frac{1}{\omega}[\lambda - a - bv]\sin(\omega t) \\ v\cos(\omega t) + \frac{1}{\omega}[c + (\lambda-a)v]\sin(\omega t) \end{bmatrix}.$$

Let $0 < t_0$ be the first time for which $\phi(t_0,(1,v))$ intersects $x \equiv 0$. Thus the point of intersection lies below where the ellipse intersects $x \equiv 0$. The ellipse intersects the line $x \equiv 0$ at $0 < t_1$ for which

$$0 = \cos(\omega t_1) - \frac{1}{\omega}[\lambda - a - bv]\sin(\omega t_1).$$

Thus,

$$\begin{aligned} y(t_1) &= v\left[\frac{1}{\omega}[\lambda - a - bv]\right.\sin(\omega t_1)] + \frac{1}{\omega}[c + (\lambda-a)v]\sin(\omega t_1) \\ &= \frac{1}{\omega}[c + 2(\lambda-a)v - bv^2]\sin(\omega t_1). \end{aligned}$$

As $\tan(\omega t_1) = \omega/(\lambda - a - bv)$ then $\sin(\omega t_1) = \omega/\sqrt{\omega^2 + (\lambda - a - bv)^2}$ and substituting into the above gives,

$$y(t_1) = \frac{c + 2(\lambda-a)v - bv^2}{\sqrt{\omega^2 + (\lambda - a - bv)^2}}.$$

As $\lambda = (a+d)/2, \omega = \sqrt{4(ad-bc) - (a+d)^2}/2$, in substitution and simplification into the fraction for $y(t_1)$ gives

$$y(t_1) = -\sqrt{v^2 - \frac{d-a}{b}v - \frac{c}{b}}.$$

Let $v' = \min\{-a/b, (-a^2/b - c)/(2\lambda)\}$. For $v \le v'$ the following inequalities hold,

$$\begin{aligned} & \frac{-a^2/b - c}{2\lambda} \ge v \\ \Rightarrow\quad & \frac{-a^2/b - c}{b} \ge \frac{2\lambda}{b}v \\ \Rightarrow\quad & -\frac{a^2}{b^2} - \frac{c}{b} \ge \left(2\frac{a}{b} + \frac{d-a}{b}\right)v \\ \Rightarrow\quad & v^2 - \frac{d-a}{b}v - \frac{c}{b} \ge v^2 + 2\frac{a}{b}v + \frac{a^2}{b^2} \\ \Rightarrow\quad & \sqrt{v^2 - \frac{d-a}{b}v - \frac{c}{b}} \ge -v - \frac{a}{b}. \end{aligned}$$

Thus,

$$y(t_1) = -\sqrt{v^2 - \frac{d-a}{b}v - \frac{c}{b}} \leq v + \frac{a}{b}.$$

Thus $\phi(t_0, (1, v)) = (0, \phi_2(t_0, (1, v)))$ where $\phi(t_0, (1, v)) < v + (a/b)$. After π/ω units of time, then by lemma 3.9,

$$\phi(t_0 + \pi/\omega, (1, v)) = (0, -e^{\frac{\lambda\pi}{\omega}}\phi_2(t_0, (1, v)))$$

where

$$-e^{\frac{\lambda\pi}{\omega}}\left(v + \frac{a}{b}\right) < -e^{\frac{\lambda\pi}{\omega}}\phi_2(t_0, (1, v)).$$

Let $\overline{v} = -e^{\frac{\lambda\pi}{\omega}}(v + a/b)$. Note that $0 \leq \overline{v}$. Consider the ellipse that passes through $(0, \overline{v})$. This ellipse intersects the line $x \equiv 1$ at a point below where the $\phi(t, (1, v))$ intersects the same line. Thus, a lower bound for $\pi(v)$ has been obtained. The ellipse that passes through $(0, \overline{v})$ has the equation given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{b}{\omega}\overline{v}\sin(\omega t) \\ \overline{v}\cos(\omega t) + \frac{1}{\omega}(\lambda - a)\overline{v}\sin(\omega t) \end{bmatrix}.$$

At $0 < t_2$ for which the ellipse intersects the line $x \equiv 1$, then $\omega/b = \sin(\omega t_2)$. Computing $y(t_2)$ gives

$$\begin{aligned} y(t_2) &= \sqrt{\overline{v}^2 - \frac{\omega^2}{b^2}} + \frac{1}{\omega}(\lambda - a)\frac{\omega}{b} \\ &= \overline{v}\sqrt{1 - \frac{\omega^2}{\overline{v}^2 b^2}} + \frac{\lambda - a}{b}. \end{aligned}$$

Let $v'' = \max\{1/2, 1/4 + \omega^2/b^2\}$, then for $\overline{v} \geq v''$ the following inequalities hold,

$$\begin{aligned} & \overline{v} \geq \frac{1}{4} + \frac{\omega^2}{b^2} \\ \Rightarrow \quad & \frac{1}{\overline{v}} \geq \frac{1}{\overline{v}^2}\left(\frac{1}{4} + \frac{\omega^2}{b^2}\right) \\ \Rightarrow \quad & -\frac{\omega^2}{\overline{v}^2 b^2} \geq -\frac{1}{\overline{v}} + \frac{1}{4\overline{v}^2} \\ \Rightarrow \quad & 1 - \frac{\omega^2}{\overline{v}^2 b^2} \geq 1 - \frac{1}{\overline{v}} + \frac{1}{4\overline{v}^2} \\ \Rightarrow \quad & \sqrt{1 - \frac{\omega^2}{\overline{v}^2 b^2}} \geq 1 - \frac{1}{2\overline{v}}. \end{aligned}$$

Thus

$$\begin{aligned} y(t_2) &\geq \overline{v}\left(1 - \frac{1}{2\overline{v}}\right) + \frac{\lambda - a}{b} \\ &= \overline{v} - \frac{1}{2} + \frac{\lambda - a}{b}. \end{aligned}$$

Finally, let $v_0 = \min\{v', -a/b - e^{-\frac{\lambda\pi}{\omega}}v''\}$. Then for $v \leq v_0 < y^*$ one has that

$$\begin{aligned} \pi(v) &> -e^{\frac{\lambda\pi}{\omega}}\left(v + \frac{a}{b}\right) - \frac{1}{2} + \frac{\lambda - a}{b} \\ &= -e^{\frac{\lambda\pi}{\omega}}v - e^{\frac{\lambda\pi}{\omega}}\frac{a}{b} - \frac{1}{2} + \frac{\lambda - a}{b} \\ &= -e^{\frac{\lambda\pi}{\omega}}v + K_0 \end{aligned}$$

where

$$K_0 = -e^{\frac{\lambda\pi}{\omega}}\frac{a}{b} - \frac{1}{2} + \frac{\lambda - a}{b}. \qquad \blacksquare$$

Lemma 3.16. *Let $(x_i, y_i), x_i < 1$ be the induced fixed point of the vector field ξ for which*

$$\frac{1}{b}(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a) < y_i < \frac{1}{b}(-d(1-x_i) - a).$$

Consider the linear vector field given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

where $k = (ax_i + by_i)/(1-x_i), l = (cx_i + dy_i)/(1-x_i)$. Let $\lambda_1 \pm i\omega_1$ be the complex eigenvalues of the matrix defining the vector field. Let $\phi(t,(x_0,y_0))$ be the solution through the point (x_0,y_0), then for $t < 0$ the point $\phi(t,(x_0,y_0))$ lies outside of the ellipse given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (x_0 - x_i)\cos(\omega_1 t) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(x_0 - x_i) - b(y_0 - y_i)]\sin(\omega_1 t) \\ (y_0 - y_i)\cos(\omega_1 t) + \frac{1}{\omega_1}[(c+l)(x_0 - x_i) + (\lambda_1 - (a+k))(y_0 - y_i)]\sin(\omega_1 t) \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

PROOF. The solution through the point (x_0, y_0) is given by

$$\phi(t,(x_0,y_0)) =$$

$$e^{\lambda_1 t}\begin{bmatrix} (x_0 - x_i)\cos(\omega_1 t) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(x_0 - x_i) - b(y_0 - y_i)]\sin(\omega_1 t) \\ (y_0 - y_i)\cos(\omega_1 t) + \frac{1}{\omega_1}[(c+l)(x_0 - x_i) + (\lambda_1 - (a+k))(y_0 - y_i)]\sin(\omega_1 t) \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

As $y_i < (-d(1-x_i)-a)/b$ then $\lambda_1 = (a+k+d)/2 < 0$. Thus for $t < 0$ the vector $\phi(t,(x_0,y_0)) - [x_i \; y_i]^t$ has a longer length while still being in the same direction as the vector

$$\begin{bmatrix} (x_0 - x_i)\cos(\omega_1 t) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(x_0 - x_i) - b(y_0 - y_i)]\sin(\omega_1 t) \\ (y_0 - y_i)\cos(\omega_1 t) + \frac{1}{\omega_1}[(c+l)(x_0 - x_i) + (\lambda_1 - (a+k))(y_0 - y_i)]\sin(\omega_1 t) \end{bmatrix}.$$

Thus, for $t < 0$ the point $\phi(t,(x_0,y_0))$ lies outside of the ellipse

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (x_0 - x_i)\cos(\omega_1 t) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(x_0 - x_i) - b(y_0 - y_i)]\sin(\omega_1 t) \\ (y_0 - y_i)\cos(\omega_1 t) + \frac{1}{\omega_1}[(c+l)(x_0 - x_i) + (\lambda_1 - (a+k))(y_0 - y_i)]\sin(\omega_1 t) \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}. \blacksquare$$

Lemma 3.17. *Let $(x_i, y_i), x_i < 1$ be the induced fixed point of the vector field ξ such that*

$$\frac{1}{b}(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a) < y_i < \frac{1}{b}(-d(1-x_i) - a).$$

To the fixed point corresponds unique values of k, l where $k = (ax_i + by_i)/(1-x_i), l = (cx_i + dy_i)/(1-x_i)$. Let $\lambda_1 \pm i\omega_1$ be the complex eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

Then there exists $y^* < v_1, K_1$ such that for $v_1 \leq v$ the return map $\pi : L(y^*, \infty) \to L(-\infty, y^*]$ satisfies

$$\pi(v) < -e^{\frac{\lambda_1 \pi}{\omega_1}} v + K_1.$$

PROOF. Let $(1, v)$ be a point with $y^* \leq v$. As $y_i < (-d(1-x_i) - a)/b$ then $\lambda_1 = (a+k+d)/2 < 0$. Let $\phi(t, (1, v))$ be the solution that passes through the point $(1, v)$. By lemma 3.16 under reverse time the solution $\phi(t, (1, v))$ lies outside of the ellipse given by the equation

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (1-x_i)\cos(\omega_1 t) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(1-x_i) - b(v-y_i)]\sin(\omega_1 t) \\ (v-y_i)\cos(\omega_1 t) + \frac{1}{\omega_1}[(c+l)(1-x_i) + (\lambda_1 - (a+k))(v-y_i)]\sin(\omega_1 t) \end{bmatrix} + \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

Let $t_0 < 0$ be the first negative time for which $\phi(t, (1, v))$ intersects the line $x \equiv x_i$. This point of intersection lies above the corresponding point where the ellipse intersects the line $x \equiv x_i$. The ellipse intersects the line $x \equiv x_i$ at some time $t_1 < 0$ for which

$$x_i = x(t_1) = (1-x_i)\cos(\omega_1 t_1) - \frac{1}{\omega_1}[(\lambda_1 - (a+k))(1-x_i) - b(v-y_i)]\sin(\omega_1 t_1) + x_i.$$

Thus,

$$y(t_1) = \frac{1}{\omega_1}\left[(c+l)(1-x_i) + 2(\lambda_1 - (a+k))(v-y_i) - \frac{b(v-y_i)^2}{1-x_i}\right]\sin(\omega_1 t_1) + y_i.$$

As

$$\tan(\omega_1 t_1) = \frac{\omega_1(1-x_i)}{(\lambda_1 - (a+k))(1-x_i) - b(v-y_i)}$$

then

$$\sin(\omega_1 t_1) = \frac{\omega_1(1-x_i)}{\sqrt{\omega_1^2(1-x_i)^2 + [(\lambda_1 - (a+k))(1-x_i) - b(v-y_i)]^2}}.$$

Thus,

$$y(t_1) = \frac{(c+l)(1-x_i)^2 + 2(\lambda_1 - (a+k))(v-y_i)(1-x_i) - b(v-y_i)^2}{\sqrt{\omega_1^2(1-x_i)^2 + [(\lambda_1 - (a+k))(1-x_i) - b(v-y_i)]^2}} + y_i.$$

Using $\lambda_1 = (a+k+d)/2, \omega_1 = \sqrt{4(ad - bc + dk - bl) - (a+k+d)^2}/2$, the formula for $y(t_1)$ simplifies to the following form,

$$y(t_1) = \sqrt{(v-y_i)^2 - \frac{(d-(a+k))(1-x_i)}{b}(v-y_i) - \frac{(c+l)^2(1-x_i)^2}{b}} + y_i.$$

Let $v' = \max\{-(a+k)(1-x_i)/b + y_i, (-(a+k)^2(1-x_i)/b - (c+l)(1-x_i))/(2\lambda_1) + y_i\}$. For $v' \leq v$ then, and remembering $\lambda_1 = (a+k+d)/2 < 0$,

$$\frac{-(a+k)^2(1-x_i)/b - (c+l)(1-x_i)}{2\lambda_1} \leq v - y_i$$

$$\Rightarrow \quad -\frac{(a+k)^2(1-x_i)^2}{b} - (c+l)(1-x_i)^2 \geq 2\lambda_1(1-x_i)(v-y_i)$$

$$\Rightarrow \quad -\frac{(a+k)^2(1-x_i)^2}{b^2} - \frac{(c+l)(1-x_i)^2}{b} \geq \left(2\frac{a+k}{b} + \frac{d-(a+k)}{b}\right)(1-x_i)(v-y_i)$$

$$-\frac{(d-(a+k))(1-x_i)}{b}(v-y_i) - \frac{(c+l)(1-x_i)^2}{b} \geq 2\frac{(a+k)(1-x_i)}{b}(v-y_i) + \frac{(a+k)^2(1-x_i)^2}{b^2}.$$

Adding $(v-y_i)^2$ to both sides and taking the square roots results in,

$$\sqrt{(v-y_i)^2-\frac{(d-(a+k))(1-x_i)}{b}(v-y_i)-\frac{(c+l)(1-x_i)^2}{b}}\geq (v-y_i)+\frac{(a+k)(1-x_i)}{b}.$$

Thus $\phi(t_0,(1,v))=(x_i,\phi_2(t_0,(1,v)))$ satisfies

$$\phi_2(t_0,(1,v))>v+\frac{(a+k)(1-x_i)}{b}.$$

By lemma 3.9, after π/ω_1 units of time it happens that

$$\phi(t_0+\pi/\omega_1,(1,v))=(x_i,-e^{\frac{\lambda_1\pi}{\omega_1}}(\phi_2(t_0,(1,v))-y_i)+y_i)$$

where

$$-e^{\frac{\lambda_1\pi}{\omega_1}}(\phi_2(t_0,(1,v))-y_i)+y_i<-e^{\frac{\lambda_1\pi}{\omega_1}}\left[(v-y_i)+\frac{(a+k)(1-x_i)}{b}\right]+y_i.$$

Let $\overline{v}=-e^{\frac{\lambda_1\pi}{\omega_1}}[(v-y_i)+(a+k)(1-x_i)/b]+y_i$. Note that $\overline{v}-y_i\leq 0$. Now look at the ellipse that passes through the point $(x_i,\overline{v})$. This ellipse intersects the line $x\equiv 1$ at a point above where $\phi(t,(1,v))$ intersects the same line in reverse time. This enables an upper bound to be put on the value of $\pi(v)$. The ellipse through $(x_i,\overline{v})$ is given by the formula

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}\frac{b}{\omega_1}(\overline{v}-y_i)\sin(\omega_1 t)\\ (\overline{v}-y_i)\cos(\omega_1 t)+\frac{1}{\omega_1}(\lambda_1-(a+k))(\overline{v}-y_i)\sin(\omega_1 t)\end{bmatrix}+\begin{bmatrix}x_i\\ y_i\end{bmatrix}.$$

At the point of intersection with the line $x\equiv 1$ at time $t_2<0$ then

$$\frac{\omega_1}{b}(1-x_i)=(\overline{v}-y_i)\sin(\omega_1 t_2).$$

Thus,

$$\begin{aligned}y(t_1)&=-\sqrt{(\overline{v}-y_i)^2-\frac{\omega_1^2}{b^2}(1-x_i)^2}+\frac{1}{b}(\lambda_1-(a+k))(1-x_i)\\&=(\overline{v}-y_i)\sqrt{1-\frac{\omega_1^2(1-x_i)^2}{b^2(\overline{v}-y_i)^2}}+\frac{1}{b}(\lambda_1-(a+k))(1-x_i).\end{aligned}$$

Let $v''=\min\{-1/2+y_i,-1/4-\omega_1^2(1-x_i)^2/b^2+y_i\}$. Then for $\overline{v}\leq v''$ the following inequalities hold,

$$\begin{aligned}&& \overline{v}-y_i&\leq-\frac{1}{4}-\frac{\omega_1^2(1-x_i)^2}{b^2}\\ \Rightarrow&& \frac{1}{\overline{v}-y_i}&\leq\frac{1}{(\overline{v}-y_i)^2}\left(-\frac{1}{4}-\frac{\omega_1^2(1-x_i)^2}{b^2}\right)\\ \Rightarrow&& \frac{1}{\overline{v}-y_i}+\frac{1}{4(\overline{v}-y_i)^2}&\leq-\frac{\omega_1^2(1-x_i)^2}{b^2(\overline{v}-y_i)^2}\\ \Rightarrow&& 1+\frac{1}{\overline{v}-y_i}+\frac{1}{4(\overline{v}-y_i)^2}&\leq 1-\frac{\omega_1^2(1-x_i)^2}{b^2(\overline{v}-y_i)}\\ \Rightarrow&& 1+\frac{1}{2(\overline{v}-y_i)}&\leq\sqrt{1-\frac{\omega_1^2(1-x_i)^2}{b^2(1-x_i)^2}}.\end{aligned}$$

Thus,

$$\begin{aligned} y(t_1) &\le (\overline{v} - y_i)\left(1 + \frac{1}{2(\overline{v} - y_i)}\right) + \frac{1}{b}(\lambda_1 - (a+k))(1 - x_i) \\ &= \overline{v} - y_i + \frac{1}{2} + \frac{1}{b}(\lambda_1 - (a+k))(1 - x_i). \end{aligned}$$

Finally, let $v_1 = \max\{v', -(a+k)(1-x_i)/b - e^{-\frac{\lambda_1\pi}{\omega_1}}(v'' - y_i) + y_i, -a/b + 1/b\}$, then for $v_1 < v$ one has that

$$\begin{aligned} \pi(v) &< -e^{\frac{\lambda_1\pi}{\omega_1}}\left[(v - y_i) + \frac{(a+k)(1-x_i)}{b}\right] + \frac{1}{2} + \frac{1}{b}(\lambda_1 - (a+k))(1 - x_i) \\ &= -e^{\frac{\lambda_1\pi}{\omega_1}} v + K_1 \end{aligned}$$

where

$$K_1 = e^{\frac{\lambda_1\pi}{\omega_1}} y_i - e^{\frac{\lambda_1\pi}{\omega_1}} \frac{(a+k)(1-x_i)}{b} + \frac{1}{2} + \frac{1}{b}(\lambda_1 - (a+k))(1 - x_i).$$

∎

Theorem 3.18. *Let $(x,y), x < 1$ be the induced virtual fixed point of the vector field ξ with*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

and $0 < \lambda_1/\omega_1 + \lambda/\omega$. Then there exists $v_2 < y^*$ such that for $v \le v_2$ the map $\pi^2 : L(-\infty, y^*] \to L(-\infty, y^*]$ satisfies $\pi^2(v) < v$.

PROOF. Let v_0, v_1, K_0, K_1 be the constants in lemma 3.15 and lemma 3.17 respectively. Let $v' = \min\{v_0, \pi^{-1}(v_1)\}$. As $v_0 < y^*$ then $v' < y^*$. For $v \le v' \le v_0$ then $-e^{\frac{\lambda\pi}{\omega}} v + K_0 < \pi(v)$. As $v_1 \le \pi(v)$ then $\pi^2(v) < -e^{\frac{\lambda_1\pi}{\omega_1}} \pi(v) + K_1$. Combining these two results give

$$\begin{aligned} \pi^2(v) &< e^{\frac{\lambda_1\pi}{\omega_1}}(e^{\frac{\lambda\pi}{\omega}} v - K_0) + K_1 \\ &= e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} v - e^{\frac{\lambda_1\pi}{\omega_1}} K_0 + K_1. \end{aligned}$$

If $\pi^2(v) < v$ then it is sufficient that

$$e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} v - e^{\frac{\lambda_1\pi}{\omega_1}} K_0 + K_1 \le v.$$

Thus,

$$v\left(e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} - 1\right) \le e^{\frac{\lambda_1\pi}{\omega_1}} K_0 - K_1.$$

Or,

$$v \le \frac{e^{\frac{\lambda_1\pi}{\omega_1}} K_0 - K_1}{e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} - 1}.$$

Let $v'' = (e^{\frac{\lambda_1\pi}{\omega_1}} K_0 - K_1)/(e^{(\frac{\lambda_1}{\omega_1} + \frac{\lambda}{\omega})\pi} - 1)$. Finally, define $v_2 = \min\{v', v''\}$. Then for $v \le v_2$ it happens that $\pi^2(v) < v$.

∎

Theorem 3.19. *Let $(x, y), x < 1$ be the induced virtual fixed point of the vector field ξ. If*

$$\frac{1}{b}(-\sqrt{1-x}(a+d) - d(1-x) - a) < y < \frac{1}{b}(-d(1-x) - a)$$

then there are no cycles.

PROOF. For these values of (x, y) one has that $0 < \lambda_1/\omega_1 + \lambda/\omega$ (c.f. theorem 3.13) where $\lambda \pm i\omega$ are the eigenvalues of the origin and $\lambda_1 \pm i\omega$ are the eigenvalues of the induced fixed point. By theorem 3.18 there exists $v_2 < y^*$ for which $v \leq v_2$ implies $\pi^2(v) < v$.

Say a limit annulus exists. As the origin in repelling the annulus is attracting. To the annulus are two boundary cycles C_1, C_2 which intersect the line $x \equiv 1$ at the points $(1, r), (1, s)$ with $v_2 < s \leq r < y^*$.

Consider the line segment $L[v_2, s]$. Under two reverse iterations of the map π one has that $\pi^{-2}(L[v_2, s]) = L[\pi^{-2}(v_2), s] \subset L[v_2, s]$. Now π^{-2} has the unique fixed point s in $L[v_2, y]$. If another fixed point existed, then maximality of the annulus would be violated. Thus s is attracting for π^{-2}. But s is also repelling for π^2. The point s cannot be both attracting for forward and reverse time. By contradiction, the limit annulus does not exist.

By the same argument as the previous two paragraphs, if an annulus whose boundary cycles both intersected the line $x \equiv 1$ existed, then it would be semi-stable. By lemma 1.12 semi-stable annuli do not exist. If annuli exist, then at least one of the two boundary cycles do not intersect the line $x \equiv 1$. This means that one of the two regions $\{(x, y) : x \leq 1\}$ or $\{(x, y) : 1 < x\}$ admits a cycle. The trace of the vector field in both regions is nonzero. Neither of the two regions mentioned admit cycles. Thus, annuli do not exist, cycles do not exist. ∎

Having considered the case for which the induced fixed point lies in the region $\{(x, y) : x < 1\}$, the next region to be considered for the induced fixed point will be the region $\{(x, y) : 1 < x\}$.

The first four results will establish a correspondence between fixed points $\{(x, y) : 1 < x\}$ and ordered pairs $\{(v, w) : v < y^* < w\}$ as the points of intersection of the invariant manifolds through (x, y) and the line $x \equiv 1$.

Lemma 3.20. *Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. Then there exists $v < y^* < w$ such that the line through $(1, v), (1, w)$ in the direction of the vector field passes through (x, y).*

PROOF. By corollary 2.8 the product of the eigenvalues is $(ad - bc)/(1 - x) < 0$, there are two eigenvalues of opposite sign $\lambda_1 < 0 < \lambda_2$. Assume that the linear invariant manifold passing through the direction of one of the eigenvectors corresponding to an eigenvalue does not intersect $x \equiv 1$. The linear invariant manifold then has the form $x \equiv K$. Let (K, y) be a point on this line. Being on the invariant manifold the x-ordinate of the vector at (K, y) is 0, thus $(a + k)K + by = 0$. As $0 < b$ this equality cannot hold independent of the value of y. Thus, the linear invariant manifolds intersect $x \equiv 1$.

Thus solve for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = K \begin{bmatrix} x-1 \\ y-v \end{bmatrix},$$

for $K = K_1, K_2, K_1 < 0 < K_2$ and v, w where v, w are the intersection of the invariant manifolds with the line $x \equiv 1$. As $a + bv = K_1(x-1) < 0, a + bw = K_2(x-1) > 0$ it then follows that $v < y^* < w$. ∎

Lemma 3.21. *Let $v < y^* < w$, then there is the induced fixed point $(x, y), 1 < x$ for which the invariant manifolds pass through $(1, v), (1, w)$.*

PROOF. The line through $(1, v)$ in the direction of the vector at that point is given by

$$\begin{bmatrix} -(c+dv) & a+bv \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(c+dv) + (a+bv)v \end{bmatrix}.$$

Similarly, the line through $(1, w)$ in the direction of the vector field is given by

$$\begin{bmatrix} -(c+dw) & a+bw \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(c+dw) + (a+bw)w \end{bmatrix}.$$

The intersection of the two lines is then given by the solution to

$$\begin{bmatrix} -(c+dv) & a+bv \\ -(c+dw) & a+bw \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(c+dv) + (a+bv)v \\ -(c+dw) + (a+bw)w \end{bmatrix}.$$

Or,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(a+bv)(c+dw) - (a+bw)(c+dv)} \begin{bmatrix} a+bw & -(a+bv) \\ c+dw & -(c+dv) \end{bmatrix} \begin{bmatrix} -(c+dv) + (a+bv)v \\ -(c+dw) + (a+bw)w \end{bmatrix}.$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(ad-bc)(w-v)} \begin{bmatrix} (w-v)(ad-bc) - (w-v)(a+bv)(a+bw) \\ (v-w)(ac+bc(v+w)+dbwv) \end{bmatrix}.$$

As $v < y^* < w$, the value of the determinant $(ad-bc)(w-v)$ is non-zero and the inversion has been validated. Finally, after division by $w - v$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc-(a+bv)(a+bw) \\ -(ac+bc(v+w)+dbwv) \end{bmatrix}.$$

Now $x = 1 - (a+bv)(a+bw)/(ad-bc)$. As $v < y^* < w$ then $a + bv < 0 < a + bw$ and it follows that $1 < x$.It remains to show that $(1, v), (1, w)$ are the points of intersection of the invariant manifolds through (x, y) with $x \equiv 1$.

Say the points of intersection are $(1, v^*), (1, w^*), v^* < y^* < w^*$ as given by the lemma 3.20 for induced fixed points with x-ordinates larger than 1. Assume that either $v \neq v^*$ or $w \neq w^*$.

If $v \neq v^*$ then using the calculation above to find the intersection of two lines, one through $(1, v)$ and $(1, v^*)$ respectively, the x-ordinate of the point of intersection is given by $1 - (a + bv)(a +$

$bv^*)/(ad-bc)$. As (x,y) lies on both lines, and the lines have different slopes, (x,y) is the unique point of intersection. Thus it follows that $x = 1-(a+bv)(a+bv^*)/(ad-bc)$. Since $y^* < v, v^*$ then $0 < a+bv, a+bv^*$ and $x < 1$. But $1 < x$, thus by contradiction $v = v^*$.

Similarly, by the same argument $w = w^*$. Thus indeed (x,y) has $(1,v),(1,w)$ as its points of intersection with the line $x \equiv 1$. Note that the points $(1,v),(1,w)$ are on the unstable and stable manifolds respectively passing through (x,y). ∎

Lemma 3.22. *Let $(x,y), 1 < x$ be the induced fixed point of the vector field ξ. Then the linear invariant manifolds through (x,y) intersect $x \equiv 1$ at points $(1,v),(1,w)$ with $v < y^* < w$.*

Proof. The eigenvalues at the induced fixed point are the eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}$$

for $k = (ax+by)/(1-x), l = (cx+dy)/(1-x)$. The eigenvalues of the matrix may be written as

$$\lambda_1 = \frac{a+by+d(1-x)+\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2(1-x)}$$
$$\lambda_2 = \frac{a+by+d(1-x)-\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2(1-x)}.$$

As $0 < 4(1-x)(ad-bc)$ the two eigenvalues have opposite signs. Note that $\lambda_1 < 0 < \lambda_2$.

First, it will be shown that the linear invariant manifolds at (x,y) must intersect the line $x \equiv 1$. Assume that $x \equiv K$ is an invariant manifold. For points (K,y) the x-ordinate of the vector at the point must satisfy $(a+k)K + by = 0$. Since $0 < b$, this equality cannot hold independent of y. Thus, the manifolds must intersect $x \equiv 1$.

If the point $(1,w)$ lies on the invariant manifold through (x,y) for the eigenvector corresponding to the eigenvalue λ_1 then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1-x \\ w-y \end{bmatrix} = \lambda_1 \begin{bmatrix} 1-x \\ w-y \end{bmatrix}.$$

Thus,

$$(a+k)(1-x)+b(w-y) = \lambda_1(1-x),$$

or,

$$\begin{aligned} w &= \frac{(\lambda_1-(a+k))(1-x)+by}{b} \\ &= -\frac{a}{b} + \frac{a+by+d(1-x)+\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2b} \\ &= -\frac{a}{b} + \lambda_1\frac{1-x}{b}. \end{aligned}$$

Similarly, if the point $(1,v)$ lies on the invariant manifold through (x,y) for the eigenvector that corresponds to the eigenvalue λ_2 then $(a+k)(1-x)+b(v-y)=\lambda_2(1-x)$ from which

$$\begin{aligned} v &= \frac{(\lambda_2-(a+k))(1-x)+by}{b} \\ &= -\frac{a}{b}+\frac{a+by+d(1-x)-\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2b} \\ &= -\frac{a}{b}+\lambda_2\frac{1-x}{b}. \end{aligned}$$

Clearly $v < y^* < w$. For points along an invariant manifold, the vector at those points have a direction parallel to the manifold. Thus, the vectors through $(1,v),(1,w)$ induce lines passing through (x,y). ∎

Theorem 3.23. *There is a C^∞ diffeomorphism $g(x,y)=(v,w)$ from the set $\{(x,y): 1 < x\}$ of induced fixed points to the set $\{(v,w): v < y^* < w\}$.*

PROOF. By lemma 3.21 and lemma 3.22 the function g is given by

$$g\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{a}{b}+\frac{a+by+d(1-x)-\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2b} \\ -\frac{a}{b}+\frac{a+by+d(1-x)+\sqrt{(a+by+d(1-x))^2-4(1-x)(ad-bc)}}{2b} \end{bmatrix}$$

with inverse

$$g^{-1}\begin{bmatrix} v \\ w \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc-(a+bv)(a+bw) \\ -(ac+bc(v+w)+dbvw) \end{bmatrix}.$$ ∎

It is now possible to consider under what circumstances cycles may not exist when the induced fixed point lies in the region $\{(x,y): 1 < x\}$. The next theorem shows that if the linear invariant manifolds at (x,y) satisfy a somewhat mild condition on their intersection with $x \equiv 1$ then there are no cycles.

Theorem 3.24. *(Figure 10.) Let $(x,y), 1 < x$ be the induced fixed point of the vector field ξ. If*

$$y \le \left(\frac{c+dy^{**}}{a+by^{**}}\right)(x-1)+y^{**}$$

where $y^{**}=\pi(y^*)$, then there are no cycles.

PROOF. Let $g(x,y)=(v,w)$. It will be shown that $w \le y^{**}$. Assume that $y^{**} < w$, then the point (x,y) lies on the line, $-(c+dw)x+(a+bw)y=-(c+dw)+(a+bw)w$.

Thus,

$$y=\frac{(c+dw)}{(a+bw)}(x-1)+w.$$

As $y^{**} < w$,

$$y > \frac{(c+dw)}{(a+bw)}(x-1)+y^{**}.$$

As $(ad-bc)w > (ad-bc)y^{**}$, then it follows that

$$cby^{**} + adw > cbw + ady^{**}.$$

Thus,

$$ca + cby^{**} + adw + dwdy^{**} > ca + cbw + ady^{**} + dwdy^{**}.$$

Or,

$$(c+dw)(a+by^{**}) > (c+dy^{**})(a+bw).$$

Finally,

$$\frac{c+dw}{a+bw} > \frac{c+dy^{**}}{a+by^{**}}.$$

The implication for y is that

$$y > \left(\frac{c+dy^{**}}{a+by^{**}}\right)(x-1) + y^{**},$$

contradicting the hypothesis of the theorem. Thus $w \leq y^{**}$.

Clearly the points on $L(w,\infty)$ cannot be on any cycle, being constrained by the invariant linear manifold passing through $(1,w)$. The point $(1,w)$ cannot be on any cycle, being on the stable manifold of (x,y). Points in $L(-\infty,y^*]$ iterate to $L[y^{**},\infty) \subset L[w,\infty)$ so cannot be on any cycles. Finally the points on $L(y^*,w)$ iterate to $L(-\infty,y^*]$ so that these points do not lie on cycles.

If a cycle existed then it must contain fixed points whose indexes sum to 1. Any cycle must then contain the origin in its interior. The cycle lies either wholly in the region $\{(x,y) : x \leq 1\}$ or intersects the line $x \equiv 1$. As $0 < a+d$, cycles cannot lie wholly in the region $\{(x,y) : x \leq 1\}$.

If cycles existed, then they must intersect the line $x \equiv 1$. By the above paragraph, since none of the points on the line $x \equiv 1$ can be on any cycles, then cycles do not exist. ∎

It would be useful to know when it is true that (x,y) satisfies $\pi(v) = w$ where $g(x,y) = (v,w)$. In the following lemmas a graph χ will be determined that will separate the points for which $\pi(v) < w$ and $w < \pi(v)$. Then points for which $\pi(v) < w$ will be exactly those points that lie above the graph χ. It will be shown that the graph χ is continuously differentiable and extends infinitely to the right.

Lemma 3.25. *Define $\chi : (-\infty, y^*) \to \{(x,y) : 1 < x\}$ by the formula $\chi(y) = g^{-1}(\pi(y), y)$. Then χ is a continuous curve with $\lim_{y\to y^*} \chi(y) = (1, y^{**})$.*

Proof. Note that by theorem 3.23 $g^{-1}(v,w) = (x,y)$ is given by the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc-(a+bv)(a+bw) \\ -(ac+bc(v+w)+dbwv) \end{bmatrix}.$$

Continuity of χ on the interval $(-\infty, y^*)$ follows from continuity of π and $g^{-1}(v,w)$.

Let $0 < \epsilon$ and consider the ball $B((1,y^{**}),\epsilon) = \{(x,y) : \sqrt{(x-1)^2+(y-y^{**})^2} < \epsilon\}$ about $(1,y^{**})$. Consider the set given by $V \times W = g(B((1,y^{**}),\epsilon) \cap \{(x,y) : 1 < x\})$. As g is a diffeomorphism by theorem 3.23 the set $V \times W$ is open, thus V, W are open subsets of $x \equiv 1$. Then V, W are sets of the form $(s, y^*), (y^{**}, t)$ respectively. Let $X = \pi^{-1}(W) \cap V = (u, y^*)$.

Say $u < y < y^*$, then $\pi(y) \in W, y \in V$ and thus $\chi(y) = g^{-1}(\pi(y), y) \in B((1,y^{**}),\epsilon)$. ∎

Lemma 3.26. *Let $\chi(y) = (\chi_1(y), \chi_2(y))$ where χ is the function defined in lemma 3.25. Then $\chi_1(y)$ is a decreasing function of y.*

Proof. Note that the function $\chi_1(y)$ is given by the formula

$$\chi_1(y) = 1 - \frac{(a+by)(a+b\pi(y))}{ad-bc}.$$

As ξ is a C^0 vector field, then π is a C^1 function. In particular χ_1 is a C^1 function on $(-\infty, y^*)$, being formed from the composition of two such functions: g^{-1}, π. Taking the derivative of χ_1,

$$\chi_1'(y) = -\frac{b(a+b\pi(y)) + b\pi'(y)(a+by)}{ad-bc}.$$

As $y < y^*$ then $a + by < 0$. Since $y^* < \pi(y)$ then $0 < a + b\pi(y)$. Also, $\pi'(y) < 0$ for all $y < y^*$. Then $\chi_1'(y) < 0$, this implies that as y decreases the x-ordinate of the function χ monotonically increases.

As $\lim_{y \to y^*} \chi_1(y) = 1$, it follows that $1 < \chi_1(y)$ for all $y < y^*$. By monotonicity, the inverse of $\chi_1(y)$ exists. Thus, it is possible to write $\chi_2(y) = \chi_2(\chi_1^{-1}(\chi_1(y)))$. In other words, $\chi_2(y) = F(\chi_1(y))$ for $F(y) = \chi_2(\chi_1^{-1}(y))$. ∎

Lemma 3.27. *Let $y < y^*$. Then $\chi(y) - \chi(y^*) = \mathrm{D}\chi(\eta)K$ where $\eta \in (y, y^*)$.*

Proof. Write $\chi(y) = (\chi_1(y), \chi_2(y)) = (\chi_1(y), F(\chi_1(y)))$ as in the end of lemma 3.26. By the mean value theorem,

$$\chi_2(y) - \chi_2(y^*) = F'(\eta^*)(\chi_1(y) - \chi_1(y^*)),$$

where $\eta^* \in (\chi_1(y^*), \chi_1(y))$. By monotonicity of χ_1, then $\eta^* = \chi_1(\eta)$ for some $\eta \in (y, y^*)$. Thus,

$$\chi(y) - \chi(y^*) = \begin{bmatrix} 1 \\ F'(\chi_1(\eta)) \end{bmatrix} (\chi_1(y) - \chi_1(y^*)).$$

And since $\chi_1'(\eta) \neq 0$,

$$\chi(y) - \chi(y^*) = \begin{bmatrix} \chi_1'(\eta) \\ F'(\chi_1(\eta))\chi_1'(\eta) \end{bmatrix} \frac{\chi_1(y) - \chi_1(y^*)}{\chi_1'(\eta)}.$$

i.e., $\chi(y) - \chi(y^*) = \mathrm{D}\chi(\eta)K$ where $K = (\chi_1(y) - \chi_1(y^*))/\chi_1'(\eta)$ and $\eta \in (y, y^*)$. ∎

Lemma 3.28. *The function χ is C^1 on the interval $(-\infty, y^*)$. If $\pi'(y^*)$ exists then*

$$\lim_{y \to y^*} \frac{\chi_2(y) - y^{**}}{\chi_1(y) - 1} = \frac{\pi'(y^*)(c + dy^*) + c + dy^{**}}{a + by^{**}}.$$

PROOF. As χ is the composition of two C^1 functions defined on the interval $(-\infty, y^*)$ it too is also C^1 on $(-\infty, y^*)$.

Now $D\chi(y) = D(g^{-1}(\pi(y), y))D(\pi(y), y)$, so that

$$D\chi(y) = \frac{1}{ad - bc}\begin{bmatrix} -(a+by)b & -(a+b\pi(y))b \\ -(c+dy)b & -(c+d\pi(y))b \end{bmatrix}\begin{bmatrix} \pi'(y) \\ 1 \end{bmatrix}.$$

Thus $D\chi(y)$ is a vector with the following slope,

$$\frac{\pi'(y)(c+dy)b + (c+d\pi(y))b}{\pi'(y)(a+by)b + (a+b\pi(y))b},$$

which reduces to,

$$\frac{\pi'(y)(c+dy) + c + d\pi(y)}{\pi'(y)(a+by) + a + b\pi(y)}.$$

By lemma 3.27 which state that $\chi(y) - \chi(y^*) = D\chi(\eta)K$ for some $\eta \in (y, y^*)$, then

$$\begin{bmatrix} \chi_1(y) - 1 \\ \chi_2(y) - y^{**} \end{bmatrix} = D\chi(\eta)K.$$

Hence,

$$\lim_{y \to y^*} \frac{\chi_2(y) - y^{**}}{\chi_1(y) - 1} = \lim_{y \to y^*} \frac{\pi'(\eta)(c+d\eta) + c + d\pi(\eta)}{\pi'(\eta)(a+b\eta) + a + b\pi(\eta)}.$$

Or,

$$\lim_{y \to y^*} \frac{\chi_2(y) - y^{**}}{\chi_1(y) - 1} = \frac{\pi'(y^*)(c+dy^*) + c + d\pi(y^*)}{\pi'(y^*)(a+by^*) + a + b\pi(y^*)}.$$

As $y^* = -a/b$ the denominator reduces to $a + by^{**} > 0$ and finally,

$$\lim_{y \to y^*} \frac{\chi_2(y) - y^{**}}{\chi_1(y) - 1} = \frac{\pi'(y^*)(c+dy^*) + c + dy^{**}}{a + by^{**}}.$$

∎

The final theorems in this section will prove regions where cycles do and do not exist.

Theorem 3.29. *(Figure 11.) Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. If*

$$\chi_2(\chi_1^{-1}(x)) < y.$$

then there exists a locally attracting annulus. Moreover, assuming conjecture 0.1 holds, there exists a locally attracting limit cycle.

PROOF. Consider $L[v, y^*]$. Under one application of π the result is $\pi(L[v, y^*]) = L[\pi(y^*), \pi(v)] = L[y^{**}, \pi(v)]$. Now $\pi(v) < w$ so that $v < \pi^2(v)$. Also $\pi^2(y^*) < y^*$. Thus the image of $L[v, y^*]$ under two iteration of π results in $\pi^2(L[v, y^*]) \subset L[v, y^*]$. By theorem 2.12 there is a locally attracting annulus for points in in $L[v, y^*]$. Note that the annulus is also attracting for points in $L(y^*, w)$.

Since the point $(1, w)$ is on the invariant manifold through (x, y), the annulus is not globally attracting.

Notice that points in $L(-\infty, \pi^{-1}(w)]$ iterate to $L[w, \infty)$. There are no cycles through these points. However, all points in $L(\pi^{-1}(w), v)$ iterate to $L(\pi(v), w)$ and again to $L(v, y^*)$ and eventually attracted to the annulus. By lemma 1.10 the attracting annulus is an attracting limit cycle. ∎

Theorem 3.30. *Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. If*

$$\frac{c + dy^{**}}{a + by^{**}}(x - 1) + y^{**} < y < \chi_2(\chi_1^{-1}(x))$$

then there are no cycles.

PROOF. Assume a limit annulus exists. Since the origin is a repelling fixed point which the annulus must encircle, the annulus is attracting.

Let $g(x, y) = (v, w)$. The annulus can be characterised by two boundary cycles C_1, C_2 which intersect the line $x \equiv 1$ at the points $(1, r), (1, s)$ with $y^* < r \le s < w$.

Consider the line segment $L[s, w]$ under two reverse iterations of π. As $y < \chi_2(\chi_1^{-1}(x))$ then $w < \pi(v)$, equivalently this means $v < \pi^{-1}(w)$ and $\pi^{-2}(w) < w$. Then $\pi^{-2}(L[s, w]) = L[s, \pi^{-2}(w)] \subset L[s, w]$. Now π^{-2} has only the point $(1, s)$ as fixed point in $L[s, w]$. If another fixed point existed in this interval then maximality of the annulus would be violated. Thus the point $(1, s)$ is attracting for π^{-2}.

The point $(1, s)$ is also attracting for π^2, it cannot be attracting for both forward and reverse time. By contradiction, the limit annuli does not exist. By the same argument, if an annulus whose boundary cycles both intersected the line $x \equiv 1$ existed, then it would be semi-stable. By lemma 1.12 semi-stable annuli do not exist. If annuli exist, then at least one of the two boundary cycles do not intersect the line $x \equiv 1$. This means that one of the two regions $\{(x, y) : x \le 1\}$ or $\{(x, y) : 1 < x\}$ admits a cycle. The trace of the vector field in both regions is nonzero. Neither of the two regions mentioned admit cycles. Thus, annuli do not exist, cycles do not exist. ∎

§4. $0 < b,\ 0 < a + d,\ (a + d)^2/4 < ad - bc,\ ad - bc + dk - bl = 0$.

The only result obtained in this case has been the following corollary.

Corollary 4.1. *If $0 \le a + k + d$ then there are no limit cycles.*

PROOF. If a limit cycle existed then it must intersect the line $x \equiv 1$. By lemma 3.1 cycles may not intersect the line $x \equiv 1$, thus limit cycles do not exist. ∎

§5. $0 < b$, $0 < a + d$, $0 < ad - bc \leq (a + d)^2/4$.

Under these conditions cycles do not exist for the vector field ξ. This will be proved in the next two propositions.

Proposition 5.1. *If $ad - bc + dk - bl = 0$ then there are no cycles.*

PROOF. Consider the linear vector field given by

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

as the extension of the vector field ξ to the right of $x \equiv 1$ to the whole plane. As the determinant of the matrix is zero, the number of fixed points is either zero or infinite. There are two cases to consider.

(*i*) :If $l \neq (d/b)k$ there are no induced fixed points. In particular there are no fixed points in the region $\{(x, y) : 1 < x\}$. Any cycle must then contain the origin. As both eigenvalues at the origin are real, there is at least one invariant linear manifold through the origin. A cycle cannot enclose a linear manifold, so cycles do not exist.

(*ii*) :If $l = (d/b)k$ then $ad - bc = 0$. This case cannot occur. ∎

Proposition 5.2. *If $ad - bc + dk - bl \neq 0$ then there are no cycles.*

PROOF. Consider the induced fixed point (x, y) that the constants k, l induce. There are two cases to analyse.

(i):The induced fixed point (x, y) satisfies $x < 1$. Note that the vector field has only one fixed point, namely at the origin $(0, 0)$. As $4(ad-bc) \leq (a+d)^2$ the eigenvalues at the origin are both real. Take the eigenvector that corresponds to one of these eigenvalues, through this eigenvector lies an invariant manifold of ξ. Any cycle must encompass the origin, being the only fixed point. However, lines cannot be encompassed by closed curves. By contradiction, cycles do not exist.

(ii):The induced fixed point (x, y) satisfies $1 < x$. The vector field has two fixed points, namely one at the origin and one at the point (x, y). By corollary 2.8 the fixed point at (x, y) has eigenvalues whose product is equal to $(ad - bc)/(1 - x) < 0$. Thus (x, y) is a fixed point of index -1. Any cycle must enclose fixed points whose index sum to 1. Thus, any cycle must enclose the origin. But, as in the previous case, this is not possible, cycles do not exist. ∎

§6. $0 < b,\ 0 < a + d,\ ad - bc = 0$.

In the situation that these conditions hold true, there are no cycles.

Proposition 6.1. *There are no cycles.*

PROOF. As ad-bc=0 the rank of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is either 0 or 1. Because $0 < b$, the rank is 1. The line $ax + by = 0$ is a line of fixed points for the linear vector field in the region $\{(x, y), x \leq 1\}$.Thus the line $y = -(a/b)x$ is a line of fixed points passing through the point $(1, -a/b)$.

If $ad - bc + dk - bl = 0$ then $dk - bl = 0$ and the linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

has a line of fixed points $y = (-ax + k(1 - x))/b$ which joins with the line $y = -(a/b)x$. The two lines form a partition of the plane in which neither region of the partition has any fixed points. Thus cycles do not exist.

If $ad - bc + dk - bl \neq 0$ then there is a unique solution (x, y) to the problem

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

at the fixed point $(1, -a/b)$. However, this fixed point is on the line of fixed points given by $y = (-ax + k(1 - x))/b$. This line of fixed points prevents cycles from forming. Thus, it follows that cycles do not exist. ∎

§7. $0 < b,\ 0 < a + d,\ ad - bc < 0$.

This rather short section assumes the origin to be a saddle point.

Proposition 7.1. *If $ad - bc + dk - bl = 0$ then there are no cycles.*

PROOF. Consider the linear vector field given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

as the extension of ξ to the right of $x \equiv 1$ to the whole plane. As the determinant of the matrix is zero, the number of fixed points is either zero or infinite. There are two cases to consider.

(*i*) :If $l \neq (d/b)k$ there are no induced fixed points. Thus, there are no fixed points to the right of $x \equiv 1$. Any cycle must then contain the origin. The origin is a saddle point with index –1, thus cycles cannot exist.

(*ii*) :If $l = (d/b)k$ then $ad - bc = 0$. This case cannot occur. ∎

For the remainder of the section it will be taken that $ad - bc + dk - bl \neq 0$. The following results will give sufficient conditions for the nonexistence of cycles.

Proposition 7.2. *Let $(x, y), x < 1$ be the induced fixed point of the vector field ξ. Then ξ has no cycles.*

PROOF. Any cycle must include a fixed point, ξ has only one fixed point, the origin which must therefore be included in the interior of the cycle. By index theory, the index of the origin is 1. But $ad - bc < 0$ so that the origin is a saddle point with index –1. Thus cycles do not exist. ∎

Proposition 7.3. *Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. If*

$$y \leq \frac{1}{b}(-d(1 - x) - a)$$

then there are no limit cycles. If strict inequality holds then there are no cycles.

PROOF. If a limit cycle existed then the cycle must intersect the line $x \equiv 1$. By using lemma 3.1, it is sufficient that $0 \leq a + k + d$ for there to be no limit cycles. Thus $0 \leq a + (ax + by)/(1 - x) + d$ which reduces to $y \leq (-d(1 - x) - a)/b$.

Consider the case of strict inequality. Any cycle must enclose fixed points whose indexes sum to 1. The cycle must contain the induced fixed point. The cycle lies either wholly in the region $\{(x, y) : 1 < x\}$ or intersects the line $x \equiv 1$. By the lemma 3.1 the cycle cannot intersect the line $x \equiv 1$. But if $y < (-d(1 - x) - a)/b$ then $0 < a + k + d$. The region $\{(x, y) : 1 < x\}$ will not admit cycles. Thus, cycles do not exist. ∎

Lemma 7.4. *Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. Then (x, y) has complex eigenvalues $\Leftrightarrow$*

$$\frac{1}{b}(-2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a).$$

PROOF. The eigenvalues are determined by the characteristic equation of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix},$$

which is $\lambda^2 + (a+k+d)\lambda + ad - bc + dk - bl = 0$. By using the values $k = (ax+by)/(1-x), l = (cx+dy)/(1-x)$ the above reduces to

$$\lambda^2 - \left(\frac{a+d-dx+by}{1-x}\right)\lambda + \frac{ad-bc}{1-x} = 0.$$

The eigenvalues are complex if and only if

$$\left(\frac{a+d-dx+by}{1-x}\right)^2 < 4\left(\frac{ad-bc}{1-x}\right).$$

Thus,

$$\frac{1}{b}(-2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a). \quad \blacksquare$$

Theorem 7.5. *Let $(x, y), 1 < x$ be the induced fixed point of the vector field ξ. If the point satisfies*

$$\frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) \leq y$$

then there are no cycles.

PROOF. By lemma 7.4, the point (x, y) has at least one real eigenvalue to which can be associated a linear invariant manifold. Any cycle must contain either the fixed point at the origin or at the point (x, y). As both have linear invariant manifolds, which cannot be contained within a cycle, cycles do not exist. $\blacksquare$

To conclude this section, the following results will show that attractive limit cycles are highly unlikely.

Lemma 7.6. *The invariant manifolds through the origin intersect the line $x \equiv 1$ at the points $(1, (-a + d - \sqrt{(a+d)^2 - 4(ad-bc)})/(2b)), (1, (-a + d + \sqrt{(a+d)^2 - 4(ad-bc)})/(2b))$.*

PROOF. The problem is the same as solving

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ v \end{bmatrix}$$

for v when $\lambda = (a+d\pm\sqrt{(a+d)^2-4(ad-bc)})/2$ are the eigenvalues at the origin. Thus $a+bv=\lambda$, from which $v=(-a+\lambda)/b$. Hence, $v=(-a+d\pm\sqrt{(a+d)^2-4(ad-bc)})/(2b)$. ∎

Let the points of intersection of the invariant manifolds with the line $x\equiv 1$ be $(1,\overline{v}),(1,\overline{w}),\overline{v}<y^*<\overline{w}$. Then the following two theorems can be proved.

Theorem 7.7. *Let $(x,y), 1<x$ be the induced fixed point of the vector field ξ. If*

$$\frac{1}{b}(-d(1-x)-a)<y<\frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)}-d(1-x)-a)$$

and $\pi(\overline{w})<\overline{v}$ then there is a repelling annulus. Moreover, assuming conjecture 0.1 holds, then there is a repelling limit cycle.

Proof. Since $(-d(1-x)-a)/b<y$ then $a+k+d<0$. Under reverse time the induced fixed point is repelling, thus $\pi^{-1}(y^*)$ is well-defined.

Consider the line segment $L[\overline{v},y^*]$. Then under two reverse iterations of π,

$$\begin{aligned}\pi^{-2}(L[\overline{v},y^*]) &= \pi^{-1}(L[\pi^{-1}(y^*),\pi^{-1}(\overline{v})])\\ &\subset \pi^{-1}(L[\pi^{-1}(y^*),\overline{w}))\\ &= L(\pi^{-1}(\overline{w}),\pi^{-2}(y^*)]\\ &\subset L[\overline{v},y^*].\end{aligned}$$

Thus $\pi^{-2}(L[\overline{v},y^*])\subset L[\overline{v},y^*]$. Applying theorem 2.12 to the function π^{-2}, there exists a locally attracting annulus for π^{-2}. Thus, there a repelling annulus for π^2. By corollary 1.11 the repelling annulus is a repelling limit cycle. ∎

Theorem 7.8. *Let $(x,y), 1<x$ be the induced fixed point of the vector field ξ. If*

$$\frac{1}{b}(-d(1-x)-a)<y<\frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)}-d(1-x)-a)$$

and $\overline{v}<\pi(\overline{w})$ then there are no cycles.

Proof. Say a limit annulus exists. The annulus may be characterised by its boundary cycles, which intersect the line $x\equiv 1$ at the points $(1,r),(1,s)$ with $y^*<r\le s<\overline{w}$.

As $(-d(1-x)-a)/b<y$ then $a+k+d<0$. The fixed point which the annulus encircles is attracting, the annulus is repelling.

Consider the line segment $L[s,\overline{w}]$ under two iterations of π,

$$\begin{aligned}\pi^2(L[s,\overline{w}]) &= \pi(L[\pi(\overline{w}),\pi(s)])\\ &\subset \pi(L(\overline{v},\pi(s)])\\ &= L[\pi^2(s),\pi(\overline{v}))\\ &\subset L[s,\overline{w}].\end{aligned}$$

The point $(1, s)$ is the only fixed point of π^2 for the line segment $L[s, \overline{w}]$. If another fixed point existed then maximality of the annulus would be violated. Thus, the point $(1, s)$ is attracting for points in $L[s, \overline{w}]$.

The point $(1, s)$ cannot be both repelling and attracting, thus limit annuli do not exist. By the same argument, if an annulus whose boundary cycles both intersected the line $x \equiv 1$ existed, then it would be semi-stable. By lemma 1.12 semi-stable annuli do not exist. If annuli exist, then at least one of the two boundary cycles do not intersect the line $x \equiv 1$. This means that one of the two regions $\{(x, y) : x \leq 1\}$ or $\{(x, y) : 1 < x\}$ admits a cycle. The trace of the vector field in both regions is nonzero. Neither of the two regions mentioned admit cycles. Thus, annuli do not exist, cycles do not exist. ∎

§8. $0 < \mathbf{b}$, $0 = \mathbf{a} + \mathbf{d}$.

In contrast with the earlier sections where the analysis has been divided into the two cases $ad - bc + dk - bl \neq 0$ and $ad - bc + dk - bl = 0$, the division in this section will be between $k \neq 0$ and $k = 0$. If $k \neq 0$ then there are no limit cycles.

Proposition 8.1. *If $k \neq 0$ then there are no limit cycles.*

Proof. Note that if a vector field is to contain a limit cycle then the cycle cannot lie wholly in either the regions $\{(x, y) : x \leq 1\}$ or $\{(x, y) : 1 < x\}$ because linear vector fields do not admit limit cycles. Thus, if a such cycle exists then it must intersect the line $x \equiv 1$ transversally at some points $(1, y_1), (1, y_2)$ with $y_1 < y^* < y_2$. The cycle will then join the points $(1, y_2), (1, y_1)$ in a clockwise orientation. Let C denote the cycle. Then by Stoke's theorem,

$$\oint_C \frac{dx}{dt}dy - \frac{dy}{dt}dx = \int_{\text{int}(C)} \frac{d}{dx}\left(\frac{dx}{dt}\right) + \frac{d}{dy}\left(\frac{dy}{dt}\right) dx\ dy.$$

Breaking up the area integral into two parts, $A = \text{int}(C) \cap \{(x, y) : x \leq 1\}$ and $B = \text{int}(C) \cap \{(x, y) : 1 < x\}$, then

$$0 = \int_A \frac{d}{dx}(ax + by) + \frac{d}{dy}(cx + dy)dx\ dy + \int_B \frac{d}{dx}((a + k)x + by - k) + \frac{d}{dy}((c + l)x + dy - l)dx\ dy.$$

Thus,

$$\begin{aligned} 0 &= \int_A (a + d)dx\ dy + \int_B (a + k + d)dx\ dy \\ &= \int_B k dx\ dy. \end{aligned}$$

As $k \neq 0$ the integral on the right is nonzero. By contradiction, limit cycles do not exist. ∎

Thus the case for which $k = 0$ will be examined. First some notation and results about linear vector fields will be needed.

Definition 8.2. $L((x_0, y_0), (x_1, y_1))$ will denote the open line segment where $L((x_0, y_0), (x_1, y_1)) = \{t(x_0, y_0) + (1-t)(x_1, y_1) : 0 < t < 1\}$.

Lemma 8.3. *Let* $\mathbf{L}$ *be a linear vector field of the form*

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, 0 < \lambda.$$

Let $(x_0, y_0) \neq (0,0)$. *Construct the line segment joining the points* $(0, 2y_0), (2x_0, 0)$. *Consider the return map* $\pi : L((0, 2y_0), (x_0, y_0)) \to L((x_0, y_0), (2x_0, 0))$ *where*

$$\pi(x, y) = (e^{\lambda t_0}x, e^{-\lambda t_0}y), t_0 = \min\{0 < t : (e^{\lambda t}x, e^{-\lambda t}y) \in L((x_0, y_0), (2x_0, 0))\}.$$

Then $\pi((x_0, y_0) + s(-x_0, y_0)) = (x_0, y_0) - s(-x_0, y_0), 0 < s < 1$.

Proof. The line tangent to the vector field at the point (x_0, y_0) is given by the equation $y = -(y_0/x_0)x + 2y_0$. Thus, $L((0, 2y_0), (2x_0, 0))$ is the portion of the line bounded by the invariant manifolds of the linear vector field.

Let $(\overline{x}, \overline{y}) = (x_0 - sx_0, y_0 + sy_0), 0 < s < 1$ be any point in the line segment $L((0, 2y_0), (2x_0, 0))$. Consider the region R bounded by $(0,0), (0, 2y_0), (2x_0, 0)$, whose interior has no fixed points. The solution through the point $(\overline{x}, \overline{y})$ moves into the afore-mentioned region. If the solution did not exit the region then the point $(\overline{x}, \overline{y})$ is attracted either to a fixed point or a cycle. As the interior of the region does not have any fixed points then the point could not have been attracted to a cycle or fixed point in the interior of the region R. By choice, the point $(\overline{v}, \overline{w})$ is not on the invariant manifolds through the origin, so the solution cannot be attracted to a fixed point on the boundary of R. Thus, the solution through $(\overline{v}, \overline{w})$ exits R in some finite time. Thus π is well defined.

Now $\phi(t, (\overline{x}, \overline{y})) = (e^{\lambda t}\overline{x}, e^{-\lambda t}\overline{y})$. Letting $(\overline{x}, \overline{y})$ lie on the line $L((0, 2y_0), (2x_0, 0))$ results in the expression

$$\phi(t, (\overline{x}, \overline{y})) = (e^{\lambda t}\overline{x}, e^{-\lambda t}\left(-\frac{y_0}{x_0}\overline{x} + 2y_0\right)).$$

If $\phi(t_0, (\overline{x}, \overline{y})) \in L((x_0, y_0), (2x_0, 0))$ then

$$(e^{\lambda t_0}\overline{x}, e^{-\lambda t_0}(-\frac{y_0}{x_0}\overline{x} + 2y_0)) \in L((x_0, y_0), (2x_0, 0)).$$

i.e.,

$$e^{-\lambda t_0}\left(-\frac{y_0}{x_0}\overline{x} + 2y_0\right) = -\frac{y_0}{x_0}e^{\lambda t_0}\overline{x} + 2y_0.$$

Solving for the values of t_0 results in the values $0, \ln((2x_0 - \overline{x})/\overline{x})/\lambda$. As $0 < t_0$ only the second of the two solutions is admissable, thus $t_0 = \ln((2x_0 - \overline{x})/\overline{x})/\lambda$.

Then $\phi(t_0, (\overline{x}, \overline{y})) = (2x_0 - \overline{x}, (y_0/x_0)\overline{x})$. Thus $\pi(\overline{x}, \overline{y}) = (2x_0 - \overline{x}, (y_0/x_0)\overline{x})$, and after simplificaton, $\pi((x_0, y_0) + s(-x_0, y_0)) = (x_0, y_0) - s(-x_0, y_0)$. ∎

Lemma 8.4. *Let* $\mathbf{L}$ *be a linear vector field of the form*

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad 0 < \omega.$$

Let $(x_0, y_0) \neq (0,0)$ *be any point. Construct the line* L *through* (x_0, y_0) *with tangent vector* $[y_0 \ -x_0]^t$. *Consider the return map* $\pi : L \to L$ *where*

$$\pi(x,y) = (\cos(\omega t_0)x + \sin(\omega t_0)y, -\sin(\omega t_0)x + \cos(\omega t_0)y),$$

$$t_0 = \min\{0 < t : (\cos(\omega t)x + \sin(\omega t)y, -\sin(\omega t)x + \cos(\omega t)y) \in L\}.$$

Then $\pi((x_0, y_0) + s(y_0, -x_0)) = (x_0, y_0) - s(y_0, -x_0)$.

Proof. In polar coordinates the solution through the point $(\overline{x}, \overline{y}) = (x_0 + sy_0, y_0 - sx_0)$ is given by,

$$r(t) = \sqrt{1+s^2}\sqrt{x_0^2 + y_0^2},$$

$$\theta(t) = -\omega t + \tan^{-1}\left(\frac{y_0 - sx_0}{x_0 + sy_0}\right).$$

The equation of the line through (x_0, y_0) with direction $[y_0 \ x_0]^t$ is given by

$$r\cos\left(\theta - \tan^{-1}\left(\frac{y_0}{x_0}\right)\right) = \sqrt{x_0^2 + y_0^2}.$$

Thus at points of intersection of the solution of the vector field and the line,

$$\sqrt{1+s^2}\sqrt{x_0^2+y_0^2}\cos\left(-\omega t + \tan^{-1}\left(\frac{y_0 - sx_0}{x_0 + sy_0}\right) - \tan^{-1}\left(\frac{y_0}{x_0}\right)\right) = \sqrt{x_0^2 + y_0^2}.$$

Solving for solutions gives the following as viable values of t,

$$t = (2n\pi \pm \tan^{-1}(s) - \tan^{-1}(s))/\omega.$$

For $s < 0, t_0 = -2\tan^{-1}(s)/\omega$. When $0 \leq s, t_0 = (2\pi - 2\tan^{-1}(s))/\omega$. Computing the value of $\phi(t_0, (\overline{x}, \overline{y}))$ results in $(x_0 - sy_0, y_0 + sx_0)$. Thus $\pi((x_0, y_0) + s(y_0, -x_0)) = (x_0, y_0) - s(y_0, -x_0)$. ∎

Lemma 8.5. *A nonsingular linear mappings maps lines into lines.*

Proof. Consider the image of the point $(x, y) = s(x_0, y_0) + (1-s)(x_1, y_1)$ under the nonsingular transformation $\mathbf{A}$, $\mathbf{A}(x,y) = s\mathbf{A}(x_0, y_0) + (1-s)\mathbf{A}(x_1, y_1)$. ∎

Lemma 8.6. *Let* $ad - bc < 0$, *then the invariant manifolds through the origin intersects the line* $x \equiv 1$ *at the points* $(1, (-a - \sqrt{-(ad-bc)})/b), (1, (-a + \sqrt{-(ad-bc)})/b)$.

Proof. The problem is the same as finding eigenvectors whose x-ordinate is equal to 1,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ v \end{bmatrix} = \pm\sqrt{ad-bc}\begin{bmatrix} 1 \\ v \end{bmatrix}.$$

i.e, $a + bv = \pm\sqrt{-(ad-bc)}$ which reduces to $v = (-a \pm \sqrt{-(ad-bc)})/b$. ∎

The purpose of the previous lemmas is to prove the following two lemmas allowing particularly simple expressions for the return map.

Lemma 8.7. *Let $ad - bc < 0$. Then the return map $\pi : L((-a-\sqrt{-(ad-bc)})/b, y^*) \to L(y^*, (-a+\sqrt{-(ad-bc)})/b, y^*)$ is given by $\pi(y) = y^* + (y^* - y)$.*

PROOF. Note that the point $(1, (-a - \sqrt{-(ad-bc)})/b)$ is on the stable manifold of the the origin, so indeed π goes from the given domain into the given range. By linear algebra it is possible to find a non-singular matrix $\mathbf{A}$ such that under the change of variables $\mathbf{X} = \mathbf{A}^{-1}\mathbf{x}$ the portion of ξ to the left of $x \equiv 1$ has the form

$$\mathbf{L}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Let $\phi, \overline{\phi}$ be the respective solution to ξ to the left of $x \equiv 1$ and the solution to the above linear vector field $\mathbf{L}$. The solutions $\overline{\phi}, \phi$ are related by $\phi(t, (x_0, y_0)) = \mathbf{A}\overline{\phi}(t, \mathbf{A}^{-1}(x_0, y_0))$.

By lemma 8.5 the line $L((-a - \sqrt{ad-bc})/b, (-a + \sqrt{ad-bc})/b)$ is mapped by $\mathbf{A}^{-1}$ to a line in the $\mathbf{X}$ plane that is tangent to $\overline{\xi}$ at the point $\mathbf{A}^{-1}(1, y^*)$. Using lemma 8.3 to compute the return map for a point $y \in L((-a - \sqrt{ad-bc})/b, y^*)$,

$$\begin{aligned} \phi(t_0, (1, y)) &= \phi(t_0, (1, y^*) + (0, y - y^*)) \\ &= \mathbf{A}\overline{\phi}(t_0, \mathbf{A}^{-1}(1, y^*) + \mathbf{A}^{-1}(0, y - y^*)) \\ &= \mathbf{A}(\mathbf{A}^{-1}(1, y^*) - \mathbf{A}^{-1}(0, y - y^*)) \\ &= (1, y^* - (y - y^*)). \end{aligned}$$

Thus, $\pi(y) = y^* - (y - y^*) = y^* + (y^* - y)$. ∎

Lemma 8.8. *Let $0 < ad - bc$. Then $\pi : L(-\infty, y^*) \to L(y^*, \infty)$ is given by $\pi(y) = y^* + (y^* - y)$.*

PROOF. Let $\mathbf{A}$ be a nonsingular matrix such that under the change of variables $\mathbf{X} = \mathbf{A}^{-1}\mathbf{x}$ the vector field ξ to the left of $x \equiv 1$ has the form of the linear vector field,

$$\mathbf{L}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Let $\overline{\phi}$ be the solution to $\mathbf{L}$. Then ϕ, the solution of ξ to the left of $x \equiv 1$ and $\overline{\phi}$ are related by $\phi(t, (x_0, y_0)) = \mathbf{A}\overline{\phi}(t, \mathbf{A}^{-1}(x_0, y_0))$.

By lemma 8.5 the line $x \equiv 1$ is mapped by $\mathbf{A}^{-1}$ to a line tangent to $\mathbf{A}^{-1}(1, y^*)$ in the $\mathbf{X}$ plane. Using lemma 8.4 to compute the return map,

$$\begin{aligned} \phi(t_0, (1, y)) &= \phi(t_0, (1, y^*) + (0, y - y^*)) \\ &= \mathbf{A}\overline{\phi}(t_0, \mathbf{A}^{-1}(1, y^*) + \mathbf{A}^{-1}(0, y - y^*)) \\ &= \mathbf{A}(\mathbf{A}^{-1}(1, y^*) - \mathbf{A}^{-1}(0, y - y^*)) \\ &= (1, y^* - (y - y^*)). \end{aligned}$$

Thus, $\pi(y) = y^* - (y - y^*) = y^* + (y^* - y)$. ∎

Theorem 8.9. *If $ad - bc \neq 0, ad - bc - bl \neq 0$ then there are no limit cycles.*

PROOF. Note that a limit cycle cannot pass through the point $(1, y^*)$ because such a cycle would lie wholly in the region $\{(x, y) : x \leq 1\}$ is which ξ is linear. It is then sufficient to show that cycles in $L(-\infty, y^*)$ cannot be limit cycles.

Since $ad - bc - bl \neq 0$, by theorem 2.6, the vector field ξ has induced fixed points (x, y). There are several cases to consider.

$(i) : 0 < ad - bc, (x, y), x < 1$. The origin is a center and by lemma 8.8 the return map defined on $L(-\infty, y^*)$ is given by $\pi(y) = y^* + (y^* - y)$. By corollary 2.8 the induced fixed point has eigenvalues whose product is $(ad - bc)/(1 - x)$ which is positive. The induced point has imaginary eigenvalues. By lemma 8.8 the return map defined on $L(y^*, \infty)$ is given by $\pi(y) = y^* + (y^* - y)$. Thus, computing π^2 for points along $L(-\infty, y^*)$ gives $\pi^2(y) = y$. As $\pi^2(y) = y$ it follows that there are infinitely many concentric cycles, the cycles cannot be attracting or repelling. This is because that arbitrarily close to any cycle there exists another cycle, i.e. the cycles are not isolated.

$(ii) : 0 < ad - bc, (x, y), 1 < x$. As in case(i) above, the return map defined on $L(-\infty, y^*)$ is given by $\pi(y) = y^* + (y^* - y)$. The induced fixed point in this case happens to have real distinct eigenvalues. Consider the line segment $L(y_0, y_1)$ of $x \equiv 1$ that contains $(1, y^*)$ and is bounded by the invariant manifolds of the induced fixed point. Under a nonsingular linear transformation the image of the point $(1, y^*)$ bisects the image of the line. The implication is that $(1, y^*)$ bisects $L(y_0, y_1)$. Thus $L(y_0, y_1)$ has the form $L(y^* - m, y^* + m)$ for some $0 < m$. By lemma 8.7 it then follows that the return map on $L(y^*, y^* + m)$ is given by $\pi(y) = y^* + (y^* - y)$. Thus, on the line segment $L(y^* - m, y^*)$ (and $L(y^*, y^* + m)$) the return map satisfies $\pi^2(y) = y$. As in case(i) above, there are no attracting or repelling cycle in $L(y^* - m, y^*)$. Points on $L[y^* + m, \infty)$ cannot induced cycles, being bounded away by the invariant manifold at $(1, y^* + m)$ from returning to $x \equiv 1$. Points on $(-\infty, y^* - m]$ iterate to $L[y^* + m, \infty)$, hence neither can they form cycles.

$(iii) : ad - bc < 0, (x, y), x < 1$. There are no cycles as the only fixed point in the plane is at the origin which has index -1.

$(iv) : ad - bc < 0, (x, y), 1 < x$. The fixed point at the origin has distinct real eigenvalues, by lemma 8.7 return map is given by $\pi(y) = y^* + (y^* - y)$ for points on $L(y^* - m, y^*)$. By corollary 2.8 the induced fixed point has eigenvalues whose product is $(ad - bc)/(1 - x)$ which is positive, implying that the eigenvalues are imaginary. By considering the induced fixed point as the origin and applying lemma 8.8 the return map is then given by $\pi(y) = y^* + (y^* - y)$ for points on $L(y^*, \infty)$. Thus on $L(y^* - m, y^*)$ the return map satisfies $\pi^2(y) = y$. There are no cycles in $L(y^* - m, y^*)$ (and $L(y^*, y^* + m)$) which are either attracting or repelling. Points on $L(-\infty, y^* - m]$ are bounded by the invariant manifold through $(1, y^* - m)$ from forming cycles. Points on $L[y^* + m, \infty)$ iterate to

$L(-\infty, y^* - m]$ and neither can they form cycles. ∎

Theorem 8.10. *If $ad - bc \neq 0$, $ad - bc - bl = 0$ then there are no limit cycles.*

PROOF. As $ad - bc - bl = 0$ it follows that $c + l = ad/b$. The linear vector field to the right of $x \equiv 1$ is then given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ \frac{ad}{b} & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{ad-bc}{b} \end{bmatrix}.$$

The solution is

$$\phi(t,(1,y_0)) = (1 + (a + by_0)t - \frac{1}{2}(ad - bc)t^2, y_0 + (c - ay_0)t + \frac{a}{2b}(ad - bc)t^2).$$

The times for which $\phi(t,(1,y_0))$ intersects the line $x \equiv 1$ are at the values $t = 0, 2(a+by_0)/(ad-bc)$. There are two cases to consider.

$(i) : 0 < ad-bc$. By lemma 8.8 the return map on the left is given by $\pi(y) = y^* + (y^* - y), y < y^*$. On the right of $x \equiv 1$ the point $(1, y), y^* < y$ iterates under positive time $2(a + by_0)/(ad - bc)$ to the point $(1, y^* + (y^* - y))$. The return map to the right of $x \equiv 1$ is given by $\pi(y) = y^* + (y^* - y), y^* < y$. Thus the return map satisfies $\pi^2(y) = y$. There are no cycles which are either attracting or repelling.

$(ii) : ad - bc < 0$. The linear vector field $\mathbf{L}$ has no fixed points so that ξ has no fixed points to the right of $x \equiv 1$. There is only one fixed point in the plane, namely at the origin. With index –1, no cycles can be formed. ∎

Proposition 8.11. *If $ad - bc = 0$ then there are no cycles.*

PROOF. There is a line of fixed points $ax + by = 0$ to the left of $x \equiv 1$. The line $y = -(a/b)x$ is a line of fixed points passing through the point $(1, -a/b)$.

If $ad - bc + dk - bl = 0$ then $dk - bl = 0$ and the linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}$$

has a line of fixed points $y = (-ax + k(1-x))/b$ which joins with the line $y = -(a/b)x$. The two lines form a partition of the plane in which neither region of the partition has any fixed points. Thus cycles do not exist.

If $ad - bc + dk - bl \neq 0$ then there are no induced fixed points. In particular, there are no fixed points in the region $\{(x, y) : 1 < x\}$. The line $y = -(a/b)x$ is semi-infinite, and cannot be enclosed in any cycle. Without invariant sets for which a cycle may enclose, cycles do not exist. ∎

§9. $0 = b$.

In the event that $b = 0$ it happens that there are no cycles.

Proposition 9.1. *There are no cycles.*

PROOF. Note that along the line $x \equiv 1$ the vector field is given by

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} a \\ c + dy \end{bmatrix}.$$

As along the line $x \equiv 1$ the x-ordinate of vectors have constant value, cycles cannot cross the line. If cycles existed then they lie in either of the two regions $\{(x, y) : x \leq 1\}$ or $\{(x, y) : 1 \leq x\}$.

Cycles cannot lies in the region $\{(x, y) : x \leq 1\}$, the only fixed point is the origin to which passes a linear invariant manifold along the y axis.

In the region $\{(x, y) : 1 \leq x\}$ the vector field ξ has the form

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + k & 0 \\ c + l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

If a cycle existed then it must enclose a fixed point of ξ. But through the fixed point lies a linear invariant manifold parallel to the y axis. No cycles can exist in this region. ∎

References.

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Appendix A.

In this appendix the equivalence of the continuous piecewise vector fields of Chua[2] and those used herein will be examined. First, the respective definitions will be recalled.

Definition A.1. (Chua)ξ is a continuous piecewise linear vector field in canonical form $\Leftrightarrow$ there exists an integer $1 \leq n$, matrix $\mathbf{B}$, vectors $\alpha, \alpha_i, \beta_i$, $1 \leq i \leq n$ and constants γ_i, $1 \leq i \leq n$ for which $\xi(\mathbf{x}) = \alpha + \mathbf{B}\mathbf{x} + \sum_{i=1}^{n} \alpha_i | < \beta_i, \mathbf{x} > -\gamma_i |$.

The following definition is repeated from the main body of the text:

Definition 1.2. ξ is a continuous piecewise linear vector field $\Leftrightarrow$ there exists constants a, b, c, d, k, l with either $k \neq 0$ or $l \neq 0$, and

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x. \end{cases}$$

The following are two lemmas showing the equivalence of the two types of continuous piecewise linear vector fields and the relationship between the defining constants that allows this equivalence.

Lemma A.2. *(i) :Let* $\xi(\mathbf{x}) = \alpha + \mathbf{B}\mathbf{x} - \alpha|[1\ 0]\mathbf{x} - 1|$ *with*

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11} + \alpha_1 & b_{12} \\ b_{21} + \alpha_2 & b_{22} \end{bmatrix}, \quad \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} -2\alpha_1 \\ -2\alpha_2 \end{bmatrix}.$$

(ii) :Let

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

be a continuous piecewise linear vector field with k, l not both zero. Then $\xi(\mathbf{x}) = \alpha + \mathbf{B}\mathbf{x} - \alpha|[1\ 0]\mathbf{x} - 1|$ *where*

$$\alpha = \begin{bmatrix} -\frac{1}{2}k \\ -\frac{1}{2}l \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a + \frac{1}{2}k & b \\ c + \frac{1}{2}l & d \end{bmatrix}.$$

PROOF. The continuous piecewise linear vector field ξ in canonical form has the following decomposition

$$\begin{aligned}\xi(\mathbf{x}) &= \alpha + \mathbf{B}\mathbf{x} - \alpha(1 - [1\ 0]\mathbf{x}) && \mathbf{x} \in \{(x,y) : x \le 1\}\\ &= (\mathbf{B} + \alpha[1\ 0])\mathbf{x}\\ \xi(\mathbf{x}) &= \alpha + \mathbf{B}\mathbf{x} - \alpha([1\ 0]\mathbf{x} - 1) && \mathbf{x} \in \{(x,y) : 1 < x\}\\ &= (\mathbf{B} - \alpha[1\ 0])\mathbf{x} + 2\alpha.\end{aligned}$$

The corresponding decomposition for a continuous piecewise linear vector field is

$$\begin{aligned}\xi\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} && \mathbf{x} \in \{(x,y) : x \le 1\}\\ \xi\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : 1 < x\}.\end{aligned}$$

Matching the two expressons givees,

$$\begin{aligned}(\mathbf{B} + \alpha[1\ 0])\mathbf{x} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} && \mathbf{x} \in \{(x,y) : x \le 1\}\\ (\mathbf{B} - \alpha[1\ 0])\mathbf{x} + 2\alpha &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : 1 < x\}.\end{aligned}$$

(i) :Given the values of

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then by the equations above, it follows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11} + \alpha_1 & b_{12} \\ b_{21} + \alpha_2 & b_{22} \end{bmatrix}, \ \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} -2\alpha_1 \\ -2\alpha_2 \end{bmatrix}.$$

(ii) :Given the values of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}, \begin{bmatrix} k \\ l \end{bmatrix},$$

then by the same set of equations above,

$$\alpha = \begin{bmatrix} -\frac{1}{2}k \\ -\frac{1}{2}l \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} a + \frac{1}{2}k & b \\ c + \frac{1}{2}l & d \end{bmatrix}.$$

∎

The concept of the induced fixed point occurs frequently and in many of the results that have have been obtained. Therefore, it would be desirable to find how the induced fixed point of a continuous piecewise linear vector field is connected with the defining constants in the canonical representation. The next two lemmas show how the induced fixed point is related to the canonical representaion.

Lemma A.3. *If*

$$\xi(\mathbf{x}) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\mathbf{x} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} |[1\ 0]\mathbf{x} - 1|$$

with $b_{22}(b_{11}-\alpha_1)-b_{12}(b_{21}-\alpha_2)\neq 0$ *then the induced fixed point of* ξ *is*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{b_{22}(b_{11}-\alpha_1)-b_{12}(b_{21}-\alpha_2)} \begin{bmatrix} b_{12}\alpha_2 - b_{22}\alpha_1 \\ b_{21}\alpha_1 - b_{11}\alpha_2 \end{bmatrix}.$$

PROOF. By using lemma A.2 the continuous piecewise linear vector field in canonical form can be rewritten as

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11}+\alpha_1 & b_{12} \\ b_{21}+\alpha_2 & b_{22} \end{bmatrix}, \quad \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} -2\alpha_1 \\ -2\alpha_2 \end{bmatrix}.$$

The induced fixed point is then the solution to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11}-\alpha_1 & b_{12} \\ b_{21}-\alpha_2 & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -2\alpha_1 \\ -2\alpha_2 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{b_{22}(b_{11}-\alpha_1)-b_{12}(b_{21}-\alpha_2)} \begin{bmatrix} b_{12}\alpha_2 - b_{22}\alpha_1 \\ b_{21}\alpha_1 - b_{11}\alpha_2 \end{bmatrix}. \quad \blacksquare$$

Lemma A.4. *Let* a, b, c, d *be given. If the vector field* ξ *in canonical form has the induced fixed point at* (x_i, y_i), $x_i \neq 1$ *then*

$$\xi(\mathbf{x}) = -\frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} + \begin{bmatrix} a + \frac{ax_i+by_i}{2(1-x_i)} & b \\ c + \frac{cx_i+dy_i}{2(1-x_i)} & d \end{bmatrix} \mathbf{x} + \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} - 1|.$$

PROOF. The vector field

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

for

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{1}{1-x_i} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix}$$

has the point (x_i, y_i) as its induced fixed point. Using lemma A.2 to convert ξ into the canonical representaion means

$$\alpha = -\frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a + \frac{ax_i+by_i}{2(1-x_i)} & b \\ c + \frac{cx_i+dy_i}{2(1-x_i)} & d \end{bmatrix}.$$

Whence,

$$\xi(\mathbf{x}) = -\frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} + \begin{bmatrix} a + \frac{ax_i+by_i}{2(1-x_i)} & b \\ c + \frac{cx_i+dy_i}{2(1-x_i)} & d \end{bmatrix} \mathbf{x} + \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} - 1|. \quad \blacksquare$$

Figure captions.

Figure 1. In this vector field there are infinitely many concentric cycles, none of which is a limit cycle.

Figure 2. This vector field does not admit any limit cycles.

Figure 3. In contrast to the vector field in example 2, this vector field has a unique attracting cycle.

Figure 4. When the defining constant b is zero, there do not exist any cycles. This example is such a case in point.

Figure 5. If $\pi^2(L[v,w]) \subset L[v,w]$ then the points $\pi^i(v), \pi^i(w)$ form decreasing and increasing sequences respectively. The limit of these sequences are points through which lie cycles, giving the existence of an annulus.

Figure 6. Through the induced virtual fixed point (x,y) a linear invariant manifold intersects the line $x \equiv 1$ at the point $(1,v)$. The line segment $L[v,y^*]$ maps into itself under application of π^2. This is sufficient to prove the existence of an attracting cycle.

Figure 7. The solution through the point $(1,v)$ meets the y-axis at a point T above where the tangent line through the same point meets the y-axis at S. The images of T,S on the next intersection with the y-axis are at V,U. The tangent line through U meets the line $x \equiv 1$ above the solution through V meeting the same line. Thus $\pi(v) < -e^{\frac{\lambda\pi}{\omega}}v + K_0$.

Figure 8. Consider the solution through $(1,v)$. The solution meets the line $x \equiv x_i$ at the point T below where the line tangent to $(1,v)$ meets $x \equiv x_i$ at the point S. The images of T and S under π/ω_1 units of time are at the points V,U. The line tangent to U meets the line $x \equiv 1$ below the solution through V meeting the line $x \equiv 1$. This enables a lower bound to be placed on the return map, $-e^{\frac{\lambda_1\pi}{\omega_1}}v + K_1 < \pi(v)$.

Figure 9. The point $(1, v_0)$ satisfies $v_0 < \pi^2(v_0)$. Under π^2 the line segment $L[v_0, y^*]$ maps into itself. The attracting annulus so formed also happens to be an attracting cycle.

Figure 10. The invariant manifolds through the induced fixed point (x,y) intersect the line $x \equiv 1$ at the points $(1,v)$ and $(1,w)$. As $w < y^{**}$, solutions starting from the line $x \equiv 1$ below the point $(1,y^*)$ intersect the line $x \equiv 1$ above the point $(1,w)$. The manifold through $(1,w)$ prevents cycles from forming.

Figure 11. The invariant manifolds through the induced fixed point (x,y) intersect the line $x \equiv 1$ at the points $(1,v)$ and $(1,w)$. As $\pi(v) < w$ it then follows that $v < \pi 2(v)$. The line segment $L[v,y^*]$ maps into itself, a sufficient condition for the existence of an attracting cycle.

Table 1. Computer aided phase portraits of piecewise-linear vector fields by the INSITE program.

a. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = 4.5 - 2.5x + y - 4.5|x-1|$$
$$\frac{dy}{dt} = 9 - 10x + 2y - 9|x-1|.$$

The vector field has an attracting cycle. The light lines are representative orbits that approach the limit cycle.

b. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = 4 - 2x + y - 4|x-1|$$
$$\frac{dy}{dt} = 8 - 9x + 2y - 8|x-1|.$$

The phase portrait given is that of the solution through the point $(0.1, 0)$. The origin is a repelling fixed point and the solution through $(0.1, 0)$ is repelled way from the origin.

c. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = 3.5 - 1.5x + y - 3.5|x-1|$$
$$\frac{dy}{dt} = 7 - 8x + 2y - 7|x-1|.$$

This vector does not have any attractors. The origin is in fact a repellor.

d. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -1 + 2y + |x-1|$$
$$\frac{dy}{dt} = -0.5 + 2.5x + y + 0.5|x-1|.$$

This is an example of a vector field globally conjugate to a linear saddle point. The bold lines have been added as the stable and unstable invariant manifolds of the vector field.

e. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = 2 + y - 2|x-1|.$$

In this interesting case, a continuum of cycles surround a fixed point. The cycles are also bounded by a homoclinic orbit. The fixed point and homoclinic orbit have been added in bold. Also in bold are the unstable and stable invariant manifolds through the origin.

f. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -1 + 2y + |x-1|$$
$$\frac{dy}{dt} = 2 + y - 2|x-1|.$$

In this example, a node-saddle connection results in orbits going to infinity. In bold are the unstable and stable invariant manifolds of the origin highlighting the node-saddle phenomenon of this vector field.

g. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = 2.5 - 1.5x + 3y - 2.5|x-1|$$
$$\frac{dy}{dt} = 2 - x + 2y - 2|x-1|.$$

This is another example of a node-saddle connection. Unlike the case in (f), the node is an attractor. Again, in bold are the unstable and stable manifolds of the origin. A quadrant of the plane is in the basin of attraction for the attractor node.

h. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -1.5 + 2.5x + 3y + 1.5|x-1|$$
$$\frac{dy}{dt} = -1 + 2x + 2y + |x-1|.$$

This is a degenerate case where the origin is extremely weakly attracting along the direction of the stable manifold. In bold is a completion of the stable invariant manifold through the origin.

i. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -0.5 + 2.5x + y + 0.5|x-1|$$
$$\frac{dy}{dt} = -1 + 2x + 2y + |x-1|.$$

An example of a vector field conjugate to the node of linear vector fields.

j. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -1 + 2y + |x-1|$$
$$\frac{dy}{dt} = -0.5 - 1.5x + y + 0.5|x-1|.$$

Despite the fact that the linear vector field to the left of the line $x \equiv 1$ would normally induce a center at the origin, the linear vector field right of $x \equiv 1$ perturbs the overall vector field into that of a repelling center.

k. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = 1 - x - |x-1|$$
$$\frac{dy}{dt} = 2 - 2x - 2|x-1|.$$

To the left of the line $x \equiv 1$ the vector field is very degenerate. The shaded region of the phase portrait represents a half plane of fixed points for the vector field.

1. This phase portrait corresponds to the piecewise linear vector field given by the pair of equations

$$\frac{dx}{dt} = -1 + 2x + y + |x - 1|$$
$$\frac{dy}{dt} = -2 + 4x + 2y + 2|x - 1|.$$

In this example a line of fixed points, in bold, divided the plane into two regions. The line of fixed points is also repelling.

TABLE 1
(a)

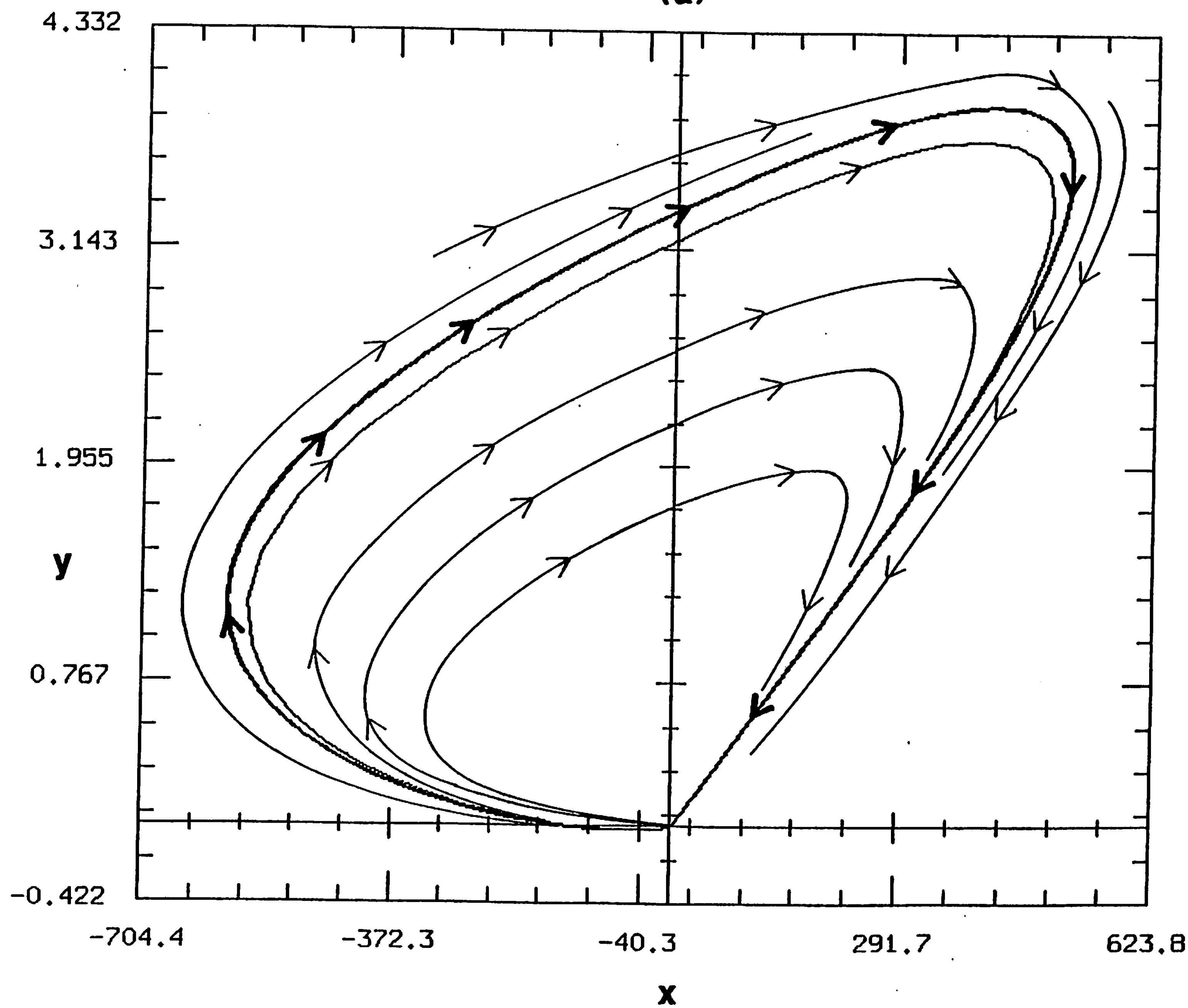


TABLE 1
(b)

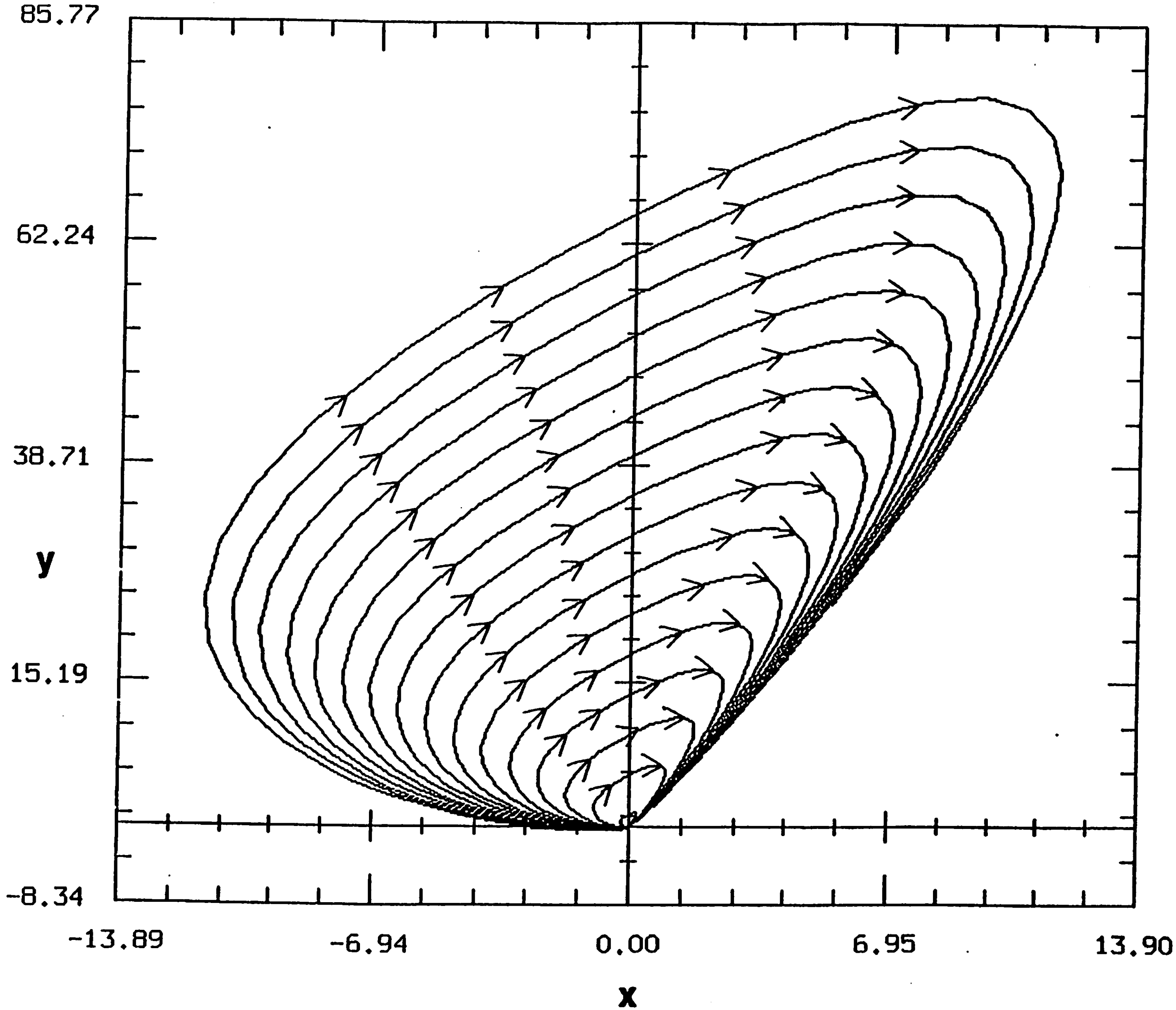


TABLE 1
(c)

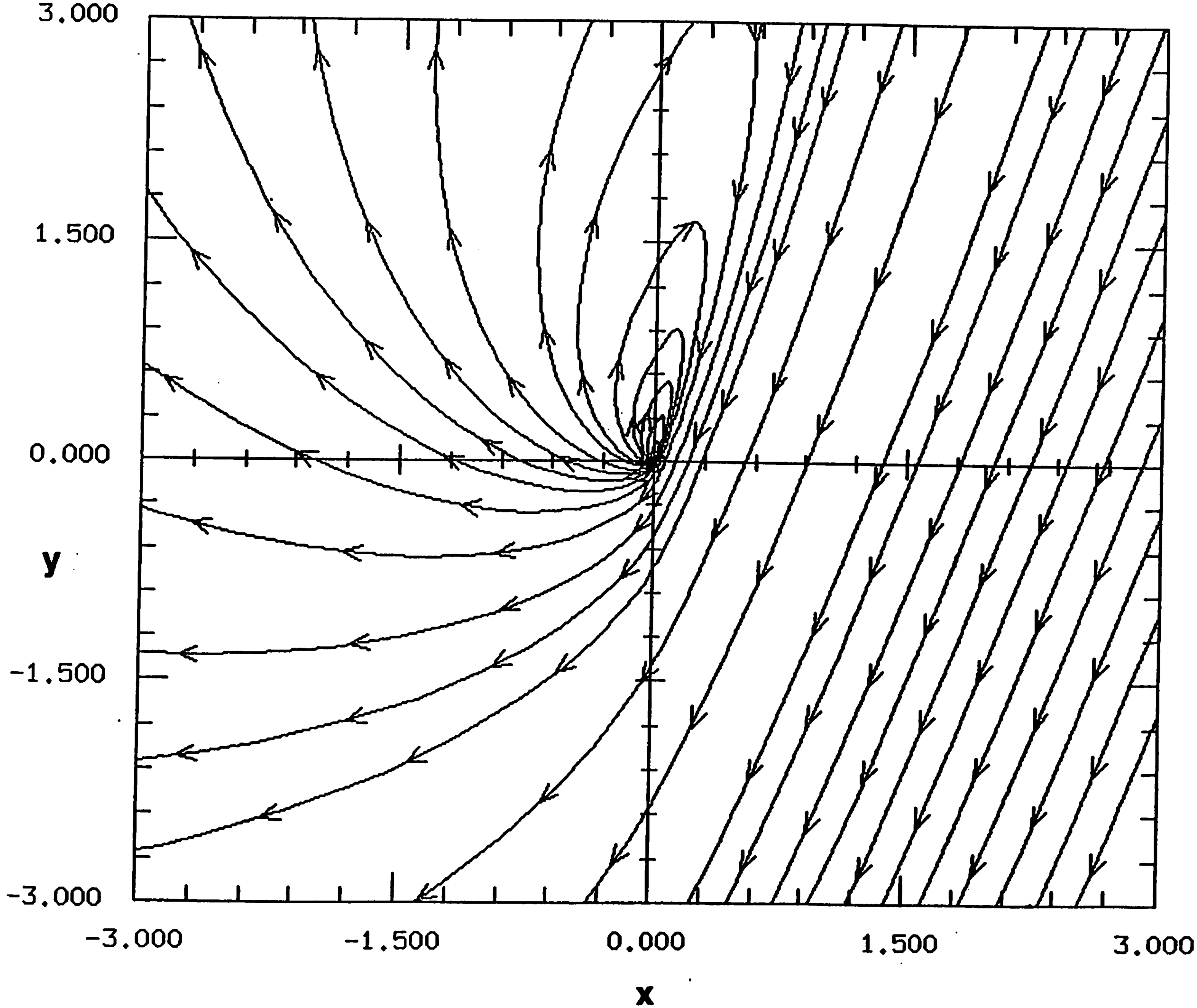
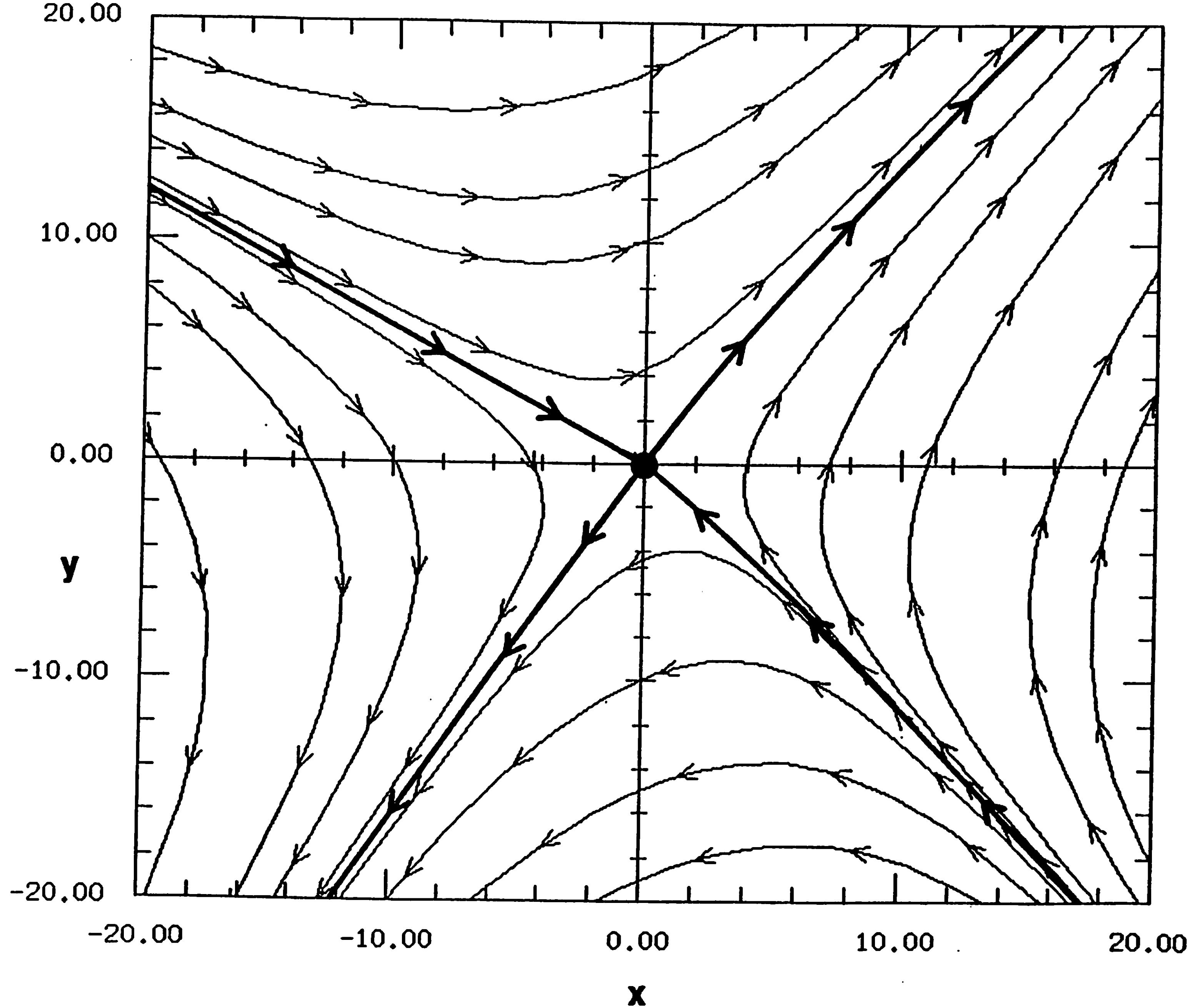


TABLE 1
(d)



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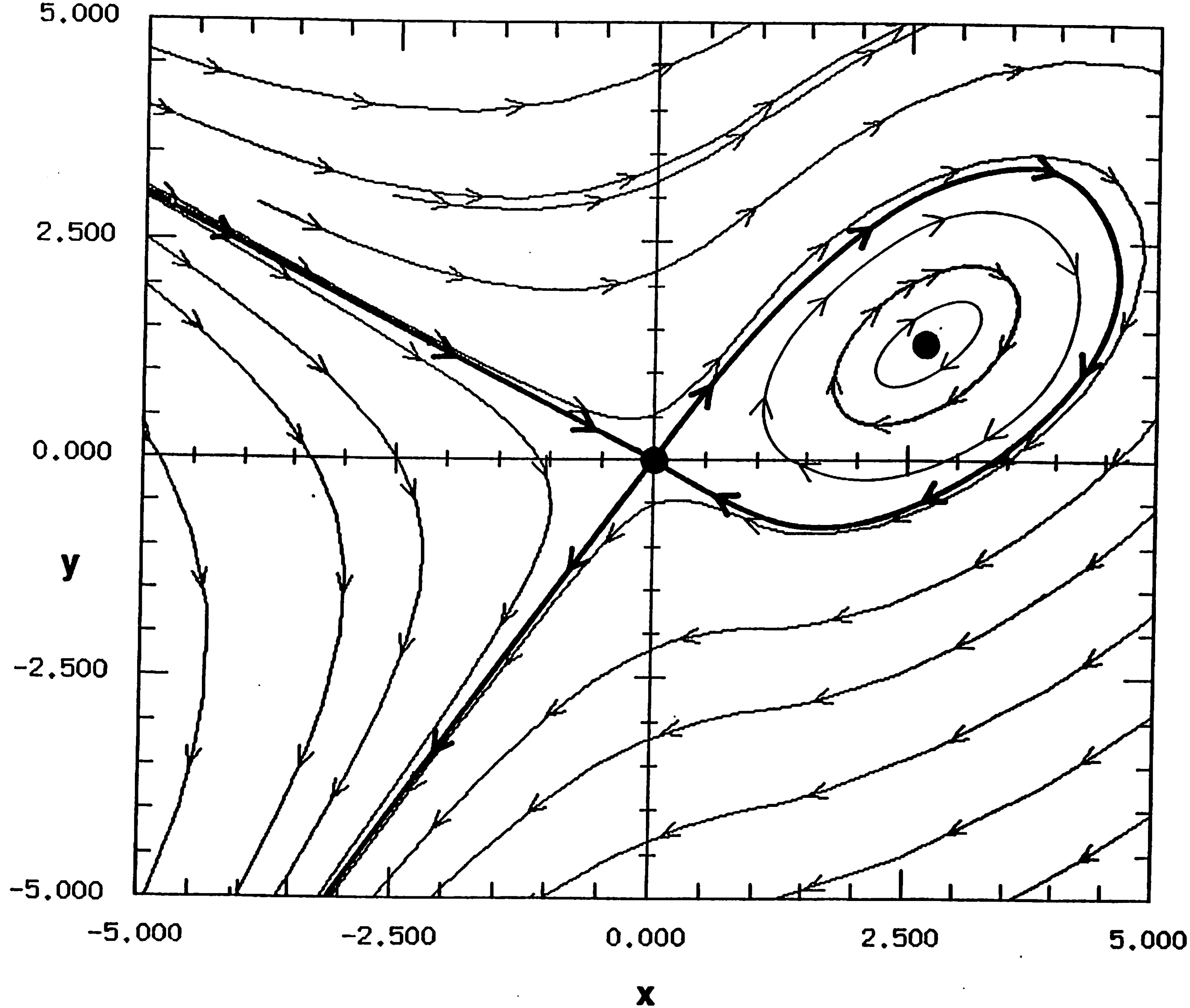

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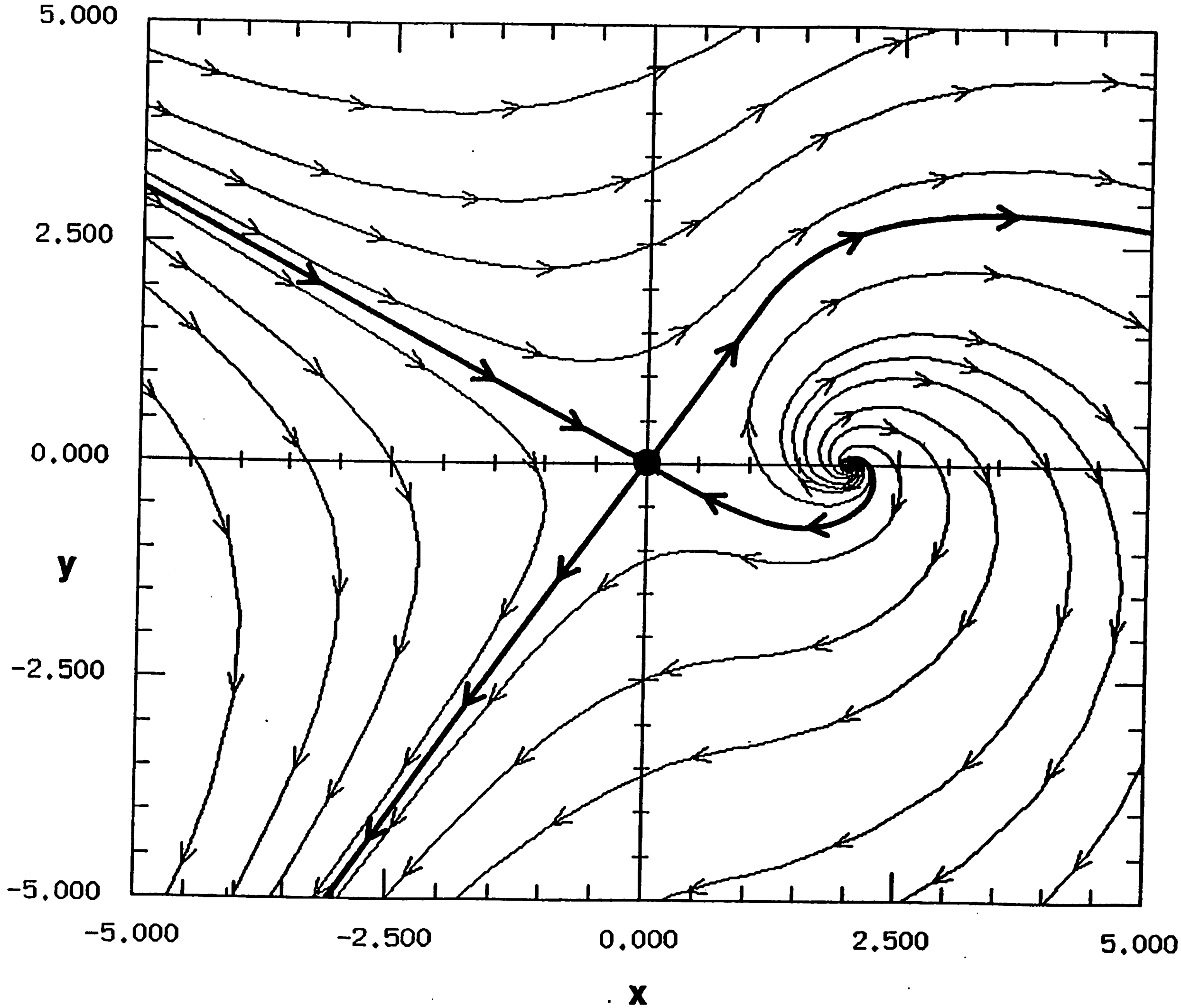
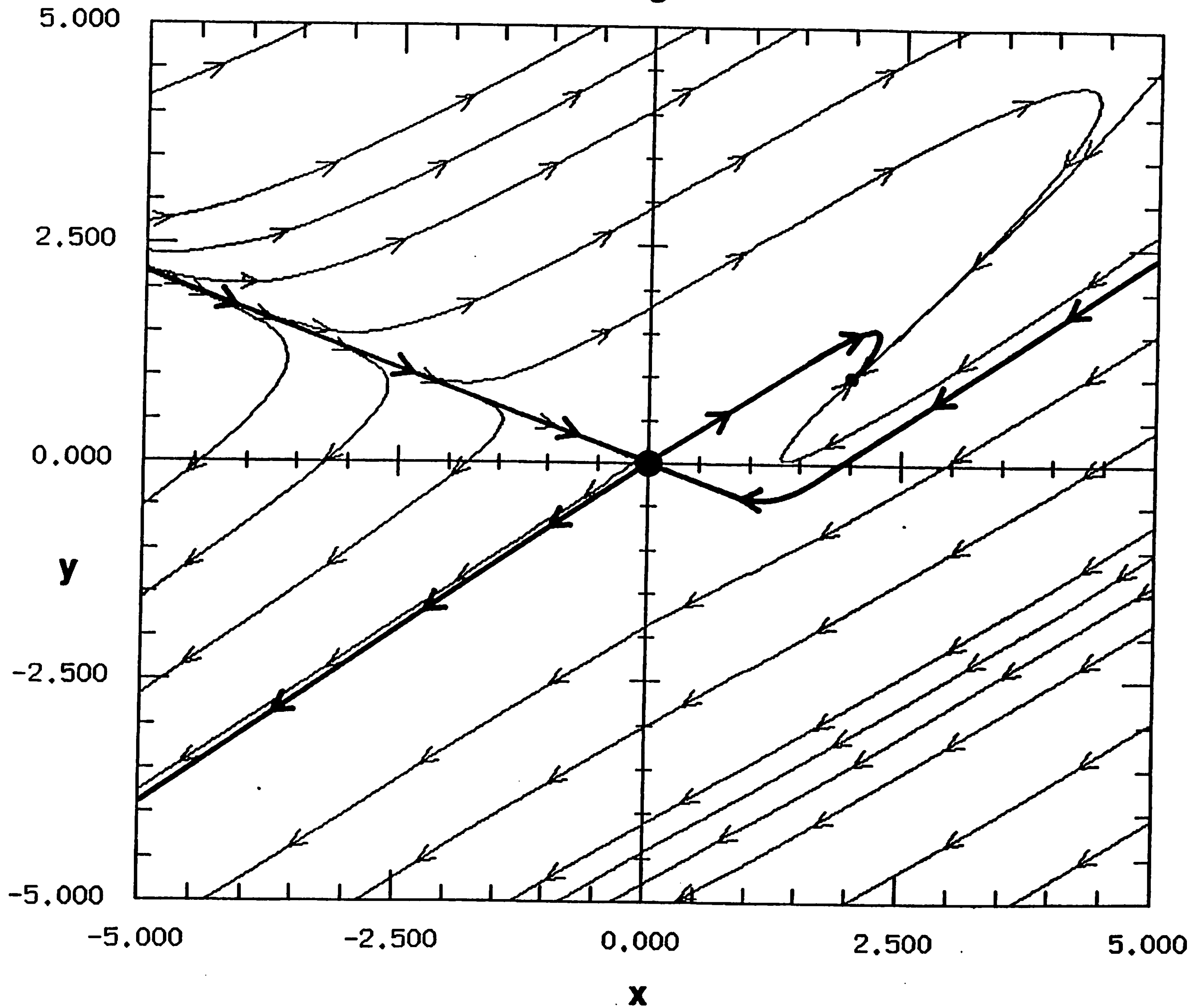


TABLE 1
(g)



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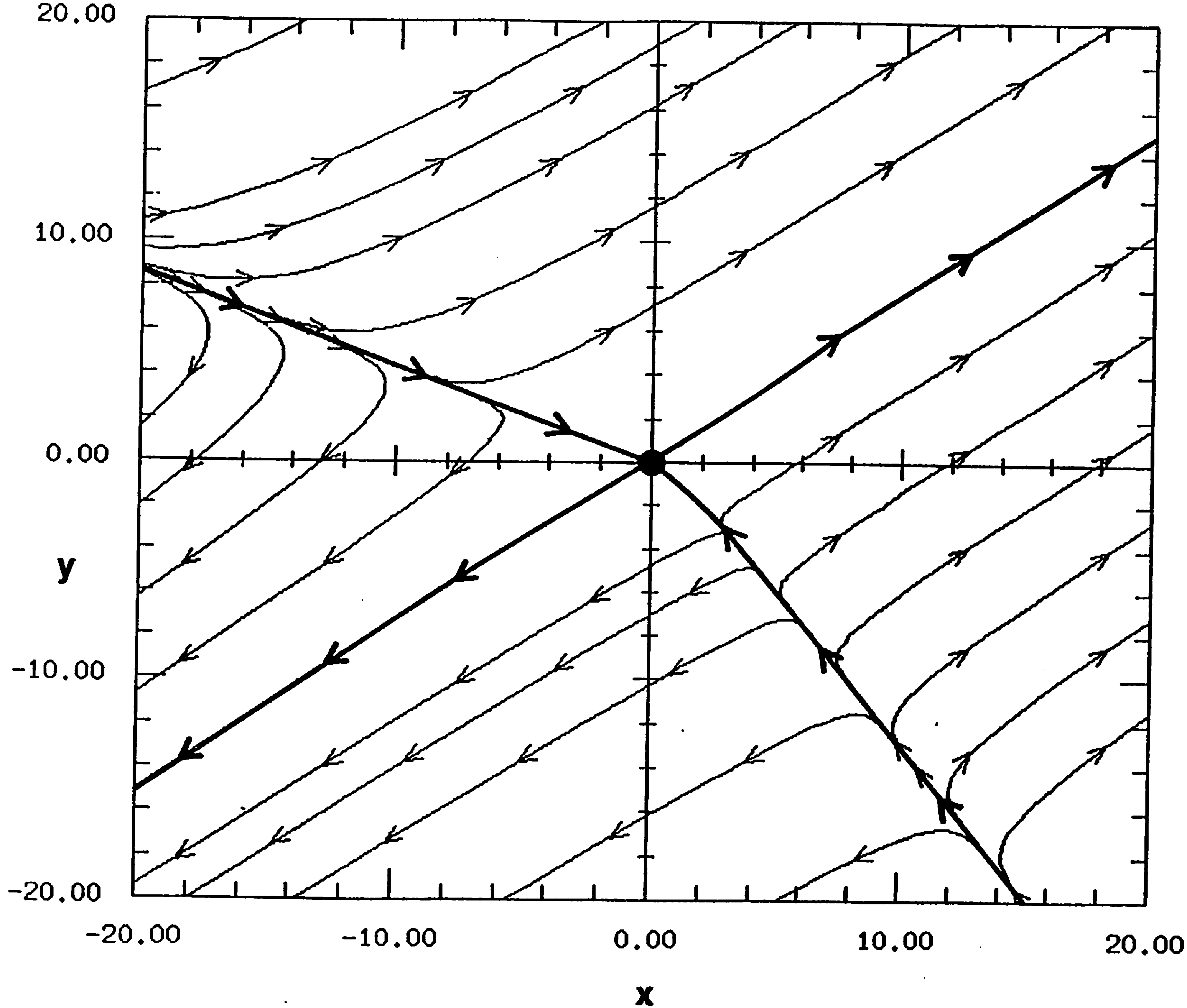

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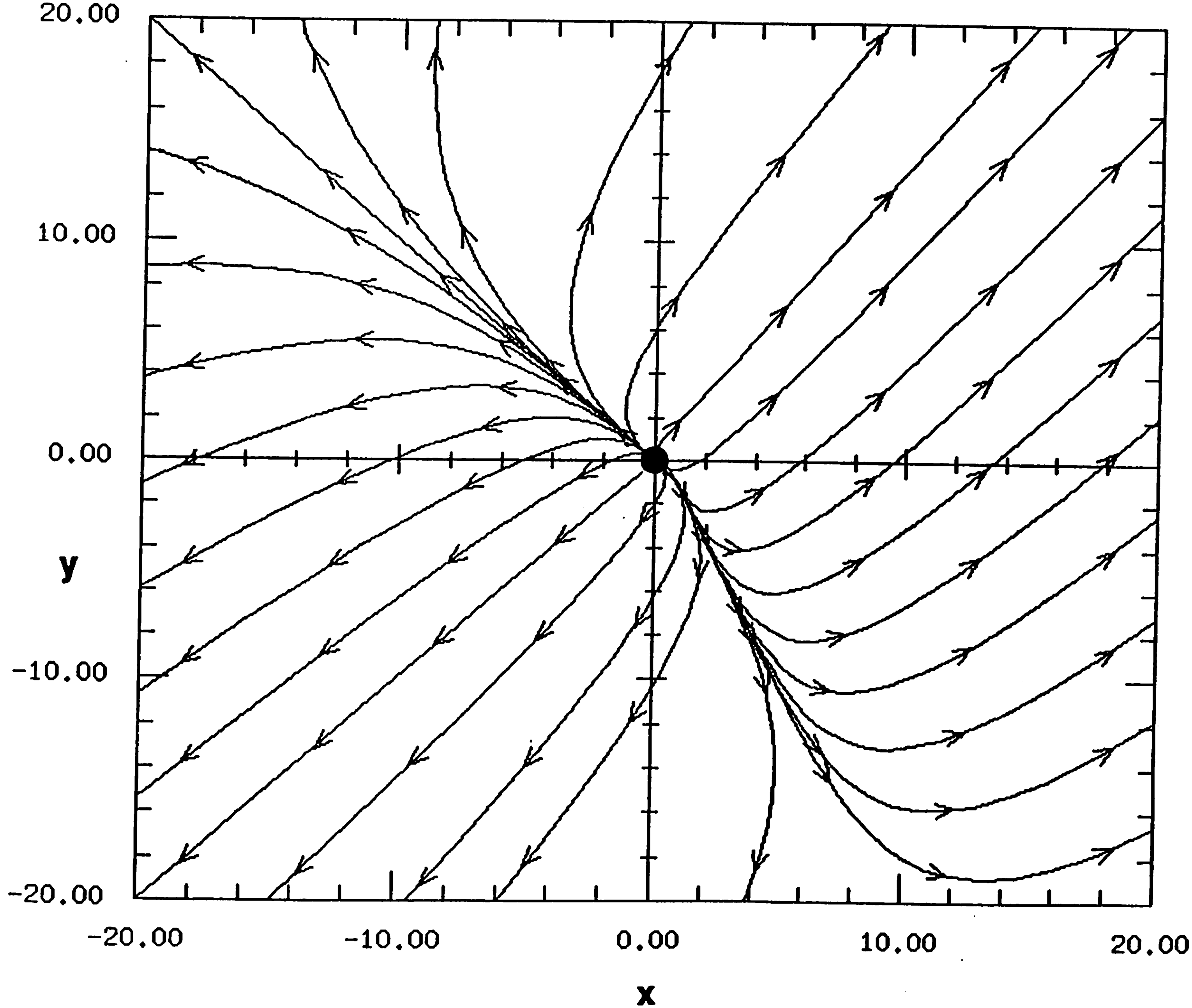

TABLE 1
(j)

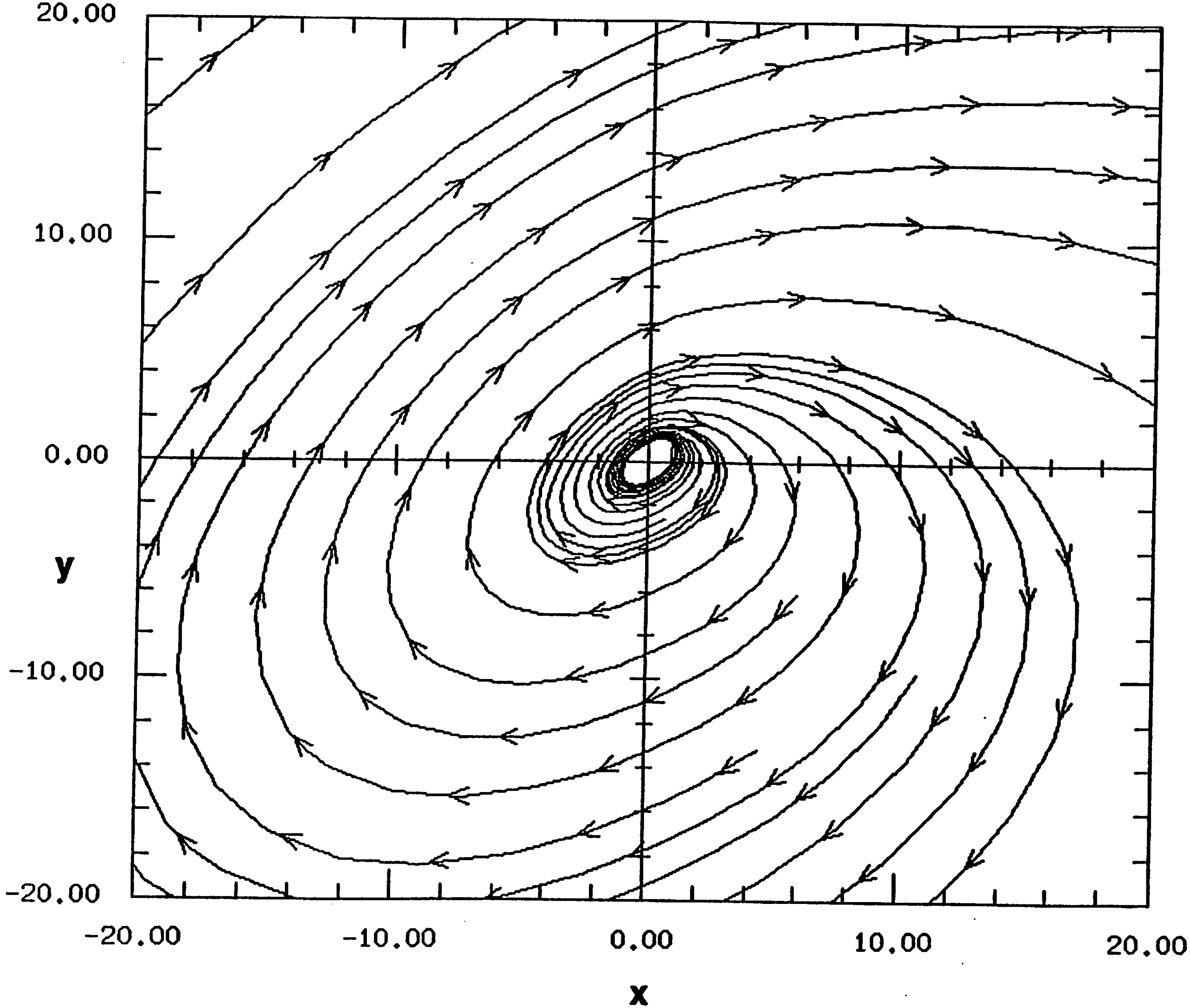

TABLE 1
(k)

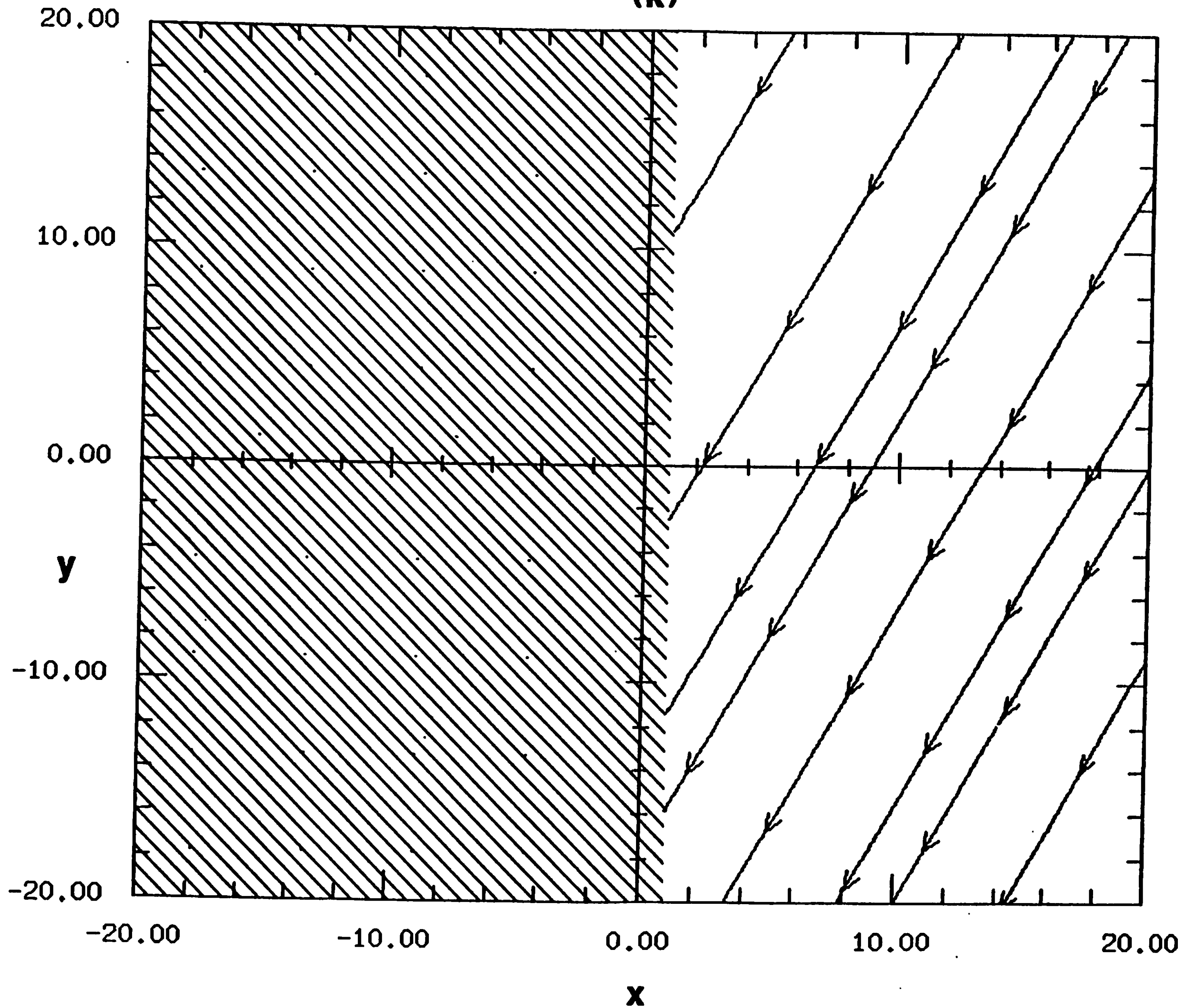

TABLE 1
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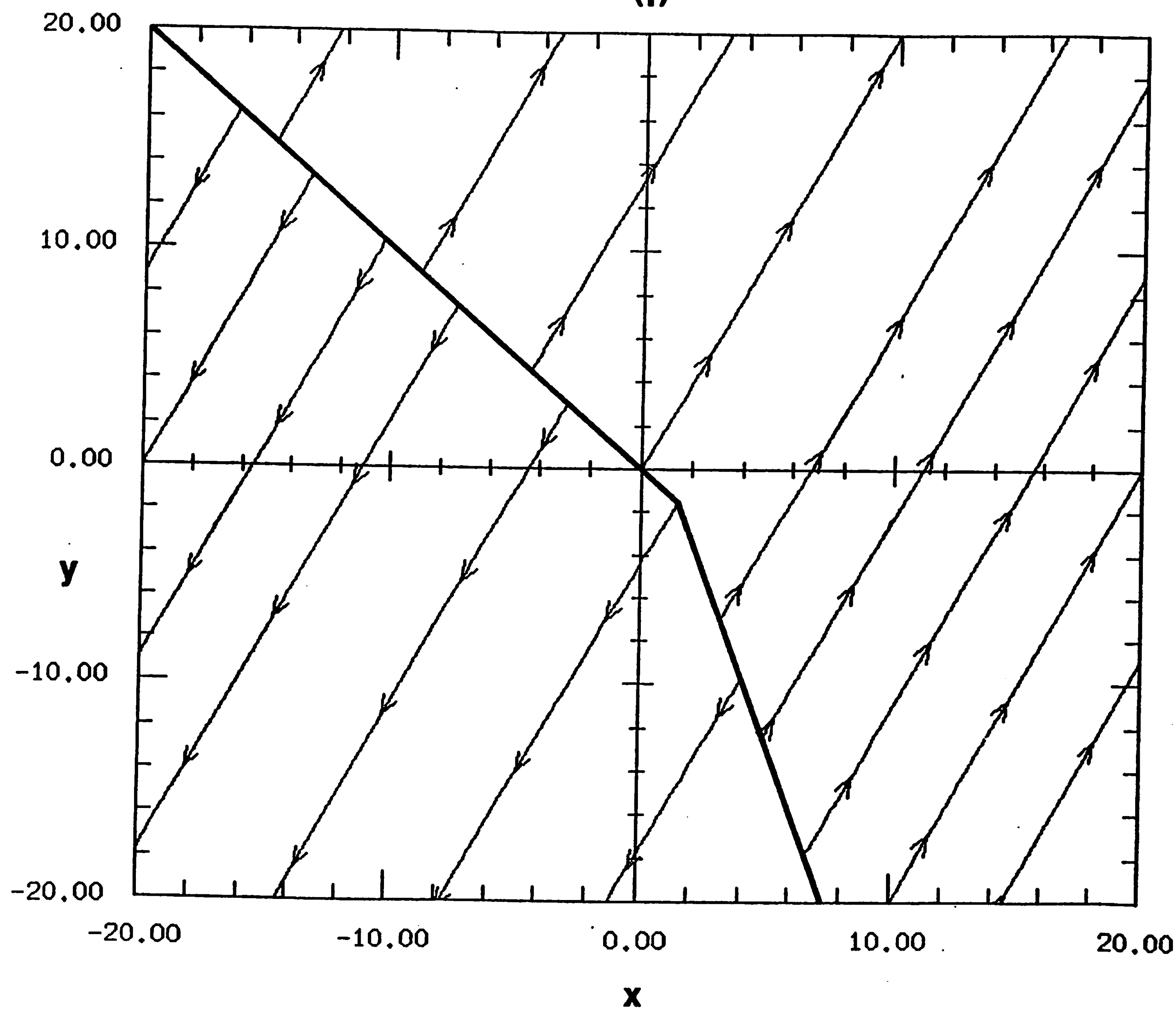

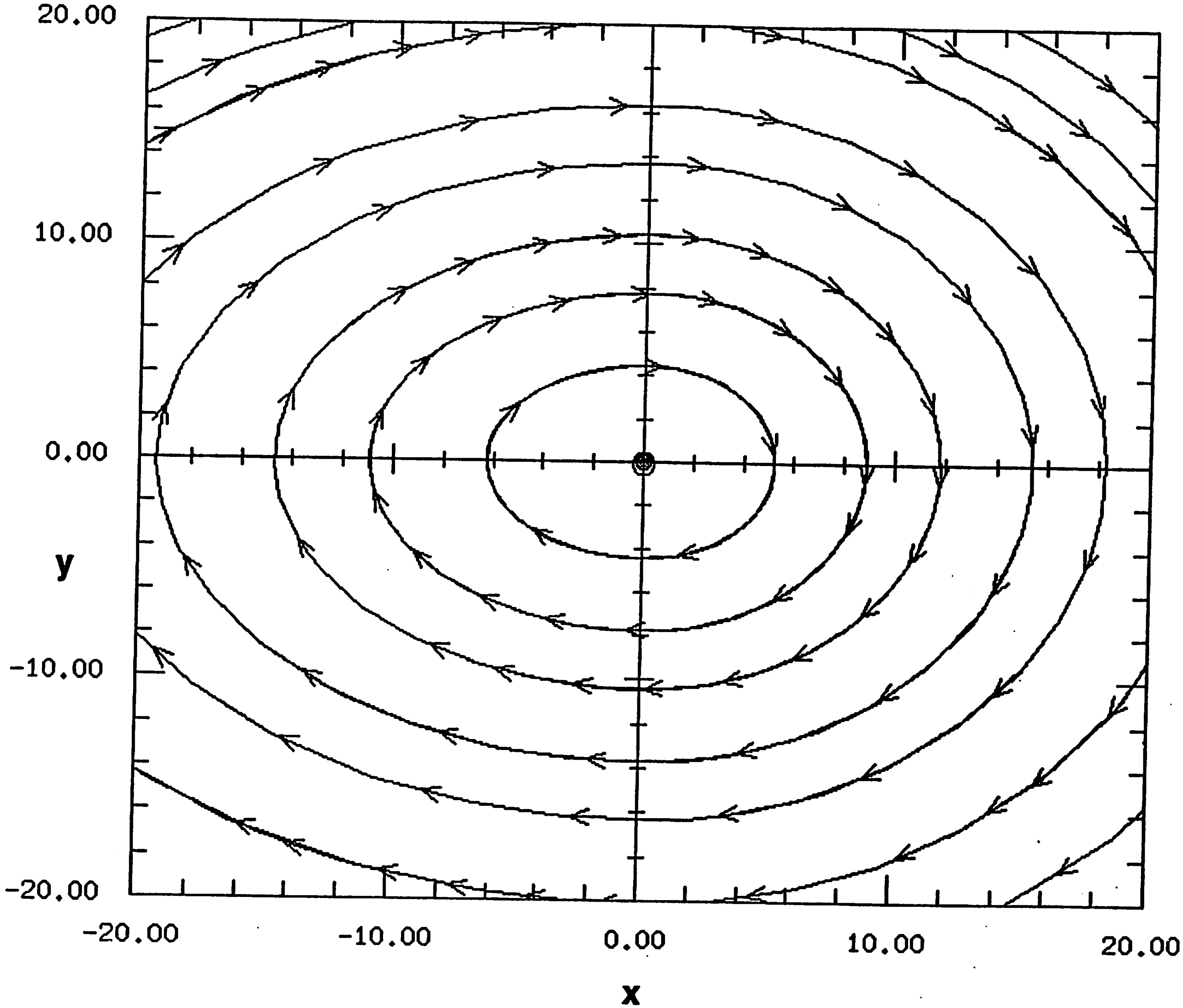

FIG. 1

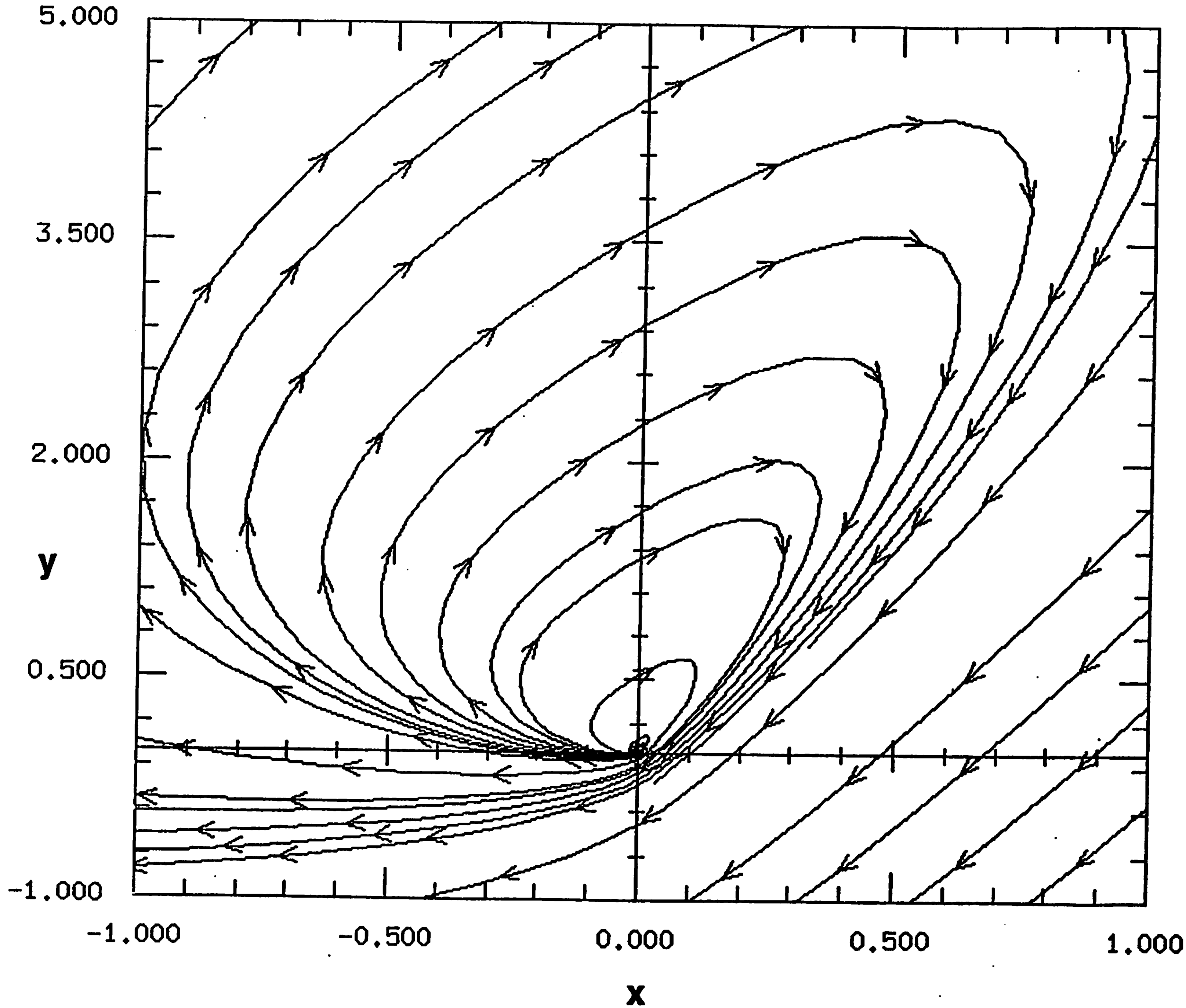

FIG. 2

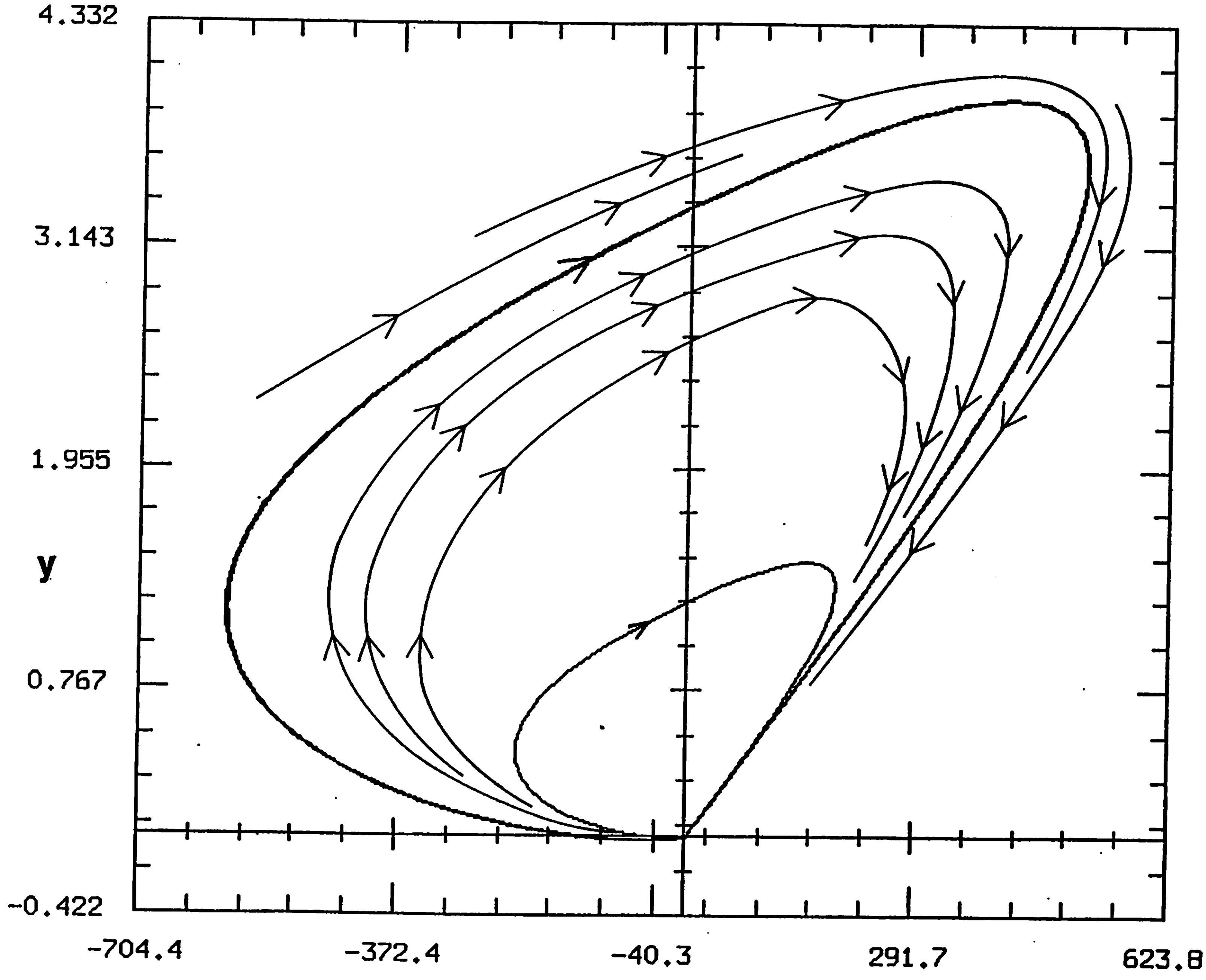

FIG. 3

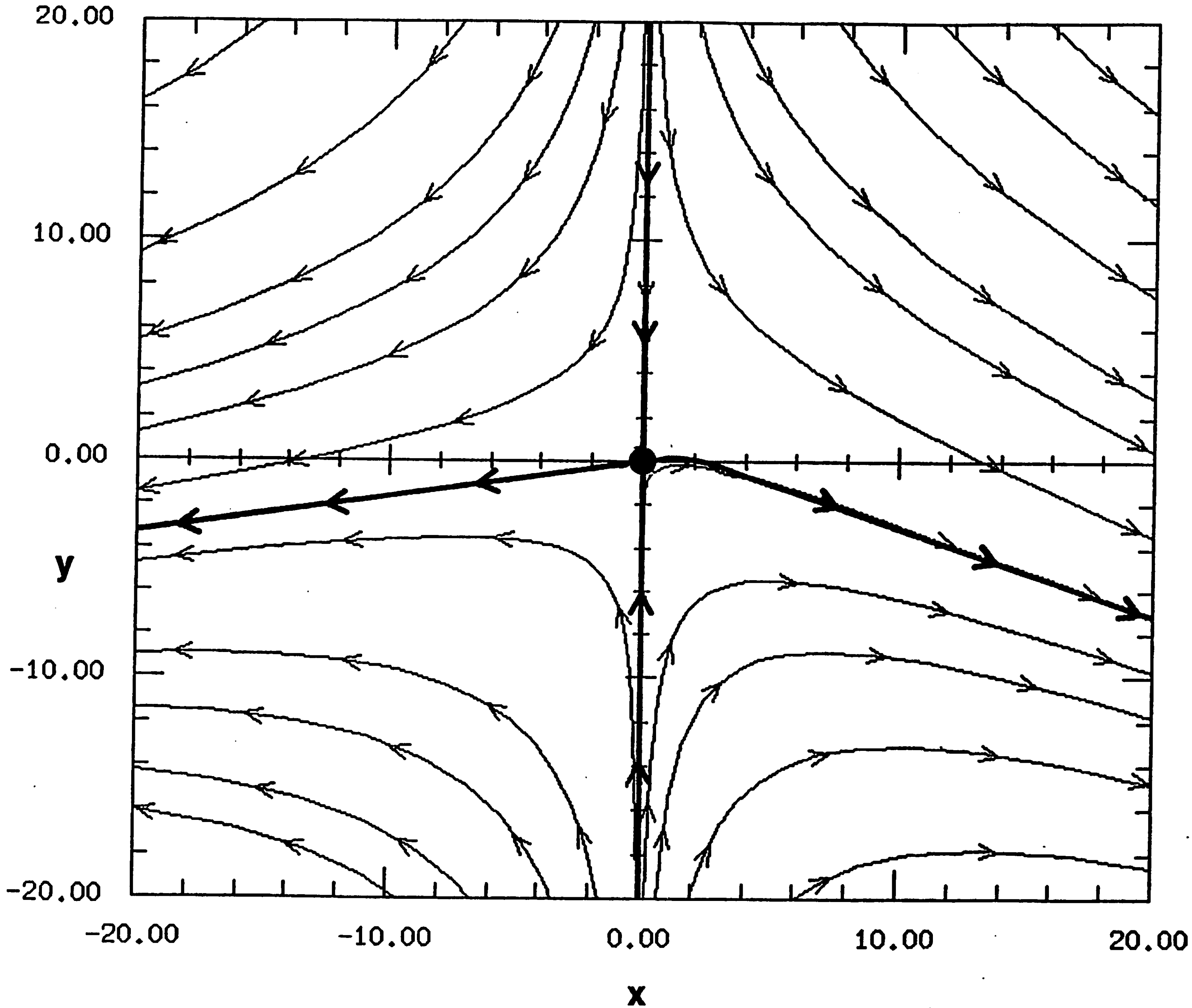

FIG. 4

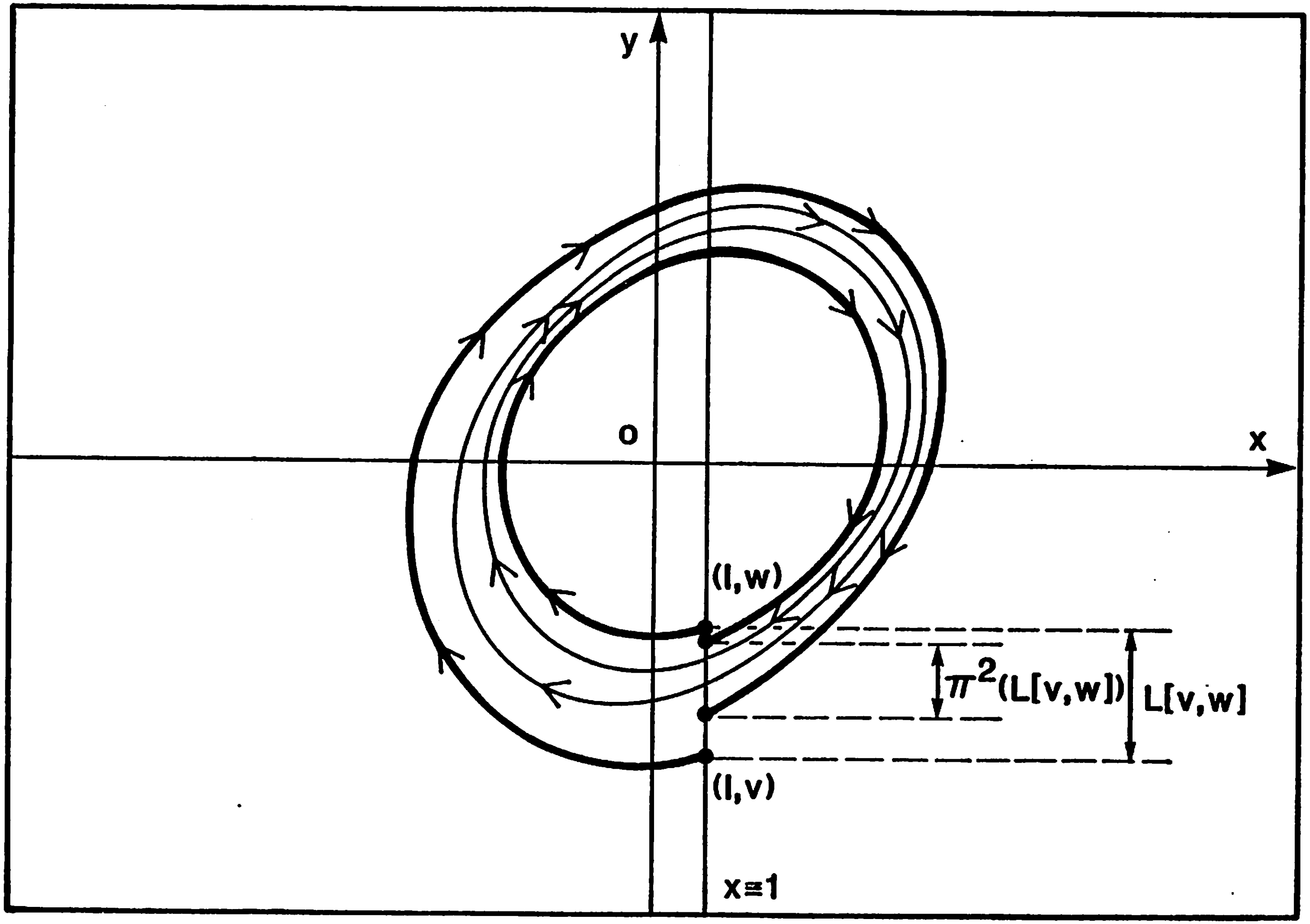

FIG. 5

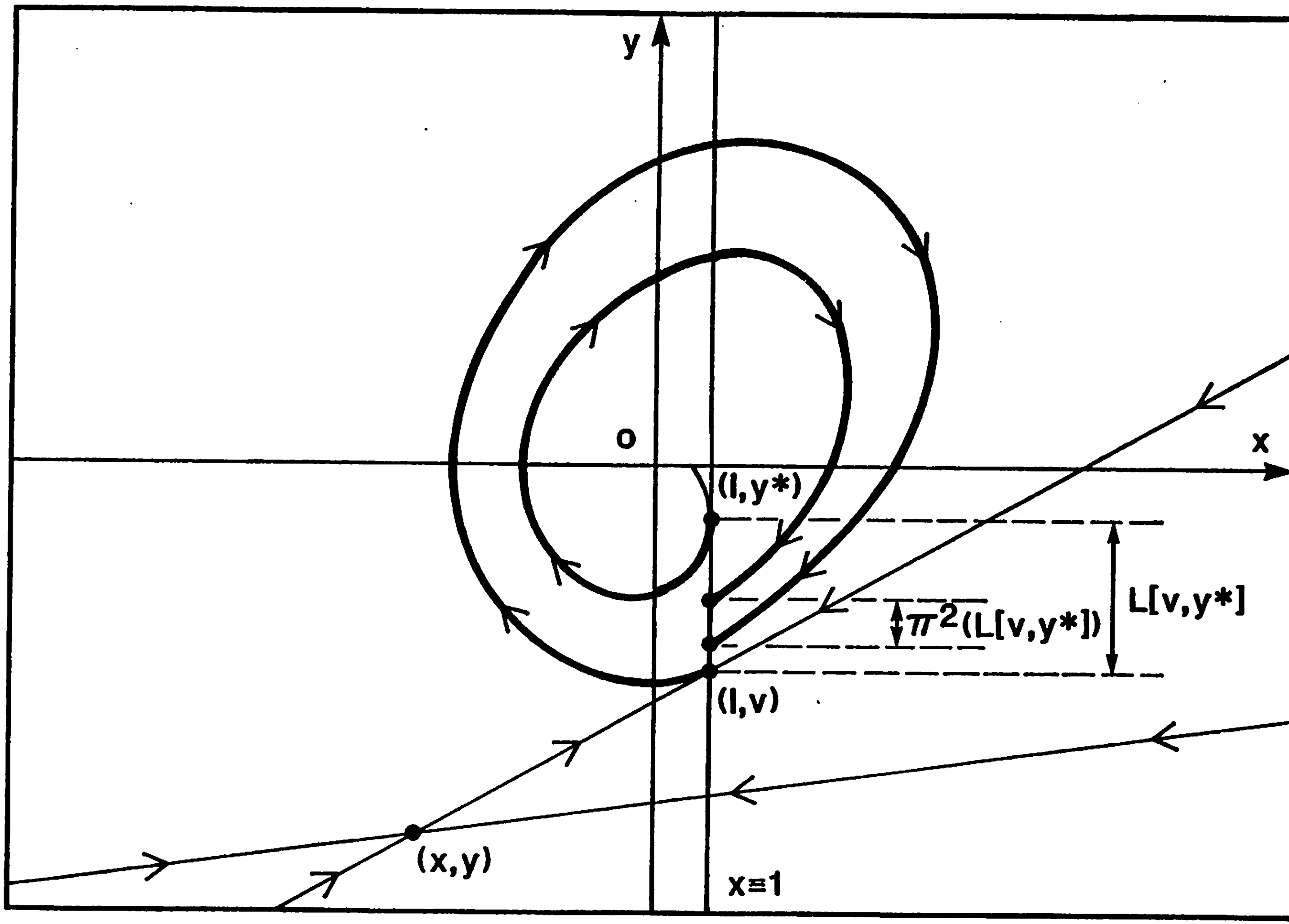

FIG. 6

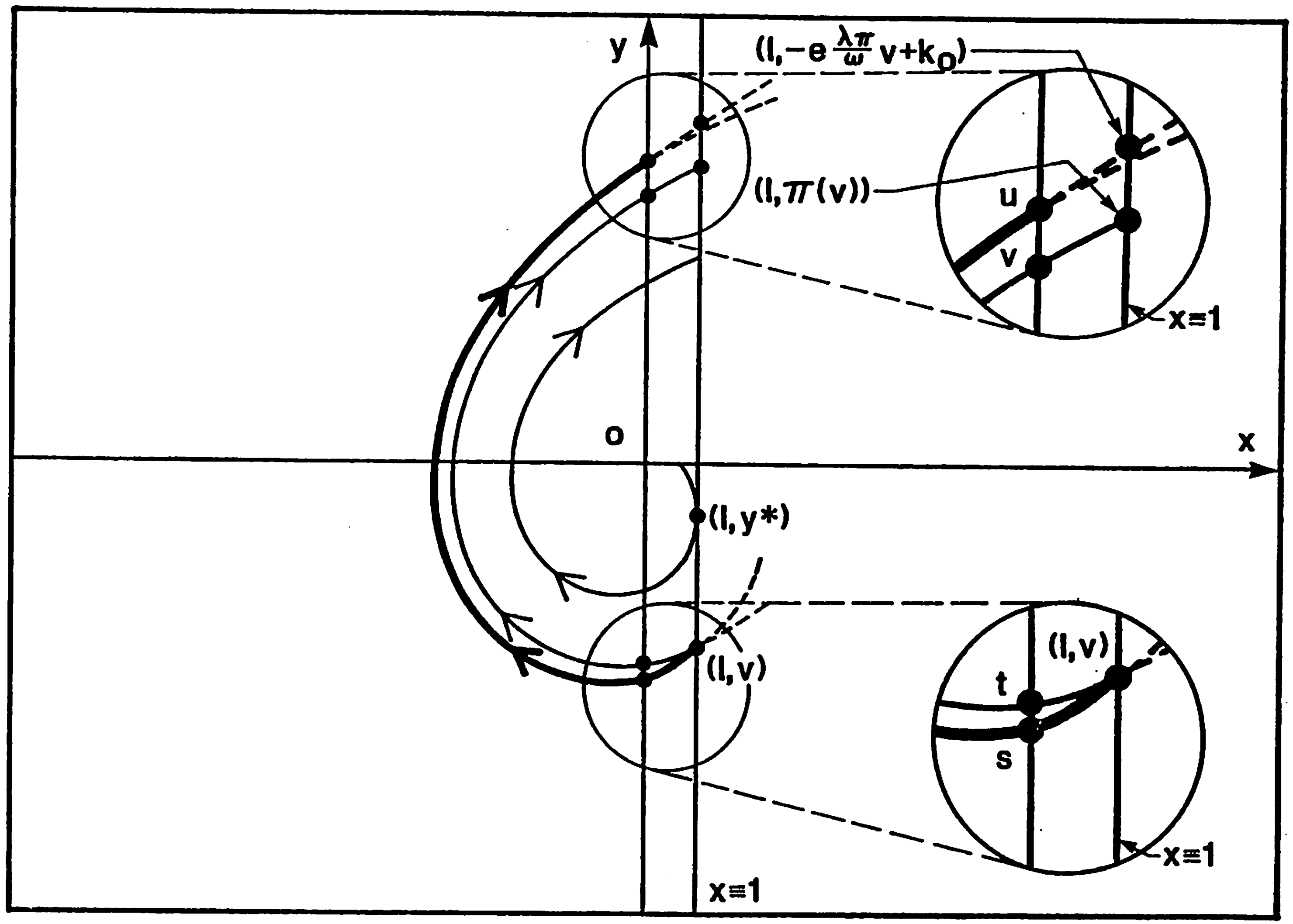

FIG. 7

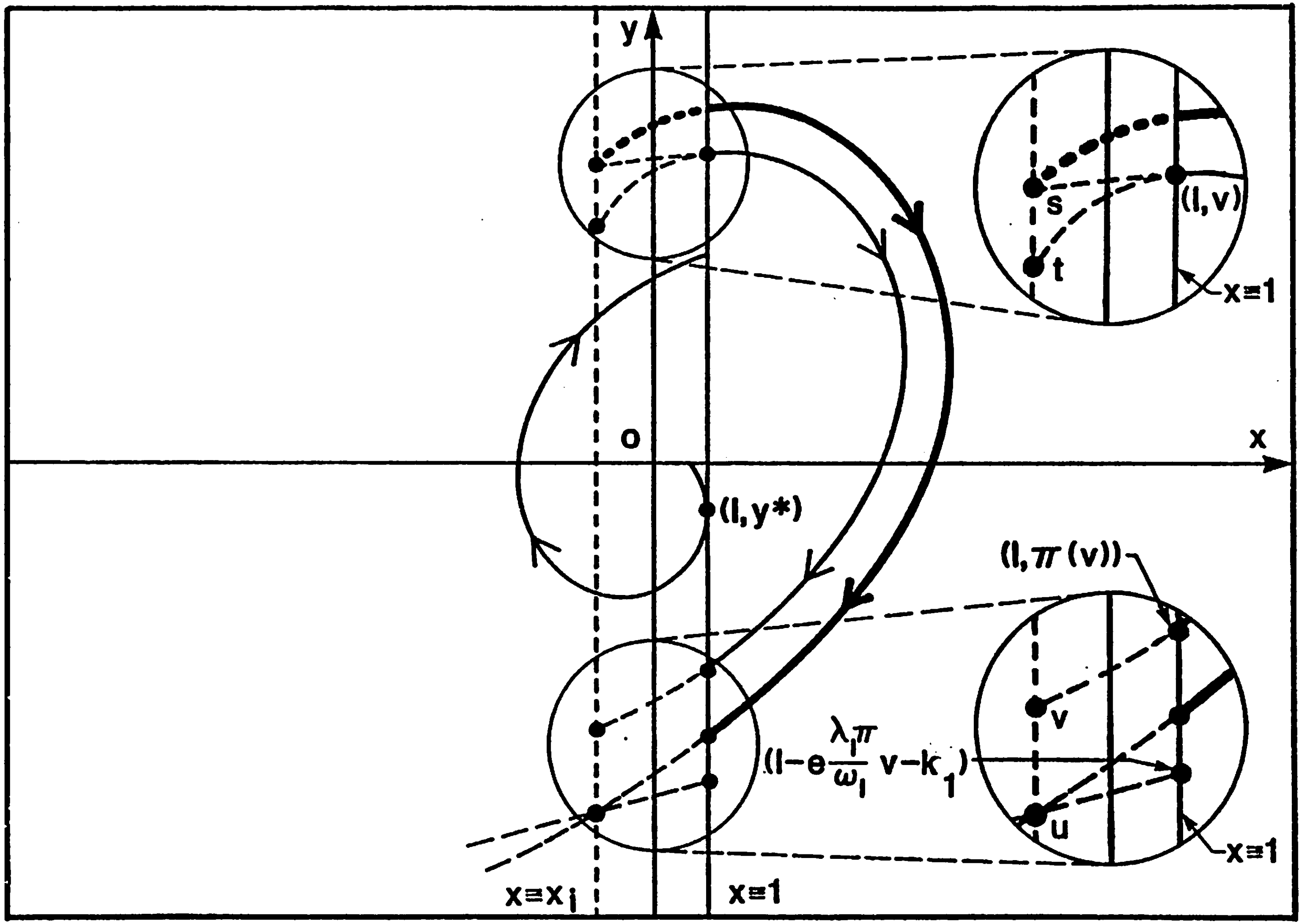

FIG. 8

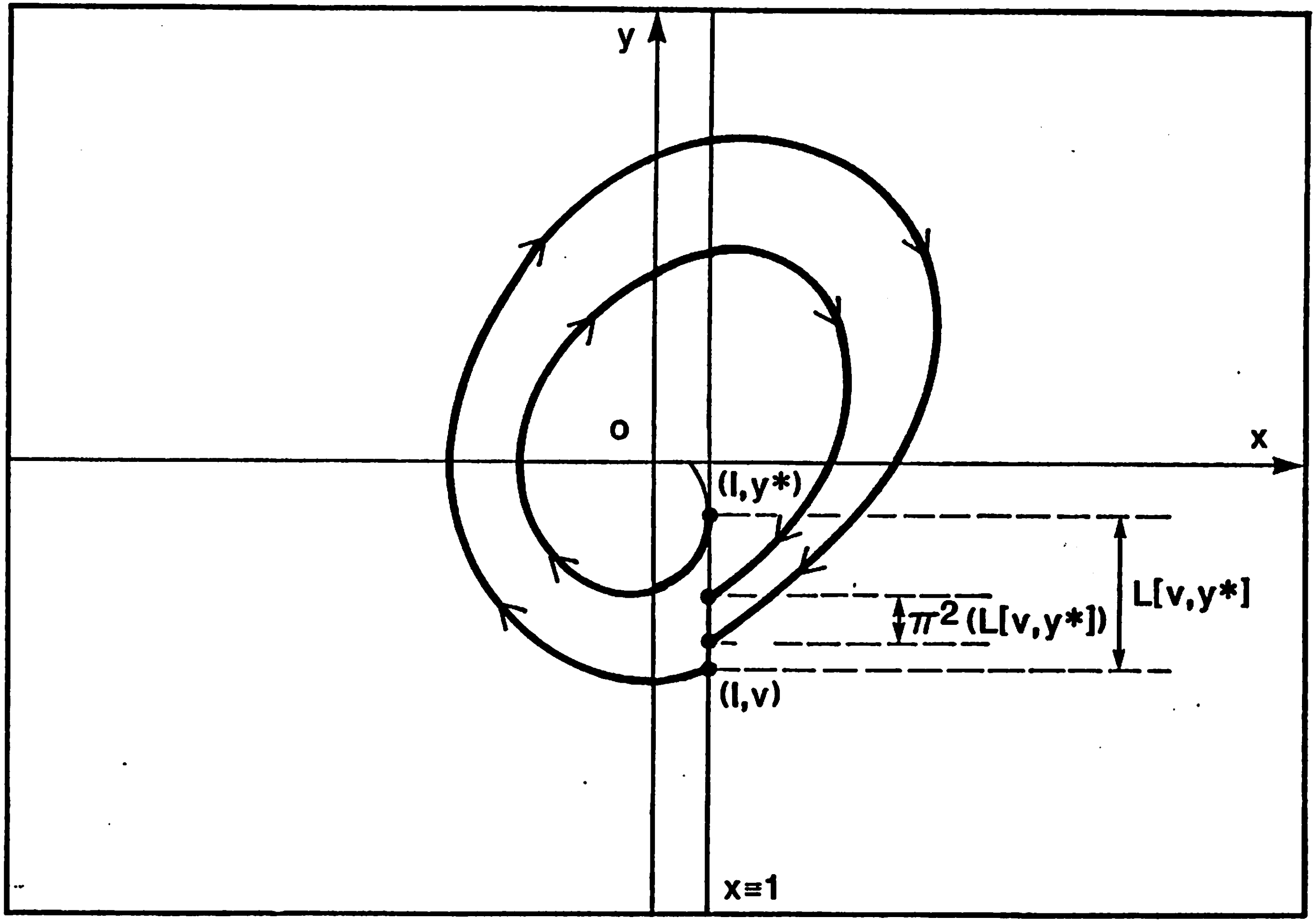

FIG. 9

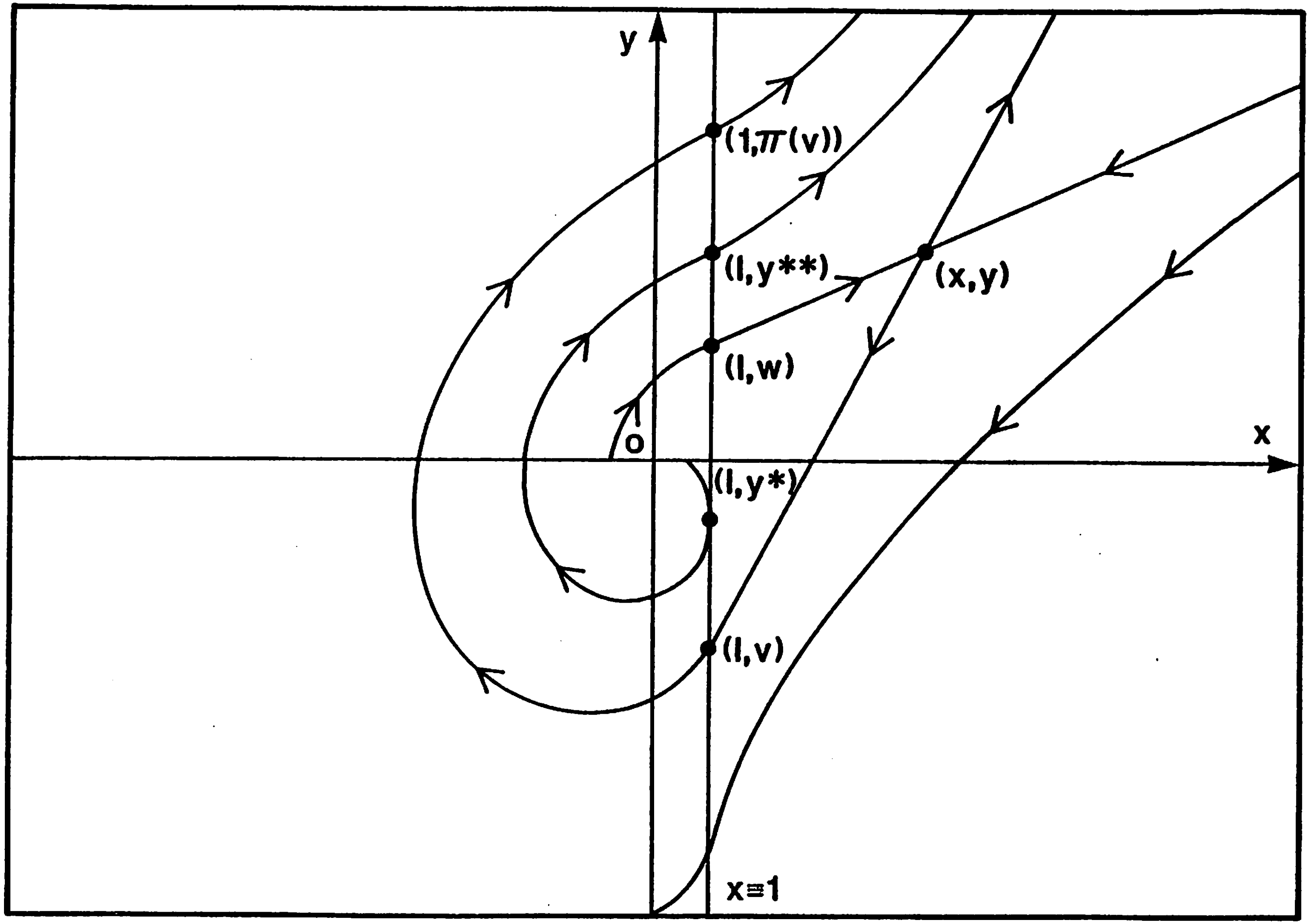

FIG. 10

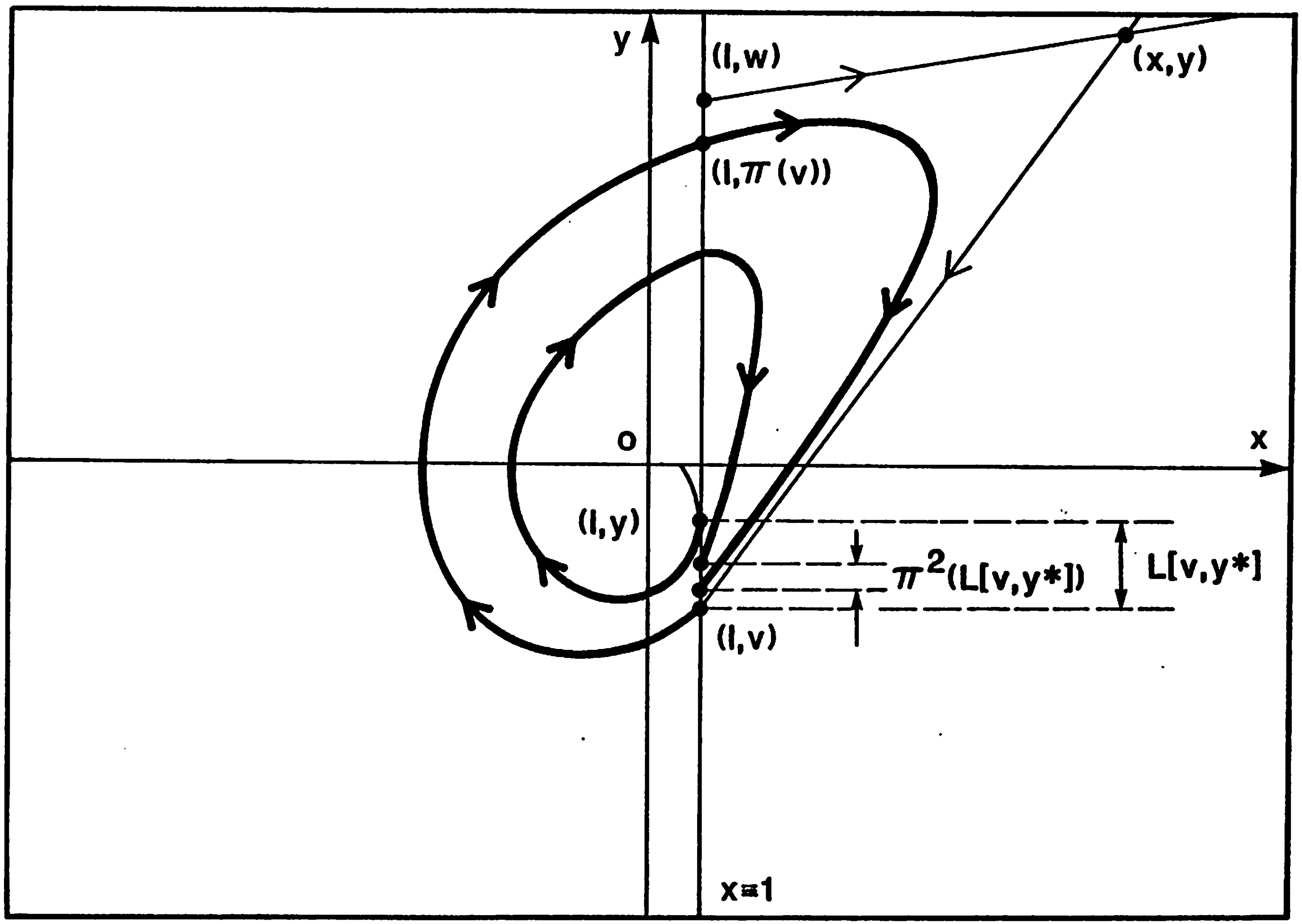

FIG. 11